STRONG APPROXIMATION OF SBV FUNCTIONS WITH PRESCRIBED JUMP DIRECTION

GIULIANO LAZZARONI, PIOTR WOZNIAK, AND CATERINA IDA ZEPPIERI

Abstract. In this note we show that SBV functions with jump normal lying in a prescribed set of directions $\mathcal{N}$ can be approximated by sequences of SBV functions whose jump set is essentially closed, polyhedral, and preserves the orthogonality to $\mathcal{N}$, moreover the functions are smooth away from their jump set. This approximation result is proven with respect to a strong convergence for which a large class of free-discontinuity functionals is continuous.

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1. Introduction

In the proof of approximation or homogenisation results of *free-discontinuity functionals* one is concerned with the construction of a sequence of functions, the so-called *recovery sequence*, along which a certain functional upper-bound inequality shall be satisfied, the so-called Γ*-limsup inequality* (see, *e.g.*, [6]). The construction of a recovery sequence is often nontrivial and in most cases it is only feasible after assuming some additional regularity of the target function. In a particular instance, if the considered object belongs to the space of *special functions of bounded variation*, SBV, it is of crucial importance to replace it with a function whose jump set is as simple as possible, typically *polyhedral*, as well as to attain sufficient smoothness of the function away from its jump set. In this respect the mathematical literature provides us with a number of approximation results for SBV functions which are, moreover, tailor-made to deal with the aforementioned upper-bound inequalities; see, *e.g.*, [2, 7], as well as [4, 5, 9] for approximants with polyhedral jump set.

However, should the target SBV functions satisfy some geometric constraint arising in the problem under examination, the available approximation results may fail to preserve this additional constraint. In the context of variational methods for fracture and image segmentation, in this paper we establish a density result for SBV functions with prescribed jump direction, describing, *e.g.*, deformations of materials with cracks appearing only along certain directions.

If $\Omega \subset \mathbb{R}^n$ is open, bounded, with Lipschitz boundary and $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$, the prototypical free-discontinuity functionals we consider are of the form

$$\mathcal{F}(u) = \int_\Omega |\nabla u| \,\mathrm{d}\mathscr{L}^n + \int_{S_u} \gamma(x, u^+, u^-, \nu_u) \,\mathrm{d}\mathscr{H}^{n-1}, \tag{1.1}$$

where the surface integrand $\gamma\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1} \to [0,+\infty]$ encodes the relevant properties of the (effective) model. We recall that an element of the space $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ is a $\mathrm{BV}(\Omega;\mathbb{R}^m)$ function

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whose Jacobi matrix is a measure satisfying

$$Du = \nabla u \, \mathrm{d}\mathscr{L}^n \llcorner \Omega \; + \; (u^+ - u^-) \otimes \nu_u \, \mathrm{d}\mathscr{H}^{n-1} \llcorner S_u, \tag{1.2}$$

where ∇u is the density of the absolutely continuous part and $\mathscr{H}^{n-1}(S_u) < +\infty$. In (1.2) the vectorial functions u^+ and u^- represent the traces of u on both sides of the discontinuity set S_u and ν_u is the measure theoretical normal to S_u [1].

If (1.1) allows only for a finite number of given jump directions, and $\nu_1, \ldots, \nu_M \in \mathbb{S}^{n-1}$ is the list of corresponding normals, we shall consider a surface energy density γ such that $\gamma(\cdot,\cdot,\cdot,\nu) \equiv +\infty$ if $\nu \notin \mathcal{N} := \{\pm\nu_1, \ldots, \pm\nu_M\}$. Therefore in this case the domain of $\mathcal{F}$ is strictly smaller than $\mathrm{SBV}(\Omega;\mathbb{R}^m)$ and the additional constraint of

$$\nu_u(x) \in \mathcal{N} \quad \text{for } \mathscr{H}^{n-1}\text{-a.e. } x \in S_u \tag{1.3}$$

is to be satisfied. We denote by $\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ the space of those $\mathrm{SBV}(\Omega;\mathbb{R}^m)$ functions satisfying (1.3) as well as $\mathscr{H}^{n-1}(S_u) < +\infty$ and in this paper we are concerned with a *strong approximation* scheme for functions therein. Namely, in the main result of this paper, Theorem 3.1, we prove that any function $u \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ can be approximated by a sequence $(u_k) \subset \mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ with $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$ satisfying the following properties:

- S_{u_k} is essentially closed, *i.e.*, $\mathscr{H}^{n-1}(\overline{S_{u_k}} \setminus S_{u_k}) = 0$;
- $\overline{S_{u_k}}$ is the union of a *finite* number of $(n-1)$-dimensional *pairwise disjoint* closed cubes;
- $u_k \in \mathrm{C}^\infty(\Omega \setminus \overline{S_{u_k}};\mathbb{R}^m) \cap W^{1,\infty}(\Omega \setminus \overline{S_{u_k}};\mathbb{R}^m)$;

The density result is regarded in the following *strong* convergence:

$$u_k \to u \ \text{ in } \mathrm{L}^1(\Omega;\mathbb{R}^m), \quad \nabla u_k \to \nabla u \ \text{ in } \mathrm{L}^1(\Omega;\mathbb{R}^{m\times n}), \quad \mathscr{H}^{n-1}(S_{u_k} \triangle S_u) \to 0, \tag{1.4}$$

and

$$\lim_{k\to+\infty} \int_{S_{u_k} \cup S_u} \big(|u_k^+ - u^+| + |u_k^- - u^-| \big) \, \mathrm{d}\mathscr{H}^{n-1} = 0, \tag{1.5}$$

where in (1.5) we choose the orientation $\nu_{u_k} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_k} \cap S_u$. As an easy consequence of (1.4) and (1.5) we also have

$$\limsup_{k\to+\infty} \int_{S_{u_k} \cap \overline{A}} \gamma\big(x, u_k^+, u_k^-, \nu_{u_k}\big) \, \mathrm{d}\mathscr{H}^{n-1} \le \int_{S_u \cap \overline{A}} \gamma\big(x, u^+, u^-, \nu_u\big) \, \mathrm{d}\mathscr{H}^{n-1},$$

for any open set $A \subset\subset \Omega$ and for any upper semicontinuous function $\gamma : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1} \to [0,+\infty]$ such that $\gamma(\cdot,\cdot,\cdot,\nu) \equiv +\infty$ whenever $\nu \notin \mathcal{N}$ and $\gamma(\cdot,a,b,\nu) = \gamma(\cdot,b,a,-\nu)$ for every $a,b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{n-1}$.

The proof strategy of Theorem 3.1 mainly relies on the approximation techniques employed by De Philippis, Fusco, and Pratelli in [7]. Yet in comparison with those, the main disparity in methodology arises from the geometric constraint of prescribed orientation of the discontinuity set, which is not preserved by the constructions implemented in [7]. This is done in the present paper by means of a fine cover lemma (cf. Lemma 3.2) which provides us with a finite number of pairwise disjoint $(n-1)$-dimensional cubes covering the major part of the discontinuity set S_u. Then, the desired sequence (u_k) is obtained by successive regularisation steps mainly relying on convolution results with variable kernels (see, *e.g.*, [7, Proposition 2.3]) and on extension results in domains with cracks (see, *e.g.*, [7, Lemma 4.1]).

We observe that our result also covers case of *infinitely many* jump directions under the sole additional assumption that $\mathscr{N}$ is totally disconnected.

The unconstrained case $\mathscr{N} = \mathbb{S}^{n-1}$ is treated by Cortesani and Toader in [5] for SBV^p functions with $p > 1$, being the assumption $p > 1$ crucial to exploit some classical regularity results for the local minimisers of the Mumford-Shah functional [8]. In the last section of this note we also prove an approximation theorem for SBV^1 without any constraints on the jump directions (Theorem 4.3). This is obtained as a corollary of the result by Cortesani and Toader [5, Theorem 3.1 and Remark 3.5], by resorting to a strong approximation of SBV^1 functions by means of SBV^p functions with $p > 1$, which is proven in Proposition 4.1 (see Section 4 below).

We conclude this introduction by mentioning that in [3] Conti, Diermeier, and Zwicknagl prove a density result for SBV^2 functions with given jump normal direction (see Section 4.2 therein). In contrast to our results, we observe that Conti, Diermeier, and Zwicknagl's proof provides the L^2 convergence of the approximant's gradients. On the other hand, it is valid exclusively in dimension two, for one prescribed jump direction, and it does not ensure the strong convergence of the traces, which may turn out crucial in a number of applications. However, the proof of a density result with respect to the "strong" convergence in SBV^2 (and more in general in SBV^p with $p > 1$) appears to be way more delicate than the approximation result proven in the present note. In fact, in the SBV^p setting an additional issue one needs to face pertains to combining the constraint in (1.3) with the strong convergence $\nabla u_k \to \nabla u$ in $\mathrm{L}^p(\Omega; \mathbb{R}^m)$. Moreover a density result in SBV^p shall rely also on deeper results in the theory of SBV functions like, *e.g.*, the regularity properties of local minimisers on the Mumford-Shah functionals [8], similarly as in [9, 4, 5]. A density result in SBV^p for functions with prescribed jump direction can be relevant in a number of applications and will be the subject of a forthcoming paper.

2. Notation, functional setup, and preliminaries

We introduce the notation and conventions present in the paper. Let $n, m \geq 1$ be integers; the symbols $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$ indicate the usual n-dimensional Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, respectively. By $Q_r(x) \subset \mathbb{R}^n$ we mean the n-dimensional open cube of side length $r > 0$, centred at $x \in \mathbb{R}^n$, and with faces parallel to the coordinate hyperplanes. Given a unit vector $\nu \in \mathbb{S}^{n-1}$ we set Π^ν_x to be the hyperplane orthogonal to ν and passing through a point $x \in \mathbb{R}^n$. Likewise $Q^\nu_r(x) \subset \mathbb{R}^n$ is understood as an open cube of side-length $r > 0$, centred at $x \in \mathbb{R}^n$, and with a face orthogonal to ν.

Throughout, the real number $c > 0$ shall be thought of as absorbing constant with dependences emphasised when being relevant.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. We use the standard notation $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ for the space of $\mathbb{R}^m$-valued *special functions of bounded variation* in Ω. We recall that a function $u\colon \Omega \to \mathbb{R}^m$ belongs to $\mathrm{SBV}(\Omega; \mathbb{R}^m)$ if u is in $\mathrm{BV}(\Omega; \mathbb{R}^m)$ and its distributional derivative satisfies

$$Du(B) = \int_B \nabla u \, \mathrm{d}\mathscr{L}^n + \int_{S_u \cap B} [u] \otimes \nu_u \, \mathrm{d}\mathscr{H}^{n-1},$$

for any Borel set $B \subset \Omega$. By ∇u we mean the density of the diffuse part of Du; the latter turns out to coincide with the approximate gradient of u. The symbol S_u denotes the approximate discontinuity set of u and is a $\mathscr{H}^{n-1}$-rectifiable set. The associated measure theoretic normal is ν_u (defined up to the sign) whereas $[u] := u^+ - u^-$ is the difference of the traces of u on both sides of S_u. We notice that (u^+, u^-) is to be replaced by (u^-, u^+) if the orientation of ν_u is reversed. Let us also recall that the BV-norm of a function $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ is given by

$$\|u\|_{\mathrm{BV}(\Omega;\mathbb{R}^m)} := \|u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + |Du|(\Omega)$$

where $|Du|$ denotes the total variation of Du, *i.e.*,

$$|Du|(B) = \int_B |\nabla u| \, \mathrm{d}\mathscr{L}^n + \int_{S_u \cap B} |[u]| \, \mathrm{d}\mathscr{H}^{n-1},$$

where B is any Borel subset of $\mathbb{R}^n$.

For the general theory of BV and SBV functions we refer the readers to the comprehensive monograph [1].

In this paper the following subspace of SBV is also taken into consideration:

$$\mathrm{SBV}^1(\Omega; \mathbb{R}^m) := \{u \in \mathrm{SBV}(\Omega; \mathbb{R}^m) \colon \mathscr{H}^{n-1}(S_u) < +\infty\}.$$

Let now $\mathcal{N}$ be a *totally disconnected subspace* of $\mathbb{S}^{n-1}$ and let us assume that

$$\nu \in \mathcal{N} \iff -\nu \in \mathcal{N}. \tag{2.1}$$

We introduce the following space of SBV^1 functions with $\mathcal{N}$-*oriented* discontinuity set

$$\mathrm{SBV}^1_{\mathcal{N}}(\Omega; \mathbb{R}^m) := \{u \in \mathrm{SBV}^1(\Omega; \mathbb{R}^m) \colon \nu_u \in \mathcal{N} \ \mathscr{H}^{n-1}\text{-a.e. in } S_u\}.$$

We notice that in view of (2.1) the definition of $\mathrm{SBV}_{\mathcal{N}}(\Omega; \mathbb{R}^m)$ is unambiguous.

A set $F \subset \Omega$ is called *polyhedral* (with respect to Ω) if it is the intersection of Ω with a *finite* number of $(n-1)$-dimensional simplices in $\mathbb{R}^n$. In the interest of our work we define a special case of polyhedral sets whose normal belongs $\mathcal{N}$.

Definition 2.1 ($\mathcal{N}$-aligned regular set)**.** We say that a set $F \subset \Omega$ is an $\mathcal{N}$-*aligned regular set* if there exists a *finite* collection of sets $F_1, \dots, F_N$ such that each F_i is a $(n-1)$-dimensional closed cube in $\mathbb{R}^n$ orthogonal to ν for some $\nu \in \mathcal{N}$ and

$$F = \Omega \cap \bigcup_{i=1}^{N} F_i.$$

If the sets $F_1, \dots, F_N$ are additionally *pairwise disjoint*, the set F is called an $\mathcal{N}$-*aligned regular disconnected set.*

We now introduce the space of approximating functions.

Definition 2.2 (The approximating space)**.** We say that u belongs to the space $\mathcal{W}_{\mathcal{N}}(\Omega; \mathbb{R}^m)$ if:

(a) $u \in \mathrm{SBV}^1(\Omega; \mathbb{R}^m)$;
(b) S_u is essentially closed, *i.e.*, $\mathscr{H}^{n-1}(\overline{S_u} \backslash S_u) = 0$;
(c) $\overline{S_u}$ is a $\mathcal{N}$-aligned regular disconnected set;
(d) $u \in \mathrm{C}^\infty(\Omega \backslash \overline{S_u}; \mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_u}; \mathbb{R}^m)$.

In accordance with [5] we consider the following notion of “strong” convergence for which a large class of free-discontinuity functionals is continuous.

Definition 2.3 ($\mathscr{S}$-convergence)**.** We say that a sequence $(u_k) \subset \mathrm{SBV}^1(\Omega; \mathbb{R}^m)$ $\mathscr{S}$-converges to $u \in \mathrm{SBV}^1(\Omega; \mathbb{R}^m)$ as $k \to +\infty$, written $u_k \xrightarrow{\mathscr{S}} u$, if:

(a) $u_k \to u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^m)$;
(b) $\nabla u_k \to \nabla u$ in $\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})$;
(c) $\mathscr{H}^{n-1}(S_{u_k}\triangle S_u) \to 0$;
(d) there holds

$$\int_{S_{u_k}\cup S_u} \big(|u_k^+ - u^+| + |u_k^- - u^-|\big)\,\mathrm{d}\mathscr{H}^{n-1} \longrightarrow 0, \tag{2.2}$$

where in (2.2) we choose the orientation $\nu_{u_k} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. on $S_{u_k}\cap S_u$.

We remark that $\mathscr{S}$-convergence is evidently stronger than the convergence induced from the $\mathrm{BV}(\Omega;\mathbb{R}^m)$-norm.

Below we recall three technical lemmas which are used to prove our main result, Theorem 3.1. These are based on the corresponding results in [7]. The first concerns a smooth approximation of functions in $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ obtained by convolutions with variable kernels.

Lemma 2.4 (Approximation by convolution)**.** *Let $u \in \mathrm{SBV}^1(\Omega)$ and let $F \subset\subset \Omega$ be a compact set. For any $\varepsilon > 0$, there exist $v_\varepsilon \in \mathrm{SBV}^1(\Omega)\cap \mathrm{C}^\infty(\Omega\backslash F)$ and $\xi_\varepsilon \in \mathrm{L}^1(\Omega;\mathbb{R}^n)\cap \mathrm{C}^\infty(\Omega\backslash F;\mathbb{R}^n)$ such that the following properties hold true:*

(1) $\|v_\varepsilon - u\|_{\mathrm{L}^1(\Omega)} + \|\xi_\varepsilon - \nabla u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} < \varepsilon$;

(2) $v_\varepsilon^\pm = u^\pm$ *in* F, *therefore* $S_{v_\varepsilon} = S_u \cap F$;

(3) there exists an $\mathbb{R}^n$-valued Radon measure μ_ε on Ω such that $|\mu_\varepsilon|(\Omega) \le 2|Du|(\Omega)$ and

$$|Du - Dv_\varepsilon|(\Omega) \le \|\nabla u - \xi_\varepsilon\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + 3|Du \llcorner (S_u\backslash F)|(\Omega) + \varepsilon|\mu_\varepsilon|(\Omega);$$

(4) if $u \in \mathrm{L}^\infty(\Omega)$, then $\|v_\varepsilon\|_{\mathrm{L}^\infty(\Omega)} \le \|u\|_{\mathrm{L}^\infty(\Omega)}$.

Proof. The results can be retrieved by combining [7, Proposition 2.3, Corollary 2.4, and Lemma 2.5]. □

Moreover, we recall the following existence result of bounded Lipschitz extensions for C^1-regular interior and boundary traces.

Lemma 2.5 (Extension)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary. Let $M \subset\subset U$ be either a compact and connected $(n-1)$-dimensional C^1 manifold with (possibly empty) C^1 boundary, or an $(n-1)$-dimensional closed cube. Then there exists a constant $c_{U,M} > 0$ with the following properties:*

(a) Given three functions $\phi \in \mathrm{L}^1(\partial U)$, $\phi^+, \phi^- \in \mathrm{L}^1(M)$, there exists $\psi \in \mathrm{W}^{1,1}(U\backslash M)$ such that $\psi^\pm = \phi^\pm$ in M, $\psi = \phi$ on ∂U (in the sense of traces) and

$$\|\psi\|_{\mathrm{W}^{1,1}(U\backslash M)} \le c_{U,M}\Big(\|\phi\|_{\mathrm{L}^1(\partial U)} + \|\phi^+\|_{\mathrm{L}^1(M)} + \|\phi^-\|_{\mathrm{L}^1(M)}\Big).$$

(b) Given three functions $\phi \in \mathrm{C}^1(\partial U)$, $\phi^+, \phi^- \in \mathrm{C}^1(M)$ satisfying $\phi^+ = \phi^-$ on ∂M, there exists $\psi \in \mathrm{W}^{1,\infty}(U\backslash M)$ such that $\psi^\pm = \phi^\pm$ in M, $\psi = \phi$ on ∂M and

$$\|\psi\|_{\mathrm{W}^{1,\infty}(U\backslash M)} \le c_{U,M}\Big(\|\phi\|_{\mathrm{C}^1(\partial U)} + \|\phi^+\|_{\mathrm{C}^1(M)} + \|\phi^-\|_{\mathrm{C}^1(M)}\Big).$$

Proof. In the case where M has C^1 boundary, the result is stated in [7, Lemma 4.1]. When M is an $(n-1)$-dimensional closed cube, the proof requires some minor adjustments, detailed below. Let $\delta > 0$ be such that

$$P := \{x + t\nu(x) \colon x \in M,\, t \in (-\delta, \delta)\} \subset\subset U$$

where $\nu(x)$ is the normal to M at x. Then $\partial P \setminus \partial M =: D^+ \cup D^-$ where $D^\pm$ are in bilipschitz correspondence with M, $D^+ \cap D^- = \emptyset$ and $\partial_{\partial P} D^\pm = \partial M$, where $\partial_{\partial P} D^\pm$ denote the boundary of $D^\pm$ in the relative topology of ∂P. Subsequently we may find a bilipschitz mapping $\Phi : U \backslash M \to U \backslash P$ such that Φ is the identity in a neighbourhood of ∂U and $\Phi^{-1}(D^\pm) = M$. Then to establish *(a)* it is enough to apply the standard extension result to the functions $\phi \circ \Phi^{-1}$, $\phi^\pm \circ \Phi^{-1}$ in the Lipschitz domain $U \setminus \overline{D}$, arguing as in the proof of [7, Lemma 4.1] to which we refer the reader for more details. To prove *(b)* one may argue in a similar way, now resorting to the McShane Theorem. □

To conclude this section we prove a vectorial truncation lemma to promote an L^∞-bound of any $\mathscr{S}$-converging sequence to a bounded SBV^1-function.

Lemma 2.6 (Vectorial truncation). *Let $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ and $(u_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ be such that $u_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$. Then there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that $v_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$, $S_{v_k} = S_{u_k}$, and $\|v_k\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}$. Moreover, if $u_k \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m)$ then the same holds for v_k.*

Proof. The proof relies on classical arguments and in the scalar case it follows as in [7, Lemma 3.2].

Let $\eta > 0$ be arbitrary and fixed; let $0 < \varepsilon \le \eta$ be fixed depending on u and η as specified later. By assumption, for $k \in \mathbb{N}$ large enough there holds

$$\begin{aligned} \|u_k - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + \|\nabla u_k - \nabla u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})} + \mathscr{H}^{n-1}(S_{u_k} \triangle S_u) \\ + \int_{S_{u_k}\cup S_u} |u_k^+ - u^+| \,\mathrm{d}\mathscr{H}^{n-1} \int_{S_{u_k}\cup S_u} |u_k^- - u^-| \,\mathrm{d}\mathscr{H}^{n-1} < \varepsilon\,. \end{aligned} \tag{2.3}$$

Now, set $a^\eta := \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} + \eta$; let moreover $\psi^\eta \colon \mathbb{R}^+ \to (0, a^\eta + \eta)$ be a C^∞ function such that

$$0 < (\psi^\eta)' \le 1 \quad \text{in} \quad \mathbb{R}^+, \qquad \psi^\eta(t) = t \quad \text{in} \quad (0, a^\eta).$$

For every $y \in \mathbb{R}^m$ set

$$\varphi^\eta(y) := \frac{y}{|y|}\,\psi^\eta(|y|) \ \text{ for } \ y \neq 0, \qquad \varphi^\eta(0) = 0.$$

By the definition of ψ^η there holds

$$\varphi^\eta(y) = y \quad \text{if} \quad |y| < a^\eta \quad \text{and} \quad \|\varphi^\eta\|_{\mathrm{L}^\infty(\mathbb{R}^m;\mathbb{R}^m)} \le a^\eta + \eta\,.$$

Hence φ^η belongs to $\mathrm{C}^\infty(\mathbb{R}^m;\mathbb{R}^m)$ and has Lipschitz constant less than or equal to one. Furthermore we observe that φ^η is injective. Indeed, $\varphi^\eta(y_1) = \varphi^\eta(y_2)$ implies that y_1 and y_2 differ by a strictly positive multiplicative constant. Therefore $y_1/|y_1| = y_2/|y_2|$ and in turn $\psi^\eta(|y_1|) = \psi^\eta(|y_2|)$, thus we get $|y_1| = |y_2|$ and finally $y_1 = y_2$.

For every $k \in \mathbb{N}$ set $w_k^\eta := \varphi^\eta(u_k)$; clearly $w_k^\eta \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, $S_{w_k^\eta} = S_{u_k}$ by the injectivity of φ^η, and $w_k^\eta \in \mathrm{C}^\infty(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega \backslash \overline{S_{u_k}};\mathbb{R}^m)$ if the same holds true for u_k. Moreover we notice that for k large enough we have

$$\|w_k^\eta - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} + \int_{S_{w_k^\eta}\cup S_u} |(w_k^\eta)^+ - u^+| \,\mathrm{d}\mathscr{H}^{n-1} + \int_{S_{w_k^\eta}\cup S_u} |(w_k^\eta)^- - u^-| \,\mathrm{d}\mathscr{H}^{n-1} < \varepsilon. \tag{2.4}$$

Indeed, define $A_k^\eta := \{x \in \Omega \colon |u_k| \geq a^\eta\}$, we then have

$$\begin{aligned}\|w_k^\eta - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)} &\leq \|u_k - u\|_{\mathrm{L}^1(\Omega\setminus A_k^\eta;\mathbb{R}^m)} + \|\varphi^\eta(u_k) - \varphi^\eta(u)\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^m)} \\ &\leq \|u_k - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^m)},\end{aligned}$$

where we have used the fact that φ^η has Lipschitz constant less than or equal to one. Furthermore, since $(w_k^\eta)^\pm = \varphi^\eta(u^\pm)$, a similar argument shows that

$$\int_{S_{w_k^\eta}\cup S_u} |(w_k^\eta)^\pm - u^\pm| \,\mathrm{d}\mathscr{H}^{n-1} \leq \int_{S_{u_k}\cup S_u} |u_k^\pm - u^\pm| \,\mathrm{d}\mathscr{H}^{n-1},$$

hence (2.4) follows by (2.3).

We now estimate the L^1 norm of $\nabla w_k^\eta - \nabla u$. To this end, we preliminarily observe that by construction $\nabla w_k^\eta = \nabla u_k$ in $\Omega \setminus A_k^\eta$ while $|\nabla w_k^\eta| \leq |\nabla u_k|$ in A_k^η. Moreover, since $|u_k - u| \geq \eta$ in A_k^η, by (2.3) and also invoking the Chebyshev Inequality we deduce that for k large enough $\eta\mathscr{L}^n(A_k^\eta) < \varepsilon$, and therefore $\mathscr{L}^n(A_k^\eta) < \varepsilon/\eta$. Hence, choosing ε so small that $\|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} < \eta$ for k large, we obtain

$$\begin{aligned}\|\nabla w_k^\eta - \nabla u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{m\times n})} &\leq \|\nabla u_k - \nabla u\|_{\mathrm{L}^1(\Omega\setminus A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla w_k^\eta - \nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} \\ &\leq \varepsilon + \|\nabla w_k^\eta\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} \\ &\leq \varepsilon + \|\nabla u_k\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} + \|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} \\ &\leq 2\varepsilon + 2\|\nabla u\|_{\mathrm{L}^1(A_k^\eta;\mathbb{R}^{m\times n})} \leq 2\varepsilon + 2\eta, \end{aligned} \tag{2.5}$$

for every k large enough.

Eventually, set

$$v_k^\eta := \frac{a^\eta - \eta}{a^\eta + \eta}\, w_k^\eta;$$

by definition $v_k^\eta \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$, $S_{v_k^\eta} = S_{w_k^\eta} = S_{u_k}$, and $v_k^\eta \in \mathrm{C}^\infty(\Omega\setminus\overline{S_{u_k}};\mathbb{R}^m) \cap \mathrm{W}^{1,\infty}(\Omega\setminus\overline{S_{u_k}};\mathbb{R}^m)$ if the same holds for u_k (and hence for w_k^η). Furthermore we have

$$\|v_k^\eta\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)} \leq a^\eta - \eta = \|u\|_{\mathrm{L}^\infty(\Omega;\mathbb{R}^m)}.$$

Finally, by combining (2.4), (2.5) and invoking a standard diagonal argument we can find $\eta_k \to 0^+$ as $k \to +\infty$ such that setting $v_k := v_k^{\eta_k}$ we get $v_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$ and thus the claim. ☐

Remark 2.7. Lemma 2.6 can be generalised observing that, if $u \in \mathcal{K}$ a.e. in Ω, where $\mathcal{K} \subset \mathbb{R}^m$ is compact, Lipschitz, and star-shaped with respect to the origin, then the sequence (v_k) can be chosen in such a way that for every $k \in \mathbb{N}$ there holds $v_k \in \mathcal{K}$ a.e. in Ω. Indeed, for $y \in \mathbb{R}^m$ set $\lambda_{\mathcal{K}}(y) := \inf\{\rho > 0 \colon y \in \rho\mathcal{K}\}$. Notice that in view of the compactness of $\mathcal{K}$ we have $\lambda_{\mathcal{K}}(y) > 0$, for every $y \in \mathbb{R}^m \setminus \{0\}$. Then to find the desired sequence (v_k) it is enough to choose

$$\varphi^\eta(y) := \frac{y}{\lambda_{\mathcal{K}}(y)}\, \psi^\eta(\lambda_{\mathcal{K}}(y)) \ \text{ for } \ y \neq 0, \qquad \varphi^\eta(0) = 0,$$

where ψ^η is as in the proof of Lemma 2.6. As for the regularity, one has $v_k \in \mathrm{W}^{1,\infty}(\Omega\setminus\overline{S_{v_k}};\mathbb{R}^m)$ if the same holds for u_k; moreover, $v_k \in \mathrm{C}^\infty(\Omega\setminus\overline{S_{v_k}};\mathbb{R}^m)$ if the same holds for u_k and $\mathcal{K}$ is C^∞.

3. The main result

In this section we state and prove the main result of this paper.

Theorem 3.1 ($\mathscr{S}$-approximation of $\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)$ functions)**.** *Let $\mathcal{N}\subset\mathbb{S}^{n-1}$ be a totally disconnected set of directions satisfying* (2.1)*. Then the space $\mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ is $\mathscr{S}$-dense in $\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Specifically, for any $u\in\mathrm{SBV}^1_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ there exists a sequence $(u_k)\subset\mathcal{W}_{\mathcal{N}}(\Omega;\mathbb{R}^m)\cap\mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ with $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)}\le\|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$ such that $u_k\xrightarrow{\mathscr{S}}u$ as $k\to+\infty$. Moreover one has*

$$\limsup_{k\to+\infty}\int_{S_{u_k}\cap\overline{A}}\gamma\big(x,u_k^+,u_k^-,\nu_{u_k}\big)\,\mathrm{d}\mathscr{H}^{n-1}\le\int_{S_u\cap\overline{A}}\gamma\big(x,u^+,u^-,\nu_u\big)\,\mathrm{d}\mathscr{H}^{n-1},\tag{3.1}$$

for any open set $A\subset\subset\Omega$ and for any upper semicontinuous function $\gamma:\Omega\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{S}^{n-1}\to[0,+\infty]$ such that $\gamma(\cdot,\cdot,\cdot,\nu)\equiv+\infty$ whenever $\nu\notin\mathcal{N}$ and $\gamma(\cdot,a,b,\nu)=\gamma(\cdot,b,a,-\nu)$ for every $a,b\in\mathbb{R}^m$ and $\nu\in\mathbb{S}^{n-1}$.

Before embarking on the proof of Theorem 3.1 we give a preliminary covering lemma for $\mathcal{N}$-aligned sets.

Lemma 3.2 (Fine cover)**.** *Let $\varepsilon>0$ be arbitrary. Let $\mathcal{N}\subset\mathbb{S}^{n-1}$ be a set of directions satisfying* (2.1) *and let $K\subset\Omega$ be a $\mathscr{H}^{n-1}$-rectifiable set with $\mathscr{H}^{n-1}(K)<+\infty$ and measure theoretic normal $\nu_K\in\mathcal{N}$ $\mathscr{H}^{n-1}$-a.e. Then there exist a set $K'\subset K$ and a finite family of open cubes $\mathcal{Q}_N=\{Q^{\nu_i}_{r_i}(x_i)\}_{i=1}^N$ with a face orthogonal to $\nu_i:=\nu_K(x_i)\in\mathcal{N}$ satisfying:*

(1) $\mathscr{H}^{n-1}(K\backslash K')<\varepsilon$;
(2) $Q^{\nu_i}_{r_i}(x_i)$ is centred in K, i.e., $x_i\in K$ for every $i=1,\dots,N$;
(3) the family $\{\overline{Q^{\nu_i}_{r_i}(x_i)}\}_{i=1}^N$ is pairwise disjoint and $Q^{\nu_i}_{r_i}(x_i)\subset\subset\Omega$ for every $i=1,\dots,N$;
(4) $K'\subset\bigcup_{i=1}^N Q^{\nu_i}_{r_i}(x_i)$ and $K'\cap Q^{\nu_i}_{r_i}(x_i)\subset\Pi^{\nu_i}_{x_i}$ for every $i=1,\dots,N$;
(5) $r_i^{n-1}\le\frac{1}{1-\varepsilon}\,\mathscr{H}^{n-1}(K'\cap Q^{\nu_i}_{r_i}(x_i))$ for every $i=1,\dots,N$;
(6) $\sum_{i=1}^N r_i^{n-1}<\frac{1}{1-\varepsilon}\,\mathscr{H}^{n-1}(K)$.

Proof. Up to a set of zero $\mathscr{H}^{n-1}$-measure we may rewrite K as $\bigcup_{j\in\mathbb{N}}K_j$ with each K_j being a C^1-image of the closed unit ball in $\mathbb{R}^{n-1}$. Using the properties of Radon measures we can find a compact set $K_\varepsilon\subset K$ such that $K_\varepsilon\subset\bigcup_{j=1}^{N_\varepsilon}K_j$ for some $N_\varepsilon\in\mathbb{N}$ and $\mathscr{H}^{n-1}(K\backslash K_\varepsilon)<\varepsilon/3$. Since each K_j is smooth and $\mathcal{N}$ is totally disconnected we have $\nu_K\equiv\mathsf{n}_j$ $\mathscr{H}^{n-1}$-a.e. in K_j, for some $\mathsf{n}_j\in\mathcal{N}$; thus $K_j\subset\Pi^{\mathsf{n}_j}_{y_j}$ for some $y_j\in K_j$, for every $j\in\{1,\dots N_\varepsilon\}$.

If the set $\mathcal{N}$ contains more than one direction, the hyperplanes $\Pi^{\mathsf{n}_j}_{y_j}$ shall have a finite number of intersections, which we are going to remove. Namely, we choose an open set $C_\varepsilon\subset\Omega$ such that $\bigcup_{j=1}^{h_\varepsilon}\Pi^{\mathsf{n}_j}_{y_j}\setminus C_\varepsilon$ has no self-intersections in Ω and $\mathscr{H}^{n-1}\big(\bigcup_{j=1}^{h_\varepsilon}\Pi^{\mathsf{n}_j}_{y_j}\setminus C_\varepsilon\big)<\varepsilon/3$.

Next, the regularity of rectifiable sets [11, Theorem 3.3] allows us to further select $K^0_\varepsilon\subset K_\varepsilon\setminus C_\varepsilon$ with $\mathscr{H}^{n-1}((K_\varepsilon\backslash C_\varepsilon)\backslash K^0_\varepsilon)=0$ such that for any $x\in K^0_\varepsilon$ and any closed cube $\overline{Q^{\nu_K(x)}_r(x)}$ there holds

$$\lim_{r\to0^+}\frac{\mathscr{H}^{n-1}\left(\overline{Q^{\nu_K(x)}_r(x)}\cap K^0_\varepsilon\right)}{r^{n-1}}=1.$$

Thus there exists a number $r(x,\varepsilon)>0$ such that for any $r\in(0,r(x,\varepsilon))$

$$\mathscr{H}^{n-1}\left(\overline{Q^{\nu_K(x)}_r(x)}\cap K^0_\varepsilon\right)\ge(1-\varepsilon)\,r^{n-1}=(1-\varepsilon)\,\mathscr{H}^{n-1}\left(\overline{Q^{\nu_K(x)}_r(x)}\cap\Pi^{\nu_K(x)}_x\right).\tag{3.2}$$

We are now in a position to define a family of suitable cubes covering K_ε^0, from which we are going to extract the desired finite cover. To this end we set

$$d := \min_{k,l\in\{1,\dots,h_\varepsilon\}}\{\mathrm{dist}(\Pi_{y_k}^{\mathfrak{n}_k}\backslash C_\varepsilon, \Pi_{y_l}^{\mathfrak{n}_l}\backslash C_\varepsilon) \colon \Pi_{y_k}^{\mathfrak{n}_k} \neq \Pi_{y_l}^{\mathfrak{n}_l}\} > 0$$

and consider the family of cubes

$$\mathcal{F} := \left\{\overline{Q_r^{\nu_K(x)}(x)} \colon x \in K_\varepsilon^0,\ 0 < r \leq \min\left\{r(x,\varepsilon),\ \frac{d}{2\sqrt{n}(1+\varepsilon)},\ \frac{1}{4\sqrt{n}}\mathrm{dist}(K_\varepsilon, \partial\Omega)\right\}\right\}.$$

Clearly, the elements of $\mathcal{F}$ are properly contained in Ω and each of them intersects only one of the hyperplanes $\Pi_{y_k}^{\mathfrak{n}_k}$, $k = 1,\dots,h_\varepsilon$. Moreover, $\mathcal{F}$ is a Vitali cover for K_ε^0. Therefore employing a variant of the Vitali covering Theorem, cf. [10, Theorem 1.10], one can find a countable and pairwise disjoint collection of cubes $\{\overline{Q_{\widetilde{r}_i}^{\nu_i}(x_i)}\}_{i\in\mathbb{N}} \subset \mathcal{F}$ with $\nu_i := \nu_K(x_i) \in \mathcal{N}$ such that

$$\mathscr{H}^{n-1}\left(K_\varepsilon^0 \setminus \bigcup_{i\in\mathbb{N}} \overline{Q_{\widetilde{r}_i}^{\nu_i}(x_i)}\right) = 0.$$

We then select an integer $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\mathscr{H}^{n-1}\left(K_\varepsilon^0 \setminus \bigcup_{i=1}^{N} \overline{Q_{\widetilde{r}_i}^{\nu_i}(x_i)}\right) < \frac{\varepsilon}{3} \tag{3.3}$$

and consequently declare the set K' to be

$$K' := K_\varepsilon^0 \cap \bigcup_{i=1}^{N} \overline{Q_{\widetilde{r}_i}^{\nu_i}(x_i)}.$$

By (3.3) and by the choice of K_ε^0 we deduce

$$\mathscr{H}^{n-1}(K\backslash K') = \mathscr{H}^{n-1}(K_\varepsilon \cap C_\varepsilon) + \mathscr{H}^{n-1}((K_\varepsilon\backslash C_\varepsilon)\backslash K_\varepsilon^0) + \mathscr{H}^{n-1}(K_\varepsilon^0\backslash K') < \varepsilon$$

and thus (1) follows. As $\overline{Q_{\widetilde{r}_i}^{\nu_i}(x_i)}$ are pairwise disjoint there exist positive numbers $r_i \in (\widetilde{r}_i, \widetilde{r}_i + \varepsilon)$ such that the cubes $\{\overline{Q_{r_i}^{\nu_i}(x_i)}\}_{i=1}^N$ remain pairwise disjoint and

$$K' \subset \bigcup_{i=1}^{N} Q_{r_i}^{\nu_i}(x_i). \tag{3.4}$$

This provides us with a finite family of open cubes satisfying (2)-(3) by construction and (4) by (3.4). Moreover, (5) follows by (3.2) and the definitions of d and $\mathcal{F}$. Eventually, (6) is an immediate consequence of (3) and (5). □

We are now equipped with all the tools to prove Theorem 3.1. The proof is of constructive nature and follows by successive approximations and regularisations.

Proof of Theorem 3.1. We notice that thanks to Lemma 2.6 it is enough to prove the existence of an approximating sequence (u_k) satisfying all the desired properties but the uniform bound $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$. Without loss of generality we may assume that $m = 1$; then the proof for $m > 1$ follows arguing componentwise.

Let $u \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ be chosen and arbitrary. We divide the proof into four steps.

Step 1: *Modification of the discontinuity set.* In this first step we approximate u with a sequence of functions whose jump sets are $\mathcal{N}$-aligned and contained in the finite union of $(n-1)$-dimensional closed cubes.

Let $(\varepsilon_k) \searrow 0$ be an arbitrary infinitesimal sequence. For every $k \in \mathbb{N}$ applying Lemma 3.2 to S_u with $\varepsilon = \varepsilon_k$ we obtain a set $K' \subset S_u$ along with a finite collection of pairwise disjoint open cubes $\{Q^{\nu_i}_{r_i^k}(x_i^k)\}_{i=1}^{N_k}$ with $\nu_i := \nu_u(x_i)$, satisfying properties (1)-(6). We define the compact sets

$$F_k := \bigcup_{i=1}^{N_k} F_k^i := \bigcup_{i=1}^{N_k} \overline{\Pi^{\nu_i}_{x_i^k} \cap Q^{\nu_i}_{r_i^k}(x_i^k)}.$$

We notice that the sets F_k^i are pairwise disjoint by construction, hence F_k is an $\mathcal{N}$-aligned regular disconnected set in the sense of Definition 2.1. From property (4) of Lemma 3.2 we deduce the inclusion $K' \subset F_k \cap S_u$. Moreover from property (1), (5), and (6) of Lemma 3.2 we infer

$$\begin{aligned}\mathscr{H}^{n-1}(F_k \triangle S_u) &\leq \mathscr{H}^{n-1}(F_k \backslash K') + \mathscr{H}^{n-1}(S_u \backslash K') \leq \sum_{i=1}^{N_k} (r_i^k)^{n-1} - \mathscr{H}^{n-1}(K') + \varepsilon_k \\ &\leq \varepsilon_k \sum_{i=1}^{N_k} (r_i^k)^{n-1} + \varepsilon_k \leq \frac{\varepsilon_k}{1-\varepsilon_k} \mathscr{H}^{n-1}(S_u) + \varepsilon_k,\end{aligned}$$

thus

$$\mathscr{H}^{n-1}(F_k \triangle S_u) \to 0 \tag{3.5}$$

as $k \to +\infty$.

Let $k \in \mathbb{N}$ be fixed and let $v_{\varepsilon_k} \in \mathrm{SBV}^1_{\mathcal{N}}(\Omega) \cap L^\infty(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash F_k)$ and $\xi_{\varepsilon_k} \in L^1(\Omega; \mathbb{R}^n) \cap \mathrm{C}^\infty(\Omega \backslash F_k; \mathbb{R}^n)$ be the functions obtained by applying Lemma 2.4 to u and F_k. By Lemma 2.4 (2) there holds $S_{v_{\varepsilon_k}} = F_k \cap S_u$, hence

$$S_u \triangle S_{v_{\varepsilon_k}} = S_u \setminus S_{v_{\varepsilon_k}} = S_u \setminus F_k. \tag{3.6}$$

Further $\|v_{\varepsilon_k}\|_{\mathrm{L}^\infty(\Omega)} \leq \|u\|_{\mathrm{L}^\infty(\Omega)}$ in which case Lemma 2.4 (3)-(4) along with (3.5) yield

$$\begin{aligned}\int_{S_{v_{\varepsilon_k}} \cup S_u} \left(|v^+_{\varepsilon_k} - u^+| + |v^-_{\varepsilon_k} - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} &= \int_{S_u \backslash F_k} \left(|v^+_{\varepsilon_k} - u^+| + |v^-_{\varepsilon_k} - u^-|\right) \mathrm{d}\mathscr{H}^{n-1} \\ &\leq 4\|u\|_{\mathrm{L}^\infty(\Omega)} \mathscr{H}^{n-1}(F_k \triangle S_u) \to 0\end{aligned} \tag{3.7}$$

as $k \to +\infty$. Moreover, by Lemma 2.4 (3) the function $\xi_{\varepsilon_k} \in \mathrm{L}^1(\Omega; \mathbb{R}^n)$ satisfies

$$\begin{aligned}|Du - Dv_{\varepsilon_k}|(\Omega) &\leq \|\nabla u - \xi_{\varepsilon_k}\|_{\mathrm{L}^1(\Omega)} + 3|Du \llcorner (S_u \backslash F_k)|(\Omega) + 2\varepsilon_k |Du|(\Omega) \\ &\leq \|\nabla u - \xi_{\varepsilon_k}\|_{\mathrm{L}^1(\Omega)} + 3\|u\|_{\mathrm{L}^\infty(\Omega)} \mathscr{H}^{n-1}(S_u \backslash F_k) + 2\varepsilon_k |Du|(\Omega).\end{aligned}$$

Using Lemma 2.4 (1) in conjunction with (3.5) amounts in the convergence $|Du - Dv_{\varepsilon_k}|(\Omega) \to 0$ and thus

$$\|\nabla u - \nabla v_{\varepsilon_k}\|_{\mathrm{L}^1(\Omega; \mathbb{R}^n)} \to 0 \tag{3.8}$$

as $k \to +\infty$. Since by Lemma 2.4 (1) $v_{\varepsilon_k} \to u$ in $\mathrm{L}^1(\Omega)$, combining (3.6), (3.7) and (3.8) we conclude $v_{\varepsilon_k} \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$.

Note that as a consequence of the construction carried out in this step we get that v_{ε_k} is smooth outside F_k. However, in the next step we may lose this property, hence we will need to perform again the regularisation by convolution with variable kernels provided by Lemma 2.4.

Step 2: *Closing the discontinuity gap.* At this stage we only know $S_{v_{\varepsilon_k}} \subset F_k$, so we modify the approximating sequence in such a way that its discontinuity set coincides with F_k.

Let us recall that $F_k = \bigcup_{i=1}^{N_k} F_k^i$ where each F_k^i is a closed $(n-1)$-dimensional cube compactly contained in Ω; moreover, the sets F_k^i are pairwise disjoint, cf. Lemma 3.2 (3). For every $i \in 1, \dots, N_k$ we may find open sets Ω_k^i, pairwise disjoint, with smooth boundary, such that $F_k^i \subset\subset \Omega_k^i \subset\subset \Omega$. Let $\varphi_k^i : F_k^i \to \mathbb{R}$ be a C^1 function such that $\varphi_k^i > 0$ in $F_k^i \setminus \partial_{\Pi^{\nu_i}_{x_i^k}} F_k^i$, $\varphi_k^i = 0$ on $\partial_{\Pi^{\nu_i}_{x_i^k}} F_k^i$ (where $\partial_{\Pi^{\nu_i}_{x_i^k}} F_k^i$ denotes the boundary of F_k^i in the relative topology induced by $\Pi^{\nu_i}_{x_i^k}$) and

$$\|\varphi_k^i\|_{\mathrm{C}^1(F_k^i)} \le \min\Big\{1, \frac{1}{N_k\, c_{i,k}}\Big\}, \tag{3.9}$$

where $c_{i,k} > 0$ is the constant from Lemma 2.5, applied to $U = \Omega_k^i$ and $M = F_k^i$. Choosing $\phi \equiv 0$, $\phi^+ = \varphi_k^i$, and $\phi^- \equiv 0$, Lemma 2.5 (b) provides us with a function $\psi_k^i \in \mathrm{W}^{1,\infty}(\Omega_k^i \backslash F_k^i)$ such that $\psi_k^i = 0$ on $\partial\Omega_k^i$, $(\psi_k^i)^+ = \varphi_k^i$ and $(\psi_k^i)^- = 0$ in F_k^i.

Now we define

$$w_k := \begin{cases} v_{\varepsilon_k} + \delta_k \psi_k^i & \text{in } \Omega_k^i \text{ for every } i \in \{1, \dots, N_k\}, \\ v_{\varepsilon_k} & \text{otherwise in } \Omega, \end{cases}$$

where $\delta_k > 0$ is to be determined in forthcoming manner. Inspecting the jump points of w_k in F_k we readily deduce that $S_{w_k} \subset F_k$ for all $k \in \mathbb{N}$ and the inequality $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) > 0$ is only true for at most countably many $\delta_k \in \mathbb{R}$. This follows from a standard argument (see *e.g.*, [4, Step 4 of the proof of Theorem 3.9]): consider the pairwise disjoint sets defined for $t \in \mathbb{R}$ by $\Sigma_t := \{x \in F_k \colon [v_{\varepsilon_k}](x) + t\varphi(x) = 0\}$; since $\mathscr{H}^{n-1}(F_k) < +\infty$ and $\{\Sigma_t\}_{t\in\mathbb{R}}$ partitions F_k, there exist at most countably many $t \in \mathbb{R}$ such that $\mathscr{H}^{n-1}(\Sigma_t) > 0$. In other words there exists an infinitesimal positive sequence (δ_k) such that $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) = 0$ for all $k \in \mathbb{N}$ and this shall be our choice in the definition of w_k.

For k large enough Lemma 2.5 (b) and (3.9) imply

$$\begin{aligned} &\|u_{\varepsilon_k} - w_k\|_{\mathrm{BV}(\Omega)} + \int_{S_{u_{\varepsilon_k}} \cup S_{w_k}} \big(|w_k^+ - (u_{\varepsilon_k})^+| + |w_k^- - (u_{\varepsilon_k})^-|\big)\, \mathrm{d}\mathscr{H}^{n-1} \\ &\le \delta_k \sum_{i=1}^{N_k} \|\psi_k^i\|_{\mathrm{BV}(\Omega_k^i)} + \delta_k \sum_{i=1}^{N_k} \|\varphi_k^i\|_{\mathrm{L}^1(F_k^i)} \\ &\le \delta_k \sum_{i=1}^{N_k} \|\psi_k^i\|_{\mathrm{W}^{1,\infty}(\Omega_k^i \backslash F_k^i)} + 2\delta_k \sum_{i=1}^{N_k} \|\varphi_k^i\|_{\mathrm{L}^1(F_k^i)} \le 3\delta_k. \end{aligned}$$

Therefore $w_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$. In addition we claim $F_k = \overline{S_{w_k}}$ which then shows that $\overline{S_{w_k}}$ is an $\mathcal{N}$-aligned regular disconnected set in the sense of Definition 2.1. Indeed suppose there exists $x \in F_k \setminus \overline{S_{w_k}} \subset \Omega \setminus \overline{S_{w_k}}$, then we can find a radius $r > 0$ such that $B_r(x) \subset \Omega$ and $B_r(x) \cap \overline{S_{w_k}} = \emptyset$. Therefore we may deduce

$$\mathscr{H}^{n-1}(F_k \cap B_r(x)) = \mathscr{H}^{n-1}(\overline{S_{w_k}} \cap B_r(x)) = 0,$$

which leads to a contradiction since by definition of F_k we clearly have $\mathscr{H}^{n-1}(F_k \cap B_r(x)) > 0$. Hence $\overline{S_{w_k}}$ is an $\mathcal{N}$-aligned regular disconnected set. Finally since $\mathscr{H}^{n-1}(F_k \backslash S_{w_k}) = 0$ and $F_k = \overline{S_{w_k}}$, we observe that S_{w_k} is essentially closed for all $k \in \mathbb{N}$.

Step 3: *Final regularisation.* In order to conclude that the constructed approximants are in the admissible set, it remains to regularise every function w_k outside F_k. Applying Lemma 2.4, cf. parts (2) and (4), with $u = w_k$, $F = F_k$, and $\varepsilon = \varepsilon_k$, we obtain $(u_k) \subset \mathrm{C}^\infty(\Omega \backslash F_k)$ such that $S_{u_k} = S_{w_k}$,

$u_k^\pm = w_k^\pm$ in S_{u_k}, and $u_k \to u$ in $\mathrm{BV}(\Omega)$. Arguing as in Step 1, in conjunction with Step 2 we conclude that $(u_k) \subset \mathcal{W}_{\mathcal{N}}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, $\overline{S_{w_k}} = F_k$, and $u_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$.

Step 4: *Convergence of surface integrals.* This final step is devoted to the proof of (3.1). To this end let $\hat\gamma : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{N} \to [0,+\infty)$ be the function defined as $\hat\gamma(x,a,b,\nu) := \gamma(x,a,b,\nu)$, for every $x \in \Omega$, $a,b \in \mathbb{R}^m$, and $\nu \in \mathcal{N}$. Since both $\overline{S_{u_k}}$ and $\overline{S_u}$ are $\mathcal{N}$-aligned, proving (3.1) is equivalent to proving that

$$\limsup_{k\to+\infty} \int_{S_{u_k}\cap\overline{A}} \hat\gamma(x,u_k^+,u_k^-,\nu_{u_k})\,\mathrm{d}\mathscr{H}^{n-1} \le \int_{S_u\cap\overline{A}} \hat\gamma(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1}, \tag{3.10}$$

for every open set $A \subset\subset \Omega$.

By assumption $\hat\gamma$ is upper semicontinuous in which case we may find a decreasing sequence of continuous functions $(\gamma_j)_{j\in\mathbb{N}}$ with $\gamma_j : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{N} \to [0,+\infty)$ and $\gamma_j \ge \hat\gamma$ for any $j \in \mathbb{N}$, such that $\gamma_j \to \hat\gamma$ pointwisely in $\Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{N}$ as $j \to +\infty$.

Let $A \subset\subset \Omega$ be a fixed open set; without loss of generality we can assume that the limsup in the left hand side of (3.10) is actually a limit. By the $\mathscr{S}$-convergence of u_k to u we can find a subsequence (not relabelled) such that $u_k^+ \to u^+$ and $u_k^- \to u^-$ $\mathscr{H}^{n-1}$-a.e. on S_u as $k \to +\infty$. Then since by construction $\nu_{u_k} = \nu_u$ $\mathcal{H}^{n-1}$-a.e. in F_k, we get

$$\begin{aligned}\lim_{k\to+\infty} \int_{S_{u_k}\cap\overline{A}} \gamma_j(x,u_k^+,u_k^-,\nu_{u_k})\,\mathrm{d}\mathscr{H}^{n-1} &= \lim_{k\to+\infty} \int_{F_k\cap\overline{A}} \gamma_j(x,u_k^+,u_k^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1}\\ \le \limsup_{k\to+\infty} \int_{S_u\cap\overline{A}} \gamma_j(x,u_k^+,u_k^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1} &+ \|\gamma_j\|_{\mathrm{L}^\infty} \lim_{k\to+\infty} \mathscr{H}^{n-1}(S_u \triangle F_k).\end{aligned}$$

Therefore appealing to the Dominated Convergence Theorem and recalling (3.5) we get

$$\begin{aligned}\limsup_{k\to+\infty} \int_{S_{u_k}\cap\overline{A}} \hat\gamma(x,u_k^+,u_k^-,\nu_{u_k})\,\mathrm{d}\mathscr{H}^{n-1} &\le \lim_{k\to+\infty} \int_{S_{u_k}\cap\overline{A}} \gamma_j(x,u_k^+,u_k^-,\nu_{u_k})\,\mathrm{d}\mathscr{H}^{n-1}\\ &\le \int_{S_u\cap\overline{A}} \gamma_j(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1},\end{aligned}$$

for every $j \in \mathbb{N}$. Eventually (3.10) follows by the Monotone Convergence Theorem taking the limit as $j \to +\infty$ and this concludes the proof. □

4. Approximation results in the unconstrained case

In this section we prove the analogue of the SBV^p density result of Cortesani and Toader [5, Theorem 3.1] for $p = 1$. Namely, we show that every element of $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ can be approximated by a sequence of functions which are regular outside their jump set, the latter being a finite union of pairwise disjoint $(n{-}1)$-dimensional simplices. This result is obtained as an immediate corollary of the following approximation statement, which is interesting in its own right.

Proposition 4.1 ($\mathrm{SBV}^p(\Omega;\mathbb{R}^m)$-approximation). *Let $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Then there exists a sequence $(u_k) \subset \mathrm{SBV}^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that $S_{v_k} =: M_k$ is a compact $(n{-}1)$-dimensional C^1 manifold in Ω with C^1 boundary ∂M_k, the traces of v_k on both sides of M_k are C^1 and coincide on ∂M_k, and $u_k \xrightarrow{\mathscr{S}} u$ as $k \to +\infty$.*

Proof. As above we may consider the scalar case $m = 1$; in the general case we argue componentwise, observing that S_{v_k} is the union of the jump sets of the components, each of them being a compact $(n{-}1)$-dimensional C^1 manifold.

The proof follows arguments in the spirit of [7, Lemma 4.5] and is divided into three steps.

Step 1: *Preliminary regularisations.* Fix $\varepsilon > 0$. By the rectifiability of S_u, we can find a compact $(n-1)$-dimensional C^1 manifold $M_\varepsilon \subset\subset \Omega$ with C^1 boundary and a finite number of connected components, such that

$$\mathscr{H}^{n-1}(S_u \triangle M_\varepsilon) = \mathscr{H}^{n-1}(S_u \backslash M_\varepsilon) < \varepsilon.$$

Moreover, by the properties of Radon measures there exists a compact set $K_\varepsilon \subset S_u \cap M_\varepsilon$ with

$$\mathscr{H}^{n-1}(S_u \triangle K_\varepsilon) = \mathscr{H}^{n-1}(S_u \backslash K_\varepsilon) < \varepsilon. \tag{4.1}$$

Therefore appealing to Lemma 2.4 with $F = K_\varepsilon$ we get a function $u_\varepsilon \in \mathrm{SBV}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ satisfying the following properties:

$$u_\varepsilon \in \mathrm{C}^\infty(\Omega \backslash K_\varepsilon), \quad \|u - u_\varepsilon\|_{\mathrm{L}^1(\Omega)} < \varepsilon, \quad u_\varepsilon^\pm = u^\pm \;\text{ in }\; K_\varepsilon, \quad S_{u_\varepsilon} = S_u \cap K_\varepsilon. \tag{4.2}$$

Moreover, gathering (4.1) and the last equality in (4.2) gives

$$\mathscr{H}^{n-1}(S_u \triangle S_{u_\varepsilon}) < \varepsilon,$$

whereas (4.1) combined with Lemma 2.4 yields

$$\|\nabla u - \nabla u_\varepsilon\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} < c\,\varepsilon.$$

We now show the smallness of the traces of $u - u_\varepsilon$. To this end we observe that in view of Lemma 2.4 (2) and (4) we have

$$\begin{aligned}\int_{S_{u_\varepsilon} \cup S_u} \big(|u_\varepsilon^+ - u^+| + |u_\varepsilon^- - u^-|\big)\, \mathrm{d}\mathscr{H}^{n-1} &= \int_{S_u \backslash K_\varepsilon} \big(|u_\varepsilon^+ - u^+| + |u_\varepsilon^- - u^-|\big)\, \mathrm{d}\mathscr{H}^{n-1} \\ &\le 4\|u\|_{\mathrm{L}^\infty(\Omega)} \mathscr{H}^{n-1}(S_u \backslash K_\varepsilon),\end{aligned}$$

hence the claim again follows by (4.1).

Step 2: C^1-*modification of inner traces.* Let $\{M_\varepsilon^i\}_{i=1}^{N_\varepsilon}$ be the connected components of M_ε and let $\{U^i\}_{i=1}^{N_\varepsilon}$ be pairwise disjoint open sets with smooth boundary such that $M_\varepsilon^i \subset\subset U^i \subset\subset \Omega$. Let $i \in \{1, \ldots, N_\varepsilon\}$ be fixed and and let $\phi_i^+, \phi_i^- \in \mathrm{C}_c^1(M_\varepsilon^i)$ be such that

$$\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)} \le \frac{\varepsilon}{(1 + c_\varepsilon^i)N_\varepsilon},$$

where $c_\varepsilon^i := c(U^i, M_\varepsilon^i) > 0$ is the constant from Lemma 2.5 applied with $U = U^i$ and $M = M_\varepsilon^i$. Subsequently by Lemma 2.5 (a) we find $\psi_{\varepsilon,i} \in \mathrm{W}^{1,1}(U^i \backslash M_\varepsilon^i) \cap \mathrm{L}^\infty(\Omega)$ such that $\psi_{\varepsilon,i} = 0$ on ∂U^i, $\psi_{\varepsilon,i}^\pm = \phi_i^\pm - u_\varepsilon^\pm$ on M_ε^i and

$$\|\psi_{\varepsilon,i}\|_{\mathrm{W}^{1,1}(U^i \backslash M_\varepsilon^i)} \le \frac{c_\varepsilon^i\,\varepsilon}{(1 + c_\varepsilon^i)N_\varepsilon}.$$

Define

$$w_\varepsilon := \begin{cases} \psi_{\varepsilon,i} + u_\varepsilon & \text{in } U^i, \\ u_\varepsilon & \text{otherwise in } \Omega, \end{cases}$$

in which case

$$\begin{aligned}&\|u_\varepsilon - w_\varepsilon\|_{\mathrm{BV}(\Omega)} + \int_{S_{u_\varepsilon}\cup S_{w_\varepsilon}} \big(|w_\varepsilon^+ - u_\varepsilon^+| + |w_\varepsilon^- - u_\varepsilon^-|\big)\,\mathrm{d}\mathscr{H}^{n-1}\\ &\le \sum_{i=1}^{N_\varepsilon} \|\psi_{\varepsilon,i}\|_{\mathrm{BV}(U^i)} + \sum_{i=1}^{N_\varepsilon}\Big(\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)}\Big)\\ &\le \sum_{i=1}^{N_\varepsilon} \|\psi_{\varepsilon,i}\|_{\mathrm{W}^{1,1}(U^i\setminus M_\varepsilon^i)} + 2\sum_{i=1}^{N_\varepsilon}\Big(\|\phi_i^+ - u_\varepsilon^+\|_{\mathrm{L}^1(M_\varepsilon^i)} + \|\phi_i^- - u_\varepsilon^-\|_{\mathrm{L}^1(M_\varepsilon^i)}\Big) \le 3\varepsilon.\end{aligned}$$

We observe that the traces $w_\varepsilon^\pm$ coincide on ∂M_ε. Arguing as in the proof of Step 2, Theorem 3.1 (see also [7, Lemma 4.3]), we can construct $\hat{w}_\varepsilon \in \mathrm{SBV}^1(\Omega)\cap \mathrm{L}^\infty(\Omega)\cap \mathrm{W}^{1,1}(\Omega\setminus M_\varepsilon)$ such that $\mathscr{H}^{n-1}(M_\varepsilon\setminus S_{\hat{w}_\varepsilon}) = 0$, $\hat{w}_\varepsilon^\pm$ are C^1-regular, they coincide on ∂M_ε, and

$$\|\hat{w}_\varepsilon - w_\varepsilon\|_{\mathrm{BV}(\Omega)} + \int_{S_{\hat{w}_\varepsilon}\cup S_{w_\varepsilon}} \big(|\hat{w}_\varepsilon^+ - w_\varepsilon^+| + |\hat{w}_\varepsilon^- - w_\varepsilon^-|\big)\,\mathrm{d}\mathscr{H}^{n-1} < c\,\varepsilon.$$

Step 3: *Final regularisation.* Since the traces $\hat{w}_\varepsilon$ are C^1 we may employ Lemma 2.5 (b), for every $i\in\{1,\dots,N_\varepsilon\}$ to find $z_{\varepsilon,i}\in \mathrm{W}^{1,\infty}(U^i\setminus M_\varepsilon^i)$ such that $z_{\varepsilon,i}=0$ on ∂U^i and $z_{\varepsilon,i}^\pm = \hat{w}_\varepsilon^\pm$ in M_ε^i for every $i\in\{1,\dots,N_\varepsilon\}$. Set $z_\varepsilon := z_{\varepsilon,i}$ in U^i, $z_\varepsilon := 0$ otherwise. Since $\hat{w}_\varepsilon - z_\varepsilon \in \mathrm{W}^{1,1}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ we find $\eta_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega)\cap \mathrm{C}^\infty(\Omega)$ such that $\|\hat{w}_\varepsilon - z_\varepsilon - \eta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} \le \varepsilon$. Consequently define $\hat{z}_\varepsilon \in \mathrm{SBV}^1(\Omega)$ by

$$\hat{z}_\varepsilon := z_\varepsilon + \eta_\varepsilon.$$

Then by construction $\hat{z}_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega'\setminus M_\varepsilon)\cap \mathrm{L}^\infty(\Omega)$ for every $\Omega'\subset\subset\Omega$, $\hat{z}_\varepsilon^\pm = \hat{w}_\varepsilon^\pm$ in M_ε and $S_{\hat{z}_\varepsilon} = S_{\hat{w}_\varepsilon}$. Thus

$$\|\hat{w}_\varepsilon - \hat{z}_\varepsilon\|_{\mathrm{BV}(\Omega)} = \|\hat{w}_\varepsilon - z_\varepsilon - \eta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} < \varepsilon.$$

Let $\varphi \in \mathrm{C}_c^\infty(\Omega;[0,1])$ be such that $\varphi = 1$ in a neighbourhood of M_ε. Since $(1-\varphi)\hat{z}_\varepsilon \in \mathrm{W}^{1,1}(\Omega)\cap \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega)$, for any $1<p<\infty$ there exists a function $\zeta_\varepsilon \in \mathrm{W}^{1,p}(\Omega)\cap \mathrm{L}^\infty(\Omega)\cap \mathrm{C}^\infty(\Omega)$ such that $\|(1-\varphi)\hat{z}_\varepsilon - \zeta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} < \varepsilon$. Eventually, define

$$v_\varepsilon := \varphi\hat{z}_\varepsilon + \zeta_\varepsilon;$$

from the construction above we have $\varphi\hat{z}_\varepsilon \in \mathrm{W}^{1,\infty}(\Omega\setminus M_\varepsilon)$ which ultimately implies the inclusion $\varphi\hat{z}_\varepsilon \in \mathrm{SBV}^p(\Omega)\cap \mathrm{L}^\infty(\Omega)$ and thus $v_\varepsilon \in \mathrm{SBV}^p(\Omega)\cap \mathrm{L}^\infty(\Omega)$. Furthermore $v_\varepsilon^\pm = \hat{z}_\varepsilon^\pm$ in M_ε, $S_{v_\varepsilon} = S_{\hat{z}_\varepsilon}$, and

$$\|v_\varepsilon - \hat{z}_\varepsilon\|_{\mathrm{BV}(\Omega)} = \|(1-\varphi)\hat{z}_\varepsilon - \zeta_\varepsilon\|_{\mathrm{W}^{1,1}(\Omega)} < \varepsilon.$$

Altogether this concludes the proof. □

Remark 4.2. For later reference we observe that, if $u\in \mathrm{SBV}^1_{\mathcal{N}}(\Omega)\cap \mathrm{L}^\infty(\Omega)$, then the approximating functions v_ε given by Proposition 4.1 can be chosen in a way so that $\nu_{v_\varepsilon}\in\mathcal{N}$, $\mathscr{H}^{n-1}$-a.e. in S_{v_ε}.

Theorem 4.3 (Approximation of $\mathrm{SBV}^1(\Omega;\mathbb{R}^m)$ functions)**.** *Let $u\in \mathrm{SBV}^1(\Omega;\mathbb{R}^m)\cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$. Then there is a sequence $(u_k)\subset \mathrm{SBV}^1(\Omega;\mathbb{R}^m)\cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ satisfying the following properties:*

(i) $\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}$;

(ii) $\mathscr{H}^{n-1}(\overline{S_{u_k}}\setminus S_{u_k}) = 0$;

(iii) $\overline{S_{u_k}}$ *is the intersection of* Ω *with a finite number of pairwise disjoint* $(n-1)$*-dimensional simplices;*

(iv) $(u_k)\subset \mathrm{W}^{\ell,\infty}(\Omega\setminus\overline{S_{u_k}};\mathbb{R}^m)$, *for any* $\ell\in\mathbb{N}$;

(v) $\|u_k - u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$;

(vi) $\|\nabla u_k - \nabla u\|_{L^1(\Omega;\mathbb{R}^m)} \to 0$;

(vii) $\mathscr{H}^{n-1}(S_{u_k}) \to \mathscr{H}^{n-1}(S_u)$;

(viii) *for any open set* $A \subset\subset \Omega$ *there holds*

$$\limsup_{k\to+\infty} \int_{S_{u_k}\cap\overline{A}} \gamma\big(x, u_k^+, u_k^-, \nu_{u_k}\big)\, \mathrm{d}\mathscr{H}^{n-1} \le \int_{S_u\cap\overline{A}} \gamma\big(x, u^+, u^-, \nu_u\big)\, \mathrm{d}\mathscr{H}^{n-1},$$

for any upper semicontinuous function $\gamma : \Omega \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^{n-1} \to [0, +\infty)$ *such that* $\gamma(\cdot, a, b, \nu) = \gamma(\cdot, b, a, -\nu)$ *for every* $a, b \in \mathbb{R}^m$ *and* $\nu \in \mathbb{S}^{n-1}$.

Proof. Let $u \in \mathrm{SBV}^1(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ and $p \in (1, 2]$ be fixed. In view of Proposition 4.1 and Lemma 2.6, for every $\varepsilon > 0$ we can find $v_\varepsilon \in \mathrm{SBV}^p(\Omega;\mathbb{R}^m) \cap \mathrm{L}^\infty(\Omega;\mathbb{R}^m)$ such that

$$\|v_\varepsilon - u\|_{L^1(\Omega;\mathbb{R}^m)} < \varepsilon, \quad \|\nabla v_\varepsilon - \nabla u\|_{L^1(\Omega;\mathbb{R}^{m\times n})} < \varepsilon, \quad \mathscr{H}^{n-1}(S_{v_\varepsilon} \triangle S_u) < \varepsilon,$$

$$\int_{S_{v_\varepsilon}\cup S_u} \big(|v_\varepsilon^+ - u^+| + |v_\varepsilon^- - u^-|\big)\, \mathrm{d}\mathscr{H}^{n-1} < \varepsilon,$$

and

$$\|v_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^m)} \le \|u\|_{L^\infty(\Omega;\mathbb{R}^m)}.$$

Applying [5, Theorem 3.1 and Remark 3.5] to v_ε we find a sequence $(v_{\varepsilon,k})_k \subset \mathrm{SBV}^p(\Omega) \cap \mathrm{L}^\infty(\Omega)$ satisfying *(i)-(viii)* with u replaced by v_ε. Finally, a standard diagonal argument readily yields the desired sequence and thus the claim. □

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(Giuliano Lazzaroni) DiMaI U. Dini, Università di Firenze, V.le G.B. Morgagni 67/A, 50134 Firenze, Italy
Email address, Giuliano Lazzaroni: giuliano.lazzaroni@unifi.it

(Piotr Wozniak, Caterina Ida Zeppieri) Applied Mathematics Münster, University of Münster, Einsteinstrasse 62, 48149 Münster, Germany
Email address, Piotr Wozniak: piotr.wozniak@uni-muenster.de
Email address, Caterina Ida Zeppieri: caterina.zeppieri@uni-muenster.de