Contents

CHAPTER 1

Introduction

If $u : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 then the Jacobian determinant of u is classically defined as the determinant of the $n \times n$ matrix which represents ∇u in a system of orthonormal coordinates with a fixed (right or left) orientation. Using the language of differential forms if $x_1, \ldots, x_n$ are the coordinate functions relative to an orthonormal system compatible with one given orientation of $\mathbb{R}^n$, the Jacobian determinant is the only function f which satisfies the relation

$$du_1 \wedge du_2 \wedge \ldots \wedge du_n = f dx_1 \wedge \ldots \wedge dx_n.$$

How can the notion of Jacobian determinant be extended to less regular maps? Since this notion involves sums of products of n partial derivatives it is straightforward that it can be extended to any function $u \in W^{1,n}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$. Of course in this case the Jacobian determinant has lost its pointwise meaning, being merely an L^1_{loc} function, but many useful integral formulas satisfied by the classical Jacobian are still valid in the appropriate formulations.

However this extension is not entirely satisfactory and one would like to define a suitable notion of Jacobian determinant for less regular functions. The relevance of this problem is underlined, just to give an example among many, by the issue of constructing an existence theory for minimizers of energies like

$$\int_\Omega \|\nabla u\|^2$$

among diffeomorphisms u subject to the incompressibility constraint $\det \nabla u = 1$.

The classical framework of Geometric Measure Theory provides a notion of $\det \nabla u$ for very irregular functions, which works as follows (see section 1 of chapter 5). Let us suppose that we have a map $u \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$: then classical results in GMT imply that for a.e. $x \in \mathbb{R}^n$ the graph of u is well approximated in $(x, u(x))$ by the graph of a linear function $L_x : \mathbb{R}^n \to \mathbb{R}^n$ (see Definition 5.1.4 for the precise meaning of "well approximated" in this framework). Hence we could define $\det(\nabla u(x)) := \det(L_x)$. This provides us with a measurable function

$\det \nabla u$ defined almost everywhere. However the object so defined is difficult to handle and many useful formulas and relations involving the classical Jacobian have no counterparts in this framework.

A second possibility for extending the notion of Jacobian determinants starts from the following observation:

$$d(u_1 du_2 \wedge \ldots \wedge du_n) = du_1 \wedge \ldots \wedge du_n$$

$$= (-1)^{k-1} d\left(u_k du_1 \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_n\right)$$

when $u \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. These identities can be regarded as the fact that the Jacobian determinant is the divergence of certain vector fields which are suitable nonlinear functions of u and ∇u. For example if we denote by $\Lambda_n(\nabla u)$ the matrix of cofactors of ∇u then

$$\det \nabla u = \frac{1}{n} \operatorname{div}(u \cdot \Lambda_n(\nabla u)).$$

We now remark that to define $u_k du_1 \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_n$ (or to define the vector $u \cdot \Lambda_n(\nabla u)$) it is sufficient that $\nabla u \in L^q_{loc}$ and $u \in L^p_{loc}$, with $1/p + (n-1)/q = 1$. Hence for such maps we can define the weak Jacobian determinant as the distributional exterior derivative of the form

$$\frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} u_k du_1 \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_n \tag{1}$$

or as the distributional divergence of $u \cdot \Lambda_n(\nabla u)/n$. Standard regularization arguments show that this definition coincides with the one given above when $u \in W^{1,n}$.

Starting from this remark, Ball in [**9**] constructed a theory of existence of minimizers for a wide range of functionals of nonlinear elasticity. Moreover many authors started to study properties of functions whose weak Jacobians satisfy appropriate conditions, connecting them with classical topics like topological degree and other branches of Geometric Measure Theory (see [**3**], [**10**], [**20**], [**22**], [**23**], [**24**], [**26**], [**27**], [**28**], [**32**], [**34**], [**35**], [**36**], [**37**], [**38**], [**39**], [**41**]).

Recently, in connection with Ginzburg-Landau models, some authors have begun to study properties of maps $u : \mathbb{R}^N \to \mathbb{R}^n$ with $n < N$ for which the distributional n–form

$$d(u_1 du_2 \wedge \ldots \wedge du_n)$$

can be defined (see [**3**], [**26**], [**27**], [**28**]). We notice that here the word "distributional n–form" means the following. First of all, given a system of orthonormal coordinates, we write

$$\omega = \sum_I T_I dx_I$$

where I runs among all sets of $n-1$ indices $\{i_1, \ldots, i_{n-1}\} \subset \{1, \ldots, N\}$, $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i-1}$ and T_I are L^1 functions. Thus we could formally write

$$d\omega = \sum_{i=1}^{m} \sum_I \partial_i T_I dx_i \wedge dx_I$$

where $\partial_i T_I$ is the partial derivative of T_I with respect to x_i in the sense of distributions. Moreover, collecting the terms of the form $dx_i \wedge dx_I$ modulo the usual equivalence relations we finally have the formal expression

$$d\omega = \sum_J T_J dx_J \tag{2}$$

where J runs among all the subset of $\{1, \ldots, N\}$ with cardinality n and every T_J is a distribution.

In [**27**] Jerrard and Soner have defined the space of functions of bounded higher variation, namely the space of functions for which such a distributional n–form is a measure (more precisely, the space of functions for which when substituting $\omega = \sum_k u_k dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n$ in (2) the T_J are all Radon measures). For the sake of simplicity in the following we will call the form $d\omega$ (or any similar object) a *weak Jacobian*. In the same paper the authors have proved many results which underline how the weak Jacobian contains a lot of informations about the level sets of u.

In this thesis we have explored the notion of weak Jacobian from several points of view. In the third chapter we propose a different way of approaching Jerrard's and Soner's BnV functions, which uses currents instead of distributional differential forms. We use this new formulation to give different simplified proofs of some theorems of [**27**] and to study in more detail the structure of the weak Jacobians. In the fourth chapter we explore the relations between different formulations of the classical chain rule and coarea formula in this new context, underlining relationships among them. In the fifth chapter we study the notion of weak Jacobians in relation with a problem in the calculus of variations and we prove the equivalence of certain class of maps, exploring the connection of them with the concept of Cartesian Currents. In the sixth chapter, still focusing on the relations with certain problems in the Calculus of variations, we show the failure of closedness of some conditions in the case of critical exponents. As a byproduct we obtain a highly non–trivial example in which the continuity of weak Jacobians fails.

In the rest of this introduction we will give more details about the results contained in the various chapters.

In the third chapter we propose a different definition of functions of higher bounded variation using the theory of currents. We start from the fact that, given a function $u : \mathbb{R}^m \to \mathbb{R}^n$ (which is sufficiently regular to allow the definition of weak Jacobian), we can think of the form $u_1 du_2 \wedge du_n$ as a current via the natural action

$$T(dw_1 \wedge \ldots \wedge dw_{m-n+1}) =$$

$$\int_{\mathbb{R}^m} u_1 \det(\nabla u_2, \ldots , \nabla u_n, \nabla w_1, \ldots , \nabla w_{m-n+1}) d\mathcal{L}^m.$$

Of course, by linearity, a similar action can be written for the form $\sum_k u_k du_1 \wedge \ldots \wedge \widehat{du_k} \wedge \ldots \wedge du_n$. Thus we can define the weak Jacobian $[Ju]$ as the boundary of the current T induced by the action of form (1) and the space BnV can be identified with those u such that $[Ju]$ is a normal current.

This new point of view allows us to use many powerful tools of Geometric Measure Theory. Moreover instead of working in the classical framework we have preferred to use the "metric currents theory" introduced by Ambrosio and Kirchheim in [**8**]. Indeed we think that this theory, besides the fact that it can be extended to more general settings, also gives a new insight in the Euclidean case and provides many simplifications in proofs and notations.

The first, second and third sections of the second chapter are then devoted to a brief introduction to Ambrosio and Kirchheim's theory. In the fourth section we propose a general decomposition for flat currents in different parts which resembles the decomposition of derivatives of BV functions into absolutely continuous, Cantor and jump parts.

The general theory of Ambrosio and Kirchheim allows us to give an easy proof of the fact that the "lower dimensional part" of this decomposition is rectifiable. This provides an extension to flat currents of a rectifiability criterion which in the case of normal currents has been proved independently by White [**42**] and Ambrosio and Kirchheim. Their proof uses a new BV estimate on the slices of currents which was first proved by Jerrard and Soner in [**27**] in the case of currents which can be realized as weak Jacobians of functions (though the result is not stated in these terms in their work).

The general decomposition for currents induces an analogous decomposition on the weak Jacobian of BnV functions. In the third chapter we study the structure of the various parts (and in particular we generalize a Theorem of Müller proved in [**34**]). Moreover, in analogy with the space of special functions of bounded variation, we define the space SBnV and we prove a Closure Theorem for it which is a generalization of the theorem given by Ambrosio and De Giorgi for SBV (see [**6**]).

The fourth chapter contains partial answers to a conjecture of by Jerrard and Soner in [**27**] together with a study of the relations between the coarea formula and chain rule for weak Jacobians.

If we define $u^y = (u-y)/|u-y|$ for every $y \in \mathbb{R}^n$ such that $\mathcal{L}^m(u^{-1}(y)) = 0$, then we have the following

THEOREM 1.0.1. (See sect. 4 of [**27**]) *If $u \in W^{1,n-1} \cap L^\infty$ then $u^y \in W^{1,n-1} \cap L^\infty$ for a.e. $y \in \mathbb{R}^n$ and*

$$\langle [Ju], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle [Ju^y], \omega \rangle dy \tag{3}$$

for every smooth $m-n$ differential form ω (with ω_n we denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$). If $u \in W^{1,n-1} \cap L^\infty$ and $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then

$$\langle [J(F(u))], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(y) \, \langle [Ju^y], \omega \rangle dy. \tag{4}$$

It is natural to think of (4) as an extension of the chain rule for Jacobians of smooth maps, hence we will refer to it as the *weak chain rule.* Of course when u is smooth we have the much stronger relation $[J(F(u))] = \det \nabla F(u)[Ju]$. For general u we can not even give a meaning to the right hand side of this equation, because $[Ju]$ is only a current. However when $[Ju]$ is a current of finite mass we might hope that there exists a pointwise representative $\overline{u}$ of u which is Borel measurable and satisfies $[J(F(u))] = \det \nabla F(\overline{u})[Ju]$. We notice that the right hand side is now well defined because $\det \nabla F(\overline{u})$ is a Borel function and $[Ju]$ can be written as $\nu\mu$ for some Borel covector–valued function ν and some finite positive measure μ. In general this identity does not hold. If it holds we say that u satisfies a *strong chain rule.*

If for every open set $A \subset \Omega$ we define

$$\|Ju\|(A) := \sup\{\langle [Ju], \omega \rangle | \omega \text{ is a } C_c^\infty(A) \text{ form and } \|\omega\|_\infty \leq 1\},$$

then a consequence of Theorem 1.0.1 is

$$\|Ju\|(A) \leq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(A)dy. \tag{5}$$

A natural question is to ask under what conditions the equality holds.

When $n = 1$ the problem reduces to the coarea formula on the level sets of a scalar function (see for example Theorem 3.40 of [**7**]). When $n > 1$ Jerrard and Soner proved that the equality holds for $u \in W^{1,p}$ with $p > n$. In this case $[Ju]$ is equivalent to the natural action as current of $du_1 \wedge \ldots \wedge du_n$ and a comparison with the classical coarea formula for approximately differentiable functions (see for example Theorem 3.2.3 of [**18**]) implies that $[Ju^y]$ is (up to a dimensional constant) the integer rectifiable current supported by $u^{-1}(y)$ with the usual orientation. Motivated by these facts we will say that equation (3) represents a *weak coarea formula* and that u satisfies the *strong coarea formula* if the equality holds in (5).

In general, to obtain equality, it is not sufficient to assume merely that $\|Ju\|$ is a Radon measure (see [**27**] for some counterexamples based on earlier ones of Giaquinta, Modica and Souček [**21**]). However it is conjectured in [**27**] that the strong coarea formula holds for $W^{1,n}$ functions and for continuous functions such that $\|Ju\|$ is a Radon measure.

In the fourth chapter we prove that if u satisfies the strong chain rule then the strong coarea formula holds (see Theorem 4.2.1). This theorem provides a different proof of the strong coarea formula when $u \in W^{1,p}$, $p > n$, which can be extended to $u \in W^{1,n}$. More generally it implies that the strong coarea formula holds whenever u is a Cartesian map in the sense of Giaquinta, Modica and Souček (see Remarks 4.3.3 and 4.4.2 of Chapter 4). Moreover in the third section of the same chapter we will give some conditions under which the strong chain rule holds.

The material of the Chapter 5 and 6 is motivated by the interest of the author and Sergio Conti in the following variational problem from nonlinear elasticity: minimizing energies of the form

$$E(u) := \int_{\Omega} |\nabla u|^2 + \varphi(\det \nabla u) \tag{6}$$

where φ is a convex function with superlinear growth and approaching infinity at zero. The minimizers are sought among deformations u which map $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$ and satisfy some notion of invertibility and a Dirichlet boundary condition.

Finding a good weak notion of invertibility is a crucial point here. In the fifth chapter we analyze two different weak notions of invertibility: one based the approach of Giaquinta, Modica and Souček (see [**25**]) which uses Cartesian Currents and the other which is a slight generalization of ideas of Müller and Spector in [**38**] (in the following we will call INV maps those maps which satisfy Müller and Spector condition of invertibility, see Section 2 of Chapter 5 for the definition). Both these notions involve computations on weak Jacobians and we will see that many proofs will pass through the characterization of the weak Jacobians of the maps considered.

In section 3 we prove that the two theories are essentially equivalent. This provides the first nontrivial example of a class of maps for which certain properties of the weak Jacobians are equivalent to properties of the graphs considered as currents, partially answering questions asked in [**25**]. In general, whereas the weak Jacobian can be easily reconstructed when one knows the action of the graph of u as current, the other way round is a difficult task. The book of Giaquinta, Modica and Souček provides many examples in which informations on the weak Jacobian fail to provide good informations on the action of the graph of u as current.

In Chapter 6 we give an example of "good" maps, bounded (for an appropriate choice of φ) in energies (6), which are converging to a map with large singularity and very bad behavior. This example shows that the problem of finding a minimizer to the energy (6) cannot be solved using extensions of Müller and Spector's approach (which is tailored to energies of the form $\int |\nabla u|^p + \varphi(\det \nabla u)$ for $p > 2$). Precisely it shows that any attempt of solving this problem via direct methods has to deal with the existence of very bad behavior in the domain of admissible maps. We emphasize that the example given provides a new type of singularity in the context of nonlinear elasticity, different from cavitation.

At the level of weak Jacobians this provides a highly nontrivial example of maps $u_n \in W^{1,n-1}(\mathbb{R}^n, \mathbb{R}^n) \cap L^\infty$ converging weakly to a map u such that the weak Jacobians of u_n do not converge in the sense of distributions to the weak Jacobian of u. Indeed in this example the weak Jacobians $[Ju_n]$ are all positive measures whereas $[Ju]$ contains a Dirac mass with minus sign. The failure of the continuity of weak Jacobians for weak convergence in $W^{1,n-1} \cap L^\infty$ was already known in the literature (see for example in [**25**]). However all the known examples involved sequences of maps which were very far from invertible and already quite singular before reaching the limit, whereas our u_n are Lipschitz maps with Lipschitz inverse.

Thanks to our characterizations of INV maps in terms of currents, we will emphasize how the failure of convergence of weak Jacobians, the non–closedness of INV condition and the necessity of introducing graphs with non–trivial boundary are deeply interrelated.

Almost all the original results contained in this thesis have been published in the papers [**13**], [**16**], [**17**].

Aknowledgements First of all I would like to thank Luigi Ambrosio, who kindly leaded me during my study of all these topics. Moreover I am greatly indebted to Sergio Conti (with whom I worked jointly on the problems of Chapter 5 and 6) and ro Giovanni Alberti, Robert Jerrard, Bernd Kirchheim, Stefan Müller and Gianni Royer for many interesting and illuminating discussions. It is thanks to Gianni that my interests were led to the variational problem from which the last chapters eventually stemmed. Finally I wish to thank Lorenza Pescia, Govind Menon, Sun–Ra Mosconi, Mario Romeo and Paolo Tilli for their continuous support and help.

CHAPTER 2

Metric currents and rectifiability

Instead of working in the framework of classical Geometric Measure Theory we prefer to use the "metric currents theory" of [**8**] because we think that it is much easier to handle and provides more powerful tools to study the structure of weak Jacobians. The main idea of this approach, suggested by De Giorgi in [**15**], is to replace the duality with differential forms with the duality with $(k+1)$-ples of Lipschitz functions. We hope to show that in this way we simplify notations and proofs. Moreover we will see as this new point of view provides a good insight and new interesting theorems even in the Euclidean setting.

Following this idea Ambrosio and Kirchheim has given suitable definitions of all the usual notions on currents, such as boundary, restriction, mass, flatness, normality and so on. In the second section of this chapter we will introduce some of these definitions and notations from the theory of "metric currents", whereas the third section will be devoted to the slicing theorem for normal and flat currents which plays a central role in the further developments of this thesis.

In the fourth section we will define a general decomposition of flat currents of finite mass. Indeed we show that it is possible to decompose every k-dimensional flat metric current T of finite mass into two currents of finite mass T_l and T_u such that:

(a) T_l is concentrated on a $\mathcal{H}^k$- rectifiable set S;
(b) the mass of T_l is absolutely continuous with respect to $\mathcal{H}^k \llcorner S$;
(c) T_u neglects all $\mathcal{H}^k$ σ-finite sets.

The "lower dimensional part" T_l can be characterized via its slices and actually it is this characterization together with a BV estimate on the slices which enforces rectifiability. One of the consequence of this decomposition is the following criterion of rectifiability for flat metric currents:

(A) a flat k-dimensional current T of finite mass on E is rectifiable if and only if for every Lipschitz function $\pi : E \to \mathbb{R}^k$ almost every slice of T with respect to π is composed of atoms (see Theorem 2.4.3).

This criterion has been already proved by Ambrosio and Kirchheim in [**8**] for normal metric currents, and by White in [**42**], with a different approach, for normal currents on Euclidean spaces with coefficients in normed groups. The proof of Ambrosio and Kirchheim is based on the idea sketched above. The BV estimate on the slicing of currents used by them is due to Jerrard and Soner in the Euclidean case and then developed in the more general case in [**8**].

One useful technical device to carry on the cited proofs of rectifiability is the notion of BV functions which take values in weakly separable metric spaces. This notion is due to Ambrosio in [**5**] and in the first section we give the basic definitions and some properties related to it.

When the metric space E has Hausdorff dimension n one can go further splitting T_u into two other parts, one such that $\|T_u\|$ is absolutely continuous with respect to $\mathcal{H}^n$ and the other, which we call T_c, such that $\|T_c\|$ is singular with respect to $\mathcal{H}^n$. The final decomposition $T = T_l + T_c + T_a$ resembles the decomposition of the derivative of (Euclidean) BV functions into jump, Cantor and absolutely continuous parts due to De Giorgi and Ambrosio (see [**7**] for details on this decomposition and related theorems). The Euclidean case $\mathbb{R}^n$ is of course an example which allows for our decomposition: in the next chapter we will apply it to weak Jacobians and it will be much clearer the analogy with De Giorgi and Ambrosio's decomposition.

1. Functions of metric bounded variation

As we will see in the third section of this chapter the slicing map of a normal current has a remarkable property. In order to state it we need the definition of map of bounded variation from an open set of $\mathbb{R}^n$ to a weakly separable metric space (M, d) (see [**8**] and [**5**]).

DEFINITION 2.1.1. *We say the metric space (M, d) is weakly separable if there exists a countable family $\mathcal{F} \subset Lip_b(M)$ such that*

$$d(x, y) = \sup_{\varphi \in \mathcal{F}} |\varphi(x) - \varphi(y)| \qquad \text{for every } x, y \in M. \tag{7}$$

Weakly separable metric spaces can be seen as a suitable generalization of separable Banach spaces. Indeed if E is a separable Banach space then we can choose as $\mathcal{F}$ in equation (7) a family of linear functionals. Hence a natural definition for map of bounded variation which take values in a weakly separable metric space would be the following:

(A) $u : \mathbb{R}^n \to E$ is MBV if for every $\varphi \in \mathcal{F}$ the function $\varphi \circ u$ is a function of bounded variation.

In particular this is one of the possible definitions of BV functions when $E = \mathbb{R}^n$. However another very natural requirement on a map of bounded variation would be that the measures $\|D(\varphi \circ u)\|$ enjoy some kind of "global control", independent of the choice of $\varphi \in \mathcal{F}$. In the Euclidean space this property is a byproduct of (A) because, endowing $\mathbb{R}^n$ with a suitable norm (which of course is equivalent to the Euclidean norm), we can choose finite families $\mathcal{F}$ satisfying (7), but in more general cases this is not true. Hence we require this "global control" in the definition. In order to define this we need to introduce the concept of supremum of a family of measures.

DEFINITION 2.1.2. *Let $\{\mu_i\}_{i\in I}$ be a family of positive measures on E. Then for every Borel subset of E we define*

$$\bigvee_{i\in I} \mu_i(B) := \sup \left\{ \sum_{i\in J} \mu_i(B_i) : \bigcup_{i\in J} B_i = B \text{ and } B_i \text{ are pairwise disjoint} \right\}$$

where J runs through all countable subsets of I.

DEFINITION 2.1.3. *Let $U \subset \mathbb{R}^k$ be an open set, (M, d) a weakly separable metric space and $u : U \to M$. We say that u is of metric bounded variation if*

(a) *$\varphi \circ u$ is of locally bounded variation for every $\varphi \in \mathcal{F}$;*
(b)

$$\|Du\|_{MBV} := \bigvee_{\varphi\in\mathcal{F}} |D(\varphi \circ u)|(\Omega) < \infty.$$

We remark that this definition does not depend on the choice of $\mathcal{F}$ and that

$$\|Du\|_{MBV} = \bigvee_{\varphi\in \mathrm{Lip}_b(M)} |D(\varphi \circ u)|(\Omega)$$

(see [**8**] or Theorem 7.3.1 in the appendix for the proofs). From now on we will denote by $\|Du\|$ the measure $\bigvee |D(\varphi \circ u)|$.

2. Brief introduction to metric currents

Throughout this thesis (E, d) will denote a complete metric space, $\mathrm{Lip}\,(E)$ and $\mathrm{Lip}_b(E)$ will be, respectively, the space of Lipschitz real functions and the space of Lipschitz and bounded real functions on E, $\mathrm{Lip}\,(f)$ will denote the Lipschitz constant f and we will put

$$\mathrm{Lip}_1(E) := \{f \in \mathrm{Lip}\,(E) | \mathrm{Lip}\,(F) \leq 1\}.$$

From now on Ω will denote an open set and $B_r(x)$ the open ball of radius r centered on x. If μ is a Radon (vector valued) measure on Ω and $A \subset \Omega$ is a Borel set then $\mu \llcorner A$ will indicate the measure ν given by $\nu(E) = \mu(A \cap E)$. Moreover $\|\mu\|$ will denote the total variation measure as usually defined in the literature.

We denote by $\mathcal{D}^k(E)$ the set of all $(k+1)$-ples $(f, g_1, \dots, g_k)$ of functions such that $f, g_1, \dots, g_k \in \operatorname{Lip}_b(E)$ and we refer to it as the space of k-dimensional differential forms (or simply k-forms). For every k-form $\omega = (f, g_1, \dots, g_k)$ we define its exterior derivative as the $(k+1)$-form

$$d\omega = (1, f, g_1, \dots, g_k). \tag{8}$$

If $\phi : F \to E$ is Lipschitz and bounded (and F is a complete metric space), we define the pull-back of ω as the k-form on F given by

$$\phi^{\#}\omega = (f \circ \phi, g_1 \circ \phi, \dots, g_k \circ \phi). \tag{9}$$

If $\omega_1 = (f, g_1, \dots, g_n)$ and $\omega_2 = (w, h_1, \dots, h_k)$ then their exterior product is the $(n+k)$-form

$$\omega_1 \wedge \omega_2 := (fw, g_1, \dots, g_n, h_1, \dots, h_k).$$

Let us fix $\omega = (f, g_1, \dots, g_n) \in \mathcal{D}^n(E)$. For every i we define

$$\mathcal{C}_i := \{C \text{ open} |\ g_i \text{ is constant on every connected component of } C\}.$$

After setting $C_i := E \setminus \left(\bigcup\{U \in \mathcal{C}_i\}\right)$ we define the closed set

$$\operatorname{supp}(\omega) = \operatorname{supp}(f) \cap \bigcap_{i=1}^{n} C_i \tag{10}$$

and we refer to it as *support* of ω.

DEFINITION 2.2.1. *Let $k \in \mathbb{N}$. A k-dimensional current in E is a functional $T : \mathcal{D}^k(E) \to \mathbb{R}$ such that*

(a) $\lim_i T(f, g_1^i, \dots, g_k^i) = T(f, g_1, \dots, g_k)$ *if $g_k^i \to g_k$ pointwise and $(Lip\,(g_k^i))$ is bounded for every k;*
(b) *T is multilinear with respect to $(f, g_1, \dots, g_k)$;*
(c) $T(f, g_1, \dots, g_k) = 0$ *if* $\operatorname{supp}((f, g_1, \dots, g_k)) = \emptyset$.

We denote by $\mathcal{M}_k(E)$ the vector space of k-dimensional currents.

REMARK 2.2.2. We could replace $\mathcal{D}^k(E)$ with $\mathcal{D}^k_b(E)$, namely the set of differential forms with bounded support, and we could define as well a k-dimensional "local current" as a linear functional that satisfies conditions (b), (c) above and condition (a') below:

(a') $\lim_i T(f, g_1^i, \dots, g_k^i) = T(f, g_1, \dots, g_k)$ if $g_k^i \to g_k$ pointwise, $(\operatorname{Lip}(g_k^i))$ is bounded for every k and $\operatorname{supp}((f, g_1^i, \dots, g_k^i))$ is contained on a bounded subset K for every i.

All the definitions and theorems of this section work as well with slight modifications. Moreover in the applications to weak Jacobians we will use local currents.

DEFINITION 2.2.3. *Let T be a k-dimensional current. If there exists a finite positive measure μ such that*

$$T(f, g_1, \dots, g_k) \le \prod_{i=1}^{k} Lip\,(g_i) \int_E |f| d\mu \tag{11}$$

then we say that T is of finite mass. *We call* mass *of the current T the minimal μ that satisfies (11) and we denote it with $\|T\|$. We say that T is concentrated on a Borel set B if $\|T\|(E \backslash B) = 0$.*

We denote by $\mathcal{M}_k(E)$ the vector space of k-dimensional currents of finite mass.

From now on, given a current T of finite mass we will denote by $\mathbf{M}(T)$ the total variation of $\|T\|$ in E. If T has not finite mass we set $\mathbf{M}(T) = \infty$.

REMARK 2.2.4. We will always assume that $\|T\|$ is concentrated on a σ-compact set. However, as observed in [**8**], this fact can be proved if E is separable or if the cardinality of E is a Ulam number. The assumption that the cardinality of any set E is a Ulam number is consistent with the standard ZFC theory.

DEFINITION 2.2.5. *Given a sequence $(T_n) \subset \mathcal{M}_k(E)$, we say that*

$$T_n \rightharpoonup T \in \mathcal{M}_k(E) \tag{12}$$

if $T_n(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{k-1}(E)$.

Sometimes we will write $\langle T, \omega \rangle$ for $T(\omega)$. As we can see in [**8**] from the assumptions of Definition 2.2.1 it follows that a k-dimensional current is always alternating in $(g_1, \dots, g_k)$; hence we use for differential forms the usual notation

$$f dg_1 \wedge \ldots \wedge dg_n.$$

Sometimes, for sake of simplicity, we will denote by g the n-tuple $(g_1, \dots, g_n)$ and we will write $f dg$ for $f dg_1 \wedge \ldots \wedge dg_n$. A trivial computation shows that if $\omega \in \mathcal{D}^n(E)$, $\nu \in \mathcal{D}^k(E)$ and $T \in \mathcal{M}_{n+k}(E)$, then

$$T(d(\omega \wedge \nu)) = T(d\omega \wedge \nu) + (-1)^n T(\omega \wedge d\nu).$$

Moreover every current satisfies the usual chain rule

$$T(f\, dg_1 \wedge \ldots \wedge dg_n) + T(g_1\, df \wedge \ldots \wedge dg_n) = T(1\, d(f g_1) \wedge \ldots \wedge dg_n).$$

If $T \in \mathcal{M}_k(\mathbb{R}^k)$, then for every $g \in C^1_c(\mathbb{R}^k, \mathbb{R}^k)$ and $f \in \mathrm{Lip}\,(\mathbb{R}^k)$ we have

$$T(f dg_1 \wedge \ldots \wedge dg_n) = T(f \det(\nabla g)\, dx_1 \wedge \ldots \wedge dx_n)$$

(with x_i we denote the projection on the i-th coordinate of the canonical system of $\mathbb{R}^k$).

We can define a boundary operator $\partial : \mathcal{M}_k \to \mathcal{M}_{k-1}$ with the duality relation $\partial T(\omega) := T(d\omega)$; it is not difficult to see that ∂T satisfies conditions (a), (b) and (c) of 2.2.1, but it can fail to be of finite mass, even if T itself has finite mass. Moreover we notice that, since $d(d\omega) = 0$ for every form ω, we have the usual relation $\partial(\partial T) = 0$.

Definition 2.2.6. *If T and ∂T are currents of finite mass then we call T* normal. *We denote by $\mathcal{N}^k(E)$ the vector space of normal k–dimensional currents.*

Remark 2.2.7. Given $T \in \mathcal{N}^k(E)$ we can define

$$\|T\|_N := \|T\|(E) + \|\partial T\|(E).$$

It is easy to check that $\mathcal{N}^k(E)$ endowed with the norm $\|\cdot\|_N$ is a Banach space.

The following Theorem is one of the most important in the theory of currents and we will see further that it will play a crucial role in the proof of the Slicing Theorem 2.3.1.

Theorem 2.2.8. *Let $(T_h) \subset \mathcal{N}^k(E)$ be a bounded sequence and assume that for any integer $p \geq 1$ there exists a compact set K_p such that*

$$\|T_h\|(E \backslash K_p) + \|\partial T_h\|(E \backslash K_p) \leq \frac{1}{p} \qquad \forall h.$$

Then there exists a subsequence $T_{h(n)}$ converging to a current $T \in \mathcal{N}^k(E)$ satisfying

$$\|T\| \left(E \backslash \bigcup_{p=1}^{\infty} K_p \right) + \|\partial T\| \left(E \backslash \bigcup_{p=1}^{\infty} K_p \right) = 0.$$

In order to avoid breaking the discussion of this section we postpone the proof of the previous theorem to the first section of the appendix, where we will follow the proof of [**8**] (Theorem 5.2).

Definition 2.2.9. *Let T be a k-dimensional current on E. We define the flat norm $\mathbf{F}(T)$ as*

$$\inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(\partial S) | S \textit{ is a } (k+1)\textit{-dimensional current}\}.$$

DEFINITION 2.2.10. *Let T be a k-dimensional current. We say that T is a flat current if there exists a sequence of normal currents (T_n) such that*

$$\lim_{n\to\infty} \mathbf{F}(T_n - T) = 0.$$

It is easy to see that a current T of finite mass is flat if and only if there exists a sequence of normal currents (T_n) such that $\mathbf{M}(T_n - T) \to 0$. Indeed one implication is trivial because, for every current S, we have $\mathbf{F}(S) \leq \mathbf{M}(S)$. Moreover if T is a flat current of finite mass then for every n there exist $T_n \in \mathcal{N}_k(E)$ and $S_n \in \mathcal{M}_k(E)$ such that

$$\mathbf{M}(T - T_n - \partial S_n) + \mathbf{M}(\partial S_n) \leq \frac{1}{n}.$$

So we have that $T_n' := T_n + \partial S_n$ is a normal current and $\mathbf{M}(T - T_n') \leq 1/n$. A useful consequence of the last statement is that for every current T we can find a sequence of normal current T_n such that

$$\lim_{n\to\infty} \mathbf{M}\left(T - \sum_{i=1}^{n} T_n\right) = 0 \tag{13}$$

and

$$\sum_{i=1}^{\infty} \mathbf{M}(T_n) < \infty. \tag{14}$$

If (13) and (14) hold we simply write

$$T = \sum_{i=1}^{\infty} T_n.$$

DEFINITION 2.2.11. *We say that a k-dimensional current T of finite mass is rectifiable if it is concentrated on a k-dimensional rectifiable set and $\|T\| << \mathcal{H}^k$.*

As for the notion of boundary, we can define by duality the push-forward of currents. Indeed, given a Lipschitz and bounded map $\phi : E \to F$ and a k-dimensional current T on E it is not difficult to check that $\phi_\# T$ defined by

$$\langle \phi_\# T, \omega \rangle := T(\phi^\# \omega) \tag{15}$$

is a k-dimensional current. Moreover if T is a current of finite mass, then $\phi_\# T$ has finite mass and

$$\|\phi_\#(T)\| \leq \phi_\# \|T\|.$$

(We recall that if μ is a measure then its push forward $\phi_\# \mu$ is defined by $\phi_\# \mu(U) = \mu(\phi^{-1}(U))$.)

From these definitions one can develop a self-contained theory of normal currents in E which is equivalent to the classical theory in the Euclidean case. Hereafter we study the aspects that are useful for our purposes. We begin with the definitions of restriction and slicing.

DEFINITION 2.2.12. *Let $T \in \mathcal{M}_k(E)$ and $\omega \in \mathcal{D}^h(E)$, with $h \leq k$. We define the* restriction of T to ω *as the $(k-h)$-dimensional current given by*

$$T \llcorner \omega\,(\nu) := T(\omega \wedge \nu).$$

REMARK 2.2.13. If T is a current of finite mass, then we can extend its action to the $(k+1)$-ples $(f, g_1, \dots, g_k)$ such that $g_i \in \mathrm{Lip}_b(E)$ and f is bounded and Borel measurable. Indeed, $T \llcorner dg$ is a 0-dimensional current of finite mass and so there exists a finite measure μ_g such that

$$\langle T, w\, dg\rangle = \langle T \llcorner dg, w\rangle = \int_E w\, d\mu_g \qquad \text{for every } w \in Lip_b(E) \tag{16}$$

and

$$\|\mu_g\|_{\text{var}} \leq \|T \llcorner dg\|(E) \leq \left(\prod_{i=1}^{k} \mathrm{Lip}(g_i)\right) \|T\|(E).$$

Using equation (16) the action of $T \llcorner dg$ can be easily extended to every Borel measurable and bounded function. Of course, if $f^k \to f$ uniformly, $g_i^k \to g_i$ pointwise and $(\mathrm{Lip}\,(g_i^k))$ is bounded for every i, then

$$\langle T, f^i dg_1^i \wedge \ldots \wedge dg_k^i\rangle \to \langle T, f dg_1 \wedge \ldots \wedge dg_k\rangle.$$

From this last last remark it follows that if T is a current of finite mass then for every Borel set A we can define the current $T \llcorner A$:

$$\langle T \llcorner A, f dg\rangle := \langle T \llcorner \chi_A, f dg\rangle = \langle T, f \chi_A\, dg\rangle.$$

Moreover $\|T \llcorner \chi_A\| \leq \|T\|$.

REMARK 2.2.14. A very useful formula whose validity can be easily checked from the definitions is the following (valid for any Lipschitz function f; later on it will be generalized to any k-form)

$$\partial(T \llcorner f) = (\partial T) \llcorner f - T \llcorner df. \tag{17}$$

3. Slicing theorems

THEOREM 2.3.1. *Let T be a k-dimensional normal current in E and π a Lipschitz function from E to $\mathbb{R}^h$, with $h \leq k$. Moreover let us suppose that T and ∂T are concentrated on the σ–compact set L. Then there exist normal $(k-h)$-dimensional currents $\langle T, \pi, x\rangle$ such that:*

(i) *$\langle T, \pi, x\rangle$ and $\partial\langle T, \pi, x\rangle$ are concentrated in $L \cap \pi^{-1}(x)$;*

(ii) *the map* $x \to \langle T, \pi, x\rangle$ *is weakly** *continuous and for every* $\psi \in C_c(\mathbb{R}^h)$ *we have*

$$\int_{\mathbb{R}^h} \langle T, \pi, x\rangle \psi(x) d\mathcal{L}^h = T \llcorner (\psi \circ \pi) d\pi; \tag{18}$$

(iii) *moreover it holds*

$$\int_{\mathbb{R}^h} \|\langle T, \pi, x\rangle\| d\mathcal{L}^h = \|T \llcorner d\pi\|; \tag{19}$$

(iv) *if* L' *is a* σ*–compact set,* $T^x \in \mathcal{M}_{k-m}(E)$ *are all concentrated on* L'*, satisfy (ii) and* $x \to \mathbf{M}(T^x)$ *is integrable on* $\mathbb{R}^k$*, then* $T^x = \langle T, \pi, x\rangle$ *for* $\mathcal{L}^m$ *a.e.* $x \in \mathbb{R}^m$*.*

(v) *if* $m = 1$*, there exists a* $\mathcal{L}^1$ *negligible set* $N \subset \mathbb{R}$ *such that*

$$\langle T, \pi, x\rangle = \lim_{y \to x} \frac{\chi_{\{x<\pi<y\}}}{y-x} d\pi = (\partial T) \llcorner \{\pi > x\} - \partial(T \llcorner \{\pi > x\})$$

for any $x \in \mathbb{R} \backslash N$*. Moreover*

$$\mathbf{M}(\langle T, \pi, x\rangle) \leq Lip\,(\pi) \frac{d}{dt}(\mathbf{M}(T \llcorner \{\pi > x\}))$$

for $\mathcal{L}^1$ *a.e.* x *and*

$$\int_{-\infty}^{+\infty} \|\langle T, \pi, x\rangle\|_N dx \leq Lip\,(\pi) \|T\|_N. \tag{20}$$

Such a map $\langle T, \pi, x\rangle$ is called *slicing of* T *with respect to* π.

REMARK 2.3.2. Before going further with our discussion we notice something that will be needed later: statement (iv) of the previous Theorem has the following interesting consequence.

Let us suppose that we have two functions $\pi_1 : E \to \mathbb{R}^{k_1}$ and $\pi_2 : E \to \mathbb{R}^{k_2}$ and a normal current $T \in \mathcal{N}_n(E)$ with $n \geq k_1 + k_2$. Then we can perform two times the operation of slicing, getting a map from $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathcal{N}_{n-k_1-k_2}(E)$ given by $(x, y) \to \langle\langle T, \pi_1, x\rangle, \pi_2, y\rangle$. Thanks to (iv) we can easily check that this map coincide (up to a $\mathcal{L}^{k_2+k_1}$ negligible set) with the map $\langle\langle T, \pi_2, y\rangle, \pi_1, x\rangle$.

Moreover if we construct the map $\pi : E \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ given by $\pi(p) = (\pi_1(p), \pi_2(p))$ then both the "double–slicing maps" coincide with $(x, y) \to \langle T, (x, y), \pi\rangle$.

Due to the importance of the Slicing Theorem in this thesis (as we will see this machinery will be the main tool in the proofs of many theorems on weak Jacobians) we will include Ambrosio and Kirchheim's proof for the reader's convenience.

This proof is based mainly on three ingredients: the Compactness Theorem for normal currents stated above (Theorem 2.2.8), the Localization Lemma proved below (Lemma 2.3.4) and the following technical lemma which allows to represent the mass of current via the supremum of a countable family of restrictions (moreover this lemma will be used also in the next section).

LEMMA 2.3.3. *Let T be a k–dimensional current of finite mass. Then there exists a countable set $D \subset Lip_1(E) \cap Lip_b(E)$ such that*

$$\|T\| = \bigvee \{\|T \llcorner d\pi\| : \pi_1, \dots, \pi_k \in D\}. \tag{21}$$

PROOF. We only sketch the main ideas of the proof, referring to [**8**] for the details. As already stated at the beginning of this section, we are supposing that every current of finite mass is concentrated on σ–compact set S. Hence we can express S as the union of a countable family of Lipschitz sets K_h. Now for every h we select a family of Lipschitz functions D_h which is dense (in the uniform convergence) in the space $\mathrm{Lip}_1(K_h) \cap \mathrm{Lip}_b(K_h)$.

Next to this we characterize the mass of a current as the minimal M which satisfies

$$\sum |T(f_i, g_1^i, g_2^i, \dots g_k^i)| \leq M$$

for every choice of functions $g_i^j \in \mathrm{Lip}_1(E) \cap \mathrm{Lip}_b(E)$ and f_i such that $\sum |f_i| \leq 1$. Finally we use a truncation argument and the density of D_h in $\mathrm{Lip}_1(K_h) \cap \mathrm{Lip}_b(K_h)$ to prove the representation formula (21) for $D = \cup D_h$. $\square$

LEMMA 2.3.4. *Let $\varphi \in Lip_b(E)$ and $T \in \mathcal{N}^k(E)$. Then $T \llcorner \{\varphi > t\} \in \mathcal{N}^k(E)$ and*

$$\|\partial(T \llcorner \{\varphi > t\})\|(\{\varphi = t\}) \leq \frac{d}{d\tau} \|T \llcorner d\varphi\|(\{\varphi \leq \tau\})\bigg|_{\tau = t} \tag{22}$$

for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$. Moreover if S is any σ–compact set where both S and ∂S are concentrated, then $T \llcorner \{\varphi > t\}$ and its boundary are concentrated on S for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$.

PROOF. Let $\mu = \|T\| + \|\partial T\|$ and (K_p) a sequence of compact sets which cover μ almost all of E. Moreover we put

$$g(t) := \mu(\{\varphi \leq t\}), \qquad g_p(t) = \mu(\{\varphi \leq t\} \cap K_p).$$

We denote by $L \subset \mathbb{R}$ the set of t where $g'_t = \sum_p g'_p(t)$ exists and is finite and the derivative in the left hand side of (22) exists. Since the real functions involved are non–decreasing we have $\mathcal{L}^1(\mathbb{R} \backslash L) = 0$.

Let $t \in L$ and for some $\varepsilon_h \downarrow 0$ define

$$f_h(s) = \begin{cases} 0 & \text{for } s \leq t \\ 1 & \text{for } s \geq t + \varepsilon_h \\ (s-t)/\varepsilon_h & \text{for } s \in [t, t+\varepsilon_h]. \end{cases}$$

Then (17) implies that

$$\partial(T \llcorner f_h \circ \varphi) = \partial T \llcorner f_h \circ \varphi - \frac{1}{\varepsilon} T \llcorner \{t < \varphi < t + \varepsilon_h\}. \qquad (23)$$

We call the last term of the right hand side R_h: again applying formula (17) we get

$$\partial R_h = \partial(\partial T \llcorner f_h \circ \varphi) = -\frac{1}{\varepsilon_h} \partial T \llcorner \chi_{\{t<\varphi<t+\varepsilon_h\}} d\varphi.$$

It can be easily checked that, since $t \in L$, the sequence (R_h) satisfies the assumptions of the Compactness Theorem 2.2.8. Hence we can assume, after possibly extracting a subsequence, that (T_h) converges to some $R \in \mathcal{N}^k(E)$, such that both R and ∂R are concentrated on $\cup K_p$.

Since $T \llcorner f_h \circ \varphi$ and $\partial T \llcorner \circ \varphi$ converge respectively to $T \llcorner \{\varphi > t\}$ and $\partial T \llcorner \{\varphi > t\}$, passing to the limit in equation (23) we conclude

$$\partial(T \llcorner \{\varphi > t\}) = \partial T \llcorner \{\varphi > t\} - R.$$

Hence $\|\partial(T \llcorner \{\varphi > t\})\|(\{\varphi = t\}) \leq \mathbf{M}(R)$ and the lower semicontinuity of mass gives:

$$\mathbf{M}(R) \leq \liminf_{h \to \infty} \mathbf{M}(R_h) \leq \frac{d}{d\tau} \|T \llcorner d\varphi\|(\{\varphi \leq \tau\}) \Big|_{\tau = t}.$$

□

Proof of Theorem 2.3.1 **Step 1** First of all we begin by considering the case $m = 1$. In this case we take statement (v) as the definition of $\langle T, \pi, x \rangle$. Reasoning as in the proof of Lemma 2.3.4 we have that the limit

$$\lim_{y \downarrow x} \frac{1}{y - x} T \llcorner \chi_{\{x<\pi<y\}} d\pi$$

exists in almost every $x \in \mathbb{R}$ and is equal to $\partial T \llcorner \{\pi > x\} - \partial(T \llcorner \{\pi > x\})$. We call S_x these currents and we notice that they are supported on $\{\pi = x\}$. Moreover Lemma 2.3.4 implies also that

$$\mathbf{M}(S_x \llcorner \omega) \leq \frac{d}{dt} \|(T \llcorner d\pi) \llcorner \omega\|(\{\pi = t\}) \Big|_{t=x} \qquad \text{for } \mathcal{L}^1 \text{ a.e. } x$$

for every p–form ω with $p \leq k-1$. Integrating with respect to x we get

$$\int_{\mathbb{R}}^{*} \mathbf{M}(S^x \llcorner \omega) dx \leq \mathbf{M}((T \llcorner d\pi) \llcorner \omega) \tag{24}$$

(where $\int^*$ denotes the upper integral: we notice that since we have not proved yet the measurability of the function $x \to \mathbf{M}(S_x \llcorner \omega)$ we cannot replace it with the ordinary integral).

Now we check (ii). Fixed a function $\psi \in C_c(\mathbb{R})$ of course we can write it as the difference of two continuous functions ψ_1, ψ_2 such that $\psi_i \geq 1$ for every i. We set $\Psi_i(t) = \int_0^t \psi_i(\tau) d\tau)$ and after fixing a $k-1$–form ω we compute

$$\begin{aligned}
& \int_0^\infty S_x(\omega)\psi_i(x) dx \\
= \quad & \int_0^\infty (\partial T \llcorner \{\pi > x\})(\omega)\psi_i(x) dx - \int_0^\infty \partial(T \llcorner \{\pi > x\})(\omega)\psi_i(x) dx \\
= \quad & \int_0^\infty (\partial T \llcorner \{\Psi_i \circ \pi > t\})(\omega) dt - \int_0^\infty \partial(T \llcorner \{\Psi_i \circ \pi > t\})(\omega) dt
\end{aligned}$$

Now, by linearity of the currents T and ∂T it can be checked (for example approximating the two last integrals with Riemann's sums) that the last expression is equal to

$$\partial T(\Psi_i^+ \circ \pi\, \omega) - T(\Psi_i^+ \circ \pi\, d\omega).$$

Using the fact that S_x is also equal to $\partial(T \llcorner \{\pi \leq x\}) - \partial T \llcorner \{\pi \leq x\}$ and reasoning as above we get

$$\int_{-\infty}^0 S_x(\omega)\psi_i(x) = -\partial T(\Psi_i^- \circ \pi\, \omega) + T(\Psi_i^- \circ \pi d\omega).$$

Summing the two relations just found we get

$$\begin{aligned}
\int_{\mathbb{R}} S_x(\omega)\psi_i(x) &= \partial T(\Psi_i \circ \pi\omega) - T(\Psi_i \circ \pi d\omega) \\
&= T(d(\Psi_i \circ \pi) \wedge \omega)) = T(\Psi_i' \circ \pi d\pi \wedge \omega) \\
&= (T \llcorner \psi_i \circ \pi)(\omega).
\end{aligned}$$

Using the fact that $\psi_1 - \psi_2 = \psi$, this implies the identity of (ii).

Now we apply this identity to prove

$$T \llcorner d\pi(f dg) = \int_{\mathbb{R}} S_x(f dg) \leq \Pi_{i=1}^{k-1} \mathrm{Lip}\,(g_i) \int_{*\mathbb{R}} \left(\int_E |g| d\|S_x\| \right) dx$$

where $\int_*$ denotes the lower integral. Since the lower integral is super-additive and $(\int_E |g|d\|S_x\|) \le \mathbf{M}(S_x \llcorner |g|)$, using (24) we get

$$\begin{aligned}\int_E |g|d\|T\llcorner d\pi\| &\le \int_{*\mathbb{R}} \mathbf{M}(S_x\llcorner |g|)dx \le \int_{\mathbb{R}}^* \mathbf{M}(S_x\llcorner |g|)dx \\ &\le \mathbf{M}((E\llcorner d\pi)\llcorner |g| \le \int_E |g|d\|T\llcorner d\pi\end{aligned}$$

for every $g \in L^1(E, \|T\llcorner d\pi\|)$. This gives the weak* measurability of $x \to \|S_x\|$ (so the weak* measurability of $x \to S_x$) and moreover proves (iv).

Finally we complete the proof of (v) using the identity

$$\partial\langle T, \pi, x\rangle = \langle \partial T, \pi, x\rangle$$

and we apply (19) to get (20).

Step 2 Now we complete the proof of existence of $\langle T, \pi, x\rangle$ reasoning by induction: we assume that the statements (i), (ii), (iii) and (v) are true for $m \in [1, h-1]$ and we prove that they are also true for $m = h$. Let $\pi = (\pi_1, \tilde{\pi})$, where $\tilde{\pi} \in \mathrm{Lip}\,(E, \mathbb{R}^{m-1})$ and $\pi_1 \in \mathrm{Lip}\,(E)$, set $x = (t, y)$ and define

$$T_t := \langle T, \pi_1, t\rangle, \qquad T_x := \langle T_t, \tilde{\pi}, y\rangle.$$

The induction assumption and (24) give

$$\int_{\mathbb{R}}^* \int_{\mathbb{R}^{m-1}} \mathbf{M}(T_x)dydt = \int_{\mathbb{R}}^* \mathbf{M}(T\llcorner d\tilde{\pi}) \le \mathbf{M}(T\llcorner d\pi) \tag{25}$$

and applying twice (ii) (one times from the induction assumption and the other from applying Step 1) we get

$$\begin{aligned}\int_{\mathbb{R}^m} T_x\psi_2(y)\psi_1(t)dydt &= \int_{\mathbb{R}} \psi_1(t)(T_t\llcorner \psi_2(\tilde{\pi})d\tilde{\pi})dt \\ &= T\llcorner(\psi_2(\tilde{\pi})\psi_1(\pi_1)d\pi)\end{aligned}$$

for every $\psi_2 \in C_c(\mathbb{R}^{m-1})$ and $\psi_1 \in C_c(\mathbb{R})$. Approximating $C_c(\mathbb{R}^m)$ with finite sums of products of type $\psi_1\psi_2$ and using the linearity and continuity of currents we get (ii). Point (iii) can be proved using its validity for the 1–d case and (25), reasoning exactly as in Step 1. Points (i) and (v) come directly from the way we have defined T_x.

Step 3 It remains to prove (v). Let us fix a $k-m$ form ω and denote by ρ_ε a sequence of mollifiers. From (ii) we get that

$$T^x(\omega) = \lim_{\varepsilon\downarrow 0} T(\rho_\varepsilon \circ \pi d\pi \wedge \omega).$$

Hence $T_x(\omega)$ is uniquely determined by (ii) for a.e. x. Using Lemma 2.3.3 we conclude that actually (ii) determines T_x uniquely for a.e. x and this proves (v).

□

Theorem 2.3.1 can be easily extended to flat currents.

THEOREM 2.3.5. *Let T be a k-dimensional flat current of finite mass on E and $\pi : E \to \mathbb{R}^h$ a Lipschitz function (with $h \leq k$). Then there exist $(k-h)$-dimensional flat currents $\langle T, \pi, x\rangle$ of finite mass such that:*

(i) *$\langle T, \pi, x\rangle$ is concentrated on $E \cap \pi^{-1}(x)$;*

(ii) *for every $\psi \in C_c(\mathbb{R}^h)$,*

$$\int_{\mathbb{R}^h} \langle T, \pi, x\rangle \psi(x) d\mathcal{L}^h = T \llcorner (\psi \circ \pi) d\pi; \tag{26}$$

(iii)

$$\int_{\mathbb{R}^h} \|\langle T, \pi, x\rangle\| d\mathcal{L}^h = \|T \llcorner d\pi\|. \tag{27}$$

PROOF. Let T_n be a sequence of normal currents such that

$$T = \sum_{i=1}^{\infty} T_n.$$

From Theorem 2.3.1 we have that there are normal $(k-h)$-dimensional currents $\langle T_n, \pi, x\rangle$ that verifies conditions (a), (b) and (c) above. Let us think of $\langle T_n, \pi, x\rangle$ as an L^1 function of x that takes values on the Banach space $\mathcal{M}_{k-h}(E)$ (endowed with the norm $\mathbf{M}$). Condition (c) and inequality (14) imply that

$$\sum_{i=1}^{\infty} \langle T_n, \pi, \cdot\rangle \tag{28}$$

is a totally convergent series in $L^1(\mathbb{R}^h, \mathcal{M}_{k-h}(E))$. We define $\langle T, \pi, \cdot\rangle$ as the sum of (28). It is easy to check that T verifies conditions (i), (ii) and (iii). Moreover we can extract a subsequence $T_{j(n)}$ such that for $\mathcal{L}^h$-a.e. $x \in \mathbb{R}^h$

$$\lim_{n\to\infty} \mathbf{M}\left(\langle T, \pi, x\rangle - \sum_{i=1}^{n} \langle T_{j(n)}, \pi, x\rangle\right) = 0.$$

This allows us to conclude that, for $\mathcal{L}^h$ a.e. x, $\langle T, \pi, x\rangle$ is a flat current of finite mass.

□

The key of the proof of Theorem 2.4.2 in the next section is the fact that the slicing map of a k-dimensional normal current T with respect to $\pi \in \mathrm{Lip}_b(E, \mathbb{R}^k)$ is a map of metric bounded variation if we endow $\mathcal{M}_0(E)$ with the flat norm

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in \mathrm{Lip}_b(E),\ \mathrm{Lip}(\phi) \leq 1\}.$$

This observation, due to Jerrard and Soner in the case of weak Jacobians ([**27**]), has been developed by Ambrosio and Kirchheim ([**8**]) in the framework of normal currents. (With a little effort one can see that the last definition of flat norm coincide with that given in Definition 2.2.10 when $E = \mathbb{R}^n$.)

THEOREM 2.3.6. *Let E be a weak separable metric space, T a normal n-dimensional current in E and $\pi : E \to \mathbb{R}^n$ a Lipschitz map. Then the slicing map*

$$\mathcal{S} : \mathbb{R}^n \ni x \to \langle T, \pi, x\rangle \in \mathcal{M}_0(E)$$

is metric bounded variation if we endow $\mathcal{M}_0(E)$ with the flat norm. Moreover the MBV semi-norm of $\langle T, \pi, x\rangle$ is bounded by the norm of T in $\mathcal{N}_k(E)$.

PROOF. With little effort one can see that there is a countable family $\mathcal{F} \subset \mathrm{Lip}_b(E)$ such that

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in \mathcal{F}\}$$

and $\mathrm{Lip}(\phi) \leq 1$ for every $\phi \in \mathcal{F}$. We can think of $\phi \in \mathcal{F}$ as a Lipschitz real function defined on $\mathcal{M}_0$. Then (recall Definition 2.1.3) we will show that

(a) for every such ϕ, $\phi \circ \mathcal{S}(x) = \langle \mathcal{S}(x), \phi\rangle$ is a function of locally bounded variation (as real–valued function of x);

(b)
$$\bigvee_{\phi \in \mathcal{F}} |D(\mathcal{S} \circ \phi)| \leq n\pi_\# \|T\| + n\pi_\# \|\partial T\|.$$

Indeed, let us fix a bounded ϕ such that $\mathrm{Lip}(\phi) \leq 1$. If we consider a test function $\psi \in C^1_c(\mathbb{R}^n)$, then

$$\begin{aligned}
(-1)^{i-1} \int_{\mathbb{R}^n} \mathcal{S}(\phi(x)) \partial_i \psi(x) dx &= (-1)^{i-1} T \llcorner d\pi(\phi\, \partial_i \psi \circ \pi) \\
&= T(\phi\, d(\psi \circ \pi) \wedge d\tilde{\pi}_i) \\
&= \partial T(\phi(\psi \circ \pi) d\tilde{\pi}_i) - T(\psi \circ \pi d\phi \wedge d\tilde{\pi}_i) \\
&\leq \|\partial T\|(\psi \circ \pi) + \|T\|(\psi \circ \pi),
\end{aligned}$$

where $d\tilde{\pi} = d\pi_1 \wedge \ldots \wedge d\pi_{i-1} \wedge d\pi_{i+1} \wedge \ldots \wedge d\pi_{m-n}$. Then $\phi \circ S$ is a function of locally bounded variation and

$$|D(\phi \circ \mathcal{S})| \leq n\pi_{\#}\|T\| + n\pi_{\#}\|\partial T\|.$$

$\square$

4. Decomposition of currents and rectifiability theorem

Given a k-dimensional current T of finite mass we can find a $\mathcal{H}^k$ σ-finite set L_T such that:

$\|T\|(F) = 0$ whenever $\mathcal{H}^k(F) < \infty$ and $L_T \cap F = \emptyset$.

We construct this set a follows. Let us consider

$$K = \sup\{\|T\|(L) | L \text{ is } \mathcal{H}^k\ \sigma\text{-finite}\}.$$

We choose a sequence (L_n) of $\mathcal{H}^k$ σ-finite sets such that $\|T\|(L_n) \uparrow K$ and we put $L_T = \cup L_n$. Then we have that L_T is $\mathcal{H}^k$ σ-finite and $\|T\|(L_T) = K$: hence L_T has the desired properties.

Definition 2.4.1. *Let T be a k-dimensional current of finite mass and L_T be defined as above. Then we define*

$$\begin{cases} T_u := T \llcorner (E \backslash L_T) \\ T_l := T \llcorner (L_T) \end{cases} \tag{29}$$

and we refer to T_l as lower dimensional part of T.

Of course $\|T_u\|$ and $\|T_l\|$ are mutually singular and $T_u + T_l = T$. Moreover $\|T_u\| \leq \|T\|$ and $\|T_l\| \leq \|T\|$. If E is $\mathcal{H}^p$ σ-finite for some $p > k$ then we define $\|T\|_a$ as the absolutely continuous part of $\|T\|$ with respect to $\mathcal{H}^p$. Of course $\|T\|_a$ and $\|T_l\|$ are mutually singular and so there exists a Borel set A_T disjoint from L_T such that $\|T\| \llcorner A_T = \|T\|_a$. Therefore we can define

$$\begin{aligned} T_a &= T_u \llcorner A_T \\ T_c &= T_u \llcorner (E \backslash A_T) \end{aligned} \tag{30}$$

and we refer to T_a and T_c as, respectively, *absolutely continuous part* and *Cantor part* of T. Notice that $T_a + T_c + T_l = T$.

When T is a flat current of finite mass it is easy to see that there is a Borel set R_T such that $\|T_l\|$ is absolutely continuous with respect to the measure $\mathcal{H}^k \llcorner R_T$. The main result of this section is that in this case T_l is a rectifiable current: to prove this we need only to check that R_T is rectifiable.

Theorem 2.4.2. *If E is separable and T is a k-dimensional flat current of finite mass on E, then T_l is a rectifiable current.*

A consequence of this fact is the following criterion of rectifiability, obtained in another framework by White in [**42**].

THEOREM 2.4.3. *Let T be a k-dimensional flat current of finite mass on a separable metric space E. Then T is rectifiable if and only if for every Lipschitz function $\pi : E \to \mathbb{R}^k$ and for $\mathcal{L}^k$ a.e. $x \in \mathbb{R}^k$ the sliced current $\langle T, \pi, x\rangle$ is supported on a finite number of points.*

We remark that one implication is trivial: if T is rectifiable and $\pi : E \to \mathbb{R}^k$ is Lipschitz, then $\langle T, \pi, x\rangle$ is concentrated on $\pi^{-1}(\{x\})$, which for almost every x consists of a finite number of points.

Before proving Theorem 2.4.2 and the other implication of Theorem 2.4.3 we need some tools.

THEOREM 2.4.4. *Let E be a separable metric space and let us endow $\mathcal{M}_0(E)$ with the norm*

$$\mathbf{F}(T) = \sup\{\langle T, \phi\rangle | \phi \in Lip_b(E),\ \ Lip(\phi) \leq 1\}.$$

If $\mathcal{S} \in MBV(\mathbb{R}^k, E)$ and $K \subset E$ is a compact set then there exists an $\mathcal{L}^k$-negligible set $A \in \mathbb{R}^k$ such that

$$S := \{y \in K | \|\mathcal{S}(x)\|(\{y\}) > 0 \quad \textit{for some } x \in \mathbb{R}^k \backslash A\}$$

is countably $\mathcal{H}^k$-rectifiable.

Before proving this Theorem we introduce the notion of maximal function for MBV mappings. Given a function $u \in MBV(\mathbb{R}^k, M)$, where M is a weakly separable metric space, we set

$$MDu(x) := \sup_{\rho > 0} \frac{\|Du\|(B_\rho(x))}{\omega_k \rho^k}$$

(MDu is known in the literature as the *maximal function* of the measure $\|Du\|$, see for example [**40**]). It is not difficult to see that this function is finite for almost every x: in fact we can estimate $\mathcal{L}^k(\{MDu > \lambda\}$ from above with a constant times $\|Du\|(\mathbb{R}^k)/\lambda$. As it happens for classical real-valued functions of bounded variation, MDu provides a Lipschitz property for u.

LEMMA 2.4.5. *Let (M, d) be weakly separable and $\mathcal{S} : \mathbb{R}^k \to M$ a map of metric bounded variation. Then there exists $N \subset \mathbb{R}^k$ of measure zero such that*

$$d(\mathcal{S}(x), \mathcal{S}(y)) \leq c(MD\mathcal{S}(x) + MD\mathcal{S}(y))|x - y|, \qquad \forall x, y \in \mathbb{R}^k \backslash N \tag{31}$$

where c depends only on k

PROOF. Let us choose a family $\mathcal{F}$ of weakly dense Lipschitz functions. Then for every $\varphi \in \mathcal{F}$ we define L_φ as the set of Lebesgue points of $\varphi \circ \mathcal{S}$ (which is a real function on $\mathbb{R}^k$). For every $x, y \in L_\varphi$ we claim that inequality (31) holds with $w = \varphi \circ \mathcal{S}$ in place of $\mathcal{S}$.

Indeed, let us choose a ball B of radius $R = |x-y|/2$ centered at $(x-y)/2$; we obtain

$$\begin{aligned}
&|w(x) - w(y)| = \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(y)|}{|x-y|} dz \\
&\leq \frac{1}{\omega_k R^k} \int_B \frac{|w(x) - w(z)|}{|x-z|} dz + \frac{1}{\omega_k R^k} \int_B \frac{|w(z) - w(y)|}{|z-y|} dz \\
&\leq \frac{1}{\omega_k R^k} \left(\int_{B_{2R}(x)} \frac{|w(x) - w(z)|}{|x-z|} dz + \int_{B_{2R}(y)} \frac{|w(y) - w(z)|}{|y-z|} dz \right)
\end{aligned}$$

Moreover we have

$$\frac{1}{\omega_k (2R)^k} \int_{B_{2R}(x)} \frac{|w(x) - w(z)|}{|x-z|} dz \leq \int_0^1 \frac{|Dw|(B_{tR}(x))}{\omega_k (tR)^k} dt \leq MDw(x),$$

and the claim easily follows.

Now, if we consider $\bigcap_{\varphi \in \mathcal{F}} L_{\varphi \circ \mathcal{S}}$, recalling that

$$d(\mathcal{S}(x), \mathcal{S}(y)) = \sup_{\varphi \in \mathcal{F}} |\varphi \circ \mathcal{S}(x) - \varphi \circ \mathcal{S}(y)|$$

and

$$\begin{aligned}
MD\mathcal{S}(x) &= \sup_{\rho > 0} \frac{\|D\mathcal{S}\|(B_\rho(x))}{\omega_k \rho^k} \\
&\geq \sup_{\rho > 0} \frac{\|D(\varphi \circ \mathcal{S})\|(B_\rho(x))}{\omega_k \rho^k} = MD(\varphi \circ \mathcal{S})(x),
\end{aligned}$$

we obtain (31).

□

PROOF OF THEOREM 2.4.4 First of all we set

$$A := N_1 \cup \{x \in \mathbb{R}^k \,|\, MD\mathcal{S}(x) = \infty\},$$

where N_1 is the set of measure zero that plays the role of N in the previous Lemma. Of course $\mathcal{H}^k(A) = 0$.

Following [**8**] we define $Z_{\varepsilon,\delta}$ as the set of points $z \in \mathbb{R}^k \backslash A$ such that

(a) $MD\mathcal{S}(z) \leq 1/(2\varepsilon)$;
(b) for every $x \in K$ such that $\|\mathcal{S}(z)\|(\{x\}) \geq \varepsilon$ there holds $\|\mathcal{S}(z)\|(B_{3\delta}(x) \backslash \{x\}) \leq \varepsilon/3$.

Next we define $R_{\varepsilon,\delta} := \{x \in E \mid \|\mathcal{S}(z)\|(\{x\}) \geq \varepsilon \text{ for any } z \in Z_{\varepsilon,\delta}\}$. Observing that

$$S = \bigcup_{\varepsilon,\delta} R_{\varepsilon,\delta},$$

we will prove that for each ε, δ the set $R_{\varepsilon,\delta}$ is $\mathcal{H}^k$-rectifiable. Indeed, for every $x, x' \in R_{\varepsilon.\delta}$ and every $z, z' \in Z_{\varepsilon,\delta}$ such that

(i) $\|\mathcal{S}(z)\|(\{x\}) \geq \varepsilon$, $\|\mathcal{S}(z)\|(\{x'\}) \geq \varepsilon$,
(ii) $d(x, x') \leq \delta$,

it holds

$$d(x, x') \leq \frac{3c(\delta+1)}{\varepsilon^2}|z - z'|. \tag{32}$$

Before proving this estimate we remark it implies that $R_{\varepsilon,\delta} \cap B$ is the image of a Lipschitz function whenever $\operatorname{diam}(B) \leq \delta$: indeed for any $z \in Z_{\varepsilon,\delta} \cap B$ there is only one $x = f(z) \in R_{\varepsilon,\delta}$ such that $\|\mathcal{S}\|(x) \geq \varepsilon$; moreover f is Lipschitz and, if D is the domain of f, $f(D) = B \cap R_{\varepsilon,\delta}$.

Now we complete the proof by showing that (32) holds. Let us set $d = d(x, x')$ and consider a Lipschitz function $\phi : E \to \mathbb{R}$ such that

(a') $\phi(y) = d(y, x)$ for every $y \in B_d(x)$;
(b') $\phi \equiv 0$ in $\mathbb{R}^m \setminus B_{2\delta}(x)$;
(c') $\sup|\phi| = d$ and $\operatorname{Lip}(\phi) \leq 1$.

We have $|(\mathcal{S}(z))(\phi)| \leq \varepsilon\delta/3$ and $|(\mathcal{S}(z'))(\phi)| \geq \varepsilon\delta - \varepsilon\delta/3$, so we get

$$\begin{aligned}\frac{\varepsilon}{3} d(x, x')| &\leq |(\mathcal{S}(z'))(\phi) - (\mathcal{S}(Z))(\phi)| \leq (\delta+1)\mathbf{F}(\mathcal{S}(z) - \mathcal{S}(z')) \\ &\leq c(\delta+1)(MD\mathcal{S}(z) + MD\mathcal{S}(z'))|z - z'|.\end{aligned}$$

Recalling that $MD\mathcal{S}(z) \leq 1/(2\varepsilon)$, we obtain the desired estimate.

□

Proof of Theorem 2.4.2 We need only to prove that T_l is concentrated on a $\mathcal{H}^k$ σ-rectifiable set. First let us fix a Lipschitz function $\pi : E \to \mathbb{R}^k$. We want to prove that $T_l \llcorner d\pi$ is concentrated on a rectifiable set. We set $\mathcal{S}_\pi(x) = \langle T, d\pi, x\rangle$ and we claim that

(S) there exists a set $N \subset \mathbb{R}^k$ such that $\mathcal{L}^k(N) = 0$ and
$$S_\pi := \{y \in E | \|\mathcal{S}_\pi(x)\|(\{y\}) > 0 \quad \text{for some } x \in \mathbb{R}^k \setminus N\}$$
is countably rectifiable.

To prove this let us choose a sequence T_n of normal currents such that

(i) $\mathbf{M}(T - T_n) \to 0$;
(ii) there exists a set $N^\infty \subset \mathbb{R}^k$ such that $\mathcal{L}^k(N^\infty) = 0$ and
$$\lim_{n\to\infty} \mathbf{M}(\mathcal{S}(x) - \langle T_n, \pi, x\rangle) = 0$$
for every $x \in \mathbb{R}^k \setminus N^\infty$.

To simplify the notation we write $\mathcal{S}_n(x) = \langle T_n, \pi, x\rangle$. We remark that if (μ_n) is a sequence of finite measures and there exists a measure μ such that $\|\mu_n - \mu\|_{var} \to 0$, then the set of atoms of μ is contained in the union of the sets of atoms of μ_n. Recalling that $\mathcal{M}_0(E)$ can be represented as the space of finite measures on E, we conclude that for almost every $x \in \mathbb{R}^k \backslash N^\infty$

$$\{z | \|\mathcal{S}_\pi(x)\|(z) > 0\} \subset \bigcup_n \{z | \|\mathcal{S}_n(x)\|(z) > 0\}.$$

Using Theorem 2.4.4 we infer that for every i there exists a set $N^i \subset \mathbb{R}^k$ of measure zero such that

$$S^i := \{z \in E | \|\mathcal{S}_i(x)\|(z) > 0 \quad \text{for some } x \in \mathbb{R}^k \backslash N_i\}$$

is countably rectifiable. If in statement (A) we set

$$N = N^\infty \cup \bigcup_i N_i$$

then we have

$$S_\pi \subset \bigcup_{i=1}^{\infty} S^i.$$

We conclude that S_π is countably rectifiable.

Now, let us prove that

$$\|T_l \llcorner d\pi\|(E \backslash S_\pi) = 0. \tag{33}$$

Recalling Definition 2.4.1 we must only check that $\|T \llcorner d\pi\|(A) = 0$ for every $\mathcal{H}^k$ σ-finite set A such that $A \cap S_\pi = \emptyset$.

Since A is $\mathcal{H}^k$ σ-finite, for a.e. $x \in \mathbb{R}^k$, $(\pi^{-1}\{x\}) \cap A$ contains at most a countable number of points; this fact combined with $A \cap S_\pi = \emptyset$ implies that, for a.e. $x \in \mathbb{R}^k$, $\|\mathcal{S}_\pi(x)\|(A) = 0$. Then we have

$$\|T \llcorner d\pi\|(A) = \int_{\mathbb{R}^n} \|\mathcal{S}_\pi(x)\|(A) d\mathcal{L}^k(x) = 0$$

and this proves (33).

Now we recall that $\|T\|$ is concentrated on a σ-compact set; because of this fact it can be proved (see [**8**], Lemma 5.4) that there exists a countable set $D \subset \mathrm{Lip}_1(E) \cap \mathrm{Lip}_b(E)$ such that

$$\|T\| = \bigvee \{\|T \llcorner d\pi\| | \pi_1, \dots, \pi_k \in D\}. \tag{34}$$

If we take the countably rectifiable set

$$S := \bigcup \{S_\pi | \pi_1, \dots, \pi_k \in D\},$$

then from (34) it follows that $\|T_l\|(E/S) = 0$.

□

Notice that if we are in the hypotheses of Theorem 2.4.3, then we can reason as in the previous case. In fact we have that for every π, $T \llcorner d\pi$ is concentrated on a $\mathcal{H}^k$ rectifiable set. Then it follows that T is concentrated on a $\mathcal{H}^k$ rectifiable set and coincides with T_l.

REMARK 2.4.6. Assuming that every set has a cardinality that is a Ulam number, we can drop the assumption that E is separable (see Remark 2.2.4).

CHAPTER 3

Weak Jacobians through currents

In this chapter we apply the tools of the previous one to the study of the space BnV. This space, which has been defined by Jerrard and Soner in [**27**], is composed, roughly speaking, by those functions such that their weak Jacobians are measures.

As already mentioned in the introduction if $u \in C^1(\mathbb{R}^m, \mathbb{R}^n)$, with $m \geq n$, then the Jacobian of u can be seen as the differential form $\omega = du_1 \wedge \ldots \wedge du_n$. Of course this notion can be easily extended to functions $u \in W^{1,n}$ but the main idea for a broader extension is based on the fact that $\omega = d(u_1 du_2 \wedge \ldots \wedge du_n)$. Indeed we need less summability on the derivatives of u to handle the form $\nu = u_1 du_2 \wedge \ldots \wedge du_n$ and we can define the weak Jacobian of u as the exterior derivative of ν in the weak sense.

In the first section we propose to think of $u_1 du_2 \wedge \ldots \wedge du_n$ as a current T via the natural action

$$T(dw_1 \wedge \ldots \wedge dw_{m-n+1}) =$$

$$\int_{\mathbb{R}^m} u_1 \det(\nabla u_2, \ldots, \nabla u_n, \nabla w_1, \ldots, \nabla w_{m-n+1}) d\mathcal{L}^m.$$

Thus we can define the weak Jacobian $[Ju]$ as the boundary of T and the space BnV can be identified with those u such that $[Ju]$ is a normal current. Using this identification we will give different proofs of some theorems of Jerrard and Soner (see for example Theorem 3.1.6 and Theorem 3.2.1) as byproducts of corresponding theorems for metric currents and we hope to show that this simplifies the notations and underlines the importance of some properties (like for example the BV estimate on slices).

In the last section we define a new class of functions, called SBnV, that is a generalization of the space of special functions of bounded variations (see [**4**], [**6**]). We prove for SBnV a closure Theorem which is a generalization of the closure Theorem for SBV (see Theorem 3.3.5 and Theorem 3.3.7).

The definition of SBnV is induced, as a particular case, by the more general decomposition of flat currents of finite mass which has been proposed in the fourth section of the previous chapter.

More precisely taken a function $u \in \mathrm{BnV}$ we single out a maximal "lower dimensional part" $[Ju]_l$ of the Jacobian and we prove that it is a rectifiable current. The remaining part of the Jacobian (namely $[Ju] - [Ju]_l$) can be split further into two currents: one that is absolutely continuous with respect to the Lebesgue measure and the other that is singular (which we call Cantor part, in analogy with the case of functions of bounded variation). Thanks to its flatness the lower dimensional part of $[Ju]$ can be represented as

$$\langle [Ju]_l, \omega \rangle = \int_{S_l} m(x) \langle \tau(x), \omega(x) \rangle d\mathcal{H}^{m-n}$$

where S_l is a $\mathcal{H}^k$-rectifiable set, $\tau(x)$ is its approximate tangent space in x and ω is any smooth $(m-n)$-form.

Then we analyze the structure of the absolutely continuous part of the Jacobian and extending a result of Müller (see [**34**]) we prove that it can be represented as

$$[Ju]_a = H(du_1 \wedge \ldots \wedge du_n)\mathcal{L}^m$$

where H is the Hodge star operator. This allows to recover the Jacobian determinant as defined in the classical Geometric Measure Theory as part of the weak Jacobian.

Thus $[Ju]_l + [Ju]_a$ can be represented as $\nu d\mu$, where ν is a simple covector and μ is a measure. To complete this picture it would be nice to have a proof that also the Cantor part can be represented as $\nu_c d\mu_c$ for some Borel function ν_c whose values are simple covectors, because it would allow to have the same representation for the full weak Jacobian. Unfortunately we have not been able to prove it but to our knowledge there are no examples showing that $[Ju]_c$ has not this structure. Hence we advance the conjecture that indeed such a property holds true. This conjecture seems similar to the conjecture proved by Alberti in [**2**].

In the last section we define the functions of special bounded higher variation as those BnV functions whose Jacobian has zero Cantor part. Finally we prove that under suitable conditions (i.e. equiintegrability of the absolutely continuous part and equiboundedness of the Hausdorff measure of the singular supports) a compactness property holds for SBnV.

The proof of this compactness property is achieved using the slicing technique developed in the previous chapter. In particular we first prove it for maps $u : \mathbb{R}^n \to \mathbb{R}^n$ and we then pass to the general case.

1. Weak Jacobians and BnV functions

In this section we are going to transpose some definitions and concepts from [**27**] in the language of the second chapter. We will work with differential forms with compact support and local currents, but this does not create any problem, as observed in Remark 2.2.2. Finally we recall that what we call differential forms in this thesis are not the usual Lipschitz differential forms: in terms of the classical theory $\mathcal{D}_b^k(\mathbb{R}^n)$ is the set of Lipschitz *simple* differential forms with bounded support. Moreover, since in the Euclidean case bounded closed sets are compact we will use the notation $\mathcal{D}_c^k$ in place of $\mathcal{D}_b^k$.

DEFINITION 3.1.1. *We define the k-dimensional local current H_k in $\mathbb{R}^k$ as*

$$H_k(f dg) = \int_{\mathbb{R}^k} f \det(\nabla g_1, ..., \nabla g_k) d\mathcal{L}^k.$$

The continuity axiom (Definition 2.2.1, condition (c)) is satisfied because the Jacobian determinant is weakly* continuous in $W^{1,\infty}$ (see Theorem 3.1.4 below). We remark that the classical Hodge star operator assigns to every $\omega \in \mathcal{D}_c^k(\mathbb{R}^n)$ the local $(n-k)$-dimensional current given by $H_n \llcorner \omega$.

In the definition of $H_n \llcorner \omega$ the regularity assumptions on ω can be weakened. In particular, let us suppose that $\omega = f dg$ satisfies

(i) $f \in L^p$;
(ii) $g \in W^{1,q}(\mathbb{R}^n, \mathbb{R}^k)$;
(iii) $1/p + k/q = 1/r < 1$.

Then it is well known (see for example [**33**], [**21**] or again Theorem 3.1.4) that the map
$F : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^{n-k}) \to L^r(\mathbb{R}^n)$ given by

$$F(u) = f \det(\nabla g_1, \dots, \nabla g_k, \nabla u_1, \dots, \nabla u_{n-k})$$

is continuous if we endow $L^r(\mathbb{R}^n)$ with the weak topology and $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^{n-k})$ with the weak* topology. Even in this case, with a slight abuse of notation, we define $H_n \llcorner \omega$ as the k-dimensional local current T given by

$$T(f dg) = \int_{\mathbb{R}^n} f \det(\nabla g_1, \dots, \nabla g_k, \nabla u_1, \dots, \nabla u_{n-k}) d\mathcal{L}^n.$$

We will see below that this is a crucial point in the definition of weak Jacobians. In the rest of this section U will denote an open set.

DEFINITION 3.1.2. *Let $u \in W^{1,p}_{loc}(U, \mathbb{R}^n) \cap L^\infty$ with $U \subset \mathbb{R}^m$, $p \geq n-1$ and $m \geq n$ (or $u \in W^{1,mn/(m+1)}$). We define $j(u)$ as the $(m-n+1)$-dimensional local current $(-1)^n H_m \llcorner (u^1 \wedge du_2 \wedge \ldots \wedge du_n)$.*

DEFINITION 3.1.3. *Let u be as in the previous definition. Then we define*

$$[Ju] := \partial j(u).$$

We say that $u \in \mathrm{BnV}(U, \mathbb{R}^n)$ if $j(u)$ is a normal local current.

The last definition is motivated as follows. Let us put $\nu = u_1 du_2 \wedge \ldots \wedge du_n$ and suppose that u is sufficiently regular (i.e. Lipschitz). Then we have

$$\begin{aligned}
\langle [Ju], \omega \rangle \quad &= \langle j(u), d\omega \rangle = (-1)^n H_m(\nu \wedge d\omega) \\
&= (-1)^n H_m((-1)^{n-1}(d(\nu \wedge \omega) - d\nu \wedge \omega))) \\
&= -\partial H_m(\nu \wedge \omega) + H_m(d\nu \wedge \omega) \\
&= H_m \llcorner d\nu(\omega).
\end{aligned}$$

We remark that in view of this fact we could have defined $j(u)$ as

$$\mathrm{sign}\,(\pi) H_m \llcorner (u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}),$$

where π is any permutation of the set $\{1, \ldots, n\}$. Indeed , if u is a smooth function, then

$$\begin{aligned}
(-1)^n \partial(H_m \llcorner u_1 du_2 \wedge \ldots \wedge du_n) &= H_m \llcorner d(u_1 du_2 \wedge \ldots \wedge du_n) \\
&= H_m \llcorner (du_1 \wedge du_2 \wedge \ldots \wedge du_n) \\
&= H_m \llcorner \mathrm{sign}\,(\pi)(du_{\pi(1)} \wedge du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}) \\
&= (-1)^n \mathrm{sign}\,(\pi) H_m \llcorner d(u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}) \\
&= (-1)^n)\mathrm{sign}\,(\pi) \partial(H_m \llcorner u_{\pi(1)} du_{\pi(2)} \wedge \ldots \wedge du_{\pi(n)}). \qquad (35)
\end{aligned}$$

These equalities follows from the fact that, if T is a k-dimensional local current and ω is an h-dimensional form with $k \leq h-1$, then

$$\partial T \llcorner \omega = (\partial T) \llcorner \omega - T \llcorner d\omega.$$

Now, approximating every $u \in \mathrm{BnV}$ by convolutions with standard mollifiers, we obtain the identity (35) in its full generality. Indeed it is easy to see that if $u_n \to u$ in the strong Sobolev topology and $\|u_n\|_\infty \leq c$, then $j(u_n)$ converges to $j(u)$ as local current.

Indeed $j(u)$ satisfies a stronger continuity result: appropriate weak convergence of the functions induces weak convergence on the Jacobians. More precisely

THEOREM 3.1.4. *Suppose that $(u_n), u$ satisfy the conditions of Definition 3.1.2 and*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc};
(b) $u_k \to u$ *strongly in* $L^{p_2}_{loc}$;
(c) $(n-1)/p_1 + 1/p_2 \geq 1$ *and* $p_1 > n-1$

(or $u_k \rightharpoonup u$ in $W^{1,n-1}_{loc}$, $u_k \in C(U)$ and $u_k \to u$ uniformly on compact sets).

Then $j(u_n) \rightharpoonup j(u)$ as local current.

For the convenience of the reader we report here a proof of this fact (compare with [**32**], [**21**]).

PROOF. We will prove the theorem by induction on n. Notice that for $n = 1$ the statement comes from the usual weak convergence in Sobolev spaces.

Let us take a sequence (u^k) satisfying, for example, the first version of condition (c). After fixing a differential form $f\,dg$, let us compute

$$\langle j(u_n), f\,dg\rangle = \int_{\mathbb{R}^m} f u_1^k \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k).$$

Now let us call $\tilde{u}^k : \mathbb{R}^m \to \mathbb{R}^{n-1}$ the map given by $(u_2^k, \dots, u_n^k)$. We can also check easily (via standard approximations argument in the strong topology of the Sobolev spaces, exactly as done above) that

$$\langle [J\tilde{u}^k], f\,dg\rangle = \int_{\mathbb{R}^m} f \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k).$$

So since $[J\tilde{u}^k] = \partial j(\tilde{u}^k)$ and thanks to the induction hypothesis (because thanks to Sobolev embeddings $\tilde{u}$ satisfies (a), (b) and the first version of (c)) we conclude that

$$\begin{aligned}&\lim_{k\to\infty} \int_{\mathbb{R}^m} \varphi \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k)\\ = &\int_{\mathbb{R}^m} \varphi \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2, \dots, \nabla u_n)\end{aligned}$$

for any smooth φ. But we also notice that

$$\det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k)$$

is uniformly bounded in $L^{p_1/(n-1)}$ and hence weakly converging (up to subsequence) to some function h. Thanks to the previous identity this function h has to be equal to $\det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2, \dots, \nabla u_n)$. Hence, since u_1^k is strongly converging in L^{p_2} to u_1 we conclude that

$$u_1^k \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k)$$

is weakly converging in L^1 to the function

$$u_1 \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2, \dots, \nabla u_n).$$

This of course gives that

$$\lim_{k\to\infty}\int_{\mathbb{R}^m} f u_1^k \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2^k, \dots, \nabla u_n^k)$$
$$= \int_{\mathbb{R}^m} f u_1 \det(\nabla g_1, \dots, \nabla g_{m-n+1}, \nabla u_2, \dots, \nabla u_n),$$

which completes the proof. We reason in a similar way to handle the second version of (c). □

If the hypotheses of the previous Theorem hold then we have

$$\langle [Ju_n], \omega\rangle = \langle \partial j(u_n), \omega\rangle = \langle j(u_n), d\omega\rangle \to \langle j(u), d\omega\rangle = \langle [Ju], \omega\rangle.$$

Now let us see how the local current $[Ju]$ behaves with respect to slicing when $u \in \mathrm{BnV}$. Let us consider a projection π of $\mathbb{R}^m$ onto a subspace of dimension $m-k \le m-n$. For the sake of simplicity we choose a system of coordinates and we suppose that π is the projection on the first $m-k$ coordinates: we will adopt the notation $\mathbb{R}^m \ni z = (x,y) \in \mathbb{R}^{m-k} \times \mathbb{R}^k$.

We notice that, for a.e. $x \in \mathbb{R}^{m-k}$, $j(u(x,\cdot))$ is a $(k-n+1)$ dimensional local current in $\mathbb{R}^k$: indeed, because of the Fubini-Tonelli Theorem, for a.e. x the map $u(x,\cdot)$ belongs to the appropriate Sobolev space that allows us to define

$$j(u(x,\cdot)) := (-1)^n H_k \llcorner u_1(x,\cdot) d_y u(x,\cdot).$$

Definition 3.1.5. *We denote by i^x the natural identification between $\mathbb{R}^k$ and the affine subspace $\{x\}\times\mathbb{R}^k$ of $\mathbb{R}^m$.*

Theorem 3.1.6. *Let u be as in Definition 3.1.2 and $\pi : \mathbb{R}^{m-k}\times\mathbb{R}^k \to \mathbb{R}^{m-k}$ a projection, with $k \ge n$. Then we have*

$$\langle j(u), d\pi, x\rangle = (-1)^l (i^x)_\# j(u(x,\cdot)), \tag{36}$$

$$\langle [Ju], d\pi, x\rangle = (-1)^r (i^x)_\# [Ju(x,\cdot)]. \tag{37}$$

with $l = (m-k)(n-1)$ and $r = (m-k)n$.

Proof. We use the notations of the previous paragraph to simplify the calculations. We observe that

$$j(u)\llcorner d\pi = H_m \llcorner (u_1 du_2 \wedge \dots \wedge du_n \wedge dx_1 \dots \wedge dx_{m-k}).$$

So, for every $f dg \in \mathcal{D}_c^{k-n+1}$, we have

$$H_m \llcorner (u_1\, du_2 \wedge \dots \wedge du_n \wedge d\pi)(f dg)$$

$$= \int_{\mathbb{R}^m} f u_1 \det(\nabla u_2, \dots, \nabla u_n, e_1, \dots, e_{m-k}\nabla g_1, \dots, \nabla g_{k-n+1}, dz \tag{38}$$

$$= (-1)^{(m-k)(n-1)} \int_{\mathbb{R}^m} f u_1 \det(e_1, \dots, e_{m-k}, \nabla\tilde{u}, \nabla g)\, dz,$$

where $e_1, \dots, e_k$ are the first $m-k$ vectors of the canonical basis and $\tilde{u}$ denotes the vector $(u_2, \dots, u_n)$.

We remark that the matrix $(e_1, \dots, e_{m-k}, \nabla\tilde{u}, \nabla g)$ can be written as

$$\begin{pmatrix} Id & \nabla_x \tilde{u} & \nabla_x g \\ 0 & \nabla_y \tilde{u} & \nabla_y g \end{pmatrix}$$

(where Id is the identical $k\times k$ matrix, 0 is the $(m-k)\times k$ null matrix). Therefore $(\nabla_y \tilde{u}, \nabla_y g)$ is a $(k \times k)$ matrix and

$$\det(e_1, \dots, e_{m-k}, \nabla\tilde{u}, \nabla g) = \det(\nabla_y \tilde{u}, \nabla_y g);$$

this means that (38) is equal to

$$(-1)^l \int_{\mathbb{R}^{m-k}} \int_{\mathbb{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\, dy\, dx. \quad (39)$$

Then the expression

$$\int_{\mathbb{R}^k} f(x,y) u_1(x,y) \det(\nabla_y \tilde{u}(x,y), \nabla_y g(x,y))\, dy$$

can be read as

$$\langle (i^x)_\# j(u(x,\cdot)), f\, dg \rangle.$$

We conclude from (39) that

$$\mathbb{R}^k \ni x \to \mathcal{S}(x) = (-1)^l (i^x)_\# j(u(x,\cdot))$$

is the slicing for $j(u)$ with respect to $d\pi$.

Notice that

$$\begin{aligned} [Ju] \llcorner d\pi &= (-1)^{m-k}(\partial(j(u)\llcorner d\pi) - j(u)\llcorner d(d\pi)) \\ &= (-1)^{m-k}\partial(j(u)\llcorner d\pi); \end{aligned}$$

hence $(-1)^r (i^x)_\# [Ju(x,\cdot)]$ is a slicing map for $[Ju]$.

□

For the sake of simplicity from now on we will identify the local current $[Ju(x,\cdot)]$ and its push-forward via i^x.

2. Structure of weak Jacobians

Using the decomposition defined in Section 2 of Chapter 2, when $u \in \mathrm{BnV}$ we can consider $[Ju]_a$, $[Ju]_c$ and $[Ju]_l$; moreover from Theorem (2.4.2) it follows that $[Ju]_l$ is a rectifiable local current. From now on we define S_u as the set on which $[Ju]_l$ is concentrated.

From the classical theory we know that there exists a Borel function ν from $\mathbb{R}^m$ to the linear space of $m-n$ covectors $\Lambda_{m-n}(\mathbb{R}^m)$ such that

$$\langle [Ju], \omega \rangle = \int_{\mathbb{R}^m} \langle \nu(x), \omega(x) \rangle d\|Ju\| \tag{40}$$

(where $\|Ju\|$ is the mass of $[Ju]$). Of course similar representations hold for the three parts of $[Ju]$:

$$\langle [Ju]_a, \omega \rangle = \int_{\mathbb{R}^m} \langle \nu_a(x), \omega(x) \rangle d\|Ju\|_a$$

$$\langle [Ju]_c, \omega \rangle = \int_{\mathbb{R}^m} \langle \nu_c(x), \omega(x) \rangle d\|Ju\|_c$$

$$\langle [Ju]_l, \omega \rangle = \int_{S_u} \langle \nu_l(x), \omega(x) \rangle d\mathcal{H}^{m-n}.$$

In fact we can say a little more: from the fact that $[Ju]$ is a flat current it follows that

$$\nu_l(x) = m(x)\tau(x) \qquad \text{for } \mathcal{H}^{m-n} \text{ a.e. } x,$$

where $\tau(x)$ is the approximate tangent plane to S_u in x and m is a Borel measurable real-valued function.

Using the machinery just outlined we easily recover here the result of [**27**], namely the rectifiability of weak Jacobians which take values into spheres. Indeed, thanks to the flatness of $[Ju]$ we also have the representation written above, which in the language of Jerrard and Soner (i.e. in the distributional forms language) means that when $u \in \mathrm{BnV}(\mathbb{R}^m, \mathbf{S}^{n-1})$ the form

$$Ju := \frac{1}{n} d \left(\sum_{i=1}^{m} u_i du_1 \wedge \ldots \wedge \widehat{du_i} \wedge \ldots \wedge du_n \right)$$

can be represented as $m df_1 \wedge \ldots \wedge df_n \mathcal{H}^{m-n} \llcorner R$ for some rectifiable set R, where m is a Borel function and f_i are Lipschitz functions such that every $\nabla f_i(x)$ exists for a.e. $x \in R$ and is perpendicular to the tangent of R in x.

THEOREM 3.2.1. *If* $u \in \mathrm{BnV}(\mathbb{R}^m, \mathbf{S}^{n-1})$ *then* $[Ju]$ *is a rectifiable current.*

PROOF. Thanks to the construction of the previous chapter (in particular thanks to Theorem 2.4.3) and to Theorem 3.1.6 of the previous section it is sufficient to prove that the theorem holds for $m = n$. Hence in this case we have to prove that the measure $[Ju]$ is indeed a sum of Dirac masses.

Let us fix $x \in \mathbb{R}^m$. If $r > 0$ we approximate the function $\chi_{B_r(x)}$ with a family of continuous radial functions φ_ε of the form

$$\varphi_\varepsilon(\rho) = \begin{cases} 1 & \text{if } \rho < r - \varepsilon \\ 0 & \text{if } \rho > r + \varepsilon \\ 1/2 + (\rho - r)/2\varepsilon & \text{otherwise.} \end{cases}$$

Then, except for a countable number of r, we have

$$[Ju](B_r(x)) = \int \chi_{B_r(x)} d[Ju] = \lim_{\varepsilon \downarrow 0} \int \varphi_\varepsilon d[Ju].$$

Integrating by parts the last integral becomes

$$\frac{1}{2\varepsilon} \int_{B_{r+\varepsilon} \setminus B_{r-\varepsilon}} \frac{1}{n} \left(u \cdot \Lambda_n(\nabla u) \cdot \frac{x}{|x|} \right) dx,$$

where $\Lambda_n(\nabla u)$ is the matrix of cofactors of ∇u (cfr. Lemma 4.1.1, Remark 4.1.2 and Definition 5.1.1). Now we notice that thanks to standard Fubini–Tonelli argument, for $\mathcal{L}^1$ a.e. r the previous integral converges to

$$\frac{1}{n} \int_{\partial B_r} u \cdot \Lambda_n(\nabla u) \cdot \nu \, d\mathcal{H}^{n-1} \tag{41}$$

where this last integral can be written because $u \in L^\infty$ (since it takes values in $\mathbf{S}^{n-1}$) and $\nabla u \in L^{n-1}(\partial B_r)$ for a.e. r, again thanks to Fubini–Tonelli Theorem.

Now, as remarked in Lemma 4.1.4 if $u \in W^{1,n-1} \cap C(\partial B_r)$ then the integral in (41) is equal to the area of $\mathbf{S}^{n-1}$ (which will be called ω_n) times the degree of u seen as map from the sphere ∂B_r to the standard unit sphere $\mathbf{S}^{n-1}$. Hence if u is continuous on ∂B_r then $[Ju](B_r)/\omega_n$ is an integer. Now given a general $u \in W^{1,n-1} \cap L^\infty(\partial B_r)$ we can approximate it strongly in $W^{1,n-1}$ and weakly in L^∞ with a sequence of continuous functions u_k. Then it is easy to check that

$$\frac{1}{n} \int_{\partial B_r} u \cdot \Lambda_n(\nabla u) \cdot \nu \, d\mathcal{H}^{n-1} = \lim_k \frac{1}{n} \int_{\partial B_r} u_k \cdot \Lambda_n(\nabla u_k) \cdot \nu \, d\mathcal{H}^{n-1}.$$

Hence, after defining $\mu := [Ju]/\omega_n$ it turns out that there exists a negligible set $L \subset \mathbb{R}^+$ such that for $r \in \mathbb{R}^+ \setminus L$ we have that $\mu(B_r)$ is integer. Since for generic r we can write

$$\mu(B_r) = \lim_{r_k \uparrow r} B_{r_k}$$

for some $(r_k) \subset \mathbb{R}^+ \setminus L$, we conclude that $\mu(B_r) \in \mathbb{Z}$ for every r.

At this point it is not difficult to conclude that μ is the sum of a finite number of Dirac masses. Indeed, since for every x

$$\lim_{r\downarrow 0}\mu(B_r(x)) = \mu(\{x\})$$

then if $\mu(\{x\}) = 0$ definitely $\mu(B_r(x)) = 0$. Hence let us take the set $N := \{x|\mu(\{x\}) = 0\}$. Take every compact set $K \subset N$ and fix ε. Thanks to Besicovitch covering argument we can find a countable collection of disjoint balls B_i which have radius less then ε, covers μ almost all K and such that $\mu(B_i) = 0$. Then call K_ε the union of these balls. Of course $\mu(K_\varepsilon) = 0$ and moreover $\mu(K) = \lim_{\varepsilon\downarrow 0}\mu(K_\varepsilon) = 0$. Thanks to the Borel regularity of μ, for every measurable set $N' \subset N$ $\mu(N')$ is the supremum of $\mu(K)$ for $K \subset N'$ compact. Hence $\mu(N') = 0$ and so μ is concentrated on $\mathbb{R}^n\backslash N$.

Of course, since for every $x \in \mathbb{R}^n\backslash N$ we have $|\mu(\{x\})| \geq 1$, the fact that $\|\mu\|$ is a finite measure implies that N has finite cardinality. We conclude that μ is the union of a finite number of Dirac masses.

□

Also for the absolutely continuous part we have a property similar to what holds of the lower singular part. Indeed let us suppose that $u \in \mathrm{BnV}(U,\mathbb{R}^n)$ with $U \subset \mathbb{R}^n$. Then we can define a notion of "pointwise determinant" of ∇u as follows. We choose a Borel function M which is a pointwise representative of ∇u and we define the pointwise Jacobian $\det(\nabla u)$ as the class of measurable functions f such that $\det M(x) = f(x)$ for $\mathcal{L}^n$-a.e. x. Then from a result of Müller ([**34**]) we know that $\det(\nabla u) \in L^1_{loc}$ and $[Ju]_a = \det(\nabla u)\mathcal{L}^n$. In the following Theorem we will prove a slight generalization of this result.

THEOREM 3.2.2. *Let u be a* BnV *function and let us choose a pointwise representative $\tilde{u}$ of u. Then there exists a Borel function $\nu_a : \mathbb{R}^m \to \Lambda_{m-n}(\mathbb{R}^m)$ such that*

$$\langle [Ju]_a, \omega\rangle = \int_{\mathbb{R}^m} \langle \nu_a(x), \omega(x)\rangle d\mathcal{L}^m \tag{42}$$

and

$$\nu_a(x) = d\tilde{u}_1(x) \wedge \ldots \wedge d\tilde{u}_n(x) \qquad \text{for } \mathcal{L}^k \text{ a.e. } x \in \mathbb{R}^m, \tag{43}$$

where $d\tilde{u}_i(x)$ is the approximate differential of u_i at x.

Before proving the Theorem we address the special case when $m = n$.

LEMMA 3.2.3. *Let* $u \in \mathrm{BnV}(U, \mathbb{R}^n)$, *with* $U \subset \mathbb{R}^n$, *and let us choose a pointwise representative* $\tilde{u}$ *of* u. *Then* $\det(\nabla u)$ *is summable and*

$$\langle [Ju]_a, \omega \rangle = \int_{\mathbb{R}^m} \omega(x) \det(\nabla u(x)) d\mathcal{L}^m \tag{44}$$

PROOF. We follow the proof of Müller in [**34**] and to simplify notations we identify u and $\tilde{u}$.

We know that $[Ju]_a$ acts on 0 dimensional forms, i.e. on bounded measurable functions (recall Remark 2.2.13 and the fact that $[Ju]_a$ has finite mass). So we can write

$$\langle [Ju]_a, f \rangle = \int_{\mathbb{R}^n} \nu(x) f(x) d\mathcal{L}^n$$

where $\nu \in L^1_{loc}(\mathbb{R}^n)$: we only have to check

$$\nu(x) = \det(\nabla u(x)) \qquad \text{for a.e. } x.$$

Therefore let us fix $x_0 \in \mathbb{R}^n$ such that

(a) x_0 is a Lebesgue point for ν, $|\nabla u|^p$ and u (where p depends on the Sobolev space chosen in Definition 3.1.2);

(b)
$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \|Ju\|_s(B_\varepsilon(x_0)) = 0$$

(we recall that $\|Ju\|_s$ is the singular part of $\|Ju\|$).

Without loss of generality we can suppose that $x_0 = 0$ and $u(x_0) = 0$ and we define the rescaled functions

$$u^\varepsilon := \frac{1}{\varepsilon} u(\varepsilon x).$$

We observe that they are BnV and they converge strongly (in the appropriate Sobolev space) to the linear function given by $\nabla u(0)$, which we denote by u^∞. So we have that $[Ju^\varepsilon]$ converges to $[Ju^\infty]$ as local current and this implies that

$$\lim_{\varepsilon \to 0} \langle [Ju^\varepsilon], \omega \rangle = \langle [Ju^\infty], \omega \rangle = \int_{\mathbb{R}^m} \omega \det(\nabla u(0))\, d\mathcal{L}^m \tag{45}$$

for every Lipschitz function ω with compact support.

On the other hand, for every $f dg = \omega \in \mathcal{D}^1_c$, we also have

$$\langle j(u^\varepsilon), f dg \rangle =$$

$$\int_{\mathbb{R}^m} \frac{1}{\varepsilon} f(x) u_1(\varepsilon x) \det(\nabla u_2(\varepsilon x), \dots, \nabla u_n(\varepsilon x), \nabla g(x)) d\mathcal{L}^m$$

and, after changing variables,

$$= \frac{1}{\varepsilon^n} \int_{\mathbb{R}^m} u_1(y) f(y/\varepsilon) \det(\nabla u_2(y), \dots, \nabla u_n(y), \frac{1}{\varepsilon} \nabla g(y/\varepsilon) dy.$$

Thus we have obtained

$$\langle j(u^\varepsilon), \omega\rangle = \frac{1}{\varepsilon^{n+1}}\langle j(u), \omega(y/\varepsilon)\rangle$$

from which it follows that

$$\begin{aligned}\langle [Ju^\varepsilon], \omega\rangle &= \langle j(u^\varepsilon), d\omega\rangle \\ &= \frac{1}{\varepsilon^{n+1}}\langle j(u), (d\omega)(y/\varepsilon)\rangle \\ &= \frac{1}{\varepsilon^{n}}\langle j(u), d(\omega(y/\varepsilon))\rangle = \frac{1}{\varepsilon^{n}}\langle [Ju], \omega(y/\varepsilon)\rangle.\end{aligned}$$

From condition (b) we have

$$\lim_{\varepsilon\to 0}\left|\frac{1}{\varepsilon^n}\langle [Ju]_s, \omega(y/\varepsilon)\rangle\right| \le \lim_{\varepsilon\to 0}\frac{\mathrm{Lip}\,(\omega)}{\varepsilon^n}\|Ju\|_s(\varepsilon\,\mathrm{supp}\,(\omega)) = 0.$$

So we can write

$$\begin{aligned}\lim_{\varepsilon\to 0}\langle [Ju^\varepsilon], \omega\rangle &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\langle [Ju]_a, \omega(y/\varepsilon)\rangle \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\nu(y)\,\omega(y/\varepsilon)\,dy \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\nu(\varepsilon x)\,\omega(x)\,dx.\end{aligned}$$

Since 0 is a Lebesgue point for ν, $\nu(\varepsilon y)$ converges in L^1_{loc} to the function $\nu(0)$. Recalling (45) we have

$$\int_{\mathbb{R}^n}\nu(0)\,\omega\,dx = \int_{\mathbb{R}^n}\det(\nabla u(0))\omega\,dx$$

for every Lipschitz function ω with compact support. Then we conclude that $\nu(0) = \det(\nabla u(0))$.

Now we remark that $\mathcal{L}^n$ a.e. x satisfies (a) and (b) and this completes the proof.

□

Before proving Theorem 3.2.2 in its full generality we put on the space of covectors $\Lambda_{m-n}(\mathbb{R}^m)$ the norm

$$|\nu| = \sup\{\langle \nu, f_1\wedge\ldots\wedge f_n\rangle\,|f_i\in\mathbb{R}^m \text{ and } |f_i|\le 1\}.$$

From the classical theory of currents we know that $|\nu_a(x)|\in L^1(\mathbb{R}^m)$. This fact and equation (43) imply that

$$\det(\nabla u_1,\ldots,\nabla u_n,\nabla g_1,\ldots,\nabla g_{m-n})\in L^1(\mathbb{R}^m)$$

for every $(m-n)$-tuple of Lipschitz functions $(g_1,\ldots,g_{m-n})$.

PROOF OF THEOREM 3.2.2 First we choose a Borel function ν'_a such that

$$\langle [Ju]_a, \omega \rangle = \int_{\mathbb{R}^m} \langle \nu'_a(x), \omega(x) \rangle d\|Ju\|_a$$

Recalling that $\|Ju\|_a$ is absolutely continuous with respect to $\mathcal{L}^m$ we set

$$\nu_a := \frac{d\|Ju\|_a}{d\mathcal{L}^m} \nu'_a.$$

Using Lemma 3.2.3 and the slicing techniques introduced above we will prove that ν_a satisfies equation (43). To simplify notations we identify u and $\tilde{u}$ and we put $m = n + k$.

First we choose an orthogonal system $x_1, \ldots, x_{n+k}$ and a particular partition of $\{1, \ldots, n+k\}$ into two disjoint sets $I = \{i_1, \ldots, i_k\}$ and $J = \{i_{k+1}, \ldots, i_{k+n}\}$. We call $y_1, \ldots, y_k$ the k coordinates $x_{i_1}, \ldots, x_{i_k}$ and $z_1, \ldots, z_n$ the remaining n; moreover we denote by π_I the projection on the coordinates $y_1, \ldots, y_k$. From Lemma 3.2.3 we know that equation (43) holds when $m = n$; hence for a.e. $y \in \mathbb{R}^k$ we have

$$\langle [Ju(y, \cdot)]_a, f(y, \cdot) \rangle = \int_{\mathbb{R}^n} \det(\nabla_z u(y, z)) f(y, z) d\mathcal{L}^n(z). \tag{46}$$

Then, from the slicing property of the Jacobians applied to π_I, it follows that

$$\begin{aligned}
\langle [Ju]_a \llcorner d\pi_I, f \rangle &= (-1)^{nk} \int_{\mathbb{R}^k} \langle [Ju(y, \cdot)]_a, f(y, \cdot) \rangle d\mathcal{L}^k(y) \\
&= \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} (-1)^{nk} \det(\nabla_z u(y, z)) f(y, z) d\mathcal{L}^n(z)\, d\mathcal{L}^k(y) \\
&= \int_{\mathbb{R}^{n+k}} f \ \det(\nabla u_1, \ldots, \nabla u_n, e_{i_1}, \ldots e_{i_k}) d\mathcal{L}^{n+k}.
\end{aligned}$$

Of course this means that $\det(\nabla u_1, \ldots, \nabla u_n, e_{i_1}, \ldots e_{i_k})$ is an L^1 function. Moreover this fact is true for every choice of I and, from the multilinearity of the determinant, we argue that

$$\det(\nabla u_1, \ldots, \nabla u_n, \nabla g_1, \ldots, \nabla g_k)$$

is summable for every Lipschitz and bounded k-tuple $(g_1, \ldots, g_k)$. The continuity of $[Ju]_a \llcorner d\pi_I$ for every choice of I and the multilinearity of the determinant give the continuity (as k-dimensional local current) of $H_{n+k} \llcorner du$, which is defined by

$$\langle H_{n+k} \llcorner du, f dg \rangle = \int_{\mathbb{R}^{n+k}} f \det(\nabla u_1, \ldots, \nabla u_n, \nabla g_1, \ldots, \nabla g_k) d\mathcal{L}^{n+k}.$$

Of course $H_{n+k} \llcorner du$ is the same local current as $[Ju]_a$.

$\square$

Unfortunately we are not able to prove that something similar holds for the Cantor part, i.e. that $\nu_c(x)$ is a simple covector for $\|Ju\|_c$ a.e. x.

3. SBnV and compactness Theorem

In analogy with the case of SBV functions (see [**4**], [**6**]) we can define the space SBnV of special functions of bounded higher variation.

DEFINITION 3.3.1. *We say that a map $u \in \mathrm{BnV}(U, \mathbb{R}^n)$ is a "special function of bounded higher variation" if $[Ju]_c = 0$.*

The next proposition provides an equivalent definition of $SBnV$ functions.

PROPOSITION 3.3.2. *Let* Ind *be the collection of all subsets I of $\{1, \dots, m\}$ such that the cardinality of I is $m - n$. For every $I \in \mathrm{Ind}$ we denote by π_I the projection on the coordinates $\{x_i | i \in I\}$. A function $u \in \mathrm{BnV}(U, \mathbb{R}^n)$ is in* SBnV *if and only if:*

(A) for every set of indices $I := \{i_1, \dots, i_{m-n}\} \in \mathrm{Ind}$ and for $\mathcal{L}^{m-n}$ a.e. x, $u(x_{i_1}, \dots, x_{i_{m-n}}, y)$ is a SBnV *function of y.*

PROOF. The "only if" part is an easy consequence of the slicing of currents. So let us suppose that (A) holds. Then for every $I \in \mathrm{Ind}$ we have $[Ju]_c \llcorner d\pi = 0$. If ω is an n-form, we can write

$$\omega = \sum_{I \in \mathrm{Ind}} g_I d\pi_I$$

and so we obtain $[Ju]_c \llcorner \omega = 0$. We conclude that $[Ju]_c = 0$ $\square$

An interesting fact is that the special functions of bounded higher variation satisfy a closure Theorem similar to that proved in [**4**] for SBV.

REMARK 3.3.3. From now on when ν is a k-covector we denote by $|\nu|$ the standard norm induced by its action on k-vectors (see the proof of Theorem 3.1.6).

First of all we prove the closure theorem in a particular case.

THEOREM 3.3.4. *Let us consider $(u_k) \subset \mathrm{BnV}(U, \mathbb{R}^n)$ and $u \in \mathrm{BnV}(U, \mathbb{R}^n)$, with $U \subset \mathbb{R}^n$. Let us suppose that:*

(a) *$u_k \rightharpoonup u$ weakly in W^{1,p_1}_{loc}, $u_k \to u$ strongly in $L^{p_2}_{loc}$, and*

$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$

(or $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{loc}$*,* $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets);*

(b) *if we write*

$$[Ju_k] = m_k(x)\mathcal{H}^0 \llcorner E_k + H_n \llcorner \nu_k(x),$$

then $|\nu_k|$ *are equiintegrable and* $\mathcal{H}^0(E_k) \le C < \infty$*.*

Then $u \in \mathrm{SBnV}(U, \mathbb{R}^n)$ *and*

$$[Ju_k]_a \rightharpoonup [Ju]_a$$

$$[Ju_k]_l \rightharpoonup [Ju]_l.$$

Proof. We follow the ideas of the proof of SBV closure in [**4**]. First we notice that in this particular case the weak Jacobians $[Ju_n]$ are distributions. From the fact that the functions ν_k are equiintegrable, we can find a subsequence that converges weakly in L^1 to a function ν. To simplify the notation we will suppose that the whole sequence (ν_k) converges to ν.

We recall that, from the continuity of the Jacobians, $[Ju_k] \rightharpoonup [Ju]$ (which means

$$\lim_{k\to\infty} \langle [Ju_k], \omega\rangle = \langle [Ju], \omega\rangle$$

for every Lipschitz function ω with compact support). We notice that

$$[Ju_k]_l = [Ju_k] - [Ju_k]_a \rightharpoonup [Ju] - \nu\mathcal{L}^n$$

in the sense of distributions. Moreover we can write, for some integer N,

$$[Ju_k]_l = \sum_{i=1}^N a_N^k \delta_{x_N^k}.$$

Then we can find a subsequence $u_{k(r)}$ such that (possibly reordering each set $\{x_1^{k(r)}, \dots, x_N^{k(r)}\}$ in a proper way):

(B) for every $j \in \{1, \dots, N\}$ either $x_j^{k(r)}$ converges to $x_j \in \overline{U}$ or $|x_j^{k(r)}|$ tends to infinity.

Recalling that $[Ju] - \nu\mathcal{L}^n$ is the limit of $[Ju_{k(r)}]_l$ we obtain that its support is a finite number of points. But we know that $[Ju] - \nu\mathcal{L}^n$ is a measure: so it is the sum of a finite number of Dirac masses. We can conclude that $[Ju]$ is the sum of an absolutely continuous measure and a finite number of Dirac masses.

Moreover we have that

$$[Ju_k]_a \rightharpoonup [Ju]_a \tag{47}$$

and

$$[Ju_k]_l = [Ju_k]_s \rightharpoonup [Ju]_s = [Ju]_l. \tag{48}$$

(Indeed we have proved these last statements only for a subsequence. However we notice that from every subsequence of u_k we can choose another subsequence such that (47) and (48) hold. Then (47) and (48) hold for the whole sequence (u_k)).

$\square$

From the slicing property of $[Ju]$ we are now able to prove the next Theorem.

THEOREM 3.3.5. *Let us consider* $(u_k) \subset \mathrm{BnV}(U, \mathbb{R}^n)$ *and* $u \in BnV(U, \mathbb{R}^n)$*, with* $U \subset \mathbb{R}^m$*. Moreover suppose that*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc}*,* $u_k \to u$ *strongly in* $L^{p_2}_{loc}$*, and*
$$\frac{n-1}{p_1} + \frac{1}{p_2} < 1$$
(or $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{loc}$*,* $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets);*

(b) *if we write*
$$[Ju_k] = m_k(x)\tau_k(x)\mathcal{H}^{m-n} \llcorner E_k + H_m \llcorner \nu_k(x),$$
then $|\nu_k|$ *are equiintegrable and* $\mathcal{H}^{m-n}(E_k) \leq C < \infty$*.*

Then u *is of special higher bounded variation.*

During the proof we will use the representations of the previous section. So the restriction of $[Ju]_a$ to dg becomes

$$\langle [Ju]_a \llcorner dg, \omega \rangle = \int_{\mathbb{R}^m} \langle \nu_a(z), \omega(w) \wedge dg(w) \rangle \, d\mathcal{L}^m(w)$$

and the slicing map with respect to the projection π on the first $m - n$ coordinates is given by

$$\langle \mathcal{S}(x), \omega \rangle = \int_{\{x\} \times \mathbb{R}^n} \langle \nu_a(x,y), \omega(x,y) \wedge d\pi(x,y) \rangle \, d\mathcal{L}^n(y).$$

From the slicing property of Jacobians we argue that for a.e. x we can find a 0-covector valued function $\xi(x, \cdot)$ (i.e. a *real function*) such that

$$\langle [J(u(x,\, \cdot))]_a, \omega \rangle = \int_{\mathbb{R}^n} \langle \xi(x,y), \omega(y) \rangle d\mathcal{L}^n(y).$$

We denote $\xi(x,y)$ by $\nu_a(x,y) \llcorner d\pi$ and we remark that $|\xi(x,y)| \leq |\nu(x,y)|$ for a.e. (x,y)

PROOF. We will prove that statement (A) in 3.3.2 holds.

Let us fix $I \in \mathrm{Ind}$: without loss of generality we can suppose that

$$I = \{1, \dots, m-n\}.$$

We denote by z the projection on the first $m-n$ coordinates. Then we can write, for a.e. $x \in \mathbb{R}^{m-n}$,

$$[J(u_k(x,\cdot))]_a = (\nu_k(x,\cdot) \llcorner dz)\mathcal{H}^n \llcorner x \times \mathbb{R}^n \tag{49}$$

$$[J(u_k(x,\cdot))]_l = m_k(x,\cdot)\mathcal{H}^0 \llcorner (S_u \cap z^{-1}\{x\}) \tag{50}$$

and of course $[Ju_k(x,\cdot)]_c = 0$.

We split the proof into several steps.

Step 1 First we suppose that $u_k \to u$ strongly in $L^{p_2}_{loc}$ and weakly in W^{1,p_1}_{loc}. Let us fix an open set $V \subset\subset U$ and set $V_x = V \cap z^{-1}\{x\}$. Let us extract a subsequence u_k of u_n such that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^{m-n}} \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)} dx < \infty.$$

From the Monotone convergence Theorem we infer that for a.e. x

$$\sum_k \|u_k(x,\cdot) - u(x,\cdot)\|_{L^{p_2}(V_x)}$$

is a convergent series: this implies that $u_k(x,\cdot) \to u(x,\cdot)$ in $L^{p_2}(V_x)$ for a.e. x. Let us choose a family of open sets $V_n \uparrow V$ such that $V_n \subset\subset V$. We reason as above for every V_n and we apply a diagonalization argument to conclude that there is a subsequence (u_k) such that $u_k(x,\cdot) \to u(x,\cdot)$ strongly in $L^{p_2}_{loc}$ for a.e. x.

Step 2 From Fatou Lemma we have that

$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} dx$$

$$\le \liminf_{j\to\infty} \int_{\mathbb{R}^{m-n}} \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty.$$

We conclude that for a.e. x we can extract a further subsequence (u_l) (possibly depending on x) such that $\|u_l(x,\cdot)\|_{W^{1,p_1}(V_x)} < \infty$ for every open set $V \subset\subset U$. Then, recalling that for a.e. x $(u_k(x,\cdot))$ converges strongly in $L^{p_2}_{loc}$ to $u(x,\cdot)$, we have that $u_l(x,\cdot)$ converges weakly in W^{1,p_1}_{loc} to $u(x,\cdot)$.

Summarizing we have proved that for a.e. $x \in \mathbb{R}^n$ we can extract a subsequence (u_k) (possibly depending on x) such that

- $u_k(x,\cdot) \rightharpoonup u(x,\cdot)$ in W^{1,p_1}_{loc} and $u_k(x,\cdot) \to u(x,\cdot)$ strongly in $L^{p_2}_{loc}$, with
$$(n-1)/p_1 + 1/p_2 < 1.$$

In a similar way we can treat the case in which $u_k(x,\cdot) \rightharpoonup u(x,\cdot)$ in $W^{1,n-1}_{loc}$ $u_k(x,\cdot) \to u(x,\cdot)$ uniformly on compact sets as continuous functions.

Step 3 From the Dunford-Pettis Theorem on L^1 weakly compact sequences (see for example [**7**]) we know that $|\nu_k|$ belong to some Orlicz space. So there exists a real convex function ϕ with superlinear growth such that
$$\int_{\mathbb{R}^m} \phi(|\nu_k|) \leq K < \infty.$$
Then we have
$$\begin{aligned} K &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^m} \phi(|\nu_k \llcorner dz|) \\ &\geq \limsup_{k\to\infty} \int_{\mathbb{R}^{m-n}} \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)\, dx\, dy \\ &\geq \int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)\, dx\, dy. \end{aligned}$$
This implies that for a.e. x we can find a subsequence $k(r)$ such that
$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \phi(|\nu_{k(r)}(x,y) \llcorner dz|)\, dx < \infty,$$
which means that $\nu_{k(r)}(x,\cdot) \llcorner dz$ are equiintegrable (we remark that the chosen subsequence depends on x).

Step 4 Reasoning as in the previous cases we have
$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}))dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} (\mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}))$$
Then for a.e. x we can extract a subsequence (u_l) (possibly depending on x) such that
$$(\mathcal{H}^0(S_{u_l} \cap z^{-1}\{x\}))$$
is bounded.

Step 5 Now we want to put together all the informations of the previous steps. We notice that the subsequence extracted on the first step does not depend on x, whereas the choices of the other steps depend on x. However for a.e. x we can extract a subsequence that fulfills all the conditions. Indeed let us define
$$f_k(x) := \mathcal{H}^0(S_{u_k} \cap z^{-1}\{x\}) + \int_{\mathbb{R}^n} \phi(|\nu_k(x,y) \llcorner dz|)dy + \|u_k(x,\cdot)\|_{W^{1,p_1}(V_x)};$$

then we have

$$\int_{\mathbb{R}^{m-n}} \liminf_{k\to\infty} f_k(x)dx \le \liminf_{k\to\infty} \int_{\mathbb{R}^{m-n}} f_k(x)dx$$

$$\le \liminf_{k\to\infty} \left(\mathcal{H}^{m-n}(S_{u_k}) + \|u_k\|_{W^{1,p_1}} + \int_{\mathbb{R}^m} \phi(\nu_k)\right) < \infty. \tag{51}$$

We conclude that for a.e. x we can choose a subsequence u_r such that $(u_r(x,\cdot))$ and $u(x,\cdot)$ satisfy all the hypotheses of Theorem 3.3.4. Then for a.e. x the Cantor part of $[J(u(x,\cdot))]$ is zero and from statement (A) it follows that u has no Cantor part.

□

We end this section by proving that, in the same hypotheses of Theorem 3.3.5 we have

$$[Ju_k]_l \rightharpoonup [Ju]_l$$

$$[Ju_k]_a \rightharpoonup [Ju]_a.$$

In order to do this we need the next Lemma.

Lemma 3.3.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$ and (ν_k) a weakly compact sequence in $L^1(\Omega, \mu)$. Then $\nu_k \rightharpoonup \nu$ if and only if*

$$\int_\Omega |w + \nu| d\mu \le \liminf_{k\to\infty} \int_\Omega |w + \nu_k| d\mu \qquad \forall\, w \in L^1(\Omega). \tag{52}$$

We follow here the proof in [**4**].

Proof. One implication is an easy consequence of semicontinuity of the L^1 norm. Hence we will prove the converse, i.e. that if (52) holds then ν_k is weakly converging to ν. First of all, we notice that we can restrict ourselves to consider the case $\nu \equiv 0$. Moreover it is sufficient to prove that

$$\lim_k \int_A \nu_k = 0 \qquad \text{for every measurable set } A.$$

Suppose for the moment that there is a constant N such that $\|\nu_k\|_{L^\infty} \le N < \infty$ and that there exists a, b , A such that

$$\lim_{k\to\infty} \int_A \nu_k = a \ne 0$$

$$\lim_{k\to\infty} \int_{\Omega\setminus A} = b.$$

Then put $w = -N\text{sign}\,(a)\chi_A - N\text{sign}\,(b)\chi_{\Omega\setminus A}$. Then it is easy to see that we have

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |w+\nu_k| &= N\mu(A) - |a| + N\mu(\Omega\setminus A) - |b| \le N\mu(\Omega) - |a| \\ &< N\mu(\Omega) = \int_\Omega |w|.\end{aligned}$$

For the general case suppose again that there exists a, b, A as above and pick $\varepsilon < |a|/2$. Thanks to equiintegrability of ν_k we can choose a measurable set Ω' and a number N such that $\|\nu_k\|_{L^\infty(\Omega')} \le N$ and

$$\int_{\Omega\setminus\Omega'} |\nu_k| \le \varepsilon.$$

Now put $w = -N\text{sign}\,(a)\chi_{A\cap\Omega'} - N\text{sign}\,(b)\chi_{\Omega'\setminus A}$. It is easy to see that with this choice we have

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |w+\nu_k| &\le N\mu(A\cap\Omega') - |a| + N\mu(\Omega'\setminus A) + \varepsilon \\ &\le N\mu(\Omega') - \varepsilon \le \int_\Omega |w|.\end{aligned}$$

$\square$

THEOREM 3.3.7. *Let us consider $u, (u_k) \subset \mathrm{BnV}(U, \mathbb{R}^n)$, with $U \subset \mathbb{R}^m$. If conditions (a) and (b) of Theorem 3.3.5 hold then*

$$\begin{gathered}[Ju_k]_l \rightharpoonup [Ju]_l \\ [Ju_k]_a \rightharpoonup [Ju]_a\end{gathered} \tag{53}$$

PROOF. We use the notations of Theorem 3.3.5 and we reduce to prove

$$[Ju_k]_a \llcorner d\pi_I \rightharpoonup [Ju]_a \llcorner d\pi_I \tag{54}$$

for every $I \in \mathrm{Ind}$ (we recall that Ind is the collection of all subsets of $\{1, \dots, n\}$ that have cardinality $(m-n)$). Indeed from this fact we could conclude that $[Ju_k]_a \rightharpoonup [Ju]_a$ and

$$[Ju_k]_l = [Ju_k] - [Ju_k]_a \rightharpoonup [Ju] - [Ju]_a = [Ju]_l.$$

Therefore we suppose that $I = \{1, \dots, m-n\}$ and we split the proof into several steps. To simplify the notation we suppose that $U = \mathbb{R}^n$ and that global convergence hold on (u_k): the proof can been adapted easily to the local case.

Step 1 Let us fix a convex real function ϕ with superlinear growth and a real number z such that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner d\pi|) d\mathcal{L}^m < \infty.$$

For every $x \in \mathbb{R}^{m-n}$ let us put

$$J_1^x(u) = \int_{\mathbb{R}^n} \phi(|z + \nu \llcorner d\pi(x,y)|)dy$$
$$J_2^x(u) = \|u(x,\cdot)\|_{W^{1,p_1}}$$
$$J_3^x(u) = \mathcal{H}^0(\pi^{-1}\{x\} \cap S_{u_k})$$

and fix a positive real number t. From Fatou Lemma we have that

$$\liminf_{k\to\infty} J_1^x(u_k) + tJ_2^x(u_k) + tJ_3^x(u_k) = K(x) < \infty$$

for almost every x and reasoning as in the last step of Theorem 3.3.5 we infer that

$$J_1^x(u) \le J_1^x(u) + tJ_2^x(u) + tJ_3^x(u) \le K(x).$$

Integrating this inequality with respect to x we obtain

$$\int_{\mathbb{R}^m} \phi(|z + \nu \llcorner d\pi|)d\mathcal{L}^m \le$$

$$\liminf_{k\to\infty} \left(\int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner d\pi|)d\mathcal{L}^m + t\|u_k\|_{W^{1,p_1}} + t\mathcal{H}^k(S_{u_k}) \right).$$

Letting $t \downarrow 0$ we obtain

$$\int_{\mathbb{R}^m} \phi(|z + \nu \llcorner d\pi|)d\mathcal{L}^m \le \liminf_{k\to\infty} \int_{\mathbb{R}^m} \phi(|z + \nu_k \llcorner d\pi|)d\mathcal{L}^m. \tag{55}$$

We notice that the same arguments work if we replace $\mathbb{R}^m$ with an open set. Let us denote by $\mathcal{C}$ the class of functions $w \in L^1(\mathbb{R}^m)$ that can be written as

$$w = \sum_{i=1}^h \alpha_i \chi_{A_i},$$

for some open sets $A_1, \dots A_h$. Hence, equation (55) holds for every function $z \in \mathcal{C}$.

Step 2 We know that there exists a convex real function ψ with superlinear growth such that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^m} \psi(|\nu_k \llcorner d\pi|) < \infty.$$

Let us take a convex real function ϕ with superlinear growth such that $\phi(0) = 0$

$$\lim_{t\to\infty} \frac{\psi(t)}{\phi(t)} = +\infty.$$

We can easily conclude that the sequence $\phi(|\nu_k \llcorner d\pi|)$ is equiintegrable. Let us put

$$\phi_n(t) = \left(\frac{1}{n}\phi(t)\right) \vee t.$$

The equiintegrability of $\psi(|\nu_k \llcorner d\pi|)$ and the fact that $\phi_n(t) \downarrow t$ imply that

$$\liminf_{k\to\infty} \int_{\mathbb{R}^m} |\nu_k \llcorner d\pi| = \lim_{n\to\infty} \liminf_{k\to\infty} \int_{\mathbb{R}^m} \phi_n(|\nu_k \llcorner d\pi|).$$

From the previous step we easily conclude that

$$\int_{\mathbb{R}^m} |z + \nu \llcorner d\pi| \leq \int_{\mathbb{R}^m} \liminf_{k\to\infty} |z + \nu_k \llcorner d\pi| \tag{56}$$

for every $z \in \mathcal{C}$.

Step 3 By a standard approximation argument we have that (56) holds for every $z \in L^1(\mathbb{R}^n)$. Then applying Lemma 3.3.6 we conclude that

$$\nu_k \llcorner d\pi \rightharpoonup \nu \llcorner d\pi.$$

$\square$

CHAPTER 4

Coarea formula and chain rule

As already mentioned in the introduction, if we define $u^y = (u-y)/|u-y|$ for every $y \in \mathbb{R}^n$ such that $\mathcal{L}^m(u^{-1}(y)) = 0$, then we have the following

THEOREM 4.0.8. (See sect. 4 of [**27**]) *If $u \in W^{1,n-1} \cap L^\infty(\Omega, \mathbb{R}^n)$ (with $\Omega \subset \mathbb{R}^m$, $m \geq n$), then $u^y \in W^{1,n-1} \cap L^\infty$ for a.e. $y \in \mathbb{R}^n$ and*

$$\langle [Ju], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle [Ju^y], \omega \rangle dy \tag{57}$$

for every smooth $m-n$ differential form ω (with ω_n we denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$). If $u \in W^{1,n-1} \cap L^\infty$ and $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then

$$\langle [J(F(u))], \omega \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(y) \, \langle [Ju^y], \omega \rangle dy. \tag{58}$$

The first part of the previous Theorem emphasizes the relationships between the level sets of a function and its weak Jacobian. In order to understand why it is so we remark that if u is sufficiently regular and $y \in \mathbb{R}^n$ is a regular value then Ju^y is given by ω_n times the current naturally supported by $u^{-1}(y)$ (which by the Implicit Function Theorem is an oriented C^1 manifold).

For a smooth function u we have a strongest result: the classical coarea formula holds and tells us that

$$\int_{\mathbb{R}^n} \mathcal{H}^{m-n}(u^{-1}(y)) d\mathcal{L}^n(y) = \int_\Omega |Ju| d\mathcal{L}^m(x)$$

where $|Ju|$ is in this case the square root of the sum of the squares of all $n \times n$ minors of ∇u. It is not difficult to rewrite the previous equation as

$$\frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(\Omega) d\mathcal{L}^y = \|Ju\|(\Omega).$$

A consequence of the previous Theorem is the inequality

$$\|Ju\|(A) \leq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(A) dy. \tag{59}$$

Hence it turns out that a natural question is asking under what conditions equality holds. When $n = 1$ the problem reduces to the coarea formula on the level sets of a scalar BV function (see for example Theorem 3.40 of [**7**]). When $n > 1$ Jerrard and Soner proved that equality holds for $u \in W^{1,p}$ with $p > n$. In this case $[Ju]$ is equal to the natural action as current of $du_1 \wedge \ldots \wedge du_n$ and a comparison with the classical coarea formula for approximately differentiable functions (see for example Theorem 3.2.3 of [**18**]) implies that for $\mathcal{L}^n$ a.e. y $[Ju^y]$ is ω_n times the integer rectifiable current supported by $u^{-1}(y)$ with the usual orientation (and thus this result generalize what can be easily proved for C^1 functions combining the Implicit Function Theorem and Sard's Lemma). Motivated by these facts we will say that equation (57) represents a *weak coarea formula* and that u satisfies the *strong coarea formula* if equality holds in (59).

In general, to have equality in (59), it is not sufficient to assume merely that $\|Ju\|$ is a Radon measure (see [**27**] for some counterexamples based on earlier ones of Giaquinta, Modica and Souček [**21**]). However it is conjectured in [**27**] that the strong coarea formula holds for $W^{1,n}$ functions and for continuous functions such that $\|Ju\|$ is a Radon measure.

Coming back to Theorem 4.0.8 it is natural to think of (58) as an extension of the chain rule for Jacobians of smooth maps, hence we will refer to it as *weak chain rule*. Of course when u is smooth we have the much stronger relation $[J(F(u))] = \det \nabla F(u)[Ju]$.

For general u we can not even give a meaning to the right hand side of this equation, because $[Ju]$ is only a current (and hence when $m = n$ just a distribution). However when $[Ju]$ is a current of finite mass we might hope that there exists a pointwise representative $\overline{u}$ of u which is Borel measurable and satisfies $[J(F(u))] = \det \nabla F(\overline{u})[Ju]$. we notice that the right hand side is now well defined because $\det \nabla F(\overline{u})$ is a Borel function. In general this identity does not hold; if it holds we say that u satisfies a *strong chain rule*.

In this chapter we prove that if u satisfies the strong chain rule then the strong coarea formula holds (see Theorem 4.2.1). This theorem provides a different proof of the strong coarea formula when $u \in W^{1,p}$, $p > n$, which can be extended to $u \in W^{1,n}$; more generally it implies that the strong coarea formula holds whenever u is a Cartesian map in the sense of Giaquinta, Modica and Souček (see Remarks 4.3.3 and 4.4.2).

In the third section we prove (using arguments involving classical degree of continuous maps and weak Jacobians) that if u is a continuous BnV function in $W^{1,p}$ with $p > n-1$ and $\|Ju^y\|(\Omega)$ is an integrable function of y then u satisfies the strong chain rule (see Theorem 4.3.1). This provides a partial answer to the conjecture of Jerrard and Soner. However we notice that in this case the strong coarea formula can be shown with a direct proof without passing through the strong chain rule. Unfortunately we are not able to prove that if $\|Ju^y\|(\Omega) \notin L^1_{loc}$ and u is continuous then $\|Ju\|(\Omega) = \infty$, which would prove the full conjecture in the case $p > n-1$.

1. Weak Jacobians and degree theory

In this section we state two lemmas which focus on the relations between classical degree theory and weak Jacobians (see section 8 of [**38**]). We have already seen these deep relations come into the game in Theorem 3.2.1 : they will be further investigated for other purposes in the next chapter.

If $\Omega \subset \mathbb{R}^n$ and $u \in C(\overline{\Omega}, \mathbb{R}^n)$ then we will denote by $\deg(u, \Omega, y)$ the degree of u at y. If M and N are smooth compact manifolds of the same dimension and $u \in C(M, N)$ then we denote by $\deg(u, M, N)$ the degree of the map u.

LEMMA 4.1.1. *Let $u \in C^\infty(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ has smooth boundary. Then for every vector field $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} \nabla \cdot \varphi(y) \deg(u, \Omega, y) dy = \int_{\partial\Omega} \varphi(u) \cdot \Lambda_n(\nabla u) \cdot \nu \, d\mathcal{H}^{n-1},$$

where ν denotes the unit normal to $\partial\Omega$.

PROOF. After a change of variables the proof of this Lemma follows from an integration by parts. We report here all the calculations since they are very similar to many others needed later.

First of all we write

$$\int_{\mathbb{R}^n} \nabla \cdot \varphi(y) \deg(u, \Omega, y) dy = \int_{\mathbb{R}^n} \nabla \cdot \varphi(y) \sum_{x \in u^{-1}(y)} \operatorname{sign}\left(\det \nabla u(x)\right) dy$$

which, after a change of variables, is equal to

$$\int_\Omega [\nabla \cdot \varphi](u(x)) \det \nabla u(x) dx.$$

Now let us compute the divergence of the vector field $v := \varphi(u) \cdot \Lambda_n(\nabla u)$. The j component of this vector field is given by $v^j :=$

$\sum_i \varphi_i(u)[\Lambda_n(\nabla u)]_i^j$. Hence

$$\begin{aligned}\sum_j \partial_j v^j &= \sum_j \sum_k \sum_i \partial_k \varphi_i(u) \partial_j u^k [\Lambda_n(\nabla u)]_i^j \\ &= \sum_k \sum_i \varphi_i(u) \sum_j \partial_j u^k [\Lambda_n(\nabla u)]_i^j \\ &= \sum_i \sum_k \partial_k \varphi_i(u) [\nabla u \cdot \Lambda_n(\nabla u)]_i^k.\end{aligned}$$

By definition we have that $\nabla u \cdot \Lambda_n(\nabla u)$ is equal to $\det(\nabla u)$ times the identity matrix. Hence we have that

$$\sum_j \partial_j v^j = \det \nabla u \sum_i \partial_i \varphi_i(u)$$

and this completes the proof. □

REMARK 4.1.2. We recall again that, when $\Omega \subset \mathbb{R}^n$, $j(u)$ can be seen as the vector field given by $v := \frac{1}{n} u \cdot \Lambda_n(\nabla u)$ (where Λ_n is the cofactor matrix of ∇u, see also Definition 5.1.1). Indeed let us take a form $\omega = g\, dx_1 \wedge \ldots \wedge dx_{n-1}$: then we have that

$$\langle j(u), \omega \rangle = \int_{\mathbb{R}^n} g(x) v_n(x) dx.$$

Hence the 0-current $[Ju]$ is the divergence of v in the sense of distributions.

Given a set $K \subset \mathbb{R}^n$ and a real number $\varepsilon > 0$ we denote by K_ε the set $\{x \in \mathbb{R}^n \,|\, \mathrm{dist}(x, K) < \varepsilon\}$.

LEMMA 4.1.3. *Let $u \in \mathrm{BnV}(\Omega, \mathbb{R}^n)$ be a continuous map (with $\Omega \subset\subset \mathbb{R}^n$). For every $\varepsilon > 0$ there exists an open set M with smooth boundary such that*

(a) *$M \subset \Omega$ and $\Omega \subset M_\varepsilon$;*
(b) *$u \in W^{1,n-1}(\partial M)$;*
(c) *it holds*

$$\frac{1}{n} \int_{\partial M} u \cdot \Lambda_n(\nabla u) \cdot \nu \, d\mathcal{H}^{n-1} = [Ju](M).$$

PROOF. First of all we choose an open set $M' \subset \Omega$ with smooth boundary such that $\Omega \subset M'_{\varepsilon/2}$. A well known theorem on tubular neighborhoods implies that there exists a $\delta < \varepsilon/2$ such that for every $0 < t < \delta$ the sets

$$S_{-t} = \{x |\, \mathrm{dist}(x, \partial M') = t\} \cap M' \qquad S_t = \{x |\, \mathrm{dist}(x, \partial M') = t\} \setminus M'$$

are smooth manifolds. Moreover we can define a signed distance function d such that the sets S_{-t}, S_t are the level sets of d. Hence for every $|s| < \delta$, $d^{-1}(s)$ is the boundary of an open set M_s which satisfies (a). Applying the coarea formula to d we conclude that for a.e. s condition (b) holds for M_s.

Now let us put $v := u \cdot \Lambda_n(\nabla u)$; we have $v \in L^1(\Omega, \mathbb{R}^n)$ and the measure $[Ju]$ is the divergence of v/n (see the previous Remark). Using standard approximation arguments we can find a sequence of C^∞ vector fields v_k such that, for a.e. s, $v_k \to v$ strongly in $L^1(\partial M_s)$ and

$$\frac{1}{n}\int_{M_s} \nabla \cdot v_n \to [Ju](M_s).$$

Hence it follows that condition (c) is satisfied by M_s for a.e. s. $\square$

LEMMA 4.1.4. *If $M \subset\subset \mathbb{R}^n$ is an open set with smooth boundary and $u \in W^{1,n-1}(\partial M, \mathbf{S}^{n-1})$ is continuous then*

$$\omega_n \deg(u, \partial M, \mathbf{S}^{n-1}) = \frac{1}{n}\int_{\partial M} u \cdot \Lambda_n(\nabla u) \cdot \nu \, d\mathcal{H}^{n-1}.$$

PROOF. A straightforward calculation shows that $\frac{1}{n} u \cdot \Lambda_n(\nabla u) \cdot \nu$ is the Jacobian determinant of u as map from ∂M to $\mathbf{S}^{n-1}$. Then the statement of the Lemma follows from the integral formula for the degree as can be found for example in Brezis and Nirenberg [**11**]. $\square$

LEMMA 4.1.5. *Let us suppose that $u \in C(\overline{\Omega}, \mathbb{R}^n) \cap W^{1,n-1}$, where $\Omega \subset \mathbb{R}^n$. Then for a.e. $y \in \mathbb{R}^n$ we have*

$$\omega_n |\deg(u, \partial\Omega, y)| \leq \|Ju^y\|(\Omega). \tag{60}$$

PROOF. As it is stated in Theorem 4.0.8, $u^y \in W^{1,n-1}(\Omega, \mathbf{S}^{n-1})$ for a.e. y (see Lemma 6 in sec. 4 of [**27**] for a proof of this fact). If $y \in u(\partial\Omega)$ then $|\deg(u, \partial\Omega, y)| = 0$ and the statement is obviously true; in an analogous way we reason if $\|Ju^y\|(\Omega) = \infty$. Hence let us fix a $y \notin u(\partial\Omega)$ such that $u^y \in W^{1,n-1}(\Omega, \mathbf{S}^{n-1})$ and $\|Ju^y\|(\Omega) < \infty$. If we take an open set M with smooth boundary such that $u^y \in W^{1,n-1} \cap C(\partial M)$ then

$$\omega_n \deg(u^y, \partial M, \mathbf{S}^{n-1}) = \frac{1}{n}\int_{\partial M} u^y \cdot \Lambda_n(\nabla u^y) \cdot \nu d\mathcal{H}^{n-1}.$$

Using Lemma 4.1.3 we conclude that for ε sufficiently small there is an open set M such that $M \subset \Omega \subset M_\varepsilon$, $u^y \in W^{1,p} \cap C(\partial M)$ and

$$\omega_n \deg(u^y, \partial M, \mathbf{S}^{n-1}) = [Ju^y](M)$$

$$\deg(u^y, \partial M, \mathbf{S}^{n-1}) = \deg(u, \partial M, y) = \deg(u, \partial\Omega, y)$$

and this completes the proof. $\square$

We end this section with a corollary of the area formula (see [**31**] or Corollary 3.2.20 in [**18**] for a proof of it; also compare with the proof of Theorem 5.1.8).

LEMMA 4.1.6. *Let Γ be an oriented C^∞ $(n-1)$-dimensional manifold and suppose that $u \in W^{1,p}(\Gamma, \mathbb{R}^n) \cap C$ with $p > n-1$. Then for any $\mathcal{H}^{n-1}$ measurable set $A \subset \Gamma$ we have*

$$\mathcal{H}^{n-1}(u(A)) \leq (n-1)^{(1-n)/2} \int_A |Du|^{n-1} d\mathcal{H}^{n-1}.$$

2. Strong coarea formula

In this section we prove that the strong chain rule implies the strong coarea formula. In particular we have the following

THEOREM 4.2.1. *Let $u \in \mathrm{BnV}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^m$, and suppose that there is a pointwise representative $\overline{u}$ of u such that $J(F(u)) = \det \nabla F(\overline{u})[Ju]$ for every $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\omega_n \int_{\mathbb{R}^n} \|Ju^y\|(\Omega) = \|Ju\|(\Omega). \tag{61}$$

In the following we deal with family of currents parameterized by variables in $\mathbb{R}^n$: for the sake of clarity if T_y is a family of currents on $\mathbb{R}^m$ parameterized by y and ω is a form sometimes we will indicate with $\langle T_y(x), \omega(x)\rangle$ the real numbers $T_y(\omega)$.

Moreover we will need the following lemma, which will state in more generality with respect to what needed now because we will use it in the next chapter.

LEMMA 4.2.2. *Suppose $A \subset\subset \mathbb{R}^n$ and $B \subset\subset \mathbb{R}^m$ and let us call $\mathcal{P}$ the set of functions $\{f(x)g(y) | f \in C^k(\overline{A}), g \in C^k(\overline{B})\}$. Then the vector space generated by $\mathcal{P}$ is dense in $C^k(\overline{A \times B})$. The statement remains true if we replace C^k with C^∞ endowed with the topology induced by the C^k seminorms. Also the statement remains true if we replace C^k or C^∞ by C^k_c or C^∞_c.*

PROOF. It is sufficient to prove that the polynomials are dense. If we take a function $u \in C^k(\overline{A \times B})$ then we can extend it to a function $\tilde{u} \in C^1_c(\mathbb{R}^n \times \mathbb{R}^m)$. Hence let us call (B_k) the sequence of Bernstein polynomials (which is the usual example of a sequence of polynomials which converge locally in the sense of distributions to a Dirac mass centered in 0); the convolutions $\tilde{u} * B_k$ are polynomials which converge strongly in $C^k_{loc}(\mathbb{R}^n \times \mathbb{R}^m)$ to $\tilde{u}$: this completes the proof. Of course if $u \in C^\infty$ the also $\tilde{u} * B_k \in C^\infty$ and the previous convergence implies the convergence of u_n to u in all C^k seminorms.

A standard truncation argument implies the same statement with C^k (or C^∞) replaced by C_c^k or C_c^∞. $\square$

Proof of Theorem 4.2.1 To simplify notations we will identify u and $\overline{u}$. Moreover we notice that it is sufficient to prove

$$\|Ju\|(\Omega) \geq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(\Omega) dy. \tag{62}$$

Let us define the current T on $\Omega_x \times \mathbb{R}^n_y$ in the following way:

$$T(\omega) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle \overline{Ju}_y, \omega \rangle dy$$

where $\overline{Ju}_y$ is the push–forward of $[Ju^y]$ via the map $i_y : \Omega \to \Omega \times \mathbb{R}^n$ given by $i_y(x) = (x, y)$. (Or, in a more explicit way, we could have defined T as

(a) if $\omega = \sum f_I(x,y) dx_I$ then

$$T(\omega) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \sum \langle [Ju^y], f_I(x,y) dx_I \rangle dy;$$

(b) if $\omega = f(x,y) \nu \wedge dy_j$ then $\langle T, \omega \rangle = 0$;

(c) T is extended by linearity to the whole space of smooth forms.)

We will prove that

$$\|Ju\|(\Omega) \geq \|T\|(\Omega \times \mathbb{R}^n) \geq \frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(\Omega). \tag{63}$$

Step 1 We prove the second inequality of (63). First of all we recall that

$$\int_{\mathbb{R}^n} \|j(u^y)\|_{L^1} dy = \omega_n \|j(u)\|_{L^1} < \infty. \tag{64}$$

Let W' be the set of points y_0 such that

$$\lim_{\varepsilon \to 0} \frac{1}{\omega_n \varepsilon^n} \int_{B_\epsilon(y_0)} \|j(u^{y_0}) - j(u^y)\|_{L^1} dy = 0.$$

Then equation (64) gives that $\mathcal{L}^n(\mathbb{R}^n \backslash W') = 0$. We claim that, fixed an $\eta > 0$, for every $z \in W'$ there is an $\varepsilon(z)$ such that

(A) For every $\delta < \varepsilon(z)$ there is a smooth form ν such that $\|\nu\|_\infty \leq 1$, $\operatorname{supp}(\nu) \subset \Omega \times B_\delta(z)$ and $T(\nu) \geq (1-\eta)^3 \delta^n \|Ju^z\|(\Omega)$.

Using (A) and standard covering arguments it is easy to find a sequence of smooth n-forms $\nu_k(x,y)$ on $\Omega \times \mathbb{R}^n$ such that $\|\nu_k\|_\infty \leq 1$ and

$$\|T\|(\Omega \times \mathbb{R}^n) \geq \limsup_{k \to \infty} T(\omega_k) \geq (1-\eta)^3 \frac{1}{\omega_n} \int_{\mathbb{R}^n} \|Ju^y\|(\Omega) dy.$$

If we let η go to 0 we have the desired inequality.

To prove (A) we reason as follows. First of all we find a smooth $\omega \in C_c^\infty(\Omega, \Lambda^{m-n}(\mathbb{R}^m))$ such that $\|\omega\|_\infty \le 1$ and $\langle [Ju^z], \omega\rangle \ge (1-\eta)\|Ju^z\|(\Omega)$. Then we notice that there is an ε such that for $\delta \le \varepsilon$

$$\begin{aligned}\frac{1}{\omega_n\delta^n}\int_{B_\delta(z)}\langle [Ju^y(x)], \omega(x)\rangle dy &= \frac{1}{\omega_n\delta^n}\int_{B_\delta(z)}\langle j(u^y)(x), d\omega(x)\rangle dy\\ &\ge (1-\eta)\langle j(u^z), d\omega\rangle\\ &= (1-\eta)\langle [Ju^z], \omega\rangle.\end{aligned}$$

Moreover with the same idea it is not difficult to find for every δ a function $\varphi \in C_c^\infty(B_\delta(z))$ such that

$$\int_{\mathbb{R}^n}\langle [Ju^y](x), \varphi(y)\omega(x)\rangle dy \ge (1-\eta)\int_{B_\delta(z)}\langle [Ju^y], \omega\rangle dy.$$

Then the form $\nu(x,y) = \varphi(y)\omega(x)$ satisfies condition (A) for δ.

Step 2 We will prove that $\|T\|(\Omega\times\mathbb{R}^n) \le \|Ju\|(\Omega)$. First we notice that the strong chain-rule and Theorem 4.3.1 imply

$$\langle \det\nabla F(y)\omega(x), T(x,y)\rangle = \frac{1}{\omega_n}\int_{\mathbb{R}^n}\det\nabla F(y)\langle [Ju^y](x), \omega(x)\rangle dy$$

$$= \langle [\det\nabla F(u(x))]\omega(x), [Ju](x)\rangle \tag{65}$$

for every ω with $\operatorname{supp}(\omega) \subset\subset \Omega$ and for every $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Now let us take a real function $\psi \in C^1(\mathbb{R}^n)$: it is easy to see that we can find two functions ψ_1, ψ_2 such that $\psi_1 + \psi_2 = \psi$, $\psi_i \in C^1$ and $\psi_1 \ge k$, $\psi_2 \le -k$ for some $k > 0$. From a result of Dacorogna and Moser (see [**14**]) it follows that for every ball $B \subset \mathbb{R}^n$ we can find functions $F_B^i \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\det\nabla F_B^i(y) = \psi_i(y) \qquad \text{for every } y \in B.$$

If we fix a form ω with compact support we can find a ball B such that $u(\operatorname{supp}(\omega)) \subset B$, hence equation (65) applied to F_B^i and the multilinearity of T give

$$\langle \psi(y)\omega(x), T(x,y)\rangle = \langle \psi(u(x))\omega(x), [Ju](x)\rangle.$$

Now we would like to use density arguments to find:

$$\langle \psi(x,y)\omega(x), T(x,y)\rangle = \langle \psi(x,u(x))\omega(x), [Ju](x)\rangle. \tag{66}$$

for every $\psi \in C_c^1(\Omega\times\mathbb{R}^n)$.

We notice that (64) and the relation $\partial j(u^y) = Ju^y$ imply

$$\lim_{k\to\infty} T(\psi_k(x,y)\omega(x)) = T(\psi(x,y)\omega(x))$$

if $\psi_k \to \psi$ in C^1. Moreover we remark that the vector space generated by

$$\{\phi \in C^1_c(\Omega \times \mathbb{R}^n) | \phi \text{ is of the form } g(x)f(y)\}$$

is dense in $C^1_c(\Omega \times \mathbb{R}^n)$ in the strong topology (this last statement follows from Lemma 4.2.2). These facts imply that (66) holds.

By the definition of T we have that if ν is a smooth form on $\Omega \times \mathbb{R}^n$ with compact support then there are a scalar function $\psi \in C^\infty(\Omega \times \mathbb{R}^n)$ and an $(m-n)$-form ω on Ω such that

(a) $T(\nu) = T(\psi(x,y)\omega(x))$;
(b) $\|\psi\omega\|_\infty \le \|\nu\|_\infty$.

We conclude that (66) gives $\|T\|(\Omega \times \mathbb{R}^n) \le \|Ju\|(\Omega)$. $\square$

3. Strong chain rule

In this section we will prove the following

THEOREM 4.3.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ be continuous, with $\Omega \subset \mathbb{R}^m$ and $p > n-1$. Moreover suppose that*

$$\int_{\mathbb{R}^n} \|Ju^y\|(\Omega) dy < \infty. \tag{67}$$

If $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then $F(u) \in \mathrm{BnV}$ and $J(F(u)) = \det \nabla F(u)[Ju]$.

First of all we will prove the Theorem when $m = n$ and then we will recover the general case with a slicing argument. The key ingredient for the case $m = n$ is the lemma stated below, where we use some ideas of Müller and Spector (see sec. 8 of [**38**]). We remark that the hypothesis $p > n-1$ is needed in the proof of the first step of Lemma 4.3.2. Indeed in this proof we use Lemma 4.1.6 which in general does not hold when $p = n-1$.

The continuity of u is used heavily in the second step of Lemma 4.3.2 and the technical hypothesis (67) is employed to derive condition (C) below. Condition (C) is crucial for passing to the limit in formula (71) because it allows to use the Dominated Convergence Theorem. Indeed it could be true that the relation

$$\int_{\mathbb{R}^n} \deg(u, A, y) dy = [Ju](A) \tag{68}$$

holds true for every continuous BnV function u and for every open set A such that $\mathcal{L}^n(u(A)) = 0$. This would imply that the strong chain rule, and hence the strong coarea formula, are true for every continuous BnV function in $W^{1,p}$, $p > n-1$. However at the present stage we do not know how to derive equality (68) without using condition (C).

LEMMA 4.3.2. *Let* $u \in \mathrm{BnV}(\Omega, \mathbb{R}^n) \cap W^{1,p}$, *where* $\Omega \subset \mathbb{R}^n$ *and* $p > n-1$. *Suppose that* u *is continuous and that there exists a function* $N \in L^1_{loc}(\mathbb{R}^n)$ *such that*

(C) *for all open* $\Omega' \subset\subset \Omega$, $|\deg(u, y, \Omega')| \leq N(y)$ *for a.e.* y.

Then for every open set $A \subset\subset \Omega$ *such that* $\mathcal{L}^n(u(\partial A)) = 0$ *we have*

$$\int_{\mathbb{R}^n} \deg(u, y, A) dy = [Ju](A)\,, \qquad \int_{\mathbb{R}^n} |\deg(u, y, A)| dy \leq \|Ju\|(A). \tag{69}$$

PROOF. We split the proof into two steps.

First step We prove the first equation of (69) when A is an open set with smooth boundary such that $u \in W^{1,p}(\partial A)$ and

$$\frac{1}{n} \int_{\partial A} u \cdot \Lambda_n(\nabla u) \cdot \nu d\mathcal{H}^{n-1} = [Ju](A). \tag{70}$$

Let us approximate u strongly in L^∞ and in $W^{1,p}(A) \cap W^{1,p}(\partial A)$ with C^∞ functions u_k. The formula of integration by parts of Lemma 4.1.1 gives

$$\int_{\mathbb{R}^n} \nabla \cdot \varphi(y) \deg(u_k, A, y) dy = \frac{1}{n} \int_{\partial A} \varphi(u) \cdot \Lambda_n(\nabla u_k) \cdot \nu d\mathcal{H}^{n-1}$$

for every C^∞_c vector field. This equation implies that $f_k := \deg(u_k, A, y)$ are functions of bounded variation and

$$\|Df_k\|(\mathbb{R}^n) \leq \frac{1}{n} \int_{\partial A} \|\Lambda_n(\nabla u_k)\| d\mathcal{H}^{n-1} \leq \frac{1}{n} \|\nabla u_k\|^{n-1}_{L^{n-1}(\partial A)}.$$

Moreover the supports of f_k are all contained in a ball big enough. From the Poincaré inequality for BV functions it follows that their BV norms are equibounded, hence the sequence f_k is weakly compact in L^1. Moreover $\deg(u_k, A, y)$ converges to $\deg(u, A, y)$ for every $y \notin u(\partial A)$ and Lemma 4.1.6 implies that $u(\partial A)$ is negligible, hence we have $f_k = \deg(u_k, A, y) \to \deg(u, A, y)$ strongly in L^1. Passing to the limit we conclude that

$$\int_{\mathbb{R}^n} \deg(u, y, A) = \frac{1}{n} \int_{\partial A} u \cdot \Lambda_n(\nabla u) \cdot \nu d\mathcal{H}^{n-1}$$

and from (70) we obtain the first equation of (69).

Second Step Let us fix an open set A with $\mathcal{L}^n(u(\partial A)) = 0$: using Lemma 4.1.3 we choose open sets $A_k \uparrow A$ such that every A_k has smooth boundary and satisfies (70). By classical arguments we have

$\deg(u, A_k, y) \to \deg(u, A, y)$ for a.e. y and the Dominated convergence Theorem allows us to conclude

$$\int_{\mathbb{R}^n} \deg(u, A, y)dy = \lim_{k\to\infty} \int_{\mathbb{R}^n} \deg(u, A_k, y)dy \tag{71}$$
$$= \lim_{k\to\infty} [Ju](A_k) = [Ju](A).$$

Now let us define for every $k \in \mathbb{Z}$

$$L_k := \{y | \deg(u, y, A) = k\} \qquad B_k = u^{-1}(L_k) \cap A.$$

Then we have that every L_k is an open set and its boundary is contained in $u(\partial A)$. It follows that every B_k is an open set and $u(\partial B_k)$ has measure zero. Moreover $|\deg(u, y, A)| = \sum_k |\deg(u, y, B_k)|$ and we conclude

$$\int_{\mathbb{R}^n} |\deg(u, y, A)|dy = \sum_k |[Ju](B_k)| \le \|Ju\|(A).$$

□

Proof of Theorem 4.3.1 **First step: $\Omega \subset \mathbb{R}^n$.** Equation (67), the weak coarea formula and the weak chain rule imply that u and $F(u)$ are BnV functions, hence the expression $\det \nabla F(u)[Ju]$ is formally correct in the weak sense because it is the product of a measure by a continuous function. Moreover condition (67) implies condition (C) of the previous Lemma (see Lemma 4.1.5). Then, thanks to standard measure-theoretic arguments, we only have to check

$$J(F(u))(B) = \int_B \det \nabla F(u(x))d[Ju](x).$$

for every ball $B \subset\subset \Omega$.

We notice that the previous lemma implies

$$J(F(u))(B) = \int_{\mathbb{R}^n} \deg(F(u), y, B)dy = \int_{\mathbb{R}^n} \det \nabla F(y) \deg(u, y, B)dy.$$

By standard covering arguments and the continuity of u we can find a countable family of pairwise disjoint balls B_i centered on x_i such that

(a) $\|Ju\|(B) = \sum_i \|Ju\|(B_i)$;
(b) $\|J(F(u))\|(B) = \sum_i \|J(F(u))\|(B_i)$;
(c) $|\det \nabla F(y) - \det \nabla F(u(x_i))| \le \varepsilon$ for every $y \in u(B_i)$.

First of all we notice that for a.e. y

$$\deg(u, y, B) = \sum_i \deg(u, y, B_i). \tag{72}$$

Indeed Lemma 4.1.6 implies $\mathcal{L}^n(u(\partial B_i)) = 0$, hence we have

$$\int_{\mathbb{R}^n} \left| \deg(u, y, B) - \sum_{i=1}^{N} \deg(u, y, B_i) \right| dy$$

$$\int_{\mathbb{R}^n} \left| \deg\left(u, y, B \setminus \bigcup_{i=1}^{N} \overline{B}_i \right) \right| \leq \|Ju\| \left(B \setminus \bigcup_{i=1}^{N} \overline{B}_i \right)$$

and if we let N go to infinity condition (b) gives (72). Moreover conditions (b) and (c) imply that

$$\Delta_i := \left| \det \nabla F(u(x_i))[Ju](B_i) - \int_{\mathbb{R}^n} \deg(F(u), y, B_i) dy \right|$$

$$\leq \int |\det \nabla F(y) - \det \nabla F(u(x_i))| |\deg(u, y, B_i)| dy \leq \varepsilon \|Ju\|(B_i)$$

and that

$$\begin{aligned} \Lambda_i : &= \left| \det \nabla F(u(x_i))[Ju](B_i) - \int_{B_i} \det \nabla F(u(x)) d[Ju](x) \right| \\ &\leq \varepsilon \|Ju\|(B_i). \end{aligned}$$

Hence we have

$$\begin{aligned} &\left| J(F(u))(B) - \int_B \det \nabla F(u(x)) d[Ju](x) \right| \\ &\leq \sum_i \left| [J(F(u))](B_i) - \int_{B_i} \det \nabla F(u(x)) d[Ju](x) \right| \\ &\leq \sum_i (\Delta_i + \Lambda_i) \leq 2\varepsilon \sum_i \|Ju\|(B_i) = 2\varepsilon \|Ju\|(B). \end{aligned}$$

If we let ε go to zero we conclude $[J(F(u))](B) = [\det \nabla F(u) Ju](B)$.

Second Step Again as in the previous step we remark that both $F(u)$ and u are functions of bounded higher variation. For the sake of simplicity we choose an orthogonal system of coordinates and we denote by $x_1, \ldots, x_{m-n}$ the first $m-n$ and by $z_1, \ldots, z_n$ the remaining ones. Moreover let us fix a form ω of type $f dx_1 \wedge \ldots \wedge dx_{m-n}$ and define $v_x(z) := u(x, z)$, $\Omega_x := \Omega \cap (\{x\} \times \mathbb{R}^n)$. Of course we have $v_x^y(z) = u^y(x, z)$, hence applying Theorem 2.3.1 and Fubini Tonelli Theorem we have

$$\begin{aligned} \int_{\mathbb{R}^{m-n}} \int_{\mathbb{R}^n} \|Jv_x^y\|(\Omega_x) dy dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{m-n}} \|Jv_x^y\|(\Omega_x) dx dy \\ &\leq \int_{\mathbb{R}^n} \|Ju^y\|(\Omega) dy < \infty. \end{aligned}$$

It follows that for a.e. x

$$\int_{\mathbb{R}^n} \|Jv_x^y\|(\Omega_x)dy < \infty.$$

From the slicing arguments and the previous step we know that

$$\begin{aligned}\langle [J(F(u))], \omega\rangle &= \int_{\mathbb{R}^{m-n}} \langle f(\cdot, x), [J(F(v_x)]\rangle dx \\ &= \int_{\mathbb{R}^{m-n}} \langle \det \nabla F(v_x) f(\cdot, x), [Jv_x]\rangle dx \\ &= \langle [Ju], \omega \det \nabla F(u)\rangle.\end{aligned}$$

We can reason in the same way for every differential form $\omega = f d\pi$, where π is a projection on $m-n$ coordinates, and this completes the proof. $\square$

REMARK 4.3.3. If $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p \geq n$ then the strong chain rule holds without any other assumption. For sake of simplicity let us suppose $\Omega \subset \mathbb{R}^n$. If we approximate u strongly in $W^{1,p}$ with C^∞ functions u_k we have that $Ju = \det \nabla u \, \mathcal{L}^n \llcorner \Omega$ and $\det \nabla u_k \to \det \nabla u$ strongly in $L^{p/n}$. Moreover if $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, then

$$[J(F(u_k))] = \det \nabla F(u_k) \, \det \nabla u_k \, \mathcal{L}^n \llcorner \Omega.$$

But $[J(F(u_k))] \to [J(F(u))]$ and passing to the limit we have

$$[J(F(u))] = \det \nabla F(u) \, [Ju].$$

In the same way we can recover the general case $\Omega \subset \mathbb{R}^m$ with $m \geq n$.

4. Further remarks

As corollary of Theorem 4.2.1 and of Theorem 4.3.1 we have

THEOREM 4.4.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^m$ and $p > n-1$. If u is continuous and $\|Ju^y\|(\Omega) \in L^1(\mathbb{R}^n)$ then the strong coarea formula holds.*

However a direct proof of Theorem 4.4.1 can be given as follows. First of all we can observe that

$$\int_{u^{-1}(B)} \varphi d[Ju] = \int_B \langle Ju^y, \varphi\rangle dy$$

for every $m-n$ form $\varphi \in C_c^1(u^{-1}(B))$ and for every ball B. Hence using the condition $\|Ju^y\|(\Omega) \in L^1$ and an approximation argument we can argue that the same formula holds for every form $\varphi \in C(u^{-1}(\overline{B}))$. At this point using an argument similar to that of step 1 of the proof of Theorem 4.2.1 we get the strong coarea formula.

REMARK 4.4.2. Another interesting consequence of Theorem 4.2.1 is the following. Let us suppose that $u \in W^{1,p} \cap L^\infty$ is a Cartesian map, i.e. the graph of u is a rectifiable current which has no boundary in the interior of $\Omega \times \mathbb{R}^n$ (see [**25**] for the theory of Cartesian maps and related questions). Then it is easy to prove that $\|Ju\|$ is an absolutely continuous measure and that its density coincides with the pointwise Jacobian. Hence using the invariance of the definition of Cartesian map by composition of smooth maps we point out that

$$[J(F(u))] = \det \nabla F(u)[Ju] \qquad \text{for every } F \in C^1(\mathbb{R}^n, \mathbb{R}^n).$$

Applying Theorem 4.2.1 we conclude that the strong coarea formula holds for u. Moreover for Sobolev functions we have a classical coarea formula involving the pointwise Jacobian (see [**25**] or Theorem 3.2.3 of [**18**]): a comparison gives that if u is a Cartesian map then there exists a set $\Omega' \subset \Omega$ of full measure such that $\|Ju^y\| = \mathcal{H}^{m-n} \llcorner (u^{-1}(y) \cap \Omega')$ for a.e. y.

For reader's convenience we include a short proof of the fact that for cartesian maps the weak Jacobian coincide with the pointwise Jacobian times the Lebesgue measure. First of all we need a short introduction to the definition of cartesian map (see also section 1 of the next chapter). Roughly speaking a cartesian map is a function such that the current naturally induced by its graph has no boundary inside Ω.

An easy approach, tailored to the purposes of this section, could be the following. Since we have maps which are at least $W^{1,n-1}_{loc}$ we can fix a Borel map G going from Ω to the space of $m \times n$ matrices which is a pointwise representative of the function ∇u, seen as matrix–valued functions after fixing a system of coordinates in both domain and target space of u.

Now, given two k–tuples of indices $\alpha_1, \alpha_2, \dots, \alpha_k$, $\beta_1, \beta_2, \dots, \beta_k$ we define M^α_β as the determinant of the $k \times k$ matrix obtained intersecting the columns α_i with the rows β_j of $G(x)$. Moreover if $\alpha_1, \dots, \alpha_i$ is a set of indices, then we call $\overline{\alpha}$ a set of indices $\overline{\alpha}_1, \dots, \overline{\alpha}_{m-i}$ such that $(\alpha, \overline{\alpha})$ is an even permutation of $\{1, \dots, m\}$.

Finally, given u we define G_u as the current on $\mathbb{R}^m \times \mathbb{R}^n$ which acts on $\omega = f(x,y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_i} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_j}$ (where $i + j = m$) as

$$G_u(\omega) := \int_\Omega f(x, u(x)) M^\beta_{\overline{\alpha}}(x) dx.$$

Now, since our $u \in W^{1,n-1}$ all the minors $G^\beta_{\overline{\alpha}}$ are summable functions when the length of α is at most $n-1$. Hence the previous definition makes sense if we impose that $G^\beta_{\overline{\alpha}}$ are L^1 functions also when the length of α is n. Modulo this imposition we will say that u is a cartesian map if $\partial G_u \llcorner \Omega \equiv 0$.

THEOREM 4.4.3. *Let us call* id *the set of indices* $\{1, \ldots, n\}$. *If* u *is a cartesian map and* $f \in C_c^1(\Omega)$ *then* $[Ju](f\, dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_n})$ *is equal to*

$$\int_\Omega f(x) M_\alpha^{id}(x) dx.$$

PROOF. Thanks to the usual slicing argument it is sufficient to prove the statement when $m = n$. Then in this case what we want to prove is that $[Ju](f\, dx_1 \wedge \ldots \wedge dx_n) = G_u(f(x)\, dy_1 \wedge \ldots \wedge dy_n)$. Now we notice that

$$\begin{aligned} & d(f(x) y_1 \wedge dy_2 \wedge \ldots \wedge dy_n)) \\ = \; & f(x) dy_1 \wedge \ldots \wedge dy_n + \sum_i y_1 \partial_i f(x) dx_i \wedge dy_2 \wedge dy_n, \end{aligned}$$

hence after writing the previous relation substituting y_1 with y_i and averaging we get

$$\begin{aligned} \frac{1}{n} d \left(\sum_i y_i f(x) dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_n \right) = f(x) dy_1 \wedge \ldots \wedge dy_n \\ + \frac{1}{n} \sum_{i,j} y_i \partial_j f(x) dx_j \wedge dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_n. \end{aligned}$$

Now, let us call ν the second term of this equation. Since G_u is a current without boundary in Ω, if $f \in C_c^1(\Omega)$ we have

$$G_u(f dy_1 \wedge \ldots \wedge dy_n) = -G_u(\nu).$$

However a straightforward calculation shows that

$$\begin{aligned} G_u(\nu) &= -\frac{1}{n} \int_\Omega u(x) \cdot \Lambda_n(\nabla u(x)) \cdot \nabla f(x) dx \\ &= \frac{1}{n} \int f \ \mathrm{div}(u \cdot \Lambda_n(\nabla u)), \end{aligned}$$

where the last integral must be read in the sense of distributions. Remark 4.1.2 allows us to conclude that this last term is equal to $[Ju](f dx_1 \wedge \ldots \wedge dx_n)$. ☐

Finally we notice that Remark 4.3.3 and Theorem 4.2.1 give a proof of the strong coarea formula in the case of $u \in W^{1,p}$ with $p > n$ which is different from the proof given in [**27**] and can be extended to the case $p = n$.

CHAPTER 5

Cartesian Currents and weak Jacobians

The starting point of this chapter will be the following question: how does one address the problem of existence of minimizers in the case of compressible Neohookean materials? The model problem is the minimizations of energies of the form

$$E(u) := \int_\Omega |\nabla u|^2 + \varphi(\det \nabla u) \tag{73}$$

where φ is a convex function with superlinear growth and approaching infinity at zero. The minimizers are sought among deformations u which map $\Omega \subset \mathbb{R}^3$ into $\mathbb{R}^3$ and satisfy some notion of invertibility and a Dirichlet boundary condition (the classical starting point is to look for minimizers in the class of orientation-preserving diffeomorphisms or in that of Bilipschitz maps which are orientation preserving).

The known existence theories, starting from the classical works of Morrey (see in particular Ball [**10**] and the bibliography in the second volume of [**25**]) give only partial answers for the case of interest here: the available works rely on relaxation techniques (see for example [**1, 9, 29, 30**], and the works of Giaquinta, Modica and Souček cited below) but they do not attack the problem of how "bad" the domain of the relaxed functional could be. In particular one of the consequences of Malý's work is the following. For every function $u \in W^{1,2}$ we define $F(u)$ as the infimum of

$$\liminf_{n\to\infty} E(u_n)$$

among all sequences of orientation-preserving Bilipschitz maps (u_n) weakly converging in $W^{1,2}$ to u. If u is itself an orientation-preserving Bilipschitz map, then $F(u) = E(u)$. Hence a minimizer of F could be regarded as a weak solution of the classical problem of minimizing E among regular admissible elastic deformations.

The approach of Giaquinta, Modica and Souček (see the second volume of [**25**] for an overview and further references) is part of a more general program of applying Geometric Measure Theory to a wide class of problems in the Calculus of variations. The main idea of this program is to consider graphs instead of functions, hence to relax functionals in the framework of “Cartesian currents” (which can be thought of as generalized graphs). The invertibility conditions can be translated into suitable properties of the graphs as currents and then existence can be obtained within the framework of generalized graphs using the theory of currents of Federer and Fleming.

On the other hand, works based on explicit exhibitions of function spaces closed under weak topologies can attack problems like showing existence of minimizers of

$$E^p(u) := \int_\Omega |\nabla u|^p + \varphi(\det \nabla u) \tag{74}$$

for p strictly bigger than 2, but the proofs fail when $p = 2$. The key problem in the latter case is to exhibit an appropriate weak notion of invertibility which is closed under the weak $W^{1,2}$ topology when one controls $\det \nabla u$. For $p > 3$ this was addressed by Ciarlet and Nečas [**12**].

In this chapter we focus our attention on the work of Müller and Spector [**38**] (inspired by previous ideas of Šverák [**41**]) and on those of Giaquinta, Modica and Souček [**25**]. The main idea of Müller and Spector is to give a condition of invertibility which strongly relies on topological arguments (called condition INV by the authors). This theory is tailored to energies of the form E^p with $p > 2$, and it provides an existence theory which allows for cavitation.

Our first result is that the points of view of Müller and Spector and of Giaquinta, Modica and Souček are closely related, as already suggested by the first authors in their work. Namely, the admissible classes of functions given in [**38**] can be described as the classes of those functions whose graph is a rectifiable current with some precise properties (see Theorem 5.4.1). Hence for functionals like (74) when $p > 2$ the closedness of the class of Müller and Spector can be seen as a byproduct of the closedness of the respective class of rectifiable currents. As already underlined in the introduction this characterization paritally answers a question raised by Giaquinta, Modica and Souček: what should one impose on maps so that conditions on the distributional Jacobians say something interesting on the structure of the graph of u as current? To our knowledge our result is the first in this direction. Basically it relies on the fact that condition INV is invariant under diffeomorphisms of the target and on a result of Dacorogna and Moser ([**14**]) which allows to represent a sufficiently rich class of divergences as Jacobian determinants of smooth maps.

Concerning the case $p = 2$ we show that we can give a definition of admissible maps which is the strict analog of that of Müller and Spector in the case $p > 2$. Again we can characterize this class of functions in terms of their graphs. However neither the techniques of Müller and Spector, nor the use of Geometric Measure Theory can prove the closedness of the class under the topology induced by the functional. In the following chapter we will prove that this is not merely a technical problem.

The chapter is organized as follows. In Section 1 we give some preliminary definitions and notations. In Section 2 we introduce the definition of condition INV, extended to maps in $W^{1,2} \cap L^\infty$, and we prove some properties of the admissible maps. In Section 3 we describe the weak Jacobian of the admissible maps using the properties proved in Section 2. This description will be crucial for proving in Section 4 the characterization of the classes of functions in terms of properties of their graphs.

For the sake of simplicity all the theorems and definitions are stated for maps $u : \mathbb{R}^3 \to \mathbb{R}^3$. However all can be proved in higher dimensions, provided that the maps are in the right Sobolev spaces (this means, for example, that every $W^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$ should be replaced by $W^{1,n-1}(\mathbb{R}^n, \mathbb{R}^n)$).

REMARK 5.0.4. *We warn the reader that in this and in the following chapter we will compute currents on usual differential forms. This could seem inconsistent with the previous notation, since currents were defined by duality on k–tuples of Lipschitz functions. Nonetheless thanks to the multilinearity of the metric current we can extend its action to the vector space generated by all the formal finite sums of k–tuples of Lipschitz maps modulo the obvious equivalence relation*

$$(f, g_1, \ldots, g_k) + (f, h_1, \ldots, h_k) \equiv (f, g_1 + h_1, \ldots, g_k)$$

and modulo all the analogous relations.

This will not complicate too much the notations since we will work only with 2 and 3–forms, but will simplify some explicit calculations.

1. Preliminaries

In this section we establish the basic notation and we give some definitions which will be used throughout the chapter. First of all, using the notation of [**38**] we introduce

DEFINITION 5.1.1. *Given an $n \times n$ matrix we call $\Lambda_{n-1}A$ the matrix of cofactors of A (which obeys $(\Lambda_{n-1}A) \cdot A^T = (\det A) \cdot \mathrm{Id}$).*

We observe that $\Lambda_{n-1}A$ is the transpose of the adjoint matrix to A, and that for gradient fields it is divergence-free, i.e.

$$\operatorname{div} \Lambda_{n-1}\nabla u = \partial_i(\Lambda_{n-1}\nabla u)_{ji} = 0\,.$$

In the following B will denote an open ball in $\mathbb{R}^k$, $B(x,r)$ the open ball of radius r centered on x and Ω a bounded open set. Given an open set A, ∂A will be its topological boundary, $\overline{A}$ its topological closure and χ_A its characteristic function.

We call density of A in $x \in \mathbb{R}^n$ the limit

$$\lim_{\rho\to 0} \frac{|B(x,\rho)\cap A|}{|B(x,\rho)|}$$

whenever it exists and we denote this number by $D(A,x)$.

Next we give a brief list of concepts and more technical objects of geometric measure theory. For proofs and for a more detailed exposition we refer to the book of Federer [**18**] and to the first volume of the book of Giaquinta, Modica and Souček [**25**].

We say that A is a *Caccioppoli set* if its characteristic function is a function of bounded variation. Moreover we denote by $\mathrm{Per}(A)$ the total variation of the distributional derivative of χ_A. The following is a well-known theorems

THEOREM 5.1.2. *Let $A \subset \mathbb{R}^n$ be a Caccioppoli set. Then there exists a rectifiable $n-1$ dimensional set (which is called* reduced boundary *and is denoted by $\partial^* A$) such that*

(i) *for every $x \in \partial^* A$ the sets defined by $(A-x)/\rho$ converge locally in measure to a half space when $\rho \downarrow 0$ (hence $D(A,x) = 1/2$);*
(ii) *for $\mathcal{H}^{n-1}$ a.e. $x \in \mathbb{R}^n \setminus \partial^* A$ the density of A in x is either 1 or 0;*
(iii) *the distributional derivative $D\chi_A$ is equal to $\nu\mathcal{H}^{n-1} \llcorner \partial^* A$, where ν is a unit vector normal to $\partial^* A$ (which is called* outward normal*).*

DEFINITION 5.1.3. *A measurable function $f : \Omega \to \mathbb{R}^n$ is called* a.e. injective *if there is a measurable set $A \subset \Omega$ such that $|\Omega \setminus A| = 0$ and $f|_A$ is injective.*

Now let us fix a Sobolev map $u \in W^{1,p}(\Omega)$. Using standard arguments involving maximal functions we can show that for every ε there is a closed set A such that $|\Omega \setminus A| \leq \varepsilon$ and u is Lipschitz on A. At this point, using Rademacher Theorem and Whitney Theorem, we can show that for every ε there is a closed set A such that $|\Omega \setminus A| \leq \varepsilon$ and $u|_A$ is C^1. Hence we define

Definition 5.1.4 (approximate differentiability). *Let u be a measurable function with domain Ω. We say that u is approximately differentiable in $x \in \Omega$ if there exists a closed set $A \subset \Omega$ such that $D(A,x) = 1$ and the restriction of u to A is a C^1 function. Moreover we call approximate differential of u in x the differential of $u|_A$ and we denote it by $apDu(x)$. Finally we will denote by Ω_d the set of points where u is approximately differentiable.*

Remark 5.1.5. We warn the reader that the previous definition does not coincide completely with the classical definition of Federer's book. However using Theorem 3.1.8 in [**18**], Rademacher Theorem and Whitney Theorem one can see that if u is approximately differentiable in A according to Federer's definition, then it is approximately differentiable a.e. in A according to our definition. This is enough for our purposes and we believe that our choice makes theorems and proofs easier.

Remark 5.1.6. If u is a function in some Sobolev space and we select a pointwise representative for ∇u, then this function coincides a.e. with $apDu$. Hence we will use both notations without distinction.

Definition 5.1.7. *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a function which is approximately differentiable almost everywhere. Given a set $A \subset \mathbb{R}^n$ we call* geometrical image of A through u *the set given by $u(\Omega_d \cap A)$. Moreover we denote it by $\mathrm{im}_{\mathrm{G}}(u, A)$.*

The following theorem gives a version of the classical area formula for approximately differentiable functions.

Theorem 5.1.8. *Let us suppose that u maps $\mathbb{R}^n$ into $\mathbb{R}^n$ and it is injective on a measurable set A. Then*

$$|\,\mathrm{im}_{\mathrm{G}}(u, A)| = \int_{A\cap\Omega_d} |\det(apDu(x))|dx\,. \tag{75}$$

For the reader's convenience we report a proof of the previous theorem in the second section of the appendix, basically following Theorem 3.2.3 of [**18**]. In the following we will need the next technical lemma, which is easily checked for functions which are restrictions of C^1 functions, and can be immediately extended to approximately differentiable functions using Definition 5.1.4.

Lemma 5.1.9. *Suppose that u maps $\mathbb{R}^n$ into $\mathbb{R}^n$ and that $x \in \Omega_d$ is such that* $\det(apDu(x)) > 0$*, and let A be a measurable subset of Ω. Then, the following holds:*

(i) *if $D(A,x) = 1$ then $D(\mathrm{im}_{\mathrm{G}}(u, A), u(x)) = 1$;*

(ii) *if* $D(A \cap \{(x'-x)\cdot\nu \geq 0\}, x) = 1/2$, *then*

$$D\Big(u(A) \cap \big\{(y-u(x))\cdot\Lambda_2(\nabla u)\nu \geq 0\big\}\,,\, u(x)\Big) = \frac{1}{2}\,.$$

We can associate a current to any approximately differentiable function $u : \Omega \to \mathbb{R}^n$ in the following way. First we choose an orthonormal basis $e_1, \ldots e_k$ for $\mathbb{R}^k \supset \Omega$. Then we split Ω (possibly neglecting a subset of Lebesgue measure zero) into a countable union of closed sets F_i such that $u|_{F_i}$ is a C^1 function. We regard $G_i := \{(x, u(x)) : x \in F_i\}$ as a closed subset of a C^1 manifold and we choose for every $x \in F_i$ the k-tuples of vectors

$$\{(e_1, \nabla u(x)\cdot e_1), \ldots, (e_k, \nabla u(x)\cdot e_k)\}$$

as orientation of the tangent plane to G_i in $(x, u(x))$. To every G_i we associate the current T_i induced by this orientation. At this point one is tempted to associate $\sum_i T_i$ to u. In general this is not possible because $\sum_i T_i(\omega)$ could be a divergent series for some form $\omega \in C^\infty_c$. However, when the minors of $apDu$ are L^1 functions, $\sum_i T_i$ actually defines a current (see below) and we denote it by G_u.

As already explained in Remark 5.0.4 in this chapters we will test currents also on formal sums of k–tuples of differential forms, modulo the appropriate equivalence relations. This extension will allow us to compute currents on what are identified in the literature as classical differential forms (whereas in the previous chapter the word "form" identified what in the literature is called "simple form"). From now on, for the sake of simplicity, the word "form" will take again its classical meaning.

DEFINITION 5.1.10. *Suppose that L is an $n \times k$ matrix. If α, β are two k-tuples of indices $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ then we call $G^\beta_\alpha(L)$ the matrix given by the intersection of rows $\beta_1, \ldots, \beta_k$ and columns $\alpha_1, \ldots, \alpha_k$ of L. Moreover we call $M^\beta_\alpha(L)$ the determinant of $G^\beta_\alpha(L)$.*

Let now $u : \mathbb{R}^k_x \supset \Omega \to \mathbb{R}^n_y$ be a C^1 function and fix two systems of coordinates for $\mathbb{R}^k_x$ and $\mathbb{R}^n_y$, namely $x_1, \ldots, x_k$ and $y_1, \ldots y_n$. We then identify ∇u with its matrix representation in these systems. Finally let us fix a k-dimensional form $f(x,y)dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_j} \wedge dy_{\beta_1} \wedge \ldots \wedge dy_{\beta_i}$, with $i + j = k$ (in the following we will use the shorthand $f dx_\alpha \wedge dy_\beta$). For any $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_i)$ such that $(\overline{\alpha}, \alpha)$ is an even permutation of $\{1, \ldots, k\}$ one can show that

$$G_u(f dx_\alpha \wedge dy_\beta) = \int_\Omega f(x, u(x)) M^\beta_{\overline{\alpha}}(\nabla u(x))dx. \qquad (76)$$

With standard techniques analogous formulas can be written when u is merely an approximately differentiable function such that $M^{\beta}_{\overline{\alpha}}(\nabla u(x))$ is absolutely integrable. The product structure of $\mathbb{R}^k_x \times \mathbb{R}^n_y$ gives a natural splitting of any differential form ω, which induces a splitting on currents by duality.

DEFINITION 5.1.11. *Let us write the differential form ω as*

$$\sum_{\alpha,\beta} f_{\alpha\beta} dx_\alpha \wedge dy_\beta .$$

Then for every integer h we define

$$\omega^{(h)} := \sum_{\alpha,\beta \text{ s.t. the length of } \beta \text{ is } h} f_{\alpha\beta} dx_\alpha \wedge dy_\beta \,. \tag{77}$$

Given a current T in $\mathbb{R}^k_x \times \mathbb{R}^n_y$ we define $(T)_h$ by

$$(T)_h(\omega) := T(\omega^{(h)}).$$

We notice that if u maps $\mathbb{R}^k$ into $\mathbb{R}^k$ then G_u is k-dimensional current. Hence ∂G_u is a $k-1$ dimensional current and $(\partial G_u)_i$ is a $k-1$-dimensional current which is different from zero only on forms of type

$$\omega^{(h)} := \sum_{\alpha,\beta \text{ s.t. the length of } \beta \text{ is } h} f_{\alpha\beta} dx_\alpha \wedge dy_\beta \,.$$

A typical example of a current which behaves like $(\partial G_u)_i$ is given by the product of a $(k-1-i)$-dimensional surface $M_x \subset \mathbb{R}^k_x$ with an i-dimensional surface $N_y \subset \mathbb{R}^k_y$. Of course when u is smooth then $(\partial G_u)_i = 0$ for every i. Intuitively, when $\partial G_u \not\equiv 0$ some holes or some fractures occur in the graph of u: the currents $(\partial G_u)_i$ measure in a certain sense the degree of verticality of such holes.

When $u \in W^{1,p}(\mathbb{R}^k, \mathbb{R}^k)$ we have that $(\partial G_u)_i = 0$ for every $i \leq (p-1)$. This can be proved by approximating u strongly in $W^{1,p}$ with C^∞ functions u_n and observing that $(\partial G_{u_n})_i$ converges weakly to $(\partial G_u)_i$ (since in the computations only products of less than $i+1$ derivatives of u_n are involved). We refer to pages 238–247 of the first volume of [**25**] for a thorough discussion: here we just report a simple proof of the previous statement (compare with Remark 3.2.3.3 in [**25**]).

THEOREM 5.1.12. *If $u \in W^{1,p}(\mathbb{R}^k, \mathbb{R}^k)$ we have that $(\partial G_u)_i = 0$ for every $i \leq (p-1)$.*

PROOF. By standard regularization arguments we can find a sequence of functions $u_n \in C^\infty(\mathbb{R}^k, \mathbb{R}^k)$ which are strongly converging in $W^{1,p}$ to the function u. We claim then that $(G_{u_n})_j$ is converging weakly, in the sense of currents, to (G_u) for every $j \leq p$. Indeed let for example be $\omega = f(x,y)dx_\alpha \wedge dy_\beta$ with the length of β being less than or equal to p (and $f \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^k)$. Then

$$G_{u_n}(\omega) = \int_{\mathbb{R}^k} f(x, u_n(x)) M_{\overline{\alpha}}^{\beta}(\nabla u_n(x))dx$$

Now $M_{\overline{\alpha}}^{\beta}(\nabla u_n)$ is the sum of products of at most p partial derivatives of components of u: so since ∇u_n is converging to ∇u strongly in L^p, $M_{\overline{\alpha}}^{\beta}(\nabla u_n)$ is converging to $M_{\overline{\alpha}}^{\beta}(\nabla u)$ strongly in L^1. Moreover u_n is converging also strongly in $W^{1,p}$ to u, hence $f(x, u_n(x))$ is converging strongly in L^p to u, and since $\|f\|_\infty < \infty$, $f(x, u_n(x))$ is then weakly* converging to $f(x, u(x))$. This implies that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^k} f(x, u_n(x)) M_{\overline{\alpha}}^{\beta}(\nabla u_n(x))dx \\ = &\int_{\mathbb{R}^k} f(x, u(x)) M_{\overline{\alpha}}^{\beta}(\nabla u(x))dx = G_u(\omega). \end{aligned}$$

Now we notice that $d(\omega^{(h)}) = (d(\omega^{(h)}))^{(h+1)}$. Hence for every $h \le p-1$ we have that

$$\lim_{n\to\infty} G_{u_n}(d(\omega^{(h)})) = G_u(d(\omega^{(h)}).$$

But we also recall that since $u_n \in C^1$ we have that $\partial G_{u_n} = 0$. Hence we conclude that $(\partial G_u)_h(\omega) = G_u(d(\omega^{(h)}) = 0$ and this completes the proof □

2. Condition INV

In this section we recall the definition of condition INV given by Müller and Spector in [**38**] following ideas of Šverák in [**41**] and we extend it to the case of functions in $W^{1,2} \cap L^\infty$. Moreover we will follow the proofs contained in [**38**] of some properties which we will use for the "graph" characterization of the next section. We start with the definition of degree for a map from the sphere $\mathbf{S}^2$ to $\mathbb{R}^3$. Of course Müller and Spector in their work use the definition of degree for continuous functions, since a a map $u \in W^{1,p}(\partial B)$ is continuous when $p > 2$. In our case we use the integral definition of the degree for $W^{1,2}$ functions, which is nowadays quite well known in the literature (compare with [**11**], see also the book [**19**]).

Definition 5.2.1. *Let us suppose that u belongs to $W^{1,2}(\partial B, \mathbb{R}^3)$ and to $L^\infty(\partial B, \mathbb{R}^3)$. Then we define* $\deg(u, \partial B, \cdot)$ *as the only L^1 function which satisfies the identity*

$$\int \deg(u, \partial B, y) \operatorname{div} g(y) dy := \int_{\partial B} (g \circ u) \cdot \Lambda_2(\nabla u) \cdot \nu \, d\mathcal{H}^2, \qquad (78)$$

for every C^∞ vector field g (ν denotes the outer unit normal to B).

For the sake of simplicity in the following we will use the notation $\Lambda_2(u)$ for $\Lambda_2(\nabla u)$ and we will suppose that the target space of a function is $\mathbb{R}^3$ when not specified.

REMARK 5.2.2. If $u \in C^\infty$, then Definition 5.2.1 coincides with the classical

$$\deg(u, \partial B, y) = \sum_{x \in u^{-1}(y) \cap B} \operatorname{sgn} \det \nabla u(x) \,. \tag{79}$$

This is an effect of the identity

$$\operatorname{div}\left((g \circ u)\Lambda_2(u)\right) = \det \nabla u \; (\operatorname{div} g) \circ u \,. :$$

we refer to Lemma 4.1.1 of which this remark is just a particular case.

REMARK 5.2.3. To show that deg is well defined we have to prove that if $\operatorname{div} g = \operatorname{div} h$ then the right hand side of Eq. (78) is the same for g and h. This is equivalent to prove that the right hand side vanishes if $\operatorname{div} g = 0$. In order to do that, take a sequence of functions u_n in $C^\infty(B)$ which are equibounded and converge to u strongly in $W^{1,2}(\partial B)$. Since $\Lambda_2(u_n) \to \Lambda_2(u)$ strongly in L^1, and $g \circ u_n \overset{*}{\rightharpoonup} g \circ u$ in L^∞,

$$\lim_{n\to\infty} \int (g \circ u_n) \cdot \Lambda_2(u_n) \cdot \nu \, dy = \int (g \circ u) \cdot \Lambda_2(u) \cdot \nu \, dy \,. \tag{80}$$

From the previous remark, we conclude that the right hand side is zero.

PROPOSITION 5.2.4. *The degree is an integer-valued function of bounded variation. Moreover for a.e. ball B we have*

$$\int \deg(u, \partial B, y) \operatorname{div} g(y) dy = \int_{u(\partial B \cap \Omega_d)} g(y) \cdot \tilde{\nu}(y) \, d\mathcal{H}^2(y) \,, \tag{81}$$

where

$$\tilde{\nu}(y) = \sum_{x \in u^{-1}(y) \cap \partial B \cap \Omega_d} \frac{\Lambda_2(u)(x) \cdot \nu(x)}{|\Lambda_2(u)(x) \cdot \nu(x)|}$$

($\nu(x)$ is the outer unit normal to B).

PROOF. From the definition it follows that

$$\left| \int \deg(u, \partial B, y) \operatorname{div} g(y) dy \right| \le \|\nabla u\|^2_{L^2(\partial B)} \|g\|_\infty \tag{82}$$

To prove the first statement take a sequence of functions u_n in $C^\infty(B)$ which are equibounded and converge to u in $W^{1,2}(\partial B)$. Then, from Remark 5.2.2 there is a large ball B' such that $\deg(u_n, \partial B, y) = 0$ if $y \in \mathbb{R}^3 \setminus B'$. Moreover, by Eq. (82) and Remark 5.2.2, we get that the

L^1 norm of $\nabla \deg(u_n)$ is equibounded. Poincaré inequality applied to the ball B' yields uniform control of the L^1 norm of $\deg(u_n)$. Then, by the compactness theorem for functions of bounded variation there is $\phi \in BV(\mathbb{R}^3, \mathbb{Z})$ such that

$$\deg(u_n, \partial B, y) \to \phi$$

strongly in L^1. Further, Eq. (80) shows that $\deg(u, \partial B, y) = \phi(y)$ in the sense of distributions, hence it is a BV function and it is integer-valued. Eq. (81) is the area formula from Corollary 3.2.20 of [**18**]. $\square$

One easy corollary of the previous statement is that for every $u \in W^{1,2}(\partial B) \cap L^\infty(\partial B)$, the set

$$A_{u,B} := \{y | \deg(u, \partial B, y) \neq 0\} \tag{83}$$

is a Caccioppoli set.

DEFINITION 5.2.5. *For $u \in W^{1,2}(\partial B) \cap L^\infty(\partial B)$, the topological image of B under u, $\mathrm{im}_{\mathrm{T}}(u, B)$, is the set of points where the density of $A_{u,B}$ is* 1.

The topological image, defined here as in [**41**] and [**38**], can be seen as the set "enclosed" by $u(\partial B)$. Indeed when $u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ for $p > 2$ then $u|_{\partial B}$ is continuous for almost every ball B and the degree of Definition 5.2.1 coincides with the classical topological degree. For those balls the topological image of u is an open set which has $u(\partial B)$ as boundary. Hence it is very natural to request that maps allowed in elasticity map the interior of balls inside the topological image of the respective spheres: this is what Müller and Spector call *condition INV*.

DEFINITION 5.2.6. *We say that $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfies property INV_L in the ball $B(a, r) \subset \Omega$ if*

(i) *its trace on ∂B is in $W^{1,2} \cap L^\infty$;*
(ii) *for a.e. $x \in B(a, r)$, $u(x) \in \mathrm{im}_{\mathrm{T}}(u, B(a, r))$;*
(iii) *for a.e. $x \in \Omega \setminus B(a, r)$, $u(x) \notin \mathrm{im}_{\mathrm{T}}(u, B(a, r))$.*

We say that $u \in W^{1,2}(\Omega) \cap L^\infty$ satisfies property INV if for every $a \in \Omega$ there is $r_a > 0$ such that for $\mathcal{L}^1$-a.e. $r \in (0, r_a)$ property INV_L holds in $B(a, r)$.

In the next lemmas we will follow the proofs of Müller and Spector (based on the work of Šverák) with slight modifications which allow us to include the case $u \in W^{1,2} \cap L^\infty$. The propositions will lead to prove the positivity of the weak Jacobian of the maps which satisfy condition INV (this description is given in the next section). This property and the invariance of condition INV under orientation-preserving diffeomorphisms of the target space will be the main ingredients in the characterization of Section 5.

LEMMA 5.2.7. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$. Suppose that condition INV holds and that* $\det \nabla u \neq 0$ *on Ω_d. Then, $u|_{\Omega_d}$ is injective.*

PROOF. We know that $|\Omega \setminus \Omega_d| = 0$ and that $D(\Omega_d, x) = 1$ for every $x \in \Omega_d$. Moreover by Lemma 5.1.9 $D(u(\Omega_d), u(x)) = 1$ for every $x \in \Omega_d$. To prove injectivity, fix x, $y \in \Omega_d$. By condition INV we can choose $r > 0$ such that the two balls $B(x,r)$ and $B(y,r)$ are disjoint, contained in Ω and satisfy condition INV_L. Therefore the images of the two balls are disjoint, possibly ruling out a set of measure zero. Since x and y are points of density 1 for the two images, $u(x) \neq u(y)$. □

LEMMA 5.2.8. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$, with* $\det \nabla u \neq 0$ *on Ω_d, and choose $B \subset \Omega$ such that condition INV_L holds. Then* $\mathrm{im}_{\mathrm{G}}(u,B) \subset \mathrm{im}_{\mathrm{T}}(u,B)$, *and* $\mathrm{im}_{\mathrm{G}}(u,\mathbb{R}^3 \setminus B) \subset \mathbb{R}^3 \setminus \mathrm{im}_{\mathrm{T}}(u,B)$.

PROOF. Let $A = \{x \in B \cap \Omega_d : u(x) \in \mathrm{im}_{\mathrm{T}}(u,B)\}$. By condition INV_L, $B \setminus A$ is a null set, and for any $x \in B \cap \Omega_d$ we get $D(A,x) = 1$. By Lemma 5.1.9 $D(u(A), u(x)) = 1$, and since $u(A) \subset \mathrm{im}_{\mathrm{T}}(u,B)$ by definition, $D(\mathrm{im}_{\mathrm{T}}(u,B), u(x)) = 1$. Hence $u(x) \in \mathrm{im}_{\mathrm{T}}(u,B)$. The converse is proved analogously. □

Since in the following we will consider only maps such that $\det \nabla u > 0$ a.e., we denote by Ω_d the set where u is approximately differentiable and $\det apDu > 0$.

LEMMA 5.2.9. *Suppose that $u \in W^{1,2}(\Omega) \cap L^\infty$ satisfies condition INV and that* $\det \nabla u > 0$ *a.e. Then,* $\deg(u, \partial B, y) \in \{0,1\}$ *for a.e. B and a.e. y.*

PROOF. Fix a ball B such that

(i) condition INV_L is satisfied for B;
(ii) $\mathcal{H}^2(\Omega \setminus \Omega_d \cap \partial B) = 0$.

We observe that these conditions are satisfied for almost every ball and that the approximate differential of the trace on ∂B coincides with the restriction to the tangent plane to ∂B of the approximate differential in Ω. From Proposition 5.2.4

$$\int \deg(u, \partial B, y) \operatorname{div} g(y) dy = \int_{u(\Omega_d \cap \partial B)} g \cdot \tilde{\nu} \, d\mathcal{H}^2 ,$$

and moreover, since u is invertible in Ω_d, $|\tilde{\nu}| = 1$. Consider the sets

$$U_k = \{y : \deg(u, \partial B, y) \geq k\}$$

for k integer and positive, and

$$U_k = \{y : \deg(u, \partial B, y) \leq k\}$$

for k integer and negative. From the fact that $u \in L^\infty$ it follows that all the sets U_k are bounded. Since the distributional derivative of $\deg(u, \partial B, \cdot)$ equals $\tilde{\nu}\mathcal{H}^2$ restricted to the jump set $u(\Omega_d \cap \partial B)$, and $\tilde{\nu}$ is a unit vector, we have

$$\partial^* U_k \cap \partial^* U_h = \emptyset$$

for any $k \neq h$. We now show that $|U_k| = 0$ for all $k \neq 1$. Indeed, if this was not the case, since U_k is bounded then $\mathcal{H}^2(\partial^* U_k) > 0$. Since the jump set is contained, apart from an $\mathcal{H}^2$-null set, in $u(\partial B \cap \Omega_d)$, we can choose $x \in \partial B \cap \Omega_d \cap u^{-1}(\partial^* U_k)$. Let $A = \Omega_d \setminus B$. By point (ii) of Lemma 5.1.9 we get that

$$u(A) \cap \{y \in \mathbb{R}^3 : (y - u(x)) \cdot \tilde{\nu} \geq 0\}$$

(with $\tilde{\nu}$ as in Proposition 5.2.4) has density 1/2 at $u(x)$. But by condition INV_L, $\deg(u, \partial B, \cdot)$ vanishes on $u(A)$. In view of the area formula of Proposition 5.2.4 this implies that the trace $\deg^+$ of the BV function deg corresponding to the orientation $\tilde{\nu}$ of the interface is zero. Since the jump is 1, this implies that the other trace is 1. Hence $u(x) \in \partial U_1$, contrary to our hypothesis. □

Definition 5.2.10. *Given $a \in \Omega$, we call R_a the set of $r > 0$ such that $B(a, r) \subset \Omega$,*

(i) *condition INV_L is satisfied for $B(a, r)$;*
(ii) $\mathcal{H}^2(\partial B(a, r) \setminus \Omega_d) = 0$.

Lemma 5.2.11. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$. Suppose that condition INV holds. Then, for any $a, b \in \Omega$ and any $r \in R_a$, $s \in R_b$,*

(i) $\mathrm{im_T}(u, B(a, r)) \cap \mathrm{im_T}(u, B(b, s)) = \emptyset$ *if* $B(a, r) \cap B(b, s) = \emptyset$;
(ii) $\mathrm{im_T}(u, B(a, r)) \subset \mathrm{im_T}(u, B(b, s))$ *whenever* $B(a, r) \subset B(a, s)$.

Proof. Let us first prove (i). Let $A = \mathrm{im_T}(u, B(a, r))$, and $C = \mathrm{im_T}(u, B(b, s))$. Except for an $\mathcal{H}^2$ null set, $\partial^* C$ equals $u(\Omega_d \cap \partial B(b, s))$. Take $x \in \Omega_d \cap \partial B(b, s)$. Then, $x \notin B(a, r)$, and by Lemma 5.2.8, $u(x) \notin A = \mathrm{im_T}(u, B(a, r))$. It follows that $\mathcal{H}^2(\partial^* C \setminus A) = 0$, and analogously $\mathcal{H}^2(\partial^* A \setminus C) = 0$. The result follows from Lemma 7.4.1 of the Appendix. Point (ii) is obtained analogously, by taking $A = \mathrm{im_T}(u, B(a, r))$, and $C = \mathbb{R}^3 \setminus \mathrm{im_T}(u, B(b, s))$. □

Before concluding this section, we define $F(x)$ as the "topological image of a point x". More precisely, DEFINITION 5.2.12. *For every x we define*

$$F(x) := \bigcap_{r>0} \mathrm{im}_{\mathrm{T}}(u, B(x,r)) \tag{84}$$

We observe that by Lemma 5.2.7, for $x \in \Omega_d$, $u(x) \in F(x)$.

3. Weak Jacobians of INV maps

The goal of this section is the following theorem, which characterizes the weak Jacobians of the class of functions considered by Müller and Spector. As in the previous section we follow their proofs with minor adjustments to include the case $p = 2$.

Since the weak Jacobian will be a 3–dimensional current we will refer to it as a distribution. Moreover when $[Ju]$ is a measure and φ a continuous function we will write $\int \varphi d[Ju]$ for the integral of φ in the measure $[Ju]$.

THEOREM 5.3.1. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$. Suppose that condition INV holds, that* $\det \nabla u > 0$ *a.e., and* $\mathrm{Per}(\mathrm{im}_{\mathrm{G}}(u,\Omega)) < \infty$. *Then,*

$$[Ju] = \det \nabla u + \sum_{x_i \in C_u} \mathcal{L}^3(F(x_i))\delta_{x_i} \tag{85}$$

where C_u is the set of points x such that $\mathcal{L}^3(F(x)) > 0$. Further,

$$\sum_{x_i} \mathrm{Per}(F(x_i)) \le \mathrm{Per}(\mathrm{im}_{\mathrm{G}}(u,\Omega)) \,. \tag{86}$$

Before proving Theorem 5.3.1 we give some partial results.

LEMMA 5.3.2. *Let $u \in W^{1,2}(\Omega) \cap L^\infty$. Suppose that condition INV holds, and that* $\det \nabla u > 0$ *a.e. Then,*

(i) *$[Ju] \ge 0$, hence it is a Radon measure;*
(ii) *the absolutely continuous part of $[Ju]$ with respect to $\mathcal{L}^3$ has density* $\det \nabla u$;
(iii) *for every $a \in \Omega$ and for a.e. $r \in R_a$,*

$$([Ju])(B(a,r)) = \mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B(a,r))) \,. \tag{87}$$

PROOF. Take $\phi \in C_0^\infty(B(0,1))$ radially symmetric, nonnegative, monotone in the radial direction and such that $\int \phi = 1$. Define the standard sequence of mollifiers as

$$\phi_\varepsilon(x) := \varepsilon^{-3} \phi\left(\frac{x}{\varepsilon}\right) \,.$$

First of all we prove positivity of $(\phi_\epsilon * [Ju])(x)$ for every $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) \ge \varepsilon$. Let $\phi(x) = f(|x|)$, with $f' \le 0$. Then, from the definition of weak Jacobian,

$$\begin{aligned}(\phi_\varepsilon * [Ju])(x) &= -\frac{1}{3}D\phi_\varepsilon * [u\cdot(\Lambda_2(u))](x) \\ &= -\frac{1}{3}\int_{B(x,\varepsilon)} D\phi_\varepsilon(x-z)u(z)(\Lambda_2(u))(z)dz \\ &= -\frac{1}{3}\int_0^\varepsilon \varepsilon^{-4}f'\left(\frac{r}{\varepsilon}\right)dr\int_{\partial B(x,r)} u\cdot(\Lambda_2(u))\nu\, d\mathcal{H}^2\,,\end{aligned}$$

where ν is the outward unit normal to $B(x,r)$. From Definition 5.2.1 and Lemma 5.2.9

$$\frac{1}{3}\int_{\partial B(x,r)} u\cdot(\Lambda_2(u))\nu\, d\mathcal{H}^2 = \int \deg(u,\partial B,y)dy = \mathcal{L}^3(\mathrm{im_T}(u,B)) \geq 0\,.$$

This concludes the proof of point (i). Point (ii) is a direct consequence of Theorem 3.2.2 of the third chapter. By standard arguments on Radon measures,

$$[Ju](B(a,r)) = \sup_{\delta>0}\int \Phi_\delta(|x-a|)\,d[Ju](x)$$

where

$$\Phi_\delta(s) = \begin{cases} 1 & \text{if } s\leq r-\delta \\ (r-s)/\delta & \text{if } r-\delta\leq s\leq r \\ 0 & \text{if } s\geq r\end{cases}$$

A computation similar to that above, gives, for δ sufficiently small,

$$\begin{aligned}\int \Phi_\delta(|x-a|)\,d[Ju](x) &= \int_{r-\delta}^r \frac{1}{3\delta}ds\int_{\partial B(a,s)} u\cdot\Lambda_2(u)\cdot\nu\, d\mathcal{H}^2 \\ &= \int_{r-\delta}^r \frac{1}{\delta}\mathcal{L}^3(\mathrm{im_T}(u,B(a,s)))ds \\ &= \mathcal{L}^3(\mathrm{im_T}(u,B(a,r))\end{aligned}$$

for $\mathcal{L}^1$-a.e. $r\in R_a$. □

REMARK 5.3.3. It is convenient to redefine the sets R_a excluding the zero-measure part in which Eq. (87) does not hold.

PROOF OF THEOREM 5.3.1. Let us split $[Ju]$ into the absolutely continuous part with respect to the Lebesgue measure μ_{ac} and the singular part μ_s. By standard arguments of measure theory we can find a countable collection of Dirac masses δ_{x_i} such that

$$\mu_s = \sum k_i\delta_{x_i} + \mu_c\,,$$

where μ_c contains no atoms. From the previous Lemma we know that $\mu_{ac} = \det\nabla u$. We have to prove that $\mu_c = 0$, and $k_i = \mathcal{L}^3(F(x_i))$.

Let $A = \mathrm{im_G}(u, \Omega)$. Fix $a \in \Omega$, $r \in R_a$, and let $C = \mathrm{im_T}(u, B(a, r))$. The previous Lemma shows that $\mathcal{L}^3(C \setminus A) = \mu_s(B(a, r))$. Now we want to show that

$$\mathcal{L}^3(C \setminus A)^{2/3} \leq c_3 \mathcal{H}^2(\partial^* A \cap C) \tag{88}$$

where $c_3 = (36\pi)^{-1/3}$ is the isoperimetric constant.

The set $C \setminus A$ is a Caccioppoli set, hence the previous equation is proved if we can show that

$$\mathcal{H}^2(\partial^*(C \setminus A)) \leq \mathcal{H}^2(\partial^* A \cap C)\,. \tag{89}$$

Take $y \in \partial^*(C \setminus A)$. Clearly $D(A, y) < 1$ and $D(C, y) > 0$. We now show that $D(C, y) < 1$ only for a $\mathcal{H}^2$-null set. Indeed, if $D(C, y) < 1$ then y belongs to $\partial^* C$, which up to a $\mathcal{H}^2$-null set coincides with $u(\Omega_d \cap \partial B(a, r))$. By Lemma 5.1.9, $D(A, u(x)) = 1$ for every $x \in \Omega_d$, which contradicts the statement $D(A, y) < 1$. Therefore $D(C, y) = 1$, which in turn implies $D(A, y) = 1/2$, hence $y \in \partial^* A$. By Definition 5.2.5, $y \in C$, and Eq. (89) is proved.

Now we use Eq. (88) to prove that $\mu_c = 0$. By definition we have $\lim_{r\downarrow 0} \mu_c(B(a, r)) = 0$ for every a. Fix $\varepsilon > 0$ and consider the family of closed balls

$$\mathcal{F}_\varepsilon = \left\{ \overline{B(a, r)} : a \in \Omega,\ r \in R_a,\ \mu_c(\partial B(a, r)) = 0,\ \mu_c(\overline{B(a, r)}) < \varepsilon \right\}.$$

By the Besicovitch covering theorem (e.g. Theorem 2.19 of [**7**]) there is a sequence of pairwise disjoint, closed balls $\overline{B_i} \in \mathcal{F}_\varepsilon$ such that

$$\mu_c(\Omega) = \mu_c\left(\bigcup_{i=1}^{\infty} \overline{B_i}\right) = \sum_{i=1}^{\infty} \mu_c(B_i)\,.$$

By (87) and (88),

$$\sum_{i=1}^{\infty} [\mu_c(B_i)]^{2/3} \leq \sum_{i=1}^{\infty} \left[\mathcal{L}^3(\mathrm{im_T}(u, B_i) \setminus A)\right]^{2/3} \leq c_3 \mathcal{H}^2(\partial^* A)\,.$$

We conclude that $\mu_c(\Omega) \leq c_3 \varepsilon^{1/3} \mathcal{H}^2(\partial^* A)$ and since ε was arbitrary, this gives $\mu_c = 0$.

To show that $k_i = \mathcal{L}^3(F(x_i))$, we observe that

$$\begin{aligned} k_i &= \lim_{r\to 0} \mu_s\left(B(x_i, r)\right) \\ &= \lim_{r\to 0} \mathcal{L}^3(\mathrm{im_T}(u, B(x_i, r)) \setminus \mathrm{im_G}(u, B(x_i, r)))\,. \end{aligned}$$

Since by the area formula $\mathcal{L}^3(\mathrm{im}_{\mathrm{G}}(u, B(x_i, r)))$ converges to zero, we get

$$\begin{aligned} k_i &= \lim_{r\to 0} \mathcal{L}^3(\mathrm{im}_{\mathrm{T}}(u, B(x_i, r))) \\ &= \mathcal{L}^3\left(\bigcap_{r>0} \mathrm{im}_{\mathrm{T}}(u, B(x_i, r))\right) = \mathcal{L}^3\left(F(x_i)\right) \end{aligned}$$

where monotonicity and the definition of $F(x)$ have been used. □

The following remark will be crucial for the next section.

REMARK 5.3.4 (Invariance). If we compose u with a diffeomorphism $H : B(0, R) \to \mathbb{R}^3$, where $B(0, R) \supset \mathrm{im}_{\mathrm{G}}(u, \Omega)$, then $H \circ u$ satisfies all the hypotheses of the previous theorem, and

$$[J(H \circ u)] = (\det \nabla H) \circ u \det \nabla u \mathcal{L}^3 + \sum_{x_i \in C_u} \delta_{x_i} \int_{F(x_i)} \det \nabla H(y) dy\,.$$

4. Graphs and currents

The goal of this section is to prove the following theorem.

THEOREM 5.4.1. *Suppose* $u \in W^{1,2}(\Omega) \cap L^\infty$, *with* $\det \nabla u > 0$ *a.e.. Then the following two conditions are equivalent:*

(i) *u satisfies condition INV and* $\mathrm{im}_{\mathrm{G}}(u, B)$ *has bounded variation in every ball where u satisfies* INV_L*;*
(ii) *there exists a countable number of bounded Caccioppoli sets F_i and of points $x_i \in \Omega$ such that*

$\partial G_u = -\sum_i \{x_i\} \times \partial^* F_i$
∂G_u *has locally finite mass in* Ω
u is injective a.e. and $|\mathrm{im}_{\mathrm{G}}(u, \Omega) \cap F_i| = 0$

In the previous statement there is a slight abuse of notation: indeed $\{x_i\} \times \partial^* F_i$ is a set and not a current. However, after having fixed an orientation for $\mathbb{R}^3$ we can orient $\partial^* F_i$ in such a way that in every point the orienting couple and the outward unit normal form a triple of vectors oriented as $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. This induces a two dimensional current in $\mathbb{R}^3$ and by mapping $\mathbb{R}^3$ into $\{x_i\} \times \mathbb{R}^3$ it gives a rectifiable 2-dimensional current in $\Omega \times \mathbb{R}^3$. We identify $\{x_i\} \times \partial^* F_i$ with this current.

PROOF OF (I) ⇒ (II). We will prove that if (i) holds then (ii) is true if we take $\{x_i\} = C_u = \{x : \mathcal{L}^3(F(x)) > 0\}$ and $F_i = F(x_i)$. From Theorem 5.1.12 we have

$$(\partial G_u)_{(k)} \llcorner \Omega \times \mathbb{R}^3 = 0$$

for $k=0$ and $k=1$. Hence we have to compute $\langle G_u, d\omega\rangle$ for ω of the form

$$\omega = h_1(x,y)dy_2\wedge dy_3 + h_2(x,y)dy_3\wedge dy_1 + h_3(x,y)dy_1\wedge dy_2$$

where $h\in C_0^\infty(\Omega\times\mathbb{R}^3)$. Namely, we want to show that

$$\langle \partial G_u, \omega\rangle = \langle G_u, d\omega\rangle \;=\; -\sum_{x_i\in C_u}\int_{\partial^* F(x_i)} h(x_i,y)\cdot\nu_i\, d\mathcal{H}^2(y) \qquad (90)$$

where ν_i is the outer normal to $F(x_i)$. Now, we notice that the vector space generated by functions of the form $\phi(x)g(y)$ is dense in C_0^∞ in the topology induced by C^k seminorms (see Lemma 4.2.2). Hence it is sufficient to prove (90) for

$$\omega = \phi(x)\,[g_1(y)dy_2\wedge dy_3 + g_2(y)dy_3\wedge dy_1 + g_3(y)dy_1\wedge dy_2]$$

where ϕ has compact support in Ω. We first show that, if $\operatorname{div} g = 0$, then $\langle G_u, d\omega\rangle = 0$. In order to do that, we observe that

$$\langle G_u, d\omega\rangle = \int (\operatorname{div} g)\circ u\,\phi\det\nabla u + \int g\circ u\cdot\Lambda_2(u)\cdot\nabla\phi\,.$$

The first term in the right hand side is zero. To show that also the second one vanishes we take a sequence of C^∞ functions u_n converging to u strongly in $W^{1,2}$ and weakly in L^∞. Then, the same calculation and the fact that the boundary of G_{u_n} is zero gives $\langle G_{u_n}, d\omega\rangle = 0$, and hence

$$\int g\circ u_n\cdot\Lambda_2(u_n)\cdot\nabla\phi = 0\,.$$

For $n\to\infty$, $\Lambda_2(u_n)\to\Lambda_2(u)$ strongly in L^1, and $g\circ u_n \overset{*}{\rightharpoonup} g\circ u$ in L^∞, and so $\langle G_u, d\omega\rangle = 0$ whenever $\operatorname{div} g = 0$.

We now prove (90) in the general case. By linearity, we can assume that $\operatorname{div} g \geq k > 0$. Now, using a result of Dacorogna and Moser [**14**], we can find a diffeomorphism $H:\Omega\to\mathbb{R}^3$ such that

$$\operatorname{div} g = \det\nabla H = \frac{1}{3}\operatorname{div}(H\cdot\Lambda_2(H))\,.$$

Let us define the form ν as

$$\begin{aligned}\nu \;:=\; & \frac{1}{3}\phi(x)\,[(H\cdot\Lambda_2(H))_1(y)dy_2\wedge dy_3 \\ & +(H\cdot\Lambda_2(H))_2(y)dy_3\wedge dy_1 + (H\cdot\Lambda_2(H))_3(y)dy_1\wedge dy_2]\;,\end{aligned}$$

where $(H\cdot\Lambda_2(H))_i$ denotes the i-th component of the vector $H\cdot\Lambda_2(H)$. Since we have shown that $\langle G_u, d\omega\rangle$ only depends on the divergence of

g, we have $\langle G_u, d\omega\rangle = \langle G_u, d\nu\rangle$. Then, a computation analogous to the previous one gives

$$\langle G_u, d\nu\rangle = \int (\det \nabla H)\circ u\, \phi \det \nabla u + \frac{1}{3}\int ((H\cdot \Lambda_2(H))\circ u)\cdot \Lambda_2(u)\cdot \nabla\phi$$

We observe that $(H\cdot\Lambda_2(H))\circ u\cdot\Lambda_2(u) = (H\circ u)\cdot\Lambda_2(H\circ u)$. Then, integrating by parts we get

$$\frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u)\cdot\Lambda_2(u)\cdot\nabla\phi = -\int \phi[J(H\circ u)]\,.$$

Finally, in view of Remark 5.3.4, we have

$$\begin{aligned}
\langle G_u, d\omega\rangle &= -\sum_{x_i\in C_u}\delta_{x_i}\phi(x_i)\int_{F(x_i)}\det\nabla H(y)dy\\
&= -\sum_{x_i\in C_u}\delta_{x_i}\phi(x_i)\int_{F(x_i)}\operatorname{div} g(y)dy\\
&= \sum_{x_i\in C_u}\delta_{x_i}\phi(x_i)\int_{\partial^* F(x_i)} g(y)\cdot\nu_i(y)d\mathcal{H}^2(y)
\end{aligned}$$

□

In order to prove the converse implication we first need some definitions.

DEFINITION 5.4.2. *Let $M\subset\Omega\times\mathbb{R}^3_y$ be a 3-dimensional smooth oriented manifold (possibly with boundary). Given a C^∞ open set $A\subset\subset\Omega$, for every point $y\in\mathbb{R}^3$ we set $M_y := (A\times\{y\})\cap M$. For every $x\in M_y$ we call $T(x)$ the tangent plane to M in x with its orientation and we call $T_p(x)$ the projection of $T(x)$ on $\mathbb{R}^3_y$ with the induced orientation. We set*

$$\chi(x) := \begin{cases} 0 & \textit{if } \dim(T_p(x)) < 3\\ 1 & \textit{if } \dim(T_p(x)) = 3 \textit{ and } T_p(x) \textit{ is oriented as } \mathbb{R}^3_y\\ -1 & \textit{otherwise}\end{cases}$$

We define

$$\deg(M, A, y) := \sum_{x\in M_y}\chi(x).$$

Exactly in the same way, using approximate tangent planes, we can define $\deg(T, A, y)$ *if T is a rectifiable current.*

REMARK 5.4.3. It is easy to see that the degree is well defined and

$$\int_{M\cap\{A\times\mathbb{R}^3_y\}} g(y)dy_1\wedge dy_2\wedge dy_3 = \int_{\mathbb{R}^3} g(y)\deg(M, A, y)dy. \tag{91}$$

Hence, if $(\partial A \times \mathbb{R}^3_y) \cap M$ is a 2-dimensional manifold then

$$\int_{\mathbb{R}^3} \operatorname{div} h \deg(M, A, y) dy = \int_{M \cap \{A \times \mathbb{R}^3_y\}} \operatorname{div} h \, dy_1 \wedge dy_2 \wedge dy_3$$

$$= \int_{\partial(A \times \mathbb{R}^3) \cap M} h_1 dy_2 \wedge dy_3 + h_2 dy_3 \wedge dy_1 + h_3 dy_1 \wedge dy_2.$$

Moreover if A is a ball B and $(A \times \mathbb{R}^3_y) \cap M$ is the trace of a function $u : \Omega \to \mathbb{R}^3$ then this last term is equal to

$$\int_{\partial B} h \circ u \cdot \Lambda_2(u) \cdot \nu \, d\mathcal{H}^2$$

and we can write

$$\int_{\mathbb{R}^3} \operatorname{div} h \deg(M, B, y) dy = \int_{\mathbb{R}^3} \operatorname{div} h \deg(u, \partial B, y) dy.$$

Since every smooth function g can be written as the divergence of a vector field we conclude that $\deg(M, B, y) = \deg(u, \partial B, y)$ a.e.

Proof of (ii) $\Rightarrow$ (i) in Theorem 5.4.1. What happens in the previous remark if we try to replace the manifold M with a normal rectifiable current T? Equation (91) remains true. Moreover if we fix a point x the Slicing Theorem 2.3.1 says that for $\mathcal{L}^1$ a.e. r such that $\operatorname{dist}(\Omega, x) > r > 0$ we have:

(i) the intersection of the rectifiable set which supports T with $\partial B(x, r) \times \mathbb{R}^3$ gives a 2-dimensional rectifiable set S_r;
(ii) $\mathcal{H}^2$ a.e. $y \in S_r$ has an orientation induced naturally by the orientation of the tangent space to $\partial B \times \mathbb{R}^3$ and the orientation of the approximate tangent space to T;
(iii) the orientation of (ii) on S_r induces a rectifiable current, which we also call S_r with a slight abuse of notation;
(iv) if we take in $\mathbb{R}^3 \times \mathbb{R}^3$ the current T_r given by the restriction of T to $B(x, r) \times \mathbb{R}^3$ we have that $\partial T_r = S_r$.

In fact we observe that the Slicing Theorem 2.3.1 only treats the case of normal currents, whereas the statement above is also saying to us that almost every slice is rectifiable. However we remark that this is an easy consequence of Theorem 2.4.3 and Remark 2.3.2.

Keeping this in mind we notice that the current given by $T = G_u + \sum_i \{x_i\} \times F_i$ is rectifiable and has no boundary in $\Omega \times \mathbb{R}^n$ (hence it is normal and we can apply all the machinery stated above). The current T is supported on the rectifiable set R given by $u(\Omega_d) \cup \bigcup_i \{x_i\} \times F_i$. Now let us fix an $x \in \Omega$. For $\mathcal{L}^1$-a.e. radius r such that $B(x, r) \subset \Omega$, we have that $(\partial B(x, r) \times \mathbb{R}^3) \cap R$ is given by the rectifiable set $u(\Omega_d \cap \partial B(x, r))$.

Hence (for a.e.) r ∂T_r is given by the current induced by the $W^{1,2} \cap L^\infty$ function $u|_{\partial B(x,r)}$.

Reasoning as in Remark 5.4.3 it is not difficult to check that

$$\deg(T, B(x,r), y) = \deg(u, \partial B(x,r), y)$$

for a.e. y. Hence, since every approximate tangent plane to T is oriented in the same way, it is easy to see that for every point $z \in B(x,r)$, $\deg(u, \partial B(x,r), u(z))$ is positive. Moreover, since for a.e. point $z \notin B(x,r)$ we have $(B(x,r) \times u(z)) \cap T = \emptyset$ we can conclude that $\deg(u, \partial B(x,r), u(z)) = 0$ for a.e. $z \in \Omega \backslash B(x,r)$. This implies that u satisfies condition INV.

Finally, let us fix a ball B such that u satisfies INV_L on it. Of course we have that

$$\mathrm{im_G}(u, B) = \{y | \deg(T, B, y) > 0\} \setminus \bigcup_{x_i \in B} F_i.$$

Since $\deg(T, B, \cdot) = \deg(u, \partial B, \cdot)$ we have that $\{y | \deg(B, T, y) > 0\} = \mathrm{im_T}(u, B)$. As we have seen in Section 3, the last set is a set of bounded variation. Moreover

$$\sum_{x_i \in B} \mathrm{Per}(F_i)$$

is equal to the mass of ∂G_u in $B \times \mathbb{R}^3$, hence it is finite. This means that $\mathrm{im_G}(u, B)$ is the difference between two Caccioppoli sets and completes the proof. $\square$

CHAPTER 6

Failure of continuity

As we have already seen in Chapter 3 the following Theorem holds (see Theorem 3.1.4 and the discussion which follows it)

THEOREM 6.0.4. *Suppose that* $(u_n), u$ *satisfy the conditions which allow to define the weak Jacobian and*

(a) $u_k \rightharpoonup u$ *weakly in* W^{1,p_1}_{loc};
(b) $u_k \to u$ *strongly in* $L^{p_2}_{loc}$;
(c) $(n-1)/p_1 + 1/p_2 \leq 1$ *and* $p_2 < \infty$;

or $u_k \rightharpoonup u$ *in* $W^{1,n-1}_{loc}$, $u_k \in C(U)$ *and* $u_k \to u$ *uniformly on compact sets.*

Then $[Ju_n] \rightharpoonup [Ju]$ *as local current.*

Now when u_k is converging to a function u weakly $W^{1,p}$ for $p > n-1$ and $\|u_k\|_\infty$ is equibounded then Sobolev embeddings tells us that we are in one of the cases state above. Hence $[Ju_k]$ converges to $[Ju]$. Keeping this in mind it is a natural question to ask whether the same happens in the borderline case $p = n-1$. Indeed in this case the convergence of weak Jacobians fail (see [**25**] or [**27**] for examples of this failure). However in the examples known in the literature the maps u_k are already quite pathological.

We will see in this chapter that the failure illustrated above is connected with the non–closedness of condition INV under the convergence induced by the functional (73) which we recall here:

$$E(u) := \int_\Omega |\nabla u|^2 + \varphi(\det \nabla u).$$

In this chapter we exhibit a sequence of nice functions u_n (Lipschitz with Lipschitz inverse) satisfying condition INV, which are equibounded in energy and which converge weakly to a function u which is not in their class. The limiting map appears to be very pathological from the point of view of elasticity: this pathology could be interpreted in physical terms as "interpenetration of matter". Moreover we emphasize the fact that the sequence of functions exhibited consists of orientation-preserving Bilipschitz maps. Since every reasonable class of admissible deformations has to include Bilipschitz maps, this means that if one wants to use the direct methods of calculus of variations one has to admit our pathology when building an existence theory.

From the point of view of Cartesian currents an interesting consequence is that new types of singularities need to be involved in the relaxation procedure. These singularities are different from those due to “cavitation” (i.e. the opening of holes in some points). This point is quite delicate and we shall discuss it in the final section.

Moreover this bad behavior is deeply related with the fact that the weak Jacobians $[Ju_n]$ fail to converge weakly to $[Ju]$. Indeed every $[Ju_n]$ is a nonnegative measure (since every u_n is Bilipschitz and orientation preserving) whereas one could notice that $[Ju]$ does not enjoy this property any more. However, instead of presenting a proof (which would be rather lengthy) of this last fact, in the second section we will give a simple argument which implies the existence of suitably modified maps v_n, v which are as nice as u_n, u but such that the weak convergence of $[Jv_n]$ to $[Jv]$ fails.

1. Bad sequence

In this section we construct a sequence of Bilipschitz functions $u_n : \Omega \to \mathbb{R}^3$ such that

(i) every u_n obeys condition INV;
(ii) $\|u_n\|_\infty$ is equibounded and

$$\sup_n \int \big(|\nabla u_n|^2 + \phi(\det \nabla u_n)\big) < \infty\,, \tag{92}$$

for some convex function $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ such that

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to 0} \phi(t) = \infty\,; \tag{93}$$

(iii) u_n converge weakly in $W^{1,2}$ to a map u which does not satisfy condition INV.

The limit u is illustrated in Figure 1. The whole construction is cylindrically symmetric with respect to the x-axis, therefore only a section is plotted. The half-ball a, centered in the origin, is mapped into the ball $\mathrm{im}_{\mathrm{G}}(u,a) = B((1/2,0,0),1/2)$. The half-ball e, centered in $(1,0,0)$, is mapped into $\mathrm{im}_{\mathrm{G}}(u,e) = B((1,0,0),1) \setminus \mathrm{im}_{\mathrm{G}}(u,a)$. The topological image of e, $\mathrm{im}_{\mathrm{T}}(u,e) = B((1,0,0),1)$, also contains $\mathrm{im}_{\mathrm{G}}(u,a)$, whereas a and e are disconnected. Hence condition INV is violated by u.

To understand how u can be reached as the limit of a sequence of functions satisfying condition INV, it is instructive to mention the following result (well known in the literature on harmonic maps)

LEMMA 6.1.1. *There is a sequence of maps $v_n : \mathbf{S}^2 \to \mathbf{S}^2$ which for any n are surjective,*

$$\sup_n \int_{\mathbf{S}^2} |\nabla v_n|^2 < \infty$$

with $v_n \to v$ in $W^{1,2}$, and v a constant function.

PROOF. The construction is cylindrically symmetric, and is based on expanding the usual projection of the sphere onto the complex plane. In polar coordinates, this gives

$$(v_\theta, v_\phi)(\theta, \phi) = \left(2 \arctan n \tan \frac{\theta}{2}, \phi\right)$$

It is clear that $v_\theta \to \pi$ as $n \to \infty$. The L^2 norm of cylindrically invariant functions is given by (see Appendix B)

$$||\nabla v||^2_{L^2(\mathbf{S}^2)} = 2\pi \int_{-1}^{1} (\partial_\theta v_\theta)^2 + \left(\frac{\sin v_\theta}{\sin \theta}\right)^2 d\cos\theta$$

and by direct substitution we obtain

$$||\nabla v||^2_{L^2(\mathbf{S}^2)} = 4\pi \int_{-1}^{1} \left(\frac{2n}{(n^2+1) + \cos\theta(1-n^2)}\right)^2 d\cos\theta = 8\pi\,.$$

This concludes the proof. □

Before presenting the actual construction, we show a simple method for checking the determinant constraint. One possible way would of course be to explicitly construct a function ϕ (for example $\phi(t) = t^p + t^{-q}$, for some $p > 1$, $q > 0$) which satisfies (93) and check (92). It is however easier to use the following lemma

LEMMA 6.1.2. *If the sequence of Bilipschitz maps $u_n : \Omega \to \mathbb{R}^n$ has the following properties:*

(i) *for any δ there exists ε such that, if $\omega \subset \Omega$ is measurable and $|\omega| \le \varepsilon$, then $|u_n(\omega)| < \delta$;*

(ii) *for any δ, there exists ε such that, if $|\omega| \le \varepsilon$, then $|u_n^{-1}(\omega)| < \delta$.*

Then, there is a convex function $\phi : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to 0} \phi(t) = \infty \tag{94}$$

such that

$$\sup_n \int_\Omega \phi(\det \nabla u_n) < \infty\,. \tag{95}$$

PROOF. Since every u_n is Bilipschitz we have

$$|u_n(\omega)| = \int_\omega \det \nabla u_n(x) dx.$$

Hence the first condition implies the equiintegrability of $\det \nabla u_n$ and using Dunford-Pettis Theorem for weakly compact L^1 sequences (see for example [**7**] for a proof) we conclude that there exists a positive, increasing, convex and superlinear function ψ_1 such that $\int \psi_1(\det \nabla u_n)$ is equibounded. Using condition (ii) in the same way we find an analogous ψ_2 such that $\int \psi_2((\det \nabla u_n)^{-1})$ is equibounded. Since $\psi_2(x)$ is increasing, $\psi_2(x^{-1})$ is still convex. It follows that $\psi_1(x) + \psi_2(1/x)$ gives the desired function. $\square$

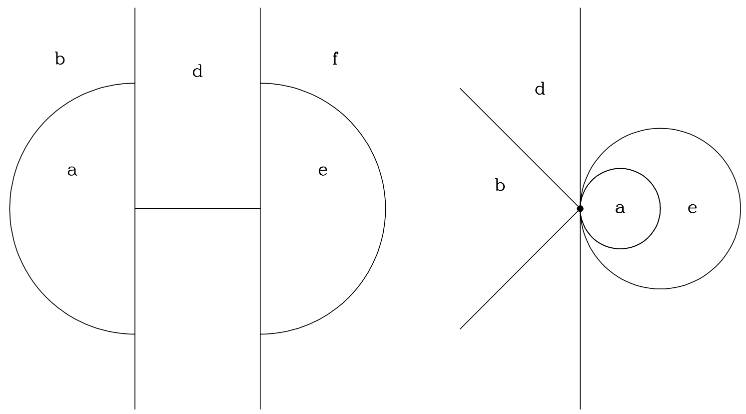

FIGURE 1: Representation of the limit function u. Left panel: subdivision of the domain. Right panel: image. The letters denote the images of the various pieces of the domain.

We now turn to the relevant construction, which is illustrated in Figures 1 and 2. The unwrapping of the sphere mentioned in Lemma 6.1.1 is used in the central section, denoted by c in Figure 2. For finite n, sections of c at constant x are mapped into spheres which are contained between the image of a and the image of e. In the limit, those spheres shrink to a point, the origin.

THEOREM 6.1.3. *There is a sequence* $(u_n) \subset W^{1,2}(\Omega, \mathbb{R}^n)$ *such that:*

(i) u_n *is Bilipschitz;*
(ii) *every* u_n *satisfies condition INV;*
(iii) $u_n \rightharpoonup u$ *in* $W^{1,2}$ *and* u *does not satisfy condition INV;*

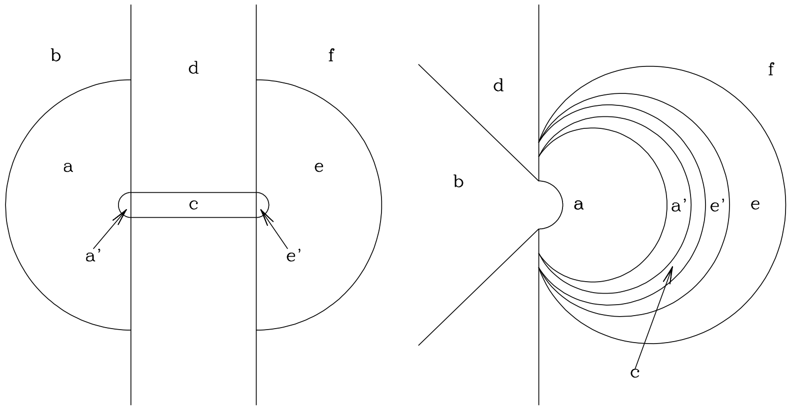

FIGURE 2: Representation of a function of the sequence u_n, which converge to the function represented in Figure 1. Left panel: subdivision of the (x, y) plane used in the construction in Theorem 6.1.3. Right panel: image. The letters denote the images of the various pieces of the domain.

(iv) $\sup_n \int_\Omega \psi(\det \nabla u_n) < \infty$ *for some convex and superlinear* ψ *approaching infinity at 0.*

PROOF. We work in cylindrical coordinates, and construct the sequence u_n using piecewise smooth functions. We shall consider $\Omega = B(0, 3)$, examples for other domains can be obtained by rescaling and translating the construction. We first construct $v_n(x, y)$, which coincides with the restriction of u_n to the $\{z = 0,\ y > 0\}$ half-plane, then extend it cylindrically,

$$u_n(x, y \cos\phi, y \sin\phi) = v_x^n(x, y)e_x + v_y^n(x, y)(\cos\phi e_y + \sin\phi e_z)\,.$$

The construction of v_n is based on the subdivision of the domain shown in Figure 2. In the limit u, the region a, which corresponds to a half-sphere, is mapped into the sphere $u(a) = B(1/2, 1/2)$, and the region e is mapped into $u(e) = B(1, 1) \setminus B(1/2, 1/2)$. The regions b and d are mapped into the $x < 0$ half-space, the region f is mapped into the rest of the $x > 0$ half-space. The remaining regions all disappear in the limit. For finite n, the regions a', c and e' (of total volume of order n^{-2}) ensure that the maps are Bilipschitz. The corresponding images also have small volume, and join continuously $u_n(a)$ with $u_n(e)$.

Let (r, θ) be polar coordinates centered in $(0, 0)$, and $(\bar{r}, \bar{\theta})$ polar coordinates centered in $(1, 0)$. We consider first the inner regions a, a', c, e' and e, and afterwards the outer regions, b, d and f. For simplicity of notation we denote here v_n by v, and use $\varepsilon := 1/n$ as small parameter. We also use the same symbols a, b, etc. to denote both the three-dimensional pieces and their two-dimensional sections.

We start from $a := \{\varepsilon \le r \le 1\,, \pi/2 \le \theta \le \pi\}$, and set

$$v_\theta := \pi - \theta \tag{96}$$

$$v_r := \frac{1-r}{1-\varepsilon}(\cos v_\theta + 2\varepsilon) + \varepsilon\frac{r-\varepsilon}{1-\varepsilon} \tag{97}$$

where v_r and v_θ are polar components of $(v_x, v_y) = (v_r \cos v_\theta, v_r \sin v_\theta)$. For $r = \varepsilon$ this reduces to $v_r = \cos v_\theta + 2\varepsilon$. In region $a' := \{0 \le r \le \varepsilon\,, \pi/2 \le \theta \le \pi\}$ we set

$$v_\theta := 2\arctan\left(\frac{r\sin\theta}{-r\cos\theta + \varepsilon}\right) \tag{98}$$

$$v_r := (1+\varepsilon)\cos v_\theta + 2\varepsilon - r\cos\theta\,. \tag{99}$$

In the strip $0 \le x \le 1$ we use cartesian coordinates in the (x, y)-plane. In $c := \{0 \le x \le 1\,,\ 0 \le y \le \varepsilon\}$,

$$\begin{aligned} v_\theta &:= 2\arctan(y/\varepsilon)\,, \\ v_r &:= (1+\varepsilon)\cos v_\theta + 2\varepsilon + x\varepsilon^2\,. \end{aligned}$$

We finally come to the region $x \ge 1$. Here we use the polar coordinates $(\bar r, \bar\theta)$ centered in $(1, 0)$, and only need to consider $0 \le \bar\theta \le \pi/2$. In $e' := \{0 \le \bar r \le \varepsilon\,, 0 \le \bar\theta \le \pi/2\}$ we set

$$\begin{aligned} v_\theta &:= 2\arctan\left(\frac{\bar r\sin\bar\theta}{\bar r\cos\bar\theta + \varepsilon}\right)\,, \\ v_r &:= (1+\varepsilon)\cos v_\theta + 2\varepsilon + \bar r\cos\bar\theta + \varepsilon^2\,. \end{aligned}$$

Finally, in $e := \{\varepsilon \le \bar r \le 1\,, 0 \le \bar\theta \le \pi/2\}$, we set

$$\begin{aligned} v_\theta &:= \bar\theta \\ v_r &:= 2\varepsilon + \varepsilon\bar r + (1 + \varepsilon + \bar r)\cos v_\theta\,. \end{aligned}$$

We now come to the outer region. Since the function here has a smooth dependence on ε, we only shortly sketch the construction, without giving all details. We start from region $b := \{x \le 0\,, x^2 + y^2 \ge 1\}$. The construction is most simply understood by composing two Bilipschitz functions. Let c_ε be the map defined, in polar coordinates, by

$$(r, \theta) \to (r - 1 + \varepsilon, (\theta + \pi)/2).$$

Let ψ_ε be a Bilipschitz function which maps $\{x \le 0\,, |y| \le \varepsilon + |x|\} \setminus B(0, \varepsilon)$ into $\{x \le 0\,, |y| \le \varepsilon + |x|\} \cup B(0, \varepsilon)$ and satisfies

(i) $\psi_\varepsilon(r, \theta) = (r, \theta)$ on the lines $\theta = \pi \pm \pi/4$, $r \ge \varepsilon$;
(ii) $\psi_\varepsilon(r, \theta) = (r, \pi - \theta)$ on the half-circumference $r = \varepsilon$, $\theta \in (\pi/2, 3\pi/2)$;

(iii) ψ_ε equals the identity outside $B(0,2\varepsilon)$.

We finally set $v_\varepsilon = \psi_\varepsilon \circ c_\varepsilon$. Region f is analogous.

We now consider region $d := \{0 \leq x \leq 1\,, y \geq \varepsilon\}$. Let g_ε be uniformly Bilipschitz functions that map d into $(0,3)\times(0,\infty)$, with $g_\varepsilon(0,1) = (0,0)$, $g_\varepsilon(0,\varepsilon) = (1,0)$, $g_\varepsilon(1,\varepsilon) = (2,0)$, $g_\varepsilon(1,1) = (3,0)$ and affine in the segments joining these points. Now we consider the additional function h given by

$$\begin{aligned} h_x &:= -y\cos(\varphi(x)) \\ h_y &:= \omega(x) + y\sin(\varphi(x)) \end{aligned}$$

where $\omega(x)$ is still to be determined and

$$\varphi(x) := \frac{\pi}{4}\left(1+\frac{x}{3}\right).$$

We finally set $v_\varepsilon = h\circ g_\varepsilon$, and choose ω so that the trace on the $x=0$ axis agrees with the trace obtained from the interior.

We leave to the reader to check that all pieces match continuously and are orientation-preserving, and focus directly on the estimate of $|\nabla u_n|^2 + \phi(\det\nabla u_n)$.

The estimate of the L^2 norm of the gradients is done with the help of the expressions in polar/cylindrical coordinates given in Appendix 5. We start from region a. From Eq. (104) we see that if the partial derivatives $\partial_{(r,\theta)}u_{(r,\theta)}$ are uniformly bounded, all terms are immediately controlled, except for the last one. The last one is also uniformly controlled, since in this region $u_\theta/\sin\theta = 1$. In region a', we get $|u_{r,\theta}|+|u_{\theta,\theta}|\leq c$, and $|u_{r,r}|+|u_{\theta,r}|\leq c/\varepsilon$. Since $|r|\leq\varepsilon$, there is nothing to be checked. Similar arguments apply to region c, by considering the formula for cylindrical coordinates in Eq. (105). Finally, regions e and e' are completely analogous to a and a'.

We now come to the determinant. This is easily done using Lemma 6.1.2. Indeed, let us check the first property. Fix $\eta\in(0,1)$, and let Ω_η be the set of points of distance less than η from the segment joining $(0,0,0)$ and $(1,0,0)$. It is clear that for ε small enough (as compared to η) the first property is satisfied outside of Ω_η. This implies the first hypothesis of the Lemma, for all $\delta > 2|u_n(\Omega_\eta)|$. Since

$$\lim_{\eta\downarrow 0}\,(\sup_n |u_n(\Omega_\eta)|) = 0$$

we only have to choose η small enough. The second hypothesis of Lemma 6.1.2 can be proved analogously.

This concludes the proof of the theorem. $\square$

2. Final Remarks on failure of continuity

We now give a brief summary of the results of the previous chapter and of the previous section and of their implications. Let us fix a bounded open set $\Omega \subset \mathbb{R}^3$ (sufficiently regular) and call $\mathcal{A}$ the class of maps $u \in W^{1,2}(\Omega) \cap L^\infty$ which satisfy condition INV and such that $[Ju] > 0$ a.e.. After fixing a diffeomorphism $g : \overline{\Omega} \to \mathbb{R}^3$ we introduce the following class of functions

$$\begin{aligned} \mathcal{A}_{nc} &:= \{u \in \mathcal{A} |\, \mathrm{im_G}(u,\Omega) = \mathrm{im_T}(u,\Omega), u|_{\partial\Omega} = g|_{\partial\Omega}\} \\ \mathcal{A}_{c} &:= \{u \in \mathcal{A} |\, \mathrm{Per}(\mathrm{im_G}(u,\Omega)) < \infty, u|_{\partial\Omega} = g|_{\partial\Omega}\}. \end{aligned}$$

Of course $\mathcal{A}_{nc} \subset \mathcal{A}_c$. Moreover it is easy to build a map which is in $\mathcal{A}_c \setminus \mathcal{A}_{nc}$ (see the first examples in the pioneering work of Ball [**10**]). Basically the first class does not allow for the opening of holes, whereas the second class does: the union of such holes gives the set $\mathrm{im_T} \setminus \mathrm{im_G}$. In the first class there exists a minimum for the functional (74) when $p > 2$ and in the second one there exists a minimum of the modified functional

$$E'(u) = \int_\Omega (|\nabla u|^p + \varphi(\det \nabla u))dx \; + \mathrm{Per}\,(\mathrm{im_G}(u,\Omega))\,.$$

We have proved that the first problem is equivalent to minimization of the same functional in the class of functions u such that their graphs are currents which have no boundary in $\Omega \times \mathbb{R}^3$ and $\det \nabla u \geq 0$ a.e.. The second one is equivalent to minimization of the energy

$$\int_\Omega (|\nabla u|^p + \varphi(\det \nabla u))dx \; + \|\partial G_u\|(\Omega \times \mathbb{R}^3)$$

among all the maps u such that:

(i) their graphs are rectifiable currents with boundary given by

$$\partial G_u \llcorner \Omega \times \mathbb{R}^3 = \sum_i \{x_i\} \times \partial F_i$$

where every F_i is a Caccioppoli set;

(ii) $G_u + \sum \{x_i\} \times F_i$ has degree 1 or 0 as generalized graph in a.e. y in the target.

In Section 1 we have shown that the domain of the first problem is not closed under the $W^{1,2}$ topology. Hence we have a sequence of good functions u_n which are converging to a function u such that the boundary ∂G_u is nontrivial in $\Omega \times \mathbb{R}^3$. Moreover u does not even fall in the class $\mathcal{A}_c$. Indeed, if we take the sequence of currents given by G_{u_n} we notice that they are converging to a current T which is composed by G_u plus a nontrivial current V. Being the limit of a sequence of currents with no boundary in $\Omega \times \mathbb{R}^3$ we have that $G_u + V$ has no boundary as well.

With respect to the sequence constructed in the proof of Theorem 6.1.3, it is not difficult to check that the boundary of G_u is given by

$$\{A\} \times \mathbf{S}^2 - \{B\} \times \mathbf{S}^2$$

where A and B are the two points given by $(0,0,0)$ and $(1,0,0)$. We notice that the "hole" opened in $(1,0,0)$ has the "wrong" sign and cannot be interpreted as the boundary given by the opening of a cavity! Hence the "vertical current" V must be a cylinder which connects $\{A\} \times \mathbf{S}^2$ and $\{B\} \times \mathbf{S}^2$ (this singularity is often called "dipole" in the literature on harmonic maps).

We conclude that in order to use Giaquinta, Modica and Souček's theory one has to deal with functions which create singularities which are more complicate than cavities.

We finally remark that our construction gives a highly nontrivial example of the failure of weak continuity of Jacobian in the weak convergence of $W^{1,2}(\mathbb{R}^3, \mathbb{R}^3) \cap L^\infty$. One way of showing this could be rigorously proving the statement of the previous paragraph on the structure of the boundary of G_u. However, since this would involve tedious computations we give below an alternative (and more theoretical) proof which, thanks to the existence of u_n shows the existence of at least another sequence v_n weakly converging to v in $W^{1,n-1}$ and in L^∞ such that $[Jv_n]$ does not converge to $[Jv]$.

THEOREM 6.2.1. *There exists a diffeomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$ such that $[J(H(u_n))]$ does not converge in the sense of distributions to $[J(H(u))]$.*

PROOF. We argue by absurd supposing that the statement of the theorem is false and we focus on a bounded open set Ω which is sufficiently big so to contain all the points where u develops singularities (e.g. $\Omega = B(0,10)$).

Then as a first thing we notice that $[Ju_n]$ would converge weakly, in the sense of distributions, to $[Ju]$. Now every u_n is Lipschitz, hence $[Ju_n] = \det(\nabla u_n)\mathcal{L}^3 \llcorner \Omega$. Moreover, we know (see (iv) of Theorem 6.1.3) that there exists a superlinear and convex real function ψ such that $\int \psi(\det \nabla u_n)$ is equibounded. Hence this means that ∇u_n converges weakly in L^1 to a function f. This implies that $[Ju] = f\mathcal{L}^3 \llcorner \Omega$ and hence, thanks to Theorem 3.2.2, that $f = \det(\nabla u)$. This tells us that we can give a meaning to the current G_u (since all the minors M_α^β are summable functions).

As already noticed in the first part of the proof of Theorem 5.4.1 if the form ν is defined as

$$\begin{aligned}\nu &:= \frac{1}{3}\phi(x)\,[(H\cdot\Lambda_2(H))_1(y)dy_2\wedge dy_3 \\ &\quad +(H\cdot\Lambda_2(H))_2(y)dy_3\wedge dy_1 + (H\cdot\Lambda_2(H))_3(y)dy_1\wedge dy_2]\ ,\end{aligned}$$

(where $(H\cdot\Lambda_2(H))_i$ denotes the i-th component of the vector $H\cdot\Lambda_2(H)$) then

$$\begin{aligned}\langle G_{u_n}, d\nu\rangle &= \int (\det\nabla H)\circ u_n\,\phi\det\nabla u_n \\ &+ \frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u_n)\cdot\Lambda_2(u_n)\cdot\nabla\phi.\end{aligned}$$

Hence (still following the first part of proof of Theorem 5.4.1) we notice that $(H\cdot\Lambda_2(H))\circ u_n\cdot\Lambda_2(u_n) = (H\circ u)\cdot\Lambda_2(H\circ u)$. Thus, integrating by parts we get

$$\frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u_n)\cdot\Lambda_2(u_n)\cdot\nabla\phi = -\int \phi[J(H\circ u_n)].$$

Moreover since a similar formula holds for u we get that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u_n)\cdot\Lambda_2(u_n)\cdot\nabla\phi \\ &= \frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u)\cdot\Lambda_2(u)\cdot\nabla\phi.\end{aligned}$$

Now for every continuous and bounded function f we also have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega f(x,u_n(x))\det(\nabla u_n(x))dx \\ &= \int_\Omega f(x,u(x))\det(\nabla u(x))dx,\end{aligned}$$

which is a consequence of the following facts:

(a) $\det(\nabla u_n(x))$ converge weakly in L^1 to $\det\nabla u(x)$ (as remarked at the beginning of the proof);

(b) $f(x,u_n(x))$ is bounded in L^∞ and converges pointwise a.e. to $f(x,u(x))$.

Hence, summarizing, we conclude that

$$\begin{aligned}&\lim_{n\to\infty}\int (\det\nabla H)\circ u_n\,\phi\det\nabla u_n + \frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u_n)\cdot\Lambda_2(u_n)\cdot\nabla\phi \\ &= \int (\det\nabla H)\circ u\,\phi\det\nabla u + \frac{1}{3}\int ((H\cdot\Lambda_2(H))\circ u)\cdot\Lambda_2(u)\cdot\nabla\phi\end{aligned}$$

which means that $G_u(d\nu) = \lim_n G_{u_n}(d\nu) = 0$. But then, again arguing as in the first part of the proof of Theorem 5.4.1, this last statement implies that $\partial G_u = 0$.

This is a contradiction, since $\det \nabla u > 0$ a.e. and u is injective a.e., which together with $\partial G_u = 0$ would imply, thanks to the second part of Theorem 5.4.1, that u satisfies condition INV. $\square$

CHAPTER 7

Appendix

1. Proof of compactness of normal currents

In this section we give the proof of [**8**] of Theorem 2.2.8. We state again the theorem for the reader's convenience.

THEOREM 7.1.1. *Let $(T_h) \subset \mathcal{N}^k(E)$ be a bounded sequence and assume that for any integer $p \geq 1$ there exists a compact set K_p such that*

$$\|T_h\|(E \backslash K_p) + \|\partial T_h\|(E \backslash K_p) \leq \frac{1}{p} \qquad \forall h.$$

Then there exists a subsequence $T_{h(n)}$ converging to a current $T \in \mathcal{N}^k(E)$ satisfying

$$\|T\| \left(E \backslash \bigcup_{p=1}^{\infty} K_p \right) + \|\partial T\| \left(E \backslash \bigcup_{p=1}^{\infty} K_p \right) = 0.$$

In order to prove it we will first need the following proposition (Proposition 5.1 in [**8**]).

PROPOSITION 7.1.2. *If T is a normal k–dimensional current, then the estimate*

$$|T(f\,d\pi) - T(f\,d\pi')| \leq \sum_{i=1}^{k} \int_E |f||\pi_i - \pi_i'|d\|\partial T\|$$

$$+Lip\,(f) \int_{\mathrm{supp}\,(f)} |\pi_i - \pi_i'|d\|T\| \qquad (100)$$

holds whenever $f, \pi_i, \pi_i' \in Lip\,(E)$ and $Lip\,(\pi_i), Lip\,(\pi_i') \leq 1$.

PROOF. Assume first that f, π_i, π_i' are bounded. We set $d\pi_0 = d\pi_2 \wedge \ldots \wedge d\pi_k$ and, using the definition of ∂T, we find

$$\begin{aligned} & T(f\,d\pi_1 \wedge d\pi_0) - T(f\,d\pi_1' \wedge d\pi_0) \\ = \; & T(1d(f\,d\pi_1) \wedge d\pi_0) - T(1d(f\pi_1') \wedge d\pi_0) \\ & -T(\pi_1 df \wedge d\pi_0) + T(\pi_1' df \wedge d\pi_0) \\ = \; & \partial T(f\pi_1 d\pi_0) - \partial T(f\pi_1' d\pi_0) - T(\pi_1 df \wedge d\pi_0) + T(\pi_1' df \wedge d\pi_0). \end{aligned}$$

This last line can be estimated by

$$\int_E |f||\pi_1 - \pi_1'|d\|\partial T\| + \mathrm{Lip}\,(f)\int_{\mathrm{supp}\,(f)} |\pi_1 - \pi_1'|d\|T\|$$

and we achieve (100) repeating $k-1$ times the same argument. For unbounded f, π_i, π_i' a truncation argument will give the desired inequality. □

Proof of Theorem 7.1.1 Possibly extracting a subsequence we can assume that there are measures μ, ν such that

$$\lim_{h\to\infty}\int_E f\,d\|T_h\| = \int_E f\,d\mu$$
$$\lim_{h\to\infty}\int_E f\,d\|\partial T_h\| = \int_E f\,d\nu$$

for any bounded continuous function $f \in C_b(E)$. Moreover it is easy to check that $(\mu+\nu)(E\backslash K_p) \le 1/p$ and hence $\mu+\nu$ is concentrated on $\cup_p K_p$.

Step 1 We will first prove that (T_h) has a pointwise converging subsequence $(T_{h(n)})$; to this aim, by a diagonal argument, we need only to show that for any integer $q \ge 1$ there exists a subsequence $(h(n))$ such that

$$\limsup_{n,m\to\infty} |T_{h(n)}(f\,d\pi) - T_{h(m)}(f\,d\pi)| \le \frac{3}{q}$$

for every $f\,d\pi \in \mathcal{D}^k(E)$ with $|f| \le q$, $\mathrm{Lip}\,(f), \mathrm{Lip}\,(\pi_i) \le 1$. In order to prove this last statement we first choose $g \in Lip(E)$ with bounded support such that

$$\sup_{h\in\mathbb{N}} \|T_h - T_h \llcorner g\|_N < \frac{1}{q^2}$$

(e.g. it is sufficient to choose $g \equiv 1$ on K_{2q^2} and such that $\mathrm{Lip}\,(g) \le 1$). Then we will prove the existence of a subsequence $h(n)$ such that $T_{h(n)} \llcorner g(f\,d\pi)$ converges whenever $f\,d\pi \in \mathcal{D}^k(E)$ satisfies $\mathrm{Lip}\,(\pi_i) \le 1$.

Endowing $Z = \mathrm{Lip}_1(\cup_p K_p)$ with a separable metric inducing uniform convergence on every K_p, we can find a countable dense set $D \subset Z$ and a subsequence $(h(n))$ such that $T_{h(n)} \llcorner g(f\,d\pi)$ converges whenever $f, \pi_1, \ldots, \pi_n \in D$. Finally using proposition 7.1.2 we can prove that $T_{h(n)} \llcorner g(f\,d\pi)$ converges for every $f, \pi_1, \ldots, \pi_n \in \mathrm{Lip}_1(E)$. Indeed for any $\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_n \in D$ we can write

$$\begin{aligned} & \limsup_{n,n' \to \infty} |T_{h(n)} \llcorner g(f d\pi) - T_{h(n')} \llcorner g(f d\pi)| \\ \leq \ & 2 \limsup_{n \to \infty} |T_{h(n)} \llcorner g(f d\pi) - T_{h(n)} \llcorner g(\tilde{f} d\tilde{\pi})| \\ \leq \ & \limsup_{h \to \infty} \sum_{i=1}^{k} \int_E (|f|+1) |\pi_i - \tilde{\pi}_i| d(\|\partial(T_h \llcorner g)\| + \|T_h \llcorner g\|) \\ & + \int_E |f - \tilde{f} d \|T_h \llcorner g\| \leq \sum_{i=1}^{k} \int_{\mathrm{supp}\,(g)} (|f|+1)|\pi_i - \tilde{\pi}_i| d\mu \\ & + \sum_{i=1}^{k} \int_E (|f|+1)|g||\pi_i - \tilde{\pi}_i| d\nu + \int_E |f - \tilde{f}||g| d\mu. \end{aligned}$$

Since $\tilde{f}$ and π can be choose arbitrarily near in $L^1(\mu+\nu)$ to f and π this proves the convergence of $T_{h(n)} \llcorner g$.

Step 2 Thanks to the previous step we can define

$$T(\omega) := \lim_{n \to \infty} T_{h(n)}(\omega) \qquad \text{for every} \omega \in \mathcal{D}^k(E). \tag{101}$$

Moreover it is straightforward that T satisfies conditions (b) and (c) of Definition 2.2.1. In order to prove that T is a k–dimensional current we have only to prove that it satisfies the continuity condition (a) of Definition 2.2.1.

We notice that $\|T\| \leq \mu$ and $\|\partial T\| \leq \nu$ (in fact we have defined the mass only for currents: nonetheless we can see that for the definition of boundary and mass we only need conditions (b) and (c) of Definition 2.2.1; hence with slight abuse of notation we will use $\|T\|$ and $\|\partial T\|$ even if we have not yet proved that T is actually a current). So in order to check the continuity, thanks to the finiteness of μ and ν, we can suppose that f is Lipschitz and has bounded support and, thanks to (100), we can write

$$\begin{aligned} & |T(f d\pi) - T(f d\pi')| = \lim_{n \to \infty} |T_{h(n)}(f d\pi) - T_{h(n)}(f d\pi')| \\ \leq \ & \int_E |f||\pi'_i - \pi_i| d\mu + \mathrm{Lip}\,(f) \int_{\mathrm{supp}\,(f)} \|\pi_i - \pi'_i| d\nu \end{aligned}$$

whenever $\mathrm{Lip}\,(\pi_i), \mathrm{Lip}\,(\pi'_i) \leq 1$. This estimate gives easily condition (c) of Definition 2.2.1 and hence completes the proof. $\square$

2. Approximate differentiability

We give here the basic ideas of the proof of Theorem 5.1.8, which we state again below.

THEOREM 7.2.1. *Let us suppose that u maps $\mathbb{R}^n$ into $\mathbb{R}^n$ and it is injective on a measurable set A. Then*

$$|\operatorname{im}_{\mathrm{G}}(u,A)| = \int_{A\cap\Omega_d} |\det(apDu(x))|dx\,. \tag{102}$$

Before proving Theorem 7.2.1 we state the following useful fact about approximate differentiable functions.

THEOREM 7.2.2. *The set Ω_d defined above can be covered by a countable family of Borel sets E such that $u|_E$ is Lipschitz.*

PROOF. We only sketch the proof, which is the proof of Theorem 3.1.8 of [**18**]. For every positive integer j we define

$$Q(x,r,j) = B_r(x)\cap\{y \in : |u(x)-u(y)| \geq j|x-y|\}.$$

and

$$C_j := \Omega_d \cap \{x : \mathcal{L}^m(Q(x,r,j)) < \rho r^m \text{for every } r < 1/j\}$$

where 2ρ is the ratio

$$\frac{\mathcal{L}^m(B_{|x-y|}(x)\cap B_{|x-y|}(y))}{|x-y|^m}$$

(which is independent of the choice of $x, y \in \mathbb{R}^m$).

Now one can check that $\cup B_j = \Omega_d$ and that if $E \subset B_j$ is a set of diameter less than $1/j$, $u|_{B_j}$ is Lipschitz. $\square$

PROOF OF THEOREM 7.2.1 We will actually prove the following generalized version of the theorem: if $u : \mathbb{R}^m \to \mathbb{R}^n$, with $m \leq n$, and $J_m(apDu(x))$ the classical Jacobian of geometric measure theory then

$$\int_A |J_m(apDu(x))| d\mathcal{L}^m(x) = \int_{\mathbb{R}^n} N(u,y,A) d\mathcal{H}^m(y), \tag{103}$$

where $N(u,y,A)$ is the cardinality of $u^{-1}(\{y\})\cap A$

First step Suppose that u is continuous and take the set

$$B = \{x|\nabla u(x) \text{ exists and is an injective linear function}\}.$$

Then for every $\lambda > 1$ we can cover G with a countable number of Borel sets E such that for every E there exists an injective linear function $s : \mathbb{R}^m \to \mathbb{R}^m$ and

(i) u is injective on E;
(ii) $\operatorname{Lip}(u|_E \circ s^{-1}), \operatorname{Lip}(s \circ (u|_E)^{-1}) \leq \lambda$;

(iii) $\lambda^{-1}|s(v)| \leq |\nabla f(x)(v)| \leq \lambda|s(v)|$ for $x \in E$ and $v \in \mathbb{R}^m$;
(iv) $\lambda^{-m}|\det(s)| \leq J_m u(x) \leq \lambda^m|\det(s(x)|$ for $x \in E$.

We only sketch the proof of this fact, which is the statement of Lemma 3.2.2 of [**18**]. We fix $\varepsilon > 0$ such that $\lambda^{-1} + \varepsilon < 1 < \lambda - \varepsilon$ and we take a countable dense set S of linear automorphisms of $\mathbb{R}^m$. We then define the sets $Z(s, i)$, with $i \in \mathbb{N}$, which consist of points $a \in \mathbb{R}^m$ such that

(a) $(\lambda^{-1} + \varepsilon)|s(v)| \leq |\nabla u(a)(v)| \leq (\lambda - \varepsilon)|s(v)|$ for every $v \in \mathbb{R}^m$;
(b) $|u(b) - u(a) - \nabla u(a)(b - a)| \leq \varepsilon|s(b - a)|$ for every b in the ball of radius $1/i$ and center x.

It can be checked that if $E \subset Z(s, i)$ and the diameter of F is less than $1/i$ then on E conditions (i)–(iv) are satisfied. Hence $Z(s, i)$ has a countable covering which satisfies the conditions (i)–(iv). Using the singular decomposition of matrices it can be also checked that the sets $Z(s, i)$ give a countable covering of B

Second step Now we prove equation (103) in case u is a Lipschitz function. Our proof will be based on the following easy estimate which can be checked directly from the definition of Hausdorff measure:

(Est) if $u : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz function, then $\mathcal{H}^k(u(A)) \leq (\mathrm{Lip}\,(u))^k \mathcal{H}^k(A)$.

Thanks to this estimate it is sufficient to prove (103) for A' such that $\mathcal{L}^m(A \backslash A') = 0$. Hence we will do it for

$$A' := \{x | \text{u is differentiable in } x\}.$$

We split A' into two parts: the set B of points x where $\nabla u(x)$ is injective and the complementary, which will be called respectively B and C.

First of all we prove the Theorem for B. We fix λ and we find a countable covering (E_i) satisfying (i)–(iv) of the previous step. Then it is a consequence of (Est) above that

$$\lambda^{-2m}\mathcal{H}^m(u(E_i)) \leq \int_{E_i} J_m(u(x))dx \leq \lambda^{2m}\mathcal{H}^m(u(E_i)).$$

Hence summing these inequalities we get

$$\begin{aligned}\lambda^{-2m}\int_{\mathbb{R}^n} N(u, y, A)d\mathcal{H}^m &\leq \int_B |J_m(apDu(x))|d\mathcal{L}^m(x)\\ &\leq \lambda^{2m}\int_{\mathbb{R}^n} N(u, y, A)d\mathcal{H}^m(y).\end{aligned}$$

Letting λ we have the desired equality.

Passing to C we notice that for every $\varepsilon > 0$ we can factor $u = p \circ g$ where

$$g : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, \qquad g(x) = (u(x), \varepsilon x)$$
$$p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \qquad p(y, z) = z.$$

Now applying the same trick which we have used above for B we can estimate $\mathcal{H}^m(u(A)) \leq \varepsilon(\mathrm{Lip}\,(u) + \varepsilon)^{m-1}$ and letting ε go to zero we find $\mathcal{H}^m(u(A)) = 0$, which completes the proof.

Third Step Using Theorem 7.2.2 and the previous step we easily conclude. □

3. Functions of metric bounded variation

We report proof the proof of [**8**] of last remark of section 1 of chapter 1.

THEOREM 7.3.1. *Let $\mathcal{F} \subset Lip_b(M)$ be as in Definition 2.1.3, $u \in MBV(\mathbb{R}^k, M)$ and $\psi \in Lip_1(M) \cap Lip_b(M)$. Then $\psi \circ u \in BV_{loc}(\mathbb{R}^k)$ and*

$$|D(\psi \circ u)| \leq \bigvee_{\varphi \in \mathcal{F}} |D(\varphi \circ u)|.$$

In particular $\|Du\| = \bigvee\{|D(\varphi \circ u)| : \varphi \in Lip_1(M) \cap Lip_b(M)\}$.

PROOF. Let us first assume $k = 1$. Let $A \subset \mathbb{R}$ be an open interval and let $v : A \to \mathbb{R}$ be a bounded function. We denote by L_v the set of Lebesgue points of v and set $|Dv|(A) = +\infty$ if $v \notin BV_{loc}(A)$. It is a classical fact (see for example [**7**]) in the BV–theory of one real variable functions that

$$|Dv|(A) = \sup\left\{\sum_{i=1}^{p-1} |v(t_{i+1}) - v(t_i)| : t_1 < \ldots < t_p \in A \backslash N\right\}$$

whenever $\mathcal{L}^1(N) = 0$ and $A \backslash L_v \subset N$. Choosing

$$N := (A \backslash L_{\psi \circ u}) \cup \bigcup_{\varphi \in \mathcal{F}} [A \backslash L_{\varphi \circ u} \cup \{t \in A : |D(\varphi \circ u)|(\{t\}) > 0\}]$$

we get

$$\begin{aligned} |\psi \circ u(t_{i+1}) - \psi \circ u(t_i)| &\leq \sup_{\varphi \in \mathcal{F}} |\varphi \circ u(t_{i+1}) - \varphi \circ u(t_i)| \\ &\leq \|Du\|((t_i, t_{i+1})) \end{aligned}$$

whenever $t_i, t_{i+1} \in A \backslash N$.

Adding with respect to i and taking the supremum we obtain that $|D(\psi \circ u)|(A)$ can be estimated with $\|Du\|(A)$. By approximation the same inequality remains true if A is an open set or a Borel set.

In the higher dimensional case the proof follows from the one dimensional case recalling the following facts (see again the book [**7**] or 4.5.9 in [**18**])

(i) If $v \in BV_{loc}(\mathbb{R}^k)$ then

$$|Dv| = \bigvee_{\nu \in \mathbf{S}^{k-1}} |D_\nu v|;$$

(ii) the directional total variations can be represented as integrals of variations on lines, i.e.

$$|D_\nu v| = \int_{\pi_\nu} V_v(x,\nu) d\mathcal{H}^{k-1}(x) \qquad \forall \nu \in \mathbf{S}^{k-1}$$

where π_ν is the hyperplane orthogonal to ν, $v_{x,\nu}(t) = v(x + t\nu)$ and

$$V_v(x,\nu)(B) := |Dv_{x,\nu}|(\{t | x + t\nu \in B\})$$

for every Borel set B.

Hence, after fixing $\nu \in \mathbf{S}^{k-1}$ one can check that

$$\begin{aligned}
|D_v v| &= \int_{\pi_v} V_v(x,\nu) d\mathcal{H}^{k-1}(x) \\
&\leq \int_{\pi_n} \bigvee_{\varphi \in \mathcal{F}} V_{\varphi \circ u}(x,\nu) d\mathcal{H}^{k-1}(x) \\
&= \bigvee_{\varphi \in \mathcal{F}} \int_{\pi_n} V_{\varphi \circ u}(x,\nu) d\mathcal{H}^{k-1}(x) \\
&= \bigvee_{\varphi \in \mathcal{F}} |D_\nu(\varphi \circ u)| \leq \|Du\|.
\end{aligned}$$

Since ν is arbitrary fact (i) implies the inequality $|Dv| \leq \|Du\|$. □

4. Lemma on Caccioppoli sets

LEMMA 7.4.1. *Let $A, C \subset \mathbb{R}^n$ be two Caccioppoli sets and let us denote by A^* and C^* the sets of points with density 1 with respect to A and C. Suppose that:*

(i) *at least one of them is not $\mathbb{R}^n$;*
(ii) $\mathcal{H}^{n-1}(\partial^* A \cap C^*) = \mathcal{H}^{n-1}(\partial^* C \cap A^*) = 0$.

Then $|A \cap C| = 0$ and hence $A^* \cap C^* = \emptyset$.

PROOF. We notice that $A \cap C$ is a Caccioppoli set and that $x \in \partial^*(A \cap C)$ if and only if

$$\lim_{r \to 0} \frac{|A \cap C \cap B(x,r)|}{\omega_n r^n} = \frac{1}{2}.$$

Furthermore we notice that condition (ii) implies that for $\mathcal{H}^{n-1}$ a.e. $x \in \mathbb{R}^n$ either one of the sets A and C has density 0 in x, or they have both density 1. It follows that $\mathcal{H}^{n-1}(\partial^*(A \cap C)) = 0$ and so either $A \cap C$ is a set of Lebesgue measure zero, or is $\mathbb{R}^n$ minus a set of measure zero. The second possibility is excluded by condition (i) and this ends the proof. □

5. $W^{1,2}$ norms in cylindrical and spherical coordinates

Let (r, θ, ϕ) be spherical coordinates, with

$$\mathbf{r} = r(\mathbf{e}_1 \cos\theta + \sin\theta(\mathbf{e}_2 \cos\phi + \mathbf{e}_3 \sin\phi)),$$

and (u_r, u_θ, u_ϕ) be spherical components of $\mathbf{u}$ (here we use boldface for vectors). Assume $u_\phi = \phi$, with u_r and u_θ independent on ϕ. Then, a simple calculation gives

$$\begin{aligned} \int_\Omega |\nabla \mathbf{u}|^2 \;&=\; 2\pi \int_\Omega \left[(\partial_r u_r)^2 + \left(\frac{\partial_\theta u_r}{r}\right)^2 + (u_r \partial_r u_\theta)^2 + \left(\frac{u_r \partial_\theta u_\theta}{r}\right)^2 \right. \\ &\qquad \left. + \left(\frac{u_r \sin u_\theta}{r \sin\theta}\right)^2 \right] r^2 \sin\theta dr d\theta\,. \end{aligned} \tag{104}$$

Now consider cylindrical coordinates (x, y, ϕ), such that $\mathbf{r} = x\mathbf{e}_1 + y(\mathbf{e}_2 \cos\phi + \mathbf{e}_3 \sin\phi)$, but still express u in spherical components. Then,

$$\begin{aligned} \int_\Omega |\nabla \mathbf{u}|^2 \;&=\; 2\pi \int_\Omega \left[(\partial_x u_r)^2 + (\partial_y u_r)^2 + \left(\frac{\partial_\theta u_r}{y}\right)^2 + (u_r \partial_y u_\theta)^2 + \right. \\ &\qquad \left. \left(\frac{u_r \partial_\theta u_\theta}{y}\right)^2 + \left(\frac{u_r \sin u_\theta}{y \sin\theta}\right)^2 \right] y dx dy\,. \end{aligned} \tag{105}$$

Bibliography

[1] Acerbi, E. and Dal Maso, G. New lower semicontinuity results for polyconvex integrals. *Calc. Var. Partial Differential Equations*, 2(3):329–371, 1994.

[2] Alberti, G. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, **123**(2):239–274, 1993.

[3] Alberti G., Baldo S., Orlandi G. In preparation.

[4] Ambrosio L. A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital., VII Ser.*, **B 3**(4):857–881, 1989.

[5] Ambrosio L. Metric space valued functions of bounded variation. *Ann. Scuola Normale Sup. Pisa Cl. Sci., IV Ser.*, **17**(3):439–478, 1990.

[6] Ambrosio L., De Giorgi E. Un nuovo tipo di funzionale del calcolo delle variazioni. *Atti Accad. Naz. Lincei Rend., VIII Ser., Cl. Sci. Fis. Mat. Natur.*, **82**(2):199–210, 1988.

[7] Ambrosio L., Fusco N., Pallara D. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. Clarendon Press, Oxford, 2000.

[8] Ambrosio L., Kirchheim B. Currents on metric spaces. *Acta Mathematica*, **185**(1):1–80, 2000.

[9] Ball, J. M. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63:337–403, 1977.

[10] Ball, J.M. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Philos. Trans. Roy. Soc. London Ser. A*, **306**(1496):557–611, 1982.

[11] Brezis H., Nirenberg L. Degree theory and BMO: Part 1, compact manifolds without boudaries. *Selecta Math. (N.S.)*, **1**(2):197–263, 1995.

[12] Ciarlet, P. G. and Nečas, J. Injectivity and self-contact in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 97(3):171–188, 1987.

[13] Conti, S., De Lellis C. Some remarks on the theory of elasticity fro compressible neohookean materials. Submitted to *Archives for Rational Mechanics and Analysis.*

[14] Dacorogna B., Moser J. On a partial differential equation involving the jacobian determinat. *Ann. IHP Anal. Non Lin.*, **7**(1):1–26, 1990.

[15] De Giorgi, E. Problema di plateau generale e funzionali geodetici. *Atti Sem. Mat. Fis. Univ. Modena*, **43**(2):285–292, 1995.

[16] De Lellis C. Fine properties of currents and applications to distributional Jacobians. To appear in *Proc. Roy. Soc. Edinburgh Sect. A.*

[17] De Lellis, C. Some remarks on the distributional jacobian. To appear in *Nonlinear Analysis: Theory, Methods and Applications.*

[18] Federer H. *Geometric measure theory.* Classics in Mathematics. Springer Verlag, Berlin, 1969.

[19] Fonseca I. and Gangbo W. *Degree theory in analysis and applications.* Oxford Lecture Series in Mathematics and its Applications, 2. Clarendon Press, Oxford, 1995.

[20] Giaquinta, M., Modica, G., Souček, J. Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, **106**(2):97–159, 1989.

[21] Giaquinta M., Modica G., Souček J. Cartesian currents, weak diffeomorphisms and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, **106**(2):97–159, 1989.

[22] Giaquinta, M., Modica, G., Souček, J. Remarks on the degree theory. *J. Funct. Anal.*, **125**(1):172–200, 1994.

[23] Giaquinta M., Modica G., Souček, J. Connectivity properties of the range of a weak diffeomorphism. *Ann. Inst. H. Poincar Anal. Non Linaire*, **12**(1):61–73, 1995.

[24] Giaquinta M., Modica G., Souček, J. Composition of weak diffeomorphisms. *Math. Z*, **224**(3):385–402, 1997.

[25] Giaquinta M., Modica G., Souček J. *Cartesian currents in the calculus of variations*, volume 1, 2. Springer Verlag, Berlin, 1998.

[26] Jerrard L. Vortex filament dynamics for gross-pitaevsky type equations. http://www.mis.mpg.de/preprints/2000/ Preprint Nr. 45/2000.

[27] Jerrard L., Soner M. Functions of higher bounded variation. Forthcoming.

[28] Jerrard L., Soner M. Rectifiability of the distributional jacobian for a class of functions. *C. R. Acad. Sci. Paris Sér. I Math.*, **329**(8):683–688, 1999.

[29] Malý, J. Weak lower semicontinuity of polyconvex integrals. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(4):681–691, 1993.

[30] Malý J. Lower semicontinuity of quasiconvex integrals. *Manuscripta Math.*, **85**(3–4):419–428, 1994.

[31] Marcus M., Mizel V. J. Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational integrals. *Bull. Amer. Math. Soc.*, **79**:790–795, 1973.

[32] Müller, S. Weak continuity of determinants and nonlinear elasticity. *C. R. Acad. Sci. Paris Sér. I Math.*, **307**(9):501–506, 1988.

[33] Müller S. Weak continuity of determinants and nonlinear elasticity. *C. R. Acad. Sci. Paris Sér I Math*, **307**(9):501–506, 1988.

[34] Müller S. Det=det. A remark on the distributional determinant. *C. R. Acad. Sci Paris Sér I*, **311**(1):13–17, 1990.

[35] Müller, S. Higher integrability of determinants and weak convergence in l^1. *J. Reine Angew. Math.*, **412**:20–34, 1990.

[36] Müller, S. On the singular support of the distributional determinant. *Ann. Inst. H. Poincar Anal. Non Linéaire*, **10**(6):657–696, 1993.

[37] Müller, S., Sivaloganathan J., Spector S. J. An isoperimetric estimate and $w^{1,p}$-quasiconvexity in nonlinear elasticity. *Calc. Var. Partial Differential Equations*, **8**(2):159–176, 1999.

[38] Müller S., Spector S. An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Rat. Mech. Anal.*, **131**(1):1–66, 1995.

[39] Müller, S., Spector S. J., Tang, Q. Invertibility and a topological property of sobolev maps. *SIAM J. Math. Anal.*, **27**(4):959–976, 1996.

[40] Stein E. M. *Singular integrals and the differentiability properties of functions.* Princeton University Press, 1970.

[41] ŠVERÁK V. Regularity properties of deformations with finite energy. *Arch. Rat. Mech. Anal.*, **100**(2):105–127, 1988.
[42] WHITE B. Rectifiability of flat chains. *Ann. Math. (2)*, **150**(1):165–184, 1999.