STRICT MONOTONICITY OF THE FIRST q-EIGENVALUE OF THE FRACTIONAL p-LAPLACE OPERATOR OVER ANNULI

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Abstract. Let $B, B' \subset \mathbb{R}^d$ with $d \geq 2$ be two balls such that $B' \subset\subset B$ and the position of B' is varied within B. For $p \in (1, \infty)$, $s \in (0, 1)$, and $q \in [1, p_s^*)$ with $p_s^* = \frac{dp}{d-sp}$ if $sp < d$ and $p_s^* = \infty$ if $sp \geq d$, let $\lambda_{p,q}^s(B \setminus \overline{B'})$ be the first q-eigenvalue of the fractional p-Laplace operator $(-\Delta_p)^s$ in $B \setminus \overline{B'}$ with the homogeneous nonlocal Dirichlet boundary conditions. We prove that $\lambda_{p,q}^s(B \setminus \overline{B'})$ strictly decreases as the inner ball B' moves towards the outer boundary ∂B. To obtain this strict monotonicity, we establish a strict Faber-Krahn type inequality for $\lambda_{p,q}^s(\cdot)$ under polarization. This extends some monotonicity results obtained by Djitte-Fall-Weth (Calc. Var. Partial Differential Equations, 60:231, 2021) in the case of $(-\Delta)^s$ and $q = 1, 2$ to $(-\Delta_p)^s$ and $q \in [1, p_s^*)$. Additionally, we provide the strict monotonicity results for the general domains that are difference of Steiner symmetric or foliated Schwarz symmetric sets in $\mathbb{R}^d$.

1. Introduction

For $d \geq 1$, $p \in (1, \infty)$, and $s \in (0, 1)$, the Gagliardo seminorm $[\cdot]_{s,p}$ in $\mathbb{R}^d$ is given by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{p}},$$

and the fractional critical Sobolev exponent p_s^* is defined as $p_s^* = \frac{dp}{d-sp}$ if $sp < d$, and $p_s^* = \infty$ if $sp \geq d$. For a bounded open set $\Omega \subset \mathbb{R}^d$ and $q \in [1, p_s^*)$, we consider the best constant $\lambda_{p,q}^s(\Omega)$ in the following family of Sobolev inequalities:

$$\left(\lambda_{p,q}^s(\Omega)\right)^{\frac{1}{p}} \|u\|_{L^q(\Omega)} \leq [u]_{s,p}, \quad \forall\, u \in W_0^{s,p}(\Omega), \tag{1.1}$$

where $W_0^{s,p}(\Omega) := \{u \in L^p(\mathbb{R}^d) : [u]_{s,p} < \infty, \text{ and } u = 0 \text{ in } \mathbb{R}^d \setminus \Omega\}$ is the fractional Sobolev space. The variational characterization for $\lambda_{p,q}^s(\Omega)$ in (1.1) is given by

$$\lambda_{p,q}^s(\Omega) := \inf\left\{ [u]_{s,p}^p : u \in W_0^{s,p}(\Omega) \text{ with } \|u\|_{L^q(\Omega)} = 1 \right\}. \tag{1.2}$$

For $1 \leq q < p_s^*$, using the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ (see [12, Corollary 2.8]) and direct variational methods, a minimizer for (1.2) exists. Without loss of generality, we can assume that the minimizer is non-negative since $[|u|]_{s,p} \leq [u]_{s,p}$ holds for $u \in W_0^{s,p}(\Omega)$. Further, any minimizer $u \in W_0^{s,p}(\Omega)$ of (1.2) satisfies the following identity: for any $\phi \in W_0^{s,p}(\Omega)$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{d+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}y \, \mathrm{d}x = \lambda_{p,q}^s(\Omega) \int_{\Omega} u(x)^{q-1} \phi(x) \, \mathrm{d}x,$$

and this implies that u is a weak solution to the following nonlinear fractional eigenvalue problem:

$$\begin{aligned} (-\Delta_p)^s u &= \lambda_{p,q}^s(\Omega) u^{q-1} \text{ in } \Omega, \\ u &= 0 \text{ in } \mathbb{R}^d \setminus \Omega \text{ with } \|u\|_{L^q(\Omega)} = 1, \end{aligned} \tag{1.3}$$

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where $(-\Delta_p)^s$ is the fractional p-Laplace operator that is defined as

$$(-\Delta_p)^s u(x) = \text{P.V.} \int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}} \, \mathrm{d}y$$

where P.V. denotes the principle value. We call the best constant $\lambda^s_{p,q}(\Omega)$ to be the first *q-eigenvalue* of the fractional p-Laplace operator, and the corresponding minimizer to be a first *q-eigenfunction* of (1.3). In Proposition 2.4, we show that the first q-eigenfunctions are in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,s]$. Further, by the strong maximum principle [11, Theorem A.1], the first q-eigenfunctions are strictly positive in Ω. In the homogeneous case of $p=q$, $\lambda^s_{p,p}(\Omega)$ is the first eigenvalue of $(-\Delta_p)^s$ in Ω with the homogeneous nonlocal Dirichlet boundary condition. In the case of $q=1$, the first 1-eigenvalue is related to the fractional p-torsional rigidity $\mathcal{T}_{s,p}(\Omega)$ of Ω, which is defined as

$$\mathcal{T}_{s,p}(\Omega) = \left(\int_\Omega w \, \mathrm{d}x \right)^{p-1} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|w(x)-w(y)|^p}{|x-y|^{d+sp}} \, \mathrm{d}y \, \mathrm{d}x \right)^{p-1},$$

where $w \in W^{s,p}_0(\Omega)$ is a positive weak solution of the following fractional p-torsion problem:

$$(-\Delta_p)^s w = 1 \text{ in } \Omega,$$
$$w = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

Using simple scaling arguments, one can verify that

$$\mathcal{T}_{s,p}(\Omega) = \left(\lambda^s_{p,1}(\Omega) \right)^{-1}. \tag{1.4}$$

In this paper, we consider the following family of annular domains in $\mathbb{R}^d$: for $0<r<R<\infty$,

$$\mathcal{A}_{r,R} := \left\{ \Omega_t := B_R(0) \setminus \overline{B}_r(te_1) \subset \mathbb{R}^d : 0 \le t < R-r \right\},$$

where $B_r(z)$ is the open ball with radius $r \ge 0$ centered at $z \in \mathbb{R}^d$, and $\overline{B}_r(z)$ is the closure of the open ball $B_r(z)$ in $\mathbb{R}^d$. Our objective is to study an optimal domain for $\lambda^s_{p,q}(\cdot)$ over $\mathcal{A}_{r,R}$, and also analyze the behavior of $\lambda^s_{p,q}(\Omega_t)$ for $0 \le t < R-r$. In the local case $s=1$, Hersch [20] considered the first eigenvalue $\lambda^1_{2,2}(\cdot)$ of the Laplace operator over $\mathcal{A}_{r,R}$ with $d=2$, and proved that '$\lambda^1_{2,2}(\Omega_t)$ *attains its maximum only at* $t=0$.' Later, many authors proved this phenomenon for $d \ge 2$ by establishing *'the strict monotonicity of $\lambda^1_{p,p}(\Omega_t)$ of the p-Laplace operator.'* For example, Harrell-Kröger-Kurata [19], and independently Kesavan [23] for $p=q=2$; and Anoop-Bobkov-Sasi [3] for $1<p=q<\infty$. Further, Bobkov-Kolonitskii [10] proved the strict monotonicity of $\lambda^1_{p,q}(\cdot)$ using the domain perturbations and polarization method for $p \in (1,\infty)$ and $q \in [1,p^*)$, where p^* is the critical Sobolev exponent. In the nonlocal case $s \in (0,1)$, Djitte-Fall-Weth [17] proved the strict monotonicity of $\lambda^s_{p,q}(\Omega_t)$ when $p=2$ and $q=1,2$. Indeed, the strict monotonicity implies that $\lambda^s_{p,q}(\Omega_t)$ attains its maximum only at $t=0$. One of the main ingredients they have used is the Hadamard perturbation formula for the derivative of $\lambda^s_{p,q}(\cdot)$. The reflection methods and the strong comparison principle give a strict sign for this derivative. For $p=2$, the authors of [19] (when $s \in (0,1)$) and the authors of [17, 23] (when $s=1$), determined a strict sign for the derivative of $\lambda^1_{p,q}(\cdot)$ using the strong comparison principle. In the local case $s=1$, the strong comparison principle is unavailable for any $p \in (1,\infty)$ except $p=2$. However, the authors of [3, 10] carefully utilized the geometry of annular domains and the qualitative properties of the first eigenfunctions to bypass the strong comparison principle. Establishing the Hadamard perturbation formula is challenging and highly depends on the structure of the differential operator involved. However, this formula for the derivative of $\lambda^s_{p,q}(\cdot)$ is established in [19, Proposition 1.1] and in [23, Section 2&3] for $p=q=2$ and $s=1$; in [10, Theorem 1.3] for $p \in (1,\infty)$, $q \in [1,p^*_s)$ and $s=1$; and in [17, Corollary 1.2] for $p=2$, $q=1,2$ and $s \in (0,1)$. To our knowledge, the Hadamard perturbation formula is not available for any p,q, and s in their natural ranges.

Recently, Anoop-Ashok [2] proved the strict monotonicity of the first eigenvalue of the p-Laplace operator for $\frac{2d+2}{d+2} < p < \infty$ via a rearrangement in $\mathbb{R}^d$ called *polarization*. This approach bypasses the usage of the Hadamard perturbation formula. One of the main tools in this approach is a strict Faber-Krahn type inequality involving polarization, which indicates that the first eigenvalue strictly decreases under polarization. The restriction on the exponent p is required to get this strict Faber-Krahn type inequality by applying a general strong comparison principle due to Sciunzi [24, Theorem 1.4] for the p-Laplace operator. Another key idea is to express the translations of the inner ball as a rearrangement by polarization and then apply the strict Faber-Krahn inequality to get the strict monotonicity of the first eigenvalue. In this paper, we follow a similar approach to establish the strict monotonicity of $\lambda^s_{p,q}(\cdot)$ over $\mathcal{A}_{r,R}$ for any $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1, p^*_s)$.

To state our results, we recall the notion of polarization for sets in $\mathbb{R}^d$, which was first introduced by Wolontis [27]. A *polarizer* is an open affine-halfspace in $\mathbb{R}^d$. For a polarizer H in $\mathbb{R}^d$, the polarization $P_H(\Omega)$ of $\Omega \subseteq \mathbb{R}^d$ is defined as

$$P_H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H] \cup [\Omega \cap \sigma_H(\Omega)],$$

where $\sigma_H(\cdot)$ is the reflection in $\mathbb{R}^d$ with respect to the affine-hyperplane ∂H. It is easily verified that P_H takes open sets to open sets, and P_H is a rearrangement (i.e., it respects the set inclusion and preserves the measure) on $\mathbb{R}^d$. Now, we state our main results as a unified theorem.

Theorem 1.1. *Let $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1, p^*_s)$.*

- (i) ***Strict Faber-Krahn type inequality:*** *Let H be a polarizer, and Ω be a bounded open set with $\mathcal{C}^{1,1}$-boundary in $\mathbb{R}^d$. Then $\lambda^s_{p,q}(\cdot)$ decreases under polarization. Further, let $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$, then $\lambda^s_{p,q}(\cdot)$ strictly decreases under polarization.*
- (ii) ***Strict monotonicity:*** *Let $0 < r < R$. Then $\lambda^s_{p,q}\left(B_R(0) \setminus \overline{B}_r(te_1)\right)$ is strictly decreasing for $0 \le t < R - r$. In particular,*

$$\lambda^s_{p,q}\left(B_R(0) \setminus \overline{B}_r(0)\right) = \max_{0\le t<R-r} \lambda^s_{p,q}\left(B_R(0) \setminus \overline{B}_r(te_1)\right).$$

Remark 1.2. (i) For $p = 2$, $s \in (0,1)$ and $q = 1, 2$, Theorem 1.1-(ii) provides an alternative proof for the strict monotonicity of $\lambda^s_{2,q}\left(B_R(0) \setminus \overline{B}_r(te_1)\right)$, for $0 \le t < R - r$, obtained by [17, Theorem 1.4]. Moreover, Theorem 1.1-(ii) extends their result for $q \in [1, 2^*_s)$.
(ii) Our Theorem 1.1-(ii) is a nonlocal analog of the monotonicity results obtained by Bobkov-Kolonitsii [10] for the local problem $-\Delta_p u = f(u)$ with a particular non-linearity $f(u) = |u|^{q-2}u$, $q \in [1, p^*)$, where $p^* = \frac{dp}{d-p}$ when $d > p$ and $p^* = \infty$ when $d \le p$.
(iii) In view of (1.4) and Theorem 1.1-(ii), the fractional p-torsional rigidity $\mathcal{T}_{s,p}(\Omega_t)$ is strictly increasing for $0 \le t < R - r$.

The strict Faber-Krahn type inequality gives the strict monotonicity of $\lambda^s_{p,q}\left(B_R(0) \setminus \overline{B}_r(te_1)\right)$ for $0 \le t < R - r$, since we can express the translations of $\overline{B}_r(te_1)$ in $B_R(0)$ as rearrangements by polarization (see Figure 2 and Proposition 3.2-(iii)). We briefly describe our procedure to prove the strict Faber-Krahn type inequality in Theorem 1.1. The effect of polarization on the L^q-norm and the Gagliardo seminorm (see Proposition 2.3) proves the Faber-Krahn type inequality for $\lambda^s_{p,q}(\cdot)$. To obtain the strict sign, we prove a version of the strong comparison principle involving a Sobolev function and its polarization (see Proposition 2.6). When $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$ and equality holds in the Faber-Krahn type inequality, we prove a contradiction to the strong comparison principle in $\Omega \cap H$ with the help of two sets $A_H(\Omega)$ and $B_H(\Omega)$ (see Figure 1 or Proposition 2.2-(v)).

The remainder of the paper is organized as follows. In Section 2, we discuss some essential properties and Pólya-Szegö type inequality of polarization. In the same section, we also prove the regularity and strong comparison principle for the solutions of (1.3). The proofs of our main results are given in Section 3. At the end of Section 3, we give some strict monotonicity results with respect to certain rotations and translations of a *'hole'* in general classes of domains.

2. Polarization, regularity, and strong comparison principle

In this section, we discuss polarization and some of its properties. Also, we prove some regularity results and a version of the strong comparison principle for the solutions of the fractional p-Laplace operator.

2.1. **Polarization and its properties.** An open affine-halfspace in $\mathbb{R}^d$ is called a *polarizer*. The set of all polarizers in $\mathbb{R}^d$ is denoted by $\mathcal{H}$. We observe that, for any polarizer $H \in \mathcal{H}$ there exist $h \in \mathbb{S}^{d-1}$ and $a \in \mathbb{R}$ such that $H = \left\{x \in \mathbb{R}^d : x \cdot h < a\right\}$. For $H \in \mathcal{H}$, let σ_H be the reflection in $\mathbb{R}^d$ with respect to ∂H. Then, for any $x \in \mathbb{R}^d$,

$$\sigma_H(x) = x - 2(x \cdot h - a)h. \tag{2.1}$$

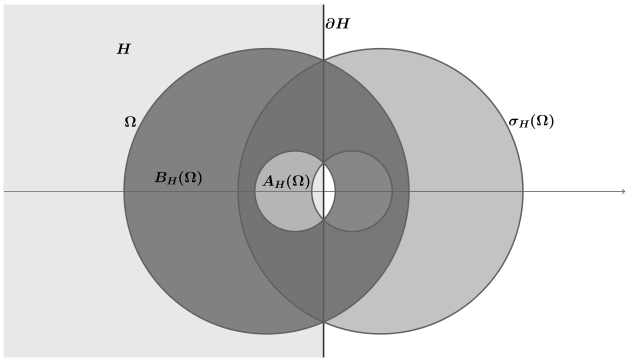

FIGURE 1. The sets $A_H(\Omega)$ and $B_H(\Omega)$ for an annular domain Ω.

The reflection of $A \subseteq \mathbb{R}^d$ with respect to ∂H is $\sigma_H(A) = \{\sigma_H(x) : x \in A\}$. It is straightforward to verify $\sigma_H(A^{\mathsf{c}}) = \sigma_H(A)^{\mathsf{c}}$, $\sigma_H(A \cup B) = \sigma_H(A) \cup \sigma_H(B)$, and $\sigma_H(A \cap B) = \sigma_H(A) \cap \sigma_H(B)$ for any $A, B \subseteq \mathbb{R}^d$. Now, we define the polarization of sets and functions, see [2, Definition 1.1].

Definition 2.1. *Let $H \in \mathcal{H}$ and $\Omega \subseteq \mathbb{R}^d$. The polarization $P_H(\Omega)$ and the dual polarization $P^H(\Omega)$ of Ω with respect to H are defined as*

$$P_H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H] \cup [\Omega \cap \sigma_H(\Omega)],$$
$$P^H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H^{\mathsf{c}}] \cup [\Omega \cap \sigma_H(\Omega)].$$

For $u : \mathbb{R}^d \longrightarrow \mathbb{R}$, the polarization $P_H(u) : \mathbb{R}^d \longrightarrow \mathbb{R}$ with respect to H is defined as

$$P_H u(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^d \setminus H. \end{cases}$$

For $u : \Omega \longrightarrow \mathbb{R}$, let $\widetilde{u}$ be the zero extension of u to $\mathbb{R}^d$. The polarization $P_H u : P_H(\Omega) \longrightarrow \mathbb{R}$ is defined as the restriction of $P_H \widetilde{u}$ to $P_H(\Omega)$.

The polarization for functions on $\mathbb{R}^d$ is introduced by Ahlfors [1] (when $d = 2$) and Baernstein-Taylor [5] (when $d \geq 2$). For further reading on polarizations and their applications, we refer the reader to [4, 8, 13, 25, 26]. Now, we list some important properties of polarization.

Proposition 2.2. *Let $H \in \mathcal{H}$, and $\Omega, \mathcal{O} \subseteq \mathbb{R}^d$. The following hold:*

(i) $P_H(\sigma_H(\Omega)) = P_H(\Omega)$ *and* $P^H(\Omega) = \sigma_H(P_H(\Omega))$*;*
(ii) $P_H(\Omega^{\mathsf{c}}) = (P^H(\Omega))^{\mathsf{c}}$*;*
(iii) $P_H(\Omega) = \Omega$ *if and only if* $\sigma_H(\Omega) \cap H \subseteq \Omega$*;*
(iv) $P_H(\Omega \setminus \mathcal{O}) = P_H(\Omega) \setminus P^H(\mathcal{O})$, *if* $\mathcal{O}, \sigma_H(\mathcal{O}) \subseteq \Omega$*;*
(v) *if Ω is open and $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$, then the sets $A_H(\Omega) := \sigma_H(\Omega) \cap \Omega^{\mathsf{c}} \cap H$ and $B_H(\Omega) := \Omega \cap \sigma_H(\Omega^{\mathsf{c}}) \cap H$ have non-empty interiors.*

Proof. The polarizations of Ω can be rewritten as

$$P_H(\Omega) = [(\Omega \cup \sigma_H(\Omega)) \cap H] \cup [\Omega \cap \sigma_H(\Omega) \cap H^{\mathsf{c}}],$$
$$P^H(\Omega) = \left[(\Omega \cup \sigma_H(\Omega)) \cap \overline{H}^{\mathsf{c}}\right] \cup [\Omega \cap \sigma_H(\Omega)].$$

(i) This follows from the fact that $\sigma_H(H) = \overline{H}^{\mathsf{c}} \in \mathcal{H}$.
(ii) We have $P_H(\Omega^{\mathsf{c}}) = [(\Omega^{\mathsf{c}} \cup \sigma_H(\Omega^{\mathsf{c}})) \cap H] \cup [\Omega^{\mathsf{c}} \cap \sigma_H(\Omega^{\mathsf{c}})]$. Therefore

$$\begin{aligned}(P_H(\Omega^{\mathsf{c}}))^{\mathsf{c}} &= [(\Omega \cap \sigma_H(\Omega)) \cup H^{\mathsf{c}}] \cap [\Omega \cup \sigma_H(\Omega)] \\ &= [\Omega \cap \sigma_H(\Omega)] \cup [(\Omega \cup \sigma_H(\Omega)) \cap H^{\mathsf{c}}] = P^H(\Omega).\end{aligned}$$

(iii) Suppose $P_H(\Omega) = \Omega$. Then the inclusion $\sigma_H(\Omega) \cap H \subseteq \Omega$ holds by noticing that $P_H(\Omega) \cap H = (\Omega \cap H) \cup (\sigma_H(\Omega) \cap H) = \Omega \cap H$. Now, assume that $\sigma_H(\Omega) \cap H \subseteq \Omega$. By applying σ_H and using the facts that $\Omega \cap \partial H = \sigma_H(\Omega) \cap \partial H$ and $H^{\mathsf{c}} = \sigma_H(H) \cup \partial H$, we arrive at $\Omega \cap H^{\mathsf{c}} \subseteq \sigma_H(\Omega)$. Hence, $P_H(\Omega) \cap H = (\Omega \cap H) \cup (\sigma_H(\Omega) \cap H) = \Omega \cap H$, and $P_H(\Omega) \cap H^{\mathsf{c}} = \Omega \cap \sigma_H(\Omega) \cap H^{\mathsf{c}} = \Omega \cap H^{\mathsf{c}}$. Therefore $P_H(\Omega) = \Omega$.

(iv) First, we observe that $(\Omega\cap\mathcal{O}^c)\cap\sigma_H(\Omega\cap\mathcal{O}^c)=(\Omega\cap\sigma_H(\Omega))\cap(\mathcal{O}^c\cap\sigma_H(\mathcal{O}^c))$. Also $\Omega\cup\sigma_H(\mathcal{O}^c)=\mathbb{R}^d$, since $\sigma_H(\mathcal{O})\subset\Omega$. Using this fact, we get $(\Omega\cap\mathcal{O}^c)\cup\sigma_H(\Omega\cap\mathcal{O}^c)=(\Omega\cup\sigma_H(\Omega))\cap(\mathcal{O}^c\cup\sigma_H(\mathcal{O}^c))$. Therefore, $P_H(\Omega\cap\mathcal{O}^c)=[((\Omega\cap\mathcal{O}^c)\cup\sigma_H(\Omega\cap\mathcal{O}^c))\cap H]\cup[(\Omega\cap\mathcal{O}^c)\cap\sigma_H(\Omega\cap\mathcal{O}^c)]=P_H(\Omega)\cap P_H(\mathcal{O}^c)$.
(v) Firstly, observe that the interiors of $A_H(\Omega)$ and $B_H(\Omega)$ are $\sigma_H(\Omega)\cap\overline{\Omega}^c\cap H$ and $\Omega\cap\sigma_H(\overline{\Omega}^c)\cap H$ respectively. If $\sigma_H(\Omega)\cap\overline{\Omega}^c\cap H=\emptyset$ then $\sigma_H(\Omega)\cap H\subseteq\Omega$, since both Ω and H are open. Hence $P_H(\Omega)=\Omega$, from (iv). Similarly, using the fact that $B_H(\Omega)=A_H(\sigma_H(\Omega))$, if $B_H(\Omega)$ has an empty interior, then $P_H(\sigma_H(\Omega))=\sigma_H(\Omega)$. This contradicts the hypothesis. Therefore, $A_H(\Omega)$ and $B_H(\Omega)$ have non-empty interiors. □

In the following proposition, we prove the invariance of L^q-norm and a fractional Pólya-Szegö type inequality under polarization. The fractional Pólya-Szegö inequality is first established by Beckner [6, page 4818]. For completeness, we give a proof motivated from [16, Section 3].

Proposition 2.3. *Let $H\in\mathcal{H}$, $\Omega\subseteq\mathbb{R}^d$ be open, and let $u:\Omega\longrightarrow\mathbb{R}^+$ be measurable. If $u\in L^p(\Omega)$ for some $p\in[1,\infty)$, then $P_H(u)\in L^p(P_H(\Omega))$ with $\|P_H(u)\|_{L^p(P_H(\Omega))}=\|u\|_{L^p(\Omega)}$. Furthermore, if $u\in W_0^{s,p}(\Omega)$ for some $s\in(0,1)$, then $P_H(u)\in W_0^{s,p}(P_H(\Omega))$ with $[P_H(u)]_{s,p}\leq[u]_{s,p}$.*

Proof. The inequality $\|P_H(u)\|_{L^p(P_H(\Omega))}=\|u\|_{L^p(\Omega)}$ follows from [26, Lemma 3.1-(i)]. To prove the inequality $[P_H(u)]_{s,p}\leq[u]_{s,p}$, we split $\mathbb{R}^d\times\mathbb{R}^d$ as the union of the regions $H\times H$, $H^c\times H^c$, $H\times H^c$, and $H^c\times H$. Now, using the change of variables, we can rewrite $[u]_{s,p}^p$ as

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\,dx=\int_H\int_H\Bigg[\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}+\frac{|u(\sigma_H(x))-u(y)|^p}{|\sigma_H(x)-y|^{d+sp}}+\frac{|u(x)-u(\sigma_H(y))|^p}{|x-\sigma_H(y)|^{d+sp}}+\frac{|u(\sigma_H(x))-u(\sigma_H(y))|^p}{|\sigma_H(x)-\sigma_H(y)|^{d+sp}}\Bigg]\,dy\,dx.$$

A similar expression can be written for $v:=P_H(u)$. To prove the inequality $[P_H(u)]_{s,p}\leq[u]_{s,p}$, it is now enough to show

$$\begin{aligned}&\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}+\frac{|u(\sigma_H(x))-u(y)|^p}{|\sigma_H(x)-y|^{d+sp}}+\frac{|u(x)-u(\sigma_H(y))|^p}{|x-\sigma_H(y)|^{d+sp}}+\frac{|u(\sigma_H(x))-u(\sigma_H(y))|^p}{|\sigma_H(x)-\sigma_H(y)|^{d+sp}}\\&\geq\frac{|v(x)-v(y)|^p}{|x-y|^{d+sp}}+\frac{|v(\sigma_H(x))-v(y)|^p}{|\sigma_H(x)-y|^{d+sp}}+\frac{|v(x)-v(\sigma_H(y))|^p}{|x-\sigma_H(y)|^{d+sp}}+\frac{|v(\sigma_H(x))-v(\sigma_H(y))|^p}{|\sigma_H(x)-\sigma_H(y)|^{d+sp}},\end{aligned}\tag{2.2}$$

for all $x,y\in H$. Firstly, we notice that

$$|\sigma_H(x)-\sigma_H(y)|=|x-y|\leq|\sigma_H(x)-y|=|x-\sigma_H(y)|,\quad\text{for } x,y\in H.\tag{2.3}$$

We consider the following four different cases.
Case 1. Let $x,y\in H$ be such that $u(x)\geq u(\sigma_H(x))$, and $u(y)\geq u(\sigma_H(y))$. Then, by the definition, $v(x)=u(x)$, $v(y)=u(y)$, $v(\sigma_H(x))=u(\sigma_H(x))$, and $v(\sigma_H(y))=u(\sigma_H(y))$. Therefore, (2.2) becomes equality.
Case 2. Let $x,y\in H$ be such that $u(x)\leq u(\sigma_H(x))$, and $u(y)\leq u(\sigma_H(y))$. Then, by the definition, $v(x)=u(\sigma_H(x))$, $v(y)=u(\sigma_H(y))$, $v(\sigma_H(x))=u(x)$, and $v(\sigma_H(y))=u(y)$. Since the right-hand side of (2.2) is invariant under σ_H, (2.2) becomes an equality in this case also.
Case 3. Let $x,y\in H$ be such that $u(x)\leq u(\sigma_H(x))$, and $u(y)\geq u(\sigma_H(y))$. Then $v(x)=u(x)$, $v(\sigma_H(x))=u(\sigma_H(x))$, $v(y)=u(\sigma_H(y))$, and $v(\sigma_H(y))=u(y)$. Hence (2.2) can be written as

$$\begin{aligned}&\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}+\frac{|u(\sigma_H(x))-u(y)|^p}{|\sigma_H(x)-y|^{d+sp}}+\frac{|u(x)-u(\sigma_H(y))|^p}{|x-\sigma_H(y)|^{d+sp}}+\frac{|u(\sigma_H(x))-u(\sigma_H(y))|^p}{|\sigma_H(x)-\sigma_H(y)|^{d+sp}}\\&\geq\frac{|u(x)-u(\sigma_H(y))|^p}{|x-y|^{d+sp}}+\frac{|u(\sigma_H(x))-u(\sigma_H(y))|^p}{|\sigma_H(x)-y|^{d+sp}}+\frac{|u(x)-u(y)|^p}{|x-\sigma_H(y)|^{d+sp}}+\frac{|u(\sigma_H(x))-u(y)|^p}{|\sigma_H(x)-\sigma_H(y)|^{d+sp}}.\end{aligned}\tag{2.4}$$

In view of (2.3), the above inequality holds provided

$$|u(x)-u(y)|^p-|u(\sigma_H(x))-u(y)|^p-|u(x)-u(\sigma_H(y))|^p+|u(\sigma_H(x))-u(\sigma_H(y))|^p\geq0.\tag{2.5}$$

This inequality is a direct consequence from [15, Lemma 1].
Case 4. Let $x,y\in H$ be such that $u(x)\geq u(\sigma_H(x))$, and $u(y)\leq u(\sigma_H(y))$. By interchanging the roles of x and y, we arrive at Case 3; we get the inequality (2.2).

Further, we have $\operatorname{supp}(P_H(u)) \subseteq P_H(\operatorname{supp}(u)) \subseteq P_H(\Omega)$ as u is non-negative, see [9, Section 2] or [2, Proposition 2.14]. Hence, $P_H(u) = 0$ in $\mathbb{R}^d \setminus P_H(\Omega)$. Therefore, we conclude that $P_H(u) \in W_0^{s,p}(P_H(\Omega))$. □

2.2. **Regularity results and strong comparison principle.** In this subsection, we discuss the regularity of the q-eigenfunctions of (1.3) and the strong comparison principle for the fractional p-Laplace operator. For $q \geq p$, Franzina [18, Theorem 1.1] showed that the minimizer of $\lambda_{p,q}^s$ defined over homogeneous fractional Sobolev space is bounded. In the following proposition, we provide an alternate proof of the global L^∞-bound for the weak solutions of (1.3).

Proposition 2.4. *Let $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1, p_s^*)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and $u \in W_0^{s,p}(\Omega)$ is a non-negative weak solution of (1.3). Then the following hold:*

(i) *$u \in L^\infty(\mathbb{R}^d)$, i.e., u is bounded on $\mathbb{R}^d$.*
(ii) *In addition, assume that Ω is of class $\mathcal{C}^{1,1}$. Then $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,s]$.*

Proof. (i) First, we show that $u \in L^\infty(\mathbb{R}^d)$. We only consider $d > sp$ and $q \in (p, p_s^*)$. The proof follows similar arguments for $d \leq sp$ and $q \in (p,\infty)$. For $q < p$, using [7, Theorem 4.1], we get $u \in L^\infty(\mathbb{R}^d)$. Set $M \geq 0$ and $\sigma \geq 1$. Define $u_M = \min\{u, M\}$ and $\phi = u_M^\sigma$. Clearly, $u_M, \phi \in L^\infty(\Omega) \cap W_0^{s,p}(\Omega)$. Choosing ϕ as a test function in the weak formulation of u, we arrive at

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}}(\phi(x)-\phi(y))\,\mathrm{d}y\,\mathrm{d}x = \lambda \int_\Omega u(x)^{q-1}\phi(x)\,\mathrm{d}x. \tag{2.6}$$

We use the following inequality given in [12, Lemma C.2]:

$$|a-b|^{p-2}(a-b)(a_M^\sigma - b_M^\sigma) \geq \frac{\sigma p^p}{(\sigma+p-1)^p}\left|a_M^{\frac{\sigma+p-1}{p}} - b_M^{\frac{\sigma+p-1}{p}}\right|^p,$$

where $a, b \geq 0$, $a_M = \min\{a, M\}$, and $b_M = \min\{b, M\}$; and the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{p_s^*}(\mathbb{R}^d)$ to estimate the left-hand side of the identity (2.6) as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}}(\phi(x)-\phi(y))\,\mathrm{d}y\,\mathrm{d}x \\
&\geq \frac{\sigma p^p}{(\sigma+p-1)^p}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\left|u_M(x)^{\frac{\sigma+p-1}{p}} - u_M(y)^{\frac{\sigma+p-1}{p}}\right|^p}{|x-y|^{d+sp}}\,\mathrm{d}x \\
&\geq \frac{C(d,s,p)\sigma p^p}{(\sigma+p-1)^p}\left(\int_{\mathbb{R}^d}\left(u_M(x)^{\frac{\sigma+p-1}{p}}\right)^{p_s^*}\,\mathrm{d}x\right)^{\frac{p}{p_s^*}}.
\end{aligned}$$

Since M is arbitrary, the monotone convergence theorem yields

$$\frac{C(d,s,p)\sigma p^p}{(\sigma+p-1)^p}\left(\int_{\mathbb{R}^d}\left(u(x)^{\frac{\sigma+p-1}{p}}\right)^{p_s^*}\,\mathrm{d}x\right)^{\frac{p}{p_s^*}} \leq \lambda\int_\Omega u(x)^{\sigma+q-1}\,\mathrm{d}x. \tag{2.7}$$

Step 1. This step shows that $u^{\sigma_1+p-1} \in L^{\frac{p_s^*}{p}}(\mathbb{R}^d)$ for $\sigma_1 := p_s^* - p + 1$. For $\sigma = \sigma_1$, (2.7) takes the following form:

$$\frac{C(d,s,p)\sigma p^p}{(p_s^*)^p}\left(\int_{\mathbb{R}^d} u(x)^{\frac{p_s^*}{p}p_s^*}\,\mathrm{d}x\right)^{\frac{p}{p_s^*}} \leq \lambda\int_\Omega u(x)^{\sigma_1+q-1}\,\mathrm{d}x. \tag{2.8}$$

We consider $A := \{x \in \Omega : u(x) \leq R\}$ and $A^c = \Omega \setminus A$ for $R > 1$ to be chosen. For $q > p$, we write $\sigma_1 + q - 1 = p_s^* + q - p$. Applying the Hölder's inequality with the conjugate pair $\left(\frac{p_s^*}{p}, \frac{p_s^*}{p_s^*-p}\right)$, the right-hand side of (2.8) can be estimated as:

$$\begin{aligned}
\int_\Omega u(x)^{\sigma_1+q-1}\,\mathrm{d}x &= \left(\int_A + \int_{A^c}\right) u(x)^{\sigma_1+q-1}\,\mathrm{d}x \\
&\leq R^{\sigma_1+q-1}|A| + \left(\int_\Omega u(x)^{\frac{p_s^*}{p}p_s^*}\,\mathrm{d}x\right)^{\frac{p}{p_s^*}}\left(\int_{A^c} u(x)^{\frac{(q-p)p_s^*}{p_s^*-p}}\,\mathrm{d}x\right)^{\frac{p_s^*-p}{p_s^*}}.
\end{aligned} \tag{2.9}$$

Observe that $\frac{(q-p)p_s^*}{p_s^*-p} < p_s^*$ since $q < p_s^*$. Then, using the Hölder's inequality with conjugate pair $\left(\frac{p_s^*-p}{q-p}, \frac{p_s^*-p}{p_s^*-q}\right)$,

$$\int_{A^c} u(x)^{\frac{(q-p)p_s^*}{p_s^*-p}} \, dx \leq |A^c|^{\frac{p_s^*-q}{p_s^*-p}} \left(\int_{A^c} u(x)^{p_s^*} \, dx \right)^{\frac{q-p}{p_s^*-p}}. \tag{2.10}$$

Now, we choose R sufficiently large so that

$$\frac{(p_s^*)^p}{C(d,s,p)\sigma p^p} \lambda |A^c|^{\frac{p_s^*-q}{p_s^*}} \left(\int_\Omega u(x)^{p_s^*} \, dx \right)^{\frac{q-p}{p_s^*}} < \frac{1}{2}.$$

Therefore, from (2.8), (2.9), and (2.10) we obtain

$$\frac{1}{2} \left(\int_{\mathbb{R}^d} u(x)^{\frac{p_s^*}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \leq \frac{(p_s^*)^p}{C(d,s,p)\sigma p^p} \lambda R^{\sigma_1+q-1} |A|.$$

Thus $u^{\sigma_1+p-1} \in L^{\frac{p_s^*}{p}}(\mathbb{R}^d)$.

Step 2. This step contains the L^∞-bound for u. Set $a = \frac{\sigma+p_s^*-1}{\sigma+q-1}$. Clearly $a > 1$, since $q < p_s^*$. Applying Young's inequality with the conjugate pair (a, a'),

$$u(x)^{\sigma+q-1} \leq \frac{u(x)^{\sigma+p_s^*-1}}{a} + \frac{1}{a'} \leq u(x)^{\sigma+p_s^*-1} + 1.$$

Moreover, $\sigma + p - 1 \leq \sigma p$. Hence, from (2.7) there exists $C = C(\lambda, d, s, p, \Omega)$ such that

$$\left(\int_{\mathbb{R}^d} u(x)^{\frac{\sigma+p-1}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \leq C \left(\frac{\sigma+p-1}{p} \right)^{p-1} \left(1 + \int_\Omega u(x)^{\sigma+p_s^*-1} \, dx \right).$$

From the above inequality, we get

$$\begin{aligned}
\left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{\sigma+p-1}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*}} &\leq 1 + \left(\int_{\mathbb{R}^d} u(x)^{\frac{\sigma+p-1}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \\
&\leq 1 + C \left(\frac{\sigma+p-1}{p} \right)^{p-1} \left(1 + \int_\Omega u(x)^{\sigma+p_s^*-1} \, dx \right) \\
&\leq \left(1 + C\,(\sigma+p-1)^{p-1} \right) \left(1 + \int_\Omega u(x)^{\sigma+p_s^*-1} \, dx \right) \\
&\leq \widetilde{C}\,(\sigma+p-1)^{p-1} \left(1 + \int_\Omega u(x)^{\sigma+p_s^*-1} \, dx \right),
\end{aligned}$$

where $\widetilde{C} = \widetilde{C}(\lambda, d, s, p, \Omega) = C + (p-1)^{1-p}$. Set $\vartheta = \sigma + p - 1$. Then, the above inequality yields

$$\left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{\vartheta}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*(\vartheta-p)}} \leq \widetilde{C}^{\frac{1}{\vartheta-p}} \vartheta^{\frac{p-1}{\vartheta-p}} \left(1 + \int_{\mathbb{R}^d} u(x)^{p_s^*+\vartheta-p} \, dx \right)^{\frac{1}{\vartheta-p}}. \tag{2.11}$$

Now, we consider the following sequences (ϑ_n) defined as:

$$\vartheta_1 = p_s^*, \vartheta_2 = p + \frac{p_s^*}{p}(\vartheta_1 - p), \cdots, \vartheta_{n+1} = p + \frac{p_s^*}{p}(\vartheta_n - p).$$

Observe that $p_s^* - p + \vartheta_{n+1} = \frac{p_s^*}{p}\vartheta_n$, and $\vartheta_{n+1} = p + \left(\frac{p_s^*}{p}\right)^n (\vartheta_1 - p)$. Since $p_s^* > p$, $\vartheta_n \to \infty$, as $n \to \infty$. Further, from (2.11),

$$\left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{\vartheta_{n+1}}{p} p_s^*} \, dx \right)^{\frac{p}{p_s^*(\vartheta_{n+1}-p)}} \leq \widetilde{C}^{\frac{1}{\vartheta_{n+1}-p}} \vartheta_{n+1}^{\frac{p-1}{\vartheta_{n+1}-p}} \left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{p_s^*}{p}\vartheta_n} \, dx \right)^{\frac{p}{p_s^*(\vartheta_n-p)}}. \tag{2.12}$$

Set $D_n := \left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{p_s^*}{p}\vartheta_n} \, dx \right)^{\frac{p}{p_s^*(\vartheta_n-p)}}$. We iterate (2.12) up to $(n+1)$-th step to get

$$D_{n+1} \leq \widetilde{C}^{\sum_{k=2}^{n+1} \frac{1}{\vartheta_k - p}} \left(\prod_{k=2}^{n+1} \vartheta_k^{\frac{1}{\vartheta_k-p}} \right)^{p-1} D_1. \tag{2.13}$$

Now $D_1 = \left(1 + \int_{\mathbb{R}^d} u(x)^{\frac{p_s^*}{p} p_s^*} \, \mathrm{d}x\right)^{\frac{p}{p_s^*(p_s^*-p)}} \leq C$ (by Step 1). Hence, from (2.13), we have

$$\|u\|_{L^{\frac{p_s^* \vartheta_{n+1}}{p}}(\mathbb{R}^d)}^{\frac{\vartheta_{n+1}}{\vartheta_{n+1}-p}} \leq \widetilde{C}^{\sum_{k=2}^{n+1} \frac{1}{\vartheta_k - p}} \left(\prod_{k=2}^{n+1} \vartheta_k^{\frac{1}{\vartheta_k - p}} \right)^{p-1} C. \tag{2.14}$$

Moreover,

$$\sum_{k=2}^{\infty} \frac{1}{\vartheta_k - p} = \frac{d}{sp(p_s^* - p)} \text{ and } \prod_{k=2}^{\infty} \vartheta_k^{\frac{1}{\vartheta_k - p}} \leq C(d, s, p).$$

Therefore, taking the limit as $n \to \infty$ in (2.14), we conclude $u \in L^{\infty}(\mathbb{R}^d)$.
(ii) Since $u \in L^{\infty}(\mathbb{R}^d)$ and Ω is of class $\mathcal{C}^{1,1}$, we apply [21, Theorem 1.1] to get $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s]$. □

Remark 2.5. For any non-negative weak solution $u \in W_0^{s,p}(\Omega)$ to (1.3), Proposition 2.4 infers that $u \in L^{\infty}(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$.

For $p \in (1, \infty)$, $s \in (0, 1)$, and an open set $\Omega \subseteq \mathbb{R}^d$ the Sobolev space $\widetilde{W}^{s,p}(\Omega)$ is defined as

$$\widetilde{W}^{s,p}(\Omega) = \left\{ u \in L^p_{loc}(\mathbb{R}^d) : \int_{\Omega} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, \mathrm{d}y \, \mathrm{d}x < \infty \right\}.$$

We remark that $W_0^{s,p}(\Omega) \subseteq \widetilde{W}^{s,p}(\Omega)$. Consider the following fractional equation

$$(-\Delta_p)^s u_1 - (-\Delta_p)^s u_2 \geq 0. \tag{2.15}$$

We say a pair (u_1, u_2) with $u_1, u_2 \in \widetilde{W}^{s,p}(\Omega)$ solves (2.15) weakly in Ω, if for any $\phi \in \mathcal{C}_c^{\infty}(\Omega)$ with $\phi \geq 0$ the following inequality holds

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y))}{|x - y|^{d+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}y \, \mathrm{d}x$$
$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y))}{|x - y|^{d+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}y \, \mathrm{d}x \geq 0.$$

Since $\mathcal{C}_c^{\infty}(\Omega') \subseteq \mathcal{C}_c^{\infty}(\Omega)$ for any $\Omega' \subseteq \Omega$, notice that the pair (u_1, u_2) solves (2.15) weakly in Ω' also if (u_1, u_2) solves (2.15) weakly in Ω. In the literature, some authors obtained a version of the strong maximum principle for fractional p-Laplace operator on the upper half-space involving antisymmetric functions. For example, by Jarohs-Weth [22, Proposition 3.6] for $p = 2$; and by Chen-Li [14, Theorem 2] for $p \neq 2$ assuming $\mathcal{C}^{1,1}_{loc}$-regularity on the functions involved. Motivated by their results, we prove a variant of the strong comparison principle involving the antisymmetric function $w := P_H(u) - u$ without $\mathcal{C}^{1,1}_{loc}$-regularity on $u \in W_0^{s,p}(\Omega)$ for $H \in \mathcal{H}$. For any set $A \subseteq \mathbb{R}^d$, $|A|$ denotes the measure of A.

Proposition 2.6 (Strong Comparison Principle)**.** *Let $H \in \mathcal{H}$, $\Omega \subset \mathbb{R}^d$ be an open set, $p \in (1, \infty)$, $s \in (0, 1)$, and $u \in W_0^{s,p}(\Omega)$ be non-negative. Assume that $P_H u$ and u satisfy the following equation weakly:*

$$(-\Delta_p)^s P_H u - (-\Delta_p)^s u \geq 0 \text{ in } \Omega \cap H. \tag{2.16}$$

Consider the following sets in $\Omega \cap H$:

$$\mathcal{A} := \{x \in \Omega \cap H : P_H u(x) = u(x)\}; \; \mathcal{B} := \{x \in \Omega \cap H : P_H u(x) > u(x)\}.$$

If $|\mathcal{B}| > 0$, then $\mathcal{A}$ has an empty interior.

Proof. Firstly, we denote $v = P_H u$ in $\mathbb{R}^d$ and $G(t) = |t|^{p-2}t$ with $G'(t) = (p-1)|t|^{p-2} \geq 0$ for $t \in \mathbb{R}$. On the contrary, we assume that the interior of $\mathcal{A}$ is nonempty. We consider a test function $\phi \in \mathcal{C}_c^{\infty}(\mathcal{A})$ with $\phi > 0$ on $K := \mathrm{supp}(\phi)$ where $|K| > 0$. From (2.16) we have

$$I := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{G(v(x) - v(y)) - G(u(x) - u(y))}{|x - y|^{d+sp}} (\phi(x) - \phi(y)) \, \mathrm{d}y \, \mathrm{d}x \geq 0. \tag{2.17}$$

For brevity, we denote the reflection point $\sigma_H(z)$ by $\overline{z}$ for a point $z \in \mathbb{R}^d$ and $H \in \mathcal{H}$. Using $\mathbb{R}^d = H \cup H^{\mathsf{c}}$ and $\overline{y} = \sigma_H(y)$, we write the inner integral of (2.17) as an integral over H as

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} (\phi(x)-\phi(y)) \,\mathrm{d}y \\
&= \left[\int_H + \int_{H^{\mathsf{c}}}\right] \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} (\phi(x)-\phi(y)) \,\mathrm{d}y \\
&= \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} \phi(x) \,\mathrm{d}y - \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} \phi(y) \,\mathrm{d}y \\
&\quad + \int_H \frac{G(v(x)-v(\overline{y})) - G(u(x)-u(\overline{y}))}{|x-\overline{y}|^{d+sp}} (\phi(x)-\phi(\overline{y})) \,\mathrm{d}y \\
&= \int_H \left(\frac{1}{|x-y|^{d+sp}} - \frac{1}{|x-\overline{y}|^{d+sp}}\right) \big(G(v(x)-v(y)) - G(u(x)-u(y))\big)\phi(x) \,\mathrm{d}y \\
&\quad - \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} \phi(y) \,\mathrm{d}y \\
&\quad + \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y)) + G(v(x)-v(\overline{y})) - G(u(x)-u(\overline{y}))}{|x-\overline{y}|^{d+sp}} \phi(x) \,\mathrm{d}y,
\end{aligned}$$

where we have used $\phi(\overline{y}) = 0$ for $y \in H$. We express the integral $I = I_1 - I_2 + I_3$, where I_1, I_2, and I_3 are given by

$$\begin{aligned}
I_1 &= \int_H \int_H \left(\frac{1}{|x-y|^{d+sp}} - \frac{1}{|x-\overline{y}|^{d+sp}}\right) \big(G(v(x)-v(y)) - G(u(x)-u(y))\big)\phi(x) \,\mathrm{d}y \,\mathrm{d}x, \\
I_2 &= \int_{\mathbb{R}^d} \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} \phi(y) \,\mathrm{d}y \,\mathrm{d}x, \\
I_3 &= \int_H \int_H \frac{G(v(x)-v(y)) - G(u(x)-u(y)) + G(v(x)-v(\overline{y})) - G(u(x)-u(\overline{y}))}{|x-\overline{y}|^{d+sp}} \phi(x) \,\mathrm{d}y \,\mathrm{d}x,
\end{aligned}$$

where in the integrals I_1 and I_3 we again used the fact that the support of ϕ lies inside H. Now, we show that $I_3 = 0$ and $I_2 \geq 0$. Using $\mathbb{R}^d = H \cup H^{\mathsf{c}}$, $\overline{x} = \sigma_H(x)$, and $v(y) = u(y)$ for $y \in K = \operatorname{supp}(\phi)$, we rewrite I_2 as

$$\begin{aligned}
I_2 = \int_H \int_K \frac{G(v(x)-u(y)) - G(u(x)-u(y))}{|x-y|^{d+sp}} \phi(y) \,\mathrm{d}y \,\mathrm{d}x \\
+ \int_H \int_K \frac{G(v(\overline{x})-u(y)) - G(u(\overline{x})-u(y))}{|\overline{x}-y|^{d+sp}} \phi(y) \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned} \tag{2.18}$$

By the definition, either $v(x) = u(x)$ or $v(x) = u(\overline{x})$ for $x \in \mathbb{R}^d$. Consider the sets $K_1 = \{x \in H : v(x) = u(\overline{x})\}$ and $K_2 = \{x \in H : v(x) = u(x)\}$. Notice that both the terms in (2.18) involving $G(\cdot)$ are zero on the set K_2. Hence, from (2.18), we get

$$I_2 = \int_{K_1} \int_K \Big(G(v(x)-u(y)) - G(u(x)-u(y))\Big) \left(\frac{1}{|x-y|^{d+sp}} - \frac{1}{|\overline{x}-y|^{d+sp}}\right) \phi(y) \,\mathrm{d}y \,\mathrm{d}x.$$

Observe that $|x-y| < |\overline{x}-y|$ for any $x, y \in H$, and $\phi \geq 0$ in H. Moreover, since $G(\cdot)$ is increasing and $v(x) \geq u(x)$ for $x \in H$, we get $G(v(x)-u(y)) - G(u(x)-u(y)) \geq 0$ for $x \in H$ and $y \in K$. Thus, we conclude that $I_2 \geq 0$. Similarly, by replacing $v(x) = u(x)$ on K we rewrite I_3 as

$$I_3 = \int_K \int_H \frac{G(u(x)-v(y)) - G(u(x)-u(y)) + G(u(x)-v(\overline{y})) - G(u(x)-u(\overline{y}))}{|x-\overline{y}|^{d+sp}} \phi(x) \,\mathrm{d}y \,\mathrm{d}x.$$

Now, the term in the numerator involving $G(\cdot)$ is zero for $x \in K$ and $y \in H = K_1 \cup K_2$. Therefore, we get $I_3 = 0$. Thus, (2.17) implies that

$$I_1 = I + I_2 \geq 0. \tag{2.19}$$

On the other hand, for a.e. $x \in K$ and $y \in H$, since $v(x) - u(x) = 0$ for $x \in K$, we get

$$(v(x)-v(y)) - (u(x)-u(y)) = u(y) - v(y) \begin{cases} < 0, & \text{when } y \in \mathcal{B}; \\ \leq 0, & \text{when } y \in H \setminus \mathcal{B}. \end{cases}$$

By the monotonicity of $G(\cdot)$: for a.e. $x \in K$ and $y \in H$,

$$G(v(x)-v(y)) - G(u(x)-u(y)) \begin{cases} <0, & \text{when } y \in \mathcal{B}; \\ \leq 0, & \text{when } y \in H \setminus \mathcal{B}. \end{cases}$$

Further, we have $\phi > 0$ on K and $|x-y| < |x-\overline{y}|$ for any $x, y \in H$. Therefore,

$$\begin{aligned} I_1 &= \int_K \int_H \left(\frac{1}{|x-y|^{d+sp}} - \frac{1}{|x-\overline{y}|^{d+sp}} \right) \left[G(v(x)-v(y)) - G(u(x)-u(y)) \right] \phi(x) \, \mathrm{d}y \, \mathrm{d}x \\ &= \left[\int_K \int_{\mathcal{B}} + \int_K \int_{H\setminus\mathcal{B}} \right] \left(\frac{1}{|x-y|^{d+sp}} - \frac{1}{|x-\overline{y}|^{d+sp}} \right) \left[G(v(x)-v(y)) - G(u(x)-u(y)) \right] \phi(x) \, \mathrm{d}y \, \mathrm{d}x \\ &:= I_{1,1} + I_{1,2} < 0, \end{aligned}$$

where the last strict inequality follows from the facts that $I_{1,1} < 0$ (since $|\mathcal{B}| > 0$ and $|K| > 0$), and $I_{1,2} \leq 0$. Thus, $I_1 < 0$, a contradiction to (2.19). Therefore, the set $\mathcal{A}$ can not have a nonempty interior. This concludes the proof. $\square$

3. Proofs for the main results

This section establishes a strict Faber-Krahn type inequality under polarization for $\lambda^s_{p,q}$. Consequently, we obtain the strict monotonicity of $\lambda^s_{p,q}$ over annular domains. Afterward, we state a remark about the strict monotonicity of $\lambda^s_{p,q}$ over other classes of Lipschitz domains.

Theorem 3.1. *Let $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1, p^*_s)$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^{1,1}$, and $H \in \mathcal{H}$ be a polarizer. Then*

$$\lambda^s_{p,q}\left(P_H(\Omega)\right) \leq \lambda^s_{p,q}(\Omega). \tag{3.1}$$

Further, if $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$ then

$$\lambda^s_{p,q}\left(P_H(\Omega)\right) < \lambda^s_{p,q}(\Omega). \tag{3.2}$$

Proof. Let $u \in W^{s,p}_0(\Omega)$ be a non-negative q-eigenfunction associated with $\lambda^s_{p,q}(\Omega)$. By Proposition 2.3 and the variational characterization of $\lambda^s_{p,q}$, we get that $P_H(u) \in W^{s,p}_0(P_H(\Omega))$ with $\|P_H(u)\|_{L^q(P_H(\Omega))} = \|u\|_{L^q(\Omega)} = 1$, and

$$\lambda^s_{p,q}(P_H(\Omega)) \leq [P_H(u)]_{s,p} \leq [u]_{s,p} = \lambda^s_{p,q}(\Omega). \tag{3.3}$$

Now, we prove the strict inequality (3.2). From Proposition 2.4, $u \in \mathcal{C}(\overline{\Omega})$ and hence $u, P_H(u) \in \mathcal{C}(\mathbb{R}^d)$. We consider the following sets

$$\mathcal{M}_u = \Big\{ x \in P_H(\Omega) \cap H : P_H u(x) > u(x) \Big\} \text{ and } \mathcal{N}_u = \Big\{ x \in P_H(\Omega) \cap H : P_H u(x) = u(x) \Big\}.$$

Observe that $\mathcal{M}_u$ is open and $\mathcal{N}_u$ is relatively closed in $P_H(\Omega) \cap H$. First, we find a set B in the open set $\Omega \cap H$ such that $B \cap \mathcal{M}_u \neq \emptyset$. Since $\Omega \neq P_H(\Omega) \neq \sigma_H(\Omega)$, from Proposition 2.2-(v) both $A_H(\Omega) = \sigma_H(\Omega) \cap \Omega^{\mathrm{c}} \cap H$ and $B_H(\Omega) = \Omega \cap \sigma_H(\Omega^{\mathrm{c}}) \cap H$ have non-empty interiors. Further, $u > 0$ in Ω. By the definition, we have $P_H u \geq u$ in $P_H(\Omega) \cap H$, and

$$\begin{aligned} &\text{in } A_H(\Omega): \quad u = 0, \ u \circ \sigma_H > 0, \text{ and hence } P_H u = u \circ \sigma_H > u; \\ &\text{in } B_H(\Omega): \quad u > 0, \ u \circ \sigma_H = 0, \text{ and hence } P_H u = u. \end{aligned} \tag{3.4}$$

Therefore, using (3.4), $P_H(\Omega) \cap H = \mathcal{M}_u \sqcup \mathcal{N}_u$ with $\mathcal{M}_u \supseteq A_H(\Omega)$, and $\mathcal{N}_u \supseteq B_H(\Omega)$, here $A_H(\Omega)$, $B_H(\Omega)$ have non-empty interiors. Also, by the definition of $A_H(\Omega)$, we get $P_H(\Omega) \cap H = (\Omega \cap H) \sqcup A_H(\Omega)$. Thus, we have

$$P_H(\Omega) \cap H = \mathcal{M}_u \sqcup \mathcal{N}_u = (\Omega \cap H) \sqcup A_H(\Omega) \text{ with } \mathcal{M}_u \supseteq A_H(\Omega).$$

Therefore, since $\mathcal{N}_u$ is relative closed in $P_H(\Omega) \cap H$ and $\Omega \cap H$ is an open set, we get $\mathcal{N}_u \subsetneq \Omega \cap H$, and hence the set $B := \mathcal{M}_u \cap \Omega \cap H$ is a non-empty open set. Therefore, the sets $\mathcal{N}_u, B \subset \Omega \cap H$ have the following properties:

$$P_H u > u \text{ in } B \text{ and } P_H u \equiv u \text{ in } \mathcal{N}_u. \tag{3.5}$$

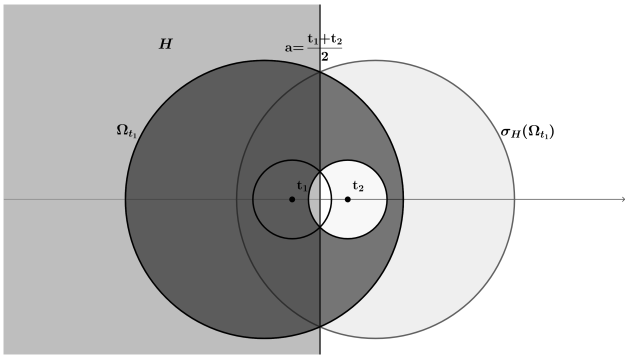

FIGURE 2. Polarization of Ω_{t_1} with respect to the polarizer $H_a = \{x \in \mathbb{R}^d : x_1 < a\}$. The dark-shaded region is $P_H(\Omega_{t_1}) = \Omega_{t_2}$.

On the contrary to (3.2), assume that $\lambda^s_{p,q}(P_H(\Omega)) = \lambda^s_{p,q}(\Omega) = \lambda$. Now, the equality holds in (3.3), and hence $P_H(u)$ becomes a non-negative minimizer of the following problem:

$$\lambda^s_{p,q}(P_H(\Omega)) = \min_{v \in W^{s,p}_0(P_H(\Omega))} \left\{ [v]^p_{s,p} : \|v\|_{L^q(P_H(\Omega))} = 1 \right\}.$$

As a consequence, the following equation holds weakly:

$$(-\Delta_p)^s P_H(u) = \lambda^s_{p,q}(P_H(\Omega))|P_H(u)|^{q-2}P_H(u) \text{ in } P_H(\Omega), \qquad P_H(u) = 0 \text{ in } \mathbb{R}^d \setminus P_H(\Omega). \quad (3.6)$$

Since $\Omega \cap H \subsetneq P_H(\Omega) \cap H$, both u and $P_H u$ are weak solutions to the equation:

$$(-\Delta_p)^s \widetilde{w} - \lambda |\widetilde{w}|^{q-2}\widetilde{w} = 0 \text{ in } \Omega \cap H,$$

where $\lambda = \lambda^s_{p,q}(P_H(\Omega))$. Using $P_H u \geq u$ in $\Omega \cap H$, we see that the following equation holds weakly:

$$(-\Delta_p)^s P_H(u) - (-\Delta_p)^s u = \lambda(|P_H(u)|^{q-2}P_H(u) - |u|^{q-2}u) \geq 0 \text{ in } \Omega \cap H.$$

Now $\mathcal{N}_u \subseteq \mathcal{A}$ with $\mathcal{N}_u$ having a non-empty interior. Moreover, since $B \subseteq \mathcal{B}$ is open, we have $|\mathcal{B}| > 0$. Therefore, applying Proposition 2.6, we get a contradiction to the statement (3.5). Thus, the strict inequality (3.2) holds. □

For $0 < r < R < \infty$ and $0 \leq t < R - r$, recall that the annular domain $\Omega_t \in \mathcal{A}_{r,R}$ is given by

$$\Omega_t = B_R(0) \setminus \overline{B}_r(te_1), \quad (3.7)$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d$, $B_r(a) = \{x \in \mathbb{R}^d : |x - a| < r\}$ is the open ball centered at $z \in \mathbb{R}^d$ with radius $r \geq 0$, and $\overline{B}_r(a)$ is the closure of $B_r(a)$ in $\mathbb{R}^d$. We are interested in studying the monotonicity of $\lambda^s_{p,q}(\Omega_t)$ for $0 \leq t < R - r$. For this, we consider the following family of polarizers

$$H_a := \left\{ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 < a \right\} \text{ for } a \in \mathbb{R}. \quad (3.8)$$

For $a \in \mathbb{R}$, we denote the reflection with respect to ∂H_a by $\sigma_a = \sigma_{H_a}$, and the polarizations in $\mathbb{R}^d$ with respect to H_a by $P_a = P_{H_a}$ and $P^a = P^{H_a}$. Recall, from (2.1), that the reflection of a point $x = (x_1, x') \in \mathbb{R}^d$ is given by $\sigma_a(x) = (2a - x_1, x')$. In the following proposition, we demonstrate the effect of polarization on balls and annular domains.

Proposition 3.2. *Let $0 < r < R$, $0 \leq t < R - r$, $a \in \mathbb{R}$, and $z = (z_1, z') \in \mathbb{R}^d$. Then:*
- (i) *$P_a(B_r(z)) = B_r(z)$ if $z_1 \leq a$, and $P_a(B_r(z)) = B_r(\sigma_a(z))$ if $z_1 \geq a$;*
- (ii) *if $0 \leq a < \frac{R-r+t}{2}$ then $\sigma_a(B_r(te_1)) \subset B_R(0)$;*
- (iii) *$P_a(\Omega_t) = \Omega_t$ for $0 \leq a \leq t$; and $P_a(\Omega_t) = \Omega_{2a-t}$ for $t \leq a < \frac{R-r+t}{2}$.*

Proof. Recall that $P_a(B_r(z)) = [(B_r(z) \cup B_r(\sigma_a(z))) \cap H_a] \cup [B_r(z) \cap B_r(\sigma_a(z))]$.
(i) To prove $P_a(B_r(z)) = B_r(z)$ for $z_1 \leq a$, from Proposition 2.2-(iv), it is enough to show $B_r(\sigma_a(z)) \cap H_a \subseteq B_r(z)$. For $x \in B_r(\sigma_a(z)) \cap H_a$, we have $|\sigma_a(z) - x| < r$ and $x_1 < a$. Since $x_1 < 2a - x_1$ and $z_1 \leq 2a - z_1$, we get $|z_1 - x_1| < |(2a - z_1) - x_1|$. Then $|z - x| \leq |\sigma_a(z) - x| < r$, and thus $B_r(\sigma_a(z)) \cap H_a \subseteq B_r(z)$. For $z_1 \geq a$, we consider P^a instead of P_a to get $P^a(B_r(z)) = B_r(z)$. Therefore, $P_a(B_r(z)) = \sigma_a(B_r(z)) = B_r(\sigma_a(z))$.
(ii) Let $x \in \sigma_a(B_r(te_1)) = B_r((2a - t)e_1)$. Since $0 \leq a < \frac{R-r+t}{2}$, we get $-(R - r) < -t \leq 2a - t <$

$R-r$. Now $|x| \le |x-(2a-t)e_1| + |(2a-t)e_1| < r+R-r = R$. Hence $x \in B_R(0)$.
(iii) Let $0 \le a < \frac{R-r+t}{2}$. From (ii) we have $\sigma_a(B_r(te_1)) \subset B_R(0)$. Now, using Proposition 2.2-(v),

$$P_a(\Omega_t) = P_a\left(B_R(0) \setminus \overline{B}_r(te_1)\right) = P_a\left(B_R(0)\right) \setminus P^a\left(\overline{B}_r(te_1)\right). \tag{3.9}$$

From (i), $P_a(B_R(0)) = B_R(0)$, $P^a\left(\overline{B}_r(te_1)\right) = \overline{B}_r(te_1)$ if $a \le t$; and $P^a\left(\overline{B}_r(te_1)\right) = \overline{B}_r((2a-t)e_1)$ if $a \ge t$. Therefore $P_a(\Omega_t) = \Omega_t$ for $0 \le a \le t$, and $P_a(\Omega_t) = B_R(0) \setminus \overline{B}_r((2a-t)e_1) = \Omega_{2a-t}$ for $t \le a < \frac{R-r+t}{2}$. ☐

Theorem 3.3. *Let $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1,p_s^*)$. Then $\lambda^s_{p,q}(\Omega_{t_2}) < \lambda^s_{p,q}(\Omega_{t_1})$ for any $0 \le t_1 < t_2 < R-r$.*

Proof. Applying Proposition 3.2-(iii) with $a = \frac{t_1+t_2}{2}$, we get $P_a(\Omega_{t_1}) = \Omega_{t_2}$. Since $\Omega_{t_1} \ne \Omega_{t_2} \ne \sigma_a(\Omega_{t_1})$, from (3.2) of Theorem 3.1 we obtain that $\lambda^s_{p,q}(\Omega_{t_2}) = \lambda^s_{p,q}(P_a(\Omega_{t_1})) < \lambda^s_{p,q}(\Omega_{t_1})$. ☐

Some monotonicity results for general domains. Our Theorem 3.1 can be applied to a broader range of $\mathcal{C}^{1,1}$-smooth domains, as discussed in [2]. Specifically, it can be used for domains of the form $\Omega \setminus \mathcal{O} \subset \mathbb{R}^d$, where $\mathcal{O} \subset\subset \Omega$ is a *'hole.'* This application yields the monotonicity of $\lambda^s_{p,q}(\Omega \setminus \mathcal{O})$ for certain rotations and translations of $\mathcal{O}$ within Ω. In particular, for the class of domains that are the difference of Steiner symmetric or foliated Schwarz symmetric sets in $\mathbb{R}^d$. Examples of Steiner symmetric planar domains are regular n-gons and ellipses. Foliated Schwarz symmetric domains in $\mathbb{R}^2$ include rhombuses, disks, eccentric annular domains, and cones.
(a) Translations in a given direction. We recall the following characterization of Steiner symmetric sets in $\mathbb{R}^d$ in terms of polarization, see for example [9, Lemma 2.2] or [2, Proposition 2.10-(i)]: *A set $\Omega \subseteq \mathbb{R}^d$ is Steiner symmetric with respect to an affine-hyperplane ∂H_{a_0}, $a_0 \in \mathbb{R}$ if and only if $P_a(\Omega) = \Omega$ for any $a \ge a_0$, and $P^a(\Omega) = \Omega$ for any $a \le a_0$.* Assume that both $\Omega, \mathcal{O}$ are Steiner symmetric with respect to ∂H_0. The translations of $\mathcal{O}$ in the e_1-direction are given by $\mathcal{O}_t := te_1 + \mathcal{O}$ for $t \in \mathbb{R}$. Consider the set $L_{\mathcal{O}}$ of translations of $\mathcal{O}$ in the e_1-direction within Ω as

$$L_{\mathcal{O}} := \left\{t \ge 0 : \mathcal{O}_t \subset \Omega\right\}.$$

Then, from [2, Lemma 4.1 and proof of Theorem 1.5], for $t_1 < t_2$ in $L_{\mathcal{O}}$ we have $P^a(\mathcal{O}_{t_1}) = \mathcal{O}_{t_2}$ with $2a = t_1 + t_2$, in particular, $P_a(\Omega \setminus \mathcal{O}_{t_1}) = \Omega \setminus \mathcal{O}_{t_2}$; and the set $L_{\mathcal{O}}$ is an interval $[0, R_{\mathcal{O}})$, where $R_{\mathcal{O}} = \sup L_{\mathcal{O}}$. Now, the proofs Theorem 3.3 and [2, Theorem 1.5] give the monotonicity of the eigenvalue $\lambda^s_{p,q}(\Omega \setminus \mathcal{O}_t)$ for $0 \le t < R_{\mathcal{O}}$, and it is stated as the following theorem.

Theorem 3.4. *Let $\mathcal{O}, \Omega \subset \mathbb{R}^d$ be two sets such that $\Omega \setminus \mathcal{O}$ is $\mathcal{C}^{1,1}$-smooth bounded domain. Let $p \in (1,\infty)$, $s \in (0,1)$, and $q \in [1,p_s^*)$. If both $\mathcal{O}$ and Ω are Steiner symmetric with respect to ∂H_0, then $L_{\mathcal{O}} = [0, R_{\mathcal{O}})$ an interval, and $\lambda^s_{p,q}(\Omega \setminus \mathcal{O}_t)$ is strictly decreasing for $0 \le t < R_{\mathcal{O}}$.*

(b) Rotations about a given point. We recall a characterization of foliated Schwarz symmetric sets in $\mathbb{R}^d$ in terms of polarizations, see for example [2, Proposition 2.10-(ii)]: *A set $\Omega \subseteq \mathbb{R}^d$ is foliated Schwarz symmetric with respect to a ray $a + \mathbb{R}^+\eta$, for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$, if and only if $P_H(A) = A$ for any $H \in \mathcal{H}_{a,\eta} := \{H \in \mathcal{H} : a + \mathbb{R}^+\eta \subset H$ and $a \in \partial H\}$.* Assume that both $\Omega, \mathcal{O}$ are foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$, $a \in \mathbb{R}^d$, $\eta \in \mathbb{S}^{d-1}$. From [2, Proposition 2.9], both $\mathcal{O}$ and Ω are axial-symmetric with respect to the straight line $a + \mathbb{R}\eta$. The rotations of $\mathcal{O}$ about the point $a \in \mathbb{R}^d$ are given by

$$\mathcal{O}_{\theta,\xi} := a + R_{\theta,\xi}(-a + \mathcal{O}) \text{ for } \theta \in [0,\pi],$$

where $\xi \in \mathbb{S}^{d-1} \setminus \{\eta\}$, and $R_{\theta,\xi}$ is the simple rotation in $\mathbb{R}^d$ with the plane of rotation $X_\xi := \operatorname{span}\{\eta,\xi\}$ and the angle of rotation θ from the ray $\mathbb{R}^+\eta$ in the counter-clockwise direction. By the axial-symmetry of both $\mathcal{O}$ and Ω, and [2, Lemma 4.3], it is enough to consider the rotations of $\mathcal{O}$ in Ω by $R_{\theta,\xi}$ for a fixed $\xi \in \mathbb{S}^{d-1} \setminus \{\eta\}$. Therefore, we set $\mathcal{O}_\theta = \mathcal{O}_{\theta,\xi}$ for $\theta \in [0,\pi]$. Consider the set of rotations of $\mathcal{O}$ about the point $a \in \mathbb{R}^d$ within Ω as

$$C_{\mathcal{O}} := \left\{\theta \in [0,\pi] : \mathcal{O}_\theta \subset \Omega\right\}.$$

Then, from [2, Lemma 4.4 and proof of Theorem 1.8], for any $\theta_1 < \theta_2$ in $C_{\mathcal{O}}$ there exists a polarization $H \in \mathcal{H}_{a,\eta}$ such that $P^H(\mathcal{O}_{\theta_2}) = \mathcal{O}_{\theta_1}$, in particular $P_H(\Omega \setminus \mathcal{O}_{\theta_2}) = \Omega \setminus \mathcal{O}_{\theta_1}$; and the set $C_{\mathcal{O}}$ is an interval. Now, the proofs Theorem 3.3 and [2, Theorem 1.8] give the monotonicity of $\lambda^s_{p,q}(\Omega \setminus \mathcal{O}_\theta)$ for $\theta \in C_{\mathcal{O}}$, and it is stated as the following theorem.

Theorem 3.5. *Let $\mathcal{O}, \Omega \subset \mathbb{R}^d$ be two sets such that $\Omega \setminus \mathcal{O}$ is $\mathcal{C}^{1,1}$-smooth bounded domain. Let $p \in (1, \infty)$, $s \in (0,1)$, and $q \in [1, p_s^*)$. If both $\mathcal{O}$ and Ω are foliated Schwarz symmetric with respect to the ray $a + \mathbb{R}^+\eta$ for some $a \in \mathbb{R}^d$ and $\eta \in \mathbb{S}^{d-1}$, then the set $C_{\mathcal{O}}$ is an interval, and $\lambda_{p,q}^s(\Omega \setminus \mathcal{O}_\theta)$ is strictly increasing for $\theta \in C_{\mathcal{O}}$.*

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