ON THE FIRST EIGENVALUE OF LIOUVILLE-TYPE PROBLEMS

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Abstract. The aim of this note is to study the spectrum of a linearized Liouville-type problem, characterizing the case in which the first eigenvalue is zero. Interestingly enough, we obtain also point-wise information on the associated first eigenfunction. To this end, we refine the Alexandrov-Bol inequality suitable for our problem and characterize its equality case.

Keywords: Liouville-type equations, non-degeneracy, Alexandrov-Bol inequality

1. Introduction

We are concerned with subsolutions of the Liouville equation, that is

$$-\Delta w \leq e^{w} \quad \text{in} \quad \Omega, \tag{1}$$

where $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and we assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded domain. We consider the eigenvalue problem associated to (1), that is,

$$\begin{cases} -\Delta \phi - e^w \phi = \hat{\nu} e^w \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

A lot of work has been done to obtain sufficient conditions to guarantee that the first eigenvalue $\hat{\nu}_1$ of (2) satisfies $\hat{\nu}_1 > 0$, which is in turn related to non-degeneracy and uniqueness properties of the associated Liouville problem, see for example [1, 4, 5, 7, 14]. In particular, it turns out that for $\int_\Omega e^w \, dx \leq 4\pi$ one has $\hat{\nu}_1 \geq 0$. However, at least to our knowledge, we do not have a characterization of the case $\hat{\nu}_1 = 0$. Our aim here is to fill this gap.

Here and in the rest of the paper a multiply connected domain is a connected but not simply connected domain and $B_\delta = \{x \in \mathbb{R}^2 : |x| < \delta\}$, which we will sometime identify with $\{z \in \mathbb{C} : |z| < \delta\}$. Moreover, let us set

$$U_\tau(x) = \ln \left(\frac{\tau}{1 + \frac{\tau^2}{8}|x|^2} \right)^2, \quad \tau > 0, \tag{3}$$

which satisfies

$$\Delta U_\tau + e^{U_\tau} = 0 \text{ in } \mathbb{R}^2, \quad \int\limits_{\mathbb{R}^2} e^{U_\tau} = 8\pi.$$

Then, we have the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain whose boundary is the union of finitely many rectifiable Jordan curves and $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (1). Let $\hat{\nu}_1$ be the first eigenvalue of (2) and assume that $\int_\Omega e^w \, dx \leq 4\pi$.*

(j) If Ω is simply connected then $\hat{\nu}_1 \geq 0$ and $\hat{\nu}_1 = 0$ happens if and only if:

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$(a)_0$ $\int_\Omega e^w\,dx = 4\pi$;
$(a)_1$ *the equality sign holds in* (1) *for any* $x \in \Omega$;
$(a)_2$ *there exists a conformal map* $\Phi : \overline{B_1} \to \overline{\Omega}$ *such that,*

$$e^{w(\Phi(z))}|\Phi'(z)|^2|dz|^2 = e^{U_{\sqrt{8}}(z)}|dz|^2, \quad z \in \overline{B_1};$$

$(a)_3$ *the first eigenfunction* ϕ*, relative to* $\hat{\nu}_1 = 0$*, takes the form* $\phi(x) = \varphi(\Phi^{-1}(x))$*, where* $\varphi(z) = \frac{1-|z|^2}{1+|z|^2}$.

Assume that w *satisfies, for* $c \in \mathbb{R}$,

$$w = c \quad \text{on } \partial\Omega,$$

then $\hat{\nu}_1 = 0$ *happens if and only if in addition to* $(a)_0, (a)_1, (a)_2, (a)_3$ *it holds:*

$(a)_4$ *there exists* $\delta > 0$ *and* $\theta \in \mathbb{R}$ *such that, up to a translation,* $\Phi(z) = \delta e^{i\theta} z$ *and then* $\Omega = B_\delta$ *and*

$$\phi(x) = \varphi(\delta^{-1}x).$$

(jj) If Ω *is a multiply connected domain of class* C^1 *and* $w \in C^1(\overline{\Omega})$ *satisfies, for* $c \in \mathbb{R}$

$$\begin{aligned} w &\geq c \quad \text{in } \Omega, \\ w &= c \quad \text{on } \partial\Omega, \end{aligned}$$

then $\hat{\nu}_1 > 0$.

Remark 1.1. *From a geometric viewpoint, in the case* $\hat{\nu}_1 = 0$, $(\Omega, e^w|dx|^2)$ *must be conformally equivalent to a hemisphere of the sphere of radius* $\sqrt{2}$*, say* $\mathbb{S}_{\sqrt{2}}$*. Actually, when* $w = c$ *on* $\partial\Omega$ *we further show that the conformal map is proportional to the identity. This in turn gives new point-wise information on the associated first eigenfunction. We refer to Remark 2.1 for further geometric interpretations.*

The proof of Theorem 1.1 is based on a refined version of the Alexandrov-Bol inequality suitable to be applied to subsolutions of the Liouville equation (1), possibly on multiply connected domains, and a careful analysis of its equality case. Indeed, such isoperimetric inequality plays a major role in the symmetrization argument needed to estimate the first eigenvalue.

We conclude the introduction with the following remark about the singular counterpart of (1).

Remark 1.2. *We briefly address here what happens in case we consider the more general Liouville problem*

$$-\Delta w \leq h(x)\, e^{w} \ \text{a.e. in } \Omega,$$

where h *is a positive weight, possibly singular. Suppose for the moment* Ω *is simply connected. Following the arguments in* [4]*, it would turn out that if* $\log(h)$ *is subharmonic in* Ω *then we always have* $\hat{\nu}_1 > 0$ *for* $\int_\Omega h(x)e^w\,dx \leq 4\pi$*. On the other hand, if* $\log(h)$ *is superharmonic then we would have a modified sharp threshold* $\int_\Omega h(x)e^w\,dx = 4\pi(1-\alpha)$*, for some* $\alpha > 0$ *depending on* h*, and it should be possible to classify the first eigenfunction, relative to* $\hat{\nu}_1 = 0$*, in terms of* $\varphi_a(z) = \frac{1-|z|^{2(1-\alpha)}}{1+|z|^{2(1-\alpha)}}$*. The case of* Ω *multiply connected is more subtle, as discussed in* [5]*, and has not been studied in full generality. We therefore postpone this discussion to a future work.*

The paper is organized as follows. In section 2 we discuss the Alexandrov-Bol inequality on simply connected domains, characterizing the equality case. The case of multiply connected domains is treated in section 3. Finally, section 4 is devoted to the proof of Theorem 1.1. Part of the proof of the Alexandrov-Bol inequality on multiply connected domains is postponed to the appendix.

2. The Alexandrov-Bol inequality on simply connected domains

In this section we introduce the Alexandrov-Bol inequality on simply connected domains, which was first derived in the analytical framework in [1] and later generalized in [14]. We further refine the argument of [14], giving a full characterization of the equality case.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a open, bounded and simply connected domain whose boundary is a rectifiable Jordan curve and $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of* (1) *which satisfies,*

$$\int_\Omega e^{w} \leq 8\pi.$$

Let $\omega \subseteq \Omega$ be any open subset whose boundary is the union of finitely many rectifiable Jordan curves. Then it holds:

$$\left(\int_{\partial\omega} (e^w)^{\frac{1}{2}}\, d\sigma\right)^2 \geq \frac{1}{2}\left(\int_\omega e^w\, dx\right)\left(8\pi - \int_\omega e^w\, dx\right). \tag{4}$$

Moreover, the equality holds in (4) *if and only if:*

$(i)_1$ *the equality sign holds in* (1) *for any $x \in \omega$;*
$(i)_2$ *ω is simply connected;*
$(i)_3$ *there exists $\tau > 0$ and a conformal map $\Phi : \overline{B_1} \to \overline{\omega}$ such that,*

$$e^{w(\Phi(z))}|\Phi'(z)|^2|dz|^2 = e^{U_\tau(z)}|dz|^2, \quad z \in \overline{B_1}. \tag{5}$$

In particular, if either the inequality in (1) *is not an equality on ω or ω is not simply connected, then the inequality in* (4) *is always strict.*
Assume that w satisfies, for $c \in \mathbb{R}$,

$$w = c \quad \text{on } \partial\omega,$$

then the equality holds in (4) *if and only if, in addition to $(i)_1$-$(i)_2$-$(i)_3$, it holds,*

$(i)_4$ *there exists $\delta > 0$ and $\theta \in \mathbb{R}$ such that, up to a translation, $\Phi(z) = \delta e^{i\theta} z$ and then in particular $\omega = B_\delta$ and $w(x) = U_{\tau\delta^{-1}}(x)$.*

Remark 2.1. *There is a well known geometric meaning behind the result, see* [2] *and references therein. In particular, in the equality case, $(i)_3$ speaks that the abstract surface $(\omega, e^w|dx|^2)$ is conformally equivalent to $(B_1, e^{U_\tau(z)}|dz|^2)$. Actually $e^{U_\tau(z)}|dz|^2$ is the local expression, after stereographic projection, of the standard metric of $\mathbb{S}_{\sqrt{2}}$. Whence, in particular the equality holds if and only if $(\omega, e^w|dx|^2)$ is conformally equivalent to a geodesic disk on $\mathbb{S}_{\sqrt{2}}$. Interestingly enough, our result shows that if $w = c$ on $\partial\omega$, then the equality holds if and only if $(\omega, e^w|dx|^2)$ coincides with the local coordinates expression of a geodesic disk, that is, the conformal map is proportional to the identity in this case.*

Proof of Proposition 2.1.
We first prove the proposition in case $\omega \subseteq \Omega$ is simply connected, whence $\partial\omega$ will be a rectifiable Jordan curve. Let

$$f := -\Delta w - e^{w} \leq 0 \text{ in } \omega,$$

and h_- be the unique solution of $-\Delta h_- = f$ in ω, $h_- = 0$ on $\partial\omega$. Next, let h_0 be the harmonic lifting of w on $\partial\omega$, that is $\Delta h_0 = 0$ in ω, $h_0 = w$ on $\partial\omega$. Since $w \in C^2(\Omega) \cap C^0(\overline{\Omega})$, then, by standard elliptic theory ([10]), h_- and h are unique, h_- is a subharmonic function of class $C^2(\omega) \cap C^0(\overline{\omega})$ and $h_0 \in C^2(\omega) \cap C^0(\overline{\omega})$. To simplify the notations let us set,

$$h = h_0 + h_- \text{ in } \omega.$$

At this point we define $u = w - h$, which satisfies,

$$-\Delta u = e^{h}e^{u} \text{ in } \omega, \quad u = 0 \text{ on } \partial\omega. \tag{6}$$

Clearly $u \in C^2(\omega) \cap C^0(\overline{\omega})$, $u \geq 0$ in ω and we define,

$$\omega(t) = \{x \in \omega \,:\, u > t\}, \quad \gamma(t) = \{x \in \omega \,:\, u = t\}, \quad t \in [0, t_+],$$

where $t_+ = \max\limits_{\overline{\omega}} u$, and

$$m(t) = \int\limits_{\omega(t)} e^h e^u dx, \qquad \mu(t) = \int\limits_{\omega(t)} e^h dx.$$

Since $|\Delta u|$ is bounded and in particular is bounded below away from zero in $\overline{\omega}$, then it is not difficult to see that actually $m(t)$ and $\mu(t)$ are continuous in $[0, t_+]$. Moreover, we notice that, by well-known arguments, the set $\{x \in \omega | \nabla u(x) = 0\} \cap u^{-1}([0, t_+])$ is of measure zero and we can use the co-area formula in ([6]) to deduce that $m(t)$ and $\mu(t)$ are absolutely continuous in $[0, t_+]$. In particular the level sets have vanishing two dimensional area $|\gamma(t)| = 0$ for any t, and we will use the fact that,

$$m(0) = \int\limits_{\omega} e^h e^u dx = \int\limits_{\omega} e^w dx \leq 8\pi, \quad m(t_+) = 0, \quad \mu(t_+) = 0.$$

By the co-area formula and the Sard Lemma we have,

$$-m'(t) = \int\limits_{\gamma(t)} \frac{e^h e^u}{|\nabla u|} d\sigma = e^t \int\limits_{\gamma(t)} \frac{e^h}{|\nabla u|} d\sigma = e^t(-\mu'(t)), \tag{7}$$

for a.a. $t \in [0, t_+]$, and, in view of (6),

$$m(t) = -\int\limits_{\omega(t)} \Delta u = \int\limits_{\gamma(t)} |\nabla u|, \tag{8}$$

for a.a. $t \in [0, t_+]$. By the Schwarz inequality we find that,

$$-m'(t)m(t) = \int\limits_{\gamma(t)} \frac{e^h e^u}{|\nabla u|} d\sigma \int\limits_{\gamma(t)} |\nabla u| d\sigma = e^t \int\limits_{\gamma(t)} \frac{e^h}{|\nabla u|} d\sigma \int\limits_{\gamma(t)} |\nabla u| d\sigma \geq$$

$$e^t \left(\int\limits_{\gamma(t)} e^{h/2} d\sigma \right)^2 \geq 4\pi e^t \mu(t), \text{ for a.a. } t \in [0, t_+], \tag{9}$$

where in the last inequality, since h is subharmonic, we used a generalization of a classical isoperimetric inequality due to Huber ([11]), which was proved in the case of open and simply connected domains. If $\omega(t)$ is multiply connected the inequality is strict. For simplicity, let assume $\omega(t) = \omega_1(t) \setminus \overline{\omega_2(t)}$, with $\omega_2(t) \subset \omega_1(t) \subset \omega$ open and simply connected domains and also $\partial\omega(t) = \partial\omega_1(t) \cup \partial\omega_2(t)$. We notice that h is well-defined and subharmonic in both $\omega_1(t)$ and $\omega_2(t)$. Then we can apply Huber's inequality to both domains:

$$\left(\int\limits_{\partial\omega(t)} e^{h/2} d\sigma \right)^2 > \left(\int\limits_{\partial\omega_1(t)} e^{h/2} d\sigma \right)^2 + \left(\int\limits_{\partial\omega_2(t)} e^{h/2} d\sigma \right)^2 \geq$$

$$\geq 4\pi \left(\int\limits_{\omega_1(t)} e^h\, dx + \int\limits_{\omega_2(t)} e^h\, dx \right) > \int\limits_{\omega(t)} e^h\, dx\, dx$$

The general case follows by induction on the number of "holes". Finally, considering the case in which $\omega(t)$ is not connected, we can assume for simplicity $\omega(t) = \omega_1(t) \cup \omega_2(t)$ with $\omega_1(t), \omega_2(t) \subseteq \omega$ open and connected subsets, $\omega_1(t) \cap \omega_2(t) = \varnothing$ and $\partial\omega(t) = \partial\omega_1(t) \cup \partial\omega_2(t)$, and make the

same calculations as done before.
Therefore we conclude that,

$$\frac{1}{8\pi}(m^2(t))' + e^t\mu(t) \leq 0, \text{ for a.a. } t \in [0, t_+]. \tag{10}$$

In particular, because of (7), we conclude that,

$$\left(\frac{1}{8\pi}m^2(t) - m(t) + e^t\mu(t)\right)' = \frac{1}{8\pi}(m^2(t))' + e^t\mu(t) \leq 0, \text{ for a.a. } t \in [0, t_+].$$

However, as mentioned above, the quantity in the parentheses in the l.h.s. of this inequality is continuous and absolute continuous in $[0, t_+]$, and then we also conclude that,

$$\frac{1}{8\pi}m^2(0) - m(0) + \mu(0) \geq \frac{1}{8\pi}m^2(t_+) - m(t_+) + e^{t_+}\mu(t_+) = 0,$$

that is, by using once more the Huber ([11]) inequality,

$$\left(\int\limits_{\partial\omega} (e^w)^{\frac{1}{2}}\, d\sigma\right)^2 = \left(\int\limits_{\partial\omega} e^{h/2} d\sigma\right)^2 \equiv \left(\int\limits_{\gamma(0)} e^{h/2} d\sigma\right)^2 \geq 4\pi \left(\int\limits_{\omega(0)} e^h dx\right) = \tag{11}$$

$$4\pi\mu(0) \geq \frac{1}{2}(8\pi m(0) - m^2(0)) = \frac{1}{2}\left(8\pi - \int\limits_{\omega} e^w dx\right)\int\limits_{\omega} e^w dx,$$

which is (4). Therefore, (4) holds for ω simply connected and we now characterize the equality sign. We first recall that the equality holds in the Huber ([11]) inequality used in (11) if and only if there exists $\alpha \in \mathbb{R}$ such that $e^{h(\eta)} = e^{-\alpha}|\Psi'(\eta)|^2$, $\eta \in \omega$, where $\Psi : \omega \to B_1$ is univalent and conformal. Since $\partial\omega$ is simple, then in particular by the Carathéodory Theorem ([13]) Ψ is continuous on $\overline{\omega}$ and maps one to one $\partial\omega$ onto ∂B_1. Let $\Phi = \Psi^{-1} : \overline{B_1} \to \overline{\omega}$ and set,

$$\xi(z) = u(\Phi(z)) - \alpha,$$

then we see that $\xi(z)$ satisfies,

$$-\Delta\xi = e^{\xi} \text{ in } B_1, \quad \xi = -\alpha \text{ on } \partial B_1,$$

and therefore it is radial ([9]) and it is well known ([14]) that it takes the form $\xi(z) = \xi(|z|) = U_\tau(z)$ for some $\tau > 0$ which solves $U_\tau(1) = -\alpha$. Also, we deduce that,

$$e^{w(\Phi(z))} = e^{h(\Phi(z))}\, e^{u(\Phi(z))} = |\Phi'(z)|^{-2} e^{U_\tau(z)}, \quad z \in \overline{B_1}, \tag{12}$$

which is (5). Since $\log(|\Phi'(z)|^2)$ is harmonic we also have,

$$-\Delta_z w(\Phi(z)) = -\Delta_z U_\tau(z) = e^{U_\tau(z)} = |\Phi'(z)|^2\, e^{w(\Phi(z))}, \quad z \in B_1,$$

which implies,

$$-\Delta w = e^w \quad x \in \omega.$$

In view of the definition of f and since $w \in C^2(\Omega)$, we deduce that necessarily $f \equiv 0$ and in particular, that necessarily the equality sign in (1) holds in ω.

Remark 2.2. *Since Φ maps ∂B_1 one to one and onto $\partial\omega$, if $w = c$, $c \in \mathbb{R}$ on $\partial\omega$, then $w(\Phi(z)) = c$ on ∂B_1. As a consequence we see from* (12) *that $|\Phi'(z)|^2$ is constant on ∂B_1 and in particular we conclude that $\log(|\Phi'(z)|^2)$ is harmonic in B_1 and constant on ∂B_1. Therefore, we have, up to a translation, $\Phi(z) = \delta e^{i\theta} z$, for some $\delta > 0$ and $\theta \in \mathbb{R}$, $\omega = B_\delta$ and in particular we infer from* (12) *that,*

$$w(x) = U_\tau(\delta^{-1}x) - 2\log(\delta) = U_{\tau\delta^{-1}}(x).$$

It follows that in this case we have $\omega = B_\delta$ and $w(x) = U_{\tau\delta^{-1}}(x)$ for some $\delta > 0$ and $\tau > 0$.

Therefore, we have shown that if ω is simply connected and the equality sign holds in (4), then necessarily the equality sign holds in (1) in ω and (5) holds as well. On the other side if these conditions are satisfied we have,

$$\int\limits_{\partial\omega} \left(e^{w}\right)^{\frac{1}{2}}\, d\sigma = \int\limits_{\partial B_\delta} \left(|\Phi^{'}(z)|^{-2} e^{U_\tau(z)}\right)^{\frac{1}{2}} |\Phi^{'}(z)|\, d\sigma(z) = \int\limits_{\partial B_\delta} e^{U_\tau(z)/2} d\sigma(z) = \frac{2\pi\delta\tau}{1+\frac{\tau^2}{8}|\delta|^2},$$

and similarly,

$$\int_\omega e^{w} = \int\limits_{B_\delta} e^{U_\tau} = \int\limits_{B_\delta} \left(\frac{\tau}{1+\frac{\tau^2}{8}|z|^2}\right)^2 = \frac{\pi\delta^2\tau^2}{1+\frac{\tau^2}{8}\delta^2},$$

and so we readily conclude that,

$$\left(\int_{\partial\omega} (e^{w})^{\frac{1}{2}}\, d\sigma\right)^2 = \left(\int\limits_{\partial B_\delta} e^{U_\tau(z)/2} d\sigma(z)\right)^2 = \frac{1}{2}\left(\int\limits_{B_\delta} e^{U_\tau}\right)\left(8\pi - \int\limits_{B_\delta} e^{U_\tau}\right) =$$

$$\frac{1}{2}\left(\int_\omega e^{w}\, dx\right)\left(8\pi - \int_\omega e^{w}\, dx\right).$$

Therefore those conditions are necessary and sufficient as far as Ω is simply connected and ω is simply connected. If Ω is simply connected and ω is an open multiply connected subdomain whose boundary is the union of finitely many Jordan curves, as observed in [7], one can use the assumption $\int_\Omega e^{w} \leq 8\pi$ and work out an induction on the number of "holes" of ω , which starts by writing ω as the difference of two simply connected domains, see also the proof of Lemma 3.3 below for further details. In particular, it turns out that the inequality (4) is always strict in this case. Finally, we consider the case of ω not connected. Also in this case the proof works out by induction on the numbers of connected components and by the same calculations of Lemma 3.3 we have the strict inequality in (4).

$\square$

Remark 2.3. *We notice that we do not need the assumption* $\int_\Omega e^{w} \leq 8\pi$ *in the case* ω *is simply connected.*

3. The Alexandrov-Bol inequality on multiply connected domains

In this section we discuss the Alexandrov-Bol inequality on multiply connected domains, which was first derived for solutions w of the Liouville equation such that $w = c$ on $\partial\Omega$, with $c \in \mathbb{R}$. We refine here the argument by treating subsolutions of the Liouville equation and characterizing the equality case.

Proposition 3.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a open, bounded and multiply connected domain of class* C^1 *and* $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *be a solution of* (1) *which satisfies, for* $c \in \mathbb{R}$,

$$w \geq c \quad \text{in } \Omega, \tag{13}$$

$$w = c \quad \text{on } \partial\Omega, \tag{14}$$

$$\int_\Omega e^{w} \leq 8\pi.$$

Let $\omega \Subset \Omega$ *be any relatively compact open subset whose boundary is the union of finitely many rectifiable Jordan curves. Then it holds:*

$$\left(\int_{\partial\omega} (e^{w})^{\frac{1}{2}}\, d\sigma\right)^2 \geq \frac{1}{2}\left(\int_\omega e^{w}\, dx\right)\left(8\pi - \int_\omega e^{w}\, dx\right). \tag{15}$$

Moreover, the equality holds in (15) *if and only if* $(i)_1$-$(i)_2$-$(i)_3$ *(and* $(i)_4$ *in case* $w = c \in \mathbb{R}$ *on* $\partial\omega$*) of Proposition 2.1 hold true.*

In particular, if either the inequality in (1) *is not an equality on* ω *or* ω *is not simply connected, then the inequality in* (15) *is always strict.*

Remark 3.1. *In the case w satisfies (1), (14) and it is also superharmonic, we can actually recover the hypothesis (13) by applying the weak minimum principle.*

Proof of Proposition 3.1.
If $\omega \Subset \Omega$ is relatively compact and simply connected, then we can apply Proposition 2.1, including the characterization of the equality sign.
We will show now that if $\omega \Subset \Omega$ is relatively compact and multiply connected, then (15) holds with the strict inequality.
We will denote by $\overline{\Omega^*}$ the closure of the union of the bounded components of $\mathbb{R}^2 \setminus \partial\Omega$ and with $\Omega^* = \overline{\Omega^*} \setminus \partial(\overline{\Omega^*})$. Clearly $\Omega \subseteq \Omega^*$ and $\Omega \equiv \Omega^*$ if and only if Ω is simply connected. Also, there is no loss of generality in assuming $c = 0$. Indeed, if $c \neq 0$, we could define,

$$w_c(x) = w(e^{-\frac{c}{2}}x) - c, \quad x \in e^{\frac{c}{2}}\Omega,$$

which satisfies (1),(13) in $e^{\frac{c}{2}}\Omega$ and (14) on $\partial\big(e^{\frac{c}{2}}\Omega\big)$ with $c = 0$, while the integrals involved in the inequality (15) are invariant. Therefore we assume in the rest of this proof that,

$$c = 0,$$

and define,

$$\widehat{w}(x) = \begin{cases} w(x) & x \in \Omega \\ 0 & x \in \Omega^* \setminus \Omega. \end{cases}$$

Lemma 3.2. *The function $\widehat{w}$ is a solution of*

$$-\Delta\widehat{w} \leq e^{\widehat{w}} \quad in \quad \Omega^*, \tag{16}$$

in the sense of distributions.

Proof. Indeed, for any $\varphi \in C^2_0(\overline{\Omega^*})$, $\varphi \geq 0$ in Ω^*, we have,

$$-\int\limits_{\Omega^*} (\Delta\varphi)\widehat{w} = -\int\limits_{\Omega} (\Delta\varphi)w = -\int\limits_{\Omega} \varphi(\Delta w) + \int\limits_{\partial\Omega} \varphi(\partial_\nu w),$$

where ν is the exterior unit normal. Since $w \geq 0$ in Ω, $w = 0$ on $\partial\Omega$ and Ω is of class C^1, $\partial_\nu w$ is well defined on $\partial\Omega$ and $\partial_\nu w \leq 0$ on $\partial\Omega$. Therefore, since $\varphi \geq 0$, we conclude that,

$$-\int\limits_{\Omega^*} \Big((\Delta\varphi)\widehat{w} + \varphi e^{\widehat{w}}\Big) = -\int\limits_{\Omega} \varphi(\Delta w) - \int\limits_{\Omega^*} \varphi e^{\widehat{w}} + \int\limits_{\partial\Omega} \varphi(\partial_\nu w) \leq$$

$$-\int\limits_{\Omega} \varphi(\Delta w + e^{w}) \leq 0,$$

as claimed. $\square$

Since $\widehat{w}$ is only Lipschitz we cannot apply directly Proposition 2.1. However in this situation (15) still holds whenever $\omega \subseteq \omega_0 \Subset \Omega^*$, ω_0 is simply connected and $\int\limits_{\omega_0} e^{\widehat{w}} \leq 8\pi$. Let us define,

$$\widehat{\ell}(\omega) = \int\limits_{\partial\omega} \Big(e^{\widehat{w}}\Big)^{\frac{1}{2}} d\sigma, \quad \widehat{m}(\omega) = \int_{\omega} e^{\widehat{w}} dx,$$

then we have,

Lemma 3.3. *Let $\widehat{w}$ be a Lipschitz continuous solution of* (16) *in the sense of distributions and let $\omega_0 \Subset \Omega^*$ be a simply connected and relatively compact subdomain such that $\int\limits_{\omega_0} e^{\widehat{w}} \leq 8\pi$. Let $\omega \subseteq \omega_0$ be any open and bounded subset whose boundary is the union of finitely many rectifiable Jordan curves. Then,*

$$\hat{\ell}^2(\omega) \geq \frac{1}{2}\hat{m}(\omega)\left(8\pi - \hat{m}(\omega)\right), \tag{17}$$

holds and if ω is not simply connected, then the inequality is strict.

Proof. By a standard approximation argument, see Lemma 2 in [5], the fact that (17) holds follows from the inequality (4) for $C^2(\omega_0) \cap C^0(\overline{\omega}_0)$ functions. If ω is connected but not simply connected we can follow the argument in [7] and conclude that the inequality in (17) is strict. Indeed, assume for simplicity that $\omega = \omega_1 \setminus \overline{\omega_2}$, $\partial\omega = \partial\omega_1 \cup \partial\omega_2$ where ω_1 and ω_2 are open and simply connected. Then, since $\omega_1 = \omega \cup \overline{\omega_2}$, by (17) we have,

$$\begin{gathered}
2\hat{\ell}^2(\omega) > 2(\hat{\ell}^2(\partial\omega_1) + \hat{\ell}^2(\partial\omega_2)) \geq \hat{m}(\omega\cup\omega_2)(8\pi - \hat{m}(\omega\cup\omega_2)) + \hat{m}(\omega_2)(8\pi - \hat{m}(\omega_2)) = \\
(\hat{m}(\omega) + \hat{m}(\omega_2))(8\pi - \hat{m}(\omega) - \hat{m}(\omega_2)) + \hat{m}(\omega_2)(8\pi - \hat{m}(\omega_2)) = \\
\hat{m}(\omega)(8\pi - \hat{m}(\omega)) + 2\hat{m}(\omega_2)(8\pi - \hat{m}(\omega) - \hat{m}(\omega_2)) \geq \hat{m}(\omega)(8\pi - \hat{m}(\omega)),
\end{gathered}$$

where we used $\hat{m}(\omega) + \hat{m}(\omega_2) = \hat{m}(\omega_1) \leq 8\pi$.
Finally, we consider the case in which ω is not connected. For simplicity, we can assume $\omega = \omega_1 \cup \omega_2$ with $\omega_1, \omega_2 \subseteq \omega_0$ open and connected subsets, $\omega_1 \cap \omega_2 = \varnothing$ and $\partial\omega = \partial\omega_1 \cup \partial\omega_2$. Then by (17) we have,

$$\begin{gathered}
2\hat{\ell}^2(\omega) > 2(\hat{\ell}^2(\partial\omega_1) + \hat{\ell}^2(\partial\omega_2)) \geq \hat{m}(\omega_1)(8\pi - \hat{m}(\omega_1)) + \hat{m}(\omega_2)(8\pi - \hat{m}(\omega_2)) = \\
8\pi\hat{m}(\omega_1) + 8\pi\hat{m}(\omega_2) - (\hat{m}(\omega_1)^2 + \hat{m}(\omega_2)^2) = \\
8\pi\hat{m}(\omega) - (\hat{m}(\omega_1) + \hat{m}(\omega_2))^2 + 2\hat{m}(\omega_1)\hat{m}(\omega_2) \geq \\
8\pi\hat{m}(\omega) - \hat{m}(\omega)^2 = \hat{m}(\omega)(8\pi - \hat{m}(\omega)).
\end{gathered}$$

□

Now, by using Lemma 3.2 and Lemma 3.3, the validity of (15) with the strict inequality can be worked out following the arguments of [5] with minor modifications. Since this argument is not well known, to be self-contained and for reader's convenience we carry it out in full details in the appendix. Finally, we consider the case of $\omega \Subset \Omega$ relatively compact and not connected. Also in this case the proof works out by induction on the numbers of connected components and by the same calculations of Lemma 3.3 we have the strict inequality in (15).

4. On the first eigenvalue

In this section we prove the main result about the first eigenvalue of the Liouville-type problem (2).

Proof of Theorem 1.1.
To avoid repetitions we work out the proof of (j) and (jj) at once. Clearly the first eigenvalue and eigenfunction $(\hat{\nu}_1, \phi)$ of (2) correspond to the first eigenvalue and eigenfunction $(\nu_1 = \hat{\nu}_1 + 1, \phi)$ of,

$$\begin{cases} -\Delta\phi = \nu_1 e^{w}\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

We recall that a nodal domain for $\phi \in C^0(\overline{\Omega})$ is the maximal connected component of a subdomain where ϕ has a definite sign. Since ϕ has only one nodal domain we assume w.l.o.g. that

$\phi \geq 0$ in Ω. In particular, by the maximum principle we have $\phi > 0$ in Ω. Recalling (3) we set $U(x) := U_1(x)$, i.e.

$$U(x) = \ln\left(\frac{1}{1+\frac{1}{8}|x|^2}\right)^2, \tag{19}$$

which satisfies,

$$\Delta U + e^U = 0 \text{ in } \mathbb{R}^2.$$

Next, let $t_+ = \max_{\overline{\Omega}} \phi$ and for $t > 0$ let us define $\Omega_t = \{x \in \Omega \,:\, \phi > t\}$ and $R(t) > 0$ such that

$$\int_{B_{R(t)}} e^U\, dx = \int_{\Omega_t} e^w\, dx.$$

Since $\phi > 0$ in Ω we put $\Omega_0 = \Omega$ and set $R_0 = \lim_{t\to 0^+} R(t)$. Clearly $\lim_{t\to (t_+)^-} R(t) = 0$. Then $\phi^* : B_{R_0} \to \mathbb{R}$, which for $y \in B_{R_0}$, $|y| = r$, is defined by,

$$\phi^*(r) = \sup\{t \in (0, t_+) \,:\, R(t) > r\},$$

is a radial, decreasing, equimeasurable rearrangement of ϕ with respect to the measures $e^w\, dx$ and $e^U dx$, and hence, in particular,

$$\begin{aligned}
&B_{R(t)} = \{x \in \mathbb{R}^2 \,:\, \phi^*(x) > t\},\\
&\int_{\{\phi^* > t\}} e^U\, dx = \int_{\Omega_t} e^w\, dx \quad t \in [0, t_+),\\
&\int_{B_{R_0}} e^U |\phi^*|^2\, dx = \int_\Omega e^w |\phi|^2\, dx.
\end{aligned} \tag{20}$$

Well known arguments (see for example [3]) show that ϕ^* is continuous and locally Lipschitz. Then, by the Sard lemma, we can apply the Cauchy-Schwartz inequality and the co-area formula to conclude that,

$$\begin{aligned}
\int_{\{\phi=t\}} |\nabla\phi|\, d\sigma &\geq \left(\int_{\{\phi=t\}} (e^w)^{\frac{1}{2}}\, d\sigma\right)^2 \left(\int_{\{\phi=t\}} \frac{e^w}{|\nabla\phi|}\, d\sigma\right)^{-1}\\
&= \left(\int_{\{\phi=t\}} (e^w)^{\frac{1}{2}}\, d\sigma\right)^2 \left(-\frac{d}{dt}\int_{\Omega_t} e^w\, dx\right)^{-1},
\end{aligned} \tag{21}$$

for a.e. t. Next, under the assumptions either of part (j) or of part (jj), we are allowed to apply the Alexandrov-Bol inequality (4),

$$\begin{aligned}
&\left(\int_{\{\phi=t\}} (e^w)^{\frac{1}{2}}\, d\sigma\right)^2 \left(-\frac{d}{dt}\int_{\Omega_t} e^w\, dx\right)^{-1}\\
&\geq \frac{1}{2}\left(\int_{\Omega_t} e^w\, dx\right)\left(8\pi - \int_{\Omega_t} e^w\, dx\right)\left(-\frac{d}{dt}\int_{\Omega_t} e^w\, dx\right)^{-1},
\end{aligned}$$

for a.e. t. Since ϕ^* is an equimeasurable rearrangement of ϕ with respect to the measures $e^w\,dx$, $e^U\,dx$, and since e^U realizes the equality in (4), we also conclude that,

$$\begin{aligned}
&\frac{1}{2}\left(\int_{\Omega_t} e^w\,dx\right)\left(8\pi-\int_{\Omega_t} e^w\,dx\right)\left(-\frac{d}{dt}\int_{\Omega_t} e^w\,dx\right)^{-1}\\
&=\frac{1}{2}\left(\int_{\{\phi^*>t\}} e^U\,dx\right)\left(8\pi-\int_{\{\phi^*>t\}} e^U\,dx\right)\left(-\frac{d}{dt}\int_{\{\phi^*>t\}} e^U\,dx\right)^{-1}\\
&=\left(\int_{\{\phi^*=t\}} (e^U)^{\frac{1}{2}}\,d\sigma\right)^2\left(-\frac{d}{dt}\int_{\{\phi^*>t\}} e^U\,dx\right)^{-1}\\
&=\int_{\{\phi^*=t\}} |\nabla\phi^*|\,d\sigma,
\end{aligned}$$

where in the last equality we used once more the co-area formula. Therefore, we have proved that,

$$\int_{\{\phi^*=t\}} |\nabla\phi^*|\,d\sigma \leq \int_{\{\phi=t\}} |\nabla\phi|\,d\sigma,$$

for a.e. t, which in turn yields,

$$\int_{B_{R_0}} |\nabla\phi^*|^2\,dx \leq \int_{\Omega} |\nabla\phi|^2\,dx. \tag{22}$$

In deriving (22) we have used the co-area formula together with the fact that $\int_{B_{R(t)}} |\nabla\phi^*|^2\,dx$ and $\int_{\Omega_t} |\nabla\phi|^2\,dx$ are Lipschitz continuous. By using (20), (22) and the variational characterization of ϕ we deduce that,

$$\int_{B_{R_0}} |\nabla\phi^*|^2\,dx - \int_{B_{R_0}} e^U|\phi^*|^2\,dx \leq \int_{\Omega} |\nabla\phi|^2\,dx - \int_{\Omega} e^w|\phi|^2\,dx = (\nu_1-1)\int_{\Omega} e^w|\phi|^2\,dx, \tag{23}$$

Moreover, $\phi^*(R_0)=0$. Now we argue by contradiction and suppose that $\hat\nu_1<0$, so we have that ϕ satisfies (18) with $\nu_1<1$ and we have that,

$$(\nu_1-1)\int_{\Omega} e^w|\phi|^2\,dx<0$$

Therefore, we conclude that the first eigenvalue of $(-\Delta-e^U)(\cdot)$ on B_{R_0} with Dirichlet boundary conditions is non-positive. Consider now $\psi(x)=\frac{8-|x|^2}{8+|x|^2}$ which satisfies,

$$-\Delta\psi-e^U\psi=0 \text{ in } \mathbb{R}^2, \quad \psi\in C^2_0(B_{\sqrt{8}}(0)).$$

Since the first eigenvalue is non-positive one can deduce that $R_0>\sqrt{8}$. Moreover,

$$8\pi\frac{R_0^2}{8+R_0^2}=\int_{B_{R_0}} e^U\,dx=\int_{\Omega} e^w\,dx\leq 4\pi, \tag{24}$$

by assumption and hence $R_0\leq\sqrt{8}$, which yields a contradiction. So, $\nu_1\geq 1$, that is $\hat\nu_1\geq 0$.

Now suppose $\hat\nu_1=0$, that is $\nu_1=1$. From (23), we deduce that

$$\int_{B_{R_0}} |\nabla\phi^*|^2\,dx-\int_{B_{R_0}} e^U|\phi^*|^2\,dx\leq 0.$$

Therefore, we conclude that the first eigenvalue of $(-\Delta-e^U)(\cdot)$ on B_{R_0} with Dirichlet boundary conditions is non-positive and, arguing as before, we conclude that $R_0\geq\sqrt{8}$. On the other hand, as pointed out in (24), $R_0\leq\sqrt{8}$. Hence we deduce that $R_0=\sqrt{8}$ and, in particular,

$$\int_{\Omega} e^w\,dx=\int_{B_{\sqrt{8}}} e^U\,dx=4\pi$$

Moreover, since $R_0 = \sqrt{8}$, then the first eigenvalue of $(-\Delta - e^U)(\cdot)$ on B_{R_0} with Dirichlet boundary conditions is 0 and ψ is its eigenfunction. In particular, from (23), we derive that

$$\int_{B_{\sqrt{8}}} |\nabla \phi^*|^2\, dx = \int_{B_{\sqrt{8}}} e^U |\phi^*|^2\, dx = \int_\Omega e^w |\phi|^2\, dx = \int_\Omega |\nabla \phi|^2\, dx$$

and that all the inequalities used to obtain (23) must be equalities. In particular, for a.e. $t \in [0, t_+)$ we have:

$$\left(\int_{\{\phi=t\}} (e^w)^{\frac{1}{2}}\, d\sigma \right)^2 = \frac{1}{2} \left(\int_{\Omega_t} e^w\, dx \right) \left(8\pi - \int_{\Omega_t} e^w\, dx \right), \tag{25}$$

Then we can choose a sequence $t_n \to 0^+$, as $n \to +\infty$, such that the equality (25) holds for any n. In both situations (j) and (jj), we can apply Proposition 2.1 or Proposition 3.1 and use the characterization of the equality sign in the Alexandrov-Bol inequality. In particular, we derive that Ω_{t_n} are simply connected. This, together with the fact that $\Omega_{t_n} \Delta \Omega \to \varnothing$, as $n \to +\infty$, implies that Ω is simply connected and proves (jj). Now, taking the $\liminf$ in (25) and using for example Theorem 2.3 in [8], we deduce that

$$\left(\int_{\partial\Omega} (e^w)^{\frac{1}{2}}\, d\sigma \right)^2 \leq \frac{1}{2} \left(\int_{\Omega} e^w\, dx \right) \left(8\pi - \int_{\Omega} e^w\, dx \right),$$

and the latter inequality turns out to be an equality by (4) in Proposition 2.1 with $\omega = \Omega$. Then, applying again the equality case of Proposition 2.1, we see from $(i)_1$ and $(i)_3$ that the equality holds in (1) in Ω and that,

$$e^{w(\Phi(z))} |\Phi'(z)|^2 |dz|^2 = e^{U_\tau(z)} |dz|^2, \quad z \in \overline{B_1},$$

where $\Phi : B_1 \to \Omega$ is conformal and univalent. Therefore we have $\int\limits_\Omega e^w = \int\limits_{B_1} e^{U_\tau}$ and in particular,

$$4\pi = \int\limits_\Omega e^w = \int\limits_{B_{R_0}} e^U = \int\limits_{B_1} e^{U_\tau} = \int\limits_{B_\tau} e^{U_1},$$

which immediately implies that $\tau = \sqrt{8}$. Therefore, we have that $\varphi(z) = \phi(\Phi(z))$ satisfies

$$\begin{cases} -\Delta\varphi = \nu_1 |\Phi'(z)|^2\, e^{w(\Phi(z))} \varphi = e^{U_{\sqrt{8}}(z)} \varphi & \text{in } B_1, \\ \varphi = 0 & \text{on } \partial B_1, \end{cases} \tag{26}$$

The function $\varphi(z) = \frac{1-|z|^2}{1+|z|^2} = \psi(\sqrt{8}z)$ is a positive solution of (26) and therefore it is the first eigenfunction for $(-\Delta - e^{U_{\sqrt{8}}(z)})_{B_1}$ with Dirichlet boundary conditions. We conclude that $\phi(x) = \varphi(\Phi^{-1}(x))$ which is $(a)_3$.
If now we assume that $(a)_0 - (a)_3$ are true, it can be easily proved that $\nu_1 = 1$, that means $\hat{\nu}_1 = 0$.
We conclude the (j) part, assuming that $w = c$ on $\partial\Omega$. Then we can apply what specified in the Remark 2.2 which implies that, up to a translation, $\Phi(z) = \delta e^{i\theta} z$ for some $\delta > 0$ and $\theta \in \mathbb{R}$, and we have $\Omega = B_\delta$ and $\phi(x) = \varphi(\delta^{-1}x)$, that is $(a)_4$. Moreover, this condition, together with $(a)_0 - (a)_3$, is sufficient to have $\hat{\nu}_1 = 0$.

□

5. Appendix

In this section we complete the proof of the Alexandrov-Bol inequality on multiply connected domains, i.e. Proposition 3.1, following the strategy in [5].

We recall that ω is assumed to be relatively compact and multiply connected and we aim to show (15) holds with the strict inequality. We start with the case where each bounded component of $\mathbb{R}^2\backslash\overline{\omega}$ contains at least one bounded component of $\mathbb{R}^2\backslash\partial\Omega$.

Let Ω_0 be the union of the bounded components of $\mathbb{R}^2\backslash\partial\Omega$ bounded by $\partial\omega$. Moreover, let ω_0 be the union of all bounded simply connected components of $\mathbb{R}^2\backslash\partial\omega$. Thus $\Omega_0 \subset \omega_0$ and we let

$$\omega^* = \omega_0\backslash\Omega_0.$$

We have $\omega^* \subset \Omega$. Moreover, $\omega^* \cup \Omega_0$ is a union of simply connected domains and $\omega \cup \overline{\omega^* \cup \Omega_0}$ is a simply connected domain. Denote by $\partial_0\omega$ the boundary of $\omega^* \cup \Omega_0$ and $\partial_1\omega = \partial\omega \setminus \partial\omega^*$. Then $\partial_1\omega = \partial(\omega \cup \overline{\omega^* \cup \Omega_0})$ and it holds

$$\partial\omega = \partial_1\omega \cup \partial_0\omega.$$

To simplify the notations we define

$$\ell(\omega) = \int\limits_{\partial\omega} \left(e^{w}\right)^{\frac{1}{2}} d\sigma, \quad m(\omega) = \int_{\omega} e^{w} dx,$$

whenever $\omega \Subset \Omega$.

Case 1. $\widehat{m}(\omega^* \cup \Omega_0) \geq 8\pi$.
Since $w \geq 0$ in Ω, by the isoperimetric inequality we have,

$$2\ell(\partial_0\omega)^2 = 2\left(\int\limits_{\partial_0\omega} \left(e^{w}\right)^{\frac{1}{2}} d\sigma\right)^2 \geq 2\left(\int\limits_{\partial_0\omega} d\sigma\right)^2 \geq 8\pi \int\limits_{\omega^*\cup\Omega_0} dx > 8\pi\widehat{m}(\Omega_0). \tag{27}$$

Since $\widehat{m}(\omega^* \cup \Omega_0) \geq 8\pi$, we have

$$\widehat{m}(\Omega_0) \geq 8\pi - \widehat{m}(\omega^*) \equiv 8\pi - m(\omega^*),$$

and then by using (27),

$$2\ell^2(\partial_0\omega) > 8\pi(8\pi - m(\omega^*)). \tag{28}$$

Similarly, since $\widehat{m}(\omega \cup \overline{\omega^* \cup \Omega_0}) > \widehat{m}(\omega^* \cup \Omega_0) \geq 8\pi$, we deduce

$$2\ell^2(\partial_1\omega) > 8\pi(8\pi - m(\omega \cup \omega^*)). \tag{29}$$

Recalling (28), (29) and the fact that since $\omega^* \subset \Omega$, then $m(\omega^*) \leq 8\pi$, we have

$$\begin{aligned}
2\ell^2(\partial\omega) &= 2(l(\partial_1\omega) + l(\partial_0\omega))^2 > 2[l^2(\partial_1\omega) + l^2(\partial_0\omega)] \\
&> m(\omega^*)(8\pi - m(\omega^*)) + m(\omega \cup \omega^*)(8\pi - m(\omega \cup \omega^*)) \\
&= m(\omega^*)(8\pi - m(\omega^*)) + (m(\omega) + m(\omega^*))(8\pi - m(\omega) - m(\omega^*)) \\
&= m(\omega)(8\pi - m(\omega)) + m(\omega^*)(16\pi - 2m(\omega) - 2m(\omega^*)).
\end{aligned}$$

Finally, since

$$m(\omega) + m(\omega^*) \leq m(\Omega) \leq 8\pi,$$

then we conclude that,

$$2\ell^2(\partial\omega) > m(\omega)(8\pi - m(\omega)),$$

which proves that (15) holds with the strict inequality.

Case 2. $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$ and $\widehat{m}(\omega \cup \overline{\omega^* \cup \Omega_0}) \geq 8\pi$.

Since $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$ and $\omega^* \cup \Omega_0 \Subset \Omega^*$ is union of simply connected domains, (17) yields,

$$\begin{aligned} 2\ell^2(\partial_0\omega) &\geq \widehat{m}(\omega^* \cup \Omega_0)(8\pi - \widehat{m}(\omega^* \cup \Omega_0)) \\ &= (m(\omega^*) + \widehat{m}(\Omega_0))(8\pi - m(\omega^*) - \widehat{m}(\Omega_0)). \end{aligned} \tag{30}$$

Observing that (27) holds, we infer that

$$\ell(\partial_0\omega) \geq \sqrt{4\pi\widehat{m}(\Omega_0)}, \qquad \ell(\partial_1\omega) \geq \sqrt{4\pi\widehat{m}(\Omega_0)}. \tag{31}$$

Observe that (29) still holds in this case and therefore, by (30), (31) we have

$$\begin{aligned} 2\ell^2(\partial\omega) &= 2[\ell^2(\partial_1\omega) + 2\ell(\partial_1\omega)\ell(\partial_0\omega) + \ell^2(\partial_0\omega)] \\ &\geq 8\pi(8\pi - m(\omega) - m(\omega^*)) \\ &\quad + (m(\omega^*) + \widehat{m}(\Omega_0))(8\pi - m(\omega^*) - \widehat{m}(\Omega_0)) + 16\pi\widehat{m}(\Omega_0) \\ &= 8\pi(8\pi - m(\omega)) - m^2(\omega^*) + \widehat{m}(\Omega_0)(24\pi - 2m(\omega^*) - \widehat{m}(\Omega_0)) \\ &= m(\omega)(8\pi - m(\omega)) + [(8\pi - m(\omega))^2 - m^2(\omega^*)] \\ &\quad + \widehat{m}(\Omega_0)(24\pi - 2m(\omega^*) - \widehat{m}(\Omega_0)). \end{aligned}$$

Now, since $m(\omega) + m(\omega^*) \leq m(\Omega) \leq 8\pi$,

$$m(\omega^*) \leq 8\pi - m(\omega).$$

On the other hand, $\widehat{m}(\omega^* \cup \Omega_0) < 8\pi$, hence

$$2m(\omega^*) + 2\widehat{m}(\Omega_0) < 16\pi.$$

We have proved that $2\ell^2(\partial\omega) > m(\omega)(8\pi - m(\omega))$, that is (15) holds with the strict inequality.

Case 3. $\widehat{m}(\omega \cup \overline{\omega^* \cup \Omega_0}) < 8\pi$.
Since $\omega \Subset \Omega$ by assumption, then $\omega \cup \overline{\omega^* \cup \Omega_0} \Subset \Omega^*$. Moreover, $\omega \cup \overline{\omega^* \cup \Omega_0}$ is simply connected. Therefore, Lemma 3.3 yields,

$$2\ell(\partial\omega)^2 \geq m(\omega)(8\pi - m(\omega)),$$

that is, (15) holds and in particular, since ω is not simply connected by assumption, it holds with the strict inequality sign.

To finish the proof, we are left with the case $\partial\omega$ bounds some simply connected subdomains $\omega_1, \ldots, \omega_k$ of Ω. Clearly, $\omega \cup \overline{\omega_1} \cup \ldots \overline{\omega_k}$ is simply connected in Ω. Therefore, using in this case the standard inequality (4) we infer

$$2\ell^2(\partial(\omega \cup \overline{\omega}_1 \cup \ldots \cup \overline{\omega}_k)) \geq m(\omega \cup \overline{\omega}_1 \cup \ldots \cup \overline{\omega}_k)[8\pi - m(\omega \cup \overline{\omega}_1 \cup \ldots \cup \overline{\omega}_k)],$$

$$2\ell^2(\partial\omega_j) \geq m(\omega_j)(8\pi - m(\omega_j)),\ j \in \{1, \cdots, k\}.$$

For $k = 1$ we readily have

$$\begin{aligned} 2\ell(\partial\omega)^2 &> 2\ell(\partial(\omega \cup \overline{\omega}_1))^2 + 2\ell(\partial\omega_1)^2 \\ &\geq m(\omega \cup \overline{\omega}_1)(8\pi - m(\omega \cup \overline{\omega}_1)) + m(\omega_1)(8\pi - m(\omega_1)) \\ &= m(\omega)(8\pi - m(\omega)) + m(\omega_1)(16\pi - 2m(\omega_1) - 2m(\omega)) \\ &\geq m(\omega)(8\pi - m(\omega)), \end{aligned}$$

and we deduce that (15) holds with the strict inequality in this case as well. The case $k > 1$ is similar.

□

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