STOCHASTIC HOMOGENISATION OF NONCONVEX ELLIPTIC INTEGRALS

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ABSTRACT. We prove a result on stochastic homogenisation of integral functionals of the form

$$\int_U f\Big(\omega, x/\varepsilon, \mathbb{A}u\Big)\mathrm{d}x$$

where ω is a random parameter, $\varepsilon > 0$ and $\mathbb{A}$ is a real elliptic vectorial differential operator. This work is intended to generalise results for the full gradient and to cover the cases of the symmetric gradient and the deviatoric operator. The homogenisation procedure is carried out by employing a variant of the blow-up method in the setting of $\mathbb{A}$-Sobolev spaces along with the Akcloglu-Krengel subadditive ergodic theorem.

Keywords: Stochastic Homogenization, Γ-convergence, elliptic operators, non-convex functionals, ergodic theory, blow-up method.

1. INTRODUCTION

This paper is aimed at deriving homogenisation limits of sequences of random integral functionals that act on vectorial differential operators. Studying the asymptotics of random nonlinear systems is an instrumental and highly effective apparatus for modelling macroscopic progressions of materials, particularly in the context of elasticity theory. In fact, essential information on measuring energies of states at small scales is on many occasions captured from the analysis of integral functionals with rapidly oscillating random integrands. Eventual homogenised formulae give rise to energy densities that describe the effective behaviour of arrangements of statistically distributed heterogeneities in a medium. In terms of variational formulations, we may demonstrate a nominal construction by considering a bounded, open region $U \subset \mathbb{R}^n$ along with scale-dependent stochastic functionals evaluated on an assigned class of deformations $u : U \subset \mathbb{R}^n \to \mathbb{R}^N$ which take the form:

$$\mathscr{F}_\varepsilon(\omega)[u; U] := \int_U f\Big(\omega, \frac{x}{\varepsilon}, \mathbb{A}u\Big)\,\mathrm{d}x. \tag{1.1}$$

Here ω is a random parameter in a probability space Ω, the mapping $\mathbb{A}$ symbolises a vectorial linear differential operator. The integral is evaluated on stochastic densities $\omega \mapsto f(\omega, \cdot, \cdot)$ subject to appropriate growth bounds. The main theme revolves around finding an explicit description of the asymptotic limit of the random sequence $(\mathscr{F}_\varepsilon(\omega))$ through Γ-convergence and thereby proving almost sure convergence of minima and minimising sequences of (1.1) as $\varepsilon \to 0$ to a minimisation problem of a precisely framed functional. Let us emphasize that introducing probabilistic compositions into the variation, grants the possibility to address statistically posed hypotheses linked to more realistic microstructured arrangements.

Currently the literature consists of numerous contributions serving detailed analysis of the above problems and to indicate an almost certainly incomplete list we reference exemplarily [6, 8, 33, 35] for deterministic motifs and [17, 18, 31, 37, 34] for the ones with stochastic realisations. As for the former variant, a customary condition assumed on integrands is periodicity in the space variable. The resulting cell-formulae is implied to satisfy homogeneity i.e. it relies only on the phase variable ξ. Interpreted physically this is ought to verify a homogenous nature of recurringly structured materials in the macroscopic perspective. In the context of random media, a notion of stochastic periodicity is pivotal for homogenisation to hold. Such analogy may be conceptualised by asserting that the spatial arrangements of cells in a region is congruent to the action of $\mathbb{P}$-preserving transformations on Ω, that is the probability of its occurrence remains unchanged. A quantitative formulation of such condition reads as follows. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability triple and let $\{\tau_z : \Omega \to \Omega : z \in \mathbb{Z}^n\}$ be a group transformation maps that satisfy $\mathbb{P}(\tau_z(E)) = \mathbb{P}(E)$ for any $E \in \mathcal{T}$ and $z \in \mathbb{Z}^n$. A volume density f is *periodic in law* or *stationary* provided:

$$f(\omega, x + z, \xi) = f(\tau_z(\omega), x, \xi) \tag{1.2}$$

for all $(x, \xi) \in \mathrm{Dom}(f(\omega, \cdot, \cdot))$, all $z \in \mathbb{Z}^n$ and all $\omega \in \Omega$. We notice that (1.2) is entirely consistent with the spatial periodicity for deterministic (i.e. not dependent on ω) integrands. Aiming for a nondegenerate behaviour in the limit, in addition a standard assumption imposed on densities $f(\omega, \cdot, \cdot)$ in such context is the p-growth condition, namely: for $p \in (1, +\infty)$ there exist constants $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha|\xi|^p \leq f(\omega, x, \xi) \leq \beta(1 + |\xi|^p) \tag{1.3}$$

for all $(x, \xi) \in \mathrm{Dom}(f(\omega, \cdot, \cdot))$ and all $\omega \in \Omega$. Early occurrences of stochastic homogenisation of integral functionals in the above setting are present for instance in the work of Dal Maso and Modica [17, 18]. It shall serve as a prototypical example in our discourse. Subject to convexity in the phase variable and superlinear growth the authors treat the case of full gradient, that is, $\mathbb{A} = \nabla = (\partial_1, \dots, \partial_n)$. It is proved that $(\mathscr{F}_\varepsilon(\omega))_{\varepsilon>0}$ Γ-converges $\mathbb{P}$-a.e. as $\varepsilon \to 0$ to a stochastic integral functional

$$F_{\mathrm{hom}}(\omega)(u; U) = \int_U f_{\mathrm{hom}}(\omega, \nabla u) \, \mathrm{d}x$$

with the limiting integrand

$$f_{\mathrm{hom}}(\omega, \xi) = \lim_{T \to +\infty} T^{-n} \inf \Big\{ \int_{(0,T)^n} f(\omega, x, \xi + \nabla v) \, \mathrm{d}x : v \in \mathrm{W}^{1,p}_0((0,T)^n) \Big\}.$$

The density f_{hom} satisfies $\mathbb{P}$-a.e. the same p-growth bounds (1.3) as f. It is worth pointing out the main disparity with the well-known periodic case lies in the task of justifying existence and the very homogeneity of the limit density which depends on the stochastic parameter $\omega \in \Omega$. This is essentially resolved by proving that the defining cell-formula gives rise to a subadditive stochastic process, see Section 2.4 for precise definitions. The claim is then shown by invoking the *ergodic theorem* as per [1] in conjunction with the integral representations in $\mathrm{W}^{1,p}$ for instance as in [19].

Inspired by the aforementioned contribution, to account for a broad spectrum of possible models where energies often do not depend on full gradients, one may look at resembling problems with differential dependence being posed. These are inscribed by themes where the full gradient in (1.1) is replaced by a potentially general linear differential operator. From the perspective of physical models, examples of high interest thereof include the *symmetric gradient* $e(u) := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$ or the *deviatoric/trace-free symmetric gradient* $e^{\mathcal{D}}(u) := e(u) - \frac{1}{n}\mathrm{div}(u) I_n$ acting on vector fields of the form $u : U \to \mathbb{R}^n$ where I_n is the $(n \times n)$-identity matrix. More formally for a pair of finite-dimensional vector spaces V, W ($\mathrm{V} \cong \mathbb{R}^K$, $\mathrm{W} \cong \mathbb{R}^L$) we consider a linear differential operator $\mathbb{A}$ from V to W defined as

$$\mathbb{A}u := \sum_{j=1}^n \mathbb{A}_j \partial_j u, \qquad u : U \to \mathrm{V} \tag{1.4}$$

where each $\mathbb{A}_j : \mathrm{V} \to \mathrm{W}$ is a fixed linear map. The desired goal is to establish an analogous Γ-convergence of energies in (1.1) acting on operators as in (1.4) up to a set of probability zero. In the interest of this discourse it is particularly intended to advance the existing results and investigate the asymptotic behaviour of energies defined on a selected family of differential operators and equally important to take into consideration integrands that are *nonconvex*. Given a potentially arbitrary differential dependence, the structure of any apparent limit shall differ from the nominal Sobolev setting of Dal Maso and Modica. As such our objective on the one hand is principally to cover numerous relevant models occurring in the related literature and on the other to give more insights into the universal notions involved in the topic thereby making a step forth in unifying the theory.

1.1. **Main Result and Strategy.** In this section we shall give an outline of the central theorem to which the content of this paper is devoted. The already said aim is to prove existence of *almost sure* Γ-limit of *nonconvex* stochastic functionals as given in (1.1) incorporating a broad class of admissible operators $\mathbb{A}$ and determine the cell-formula for homogenised densities. In doing so, striving for a more universal differential condition we wish to obtain results that are not equivalent to the $\mathrm{W}^{1,p}$-theory. By this we mean the situation where one may infer uniform gradient bounds from equi-coercive energies. In fact the works of Aronszajn and Smith [4, 40], see also [30, 27], tell us that estimates of the form

$$\|\nabla u\|_{\mathrm{L}^p(U)} \leq c\Big(\|u\|_{\mathrm{L}^p(U)} + \|\mathbb{A}u\|_{\mathrm{L}^p(U)}\Big) \tag{1.5}$$

are true to hold for operators $\mathbb{A}$ with finite-dimensional nullspace i.e. such that $\dim\{u \in \mathscr{D}'(\mathbb{R}^n; \mathrm{V}) : \mathbb{A}u = 0\} < +\infty$ where $\mathscr{D}'(\mathbb{R}^n; \mathrm{V})$ denotes the class of all V-valued distributions on $\mathbb{R}^n$. With coercivity estimates as in (1.5) within the reach and considering the p-growth assumption one may routinely recast the original problem to the known Sobolev case. Introducing a larger constellation of operators such as those having possibly infinite-dimensional nullspaces thus affects the methodology and demands an independent argumentation. Thereby in search of appropriate framework, the variational formulation of objects in (1.1) is posed in the space of weakly $\mathbb{A}$-differentiable maps or $\mathbb{A}$-Sobolev spaces denoted by $\mathrm{W}^{\mathbb{A},p}$ cf. [10, 27, 28]. These classes comprise of all $u \in \mathrm{L}^p(U; \mathrm{V})$ such that $\mathbb{A}u \in \mathrm{L}^p(U; \mathrm{W})$. We refer to Section 2.3 for a detailed discussion on the matter. In view of the growth assumption requiring admissible competitors to possess p-integrable $\mathbb{A}$-gradient and not necessarily the full derivative renders the minimisation problem well-defined. Let us recount that a number of functional properties of $\mathrm{W}^{\mathbb{A},p}$-spaces heavily depend on the type of differential operators $\mathbb{A}$ they are assigned to. It comes to light that the essential features in the interest of homogenisation is captured by the so called *ellipticity* condition coming from the study of overdetermined systems e.g. Hörmander's work [29]. By real ellipticity of $\mathbb{A}$ we mean that its Fourier Symbol $\mathbb{A}[\xi] : \mathrm{V} \to \mathrm{W}$ given by the linear map

$$\mathbb{A}[\xi]v := \sum_{j=1}^{n} \xi_j \mathbb{A}_j v$$

is injective for all $\xi \in \mathbb{R}^n \setminus \{0\}$. This algebraic notion is shared among many widely studied operators including the full and symmetric gradients or the Maxwell operator to name a few (see Example 2.2 for a larger list). Throughout our work all objects of consideration are going to assume this property. Real ellipticity (see also Definition 2.1) indeed appears to be a determining condition in conceptualising relevant machinery with the available tools of homogenisation analysis. We shall comment its influence on the strategy in more precise terms at the end of this section.

Regarding the question of Γ-convergence, based on the foundational works of Dal Maso and Modica [17, 18] we will contemplate the subadditive process defined on the probability space $(\Omega, \mathcal{T}, \mathbb{P})$ that assumes the form

$$\mu^A(\omega, U) := \inf\Big\{ \int_U f(\omega, x, A + \mathbb{A}v)\,\mathrm{d}x : v \in \mathrm{W}_0^{\mathbb{A},p}(U) \Big\}$$

where $U \subset \mathbb{R}^n$ is a bounded and open Lipschitz set, $A \in \mathrm{W}$ and $\mathrm{W}_0^{\mathbb{A},p}(U)$ denotes the closure of $\mathrm{C}_c^\infty(U; \mathrm{V})$ under the seminorm $\|\mathbb{A}(\cdot)\|_{\mathrm{L}^p(U;\mathrm{W})}$. Appealing to the ergodic theorem of Akcoglu and Krengel [1] this process is instrumental in deriving existence of homogenised densities. In particular subject to all said conceptions the hypotheses of Theorem 3.1 state that for any $U \subset \mathbb{R}^n$ open, bounded with Lipschitz boundary the energies $\mathscr{F}_\varepsilon(\omega)[\cdot, U]$ Γ-converge almost surely as $\varepsilon \to 0+$ to the limit given by:

$$\mathscr{F}_{\hom}(\omega)[u; U] := \int_U f_{\hom}(\omega, \mathbb{A}u)\,\mathrm{d}x \qquad \text{for } u \in \mathrm{W}^{1,p}(U; \mathrm{V}),$$

where

$$f_{\hom}(\omega, A) := \lim_{T\to+\infty} T^{-n} \inf\Big\{ \int_{(0,T)^n} f(\omega, x, A + \mathbb{A}v)\,\mathrm{d}x : v \in \mathrm{W}_0^{\mathbb{A},p}((0,T)^n) \Big\}.$$

We make an observation that the proposed result yields the homogenised limit defined in the $\mathbb{A}$-Sobolev regimes which analogously reciprocates preceding outcomes for the $\mathrm{W}^{1,p}$-setting. Considering the available theory for general Sobolev spaces it is necessary to alter the prototypical methods derived for the full gradient. Indeed in contrast to finite-dimensional nullspace, any uniform control of the full gradients of minimising sequences is not automatically guaranteed therefore making the classical results nonapplicable. To meet our objective, the strategy shall adhere to the blow-up argument as elaborated in [35, 7]. This choice confines one to materialise a strong notion of differentiability but simultaneously it is powerful enough to encompass many nonconvex energies. Here the assumed *ellipticity* goes hand in hand with the requisite apparatus. Indeed in verifying the ansatz-free inequality, the blow-up technique exploits the L^p-differentiability of the competitor maps. On account of the works of Alberti *et al.* [3, Lem. 3.9] the essential differentiability is inquired as to the existence of a $(1-n)$-homogeneous kernel $\mathrm{K}_\mathbb{A}$ such that

$$u = \mathrm{K}_\mathbb{A} * \mathbb{A}u \tag{1.6}$$

for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathrm{V})$. Furthermore along with boundedness of Riesz potentials (1.6) implies a variant of Poincaré inequality for zero boundary trace, cf. Proposition 2.4 which will come in useful when preserving uniform bounds of modified sequences. Such convolution representations with a Riesz-type kernels are to author's awareness, known for elliptic operators $\mathbb{A}$ and we recount here [12, Lem. 2.1]. With regards to the lim-sup inequality it is settled with the aid of am approximation result which allows for a reduction of the claim to piecewise affine maps in the spirit of [35].

1.2. **Structure of the paper.** In Section 2 we gather some key concepts of vectorial differential operators and the ellipticity condition with examples of such operators. We also introduce the notion of $\mathbb{A}$-weakly differentiable maps and present a number of properties thereof including the L^p-differentiability and approximations by piecewise affine maps. Moreover we dedicate a subsection to lay out the stochastic setting and go through all relevant aspects of ergodic theory being used in this context. In Section 3 we address the Γ-convergence of $(\mathscr{F}_\varepsilon(\omega))$ and give a proof of Theorem 3.1

which is divided into two parts, see Proposition 3.2 and Proposition 3.5. The final Section 4 deals with convergence of minima subject to Dirichlet boundary conditions.

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2. Preliminaries and Contextualisation

2.1. **Notation and general conventions.** Let us concisely mention the notation present in the paper. Throughout $n \geq 2$ is a fixed integer. We set $Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ to be the open unit cube centred at the origin. The set $Q_r(x) := rQ + x$ represents the open cube centred at $x \in \mathbb{R}^n$ with side length $r > 0$. We denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure. The collection $\mathscr{A}(\mathbb{R}^n)$ denotes all open and bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary. For a topological space $\mathcal{S}$, the symbol $\mathscr{B}(\mathcal{S})$ signifies the Borel sigma algebra of $\mathcal{S}$. For $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n; \mathrm{V})$ and $O \subset \mathbb{R}^n$ s.t. $\mathscr{L}^n(O) < +\infty$ we set $\langle u \rangle_O := ⨍_O u \,\mathrm{d}x = \mathscr{L}^n(O)^{-1} \int_O u \,\mathrm{d}x$. For a matrix $\xi \in \mathbb{R}^{K \times n}$ and a vector $b \in \mathbb{R}^K$, by ℓ_ξ we mean the affine map $x \in \mathbb{R}^n \mapsto \xi x$. We denote by $\langle \cdot, \cdot \rangle$ the euclidean inner product and by $|\cdot|$ the induced norm. The space of linear maps between two vector spaces $\mathcal{V}$ and $\mathcal{W}$ say is labelled as $\mathcal{L}(\mathcal{V}, \mathcal{W})$. Moreover for two vectors v and w the symbol $v \otimes w$ is thought of as the euclidean tensor product of v and w. The letter $c > 0$ is designated for absorbing constants where any significant dependence will be specified.

Let us fix once and for all a pair of finite-dimensional vector spaces V, W of dimension at least two (i.e. $\mathrm{V} \cong \mathbb{R}^K$ and $\mathrm{W} \cong \mathbb{R}^L$ for some integers $K, L \geq 2$).

2.2. **The class of differential operators.** A vectorial differential operator $\mathbb{A}$ from V to W of order $m \in \mathbb{N}$ and with constant coefficients is determined via the action

$$\mathbb{A}u := \sum_{j=1}^n \mathbb{A}_j \partial_j u, \qquad u : U \subset \mathbb{R}^n \to \mathrm{V} \tag{2.1}$$

for $\mathbb{A}_j \in \mathcal{L}(\mathrm{V}; \mathrm{W})$ linear maps from V to W. An operator $\mathbb{A}$ of such form can be interpreted through the linear coupling $\mathbb{A}u = \mathcal{A}(\nabla u)$ where $\mathcal{A} \in \mathcal{L}(\mathrm{V} \otimes \mathbb{R}^n; \mathrm{W})$. Explicitly we illustrate the relation by means of a commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n \times \mathrm{V} & \xrightarrow{\ \otimes\ } & \mathrm{V} \otimes \mathbb{R}^n \\ & \searrow^{\otimes_{\mathbb{A}}} & \downarrow \mathcal{A} \\ & & \mathbb{R}^n \otimes_{\mathbb{A}} \mathrm{V} \end{array} \tag{2.2}$$

Here $\otimes$ is the usual euclidean tensor pairing whereas for $\xi \in \mathbb{R}^n$ and $\sigma \in \mathrm{V}$ we declare $\xi \otimes_{\mathbb{A}} \sigma := \sum_{j=1}^n \xi_j \mathbb{A}_j \sigma$. The space $\mathscr{C}(\mathbb{A}) := \{\xi \otimes_{\mathbb{A}} \sigma : \xi \in \mathbb{R}^n, \sigma \in \mathrm{V}\}$ is called the $\mathbb{A}$-*rank-one cone*. Moreover we associate a Fourier symbol mapping to $\mathbb{A}$ by writing the linear combination

$$\mathbb{A}[\xi]\sigma := \xi \otimes_{\mathbb{A}} \sigma = \sum_{j=1}^n \xi_j \mathbb{A}_j \sigma, \qquad \text{for } \xi \in \mathbb{R}^n \text{ and } \sigma \in \mathrm{V}.$$

Definition 2.1 (Ellipticity)**.** An operator $\mathbb{A}$ from V to W is said to be *real elliptic* or $\mathbb{R}$-elliptic as long as for all ξ in $\mathbb{R}^n \setminus \{0\}$, the Fourier symbol $\mathbb{A}[\xi] : \sigma \mapsto \mathbb{A}[\xi]\sigma$ is an injective map from V to W.

By the nullspace of $\mathbb{A}$ we mean the vector subspace $\ker(\mathbb{A}) := \{u \in \mathscr{D}'(\mathbb{R}^n; \mathrm{V}) : \mathbb{A}u = 0\}$. Here $\mathscr{D}'(\mathbb{R}^n; \mathrm{V})$ denotes the class of all V-valued tempered distributions on $\mathbb{R}^n$ i.e. all bounded functionals on the space $\mathscr{D}(\mathbb{R}^n; \mathrm{V}) := \mathrm{C}_c^\infty(\mathbb{R}^n; \mathrm{V})$. The equality in the parenthesis is regarded in the distributional sense. An analytic perspective assumed in [27, Prop. 3.1] provides a sufficient relation for ellipticity and the nullspace. More precisely if $\dim \ker(\mathbb{A}) < +\infty$, then $\mathbb{A}$ is $\mathbb{R}$-elliptic. Let us note that in the latter relation one can only afford the forward implication, for there exist $\mathbb{R}$-elliptic operators which do not admit finite-dimensional nullspace. For instance if we identify $\mathbb{R}^2$ with $\mathbb{C}$, the trace-free symmetric gradient $e^{\mathcal{D}}$ contains all holomorphic maps in its kernel. Nonetheless let us recount from [15, Cor. 4.3] that for $\mathbb{A}$ elliptic there holds $\ker(\mathbb{A}) \cap \mathrm{L}^1_{\mathrm{loc}} \subset \mathrm{C}^\infty$.

A differential operator $\mathbb{B}$ from W to some finite-dimensional vector space Z is an *annihilator* of $\mathbb{A}$ if the corresponding Fourier symbol maps satisfy the equality $\ker \mathbb{B}[\xi] = \mathbb{A}[\xi](\mathrm{V})$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. In other words one requires exactness of the symbol complex:

$$\mathrm{V} \xrightarrow{\mathbb{A}[\xi]} \mathrm{W} \xrightarrow{\mathbb{B}[\xi]} 0. \tag{2.3}$$

In such an occurrence we also say that $\mathbb{A}$ is the *potential* of $\mathbb{B}$. Assuming *real ellipticity*, the existence of annihilators that obey the constant-rank condition is ensured, see e.g. [39]. However let us emphasize that annihilators of first order potentials do not necessarily have to be of the same order e.g the annihilator of the symmetrised gradient, see Example 2.2 (ii), is $\operatorname{curl} \circ \operatorname{curl}$. In terms of the tensoric formulation, by combining with the exact sequence condition it is immediate that the $\mathbb{A}$-*rank one cone* $\mathscr{C}(\mathbb{A})$ must coincide with the characteristic cone of $\mathbb{B}$, that is $\Lambda_{\mathbb{B}} := \cup_{\xi \neq 0} \ker \mathbb{B}[\xi]$:

$$\mathscr{C}(\mathbb{A}) = \Lambda_{\mathbb{B}} \tag{2.4}$$

Let us fix throughout the course of our analysis the target vector space W to be $\mathscr{R}(\mathbb{A}) := \operatorname{span}\{\mathscr{C}(\mathbb{A})\}$, the *effective range* associated to the operator $\mathbb{A}$. This represents a linear combination of all $\mathbb{A}$-tensoric pairings which in turn encapsulates the minimal cone containing the codomain of the Fourier symbol of $\mathbb{A}$ as linear mapping. Let us conclude this section by enumerating a selection of differential operators that are *real elliptic*.

Example 2.2. (i) *Full gradient*: for $\mathrm{V} = \mathbb{R}^m$ and $\mathrm{W} = \mathbb{R}^m$, the set $\ker(\nabla)$ is made out of all real $(m \times n)$-matrices.
(ii) *Symmetric gradient*: $e(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ for $\mathrm{V} = \mathbb{R}^n$ and $\mathrm{W} = \mathbb{R}^{n\times n}_{\mathrm{sym}}$. The nullspace $\ker(e)$ is given by the space of rigid deformations $\mathcal{R} := \{x \mapsto Ax + b : A \in \mathbb{R}^{n\times n}_{\mathrm{skew}}, b \in \mathbb{R}^n\}$ which is finite-dimensional.
(iii) *Trace-free symmetric gradient*: $e^{\mathcal{D}}u := e(u) - \frac{1}{n}\operatorname{div}(u) I_n$ where I_n is the $(n \times n)$-identity matrix.
(iv) For $\mathrm{V} = \mathbb{R}^2$ consider the operator $\mathbb{B}u = (\partial_1 u_2 + \partial_2 u_1, \partial_2 u_2, \partial_1 u_1)$. Then by inspecting the nullity $\mathbb{B}[\xi]v = 0$ it follows that $\mathbb{B}$ is $\mathbb{R}$-elliptic.
(v) *Maxwell operator*: $\mathbb{M}u = (\operatorname{div} u, \operatorname{curl} u)$ in the case of $\mathrm{V} = \mathbb{R}^2$ and $\mathrm{W} = \mathbb{R}^3$ is elliptic which follows from the coupled vanishing condition on div and curl.
(vi) If $\mathbb{L}$ is any of the above operators and $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^{N-1}$, then the operator $\mathbb{K} := (\lambda_1 \mathbb{L}, \ldots, \lambda_N \mathbb{L})$ has a finite-dimensional nullspace.

2.3. **Function spaces.** Let $U \subset \mathbb{R}^n$ be open and bounded and let $\mathbb{A}$ be a fixed $\mathbb{R}$-elliptic differential operator from V to W of order one. The generalised $\mathbb{A}$-Sobolev spaces are formulated as:

$$\mathrm{W}^{\mathbb{A},p}(U) := \{u : U \to \mathrm{V} : u \in \mathrm{L}^p(U;\mathrm{V}), \mathbb{A}u \in \mathrm{L}^p(U;\mathrm{W})\}.$$

Endowed with the norm $\|\cdot\|_{\mathrm{W}^{\mathbb{A},p}(U)} := \|\cdot\|_{\mathrm{L}^p(U;\mathrm{V})} + \|\mathbb{A}(\cdot)\|_{\mathrm{L}^p(U;\mathrm{W})}$, for $p > 1$ this becomes reflexive and separable Banach space i.e. we have an isometry $\mathrm{W}^{\mathbb{A},p}(U) \hookrightarrow \mathrm{L}^p(U;\mathrm{V}) \times \mathrm{L}^p(U;\mathrm{W})$ given by $u \mapsto (u, \mathbb{A}u)$. Let us denote by $\mathrm{W}_0^{\mathbb{A},p}(U)$ the closure of $\mathrm{C}_c^\infty(U;\mathrm{V})$ in the $\|\mathbb{A}(\cdot)\|_{\mathrm{L}^p(U;\mathrm{W})}$ seminorm.

Let us enumerate a couple of essential function theoretic results that are reminiscent of the $\mathrm{W}^{1,p}$-theory. Firstly we bring up the statement describing approximation of $\mathrm{W}^{\mathbb{A},p}$-spaces by smooth maps up to the boundary.

Proposition 2.3 (Global smooth approximation)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. Then $\mathrm{C}^\infty(\overline{U};\mathrm{V})$ is a dense subset of $\mathrm{W}^{\mathbb{A},p}(U)$ in the norm topology.*

Proof. The proof is standard and we include an outline for convenience. We begin by observing that ∂U can be covered by a finite collection of balls $\{B_i\}_{i=1}^{\ell}$ such that for each $i \in \{1,\dots,\ell\}$ there exists a Lipschitz coordinate function $f_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $U \cap B_i = \{(x', x_n) \,|\, 0 < x_n < f_i(x')\} \cap B_i$. We choose an associated partition of unity $\varphi_i \in \mathrm{C}_c^\infty(B_i;[0,1])$ and $\varphi_0 \in \mathrm{C}_c^\infty(U;[0,1])$ such that $\sum_{i=0}^{\ell} \varphi_i = 1$. Let $\delta_k > 0$ be such that $\delta_k \to 0$ as $k \to +\infty$ and let $\eta_{\delta_k} : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifying kernel. For every $i = 1,\dots \ell$ setting $u_k^i(x', x_n) := (\eta_{\delta_k} * \varphi_i u)(x', x_n - \delta_k) \in \mathrm{C}^\infty(\overline{U};\mathrm{V})$ it follows that for k large enough $\mathrm{supp}(u_k^i) \subset \overline{U} \cap B_i$ and $\|u_k^i - \varphi_i u\|_{\mathrm{W}^{\mathbb{A},p}(U\cap B_i)} < 2^{-k}/\ell$. On the other hand since the support of $\varphi_0 u$ is compactly contained in U by mollification we may find a map $u_k^0 \in \mathrm{C}_c^\infty(U;\mathrm{V})$ such that $\|u_k^0 - \varphi_0 u\|_{\mathrm{W}^{\mathbb{A},p}(U)} < 2^{-k}/\ell$. Altogether if we define $u_k = \sum_{i=0}^{\ell} u_k^i \in \mathrm{C}^\infty(\overline{U};\mathrm{V})$, then

$$\|u - u_k\|_{\mathrm{W}^{\mathbb{A},p}(U)} < \frac{1}{2^k}$$

and this finishes the proof by passing to $k \to +\infty$. □

Proposition 2.4 (Poincaré for $\mathrm{W}_0^{\mathbb{A},p}$)**.** *Let $\mathbb{A}$ be as in (2.1) be elliptic and let $U \subset \mathbb{R}^n$ be open and bounded. Then there exists a constant $C = C(n,\mathbb{A})$ such that*

$$\|u\|_{\mathrm{L}^p(U;\mathrm{V})} \le C(\mathrm{diam}\, U)\|\mathbb{A}u\|_{\mathrm{L}^p(U;\mathrm{W})}.$$

for all $u \in \mathrm{W}_0^{\mathbb{A},p}(U)$.

Proof. By smooth approximation in the norm convergence it suffices to consider $u \in \mathrm{C}_c^\infty(U;\mathrm{V})$. Applying the Fourier transform we may write $\widehat{u}(\xi) = \mathrm{k}_{\mathbb{A}}(\xi)\mathbb{A}[\xi]\widehat{u}(\xi)$ with the multiplier $\mathrm{k}_{\mathbb{A}}(\xi) = \mathbb{A}[\xi]^* \circ (\mathbb{A}[\xi] \circ \mathbb{A}[\xi]^*)^{-1}$. Thus in view of [12, Lem 2.1] the representation $u = \mathrm{K}_{\mathbb{A}} * \mathbb{A}u$ holds where $\mathrm{K}_{\mathbb{A}} \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(\mathrm{W}; \mathrm{V} \otimes \mathbb{R}^n))$ is an $(1-n)$-homogeneous kernel induced from the Fourier inversion. Therefore using the Young convolution inequality along with L^p-boundedness of Riesz potentials on bounded domains [41, Chpt. V] we calculate the L^p-bounds through

$$\begin{aligned}\|u\|_{\mathrm{L}^p(U;\mathrm{V})} &= \|\mathrm{K}_{\mathbb{A}} * \mathbb{A}u\|_{\mathrm{L}^p(U;\mathrm{V})} \le \|\mathrm{K}_{\mathbb{A}}\|_{\mathrm{L}^1(U)}\|\mathbb{A}u\|_{\mathrm{L}^p(U;\mathrm{W})} \\ &\le C\||\cdot|^{1-n}\|_{\mathrm{L}^1(U)}\|\mathbb{A}u\|_{\mathrm{L}^p(U;\mathrm{W})} \le C(\mathrm{diam}\, U)\|\mathbb{A}u\|_{\mathrm{L}^p(U;\mathrm{W})}.\end{aligned}$$

□

In the context of homogenisation of integral functionals, it is essential to address the issue of possible differentiability notion of the blow-up sequence elements. Following Alberti's contributions [3], we give a framework to interpret Lebesgue differentiability for higher integrability regimes.

Proposition 2.5 (L^p-differentiability)**.** *Let $\mathbb{A}$ as in (2.1) be an elliptic operator and let $p \in (1,+\infty)$ be fixed. Then any $u \in \mathrm{W}^{\mathbb{A},p}_{\mathrm{loc}}(\mathbb{R}^n)$ is L^p-differentiable for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$, that is,*

$$u(y) = \nabla u(x)(y-x) + u(x) + \mathrm{R}(x,y)$$

such that $\langle |\mathrm{R}(x,\cdot)|^p\rangle_{Q_r(x)} = o(r^p)$ as $r \to 0+$ is the first order term L^p-Taylor expansion of u at the point x.

Proof. We argue by means of localisation and regularisation. Let $B_r \subset \mathbb{R}^n$ be a ball. Take $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ such that $\varphi = 1$ in B_r. Extending φu trivially by zero yields a mapping in $\mathrm{W}^{\mathbb{A},p}_0(\mathbb{R}^n)$ with support being compactly contained in $\mathbb{R}^n$. Therefore there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\mathbb{R}^n;\mathrm{V})$ such that $u_k \to \varphi u$ as $k \to +\infty$ in the norm topology of $\mathrm{W}^{\mathbb{A},p}$. Hence by the said smooth approximation and an application of Fourier transform in a similar fashion to Proposition 2.4 we obtain

$$\varphi u = \mathrm{K}_{\mathbb{A}} * \mathbb{A}(\varphi u)$$

pointwisely $\mathscr{L}^n$-a.e. in $\mathbb{R}^n$. But since $\mathbb{A}(\varphi u)$ is in $\mathrm{L}^p(\mathbb{R}^n;\mathrm{W})$ and $\mathrm{K}_{\mathbb{A}} \in \mathrm{C}^\infty(\mathbb{R}^n\setminus\{0\};\mathcal{L}(\mathrm{W};\mathrm{V}\otimes\mathbb{R}^n))$ is an $(1-n)$-homogeneous kernel, from [3, Lem. 3.9] we conclude that φu is L^p-differentiable in $\mathbb{R}^n$ and so u is L^p-differentiable for $\mathscr{L}^n$-a.e. point in B_r. Since the initial ball was arbitrarily chosen this proves the claim. □

2.4. **Probabilistic setting and admissible integrands.** In this section we fix $\mathbb{A}$ to be a real elliptic operator of order one. Let $(\Omega,\mathcal{T},\mathbb{P})$ be a probability space and let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of $\mathbb{P}$-preserving transformations. More precisely we consider a collection of maps $\tau_z : \Omega \to \Omega$ such that for all $z \in \mathbb{Z}^n$

- τ_z is $\mathcal{T}$-measurable
- τ_z is a bijection
- $\mathbb{P}(\tau_z(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{T}$
- $\tau_z \circ \tau_{z'} = \tau_{z+z'}$ and $\tau_0 = \mathrm{id}_\Omega$.

We say that the group $(\tau_z)_{z\in\mathbb{Z}^n}$ is *ergodic* if every element $E \in \mathcal{T}$ such that $\tau_z(E) = E$ for all $z \in \mathbb{Z}^n$ is of probability 0 or 1. The admissible variety of integrands for homogenisation shall exhibit selected stochastic properties.

Definition 2.6 (Random integrand)**.** A function $f : \Omega \times \mathbb{R}^n \times \mathscr{R}(\mathbb{A}) \longrightarrow [0,+\infty)$ is a *random integrand* if

(1) f is $(\mathcal{T}\otimes\mathscr{B}(\mathbb{R}^n)\otimes\mathscr{B}(\mathscr{R}(\mathbb{A})))$-measurable
(2) for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have that $f(\omega,\cdot,\cdot)$ satisfies the p-growth condition: there exist $0 < \alpha \le \beta < +\infty$ such that

$$\alpha|A|^p \le f(\omega,x,A) \le \beta(1+|A|^p)$$

for all $x \in \mathbb{R}^n$ and all $A \in \mathscr{R}(\mathbb{A})$,

(3) for $\mathbb{P}$-a.e. $\omega \in \Omega$ $f(\omega, \cdot, \cdot)$ satisfies a continuity criterion: there exist a constant $c_1 > 0$ as well as a continuous, nondecreasing function $\rho : [0, +\infty) \to [0, +\infty)$ such that $\rho(0) = 0$ and

$$|f(x, A_1) - f(x, A_2)| \leq \rho(|A_1 - A_2|)(f(x, A_1) + f(x, A_2)) + c_1|A_1 - A_2|$$

for all $x \in \mathbb{R}^n$, $A_1, A_2 \in \mathscr{R}(\mathbb{A})$.

Definition 2.7 (Stationary random integrands)**.** For a given group $(\tau_z)_{z\in\mathbb{Z}^n}$ of $\mathbb{P}$-preserving transformations a random integrand $f : \Omega \times \mathbb{R}^n \times \mathscr{R}(\mathbb{A}) \longrightarrow [0, +\infty)$ is said to be *stationary* with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$ if the equality

$$f(\omega, x + z, A) = f(\tau_z(\omega), x, A)$$

holds for all $x \in \mathbb{R}^n$, all $A \in \mathscr{R}(\mathbb{A})$, $z \in \mathbb{Z}^n$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.
Moreover f is *ergodic* provided that $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic.

Let us recount the notion of subadditive stochastic process as per [1].

Definition 2.8 (Subadditive process)**.** A subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$ is a set function $\mu : \mathscr{A}(\mathbb{R}^n) \to \mathrm{L}^1(\Omega, \mathcal{T}, \mathbb{P})$ satisfying:
(a) measurability: for any $U \in \mathscr{A}(\mathbb{R}^n)$ the function $\omega \mapsto \mu(\omega, U)$ is $\mathcal{T}$-measurable

(b) subadditivity: for any $U \in \mathscr{A}(\mathbb{R}^n)$ and any finite, pairwise disjoint collection of subsets of U $\{U_i\}_{i=1}^M \in \mathscr{A}(\mathbb{R}^n)$ such that $\mathscr{L}^n(U \setminus \cup_{i=1}^M U_i) = 0$ there holds $\mu(\omega, U) \leq \sum_{i=1}^M \mu(\omega, U_i)$ for all $\omega \in \Omega$

(c) covariance: for all $U \in \mathscr{A}(\mathbb{R}^n)$, $\omega \in \Omega$ and $z \in \mathbb{Z}^n$, $\mu(\omega, U + z) = \mu(\tau_z\omega, U)$.

(d) upper bound: there exists a constant $C > 0$ such that $0 \leq \mu(\omega, U) \leq C\mathscr{L}^n(U)$ for every $U \in \mathscr{A}(\mathbb{R}^n)$ and $\omega \in \Omega$.
In case $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic, then we say that the process μ is ergodic.

The following result due to [1, 18] depicts the pointwise asymptotic behaviour of subadditive processes.

Theorem 2.9. [18, Prop. 1] *Let $\mu : \mathscr{A}(\mathbb{R}^n) \to \mathrm{L}^1(\Omega, \mathcal{T}, \mathbb{P})$ be a subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$. Then there exists a $\mathcal{T}$-measurable function $h : \Omega \to [0, +\infty)$ such that for every $x \in \mathbb{R}^n$, $r > 0$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have*

$$h(\omega) = \lim_{t\to+\infty} \frac{\mu(\omega, tQ_r(x))}{\mathscr{L}^n(tQ_r(x))}.$$

If μ is ergodic, then the map h is constant $\mathbb{P}$-a.e..

For each $A \in \mathscr{R}(\mathbb{A})$ and $U \in \mathscr{A}(\mathbb{R}^n)$ we define $\mu^A(\cdot, U) \in \mathrm{L}^1(\Omega, \mathcal{T}, \mathbb{P})$ by

$$\mu^A(\omega, U) := m_f(\omega, A, U) := \inf\left\{ \int_U f(\omega, x, A + \mathbb{A}v)\, \mathrm{d}x : v \in \mathrm{W}_0^{\mathbb{A},p}(U) \right\} \tag{2.5}$$

Crucially for admissible integrands f the density process μ^A is subadditive for every $A \in \mathscr{R}(\mathbb{A})$:

Proposition 2.10. *Let $f : \Omega \times \mathbb{R}^n \times \mathscr{R}(\mathbb{A}) \to [0, +\infty)$ be a stationary random integrand. Then for all $A \in \mathscr{R}(\mathbb{A})$ the mapping $\mu^A : \mathscr{A}(\mathbb{R}^n) \to \mathrm{L}^1(\Omega, \mathcal{T}, \mathbb{P})$ is a subadditive process and in addition $0 \leq \mu^A(\cdot, U) \leq \beta(1 + |A|^p)\mathscr{L}^n(U)$.*

Proof. Let $A \in \mathscr{R}(\mathbb{A})$ and $U \in \mathscr{A}(\mathbb{R}^n)$ be fixed. Firstly observe that $\omega \mapsto m_f(\omega, A, U)$ is measurable since the separability of $\mathrm{W}^{\mathbb{A},p}$-spaces and continuity of the integral in the parenthesis of (2.5) ensures existence of a set $\mathcal{D} \subset \mathrm{W}^{\mathbb{A},p}(U)$ such that

$$\mu^A(\cdot, U) = \inf\Big\{ \int_U f(\cdot, x, A + \mathbb{A}v)\,\mathrm{d}x : v \in \mathcal{D}\Big\}. \tag{2.6}$$

We now proceed in verifying all conditions of Definition 2.8.

For property (b) we take a collection $U_1, \dots, U_M \in \mathscr{A}(\mathbb{R}^n)$ of pairwise disjoint subsets of U such that $\mathscr{L}^n(U \setminus \cup_{i=1}^M U_i) = 0$. Now for any $\delta > 0$ let $u_i^\delta \in \mathrm{W}_0^{\mathbb{A},p}(U_i)$ be such that $\int_{U_i} f(\omega, x, \mathbb{A}u_i + A)\,\mathrm{d}x \leq m_f(\omega, A, U_i) + \delta/M$. Then defining $u := \sum_{i=1}^M u_i \chi_{U_i}$ it follows that

$$\mu^A(\omega, U) \leq \sum_{i=1}^M \int_{U_i} f(\omega, x, \mathbb{A}u_i + A)\,\mathrm{d}x \leq \sum_{i=1}^M \mu^A(\omega, U_i) + \delta.$$

and the claim follows by passing to $\delta \to 0$.

As to (c), applying the change of variable and invoking stationarity property of f we see that

$$m_f(\omega, A, U + z) = \inf\Big\{ \int_U f(\tau_z\omega, x, A + \mathbb{A}v)\,\mathrm{d}x : v \in \mathrm{W}_0^{\mathbb{A},p}(U)\Big\} = m_f(\tau_z\omega, A, U)$$

and therefore $\mu^A(\cdot, U)$ is covariant. Lastly the pointwise bounds are clear from the growth bounds on the integrand f appearing in the parenthesis of the infimum (2.5). Hence $\mu^A(\cdot, U)$ is a subadditive process. □

Following the cases of full and symmetric gradients, the dependence on $\mathbb{A}$ provokes an appropriated notion of quasiconvexity . An integrand $g : \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is said to be $\mathbb{A}$-quasiconvex if for all open sets $O \subset \mathbb{R}^n$ and all $A \in \mathscr{R}(\mathbb{A})$

$$g(A) = \inf\big\{ \fint_O g(A + \mathbb{A}\varphi)\,\mathrm{d}\mathscr{L}^n : \varphi \in \mathrm{C}_c^\infty(O; \mathrm{V})\big\}. \tag{2.7}$$

As noted for instance in [38] in view of the exactness property (2.3), the condition (2.7) is equivalent to the classical formulation of Fonseca and Müller [25]:

Proposition 2.11. [38, Cor. 6] *Let $\mathbb{B}$ be an annihilator of $\mathbb{A}$ as per (2.3). Suppose that $h : \mathscr{R}(\mathbb{A}) \to \mathbb{R}$ is a $\mathscr{B}(\mathscr{R}(\mathbb{A}))$-measurable and locally bounded integrand. Then for any $A \in \mathscr{R}(\mathbb{A})$*

$$\begin{aligned} &\inf\Big\{ \fint_Q h(A + \mathbb{A}\varphi)\,\mathrm{d}\mathscr{L}^n : \varphi \in \mathrm{C}_c^\infty(Q; \mathrm{V})\Big\} \\ &= \inf\Big\{ \int_Q h(A + \psi)\,\mathrm{d}\mathscr{L}^n : \psi \in \mathrm{C}^\infty(\mathbb{T}^n; \mathscr{R}(\mathbb{A})),\, \mathbb{B}\psi = 0,\, \int_{\mathbb{T}^n} \psi\,\mathrm{d}\mathscr{L}^n\Big\} \end{aligned}$$

where $\mathbb{T}^n$ denotes the n-dimensional torus.

We denote by $\mathcal{T}'$ the sub-algebra of $\mathcal{T}$ containing all $(\tau_z)_{z\in\mathbb{Z}^n}$-invariant sets, meaning that $\mathbb{P}(\tau_z(A)) = \mathbb{P}(A)\ \forall A \in \mathcal{T}'$. As m_f satisfies the hypotheses of Theorem 2.9 a direct consequence emerges:

Proposition 2.12. *Let $f : \Omega \times \mathbb{R}^n \times \mathscr{R}(\mathbb{A}) \to [0, +\infty)$ be a stationary random integrand. There exists a homogeneous random integrand $f_{\mathrm{hom}} : \Omega \times \mathscr{R}(\mathbb{A}) \to [0, +\infty)$ realised by the limit*

$$f_{\mathrm{hom}}(\omega, A) = \lim_{t\to+\infty} \frac{m_f(\omega, A, tQ_r(x))}{\mathscr{L}^n(tQ_r(x))} = \lim_{t\to+\infty} \frac{\mu^A(\omega, Q_t(0))}{t^n}$$

for all $r > 0$, $x \in \mathbb{R}^n$ and for $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover the volume integrand $f_{\hom}$ exhibits the following characterisation:

$$f_{\hom}(\omega, A) = \inf_{k\in\mathbb{N}} k^{-n}\mathbb{E}\big[m_f(\omega, A, Q_k(0))|\mathcal{T}'\big] \tag{2.8}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ where the term $\mathbb{E}\big[m_f(\omega, A, kQ)|\mathcal{T}'\big]$ denotes the conditional expectation of the random variable $\omega \mapsto m_f(\omega, A, kQ)$ given $\mathcal{T}'$.
If the integrand is f ergodic, then $f_{\hom}$ does not depend on ω and

$$f_{\hom}(A) = \lim_{t\to+\infty} t^{-n} \int_\Omega m_f(\omega, A, Q_t(0))\,\mathrm{d}\mathbb{P}. \tag{2.9}$$

Proof. Firstly let $A \in \mathscr{R}(\mathbb{A})$. Then as $\mu^A(\cdot, U)$ is a subadditive process, cf. Proposition 2.10, we may apply Theorem 2.9 to find a set $\Omega_A \in \mathcal{T}$ such that $\mathbb{P}(\Omega_A) = 1$ as well as a $\mathcal{T}$-measurable function $h_A : \Omega \to [0, +\infty)$ with

$$h_A(\omega) = \lim_{t\to+\infty} \frac{m_f(\omega, A, Q_t(0))}{t^n} \qquad \forall \omega \in \Omega_A. \tag{2.10}$$

Consequently let us define $f_{\hom} : \Omega \times \mathscr{R}(\mathbb{A}) \to [0, +\infty)$ by

$$f_{\hom}(\omega, A) = \limsup_{t\to+\infty} \frac{m_f(\omega, A, Q_t(0))}{t^n}. \tag{2.11}$$

As for the continuity criterion let $A, B \in \mathscr{R}(\mathbb{A})$ and $u \in \mathrm{W}_0^{\mathbb{A},p}(Q_t(0))$. Then

$$\begin{aligned}\int_{Q_t(0)} f(\omega, x, \mathbb{A}u + B)\,\mathrm{d}x \le \int_{Q_t(0)} f(\omega, x, \mathbb{A}u + A)\,\mathrm{d}x + c_1|A - B|t^n \\ + \rho(|A - B|)\Big(\int_{Q_t(0)} f(\omega, x, \mathbb{A}u + A)\,\mathrm{d}x + \int_{Q_t(0)} f(\omega, x, \mathbb{A}u + B)\,\mathrm{d}x\Big).\end{aligned}$$

In the case of $\rho(|A - B|) < 1$ rearranging the terms and taking the infimum through $\mathrm{W}_0^{\mathbb{A},p}(Q_t(0))$ we obtain

$$(1 - \rho(|A - B|))m_f(\omega, B, Q_t(0)) \le (1 + \rho(|A - B|))m_f(\omega, A, Q_t(0)) + c_1 t^n|A - B|.$$

or in other words

$$\begin{aligned}m_f(\omega, B, Q_t(0)) - m_f(\omega, A, Q_t(0)) \le \rho(|A - B|)(m_f(\omega, B, Q_t(0)) + m_f(\omega, A, Q_t(0))) \\ + c_1 t^n|A - B|.\end{aligned} \tag{2.12}$$

Since (2.12) is clear for $\rho(|A - B|) \ge 1$ we may exchange the roles of A, B and divide through by t^n to arrive at

$$\begin{aligned}\Big|\frac{m_f(\omega, A, Q_t(0))}{t^n} - \frac{m_f(\omega, B, Q_t(0))}{t^n}\Big| \le \rho(|A - B|)\Big(\frac{m_f(\omega, A, Q_t(0))}{t^n} + \frac{m_f(\omega, B, Q_t(0))}{t^n}\Big) \\ + c_1|A - B|\end{aligned}$$

Letting $t \to +\infty$ verifies the desired continuity condition for $f_{\hom}$.

Now since $\mathscr{R}(\mathbb{A})$ is a finite-dimensional real vector space, we may find a countable subset $\mathcal{W} \subset \mathscr{R}(\mathbb{A})$ such that $\mathcal{W}$ which is a bijection with $\mathbb{Q}^{\dim \mathscr{R}(\mathbb{A})}$. Define $\widehat{\Omega} := \cap_{A\in\mathcal{W}}\Omega_A$ so that $\mathbb{P}(\widehat{\Omega}) = 1$ and

(2.10) is valid for every $\omega \in \widehat{\Omega}$ and every $w \in \mathcal{W}$. Now take an arbitrary $A \in \mathscr{R}(\mathbb{A})$ and consider a sequence $(A_k)_{k\in\mathbb{N}} \subset \mathcal{W}$ such that $A_k \to A$. Then

$$\begin{aligned}\Big|f_{\mathrm{hom}}(\omega,A) - \frac{m_f(\omega,A,Q_t(0))}{t^n}\Big| &\le |f_{\mathrm{hom}}(\omega,A) - f_{\mathrm{hom}}(\omega,A_k)| \\ &\quad + \Big|f_{\mathrm{hom}}(\omega,A_k) - \frac{m_f(\omega,A_k,Q_t(0))}{t^n}\Big| \\ &\quad + \Big|\frac{m_f(\omega,A,Q_t(0))}{t^n} - \frac{m_f(\omega,A_k,Q_t(0))}{t^n}\Big| \\ &\le c'\rho(|A-A_k|)\Big(\Big|\frac{m_f(\omega,A,Q_t(0))}{t^n}\Big| + \Big|\frac{m_f(\omega,A_k,Q_t(0))}{t^n}\Big|\Big) \\ &\quad + c'|A-A_k|\end{aligned} \tag{2.13}$$

for some constant $c' > 0$. Passing to $t \to +\infty$ followed by $k \to +\infty$ gives the desired identification of f_{hom} for all $A \in \mathscr{R}(\mathbb{A})$ and all $\omega \in \widehat{\Omega}$:

$$f_{\mathrm{hom}}(\omega,A) = \lim_{t\to+\infty} \frac{\mu^A(\omega,Q_t(0))}{t^n}.$$

Let us observe that in view of (2.10) the integrand f_{hom} is $(\mathcal{T} \otimes \mathscr{B}(\mathscr{R}(\mathbb{A})))$-measurable.

As to the p-growth condition of f_{hom} we observe that the properties of f and the strict convexity of $|\cdot|^p$ along with the equivalence of Proposition 2.11 imply:

$$\min_{u\in \mathrm{W}_0^{\mathbb{A},p}(Q_t(0))} \fint_{Q_t(0)} f(\omega,x,\mathbb{A}u + A)\,\mathrm{d}x \ge \min_{u\in \mathrm{W}_0^{\mathbb{A},p}(Q_t(0))} \alpha \fint_{Q_t(0)} |\mathbb{A}u + A|^p\,\mathrm{d}x \ge \alpha|A|^p. \tag{2.14}$$

Likewise the upper bound is a direct consequence of the second assertion in the statement of Proposition 2.10.

The characterisation of f_{hom} in (2.8) follows from [5, Thm. 12.4.3] see also [1, Lem. 3.4].

Finally the ergodicity of f implies the ergodicity of μ^A cf. (2.5) and so Theorem 2.9 tells us that f_{hom} is independent of the random parameter ω. Thereafter the formula (2.9) follows from the bound in the assertions of Proposition 2.10 along with dominated convergence theorem. □

3. The Main Theorem

Let $f : \Omega\times\mathbb{R}^n\times\mathscr{R}(\mathbb{A}) \longrightarrow [0,+\infty)$ be an integrand as in Definition 2.6. The associated parametrised functionals in the interest of homogenisation are given by $\mathscr{F}_\varepsilon(\omega) : \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V}) \times \mathscr{A}(\mathbb{R}^n) \to [0,+\infty]$ such that

$$\mathscr{F}_\varepsilon(\omega)[u;U] := \begin{cases} \displaystyle\int_U f\Big(\omega,\frac{x}{\varepsilon},\mathbb{A}u\Big)\mathrm{d}x, & \text{for } u \in \mathrm{W}^{\mathbb{A},p}(U) \\ +\infty & \text{otherwise in } \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V}). \end{cases} \tag{3.1}$$

for $\omega \in \Omega$. Subject to all said assumptions let us now state the main theorem in the interest of our discourse.

Theorem 3.1 (Almost sure Γ-convergence)**.** *Let $\mathbb{A}$ be a differential operator as in* (2.1) *which is real elliptic. Then for any $U \in \mathscr{A}(\mathbb{R}^n)$ the functionals $\mathscr{F}_\varepsilon(\omega)[\cdot,U]$ defined in* (3.1) *Γ-converge almost surely in Ω as $\varepsilon \to 0$ in the $\mathrm{L}^p_{\mathrm{loc}}$-topology to $\mathscr{F}_{\mathrm{hom}}(\omega)[\cdot,U]$ where $\mathscr{F}_{\mathrm{hom}}(\omega) : \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})\times\mathscr{A}(\mathbb{R}^n) \longrightarrow$*

$[0, +\infty]$ *is an integral functional given by*

$$\mathscr{F}_{\mathrm{hom}}(\omega)[u; U] := \begin{cases} \int_U f_{\mathrm{hom}}(\omega, \mathbb{A}u)\,\mathrm{d}\mathscr{L}^n & \text{if } u \in \mathrm{W}^{\mathbb{A},p}(U) \\ +\infty & \text{otherwise in } \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n; \mathrm{V}). \end{cases} \tag{3.2}$$

The homogenised density is represented as

$$f_{\mathrm{hom}}(\omega, A) := \lim_{t\to+\infty} T^{-n}\mathbb{E}\big[m_f(\omega, A, Q_t(0))|\mathcal{T}'\big] \tag{3.3}$$

where $A \in \mathscr{R}(\mathbb{A})$.

Further, if $(\tau_z)_{z\in\mathbb{Z}^n}$ *is in addition ergodic, then* $\mathscr{F}_{\mathrm{hom}}$ *becomes deterministic (independent of the random parameter* ω*) and*

$$f_{\mathrm{hom}}(A) := \lim_{t\to+\infty} t^{-n} \int_\Omega \inf\Big\{ \int_{Q_t(0)} f(\omega, x, A + \mathbb{A}v)\,\mathrm{d}x : v \in \mathrm{W}^{\mathbb{A},p}_0(Q_t(0))\Big\}\,\mathrm{d}\mathbb{P}.$$

3.1. **The lim-inf inequality.** Here we turn to the proof of the lower bound in Theorem 3.1. The line argument follows the method of "blow-up", scheme of which is thoroughly explained for instance in [7, 24]. Let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0, 1)$ be an infinitesimal sequence.

Proposition 3.2. *Let* $U \in \mathscr{A}(\mathbb{R}^n)$ *and let* $u \in \mathrm{W}^{\mathbb{A},p}(U)$*. For any sequence* $(u_k)_{k\in\mathbb{N}} \subset \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n; \mathrm{V})$ *such that* $u_k \to u$ *in* $\mathrm{L}^p(U; \mathrm{V})$*, the lim-inf inequality*

$$\mathscr{F}_{\mathrm{hom}}(\omega)[u; U] \le \liminf_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k; U] \tag{3.4}$$

holds for $\mathbb{P}$*-a.e.* $\omega \in \Omega$.

Proof. Let us assume that the right hand side of (3.4) is finite as otherwise there is nothing to prove, so that $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}^{\mathbb{A},p}(U)$. Up to extraction of a subsequence let us suppose that $\lim_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k; U]$ exists. We begin by contemplating the induced sequence of Radon measures $\nu_k := \mathscr{F}_{\varepsilon_k}(\omega)[u_k; \cdot]$. Taking the asserted boundedness into account forces $(\nu_k)_{k\in\mathbb{N}}$ to be uniformly bounded in $\mathcal{M}(U; \mathrm{W})$, the space of bounded Radon measures, and whence one may select a subsequence $(\nu_{k_j})_{j\in\mathbb{N}}$ such that $\nu_{k_j} \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}(U; \mathrm{W})$. In particular the bound (3.4) may be congruently rephrased by inquiring that $\nu(\cdot) \ge \mathscr{F}_{\mathrm{hom}}(\omega)[u; \cdot]$ as measures. To this end notice that by the Lebesgue-Radon-Nikodym decomposition one has the representation $\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}\mathrm{d}\mathscr{L}^n + \nu^s$ with respect to the Lebesgue measure. Since $\nu^s \ge 0$ the claimed inequality is accessible by comparing Radon-Nikodym densities of the two terms:

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x) \ge f_{\mathrm{hom}}(\omega, \mathbb{A}u(x)) \qquad \text{for } \mathscr{L}^n\text{-a.e. } x \in U. \tag{3.5}$$

By a variant of the Besicovitch Differentiation Theorem [23, Thm 1.30] for $\mathscr{L}^n$-a.e. $x \in U$

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{r\to0+} \frac{\nu(Q_r(x))}{r^n}. \tag{3.6}$$

Furthermore in view of Proposition 2.5 we may regard x as an L^p-differentiability point of u. As ν is a finite Radon measure, we have that $\nu(\partial Q_r(x)) = 0$ for a.e. $r > 0$. Thereby for such $r > 0$ the equality

$$\nu(Q_r(x)) = \lim_{k\to+\infty} \nu_k(Q_r(x)) \tag{3.7}$$

holds true. Let $\delta, \tau \in (0, 1)$, $N \in \mathbb{N}$ be arbitrary and label $Q_i := Q_{\tau r + \frac{i}{N}r(1-\tau)}(x)$ for indices $i \in \{0, \dots, N\}$. Fix one such i and take a cut-off function $\eta_i \in \mathrm{C}^\infty_c(Q_i; [0, 1])$ such that $\eta_i = 1$ in Q_{i-1}

and $|\nabla\eta_i| \leq N/(r(1-\tau))$. Now set $u_0(y) = u(x) + \nabla u(x)(y-x)$ and define $u_{i,k} := u_0 + \eta_i(u_k - u_0)$. Then clearly $u_{i,k} \in \mathrm{W}^{\mathbb{A},p}(Q_r(x))$ and crucially the $\mathbb{A}$-gradients split as

$$\mathbb{A}u_{i,k} = \begin{cases} \mathbb{A}u_k, \quad \text{in } Q_{i-1} \\ \mathbb{A}u(x) + \eta_i\mathbb{A}(u_k - u_0) + \nabla\eta_i \otimes_{\mathbb{A}} (u_k - u_0), \quad \text{in } Q_i \setminus Q_{i-1} \\ \mathbb{A}u(x), \quad \text{in } Q_r(x) \setminus Q_i. \end{cases} \tag{3.8}$$

Thus $u_{i,k} - u_0$ is an admissible test map for infimisation in (2.5) and whence inserting it as a competitor we compute

$$\begin{aligned} &\inf\Big\{ \fint_{\frac{1}{\varepsilon_k}Q_r(x)} f\Big(\omega, y, \mathbb{A}u(x) + \mathbb{A}v\Big)\,\mathrm{d}y : v \in \mathrm{W}_0^{\mathbb{A},p}\Big(\frac{1}{\varepsilon_k}Q_r(x)\Big)\Big\} \\ &\leq \frac{1}{r^n}\int_{Q_r(x)} f\Big(\omega, \frac{y}{\varepsilon_k}, \mathbb{A}u_{i,k}\Big)\,\mathrm{d}y \\ &= \frac{1}{r^n}\int_{Q_{i-1}} f\Big(\omega, \frac{y}{\varepsilon_k}, \mathbb{A}u_k\Big)\,\mathrm{d}y + \frac{1}{r^n}\int_{Q_i\setminus Q_{i-1}} f\Big(\omega, \frac{y}{\varepsilon_k}, \mathbb{A}u_{i,k}\Big)\,\mathrm{d}y \\ &+ \frac{1}{r^n}\int_{Q_r(x)\setminus Q_i} f\Big(\omega, \frac{y}{\varepsilon_k}, \mathbb{A}u(x)\Big)\,\mathrm{d}y =: \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned} \tag{3.9}$$

Regarding bounds on $\mathbf{I}, \mathbf{II}, \mathbf{III}$ we argue as follows:

$$\begin{aligned} \mathbf{I} &\leq \fint_{Q_r(x)} f\Big(\omega, \frac{y}{\varepsilon_k}, \mathbb{A}u_k\Big)\,\mathrm{d}y = \frac{\nu_k(Q_r(x))}{\mathscr{L}^n(Q_r(x))} \\ \mathbf{II} &\leq \beta(1 + |\mathbb{A}u(x)|^p)(1-\tau^n) + \frac{\beta}{r^n}\int_{Q_i\setminus Q_{i-1}} |\mathbb{A}(u_k - u_0)|^p\,\mathrm{d}y \\ &+ \frac{\beta N}{(1-\tau)r^n}\int_{Q_i\setminus Q_{i-1}} \frac{|u_k - u_0|^p}{r^p}\,\mathrm{d}y \\ \mathbf{III} &\leq \beta(1 + |\mathbb{A}u(x)|^p)(1-\tau^n). \end{aligned} \tag{3.10}$$

Applying the change of variable on the right-hand side of $\mathbf{II}$, because $Q_i \setminus Q_{i-1}$ has its annular "thickness" comparable to N^{-1} the expression transforms to

$$\begin{aligned} \mathbf{II} &\leq \beta(1 + |\mathbb{A}u(x)|^p)(1-\tau^n) + \frac{\beta}{Nr^n}\int_{Q_r(x)\setminus Q_{\tau r}(x)} |\mathbb{A}(u_k - u_0)|^p\,\mathrm{d}y \\ &+ \frac{\beta}{(1-\tau)r^n}\int_{Q_r(x)} \frac{|u_k - u_0|^p}{r^p}\,\mathrm{d}y \end{aligned} \tag{3.11}$$

Altogether passing to $N \to +\infty$ first, followed by $k \to +\infty, r \to 0$ as well as $\tau \to 1$, in view of Proposition 2.5 and $u_k|_U \to u|_U$ in $\mathrm{L}^p(U;\mathrm{V})$, the terms $\mathbf{II}$ and $\mathbf{III}$ vanish. Hence invoking Proposition 2.12 in conjunction with (3.6) it amounts

$$f_{\mathrm{hom}}(\omega, \mathbb{A}u(x)) \overset{(3.7)}{\leq} \lim_{r\to 0}\lim_{k\to+\infty} \frac{\nu_k(Q_r(x))}{r^n} = \frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x)$$

and this concludes the proof. □

3.2. **Proof of the upper bound.** In this final section we verify the almost sure existence of recovery sequences for the functionals $(\mathscr{F}_{\varepsilon_k}(\omega))_{k\in\mathbb{N}}$. Let us recount a result based on [22, Chpt. X, Prop. 2.1] that illustrates approximation by piecewise-affine maps in the vectorial setting. We say that $v \in \mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ if there exists a finite collection of subsets $\{U_i\}_{i\in\mathcal{I}}$ such that $\cup_{i\in\mathcal{I}}U_i = U$, $\mathscr{L}^n(U_i \cap U_j) = 0$ for $i \neq j$ such that each restriction $v|_{U_i}$ is an affine map and $v \in \mathrm{C}^0(U;\mathrm{V})$.

Lemma 3.3. *Suppose that* $u \in \mathrm{C}^\infty(U;\mathrm{V}) \cap \mathrm{C}^0(\overline{U};\mathrm{V})$. *Then there exists* $(u_k)_{k\in\mathbb{N}} \subset \mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ *such that*

- $\|\nabla u_k\|_{\mathrm{L}^\infty(U)} \le \|\nabla u\|_{\mathrm{L}^\infty(U)}$
- $\|u_k - u\|_{\mathrm{L}^\infty(U)} \to 0$ *as* $k \to +\infty$ *in* $\mathrm{L}^\infty(U;\mathrm{V})$
- $\|\partial_i u_k - \partial_i u\|_{\mathrm{L}^\infty(K)} \to 0$ *as* $k \to +\infty$ *for all* $1 \le i \le n$ *and all compact* $K \subset U$.

From these conditions we infer by linearity that the convergence $\mathbb{A}u_k \to \mathbb{A}u$ holds locally in $\mathrm{L}^\infty(U;\mathrm{W})$ as well. Consequently in view of Proposition 2.3 the density of piecewise-affine maps manifests itself in the norm topology of $\mathrm{W}^{\mathbb{A},p}$.

Lemma 3.4. *The space* $\mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ *is dense in* $\mathrm{W}^{\mathbb{A},p}(U)$ *with respect to the* $\mathrm{W}^{\mathbb{A},p}$*-norm.*

Proof. Let $u \in \mathrm{W}^{\mathbb{A},p}(U)$. By Proposition 2.3 there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty(\overline{U};\mathrm{V})$ such that $\|v_k - u\|_{\mathrm{W}^{\mathbb{A},p}(U)} \to 0$ as $k \to +\infty$. However for every element v_k there exists a sequence $(u^k_j)_{j\in\mathbb{N}} \subset \mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ satisfying hypotheses of Lemma 3.3. As noted above we may conclude that for any compact $K \subset U$ $\mathbb{A}u^k_j \to \mathbb{A}v_k$ as $j \to +\infty$ in $\mathrm{L}^\infty(K;\mathrm{V})$. This fact combined with the dominated convergence, since $\mathrm{W}^{1,\infty}(U;\mathrm{V}) \subset \mathrm{W}^{\mathbb{A},p}(U)$, yields $\|v_k - u^k_j\|_{\mathrm{W}^{\mathbb{A},p}(U)} \to 0$ as $j \to +\infty$. Finally extracting a diagonal subsequence $(u^k_{j_k})_{k\in\mathbb{N}}$ eventually leads to

$$\|u - u^k_{j_k}\|_{\mathrm{W}^{\mathbb{A},p}(U)} \le \|v_k - u^k_{j_k}\|_{\mathrm{W}^{\mathbb{A},p}(U)} + \|v_k - u\|_{\mathrm{W}^{\mathbb{A},p}(U)} \longrightarrow 0$$

as $k \to +\infty$. □

Such an approximation procedure is instrumental because in conjunction with the continuity of $\mathscr{F}_{\mathrm{hom}}$ in the norm topology of $\mathrm{W}^{\mathbb{A},p}$ it allows one to transpose the question of the upper bound in the way that it is sufficient to find recovery sequences for maps in $\mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$.

Proposition 3.5. *Let* $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})$. *Then there exists sequence* $(u_k)_{k\in\mathbb{N}} \subset \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})$ *such that* $u_k \to u$ *as* $k \to +\infty$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})$ *and*

$$\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k;U] \le \mathscr{F}_{\mathrm{hom}}(\omega)[u;U]$$

for every $U \in \mathscr{A}(\mathbb{R}^n)$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$.

Proof. Let $u \in \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})$, $U \in \mathscr{A}(\mathbb{R}^n)$ and assume without loss of generality that $u \in \mathrm{W}^{\mathbb{A},p}(U)$
Step 1: Assume first that $u|_U = \ell_A + b$ is an affine map $x \mapsto Ax + b$ for $A \in \mathrm{V}\otimes\mathbb{R}^n$ and $b \in \mathrm{V}$ so that $\mathbb{A}u = \mathcal{A}(A) \in \mathscr{C}(\mathbb{A})$. Let $\delta > 0$ be fixed and let $\{Q_i\}_{i\in\mathcal{I}_\delta}$ be a subcollection of U of pairwise-disjoint cubes such that each Q_i has side length comparable to δ and $\mathscr{L}^n(U \setminus \cup_{i\in\mathcal{I}_\delta} Q_i) \le \delta$. Letting $t := \varepsilon_k^{-1}$ for each $i \in \mathcal{I}_\delta$ take $w^i_\delta \in \mathrm{W}^{\mathbb{A},p}_0(tQ_i)$ such that

$$⨍_{tQ_i} f(x, \mathcal{A}(A) + \mathbb{A}w^i_\delta;\omega)\,\mathrm{d}x \le \frac{m_f(\omega,\mathcal{A}(A),tQ_i)}{t^n} + \frac{\delta}{|\mathcal{I}_\delta|}$$

where $|\mathcal{I}_\delta|$ denotes the cardinality of $\mathcal{I}_\delta$ and define $u^i_{k,\delta} := \ell_A + b + w^i_{k,\delta}$ for $w^i_{k,\delta} := \varepsilon_k w^i_\delta\left(\frac{\cdot}{\varepsilon_k}\right)$. After applying the change of variable and letting $k \to +\infty$, in view of the subadditive theorem Prop. 2.12 the estimate reads

$$\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u^i_{k,\delta};Q_i] \le \mathscr{F}_{\mathrm{hom}}(\omega)[u;Q_i] + \frac{\delta}{|\mathcal{I}_\delta|}. \tag{3.12}$$

We let $u_{k,\delta} \in \mathrm{W}^{\mathbb{A},p}(U)$ be defined by

$$\begin{cases} u_{k,\delta} = u^i_{k,\delta} & \text{in } Q_i,\ i \in \mathcal{I}_\delta \\ u_{k,\delta} = \ell_A + b & \text{otherwise in } U \setminus Q_i. \end{cases}$$

Thus using the fact that both f as well as f_{hom} are of p-growth we deduce

$$\begin{aligned} \limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_{k,\delta}; U] &\leq \limsup_{k\to+\infty} \sum_{\mathcal{I}_\delta} \mathscr{F}_{\varepsilon_k}(\omega)[u^i_{k,\delta}; Q_i] + \beta(1+|\mathcal{A}(A)|^p)\delta + \delta \\ &\overset{(3.12)}{\leq} \limsup_{k\to+\infty} \sum_{\mathcal{I}_\delta} \mathscr{F}_{\mathrm{hom}}(\omega)[\ell_A + b; Q_i] + \beta(1+|\mathcal{A}(A)|^p)\delta + \delta \\ &\leq \mathscr{F}_{\mathrm{hom}}(\omega)[\ell_A + b; U] + \beta(1+|\mathcal{A}(A)|^p)\delta + \delta. \end{aligned}$$

Since $\delta > 0$ was arbitrary we may apply the diagonalisation argument with some $\delta = \delta_k \to 0$ as $k \to +\infty$ and verify the convergence $u_{k,\delta_k} \to \ell_A + b$ in $\mathrm{L}^p(U;\mathrm{V})$. As $w^i_{\delta_k} \in \mathrm{W}^{\mathbb{A},p}_0(\varepsilon_k^{-1}Q_i)$, Proposition 2.4 tells us that

$$\|w^i_{\delta_k}\|_{\mathrm{L}^p(\varepsilon_k^{-1}Q_i;\mathrm{V})} \leq c_{n,\mathbb{A}} \delta_k \varepsilon_k^{-1} \|\mathbb{A} w^i_{\delta_k}\|_{\mathrm{L}^p(\varepsilon_k^{-1}Q_i;\mathrm{W})}.$$

and therefore using the change of variable and the definition of u_{k,δ_k} yields

$$\begin{aligned} \|u_{k,\delta_k} - \ell_A - b\|^p_{\mathrm{L}^p(U;\mathrm{V})} &\leq \sum_{i\in\mathcal{I}_{\delta_k}} \|u^i_{k,\delta_k} - \ell_A - b\|^p_{\mathrm{L}^p(Q_i;\mathrm{V})} \\ &= \sum_{i\in\mathcal{I}_{\delta_k}} \|w^i_{k,\delta_k}\|^p_{\mathrm{L}^p(Q_i;\mathrm{V})} \leq c_{n,A,p}\delta_k^p \sum_{i\in\mathcal{I}_{\delta_k}} \|\mathbb{A} w^i_{k,\delta_k}\|_{\mathrm{L}^p(Q_i;\mathrm{W})} \end{aligned} \tag{3.13}$$

Observe that from (3.12) and the p-growth of f there holds

$$\|\mathbb{A} w^i_{k,\delta_k} + \mathcal{A}(A)\|_{\mathrm{L}^p(Q_i;\mathrm{W})} \leq \frac{\beta}{\alpha}\Big(1 + |\mathcal{A}(A)|^p\Big)\mathscr{L}^n(Q_i) + \frac{\delta_k}{|\mathcal{I}_{\delta_k}|}.$$

Combining the above bound with (3.12) and (3.13) ultimately implies

$$\begin{aligned} \|u_{k,\delta_k} - \ell_A - b\|_{\mathrm{L}^p(U;\mathrm{V})} &\leq \sum_{i\in\mathcal{I}_{\delta_k}} C\delta_k\Big(\mathscr{L}^n(Q_i) + \frac{\delta_k}{|\mathcal{I}_{\delta_k}|}\Big) \\ &\leq C\delta_k(\mathscr{L}^n(U) + \delta_k) \end{aligned}$$

from which the claim follows if we let $k \to +\infty$.

Step 2: Suppose now that $u \in \mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ so that there is a finite partition $\{U_j\}_{j\in\mathcal{J}}$ of U such that $u = \ell_{A_j} + b_j$ in U_j for $A_j \in \mathrm{V}\otimes\mathbb{R}^n$ and $b_j \in \mathrm{V}$. Hence by **Step 1** for each subset U_j there exist $u^j_k \in u + \mathrm{W}^{\mathbb{A},p}_0(U_j)$ such that $u^j_k \to u|_{U_j}$ in $\mathrm{L}^p(U;\mathrm{V})$ and $\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u^j_k; U_j] \leq \mathscr{F}_{\mathrm{hom}}(\omega)[u; U_j]$. We define $u_k := u + \sum_{j\in\mathcal{J}} u^j_k - u$ in which case $u_k \in \mathrm{W}^{\mathbb{A},p}(U)$ and $u_k \to u$ in $\mathrm{L}^p(U;\mathrm{V})$. Because $\mathscr{L}^n(U_i \cap U_j) = 0$ if $i \neq j$, it follows that

$$\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k; U] \leq \mathscr{F}_{\mathrm{hom}}(\omega)[u; U].$$

Step 3: Let $u \in \mathrm{W}^{\mathbb{A},p}(U)$, by Lemma 3.4 for any $\tau > 0$ we find $u_\tau \in \mathrm{Aff}^{\mathrm{pc}}(U;\mathrm{V})$ such that $\|u - u_\tau\|_{\mathrm{W}^{\mathbb{A},p}(U)} \leq \tau$ where $\tau > 0$. As every u_τ is piecewise-affine there exist recovery sequences

(u_τ^k) for $\mathscr{F}_{\hom}(\omega)[u_\tau;U]$. But now from the continuity condition of $f_{\hom}$, cf. Proposition 2.12, the bounds

$$|\mathscr{F}_{\hom}(\omega)[w;U]-\mathscr{F}_{\hom}(\omega)[v;U]| \le \int_U \rho(|\mathbb{A}w-\mathbb{A}v|)\Big(f_{\hom}(\omega,\mathbb{A}w)+f_{\hom}(\omega,\mathbb{A}v)\Big)\,\mathrm{d}x + c_1\mathscr{L}^n(U)\|\mathbb{A}w-\mathbb{A}v\|_{\mathrm{L}^p(U;\mathrm{W})} \tag{3.14}$$

hold true for all $w,v\in \mathrm{W}^{\mathbb{A},p}(U)$. From **Step 2** for any $\tau>0$ we may find a sequence $(u_\tau^k)_{k\in\mathbb{N}}\subset \mathrm{W}^{\mathbb{A},p}(U)$ such that

$$\limsup_{k\to+\infty}\mathscr{F}_{\varepsilon_k}(\omega)[u_\tau^k;U]\le \mathscr{F}_{\hom}[u_\tau;U] \tag{3.15}$$

and in view of the continuity condition (3.14) there holds

$$\limsup_{\tau\to 0}\limsup_{k\to+\infty}\mathscr{F}_{\varepsilon_k}(\omega)[u_\tau^k;U]\le \mathscr{F}_{\hom}[u;U]. \tag{3.16}$$

With that being brought forward, for an infinitesimal sequence $(\tau_k)_{k\in\mathbb{N}}$, we extract a diagonal sequence $v_{\tau_k}^k=u_{\tau_k}^k+(u-u_{\tau_k})$ that converges to u in $\mathrm{L}^p(U;\mathrm{V})$ and after passing to a subsequence such that $v_{\tau_k}^k\to u$ pointwise $\mathscr{L}^n$-a.e. Finally for any $k\in\mathbb{N}$ setting

$$u_k=\begin{cases} v_{\tau_k}^k & \text{in } U\\ u & \text{otherwise in } \mathbb{R}^n\end{cases}$$

yields $u_k\to u$ as $k\to+\infty$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^n;\mathrm{V})$ and from the continuity of $\mathscr{F}_{\hom}$ we have

$$\begin{aligned}\limsup_{k\to+\infty}\mathscr{F}_{\varepsilon_k}(\omega)[u_k;U] &\le \limsup_{k\to+\infty}\Big(\mathscr{F}_{\varepsilon_k}(\omega)[v_{\tau_k}^k;U]+C\tau_k\Big)\\ &\le \mathscr{F}_{\hom}(\omega)[u;U].\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. □

4. Convergence of minimisation problems with boundary conditions

In this section we utilise the almost sure Γ-convergence in Theorem 3.1 to prove convergence of minimisers of subject to Dirichlet-type conditions. To this end we will verify a homogenisation result for functionals with prescribed boundary datum and derive an appropriate compactness property. Let $U\in\mathscr{A}(\mathbb{R}^n)$ and let $u_0\in \mathrm{W}^{\mathbb{A},p}(U)$. We consider the sequence of functionals $\mathscr{F}_{\varepsilon_k}^{u_0}:\mathrm{L}^p(U;\mathrm{V})\to[0,+\infty]$ given by

$$\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u]:=\begin{cases}\mathscr{F}_{\varepsilon_k}(\omega)[u;U] & \text{if } u\in \mathrm{W}_0^{\mathbb{A},p}(U)+u_0\\ +\infty & \text{otherwise in } \mathrm{L}^p(U;\mathrm{V}).\end{cases} \tag{4.1}$$

Proposition 4.1. *The functionals $\mathscr{F}_{\varepsilon_k}^{u_0}(\omega):\mathrm{L}^p(U;\mathrm{V})\longrightarrow[0,+\infty]$ defined in (4.1) Γ-converge almost surely in Ω as $k\to+\infty$ in the $\mathrm{L}^p(U;\mathrm{V})$-topology to the limiting functional $\mathscr{F}_{\hom}^{u_0}(\omega):\mathrm{L}^p(U;\mathrm{V})\longrightarrow[0,+\infty]$ defined as*

$$\mathscr{F}_{\hom}^{u_0}(\omega)[u]:=\begin{cases}\mathscr{F}_{\hom}(\omega)[u;U] & \textit{if } u\in \mathrm{W}_0^{\mathbb{A},p}(U)+u_0\\ +\infty & \textit{otherwise in } \mathrm{L}^p(U;\mathrm{V}).\end{cases} \tag{4.2}$$

Proof. Let $u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ be chosen.
Step 1: Ansatz-free bound. Let $u \in \mathrm{W}^{\mathbb{A},p}(U)$ and $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}^{\mathbb{A},p}(U)$ be such that $u_k \to u$ in $\mathrm{L}^p(U;\mathrm{V})$. Let us assume that $\sup_{k\in\mathbb{N}} \mathscr{F}_{\varepsilon_k}^{u_0}[u_k] < +\infty$ in which case $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$.Since $\mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ is a closed subset of $\mathrm{W}^{\mathbb{A},p}(U)$ in the norm topology we have $u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$. By Theorem 3.1 there holds

$$\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[u] = \mathscr{F}_{\mathrm{hom}}(\omega)[u;U] \leq \liminf_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k;U] = \liminf_{k\to+\infty} \mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k].$$

Step 2: Existence of recovery sequences. Let $u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$. From Proposition 3.5 there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset \mathrm{W}^{\mathbb{A},p}(U)$ such that $v_k \to u$ in $\mathrm{L}^p(U;\mathrm{V})$ and

$$\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[v_k;U] \leq \mathscr{F}_{\mathrm{hom}}(\omega)[u;U] = \mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[u]. \tag{4.3}$$

In particular we may assume that $\sup_{k\in\mathbb{N}} \|\mathbb{A}v_k\|_{\mathrm{L}^p(U)} < +\infty$. Let $U', U'' \in \mathscr{A}(U)$ be such that $U' \subset\subset U'' \subset\subset U$. For a fixed integer $N \in \mathbb{N}$ consider a collection of open sets $U_0, \dots, U_N$ such that

$$U' = U_0 \subset\subset U_1 \subset\subset \cdots \subset\subset U_N = U''$$

and for each $j \in \{1,\dots,N\}$ let $\varphi_j \in \mathrm{C}_c^\infty(U_j;\mathrm{V})$ be such that $\varphi_j = 1$ in U_{j-1}, $\varphi_j = 0$ in U_j. Then defining $w_k^j := \varphi_j v_k + (1-\varphi_j)u$ we have $w_k^j \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ and for each $j \in \{1,\dots,N\}$ it holds

$$\begin{aligned}\mathscr{F}_{\varepsilon_k}(\omega)[w_k^j;U] \leq \mathscr{F}_{\varepsilon_k}(\omega)[v_k;U''] + c_{n,\mathbb{A}}\mathscr{F}_{\varepsilon_k}(\omega)[u;U\setminus U'] \\ + c_{n,\mathbb{A}} \int_{U_j\setminus U_{j-1}} |\mathbb{A}v_k|^p + \sigma N^p |v_k - u|^p \,\mathrm{d}\mathscr{L}^n.\end{aligned}$$

where $\sigma := \max_{j\in\{1,\dots,N\}} \|\nabla\varphi_j\|_{\mathrm{L}^\infty(U_j)}^p$. Therefore

$$\begin{aligned}\sum_{j=1}^N \mathscr{F}_{\varepsilon_k}(\omega)[w_k^j;U] \leq N\mathscr{F}_{\varepsilon_k}(\omega)[v_k;U''] + c_{n,\mathbb{A}}N\mathscr{F}_{\varepsilon_k}(\omega)[u;U\setminus U'] \\ + c_{n,\mathbb{A}} \int_{U''\setminus U'} |\mathbb{A}v_k|^p + \sigma N^p |v_k - u|^p \,\mathrm{d}\mathscr{L}^n.\end{aligned}$$

Consequently there exists $j_k \in \{1,\dots,N\}$ such that

$$\begin{aligned}\mathscr{F}_{\varepsilon_k}(\omega)[w_k^{j_k};U] \leq \mathscr{F}_{\varepsilon_k}(\omega)[v_k;U''] + c_{n,\mathbb{A}}\mathscr{F}_{\varepsilon_k}(\omega)[u;U\setminus U'] \\ + \frac{c_{n,\mathbb{A}}}{N} \int_{U''\setminus U'} |\mathbb{A}v_k|^p + \sigma N^p |v_k - u|^p \,\mathrm{d}\mathscr{L}^n.\end{aligned} \tag{4.4}$$

Now taking $u_k := w_k^{j_k}$ implies $u_k \to u$ in $\mathrm{L}^p(U;\mathrm{V})$ and in view of (4.3) in conjunction with (4.4) we have

$$\begin{aligned}\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k;U] \leq \mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[u] + c_{n,\mathbb{A}} \int_{U\setminus U'} 1 + |\mathbb{A}u|^p \,\mathrm{d}\mathscr{L}^n \\ + \frac{c_{n,\mathbb{A}}}{N} \sup_{k\in\mathbb{N}} \int_U |\mathbb{A}v_k|^p \,\mathrm{d}\mathscr{L}^n.\end{aligned}$$

Finally passing to $N \to +\infty$ and $U' \nearrow U$ gives

$$\limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] = \limsup_{k\to+\infty} \mathscr{F}_{\varepsilon_k}(\omega)[u_k;U] \leq \mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[u].$$

□

As the next step we state a compactness result of sequences $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}^{\mathbb{A},p}(U)+u_0$ with bounded energies $\mathscr{F}_{\varepsilon_k}(\omega)$. Since the underlying topology for Γ-convergence is induced from L^p we need an appropriate embedding property.

Lemma 4.2. *Let $U \in \mathscr{A}(\mathbb{R}^n)$ and $p \in (1,+\infty)$. The embedding $\mathrm{W}_0^{\mathbb{A},p}(U) \hookrightarrow \mathrm{L}^p(U;\mathrm{V})$ is compact.*

Proof. The proof is standard but we include details for the reader's convenience. Let $\mathcal{B} \subset \mathrm{W}_0^{\mathbb{A},p}(U)$ be a bounded set and let $\varepsilon > 0$ be fixed. Clearly $\mathcal{B}$ is a bounded subset of $\mathrm{L}^p(U;\mathrm{V})$. Let $s \in (0,1)$ be fixed and let $R = R(U) > 0$ be such that $U \subset B_R(0)$. Arguing as in the proof of [27, Lem. 4.7] there exists a constant $c = c(n,\mathbb{A},s) > 0$ such that for all $\varphi \in \mathrm{C}_c^\infty(B_R(0);\mathrm{V})$ if $|h| < R$, then

$$\|\varphi(\cdot - h) - \varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|\varphi(\cdot - h) - \varphi\|_{\mathrm{L}^p(B_{3R}(0))} \leq c|h|^s \|\mathbb{A}u * |\cdot|^{1-s-n}\|_{\mathrm{L}^p(B_{9R}(0))}.$$

Consequently applying Young's convolution inequality and using the fact that $|\cdot|^{1-s-n} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ we find a constant $\kappa = \kappa(c,U) > 0$ such that

$$\|\varphi(\cdot - h) - \varphi\|_{\mathrm{L}^p(\mathbb{R}^n)} \leq \kappa |h|^s \|\mathbb{A}\varphi\|_{\mathrm{L}^p(\mathbb{R}^n)}. \tag{4.5}$$

For every $v \in \mathcal{B}$ there exists $\varphi_\varepsilon \in \mathrm{C}_c^\infty(U;\mathrm{V})$ such that $\|\varphi_\varepsilon - v\|_{\mathrm{W}^{\mathbb{A},p}(U)} < \varepsilon$. Now extend each v and φ_ε, without relabelling, by zero outside of U to maps defined on the entire $\mathbb{R}^n$ so in particular $\mathcal{B}$ is bounded in $\mathrm{L}^p(\mathbb{R}^n;\mathrm{V})$. Subsequently we may find $\delta = \delta(\kappa,s,\varepsilon,\operatorname{diam}\mathcal{B},U) > 0$ such that for any $h \in \mathbb{R}^n$ with $|h| < \delta$ and any $v \in \mathcal{B}$ one has

$$\begin{aligned}\|v(\cdot - h) - v\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leq \|\varphi_\varepsilon(\cdot - h) - \varphi_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^n)} + \varepsilon \overset{(4.5)}{\leq} \kappa|h|^s \|\mathbb{A}\varphi_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^n)} + \varepsilon \\ &\leq \kappa|h|^s\Big(\|\mathbb{A}v\|_{\mathrm{L}^p(U)} + \varepsilon\Big) + \varepsilon < \varepsilon.\end{aligned}$$

Thus by the Riesz-Kolmogorov-Fréchet criterion, see e.g. [11, Thm 4.26] $\mathcal{B}$ is compact in $\mathrm{L}^p(U;\mathrm{V})$. □

Proposition 4.3 (Compactness)**.** *Let $U \in \mathscr{A}(\mathbb{R}^n)$ be fixed. Moreover let $\omega \in \Omega$, $u_0 \in \mathrm{W}^{\mathbb{A},p}(U)$ and $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ be such that*

$$\sup_{k\in\mathbb{N}} \mathscr{F}^{u_0}_{\varepsilon_k}(\omega)[u_k] < +\infty.$$

Then there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ and a function $u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ such that $u_{k_j} \to u$ in $\mathrm{L}^p(U;\mathrm{V})$ as $j \to +\infty$.

Proof. From the p-growth condition on f we obtain the bound

$$\sup_{k\in\mathbb{N}} \|\mathbb{A}u_k - \mathbb{A}u_0\|_{\mathrm{L}^p(U)} < +\infty.$$

Since $u_k - u_0 \in \mathrm{W}_0^{\mathbb{A},p}(U)$ by Proposition 2.4 we may further assert boundedness in $\mathrm{W}^{\mathbb{A},p}(U)$, i.e.

$$\sup_{k\in\mathbb{N}} \|u_k - u_0\|_{\mathrm{W}^{\mathbb{A},p}(U)} < +\infty.$$

As $\mathrm{W}^{\mathbb{A},p}(U)$ is a reflexive Banach space, there exists a map $v \in \mathrm{W}^{\mathbb{A},p}(U)$ and a subsequence $(u_{k_j} - u_0)_{j\in\mathbb{N}}$ such that $u_{k_j} - u_0$ converges weakly in $\mathrm{W}^{\mathbb{A},p}(U)$ to v. Since $\mathrm{W}_0^{\mathbb{A},p}(U)$ is convex and a closed subspace of $\mathrm{W}^{\mathbb{A},p}(U)$ in the norm-topology one concludes $v \in \mathrm{W}_0^{\mathbb{A},p}(U)$. Moreover from the inclusion $(\mathrm{L}^p(U;\mathrm{V}))' \subset (\mathrm{W}^{\mathbb{A},p}(U))'$ one concludes that $u_{k_j} - u_0 \to v$ weakly in $\mathrm{L}^p(U;\mathrm{V})$. On the other hand by compactness of the embedding $\mathrm{W}_0^{\mathbb{A},p}(U) \hookrightarrow \mathrm{L}^p(U;\mathrm{V})$, cf. Lemma 4.2, and the boundedness of

$(u_k - u_0)_{k\in\mathbb{N}}$ in $\mathrm{W}_0^{\mathbb{A},p}(U)$ implies existence of a map $w \in \mathrm{L}^p(U;\mathrm{V})$ such that $u_k - u_0 \to w$ in $\mathrm{L}^p(U;\mathrm{V})$ as $k \to +\infty$. In view of the above convergences the uniqueness of limits gives $v = w$ pointwisely $\mathscr{L}^n$-a.e. in U. Altogether setting $u := v + u_0$ yields $u_{k_j} \to u$ in $\mathrm{L}^p(U;\mathrm{V})$ as $j \to +\infty$. □

As an immediate consequence of Proposition 4.1 as well as the compactness property of Proposition 4.3 we obtain the following

Corollary 4.4 (Convergence of minimisers)**.** *Let $U \in \mathscr{A}(\mathbb{R}^n)$ and $u_0 \in \mathrm{W}^{\mathbb{A},p}(U)$. Moreover fix $\omega \in \Omega$ such that the hypotheses of Proposition 4.1 hold. Define*

$$m_k^\omega := \inf\Big\{\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u] \,:\, u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0\Big\}.$$

Then $m_k^\omega \to m^\omega$ as $k \to +\infty$ where

$$m^\omega := \min\Big\{\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[u] \,:\, u \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0\Big\}.$$

If $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ is a sequence such that

$$\lim_{k\to+\infty}\Big(\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] - m_k^\omega\Big) = 0,$$

then there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ and a map $\hat{u} \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ such that $u_{k_j} \to v$ in $\mathrm{L}^p(U;\mathrm{V})$ and

$$\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[v] = m^\omega.$$

Proof. Suppose that $\hat{u} \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ is such that

$$\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[\hat{u}] = m^\omega.$$

By Proposition 4.1 there exists a recovery sequence $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ for $\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[\hat{u}]$ so that

$$m^\omega = \mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[\hat{u}] \geq \limsup_{k\to+\infty}\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] \geq \limsup_{k\to+\infty} m_k^\omega. \tag{4.6}$$

Now let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ be a sequence such that

$$\lim_{k\to+\infty}\Big(\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] - m_k^\omega\Big) = 0. \tag{4.7}$$

In particular $\sup_{k\in\mathbb{N}}\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] < +\infty$ and so applying Proposition 4.3 there exists a subsequence (not relabelled) converging in $\mathrm{L}^p(U;\mathrm{V})$ to some $v \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$. The ansatz-free bound of Proposition 4.1 and (4.6) yield

$$m^\omega \leq \mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[v] \leq \liminf_{k\to+\infty}\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] \overset{(4.7)}{\leq} \liminf_{k\to+\infty} m_k^\omega \leq \limsup_{k\to+\infty} m_k^\omega \leq m^\omega. \tag{4.8}$$

As for the second part, passing to a subsequence we may assume that $\sup_{k\in\mathbb{N}}\mathscr{F}_{\varepsilon_k}^{u_0}(\omega)[u_k] < +\infty$. By Proposition 4.3 there exists a further subsequence $(u_{k_j})_{j\in\mathbb{N}} \subset (u_k)_{k\in\mathbb{N}}$ and an element $v \in \mathrm{W}_0^{\mathbb{A},p}(U) + u_0$ such that $u_{k_j} \to v$ in $\mathrm{L}^p(U;\mathrm{V})$. On the other hand from the ansatz free bound in Proposition 4.1 and the first part of the claim we infer

$$\mathscr{F}_{\mathrm{hom}}^{u_0}(\omega)[v] \leq \liminf_{j\to+\infty}\mathscr{F}_{\varepsilon_{k_j}}^{u_0}(\omega)[u_{k_j}] \leq \liminf_{j\to+\infty} m_{k_j}^\omega \leq m^\omega$$

and since v is an admissible competitor the desired equality follows. □

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