STRONG APPROXIMATION OF SBV FUNCTIONS WITH PRESCRIBED JUMP DIRECTION

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Abstract. In this note we show that SBV functions with jump normal lying in a prescribed set of directions $\mathcal{N}$ can be approximated by sequences of SBV functions whose jump set is essentially closed, polyhedral, and preserves the orthogonality to $\mathcal{N}$, moreover the functions are smooth away from their jump set.

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1. Introduction

In the proof of approximation or homogenisation results of *free-discontinuity functionals* one is concerned with the construction of a sequence of functions, the so-called *recovery sequence*, along which a certain functional upper-bound inequality shall be satisfied, the so-called Γ*-limsup inequality* (see, *e.g.*, [8]). The construction of a recovery sequence is often nontrivial and in most cases it is only feasible after assuming some additional regularity of the target function. In a particular instance, if the considered object belongs to the space of *special functions of bounded variation*, SBV, it is of crucial importance to replace it with a function whose jump set is as simple as possible, typically *polyhedral*, as well as to attain sufficient smoothness of the function away from its jump set. In this respect the mathematical literature provides us with a number of approximation results for SBV functions which are, moreover, tailor-made to deal with the aforementioned upper-bound inequalities; see, *e.g.*, [4, 9], as well as [6, 7, 11] for approximants with polyhedral jump set.

However, should the target SBV functions satisfy some geometric constraint arising in the problem under examination, the available approximation results may fail to preserve this additional constraint. In the context of variational methods for fracture and image segmentation, in this paper we establish a density result for SBV functions with prescribed jump direction, describing, *e.g.*, deformations of materials with cracks appearing only along certain directions.

If $\Omega \subset \mathbb{R}^n$ is open, bounded, with Lipschitz boundary and $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$, the prototypical free-discontinuity functionals we consider are of the form

$$\mathcal{F}(u) = \int_\Omega |\nabla u| \,\mathrm{d}\mathscr{L}^n + \int_{S_u} \gamma(x, u^+, u^-, \nu_u)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{1.1}$$

where the surface integrand $\gamma\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1} \to [0, +\infty]$ encodes the relevant properties of the (effective) model. We recall that an element of the space $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ is a $\mathrm{BV}(\Omega;\mathbb{R}^m)$ function

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whose Jacobi matrix is a measure satisfying

$$Du = \nabla u \, \mathrm{d}\mathscr{L}^n \llcorner \Omega \; + \; (u^+ - u^-) \otimes \nu_u \, \mathrm{d}\mathscr{H}^{n-1} \llcorner S_u, \tag{1.2}$$

where ∇u is the density of the absolutely continuous part and $\mathscr{H}^{n-1}(S_u) < +\infty$. In (1.2) the vectorial functions u^+ and u^- represent the traces of u on both sides of the discontinuity set S_u and ν_u is the measure theoretical normal to S_u [2].

If (1.1) allows only for a finite number of given jump directions, and $\nu_1, \dots, \nu_M \in \mathbb{S}^{n-1}$ is the list of corresponding normals, we shall consider a surface energy density γ such that $\gamma(\cdot, \cdot, \cdot, \nu) \equiv +\infty$ if $\nu \notin \mathcal{N} := \{\pm\nu_1, \dots, \pm\nu_M\}$. Therefore in this case the domain of $\mathcal{F}$ is strictly smaller than $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ and the additional constraint of

$$\nu_u(x) \in \mathcal{N} \quad \text{for} \;\; \mathscr{H}^{n-1}\text{-a.e. } x \in S_u \tag{1.3}$$

is to be satisfied. We denote by $\mathrm{SBV}^1_{\mathcal{N}}(\Omega; \mathbb{R}^m)$ the space of those $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ functions satisfying (1.3) as well as $\mathscr{H}^{n-1}(S_u) < +\infty$ and in this paper we are concerned with a *strong approximation* scheme for functions therein. Namely, in the main result of this paper, Theorem 3.1, we prove that any function $u \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega; \mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega; \mathbb{R}^m)$ can be approximated by a sequence $(u_k) \subset \mathrm{SBV}^1_{\mathcal{N}}(\Omega; \mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega; \mathbb{R}^m)$ with $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$ satisfying the following properties:

- S_{u_k} is essentially closed, *i.e.*, $\mathscr{H}^{n-1}(\overline{S_{u_k}} \backslash S_{u_k}) = 0$;
- $\overline{S_{u_k}}$ is the union of a *finite* number of $(n-1)$-dimensional *pairwise disjoint* closed cubes;
- $u_k \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{u_k}}; \mathbb{R}^m) \cap W^{1,\infty}(\Omega \backslash \overline{S_{u_k}}; \mathbb{R}^m)$;

The density result is regarded in the following convergence:

$$u_k \to u \;\text{ in } \mathrm{L}^1(\Omega; \mathbb{R}^m), \quad \nabla u_k \to \nabla u \;\text{ in } \mathrm{L}^1(\Omega; \mathbb{R}^{m \times n}), \quad \mathscr{H}^{n-1}(S_{u_k}) \to \mathscr{H}^{n-1}(S_u). \tag{1.4}$$

Moreover,

$$\limsup_{k \to +\infty} \int_{S_{u_k} \cap \overline{A}} \gamma\big(x, u_k^+, u_k^-, \nu_{u_k}\big) \, \mathrm{d}\mathscr{H}^{n-1} \le \int_{S_u \cap \overline{A}} \gamma\big(x, u^+, u^-, \nu_u\big) \, \mathrm{d}\mathscr{H}^{n-1},$$

for any open set $A \subset\subset \Omega$ and for any bounded upper semicontinuous function $\gamma : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{N} \to [0, +\infty)$ such that $\gamma(\cdot, a, b, \nu) = \gamma(\cdot, b, a, -\nu)$ for every $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{n-1}$.

The proof strategy of Theorem 3.1 mainly relies on the approximation techniques employed by De Philippis, Fusco, and Pratelli in [9] and Cortesani and Toader in [7]. Yet in comparison with those, the main disparity in methodology arises from the geometric constraint of prescribed orientation of the discontinuity set, which is not preserved by the constructions implemented in [9]. For instance, in the aforementioned literature a typical procedure is to remove singularities by setting the approximants to zero in certain regions and introducing a small jump on the boundary of such regions; however, this may violate the constraints on the jump direction, for example if the number of admissible directions is strictly less than n. In the present paper the approximation result is achieved by means of a fine cover lemma (cf. [7, Lemma 4.2]) which provides us with a finite number of pairwise disjoint $(n-1)$-dimensional cubes covering the major part of the discontinuity set S_u. Then, the desired sequence (u_k) is obtained by successive regularisation steps mainly relying on convolution results with variable kernels (see, *e.g.*, [9, Proposition 2.3]) and on extension results in domains with cracks (see, *e.g.*, [9, Lemma 4.1]).

We remark that rectifiable sets with the constraint that their measure-theoretic normal lies in $\mathcal{N}$ may be irregular even in the simple case where $\mathcal{N}$ contains a single vector. An example can be constructed as follows. Let $C \subset (0,1)$ be a Smith-Volterra-Cantor set (that is, a Cantor set with strictly positive one-dimensional measure). Let $f(x) := \operatorname{dist}(x, C)$ and $F(x) := \int_0^x f(t)\,\mathrm{d}t$. Then the graph of F, $G(F)$, is a C^1 curve and $S := \{(x,y) \in G(F) \colon x \in C\}$ is a $\mathscr{H}^1$-rectifiable set in $\mathbb{R}^2$ with normal e_2 at every point. However, S is a totally disconnected set with uncountably many connected components each lying on a different horizontal hyperplane.

We observe that our result also covers the case of *infinitely many* jump directions, including the unconstrained case $\mathcal{N} = \mathbb{S}^{n-1}$.

Such case is treated by Cortesani and Toader in [7] for SBV^p functions with $p > 1$, being the assumption $p > 1$ crucial to exploit some classical regularity results for the local minimisers of the Mumford-Shah functional [10]. In the last section of this note we prove the density in SBV^1 of local minimisers of the Mumford-Shah functional by resorting to a strong approximation of SBV^1 functions by means of SBV^p functions with $p > 1$, (cf. Proposition 4.1).

We conclude this introduction by mentioning that in [5] Conti, Diermeier, and Zwicknagl prove a density result for SBV^2 functions with given jump normal direction (see Section 4.2 therein). In contrast to our results, we observe that Conti, Diermeier, and Zwicknagl's proof provides the L^2 convergence of the approximant's gradients. On the other hand, it is valid exclusively in dimension two, for one prescribed jump direction. However, the proof of a density result in SBV^2 (and more in general in SBV^p with $p > 1$) appears to be way more delicate than the approximation result proven in the present note. In fact, in the SBV^p setting an additional issue one needs to face pertains to combining the constraint in (1.3) with the strong convergence $\nabla u_k \to \nabla u$ in $\mathrm{L}^p(\Omega;\mathbb{R}^m)$. Moreover a density result in SBV^p shall rely also on deeper results in the theory of SBV functions like, *e.g.*, the regularity properties of local minimisers on the Mumford-Shah functionals [10], similarly as in [6, 7, 11]. A density result in SBV^p for functions with prescribed jump direction can be relevant in a number of applications and will be the subject of a forthcoming paper.

2. Notation, functional setup, and preliminaries

We introduce the notation and conventions present in the paper. Let $n \geq 2, m \geq 1$ be integers; the symbols $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$ indicate the usual n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. By $B_r(x) \subset \mathbb{R}^n$ we mean the open ball centred at x with radius $r > 0$; moreover, $B_r := B_r(0)$. By $Q_r(x) \subset \mathbb{R}^n$ we mean the n-dimensional open cube of side length $r > 0$, centred at $x \in \mathbb{R}^n$, and with faces parallel to the coordinate hyperplanes. Given a unit vector $\nu \in \mathbb{S}^{n-1}$ we set Π^ν_x to be the hyperplane orthogonal to ν and passing through a point $x \in \mathbb{R}^n$. Likewise $Q^\nu_r(x) \subset \mathbb{R}^n$ is understood as an open cube of side-length $r > 0$, centred at $x \in \mathbb{R}^n$, and with a face orthogonal to ν.

Throughout, the real number $c > 0$ shall be thought of as absorbing constant with dependences emphasised when being relevant.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. We use the standard notation $\mathrm{SBV}(\Omega;\mathbb{R}^m)$ for the space of $\mathbb{R}^m$-valued *special functions of bounded variation* in Ω. We recall that a function $u \colon \Omega \to \mathbb{R}^m$ belongs to $\mathrm{SBV}(\Omega;\mathbb{R}^m)$ if u is in $\mathrm{BV}(\Omega;\mathbb{R}^m)$ and its distributional derivative satisfies

$$Du(B) = \int_B \nabla u \,\mathrm{d}\mathscr{L}^n + \int_{S_u \cap B} [u] \otimes \nu_u \,\mathrm{d}\mathscr{H}^{n-1},$$

for any Borel set $B \subset \Omega$. By ∇u we mean the density of the diffuse part of Du; the latter turns out to coincide with the approximate gradient of u. The symbol S_u denotes the approximate discontinuity set of u and is a $\mathscr{H}^{n-1}$-rectifiable set. The associated measure theoretic normal is ν_u (defined up to the sign) whereas $[u] := u^+ - u^-$ is the difference of the traces of u on both sides of S_u. We notice that (u^+, u^-) is to be replaced by (u^-, u^+) if the orientation of ν_u is reversed. Let us also recall that the BV-norm of a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ is given by

$$\|u\|_{\mathrm{BV}(\Omega;\mathbb{R}^m)} := \|u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + |Du|(\Omega)$$

where $|Du|$ denotes the total variation of Du, *i.e.*,

$$|Du|(B) = \int_B |\nabla u| \,\mathrm{d}\mathscr{L}^n + \int_{S_u \cap B} |[u]| \,\mathrm{d}\mathscr{H}^{n-1},$$

where B is any Borel subset of $\mathbb{R}^n$.

For the general theory of BV and SBV functions we refer the readers to the comprehensive monograph [2].

In this paper the following subspace of SBV is also taken into consideration:

$$\mathrm{SBV}^1(\Omega;\mathbb{R}^m) := \{u \in \mathrm{SBV}(\Omega;\mathbb{R}^m) \colon \mathscr{H}^{n-1}(S_u) < +\infty\}.$$

Let now $\mathcal{N}$ be a *Borel subset* of $\mathbb{S}^{n-1}$ and let us assume that

$$\nu \in \mathcal{N} \iff -\nu \in \mathcal{N}. \tag{2.1}$$

We introduce the following space of SBV^1 functions with $\mathcal{N}$-*oriented* discontinuity set

$$\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m) := \{u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \colon \nu_u \in \mathcal{N} \ \mathscr{H}^{n-1}\text{-a.e. in } S_u\}.$$

We notice that in view of (2.1) the definition of $\mathrm{SBV}_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ is unambiguous.

A set $F \subset \Omega$ is called *polyhedral* (with respect to Ω) if it is the intersection of Ω with a *finite* number of $(n-1)$-dimensional simplices in $\mathbb{R}^n$. In the interest of our work we define a special case of polyhedral sets whose normal belongs $\mathcal{N}$.

Definition 2.1 ($\mathcal{N}$-aligned regular set)**.** We say that a set $F \subset \Omega$ is an $\mathcal{N}$-*aligned regular set* if there exists a *finite* collection of sets $F_1, \dots, F_N$ such that each F_i is a $(n-1)$-dimensional closed cube in $\mathbb{R}^n$ orthogonal to ν for some $\nu \in \mathcal{N}$ and

$$F = \Omega \cap \bigcup_{i=1}^N F_i.$$

If the sets $F_1, \dots, F_N$ are additionally *pairwise disjoint*, the set F is called an $\mathcal{N}$-*aligned regular disconnected set*.

We now introduce the space of approximating functions.

Definition 2.2 (The approximating space)**.** We say that u belongs to the space $\mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ if:

(a) $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$;
(b) S_u is essentially closed, *i.e.*, $\mathscr{H}^{n-1}(\overline{S_u} \backslash S_u) = 0$;
(c) $\overline{S_u}$ is a $\mathcal{N}$-aligned regular disconnected set;
(d) $u \in \mathrm{C}^\infty(\Omega\backslash\overline{S_u};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega\backslash\overline{S_u};\mathbb{R}^m)$.

In accordance with [7] we consider the following notion of convergence.

Definition 2.3 ($\mathscr{S}$-convergence)**.** We say that a sequence $(u_k) \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ $\mathscr{S}$-converges to $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ as $k \to +\infty$, written $u_k \xrightarrow{\mathscr{S}} u$, if:

(a) $u_k \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$;
(b) $\nabla u_k \to \nabla u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})$;
(c) $\mathscr{H}^{n-1}(S_{u_k}) \to \mathscr{H}^{n-1}(S_u)$.

For later purposes we also introduce the following stronger convergence.

Definition 2.4 ($\overline{\mathscr{S}}$-convergence)**.** We say that a sequence $(u_k) \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ $\overline{\mathscr{S}}$-converges to $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ as $k \to +\infty$, written $u_k \xrightarrow{\overline{\mathscr{S}}} u$, if:

(a) $u_k \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$;
(b) $\nabla u_k \to \nabla u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})$;
(c) $\mathscr{H}^{n-1}(S_{u_k} \triangle S_u) \to 0$;
(d) there holds

$$\int_{S_{u_k}\cup S_u} \left(|u_k^+ - u^+| + |u_k^- - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} \longrightarrow 0, \tag{2.2}$$

where in (2.2) we choose the orientation $\nu_{u_k} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_k} \cap S_u$.

We remark that $\overline{\mathscr{S}}$-convergence is evidently stronger than the convergence induced from the $\mathrm{BV}(\Omega;\mathbb{R}^m)$-norm.

Below we recall three technical lemmas which are used to prove our main result, Theorem 3.1. These are based on the corresponding results in [9]. The first concerns a smooth approximation of functions in $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ obtained by convolutions with variable kernels.

Lemma 2.5 (Approximation by convolution)**.** *Let $u \in \mathrm{SBV}^1(\Omega)$ and let $F \subset\subset \Omega$ be a compact and $\mathscr{H}^{n-1}$-rectifiable set. For any $\varepsilon > 0$, there exists $v_\varepsilon \in \mathrm{SBV}^1(\Omega) \cap \mathrm{C}^\infty(\Omega\backslash F)$ such that the following properties hold true:*

(1) $\|v_\varepsilon - u\|_{\mathrm{L}^1(\Omega)} < \varepsilon$*;*
(2) $v_\varepsilon^\pm = u^\pm$ *in* F*, therefore* $S_{v_\varepsilon} = S_u \cap F$*;*
(3) $|Du - Dv_\varepsilon|(\Omega) \le 3|Du \llcorner (S_u\backslash F)|(\Omega) + (2|Du|(\Omega) + 1)\,\varepsilon$*;*
(4) if $u \in \mathrm{L}^\infty(\Omega)$*, then* $\|v_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \le \|u\|_{\mathrm{L}^\infty(\Omega)}$*.*

Proof. The results can be retrieved by combining [9, Proposition 2.3, Corollary 2.4, Lemma 2.5]. □

Remark 2.6. We will often apply Lemma 2.5 in the following way. Let $u \in \mathrm{SBV}^1(\Omega)$ and $\varepsilon_k \to 0$. For every k, let $F_k \subset\subset \Omega$ be a compact and $\mathscr{H}^{n-1}$-rectifiable set such that

$$F_k \subset S_u \quad \text{and} \quad \mathscr{H}^{n-1}(S_u\backslash F_k) < \varepsilon_k.$$

Applying Lemma 2.5 to u and to $F = F_k$, one finds $u_k \in \mathrm{SBV}^1(\Omega)\cap\mathrm{C}^\infty(\Omega\backslash F_k)$ such that $\|u_k\|_{\mathrm{L}^\infty} \le \|u\|_{\mathrm{L}^\infty}$, $S_{u_k} = F_k$,

$$\|u - u_k\|_{\mathrm{L}^1(\Omega)} + |Du - Du_k|(\Omega) \le \varepsilon_k + 3|Du \llcorner (S_u\backslash F_k)|(\Omega) + \varepsilon_k\,(2|Du|(\Omega) + 1)$$

and

$$\int_{S_u} \left(|u_k^+ - u^+| + |u_k^- - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} \le 4\|u\|_{\mathrm{L}^\infty} \mathscr{H}^{n-1}(S_u \backslash F_k).$$

Therefore, u_k $\overline{\mathscr{S}}$-converges to u as $k \to +\infty$.

Moreover, we recall the following existence result of bounded Lipschitz extensions for C^1-regular interior and boundary traces.

Lemma 2.7 (Extension). *Let $U \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary. Let $M \subset\subset U$ be either a compact and connected $(n-1)$-dimensional C^1 manifold with (possibly empty) C^1 boundary, or an $(n-1)$-dimensional closed cube. Then there exists a constant $c_{U,M} > 0$ with the following properties:*

(a) Given three functions $\phi \in \mathrm{L}^1(\partial U)$, $\phi^+, \phi^- \in \mathrm{L}^1(M)$, there exists $\psi \in \mathrm{W}^{1,1}(U \backslash M)$ such that $\psi^\pm = \phi^\pm$ in M, $\psi = \phi$ on ∂U (in the sense of traces) and

$$\|\psi\|_{\mathrm{W}^{1,1}(U\backslash M)} \le c_{U,M}\Big(\|\phi\|_{\mathrm{L}^1(\partial U)} + \|\phi^+\|_{\mathrm{L}^1(M)} + \|\phi^-\|_{\mathrm{L}^1(M)}\Big).$$

(b) Given three functions $\phi \in \mathrm{C}^1(\partial U)$, $\phi^+, \phi^- \in \mathrm{C}^1(M)$ satisfying $\phi^+ = \phi^-$ on ∂M, there exists $\psi \in \mathrm{W}^{1,\infty}(U \backslash M)$ such that $\psi^\pm = \phi^\pm$ in M, $\psi = \phi$ on ∂U and

$$\|\psi\|_{\mathrm{W}^{1,\infty}(U\backslash M)} \le c_{U,M}\Big(\|\phi\|_{\mathrm{C}^1(\partial U)} + \|\phi^+\|_{\mathrm{C}^1(M)} + \|\phi^-\|_{\mathrm{C}^1(M)}\Big).$$

Proof. In the case where M has C^1 boundary, the result is stated in [9, Lemma 4.1]. When M is an $(n-1)$-dimensional closed cube, the proof requires some minor adjustments, detailed below. Let $\delta > 0$ be such that

$$P := \{x + t\nu \colon x \in M,\, t \in [-\delta, \delta]\} \subset\subset U$$

where $\nu \in \mathbb{S}^{n-1}$ is the normal to M. Then $\partial P \setminus \partial M =: D^+ \cup D^-$ where $\overline{D^\pm}$ are in bilipschitz correspondence with M, $D^+ \cap D^- = \emptyset$ and $\partial_{\partial P} D^\pm = \partial M$, where $\partial_{\partial P} D^\pm$ denote the boundary of $D^\pm$ in the relative topology of ∂P. Subsequently we may find a mapping $\Phi : U \backslash M \to U \backslash P$, bilipschitz with respect to the geodesic distance in $U \backslash M$ and extending to the boundary, such that Φ is the identity in a neighbourhood of ∂U and $\Phi^{-1}(\overline{D^\pm}) = M$. Next, set $\varphi := \phi \circ \Phi^{-1}$ on ∂U and let $\varphi_0 \in \mathrm{L}^1(\partial P)$ be defined on as $\varphi_0 := \phi^\pm \circ \Phi^{-1}$ on $D^\pm$. Then to establish (a) it is enough to apply the standard extension result to the functions φ, φ_0 in the Lipschitz domain $U \setminus P$, arguing as in the proof of [9, Lemma 4.1] to which we refer the reader for more details.

To prove (b) we may argue as follows. We notice that φ and φ_0 defined above are Lipschitz, since $\phi^+ = \phi^-$ on $\partial M = \partial_{\partial P} D^\pm$. Therefore the Kirszbraun Theorem ensures the existence of a Lipschitz function $\overline{\varphi} \in \mathrm{W}^{1,\infty}(U \backslash P)$ extending φ and φ_0. Hence the claim follows setting $\psi := \overline{\varphi} \circ \Phi$. □

To conclude this section we prove a vectorial truncation lemma to promote an L^∞-bound of any $\mathscr{S}$-converging sequence to a bounded SBV^1-function.

Lemma 2.8 (Vectorial truncation). *Let $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ and $(u_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ be such that $u_k \xrightarrow{\mathscr{S}} u$ (resp., $u_k \xrightarrow{\overline{\mathscr{S}}} u$) as $k \to +\infty$. Then there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that $v_k \xrightarrow{\mathscr{S}} u$ (resp., $v_k \xrightarrow{\overline{\mathscr{S}}} u$) as $k \to +\infty$, $S_{v_k} = S_{u_k}$, and $\|v_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$. Moreover, if $u_k \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m)$ then the same holds for v_k.*

Finally, if $\|u_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} + \delta_k$ *for some* $\delta_k > 0$ *such that* $\delta_k \to 0$ *as* $k \to +\infty$*, then also* $\|u_k - v_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le \delta_k$.

Proof. The proof relies on classical arguments and in the scalar case it follows as in [9, Lemma 3.2].

Let $\eta > 0$ be arbitrary and fixed; let $0 < \varepsilon \le \eta$ be fixed depending on u and η as specified later. Set $a^\eta := \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} + \eta$; we start constructing a sequence (w_k^η) such that $\|w_k^\eta\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le a^\eta + \eta$. To this end let $\psi^\eta \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a C^∞ function such that

$$0 < \psi^\eta < a^\eta + \eta \quad \text{in } \mathbb{R}^+, \quad 0 < (\psi^\eta)' \le 1 \quad \text{in } \mathbb{R}^+, \quad \psi^\eta(t) = t \quad \text{in } (0, a^\eta). \tag{2.3}$$

For every $y \in \mathbb{R}^m$ set

$$\varphi^\eta(y) := \frac{y}{|y|}\,\psi^\eta(|y|) \quad \text{for } y \ne 0, \qquad \varphi^\eta(0) = 0.$$

By the definition of ψ^η there holds

$$\varphi^\eta(y) = y \quad \text{if} \quad |y| < a^\eta \quad \text{and} \quad \|\varphi^\eta\|_{\mathrm{L}^\infty(\mathbb{R}^m;\mathbb{R}^m)} \le a^\eta + \eta.$$

Hence φ^η belongs to $\mathrm{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$ and has Lipschitz constant less than or equal to one. Furthermore we observe that φ^η is injective. Indeed, $\varphi^\eta(y_1) = \varphi^\eta(y_2)$ implies that y_1 and y_2 differ by a strictly positive multiplicative constant. Therefore $y_1/|y_1| = y_2/|y_2|$ and in turn $\psi^\eta(|y_1|) = \psi^\eta(|y_2|)$, thus we get $|y_1| = |y_2|$ and finally $y_1 = y_2$.

For every $k \in \mathbb{N}$ set $w_k^\eta := \varphi^\eta(u_k)$; clearly $w_k^\eta \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, $S_{w_k^\eta} = S_{u_k}$ by the injectivity of φ^η, and $w_k^\eta \in \mathrm{C}^\infty(\Omega\backslash\overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega\backslash\overline{S_{u_k}};\mathbb{R}^m)$ if the same holds true for u_k. Moreover, set $A_k^\eta := \{x \in \Omega \colon |u_k| \ge a^\eta\}$, we have

$$\begin{aligned}\|w_k^\eta - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} &\le \|u_k - u\|_{\mathrm{L}^1(\Omega\backslash A_k^\eta;\mathbb{R}^m)} + \|\varphi^\eta(u_k) - \varphi^\eta(u)\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^m)}\\ &\le \|u_k - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)},\end{aligned}$$

where we have used the fact that φ^η has Lipschitz constant less than or equal to one. Now recall that by assumption, for $k \in \mathbb{N}$ large enough there holds

$$\|u_k - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + \|\nabla u_k - \nabla u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})} < \varepsilon. \tag{2.4}$$

We now estimate the L^1 norm of $\nabla w_k^\eta - \nabla u$. We observe that by construction $\nabla w_k^\eta = \nabla u_k$ in $\Omega \setminus A_k^\eta$ while $|\nabla w_k^\eta| \le |\nabla u_k|$ in A_k^η. Moreover, since $|u_k - u| \ge \eta$ in A_k^η, by (2.4) and also invoking the Chebyshev Inequality we deduce that for k large enough $\eta\,\mathscr{L}^n(A_k^\eta) < \varepsilon$, and therefore $\mathscr{L}^n(A_k^\eta) < \varepsilon/\eta$. Hence, choosing ε so small that $\|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} < \eta$ for k large, we obtain

$$\begin{aligned}\|\nabla w_k^\eta - \nabla u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})} &\le \|\nabla u_k - \nabla u\|_{\mathrm{L}^1(\Omega\backslash A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla w_k^\eta - \nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})}\\ &\le \varepsilon + \|\nabla w_k^\eta\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})}\\ &\le \varepsilon + \|\nabla u_k\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})}\\ &\le 2\varepsilon + 2\|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} \le 2\varepsilon + 2\eta,\end{aligned} \tag{2.5}$$

for every k large enough.

As a consequence, if $u_k \xrightarrow{\mathscr{S}} u$ we readily obtain that $w_k^\eta \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$. Moreover, since $(w_k^\eta)^\pm = \varphi^\eta(u_k^\pm)$, arguing as above one can show that

$$\int_{S_{w_k^\eta} \cup S_u} |(w_k^\eta)^\pm - u^\pm| \,\mathrm{d}\mathscr{H}^{n-1} \le \int_{S_{u_k} \cup S_u} |u_k^\pm - u^\pm| \,\mathrm{d}\mathscr{H}^{n-1}.$$

Hence, $w_k^\eta \xrightarrow{\overline{\mathscr{S}}} u$ if $u_k \xrightarrow{\overline{\mathscr{S}}} u$.

Then, to complete the proof, we only need to modify w_k^η in order to obtain functions v_k^η with $\|v_k^\eta\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$. To this end, set

$$v_k^\eta := \frac{a^\eta - \eta}{a^\eta + \eta}\, w_k^\eta;$$

by definition $v_k^\eta \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, $S_{v_k^\eta} = S_{w_k^\eta} = S_{u_k}$, and $v_k^\eta \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m)$ if the same holds for u_k (and hence for w_k^η). Furthermore we have

$$\|v_k^\eta\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq a^\eta - \eta = \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}.$$

In view of the convergence properties of w_k^η, invoking a standard diagonal argument we can find $\eta_k \to 0^+$ as $k \to +\infty$ such that setting $v_k := v_k^{\eta_k}$ we get $v_k \xrightarrow{\mathscr{S}} u$ (resp., $v_k \xrightarrow{\overline{\mathscr{S}}} u$) as $k \to +\infty$ and thus the claim.

We finally prove that if $\|u_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} + \delta_k$ for some $\delta_k > 0$ such that $\delta_k \to 0$ as $k \to +\infty$, then $\|u_k - v_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq \delta_k$. Setting $\eta = \delta_k/5$ by construction we have $\|u_k\|_{\mathrm{L}^\infty} \leq a^\eta + 4\eta$. Now if $x \in \Omega$ is such that $|u_k(x)| < a^\eta$, then $w_k(x) = \varphi^\eta(u_k(x)) = u_k(x)$, cf. (2.3), and so

$$|u_k(x) - v_k(x)| = \Big| u_k(x) - \frac{a^\eta - \eta}{a^\eta + \eta} u_k(x) \Big| = \Big| \frac{2\eta}{a^\eta + \eta} u_k(x) \Big| \leq 2\eta \leq \delta_k.$$

If on the other hand $x \in \Omega$ is such that $|u_k(x)| \geq a^\eta$, then $a^\eta \leq \psi^\eta < a^\eta + \eta$, thus it holds $(a^\eta + \eta)|u_k(x)| \geq (a^\eta - \eta)\, \psi^\eta(|u_k(x)|)$, cf. (2.3) again. Hence, using the definition of v_k we arrive at

$$\begin{aligned} |u_k(x) - v_k(x)| &= \Big| u_k(x) - \frac{(a^\eta - \eta)\, u_k(x)\, \psi^\eta(|u_k|)}{(a^\eta + \eta)|u_k(x)|} \Big| = \frac{(a^\eta + \eta)|u_k(x)| - (a^\eta - \eta)\, \psi^\eta(|u_k|)}{a^\eta + \eta} \\ &\leq \frac{(a^\eta + \eta)(a^\eta + 4\eta) - a^\eta(a^\eta - \eta)}{a^\eta + \eta} \leq 5\eta = \delta_k. \end{aligned}$$

Altogether we have shown that $\|v_k - u_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq \delta_k$ and thus the claim. □

Remark 2.9. Lemma 2.8 can be generalised observing that, if $u \in \mathcal{K}$ a.e. in Ω, where $\mathcal{K} \subset \mathbb{R}^m$ is compact, Lipschitz, star-shaped with respect to the origin and containing a neighbourhood of the origin, then the sequence (v_k) can be chosen in such a way that for every $k \in \mathbb{N}$ there holds $v_k \in \mathcal{K}$ a.e. in Ω. Indeed, for $y \in \mathbb{R}^m$ set $\lambda_{\mathcal{K}}(y) := \inf\{\rho > 0 \colon y \in \rho\mathcal{K}\}$. Notice that in view of the compactness of $\mathcal{K}$ we have $\lambda_{\mathcal{K}}(y) > 0$, for every $y \in \mathbb{R}^m \backslash \{0\}$. Then to find the desired sequence (v_k) it is enough to choose

$$\varphi^\eta(y) := \frac{y}{\lambda_{\mathcal{K}}(y)}\, \psi^\eta(\lambda_{\mathcal{K}}(y)) \ \text{ for } \ y \neq 0, \qquad \varphi^\eta(0) = 0,$$

where ψ^η is as in the proof of Lemma 2.8. As for the regularity, one has $v_k \in \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_{v_k}};\mathbb{R}^m)$ if the same holds for u_k; moreover, $v_k \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{v_k}};\mathbb{R}^m)$ if the same holds for u_k and $\mathcal{K}$ is C^∞.

3. The main result

In this section we state and prove the main result of this paper.

Theorem 3.1 ($\mathscr{S}$-approximation of $\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ functions)**.** *Let $\mathcal{N}\subset\mathbb{S}^{n-1}$ be a Borel set of directions satisfying* (2.1)*. Then the space $\mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ is $\mathscr{S}$-dense in $\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Specifically, for any $u\in\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ there exists a sequence $(u_k)\subset\mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ with $\|u_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}\le\|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$ such that $u_k\xrightarrow{\mathscr{S}}u$ as $k\to+\infty$. Moreover one has*

$$\limsup_{k\to+\infty}\int_{S_{u_k}\cap\overline{A}}\gamma\big(x,u_k^+,u_k^-,\nu_{u_k}\big)\,\mathrm{d}\mathscr{H}^{n-1}\le\int_{S_u\cap\overline{A}}\gamma\big(x,u^+,u^-,\nu_u\big)\,\mathrm{d}\mathscr{H}^{n-1},\tag{3.1}$$

for any open set $A\subset\subset\Omega$ and for any bounded upper semicontinuous function $\gamma:\Omega\times\mathbb{R}^m\times\mathbb{R}^m\times\mathcal{N}\to[0,+\infty)$ with $\gamma(\cdot,a,b,\nu)=\gamma(\cdot,b,a,-\nu)$ for every $a,b\in\mathbb{R}^m$ and $\nu\in\mathcal{N}$.

Before embarking on the proof of Theorem 3.1 we give a preliminary covering lemma which can be directly recovered from [7, Lemma 4.2].

Lemma 3.2 (Fine cover)**.** *Let $\mathcal{N}\subset\mathbb{S}^{n-1}$ be a Borel set, $u\in\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, and $\varepsilon\in(0,1/(6\sqrt{n}))$. Assume that $S_u\subset K$ for some compact $\mathscr{H}^{n-1}$-rectifiable set K. Then there exist a set $K'\subset S_u$ and a finite family of open cubes $\mathcal{Q}_N=\{Q^{\nu_i}_{r_i}(x_i)\}_{i=1}^N$ with $r_i\le\varepsilon$ and a face orthogonal to $\nu_i:=\nu_u(x_i)\in\mathcal{N}$ satisfying:*

(1) $\mathscr{H}^{n-1}(S_u\backslash K')<c\,\varepsilon$;

(2) $Q^{\nu_i}_{r_i}(x_i)$ is centred in S_u, i.e., $x_i\in S_u$ for every $i=1,\dots,N$;

(3) the family $\{\overline{Q^{\nu_i}_{r_i}(x_i)}\}_{i=1}^N$ is pairwise disjoint and $Q^{\nu_i}_{r_i}(x_i)\subset\subset\Omega$ for every $i=1,\dots,N$;

(4) $K'\subset\bigcup_{i=1}^N Q^{\nu_i}_{r_i}(x_i)$ and

$$K'\cap Q^{\nu_i}_{r_i}(x_i)\subset\big\{x+t\nu_i:x\in\Pi^{\nu_i}_{x_i}\cap Q^{\nu_i}_{r_i}(x_i),t\in(-\sqrt{n}\varepsilon r_i,\sqrt{n}\varepsilon r_i)\big\}=:P^{\nu_i}_{\varepsilon r_i}(x_i)$$

for every $i=1,\dots,N$;

(5) $r_i^{n-1}\le\frac{1}{1-\varepsilon}\,\mathscr{H}^{n-1}(K'\cap Q^{\nu_i}_{r_i}(x_i))$ for every $i=1,\dots,N$;

(6) $\sum_{i=1}^N r_i^{n-1}<\frac{1}{1-\varepsilon}\,\mathscr{H}^{n-1}(S_u)$;

(7) for $\mathscr{H}^{n-1}$-a.e. $x\in K'\cap Q^{\nu_i}_{r_i}(x_i)$ there holds

$$|\nu_u(x)-\nu_i|<\varepsilon\quad\text{and}\quad|u^\pm(x)-u^\pm(x_i)|<\varepsilon;\tag{3.2}$$

(8) there exist two sets $S_i^+\subset(2\sqrt{n}\varepsilon r_i,6\sqrt{n}\varepsilon r_i)$, $S_i^-\subset(-6\sqrt{n}\varepsilon r_i,-2\sqrt{n}\varepsilon r_i)$ with $\mathscr{L}^1(S_i^\pm)>0$ such that

$$x+t\nu_i\notin S_u\quad\text{for every } t\in S_i^\pm \text{ and } \mathscr{H}^{n-1}\text{-a.e. } x\in\Pi^{\nu_i}_{x_i}\cap Q^{\nu_i}_{r_i}(x_i)\tag{3.3}$$

and further for every $t\in S_i^\pm$ there exists a set T_i^t satisfying $\mathscr{H}^{n-1}(T_i^t)<c\,\varepsilon r_i^{n-1}$ such that

$$|u^*(x+t\nu_i)-u^\pm(x_i)|<\varepsilon\quad\text{for every } x\in\Pi^{\nu_i}_{x_i}\cap Q^{\nu_i}_{r_i}(x_i)\setminus T_i^t\tag{3.4}$$

where u^* is the precise representative of u.

We are now equipped with all the tools to prove Theorem 3.1. The proof is of constructive nature and follows by successive approximations and regularisations.

Proof of Theorem 3.1. Let $u \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ be chosen and arbitrary. We divide the proof into a number of steps.

We preliminarily assume that $S_u \subset K$ for some compact $\mathscr{H}^{n-1}$-rectifiable set K in order to apply the fine cover of Lemma 3.2 to u. We postpone the general case to the conclusion of the proof.

Step 1: *Reflection argument.* In this step we approximate u with a sequence of functions such that most of their jump set is contained in a finite union of $(n-1)$-dimensional cubes.

Let $\varepsilon = \varepsilon_k = 1/k$. By Lemma 3.2, we obtain a set $K' \subset S_u$ along with a finite collection of pairwise disjoint open cubes $\{Q^{\nu_i}_{r_i}(x_i)\}_{i=1}^N$ and sets $\{P^{\nu_i}_{\varepsilon r_i}(x_i)\}_{i=1}^N$ with $\nu_i := \nu_u(x_i)$, satisfying properties (1)-(8). In particular, we may find $t_i^\pm \in S_i^\pm$ as well as two sets $T_i^\pm$ such that $\mathscr{H}^{n-1}(T_i^\pm) < c\,\varepsilon\, r_i^{n-1}$ and

$$|u(x \pm t_i^\pm \nu_i) - u^\pm(x_i)| < \varepsilon \qquad \text{for every } x \in \Pi^{\nu_i}_{x_i} \cap Q^{\nu_i}_{r_i}(x_i) \setminus T_i^\pm, \tag{3.5}$$

whereby we have identified u with its precise representative. For each $i = 1, \dots, N$ define

$$R_i^+ := \big\{x \in Q^{\nu_i}_{r_i}(x_i) : x\cdot\nu_i \in (0, t_i^+)\big\} \qquad \text{and} \qquad R_i^- := \big\{x \in Q^{\nu_i}_{r_i}(x_i) : x\cdot\nu_i \in (t_i^-, 0)\big\}$$

and let $\psi_i^\pm$ be the respective reflection mappings in $R_i^\pm$ along $\Pi^{\nu_i}_{x_i+\frac{1}{2}t_i^\pm\nu_i} \cap Q^{\nu_i}_{r_i}(x_i)$. Consequently let us define $y_k \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ by

$$y_k(x) := \begin{cases} u(\psi_i^+(x)) & \text{if } x \in R_i^+ \text{ and } x\cdot\nu_i \in (0, \frac{t_i^+}{2}) \text{ for each } i, \\ u(\psi_i^-(x)) & \text{if } x \in R_i^- \text{ and } x\cdot\nu_i \in (\frac{t_i^-}{2}, 0) \text{ for each } i, \\ u(x) & \text{otherwise in } \Omega. \end{cases}$$

Then

$$S_{y_k} \subset \Big(\bigcup_{i=1}^N \Pi^{\nu_i}_{x_i} \cap Q^{\nu_i}_{r_i}(x_i)\Big) \cup \Big(\bigcup_{i=1}^N D^i_\varepsilon\Big) \cup \big(S_u \backslash K'\big) \cup \widehat{K}, \tag{3.6}$$

where $D^i_\varepsilon := \{x + t\nu_i : x \in \partial(\Pi^{\nu_i}_{x_i} \cap Q^{\nu_i}_{r_i}(x_i)),\ t \in (t_i^-/2, t_i^+/2)\}$ and $\widehat{K}$ is the reflected part of $S_u \setminus K'$ by the mappings $\psi_i^\pm$. By construction and by Lemma 3.2 (6)

$$\mathscr{H}^{n-1}\Big(\bigcup_{i=1}^N D^i_\varepsilon\Big) \le c\,\varepsilon \sum_{i=1}^N r_i^{n-1} \le c\,\varepsilon\,\mathscr{H}^{n-1}(S_u). \tag{3.7}$$

Also, $\sup_{k\in\mathbb{N}} \|y_k\|_{\mathrm{BV}(\Omega;\mathbb{R}^m)} < +\infty$, $\|y_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$, and

$$\|\nabla u - \nabla y_k\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})} \le 2\|\nabla u\|_{L^1(\cup_{i=1}^N R_i;\mathbb{R}^{m\times n})},$$

where $R_i := R_i^+ \cup R_i^-$. In view of Lemma 3.2 (6) and recalling that $\varepsilon = 1/k$, we get $\mathscr{L}^n(\cup_{i=1}^N R_i) \le c\,\varepsilon^2 \sum_{i=1}^N r_i^{n-1} \to 0$ as $k \to +\infty$, hence $\nabla y_k \to \nabla u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})$. Similarly we find that $y_k \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$.

Define the $\mathcal{N}$-regular set

$$F_k := \bigcup_{i=1}^N F_k^i := \bigcup_{i=1}^N \overline{\Pi^{\nu_i}_{x_i} \cap Q^{\nu_i}_{r_i}(x_i)}.$$

Note that each F_k^i is a closed $(n-1)$-dimensional cube compactly contained in Ω and the sets F_k^i are pairwise disjoint, cf. Lemma 3.2 (3). Moreover, observe that by (3.6), (3.7), and Lemma 3.2 (1)

$$\mathscr{H}^{n-1}(S_{y_k} \backslash F_k) \le c\,\varepsilon\Big(1 + \mathscr{H}^{n-1}(S_u)\Big). \tag{3.8}$$

Step 2: *Convergence of surface integrals.* Let $A \subset\subset \Omega$ open. This step is devoted to prove that the sequence (y_k) satisfies

$$\limsup_{k\to+\infty} \int_{F_k \cap S_{y_k} \cap \overline{A}} \gamma(x, y_k^+, y_k^-, \nu_{y_k}) \, d\mathscr{H}^{n-1} \leq \int_{S_u \cap \overline{A}} \gamma(x, u^+, u^-, \nu_u) \, d\mathscr{H}^{n-1}, \tag{3.9}$$

for any function γ as in the statement of Theorem 3.1. We notice that (3.9) is an approximated version of (3.1).

We recall that the function γ can be approximated by a decreasing sequence of uniformly continuous functions, see for instance [3, Corollary 1.34]; therefore we now prove (3.9) in the case where γ is uniformly continuous.

To this end fix three open sets $A \subset\subset A' \subset\subset A'' \subset\subset \Omega$ and consider the set of indices $\mathcal{I}_A := \{i \in \{1, \dots N\} \colon Q_{r_i}^{\nu_i}(x_i) \cap \overline{A} \neq \emptyset\}$. Without loss of generality let us choose k large enough so that $F_k^i \subset A'$ for any $i \in \mathcal{I}_A$. In addition for $\delta > 0$ let us consider the modulus of uniform continuity of γ:

$$\omega_\gamma(\delta) := \sup \big\{ |\gamma(x,a,b,\nu) - \gamma(x',a',b',\nu')| \colon (x,a,b,\nu),\ (x',a',b',\nu') \in \overline{A'} \times B_{\|u\|_\infty} \times B_{\|u\|_\infty} \times \mathcal{N}, \\ |x-x'| < \delta\,,\ |a-a'| < \delta\,,\ |b-b'| < \delta\,,\ |\nu-\nu'| < \delta \big\}.$$

Let us also recall that Lemma 3.2 (7) ensures

$$|u^\pm(x) - u^\pm(x_i)| < \varepsilon \quad \text{and} \quad |\nu_u(x) - \nu_i| < \varepsilon \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in K' \cap Q_{r_i}^{\nu_i}(x_i).$$

Therefore using the above inequalities along with Lemma 3.2 (4)-(5) and the fact that $K' \subset S_u$ we obtain

$$\begin{aligned} &\sum_{i\in\mathcal{I}_A} r_i^{n-1}\, \gamma(x_i, u^+(x_i), u^-(x_i), \nu_i) \\ &\leq \frac{1}{1-\varepsilon} \sum_{i\in\mathcal{I}_A} \mathscr{H}^{n-1}(K' \cap Q_{r_i}^{\nu_i}(x_i))\, \gamma(x_i, u^+(x_i), u^-(x_i), \nu_i) \\ &\leq \frac{1}{1-\varepsilon} \sum_{i\in\mathcal{I}_A} \int_{K'\cap Q_{r_i}^{\nu_i}(x_i)\cap\overline{A'}} \Big(\gamma(x, u^+, u^-, \nu_u) + \omega_\gamma(\sqrt{n}\,\varepsilon)\Big)\, d\mathscr{H}^{n-1} \\ &\leq \frac{1}{1-\varepsilon} \left(\int_{S_u\cap\overline{A'}} \gamma(x, u^+, u^-, \nu_u)\, d\mathscr{H}^{n-1} + \sum_{i\in\mathcal{I}_A} c\, r_i^{n-1} \omega_\gamma(\sqrt{n}\,\varepsilon) \right). \end{aligned} \tag{3.10}$$

On the other hand we know from Lemma 3.2 (8) as well as from the very construction of y_k that

$$|y_k^\pm(x) - u^\pm(x_i)| < \varepsilon \quad \text{for every } x \in F_k^i \setminus (T_i^+ \cup T_i^-).$$

Consequently,

$$\begin{aligned} &\int_{F_k\cap S_{y_k}\cap\overline{A}} \gamma(x, y_k^+, y_k^-, \nu_{y_k})\, d\mathscr{H}^{n-1} \leq \sum_{i\in\mathcal{I}_A} \int_{F_k^i} \gamma(x, y_k^+, y_k^-, \nu_{y_k})\, d\mathscr{H}^{n-1} \\ &\leq \sum_{i\in\mathcal{I}_A} \int_{F_k^i\setminus(T_i^+\cup T_i^-)} \gamma(x, y_k^+, y_k^-, \nu_{y_k})\, d\mathscr{H}^{n-1} + \|\gamma\|_{\infty,A''} \sum_{i=1}^N \mathscr{H}^{n-1}(T_i^+ \cup T_i^-) \\ &\leq \sum_{i\in\mathcal{I}_A} r_i^{n-1} \Big(\gamma(x_i, u^+(x_i), u^-(x_i), \nu_u(x_i)) + \omega_\gamma(\sqrt{n}\,\varepsilon)\Big) + c\|\gamma\|_{\infty,A''}\, \varepsilon \sum_{i=1}^N r_i^{n-1} \\ &\leq \sum_{i\in\mathcal{I}_A} r_i^{n-1} \Big(\gamma(x_i, u^+(x_i), u^-(x_i), \nu_u(x_i)) + \omega_\gamma(\sqrt{n}\,\varepsilon)\Big) + c\|\gamma\|_{\infty,A''} \frac{\varepsilon}{1-\varepsilon} \mathscr{H}^{n-1}(S_u), \end{aligned}$$

where $\|\gamma\|_{\infty,A''} := \|\gamma\|_{\mathrm{L}^\infty(\overline{A''}\times B_{\|u\|_\infty}\times B_{\|u\|_\infty}\times\mathcal{N})}$ and we used Lemma 3.2 (6) and (8) in the last two inequalities. Altogether using the above estimate in conjunction with (3.10) followed by taking the limit superior and $A' \searrow A$ yields (3.9) as desired.

In the following steps we perform a modification of each component of y_k. For brevity, abusing notation we suppress the indication of the component, regarding u and y_k as scalar functions. Also note that the approximants we are going to introduce are close to y_k in the $\mathrm{BV}(\Omega)$-norm, as we shall detail below.

Step 3: *First regularisation.* In this step we replace $y_k \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega)$ with a sequence of functions whose jump sets are $\mathcal{N}$-aligned and contained in the finite union of $(n-1)$-dimensional closed cubes.

Let $k \in \mathbb{N}$ be fixed, recall that $\varepsilon = \varepsilon_k = 1/k$, and let $v_{\varepsilon_k} \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega)\cap \mathrm{L}^\infty(\Omega)\cap \mathrm{C}^\infty(\Omega\backslash F_k)$ be the functions obtained by applying Lemma 2.5 and Remark 2.6 to $u = y_k$ and $F = F_k$. Then Lemma 2.5 (1) implies $\|v_{\varepsilon_k} - y_k\|_{\mathrm{L}^1(\Omega)} < \varepsilon_k$ while (2) yields $S_{v_{\varepsilon_k}} = S_{y_k} \cap F_k$, and therefore $S_{y_k} \setminus S_{v_{\varepsilon_k}} = S_{y_k} \setminus F_k$. Moreover by Lemma 2.5 (3) and (3.8) we have

$$\begin{aligned} |Dy_k - Dv_{\varepsilon_k}|(\Omega) &\leq 3|Dy_k \llcorner (S_{y_k}\backslash F_k)|(\Omega) + (2|Dy_k|(\Omega)+1)\,\varepsilon_k \\ &\leq 3\|u\|_{\mathrm{L}^\infty(\Omega)}\,\mathscr{H}^{n-1}(S_{y_k}\backslash F_k) + (2|Dy_k|(\Omega)+1)\,\varepsilon_k \\ &\leq c\varepsilon_k\|u\|_{\mathrm{L}^\infty(\Omega)}\Big(1+\mathscr{H}^{n-1}(S_u)\Big) + (2|Dy_k|(\Omega)+1)\,\varepsilon_k. \end{aligned}$$

Since $\sup_{k\in\mathbb{N}}|Dy_k|(\Omega) < +\infty$, as $k\to+\infty$ the convergence $|Dy_k - Dv_{\varepsilon_k}|(\Omega) \to 0$ holds and in particular

$$\|\nabla y_k - \nabla v_{\varepsilon_k}\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0. \tag{3.11}$$

On the other hand we have

$$\int_{F_k\cap\overline{A}} \gamma(x, v^+_{\varepsilon_k}, v^-_{\varepsilon_k}, \nu_{v_{\varepsilon_k}})\,\mathrm{d}\mathscr{H}^{n-1} = \int_{F_k\cap S_{y_k}\cap\overline{A}} \gamma(x, y^+_k, y^-_k, \nu_{y_k})\,\mathrm{d}\mathscr{H}^{n-1}, \tag{3.12}$$

since $v^\pm_{\varepsilon_k} = y^\pm_k$ in F_k by Lemma 2.5 (2).

Note that as a consequence of the construction carried out in this step we know that v_{ε_k} is smooth outside F_k. However, in the next step we may lose this property, hence we will need to perform again the regularisation by convolution with variable kernels provided by Lemma 2.5.

Step 4: *Closing the discontinuity gap.* At this stage we only know $S_{v_{\varepsilon_k}} \subset F_k$, so we modify the approximating sequence in such a way that (the closure of) its discontinuity set coincides with F_k.

For every $i = 1,\dots,N_k$ we may find open sets Ω^i_k, pairwise disjoint, with smooth boundary, such that $F^i_k \subset\subset \Omega^i_k \subset\subset \Omega$. Let $\varphi^i_k : F^i_k \to \mathbb{R}$ be a C^1 function such that $\varphi^i_k > 0$ in $F^i_k \setminus \partial_{\Pi^{\nu_i}_{x^k_i}} F^i_k$, $\varphi^i_k = 0$ on $\partial_{\Pi^{\nu_i}_{x^k_i}} F^i_k$ (where $\partial_{\Pi^{\nu_i}_{x^k_i}} F^i_k$ denotes the boundary of F^i_k in the relative topology induced by $\Pi^{\nu_i}_{x^k_i}$) and

$$\|\varphi^i_k\|_{\mathrm{C}^1(F^i_k)} \leq \min\Big\{1, \frac{1}{N_k\, c_{i,k}}\Big\}, \tag{3.13}$$

where $c_{i,k} > 0$ is the constant from Lemma 2.7, applied to $U = \Omega^i_k$ and $M = F^i_k$. Choosing $\phi \equiv 0$, $\phi^+ = \varphi^i_k$, and $\phi^- \equiv 0$, Lemma 2.7 (b) provides us with a function $\psi^i_k \in \mathrm{W}^{1,\infty}(\Omega^i_k\backslash F^i_k)$ such that $\psi^i_k = 0$ on $\partial\Omega^i_k$, $(\psi^i_k)^+ = \varphi^i_k$ and $(\psi^i_k)^- = 0$ in F^i_k. Note that $\|\psi^i_k\|_{\mathrm{W}^{1,\infty}(\Omega^i_k\backslash F^i_k)} \leq \frac{1}{N_k}$.

Now we define

$$w_k := \begin{cases} v_{\varepsilon_k} + \delta_k \psi_k^i & \text{in } \Omega_k^i \text{ for every } i \in \{1, \dots, N_k\}, \\ v_{\varepsilon_k} & \text{otherwise in } \Omega, \end{cases}$$

where $\delta_k > 0$ is to be determined in forthcoming manner. Inspecting the jump points of w_k in F_k we readily deduce that $S_{w_k} \subset F_k$ for all $k \in \mathbb{N}$ and the inequality $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) > 0$ is only true for at most countably many $\delta_k \in \mathbb{R}$. This follows from a standard argument (see, *e.g.*, [6, Step 4 of the proof of Theorem 3.9]): consider the pairwise disjoint sets defined for $t \in \mathbb{R}$ by $\Sigma_t^i := \{x \in F_k^i \colon [v_{\varepsilon_k}](x) + t\psi_k^i(x) = 0\}$ for every i; since $\mathscr{H}^{n-1}(F_k^i) < +\infty$ and $\{\Sigma_t^i\}_{t\in\mathbb{R}}$ partitions F_k^i, there exist at most countably many $t \in \mathbb{R}$ such that $\mathscr{H}^{n-1}(\Sigma_t^i) > 0$. In other words there exists an infinitesimal positive sequence (δ_k) such that $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) = 0$ for all $k \in \mathbb{N}$ and this shall be our choice in the definition of w_k.

For k large enough Lemma 2.7 (b) and (3.13) imply

$$\begin{aligned} &\|w_k - v_{\varepsilon_k}\|_{\mathrm{BV}(\Omega)} + \int_{S_{v_{\varepsilon_k}} \cup S_{w_k}} \big(|w_k^+ - (v_{\varepsilon_k})^+| + |w_k^- - (v_{\varepsilon_k})^-|\big)\, \mathrm{d}\mathscr{H}^{n-1} \\ &\le \delta_k \sum_{i=1}^{N_k} \|\psi_k^i\|_{\mathrm{BV}(\Omega_k^i)} + \delta_k \sum_{i=1}^{N_k} \|\varphi_k^i\|_{\mathrm{L}^1(F_k^i)} \\ &\le \delta_k \sum_{i=1}^{N_k} \|\psi_k^i\|_{\mathrm{W}^{1,\infty}(\Omega_k^i \backslash F_k^i)} + 2\delta_k \sum_{i=1}^{N_k} \|\varphi_k^i\|_{\mathrm{L}^1(F_k^i)} \le 3\delta_k. \end{aligned}$$

Also note that $\|w_k\|_{\mathrm{L}^\infty(\Omega)} \le \|u\|_{\mathrm{L}^\infty(\Omega)} + \delta_k$; hence, recalling that by construction $\nu_{w_k} = \nu_{v_{\varepsilon_k}}$ and $|w_k^\pm - v_{\varepsilon_k}^\pm| \le \delta_k$ on F_k, we have

$$\int_{F_k \cap \overline{A}} \Big(\gamma(x, w_k^+, w_k^-, \nu_{w_k}) - \gamma(x, v_{\varepsilon_k}^+, v_{\varepsilon_k}^-, \nu_{v_{\varepsilon_k}})\Big)\, \mathrm{d}\mathscr{H}^{n-1} \le \omega_\gamma(\delta_k)\, \mathscr{H}^{n-1}(F_k \cap \overline{A}), \tag{3.14}$$

where ω_γ is modulus of uniform continuity of γ as in Step 2.

In addition we claim $F_k = \overline{S_{w_k}}$ which then shows that $\overline{S_{w_k}}$ is an $\mathcal{N}$-aligned regular disconnected set in the sense of Definition 2.1. Indeed suppose there exists $x \in F_k \backslash \overline{S_{w_k}} \subset \Omega \backslash \overline{S_{w_k}}$, then we can find a radius $r > 0$ such that $B_r(x) \subset \Omega$ and $B_r(x) \cap \overline{S_{w_k}} = \emptyset$. Therefore we may deduce

$$\mathscr{H}^{n-1}(F_k \cap B_r(x)) = \mathscr{H}^{n-1}(\overline{S_{w_k}} \cap B_r(x)) = 0,$$

which leads to a contradiction since by definition of F_k we clearly have $\mathscr{H}^{n-1}(F_k \cap B_r(x)) > 0$. Hence $\overline{S_{w_k}}$ is an $\mathcal{N}$-aligned regular disconnected set. Finally since $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) = 0$ and $F_k = \overline{S_{w_k}}$, we observe that S_{w_k} is essentially closed for all $k \in \mathbb{N}$.

Step 5: *Final regularisation.* In order to conclude that the constructed approximants are in the admissible set, it remains to regularise every function w_k outside F_k. Applying Lemma 2.5, cf. parts (2) and (4), with $u = w_k$, $F = F_k$, and $\varepsilon = 1/k$, we obtain $(u_k) \subset \mathrm{C}^\infty(\Omega \backslash F_k)$ such that $S_{u_k} = S_{w_k}$, $(u_k)^\pm = (w_k)^\pm$ in S_{u_k}, $u_k \to u$ in $\mathrm{L}^1(\Omega)$, and $\nabla u_k \to \nabla u$ in $\mathrm{L}^1(\Omega)$ (see also Remark 2.6). Arguing as in Step 3, in conjunction with Step 4 we conclude that $(u_k) \subset \mathcal{W}_\mathcal{N}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\overline{S_{u_k}} = F_k$. Moreover, by (3.9), (3.12), and (3.14), we obtain (3.1). Therefore, in particular, choosing $\gamma \equiv 1$ we get

$$\limsup_{k \to +\infty} \mathscr{H}^{n-1}(S_{u_k} \cap \overline{A}) \le \mathscr{H}^{n-1}(S_u \cap \overline{A}).$$

The latter, in combination with the Compactness Theorem in SBV, see *e.g.* [1, p. 128], gives $\mathscr{H}^{n-1}(S_{u_k}) \to \mathscr{H}^{n-1}(S_u)$, and hence u_k $\mathscr{S}$-converges to u as $k \to +\infty$.

Returning to the vectorial setup, now writing $u = (u^1, \dots, u^m)$ with a slight abuse of notation, for every $l \in \{1, \dots, m\}$ we have found a sequence u_k^l so that the equality $\overline{S_{u_k^l}} = F_k$ is true for all l, cf. Step 4. Hence, setting $u_k := (u_k^1, \dots, u_k^m)$, there holds $u_k \in \mathcal{W}_{\mathcal{N}}(\Omega; \mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega; \mathbb{R}^m)$ and u_k $\mathscr{S}$-converges to u as $k \to +\infty$ and (3.1) holds. By now we have $\|u_k\|_{\mathrm{L}^\infty(\Omega)} \leq \|u\|_{\mathrm{L}^\infty(\Omega)} + \delta_k$, cf. Step 4.

Finally, appealing to the truncation of Lemma 2.8 we find a sequence $(\hat{u}_k)$ enjoying all previous properties, as well as $\|\hat{u}_k\|_{\mathrm{L}^\infty} \leq \|u\|_{\mathrm{L}^\infty}$ and $\|\hat{u}_k - u_k\|_{\mathrm{L}^\infty} < \delta_k$. Moreover we have

$$\int_{F_k \cap \overline{A}} \Big(\gamma(x, \hat{u}_k^+, \hat{u}_k^-, \nu_{\hat{u}_k}) - \gamma(x, u_k^+, u_k^-, \nu_{u_k}) \Big) \, \mathrm{d}\mathscr{H}^{n-1} \leq \omega_\gamma(\delta_k) \, \mathscr{H}^{n-1}(F_k \cap \overline{A}).$$

Thus without loss of generality we may assume that $\|w_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$.

Step 6: *Conclusion.* In this final step we remove the assumption that $S_u \subset K$ for some compact $\mathscr{H}^{n-1}$-rectifiable set K. In fact, if this is not the case, we may reduce to the aforementioned scenario by appealing to a convolution argument as in Lemma 2.5. Namely, given $\eta > 0$, from the properties of Radon measures, we may find a compact, $\mathscr{H}^{n-1}$-rectifiable set $K_\eta \subset S_u$ such that $\mathscr{H}^{n-1}(S_u \backslash K_\eta) < \eta$. Arguing as in Remark 2.6 for each component of u, we obtain functions u_η whose jump set is contained in a compact $\mathscr{H}^{n-1}$-rectifiable set. Given $\varepsilon = 1/k$, we apply the above construction to u_η, by suitably choosing $\eta = \eta(\varepsilon) > 0$ in such a way the properties of Lemma 3.2 hold. We conclude by recalling that u_η $\overline{\mathscr{S}}$-converges to u as $\eta \to 0$ and arguing as in Step 3 to obtain (3.1). □

Remark 3.3. Theorem 3.1 provides us with a sequence $u_k := (u_k^1, \dots, u_k^m)$ such that $u_k \overset{\mathscr{S}}{\to} u$ as $k \to +\infty$. However, it does *not* guarantee the componentwise convergence $u_k^l \overset{\mathscr{S}}{\to} u^l$ since by construction $\overline{S_{u_k^1}} = \dots = \overline{S_{u_k^m}} = F_k$. We notice that the same happens in [6].

On the other hand, Theorem 3.1 applied to each component of u provides us with m sequences $(u_k^l)_k$, $l = 1, \dots, m$, such that $u_k^l \overset{\mathscr{S}}{\to} u^l$. However, the jump sets $S_{u_k^l}$ may overlap; hence, $\overline{S_{u_k}}$ is an $\mathcal{N}$-aligned regular set, but in general not disconnected.

4. Approximation results in the unconstrained case

For the specific choice of $\mathcal{N} = \mathbb{S}^{n-1}$ Theorem 3.1 corresponds to the SBV^1-variant of the celebrated approximation result of Cortesani and Toader [7]. In this section we establish a link between the two aforementioned results by showing that it is possible to construct approximations of elements in $\mathrm{SBV}^1(\Omega; \mathbb{R}^m)$ by local minimisers of the Mumford-Shah functional. Such approximation is the core of the proof strategy in [6, 7].

We first prove that SBV^1 can be approximated by SBV^p functions with respect to the $\overline{\mathscr{S}}$-convergence. For simplicity, we consider here only the scalar case. In the vectorial case, the result may be applied componentwise, thus the jump set will be the union of C^1 manifolds.

Proposition 4.1 ($\mathrm{SBV}^p(\Omega)$-approximation)**.** *Let* $u \in \mathrm{SBV}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *and* $p \in (1, +\infty)$. *Then there exists a sequence* $(v_k) \subset \mathrm{SBV}^p(\Omega) \cap \mathrm{L}^\infty(\Omega)$ *such that* $M_k := \overline{S_{v_k}}$ *is a compact* $(n{-}1)$*-dimensional* C^1 *manifold in* Ω *with* C^1 *boundary* ∂M_k, *the traces of* v_k *on both sides of* M_k *are* C^1 *and coincide on* ∂M_k, *and* $v_k \xrightarrow{\mathscr{S}} u$ *as* $k \to +\infty$.

Proof. The proof follows arguments in the spirit of [9, Lemma 4.5] and is divided into three steps.

Step 1: *Preliminary regularisations.* Fix $\varepsilon > 0$. By the rectifiability of S_u, we can find a compact $(n{-}1)$-dimensional C^1 manifold $M_\varepsilon \subset\subset \Omega$ with C^1 boundary and a finite number of connected components, such that

$$\mathscr{H}^{n-1}(S_u \triangle M_\varepsilon) < \varepsilon. \tag{4.1}$$

Moreover, by the properties of Radon measures there exists a compact set $K_\varepsilon \subset S_u \cap M_\varepsilon$ with

$$\mathscr{H}^{n-1}(S_u \triangle K_\varepsilon) = \mathscr{H}^{n-1}(S_u \backslash K_\varepsilon) < \varepsilon. \tag{4.2}$$

Therefore appealing to Lemma 2.5 with $F = K_\varepsilon$ we get a function $u_\varepsilon \in \mathrm{SBV}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ satisfying the following properties:

$$u_\varepsilon \in \mathrm{C}^\infty(\Omega \backslash K_\varepsilon), \quad \|u - u_\varepsilon\|_{\mathrm{L}^1(\Omega)} < \varepsilon, \quad u_\varepsilon^\pm = u^\pm \text{ in } K_\varepsilon, \quad S_{u_\varepsilon} = S_u \cap K_\varepsilon. \tag{4.3}$$

Moreover, gathering (4.2) and the last equality in (4.3) gives

$$\mathscr{H}^{n-1}(S_u \triangle S_{u_\varepsilon}) < \varepsilon,$$

whereas (4.2) combined with Lemma 2.5 yields

$$\|\nabla u - \nabla u_\varepsilon\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} < c\,\varepsilon.$$

We now show the smallness of the traces of $u - u_\varepsilon$. To this end we observe that in view of Lemma 2.5 (2) and (4) we have

$$\begin{aligned}\int_{S_{u_\varepsilon} \cup S_u} \left(|u_\varepsilon^+ - u^+| + |u_\varepsilon^- - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} &= \int_{S_u \backslash K_\varepsilon} \left(|u_\varepsilon^+ - u^+| + |u_\varepsilon^- - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} \\ &\leq 4\|u\|_{\mathrm{L}^\infty(\Omega)} \mathscr{H}^{n-1}(S_u \backslash K_\varepsilon),\end{aligned}$$

hence the claim again follows by (4.2).

Step 2: C^1*-modification of inner traces.* Let $\{M_\varepsilon^i\}_{i=1}^{N_\varepsilon}$ be the connected components of M_ε and let $\{U^i\}_{i=1}^{N_\varepsilon}$ be pairwise disjoint open sets with smooth boundary such that $M_\varepsilon^i \subset\subset U^i \subset\subset \Omega$. Let $i \in \{1, \ldots, N_\varepsilon\}$ be fixed and and let $\phi_i^+, \phi_i^- \in \mathrm{C}_c^1(M_\varepsilon^i)$ be such that

$$\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)} \leq \frac{\varepsilon}{(1 + c_\varepsilon^i)N_\varepsilon},$$

where $c_\varepsilon^i := c(U^i, M_\varepsilon^i) > 0$ is the constant from Lemma 2.7 applied with $U = U^i$ and $M = M_\varepsilon^i$. Subsequently by Lemma 2.7 (a) we find $\psi_{\varepsilon,i} \in \mathrm{W}^{1,1}(U^i \backslash M_\varepsilon^i) \cap \mathrm{L}^\infty(\Omega)$ such that $\psi_{\varepsilon,i} = 0$ on ∂U^i, $\psi_{\varepsilon,i}^\pm = \phi_i^\pm - u_\varepsilon^\pm$ on M_ε^i and

$$\|\psi_{\varepsilon,i}\|_{\mathrm{W}^{1,1}(U^i \backslash M_\varepsilon^i)} \leq \frac{c_\varepsilon^i\,\varepsilon}{(1 + c_\varepsilon^i)N_\varepsilon}.$$

Define

$$w_\varepsilon := \begin{cases} \psi_{\varepsilon,i} + u_\varepsilon & \text{in } U^i, \\ u_\varepsilon & \text{otherwise in } \Omega, \end{cases}$$

in which case

$$\begin{aligned}
&\|u_\varepsilon - w_\varepsilon\|_{\mathrm{BV}(\Omega)} + \int_{S_{u_\varepsilon}\cup S_{w_\varepsilon}} \big(|w_\varepsilon^+ - u_\varepsilon^+| + |w_\varepsilon^- - u_\varepsilon^-|\big)\,\mathrm{d}\mathscr{H}^{n-1} \\
&\leq \sum_{i=1}^{N_\varepsilon} \|\psi_{\varepsilon,i}\|_{\mathrm{BV}(U^i)} + \sum_{i=1}^{N_\varepsilon} \Big(\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)}\Big) \\
&\leq \sum_{i=1}^{N_\varepsilon} \|\psi_{\varepsilon,i}\|_{\mathrm{W}^{1,1}(U^i\backslash M_\varepsilon^i)} + 2\sum_{i=1}^{N_\varepsilon} \Big(\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)}\Big) \leq 3\varepsilon.
\end{aligned}$$

We observe that the traces $w_\varepsilon^\pm$ coincide on ∂M_ε since $\phi_i^\pm$ are compactly supported and thus coincide on ∂M_ε. Arguing as in the proof of Step 4, Theorem 3.1 (see also [9, Lemma 4.3]), we can construct $\hat{w}_\varepsilon \in \mathrm{SBV}^1(\Omega)\cap \mathrm{L}^\infty(\Omega)\cap \mathrm{W}^{1,1}(\Omega\backslash M_\varepsilon)$ such that $\mathscr{H}^{n-1}(M_\varepsilon\backslash S_{\hat{w}_\varepsilon}) = 0$, $\hat{w}_\varepsilon^\pm$ are C^1-regular, they coincide on ∂M_ε, and

$$\|\hat{w}_\varepsilon - w_\varepsilon\|_{\mathrm{BV}(\Omega)} + \int_{S_{\hat{w}_\varepsilon}\cup S_{w_\varepsilon}} \big(|\hat{w}_\varepsilon^+ - w_\varepsilon^+| + |\hat{w}_\varepsilon^- - w_\varepsilon^-|\big)\,\mathrm{d}\mathscr{H}^{n-1} < c\,\varepsilon.$$

Therefore $\hat{w}_\varepsilon$ $\overline{\mathscr{S}}$-converges to u as $\varepsilon\to 0$.

Step 3: *Final regularisation.* In this step we apply successive modifications of $\hat{w}_\varepsilon$ to obtain higher integrability of the approximate gradients. The goal is to first obtain a sequence which is locally Lipschitz outside of M_ε and then apply Meyers-Serrin type regularisation.

Since the traces $\hat{w}_\varepsilon$ are C^1 we may employ Lemma 2.7 (b), for every $i\in\{1,\dots,N_\varepsilon\}$ to find $z_{\varepsilon,i}\in \mathrm{W}^{1,\infty}(U^i\backslash M_\varepsilon^i)$ such that $z_{\varepsilon,i}=0$ on ∂U^i and $z_{\varepsilon,i}^\pm = \hat{w}_\varepsilon^\pm$ in M_ε^i for every $i\in\{1,\dots,N_\varepsilon\}$. Set $z_\varepsilon := z_{\varepsilon,i}$ in U^i, $z_\varepsilon := 0$ otherwise. Since $\hat{w}_\varepsilon - z_\varepsilon \in \mathrm{W}^{1,1}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ we find $\eta_\varepsilon\in \mathrm{W}^{1,1}(\Omega)\cap \mathrm{C}^\infty(\Omega)$ such that $\|\hat{w}_\varepsilon - z_\varepsilon - \eta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)}\leq\varepsilon$. Consequently define $\hat{z}_\varepsilon\in\mathrm{SBV}^1(\Omega)$ by

$$\hat{z}_\varepsilon := z_\varepsilon + \eta_\varepsilon.$$

Then by construction $\hat{z}_\varepsilon\in \mathrm{W}^{1,\infty}(\Omega'\backslash M_\varepsilon)\cap \mathrm{L}^\infty(\Omega)$ for every $\Omega'\subset\subset\Omega$, $S_{\hat{z}_\varepsilon} = S_{\hat{w}_\varepsilon}$, $[\hat{z}_\varepsilon] = [\hat{w}_\varepsilon]$ and

$$\|\hat{w}_\varepsilon - \hat{z}_\varepsilon\|_{\mathrm{BV}(\Omega)} = \|\hat{w}_\varepsilon - z_\varepsilon - \eta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} < \varepsilon.$$

Now let $\varphi\in \mathrm{C}_c^\infty(\Omega;[0,1])$ be such that $\varphi = 1$ in a neighbourhood of M_ε. Since $(1-\varphi)\hat{z}_\varepsilon\in \mathrm{W}^{1,1}(\Omega)\cap \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$, for any $1<p<\infty$ there exists a function $\zeta_\varepsilon\in \mathrm{W}^{1,p}(\Omega)\cap\mathrm{L}^\infty(\Omega)\cap\mathrm{C}^\infty(\Omega)$ such that $\|(1-\varphi)\hat{z}_\varepsilon - \zeta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)}<\varepsilon$. Eventually, define

$$v_\varepsilon := \varphi\hat{z}_\varepsilon + \zeta_\varepsilon;$$

from the construction above we have $\varphi\hat{z}_\varepsilon\in\mathrm{W}^{1,\infty}(\Omega\backslash M_\varepsilon)$ which ultimately implies the inclusion $\varphi\hat{z}_\varepsilon\in\mathrm{SBV}^p(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and thus $v_\varepsilon\in\mathrm{SBV}^p(\Omega)\cap\mathrm{L}^\infty(\Omega)$. Furthermore $S_{v_\varepsilon} = S_{\hat{z}_\varepsilon}$, $[v_\varepsilon] = [\hat{z}_\varepsilon]$ and

$$\|v_\varepsilon - \hat{z}_\varepsilon\|_{\mathrm{BV}(\Omega)} = \|(1-\varphi)\hat{z}_\varepsilon - \zeta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} < \varepsilon.$$

Altogether this concludes the proof. □

Remark 4.2. For later reference we observe the following variant of the previous proposition, adapted to the constrained case where $u\in\mathrm{SBV}^1_{\mathcal{N}}(\Omega)$, as in Theorem 3.1.

Let $\mathcal{N}\subset\mathbb{S}^{n-1}$ be a Borel set of directions satisfying (2.1). Let $u\in\mathrm{SBV}^1_{\mathcal{N}}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ and $p\in(1,+\infty)$. Then there exists a sequence $(v_k)\subset\mathrm{SBV}^p(\Omega)\cap\mathrm{L}^\infty(\Omega)$ such that $M_k := \overline{S_{v_k}}$ is a compact $(n-1)$-dimensional C^1 manifold in Ω with C^1 boundary ∂M_k and normal in $\mathcal{N}$, the traces of v_k on both sides of M_k are C^1 and coincide on ∂M_k, and $v_k \xrightarrow{\mathscr{S}} u$ as $k\to+\infty$.

In order to prove this statement, we first apply Theorem 3.1 and find an approximating sequence $(u_k) \subset \mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ such that $u_k \xrightarrow{\mathscr{S}} u$. Since $\overline{S_{u_k}}$ is a $\mathcal{N}$-aligned regular disconnected set, it is possible to repeat the proof of Proposition 4.1 applied to u_k, now choosing $M_\varepsilon \subset S_{u_k}$ in (4.1). A diagonal argument yields the desired statement.

We conclude this section observing that Proposition 4.1 combined with [4, Lemma 5.2], readily gives an approximation of $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$-functions by means of local minimisers of the Mumford-Shah functional:

$$\mathrm{MS}(u,U) := \int_U |\nabla u|^2 \,\mathrm{d}x + \mathscr{H}^{n-1}(S_u \cap U),$$

with $U \subset \Omega$ open and bounded.

We recall that $v \in \mathrm{SBV}^2(\Omega) \cap \mathrm{L}^\infty(\Omega)$ is a local minimiser for $\mathrm{MS}(\cdot,\Omega)$ if $\mathrm{MS}(v,U) \leq \mathrm{MS}(w,U)$ for every open set $U \subset\subset \Omega$, whenever $w \in \mathrm{SBV}^2(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\{w \neq v\} \subset\subset U \subset\subset \Omega$.

Then, the following approximation result holds.

Corollary 4.3. *Let $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Then there exists a sequence $(u_k) \subset \mathrm{SBV}^2(\Omega) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that each u_k is a local minimiser for $\mathrm{MS}(\cdot,\Omega)$ and $u_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$.*

Proof. By Proposition 4.1 applied to (each component of) u with $p = 2$ there exists a sequence $(v_k) \subset \mathrm{SBV}^2(\Omega) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that v_k $\overline{\mathscr{S}}$-converges to u as $k \to +\infty$. Now invoking [4, Lemma 5.2], for each v_k there exists a sequence $(u_k^j) \subset \mathrm{SBV}^2(\Omega) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ of local minimisers for $\mathrm{MS}(\cdot,\Omega)$ such that u_k^j $\overline{\mathscr{S}}$-converges to v_k as $j \to +\infty$. A standard diagonal argument yields the desired sequence and therefore the result. □

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