RULED HYPERSURFACES IN HIGHER DIMENSIONAL HEISENBERG GROUPS

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Abstract. In this paper we introduce a notion of ruled hypersurface in the Heisenberg group $\mathbb{H}^n$, which generalizes the corresponding one in $\mathbb{H}^1$. We show two rigidity results in the classes of non-characteristic C^1-hypersurfaces and conical C^2-hypersurfaces, highlighting the main differences between $\mathbb{H}^1$ and higher dimensional Heisenberg groups.

1. Introduction

The aim of this paper is to make some progresses in the understanding of an intriguing topic in Geometric Measure Theory, and particularly in the study of minimal surfaces, that is the so-called *Bernstein problem* in the setting of the sub-Riemannian Heisenberg group $\mathbb{H}^n$. The classical Euclidean Bernstein problem consists in the characterization of those sets which globally minimize De Giorgi's perimeter. Thanks to decades of research in this direction ([B, Fl, A, S, DG, BDGG], we know that any non-empty perimeter minimizer in $\mathbb{R}^n$ is an hyperplane provided that $n \leq 8$. Moreover, the bound on the dimension is sharp. We refer to [G] for a wonderful survey of this problem. More recently, an increasing interest in Geometric Measure Theory in the setting of the sub-Riemannian *Heisenberg group* $\mathbb{H}^n$ has developed (cf. [FSSC, GN] and references therein). To introduce this framework, we recall that the n-th Heisenberg group is $\mathbb{R}^{2n+1}$ endowed with the group law

$$p \cdot p' = (\bar{x}, \bar{y}, t) \cdot (\bar{x}', \bar{y}', t') = (\bar{x} + \bar{x}', \bar{y} + \bar{y}', t + t' + Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}')),$$

where

$$Q((\bar{x}, \bar{y}), (\bar{x}', \bar{y}')) = \sum_{j=1}^{n} (x_j' y_j - x_j y_j')$$

and where we denoted points $p \in \mathbb{R}^{2n+1}$ by $p = (z, t) = (\bar{x}, \bar{y}, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$. With this operation, $\mathbb{H}^n$ is a *Carnot group*, whose associated *horizontal distribution*, which we denote by H, is generated by the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \qquad \text{and} \qquad Y_j = \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial t}$$

for $j = 1, \ldots, n$. In the following we denote by T the left-invariant vector field $\frac{\partial}{\partial t}$. In this way $X = (X_1, \ldots, X_n, Y_1, \ldots, Y_n, T)$ is a basis of left-invariant vector fields. Moreover, the only nontrivial commutation relationships are

$$[X_j, Y_j] = -[Y_j, X_j] = -2T$$

for any $j = 1, \ldots, n$. As customary in this framework, for given $q \in \mathbb{H}^n$ and $\lambda > 0$, we define the *left-translation* $\tau_q : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ and the *intrinsic dilation* $\delta_\lambda : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\tau_q(p) := q \cdot p \qquad \text{and} \qquad \delta_\lambda(p) := (\lambda z, \lambda^2 t)$$

for any $p = (z, t) \in \mathbb{H}^n$. It is well known that both τ_q and δ_λ are global diffeomorphisms, and that δ_λ is a Lie group isomorphism. If $T\mathbb{H}^n$ is endowed with the only Riemannian

Key words. Bernstein problem, Heisenberg group, ruled hypersurface, minimal hypersurface, cones.
MSC. 53C17, 49Q05.

metric $\langle\cdot,\cdot\rangle$ which makes X an orthonormal basis, and whose associated norm we denote by $|\cdot|$, $\mathbb{H}^n$ is equipped with a sub-Riemannian structure. Notice that, as soon as a suitable notion of *horizontal perimeter* is defined, it is meaningful to transpose many classical problems of Geometric Measure Theory to this anisotropic setting, such as for instance the study of minimal hypersurfaces. To this aim, if $\Omega \subseteq \mathbb{H}^n$ is open and $E \subseteq \mathbb{H}^n$ is measurable with $\chi_E \in L^1_{loc}(\Omega)$, we recall (cf. e.g. [FSSC, GN]) that the $\mathbb{H}$-*perimeter* of E in Ω is defined by

$$P_{\mathbb{H}}(E,\Omega) := \sup\left\{\int_E \operatorname{div}_{\mathbb{H}}(\bar{\varphi})\, d\mathcal{L}^{2n+1} \,:\, \bar{\varphi} \in C^1_c(\Omega,H),\ |\bar{\varphi}|_p \le 1 \text{ for any } p \in \Omega\right\},$$

where $C^1_c(\Omega,H)$ is the class of C^1 sections of the horizontal distribution H, and $\operatorname{div}_{\mathbb{H}}$ is the so called *horizontal divergence*, which is defined by

$$\operatorname{div}_{\mathbb{H}}\left(\sum_{j=1}^{n}(\varphi_j X_j + \varphi_{n+j}Y_j)\right) := \sum_{j=1}^{n}(X_j\varphi_j + Y_j\varphi_{n+j})$$

for any $\sum_{j=1}^n(\varphi_j X_j + \varphi_{n+j}Y_j) \in C^1(\Omega,H)$. Moreover, we say that a set E as above is an $\mathbb{H}$-*Cacioppoli set* whenever $P_{\mathbb{H}}(E,\Omega) < +\infty$ for any bounded open set $\Omega \subseteq \mathbb{H}^n$. Finally, we recall (cf. e.g. [SC]) that an $\mathbb{H}$-Cacioppoli set E is an $\mathbb{H}$-*perimeter minimizer* whenever

$$P_{\mathbb{H}}(E,\Omega) \le P_{\mathbb{H}}(F,\Omega)$$

for any $\Omega \Subset \mathbb{H}^n$ and for any $\mathbb{H}$-Cacioppoli set F such that $E\Delta F \Subset \Omega$. Following this definition, according for instance to [NGR], we say that an hypersurface of class C^1 is *minimal* whenever it coincides with the boundary of an $\mathbb{H}$-perimeter minimizer. One of the key differences between the Euclidean and the Heisenberg setting is that, as pointed out in [AK], the classical Federer's notion of rectifiability in metric spaces (cf. [Fe]) is not suitable for the Heisenberg group, since the latter turns out to be purely unrectifiable. To solve this issue, B. Franchi, R. Serapioni and F. Serra Cassano introduced in [FSSC] the intrinsic notion of $\mathbb{H}$-*regular hypersurface*, and showed that these objects are the right ones to deal with a more suitable notion of *intrinsic rectifiability*. We recall that an $\mathbb{H}$-regular hypersurface is a subset of $\mathbb{H}^n$ which can be described locally as the zero locus of a $C^1_{\mathbb{H}}$-function, i.e. a continuous function whose *horizontal gradient* is continuous and locally non-vanishing (cf. [FSSC] for more precise definitions). When an $\mathbb{H}$-regular hypersurface is of class C^1, more can be said about its structure. To clarify this point, let us fix some notation. In the following, we call a C^k-hypersurface, with $k \ge 1$, an hypersurface of class C^k which is closed and without boundary. If S is a C^1-hypersurface, we define

$$S_0 := \{p \in S : H_p = T_pS\}$$

and we call it the *characteristic set* of S. Notice that, since S is closed and of class C^1 and H is a smooth distribution, then S_0 is closed. Moreover, let us define

$$HT_pS := H_p \cap T_pS.$$

When $p \in S_0$, then $\dim(HT_pS) = 2n$. On the contrary, when $p \in S \setminus S_0$, we have $\dim(HT_pS) = 2n-1$. In this case, we define the *horizontal normal* to S at p by

$$\nu_{\mathbb{H}}(p) := \frac{N_{\mathbb{H}}(p)}{|N_{\mathbb{H}}(p)|_p}$$

for any $p \in S \setminus S_0$, where $N_{\mathbb{H}}(p)$ is the a section of the horizontal bundle defined by

$$N_{\mathbb{H}}(p) := \sum_{j=1}^{n}(\langle N(p), X_j|_p\rangle_{\mathbb{R}^{2n+1}} X_j|_p + \sum_{j=1^n}\langle N(p), Y_j|_p\rangle_{\mathbb{R}^{2n+1}} Y_j|_p,$$

being $N(p)$ the Euclidean unit normal to S at p. It is clear that a C^1-hypersurface which has empty characteristic set is $\mathbb{H}$-regular. In this case, we will also refer to it as *non-characteristic* C^1-hypersurface. After [FSSC], it was clear that the importance of $\mathbb{H}$-regular hypersurfaces went beyond rectifiability. In particular, the study of minimal hypersurfaces in this and related settings (cf. for instance [CDPT, CHMY1, CHMY2, CHY, DGN1, DLPT, GPPV, GaR2, NGR, PSTV, R, SCVi] and references therein) has highlighted many interesting differences between the behavior of characteristic and non-characteristic hypersurfaces, as it will be clearer in a while. In the effort to understand minimal hypersurfaces in the Heisenberg group, it is natural to wonder weather an analogous of Bernstein Theorem holds. In particular it is important to understand which is the right class of hypersurfaces to consider and which are the candidate counterparts of hyperplanes. A first study of the Bernstein problem was carried out by [CHMY1, DGN2, RR] in the class of *t-graphs* of class C^2. We recall that a C^k-hypersurface S is a *t-graph* whenever there exists $u \in C^k(\mathbb{R}^{2n})$ such that

$$\operatorname{graph}(u) := S = \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y}) \,:\, (\bar{x}, \bar{y}) \in \mathbb{R}^{2n}\}.$$

In the previous set of papers, the authors classified minimal t-graphs of class C^2 in the first Heisenberg group $\mathbb{H}^1$, finding examples of minimal smooth t-graphs which are not planes. These results were generalized in [HRR], where the authors classified minimal complete C^2-hypersurfaces in $\mathbb{H}^1$. Moreover, as pointed out in [MSCV, R], if one consider hypersurfaces with low regularity, the examples of minimal hypersurfaces which are not hyperplanes increase considerably. It was evident from these works that, unlike in the Euclidean setting, it is impossible to have rigidity for general minimal hypersurfaces. However, the situation is different when considering non-characteristic hypersurfaces. In this context, a meaningful counterpart of hyperplanes in the Euclidean setting is the class of *vertical hyperplanes*. Let us recall that a vertical hyperplane is a set S of the form

$$S = \{p \in \mathbb{H}^n \,:\, \langle (\bar{x}, \bar{y}), (\bar{a}, \bar{b}) \rangle = c\},$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. An easy computation (cf. Section 2) shows that S is non-characteristic. Moreover, every hyperplane which is not vertical is characteristic (cf. again Section 2). A first result in this direction was achieved in [BASCV] in the class of *intrinsic graphs* (cf. [BASCV] for a proper definition). Indeed, the authors showed that the only minimal intrinsic graphs of class C^2 in $\mathbb{H}^1$ are vertical hyperplanes. This result was generalized in [GaR1] to the class of non-characteristic minimal C^1-hypersurfaces of $\mathbb{H}^1$, in [NGSC] to the class of minimal intrinsic graphs with Euclidean Lipschitz regularity in $\mathbb{H}^1$, and in [GiR] to the class of (X, Y)-Lipschitz surfaces in the sub-Finsler Heisenberg group $\mathbb{H}^1$. While the Bernstein problem is well understood in $\mathbb{H}^1$, very few results are known in higher dimensions, and they are all negative answers. On one hand, as in $\mathbb{H}^1$, there is no rigidity in the class of smooth t-graphs ([SCVe]). On the other hand, when $n \geq 5$, there are counterexamples even in the class of smooth intrinsic graphs (cf. [BASCV]). Finally, the Bernstein problem for non-characteristic hypersurfaces is still open when $n = 2, 3, 4$. In the first Heisenberg group $\mathbb{H}^1$, a key step in the study of minimal surfaces which is common to all the aforementioned works consists in understanding that the non-characteristic part $S \setminus S_0$ of an area-stationary surface S is foliated by *horizontal line* segments in the following sense.

Ruling Property. [CHMY2, GaR1] *Let S be an area-stationary C^1-surface in $\mathbb{H}^1$. Then, S is foliated by horizontal line segments with endpoints in S_0.*

Here, by horizontal line, we mean an Euclidean line γ such that

$$\dot\gamma(t)=\sum_{j=1}^n a_jX_j(\gamma(t))+\sum_{j=1}^n b_jY_j(\gamma(t)),$$

for some $a_1,\ldots,a_n,b_1,\ldots,b_n\in\mathbb{R}$. The importance of this ruling property, which was already clear in the previous set of papers, became even more evident in [Y], where the author showed a Bernstein Theorem in the class of those minimal intrinsic graphs which present the aforementioned ruling property, thus without assuming any regularity on the surfaces.

Motivated by these considerations, in the effort to reach a better comprehension of the remaining open problems in $\mathbb{H}^n$, in this paper we propose a possible generalization of the notion of ruled surface arisen in $\mathbb{H}^1$ to higher dimensional Heisenberg groups. The core definition of this work is the following.

Definition 1.1 (Ruled Hypersurface)**.** *Let S be a C^1-hypersurface in $\mathbb{H}^n$. We say that S is* ruled *if for any $p\in S\setminus S_0$, for any $v\in HT_pS$ and for any $s>0$, the following property holds. If s is maximal with the property that*

$$p\cdot\delta_\tau(v)\in S$$

for any $\tau\in[0,s]$, it holds that

$$p\cdot\delta_s(v)\in S_0.$$

It is clear that this definition reduces to the aforementioned ruling property in $\mathbb{H}^1$. Moreover, as we will show in Section 2, it behaves well with respect many other intrinsic notions. For instance, we will prove that the class of ruled C^1-hypersurfaces is closed under the action of intrinsic dilations (cf. Proposition 2.7). and the action of the so-called *pseudohermitian transformations* (cf. Theorem 2.11). In addition, we will discuss this definition in connection with the nature of the characteristic set (cf. Proposition 2.2). Nevertheless, an heuristic interpretation of this property suggests that it could be more rigid in higher dimensional Heisenberg groups. To explain this consideration, consider a non-characteristic C^1-surface S in $\mathbb{H}^1$ and let $p\in S$. Then it is easy to see that

$$HT_qS=HT_pS$$

for any $q\in p\cdot HT_pS$. This means, roughly speaking, that one cannot exploit the ruling property to escape from an horizontal line in S. This fact is *a priori* no longer true in higher dimensional Heisenberg groups. The aim of Section 3 and Section 4 is to provide some relevant examples of classes of hypersurfaces in which this rigidity emerges. More specifically, in Section 3 we focus on the class of non-characteristic ruled C^1-hypersurfaces. In $\mathbb{H}^1$, this class is large enough to contain surfaces which are not vertical planes (cf. Section 2). On the contrary, this is no longer true when $n\geq 2$, as we will show with the following result.

Theorem 1.2. *Let $n\geq 2$ and let S be a non-characteristic ruled C^1-hypersurface. Then S is a vertical hyperplane.*

In light of this result, inspired by the $\mathbb{H}^1$ case, it could be natural to wonder if it is the case that every minimal C^1-hypersurface is ruled. We already know that this fact is true in $\mathbb{H}^1$. Moreover, combining [BASCV] and Theorem 1.2, we see that when $n\geq 5$ this claim is false even in the class of non-characteristic C^2-hypersurfaces. It is clear that a positive answer to this claim when $n=2,3,4$, together with Theorem 1.2, would solve the Bernstein problem in the non-characteristic setting. However, it is not clear whether this hope could be reasonable or not. This perplexity is motivated by the results of Section

4. Indeed, in Section 4, we study the ruling property among the class of intrinsic *conical* C^1-hypersurfaces (cf. Section 4 for a proper definition), and we prove, among the other things, another rigidity result a soon as we restrict to the C^2 case.

Theorem 1.3. *Let $n \geq 2$ and let S be a ruled conical C^2-hypersurface. If $S_0 = \emptyset$, then S is a vertical hyperplane. If $S_0 \neq \emptyset$, then S is the horizontal hyperplane H_0.*

As a corollary of the previous characterization, it is easy to provide counterexamples to the validity of the Ruling Property for minimal hypersurfaces in the characteristic setting when $n \geq 2$.

Theorem 1.4. *Let $n \geq 2$ and let $S := \mathrm{graph}(u)$, where $u(\bar{x}, \bar{y}) = \frac{1}{2}x_1^2 - \frac{1}{2}y_1^2$. Then S is a minimal smooth hypersurface which is not ruled.*

These sets of results highlights once more some interesting differences between $\mathbb{H}^1$ and higher dimensional Heisenberg groups and, according to the author's hope, could give a little burst in the grasp of such an interesting open problem as the Bernstein problem in this anisotropic setting.

Plan of the paper. The paper is organized as follows. In section 2 we begin the study of ruled hypersurfaces, introducing some first properties and examples. In Section 3 we focus on non-characteristic ruled C^1-hypersurfaces, and we prove Theorem 1.2. In Section 4 we move our attention on ruled conical hypersurfaces, first introducing some basic materials about intrinsic cones, and then proving Theorem 1.3 and Theorem 1.4.

Acknowledgements. The author would like to thank Davide Vittone, Gianmarco Giovannardi, Andrea Pinamonti, Francesco Serra Cassano and Robert Young for stimulating conversations and discussions about these topics.

2. Ruled Hypersurfaces

This section is devoted to the discussion of some first properties of ruled hypersurfaces. First we study the relationship between the ruling property and the characteristic set. Then we provide some examples and show that the class of ruled hypersurfaces is closed under the action of many reasonable maps.

Proposition 2.1. *Let S be a ruled C^1-hypersurface. Then, for any $p \in S \setminus S_0$, there exists an open neighborhood U of p such that*

$$p \cdot HT_pS \cap U \subseteq S.$$

Proof. Assume by contradiction that there exists $p \in S \setminus S_0$ and a sequence $(p_h)_h \subseteq p \cdot HT_pS \setminus S$ converging to p as $h \to +\infty$. Then, for any $h \in \mathbb{N}$, there exists $\lambda_h > 0$ and $v_h \in HT_pS$ such that $p_h = p \cdot \delta_{\lambda_h}(v_h)$. Since $p_h \notin S$ and S is closed, there exists $s_h > 0$ maximal such that $p \cdot \delta_\tau(v_h) \in S$ for any $\tau \in [0, s_h]$. Clearly $s_h \leq \lambda_h$. Therefore, being S ruled, then $q_h := p \cdot \delta_{s_h}(v_h) \in S_0$. But then, by construction, $(q_h)_h$ converges to p as $h \to +\infty$, and so, being S_0 closed, we conclude that $p \in S_0$, a contradiction. $\square$

Proposition 2.2. *Let S be a ruled C^1-hypersurface. Let $p \in S \setminus S_0$ be such that*

$$p \cdot HT_pS \cap S_0 = \emptyset.$$

Then it holds that

$$p \cdot HT_pS \subseteq S.$$

Proof. Let $p \in S \setminus S_0$ satisfy $p \cdot HT_pS \cap S_0 = \emptyset$, and assume by contradiction that there exists $q \in p \cdot HT_pS \setminus S$. Then $q = p \cdot \delta_\lambda(v)$ for some $\lambda > 0$ and $v \in HT_pS$. Then we can argue as in the proof of Proposition 2.1 to find $s > 0$ such that $p \cdot \delta_s(v) \in S_0$, which is a contradiction. $\square$

Proposition 2.3. *Let S be a C^1-hypersurface. Assume that*

$$p \cdot HT_pS \subseteq S \tag{2.1}$$

for any $p \in S$. Then S is ruled.

Proof. It suffices to observe that it is impossible to find $p \in S \setminus S_0$, $v \in HT_pS$ and $s \in \mathbb{R}$ as in Definition 1.1, as this would contradict (2.1). Hence S is trivially ruled. □

Notice that, in view of Proposition 2.2, the notion of ruled hypersurface becomes much more simpler in the case of non-characteristic hypersurfaces. Indeed, if S is a non-characteristic ruled C^1 hypersurface and $p \in S$, then clearly $p \cdot HT_pS \cap S_0 = \emptyset$. Therefore a non-characteristic C^1-hypersurface is ruled if and only if it satisfies (2.1). Now let us discuss some instances of ruled hypersurfaces. We begin with the simplest non-characteristic smooth hypersurface.

Example 2.4 (Vertical Hyperplanes)**.** Let S be a vertical hyperplane of the form

$$S = \{p \in \mathbb{H}^n \, : \, \langle(\bar{x}, \bar{y}), (\bar{a}, \bar{b})\rangle = c\}$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$. Without loss of generality, we assume that $a_1 \neq 0$. It is easy to see that

$$T_pS = \operatorname{span}\{(a_2, -a_1, 0, \ldots, 0), (a_3, 0, -a_1, 0, \ldots, 0), \ldots, (b_n, 0, \ldots, 0, -a_1), T\}$$

for any $p \in S$. Notice that $S_0 = \emptyset$. We show that S is ruled. Indeed, noticing that $T \in T_pS$ for any $p \in S$, it follows that

$$HT_pS = \operatorname{span}\{Z_2|_p, \ldots, Z_n|_p, W_1|_p, \ldots, W_n|_p\}$$

for any $p \in S$, where

$$Z_i = a_iX_1 - a_1X_i \qquad \text{and} \qquad W_j = b_jX_1 - a_1Y_j \tag{2.2}$$

for any $i = 2, \ldots, n$ and $j = 1, \ldots, n$. Let now $p = (\bar{x}, \bar{y}, t) \in S$, and let $w = (\bar{x}', \bar{y}', 0) \in HT_pS$. Then there exists $\alpha_j, \beta_j \in \mathbb{R}$ such that

$$w = \left(\sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j, -\alpha_2 a_1, \ldots, -\beta_n a_1, 0\right).$$

We conclude noticing that

$$\langle(\bar{x}', \bar{y}'), (\bar{a}, \bar{b})\rangle = a_1 \sum_{j=2}^n \alpha_j a_j + a_1 \sum_{j=1}^n \beta_j b_j - \sum_{j=2}^n \alpha_j a_1 a_j - \sum_{j=1}^n \beta_j a_1 b_j = 0.$$

Next we consider an instance in the characteristic case.

Example 2.5 (Horizontal Hyperplane)**.** Let S be the horizontal hyperplane H_0. In view of Proposition 2.3, it suffices to show (2.1). Notice that

$$T_pS = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial y_n}\right\} = \operatorname{span}\{X_1 - y_1T, \ldots, X_n - y_nT, Y_1 + x_1T, \ldots, Y_n + x_nT\}$$

for any $p \in S$. This in particular implies that $S_0 = \{0\}$. Therefore, let $p = (\bar{x}, \bar{y}, t) \neq 0$, and assume without loss of generality that $y_1 \neq 0$. This implies that

$$HT_pS = \operatorname{span}\{y_2X_1 - y_1X_2, \ldots, y_nX_1 - y_1X_n, x_1X_1 + y_1Y_1, \ldots, x_nX_1 + y_1Y_n\}.$$

Therefore, let $w = (z, 0) \in HT_pS$, and let $\alpha_j, \beta_j \in \mathbb{R}$ be such that

$$z = \left(\sum_{j=2}^n \alpha_j y_j + \sum_{j=1}^n \beta_j x_j, -\alpha_2 y_1, \ldots, -\alpha_n y_1, \beta_1 y_1, \ldots, \beta_n y_1\right).$$

Hence it follows that

$$Q((\bar{x},\bar{y}),z) = y_1\sum_{j=2}^{n}\alpha_j y_j + y_1\sum_{j=1}^{n}\beta_j x_j - \sum_{j=2}^{n}\alpha_j y_1 y_j - \sum_{j=1}^{n}\beta_j y_1 x_j = 0.$$

With the next couple of propositions we show that the class of ruled C^1-hypersurfaces is closed under the action of left translations and intrinsic dilations.

Proposition 2.6. *Let S be a ruled C^1-hypersurface. Then $\tau_q(S)$ is a ruled C^1-hypersurface for any $q \in \mathbb{H}^n$.*

Proof. Fix $q = (\bar{x}^q, \bar{y}^q, t) \in \mathbb{H}^n$, define $\tilde{S} := \tau_q(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p \in S$ be such that $\tilde{p} = \tau_q(p)$. Being $\tau_q : S \longrightarrow \tilde{S}$ a diffeomorphism, then $d\tau_q|_p : T_pS \longrightarrow T_{\tilde{p}}\tilde{S}$ is an isomorphism. Therefore we have that

$$d\tau_q|_p(T_pS) = T_{\tilde{p}}\tilde{S}.$$

Moreover, by definition of H, it is also the case that

$$d\tau_q|_p(H_p) = H_{\tilde{p}}.$$

Hence we infer that

$$d\tau_q|_p(HT_pS) = d\tau_q|_p(H_p \cap T_pS) = d\tau_q|_p(H_p) \cap d\tau_q|_p(T_pS) = H_{\tilde{p}} \cap T_{\tilde{p}}\tilde{S} = HT_{\tilde{p}}\tilde{S}.$$

In particular, notice that $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot HT_{\tilde{p}}S$ and assume that there exists $s > 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in HT_pS$ be such that $d\tau_q|_p(v) = w$. By the left-invariance of the horizontal distribution, it follows that $w = (\bar{a}, \bar{b}, 0)$. Therefore s is maximal with the property that $p \cdot \delta_\tau(v) \in S$ for any $\tau \in [0, s]$. Hence $p \cdot \delta_s(v) \in S_0$, and so, since

$$\tilde{p} \cdot \delta_s(w) = \tilde{p} \cdot (s\bar{a}, s\bar{b}, 0) = q \cdot p \cdot (s\bar{a}, \bar{b}, 0) = q \cdot (p \cdot \delta_s(v))$$

and observing that $\tau_q(S_0) = \tilde{S}_0$, we conclude that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$.

□

Proposition 2.7. *Let S be a ruled C^1-hypersurface. Then $\delta_\lambda(S)$ is a ruled C^1-hypersurface for any $\lambda > 0$.*

Proof. Fix $\lambda > 0$, define $\tilde{S} := \delta_\lambda(S)$ and, given a point $\tilde{p} \in \tilde{S} \setminus \tilde{S}_0$, let $p = (\bar{x}, \bar{y}, t) \in S$ be such that $\tilde{p} = \delta_\lambda(p)$. Arguing as in the proof of Proposition 2.6, we get that

$$d\delta_\lambda|_p(HT_pS) = HT_{\tilde{p}}\tilde{S}. \tag{2.3}$$

Therefore, again, $p \in S \setminus S_0$. Let $w \in \tilde{p} \cdot HT_{\tilde{p}}S$ and assume that there exists $s > 0$ maximal with the property that $\tilde{p} \cdot \delta_\tau(w) \in \tilde{S}$ for any $\tau \in [0, s]$. We claim that $\tilde{p} \cdot \delta_s(w) \in \tilde{S}_0$. Let $v = (\bar{a}, \bar{b}, 0) \in HT_pS$ be such that $d\delta_\lambda|_p(v) = w$. We claim that that $w = \delta_\lambda(v)$. Indeed, recalling that the Jacobian matrix of δ_λ is a diagonal matrix with diagonal $(\lambda, \dots, \lambda, \lambda^2)$, then

$$\begin{aligned} w(f)(q) &= \sum_{j=1}^{n} a_j \frac{\partial(f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^{n} b_j \frac{\partial(f \circ \delta_\lambda)}{\partial x_j}(p) + \sum_{j=1}^{n} (a_j y_j - b_j x_j)\, T(f \circ \delta_\lambda)(p) \\ &= \sum_{j=1}^{n} \lambda a_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^{n} \lambda b_j \frac{\partial f}{\partial x_j}(\tilde{p}) + \sum_{j=1}^{n}((\lambda a_j)(\lambda y_j) - (\lambda b_j)(\lambda x_j)) Tf(\tilde{p}). \end{aligned} \tag{2.4}$$

The conclusion then follows as in the previous proof, just noticing that

$$\delta_\lambda(p \cdot \delta_\tau(v)) = \delta_\lambda(p) \cdot \delta_\lambda(\delta_\tau(v)) = \tilde{p} \cdot \delta_{\lambda\tau}(v) = \tilde{p} \cdot \delta_\tau(\delta_\lambda(v)) = \tilde{p} \cdot \delta_\tau(w)$$

for any $\tau \in \mathbb{R}$, and that $\delta_\lambda(S_0) = \tilde{S}_0$. □

In view of Proposition 2.6, we can enlarge the class of examples of ruled hypersurfces.

Example 2.8 (Non-Vertical Hyperplanes)**.** We already know that H_0 is a characteristic ruled smooth hypersurface. For any fixed $q = (\bar{x}_q, \bar{y}_q, t_q) \in \mathbb{H}^n$, we know from Proposition 2.6 that $\tau_q(H_0)$ is a characteristic ruled smooth hypersurface. Moreover, an easy computation shows that

$$\tau_q(H_0) = \{(\bar{x}, \bar{y}, t) \in \mathbb{H}^n \,:\, \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle + t + d = 0\},$$

where $(\bar{a}, \bar{b}) = (-\bar{y}_q, \bar{x}_q)$ and $d = -t_q$. Finally, notice that any hyperplane which is not vertical can be obtained as left-translation of the horizontal hyperplane H_0. Hence we conclude that every hyperplane of $\mathbb{H}^n$ is ruled, and it is non-characteristic if and only if it is vertical. Finally, notice that we cannot exploit Proposition 2.7 to obtain more ruled hypersurfaces, since dilations of hyperplanes are hyperplanes.

To conclude this section, we show that the class of ruled hypersurfaces is closed under the action of the so-called *pseudohermitian transformations* of $\mathbb{H}^n$. To introduce this notion, we define the map $J : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$J(\bar{x}, \bar{y}, t) := (-\bar{y}, \bar{x}, t)$$

for any $p = (\bar{x}, \bar{y}, t) \in \mathbb{H}^n$. The map J is a global diffeomorphism which preserves the horizontal distribution, and it is usually known as *CR structure*. A global diffeomorphism $\varphi : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is said to be a pseudohermitian transformation of $\mathbb{H}^n$ if it preserves the horizontal distribution and it commutes with the CR structure J, that is

$$d\varphi(H) \subseteq H \qquad \text{and} \qquad \varphi \circ J = J \circ \varphi.$$

Let us begin by considering a special subclass of pseudohermitian transformations. To this aim, let us define the map $\varphi_R : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ by

$$\varphi_R(\bar{x}, \bar{y}, t) := (R(\bar{x}, \bar{y}), t), \tag{2.5}$$

where R is an orthogonal matrix of the form

$$R = \begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

where A and B are real-valued $n \times n$ matrices.

Proposition 2.9. *Let φ_R be as in (2.5). Then φ_R is a pseudohermitian transformation. Moreover, it holds that*

$$d\varphi_R|_p(\bar{a}, \bar{b}, 0) = (R(\bar{a}, \bar{b}), 0)$$

for any $p \in \mathbb{H}^n$ and any $(\bar{a}, \bar{b}, 0) \in H_p$.

Proof. Let $p = (\bar{x}, \bar{y}, t)$ and $(\bar{a}, \bar{b}, 0)$ as in the statement, and let $\tilde{p} := \varphi_R(p) = (\bar{\tilde{x}}, \bar{\tilde{y}}, t)$. We first claim that

$$d\varphi_R|_p(X_j|_p) = \sum_{k=1}^n \left(R_{kj} X_k|_{\tilde{p}} + R_{(n+k)j} Y_k|_{\tilde{p}}\right)$$

and

$$d\varphi_R|_p(Y_j|_p) = \sum_{k=1}^n \left(R_{k(n+j)} X_k|_{\tilde{p}} + R_{n+k)(n+j)} Y_k|_{\tilde{p}}\right)$$

for any $j = 1, \ldots, n$. Indeed, let ψ be a C^1 function defined in a neighborhood of $\tilde{p}$. Let us recall that, since $(\bar{\tilde{x}}, \bar{\tilde{y}}) = R(\bar{x}, \bar{y})$ and R is orthogonal, then $(\bar{x}, \bar{y}) = R^T(\bar{\tilde{x}}, \bar{\tilde{y}})$, which means, recalling also the special block shape of R, that

$$-x_j = \sum_{k=1}^n \left(-R_{kj}\tilde{x}_k - R_{(n+k)j}\tilde{y}_k\right) = \sum_{k=1}^n \left(-R_{(n+k)(n+j)}\tilde{x}_k + R_{k(n+j)}\tilde{y}_k\right)$$

and

$$y_j = \sum_{k=1}^{n} \left(R_{k(n+j)} \tilde{x}_k + R_{(n+k)(n+j)} \tilde{y}_k \right) = \sum_{k=1}^{n} \left(-R_{(n+k)j} \tilde{x}_k + R_{kj} \tilde{y}_k \right).$$

for any $j = 1, \ldots, n$. Then it holds that

$$\begin{aligned}
d\varphi_R|_p(X_j|_p)(\psi)(\tilde{p}) &= X_j|_p(\psi \circ \varphi_R)(p) \\
&= \frac{\partial}{\partial x_j}(\psi \circ \varphi_R)(p) + y_j T(\psi \circ \varphi_R)(p) \\
&= \sum_{k=1}^{n} \left(R_{kj} \frac{\partial \psi}{\partial x_k}(\tilde{p}) + R_{(n+k)j} \frac{\partial \psi}{\partial y_k}(\tilde{p}) \right) + y_j T(\psi)(\tilde{p}) \\
&= \sum_{k=1}^{n} \left(R_{kj} \left(\frac{\partial \psi}{\partial x_k}(\tilde{p}) + \tilde{y}_k T(\psi(\tilde{p}) \right) + R_{(n+k)j} \left(\frac{\partial \psi}{\partial y_k}(\tilde{p}) - \tilde{x}_k T(\psi)(\tilde{p}) \right) \right) \\
&= \sum_{k=1}^{n} \left(R_{kj} X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)j} Y_k|_{\tilde{p}}(\psi)(\tilde{p}) \right)
\end{aligned}$$

and, similarly,

$$d\varphi_R|_p(Y_j|_p)(\psi)(\tilde{p}) = \sum_{k=1}^{n} \left(R_{k(n+j)} X_k|_{\tilde{p}}(\psi)(\tilde{p}) + R_{(n+k)(n+j)} Y_k|_{\tilde{p}}(\psi)(\tilde{p}) \right)$$

for any $j = 1, \ldots, n$. Hence we conclude that

$$\begin{aligned}
d\varphi_R|_p(\bar{a}, \bar{b}, 0) &= \sum_{j=1}^{n} (a_j d\varphi_R|_p(X_j|_p) + b_j d\varphi_R|_p(Y_j|_p)) \\
&= \sum_{j,k=1}^{n} \left(a_j \left(R_{kj} X_k|_{\tilde{p}} + R_{(n+k)j} Y_k|_{\tilde{p}} \right) + b_j \left(R_{k(n+j)} X_k|_{\tilde{p}} + R_{n+k)(n+j)} Y_k|_{\tilde{p}} \right) \right) \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \left(R_{kj} a_j + R_{k(n+j)} b_j \right) X_k|_{\tilde{p}} + \sum_{j=1}^{n} \left(R_{(n+k)j} a_j + R_{(n+k)(n+j)} b_j \right) Y_k|_{\tilde{p}} \right) \\
&= (R(\bar{a}, \bar{b}), 0).
\end{aligned}$$

□

As a consequence of the previous result, it is easy to see that the class of ruled hypersurfaces is closed under the action of maps of the form (2.5).

Proposition 2.10. *Let S be a ruled C^1-hypersurface. Then $\varphi_R(S)$ is a ruled C^1-hypersurface for any φ_R as in* (2.5)*.*

Proof. The proof of this result, with the help of Proposition 2.9, follows as the proof of Proposition 2.6 and Proposition 2.7, noticing that $\varphi_R(S_0) = (\varphi_R(S))_0$ and that, for a given $p = (z, t) \in S \setminus S_0$, $(v, 0) \in HT_pS$ and $s \in \mathbb{R}$, it holds that

$$\begin{aligned}
\varphi_R(p \cdot \delta_s(v, 0)) &= \varphi_R(z + sv, t + Q(z, sv)) \\
&= (R(z + sv), t + sQ(z, v)) \\
&= (Rz + sRv, t + sQ(Rz, Rv)) \\
&= (Rz, t) \cdot (sRv, 0)) \\
&= \varphi_R(p) \cdot \delta_s(Rv, 0).
\end{aligned}$$

□

As a corollary of Proposition 2.9, we can conclude our initial statement.

Theorem 2.11. *Let S be a ruled C^1-hypersurface. Then $\varphi(S)$ is a ruled C^1-hypersurface for any pseudohermitian transformation φ.*

Proof. It follows combining Proposition 2.6, Proposition 2.10 and [CL, Theorem 4.1] $\square$

3. Non-Characteristic Ruled Hypersurfaces

The aim of this section is to characterise non-characteristic ruled C^1-hypersurfaces of $\mathbb{H}^n$, when $n \geq 2$. In the first Heisenberg group $\mathbb{H}^1$ there are examples of ruled, non-characteristic, smooth surfaces which are not vertical planes. As an instance, let us consider the surface S parametrized by the map $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{H}^1$ defined by

$$\varphi(t, \theta) := (t\cos\theta, t\sin\theta, \theta).$$

Notice that φ is smooth and injective. Moreover,

$$\frac{\partial\varphi}{\partial t}(t,\theta) = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} = \cos\theta X|_{\varphi(t,\theta)} + \sin\theta Y|_{\varphi(t,\theta)}$$

and

$$\frac{\partial\varphi}{\partial\theta}(t,\theta) = -t\sin\theta\frac{\partial}{\partial x} + t\cos\theta\frac{\partial}{\partial y} + T = -t\sin\theta X|_{\varphi(t,\theta)} + t\cos\theta Y|_{\varphi(t,\theta)} + (1+t^2)T.$$

These computation implies that S is a smooth, non-characteristic surface, and moreover

$$HT_{\varphi(t,\theta)}S = \operatorname{span}\left\{\frac{\partial\varphi}{\partial t}(t,\theta)\right\}$$

for any $(t,\theta) \in \mathbb{R}^2$. Finally, for given $t, \theta, s \in \mathbb{R}$, it holds that

$$(t\cos\theta, t\sin\theta, \theta)\cdot(s\cos\theta, s\sin\theta, 0) = ((t+s)\cos\theta, (t+s)\sin\theta, \theta) \in S,$$

and so S is ruled. However, the situation in higher dimensional Heisenberg groups is quite different, and the ruling condition turns out to be more restrictive. Indeed, we are going to prove that the only ruled, non-characteristic, C^1-hypersurfaces in $\mathbb{H}^n$, with $n \geq 2$, are vertical hyperplanes.

Proposition 3.1. *Assume that $n \geq 2$. Let S be a non-characteristic ruled C^1-hypersurface such that $0 \in S$. Then it holds that*

$$HT_0S = H_0 \cap S.$$

Proof. Since $S_0 = \emptyset$ and S is ruled, then $HT_0S \subseteq S \cap H_0$. Assume by contradiction that there exists $q = (z_q, 0) \in (S \cap H_0) \setminus HT_0S$. Again, being S ruled, it holds that $q \cdot HT_qS \subseteq S$, and so $q \cdot HT_qS \cap H_0 \subseteq S \cap H_0$. Note that both HT_0S and $q \cdot HT_qS \cap H_0$ are affine subspaces of H_0. Moreover, $\dim(HT_0S) = 2n-1$ and $\dim(q \cdot HT_qS \cap H_0) \geq 2n-2$. Therefore we conclude that

$$\dim(HT_0S \cap (q \cdot HT_qS \cap H_0)) \geq \dim(HT_0S) + \dim(q \cdot HT_qS \cap H_0) - 2n = 2n-3 \geq 1,$$

since $n \geq 2$. Therefore $(HT_0S)\cap(q\cdot HT_qS\cap H_0)$ contains a one-dimensional affine subspace of H_0. In particular, let $p = (z_p, 0) \in HT_0S \cap (q \cdot HT_qS \cap H_0)$. Let $v \in HT_pS$ be such that $p \cdot tv = q$ for some $t \in \mathbb{R}$, and let $\gamma_p(t) := (tz_p, 0)$. Notice that, as $p \in HT_0S$, S is ruled and $S_0 = \emptyset$, then $\gamma_p(t) \in S$ for any $t \in \mathbb{R}$. Moreover, $\dot\gamma_p(1) = (z_p, 0) \in H_p$, and so $w := (z_p, 0) \in HT_pS$. Again, being S ruled and $S_0 = \emptyset$, then $p \cdot HT_pS \subseteq S$. Therefore, in particular, it holds that

$$p \cdot (\alpha v + \beta w) \in S$$

for any $\alpha, \beta \in \mathbb{R}$. Hence, if we let $\gamma_q(t) := (tz_q, 0)$, we conclude that $\gamma(t) \in S \cap H_0$ for any $t \in \mathbb{R}$, and so $\dot\gamma_q(0) = (z_q, 0) \in T_0S$. Since clearly $(z_q, 0) \in H_0$, then $q \in HT_0S$, which is a contradiction. $\square$

Proposition 3.2. *Assume that $n \geq 2$. Let S be a non-characteristic ruled C^1-hypersurface such that $0 \in S$. Assume in addition that $T \in T_0S$. Then S is a vertical hyperplane.*

Proof. We divide the proof into some steps.
Step 1. Thanks to Proposition 3.1, we know that there exists $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$ such that

$$HT_0S = H_0 \cap S = \{(\bar{x}, \bar{y}, 0) \in \mathbb{H}^n \,:\, \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle = 0\}.$$

We assume without loss of generality that $a_1 \neq 0$, and we let $f(\bar{x}, \bar{y}) := \langle(\bar{a}, \bar{b}), (\bar{x}, \bar{y})\rangle$. We claim that

$$\pi(p \cdot HT_pS) \subseteq \pi(HT_0S)$$

for any $p \in HT_0S$, where here and in the following the map $\pi : \mathbb{H}^n \longrightarrow \mathbb{R}^{2n}$ is defined by

$$\pi(\bar{x}, \bar{y}, t) := (\bar{x}, \bar{y}).$$

Assume by contradiction that there exists $p = (z_p, 0) \in HT_0S$ and $v = (v, 0) \in HT_pS$ such that $z_p + v \notin \pi(HT_0S)$. This is equivalent to say that $f(z_p + v) \neq 0$. Let us define $q := p \cdot v = (z_p + v, Q(z_p, v))$. Being S ruled, then $q \in S$. Moreover, $Q(z_p, v) \neq 0$, since otherwise $q \in HT_0S$ and consequently $f(z_p + v) = 0$. Moreover, since $z_p \in HT_0S$, then, letting $\gamma(t) := (tz_p, 0)$, it holds that $\gamma(t) \in S$ for any $t \in \mathbb{R}$, and so $(z_p, 0) \in HT_pS$. Hence, since $p \cdot HT_pS \subseteq S$, we conclude in particular that

$$P := \{(z_p, 0) + \alpha(z_p, 0) + \beta(v, Q(z_p, v)) \,:\, \alpha, \beta \in \mathbb{R}\} \subseteq S.$$

Notice that P is a vector subspace of $\mathbb{R}^{2n+1}$. Then in particular $0 \in P$ and $(v, Q(z_p, v)) \in T_0S$. Therefore, as $T \in T_0S$, then $(v, 0) \in T_0S$, and so, since $(v, 0) \in H_0$, we conclude that $(v, 0) \in HT_0S$. Then $f(v) = 0$, and so, as $p \in HT_0S$, $f(z_p + v) = f(z_p) + f(v) = 0$, a contradiction.
Step 2. Let $p = (z_p, 0) \in HT_0S$. Thanks to Step 1, we know that $\pi(p \cdot HT_pS) \subseteq \pi(HT_0S)$. Therefore, if $v \in HT_pS$, then $f(z_p + v) = 0$. Since $f(z_p) = 0$, we conclude that $f(v) = 0$, which implies that

$$HT_pS = HT_0S \tag{3.1}$$

for any $p \in HT_0S$. Moreover, an easy computation shows that

$$HT_0S = \operatorname{span}\{Z_2|_0, \ldots, Z_n|_0, W_1|_0, \ldots, W_n|_0\},$$

where $Z_2, \ldots, Z_n, W_1, \ldots, W_n$ are as in (2.2). Then (3.1) allows to conclude that

$$HT_pS = \operatorname{span}\{Z_2|_p, \ldots, Z_n|_p, W_1|_p, \ldots, W_n|_p\}. \tag{3.2}$$

Step 3. Let us define

$$\mathcal{Z} := \{z \in \pi(HT_0S) \,:\, Q(z, w) = 0 \text{ for any } w \in \pi(HT_0S)\}.$$

Notice that, being Q a bilinear map, then $\mathcal{Z}$ is a vector subspace of $\pi(HT_0S)$. We claim that $\dim(\mathcal{Z}) \leq 2n-2$. Indeed, assume by contradiction that $\dim(\mathcal{Z}) \geq 2n-1$. Then, since $\mathcal{Z} \subseteq \pi(HT_0S)$ and $\dim(\pi(HT_0S)) = 2n-1$, we conclude that $\mathcal{Z} = \pi(HT_0S)$. We show that this leads to a contradiction. Assume first that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, and set $z_1 = (0, -1, 0 \ldots, 0)$ and $z_2 = (\bar{0}, 0, 1, 0, \ldots, 0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1, z_2) = 1 \neq 0$, which implies that $z_1, z_2 \notin \mathcal{Z}$. If it is not the case that $a_2 = \ldots = a_n = b_2 = \ldots = b_n = 0$, then assume without loss of generality that $a_2 \neq 0$. Let $z_1 = (-a_2, a_1, 0, \ldots, 0)$ and $z_2 = (-b_1, 0, \ldots, 0, a_1, 0, \ldots, 0)$. Then $f(z_1) = f(z_2) = 0$ and $Q(z_1, z_2) = a_1 a_2 \neq 0$, which implies again that $z_1, z_2 \notin \mathcal{Z}$. Therefore we conclude that $\dim(\mathcal{Z}) \leq 2n-2$, and so in particular

$$\overline{\pi(HT_0S) \setminus \mathcal{K}} = \pi(HT_0S). \tag{3.3}$$

Step 4. We claim that for any $q = (z_q, t_q) = (x_1^q, \ldots, x_n^q, y_1^q, \ldots, y_n^q, t_q)$ such that $z_q \in \pi(HT_0S) \setminus \mathcal{Z}$ there exists $p = (z_p, 0) = (x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p, 0) \in HT_0S$ and $v \in HT_pS$ such that

$$q = p \cdot v. \tag{3.4}$$

Indeed, let q as above, and let $p \in HT_0S$ and $v \in HT_pS$ to be chosen later. In view of (3.2), we can express v as

$$v = \sum_{j=2}^n \alpha_j Z_j|_p + \sum_{j=1}^n \beta_j W_j|_p = \left(\sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j\right) X_1|_p - \sum_{j=2}^n \alpha_j a_1 X_j|_p - \sum_{j=1}^n \beta_j a_1 Y_j|_p.$$

for some $\alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}$. Therefore, we infer that

$$p \cdot v = \left(x_1^p + \sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j, x_2^p - \alpha_2 a_1, \ldots, y_n^p - \beta_n a_1, Q(z_p, v)\right).$$

Let us choose

$$\alpha_i = \frac{x_i^p - x_i^q}{a_1} \qquad \textit{and} \qquad \beta_j = \frac{y_j^p - y_j^q}{a_1}$$

for any $i = 2, \ldots, n$ and any $j = 1, \ldots, n$. This choice implies that $(p \cdot v)_i = x_i^q$ and $(p \cdot v)_j = y_{j-n}^q$ for any $i = 2, \ldots, n$ and any $j = n+1, \ldots, 2n$. Moreover, since $f(z_p) = f(z_q) = 0$, it holds that

$$(p \cdot v)_1 = x_1^p + \sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j = \frac{1}{a_1}\left(\sum_{j=1}^n (a_j x_j^p + b_j y_j^p) - \sum_{j=2}^m a_j x_j^q + \sum_{j=1}^n b_j y_j^q\right) = x_1^q.$$

Finally, notice that

$$\begin{aligned}
Q(z_p, v) &= \left(\sum_{j=2}^n \alpha_j a_j + \sum_{j=1}^n \beta_j b_j\right) y_1^p - \sum_{j=2}^n \alpha_j a_1 y_j^p + \sum_{j=1}^n \beta_j a_1 x_j^p \\
&= \frac{1}{a_1}\Bigg(\sum_{j=2}^n a_j x_j^p y_1^p - \sum_{j=2}^n a_j x_j^q y_1^p + \sum_{j=1}^n b_j y_j^p y_1^p - \sum_{j=1}^n b_j y_j^q y_1^p \\
&\quad - \sum_{j=2}^n a_1 x_j^p y_j^p + \sum_{j=2}^n a_1 x_j^q y_j^p + a_1 x_1^p y_1^p + \sum_{j=2}^n a_1 x_j^p y_j^p - \sum_{J=1}^n a_1 x_j^p y_j^q\Bigg) \\
&= \frac{1}{a_1}\Bigg(-\sum_{j=1}^n a_j x_j^q y_1^p - \sum_{j=1}^n b_j y_j^q y_1^p + \sum_{j=1}^n a_1 x_j^q y_j^p - \sum_{j=1}^n a_1 x_j^p y_j^q\Bigg) \\
&= Q(z_p, z_q),
\end{aligned}$$

where in the third equality we exploited the fact that $f(z_p) = 0$, while the fourth equality follows from $f(z_q) = 0$. Since we assumed $z_q \notin \mathcal{Z}$, then there exists $w \in \pi(HT_0S)$ such that $Q(w, z_q) \neq 0$. Therefore, if we choose

$$z_p = \frac{t_q}{Q(w, z_q)} w,$$

we conclude that $p \in HT_0S$ and that $Q(z_p, z_q) = t_q$.

Step 5. We are now able to conclude. Indeed, thanks to (3.4) we infer that

$$\pi(HT_0S \setminus \mathcal{K}) \times \mathbb{R} \subseteq S.$$

But then, being S closed and recalling (3.3), we conclude that

$$\pi(HT_0S) \times \mathbb{R} = \overline{\pi(HT_0S) \setminus \mathcal{K}} \times \mathbb{R} = \overline{\pi(HT_0S) \times \mathbb{R}} \subseteq \overline{S} = S.$$

Therefore S contains the vertical hyperplane $\pi(HT_0S) \times \mathbb{R}$. The thesis then follows in view of the topological assumptions on S. $\square$

Proposition 3.3. *Assume that $n \geq 2$. Let S be a non-characteristic ruled C^1-hypersurface. Then there exists $q \in S$ such that $T \in T_0\tau_{q^{-1}}(S)$.*

Proof. Let S be as in the statement. We first claim that there exists $q \in S$ such that $T \in T_qS$. Indeed, in this case the left-invariance of T would imply that

$$d\tau_{q^{-1}}|_q(T|_q) = T|_0.$$

We split the proof of the claim into some steps.
Step 1. Assume by contradiction that

$$T|_q \notin T_qS \tag{3.5}$$

for any $q \in S$. By the left-invariance of T and Proposition 2.6, we can preserve (3.5) assuming in addition that $0 \in S$. Therefore, thanks to Proposition 3.1, we infer that

$$HT_0S = H_0 \cap S = \{(\bar{x}, \bar{y}, 0) \in \mathbb{H}^n \,:\, \langle(\bar{x}, \bar{y}), (\bar{a}, \bar{b})\rangle = 0\}$$

for some $0 \neq (\bar{a}, \bar{b}) \in \mathbb{R}^{2n}$. As usual, we assume that $a_1 \neq 0$. Moreover, (3.5) implies that S is an entire t-graph. Since $T|_0 \notin T_0S$ and S is non-characteristic, there exists $v = (z_v, t_v) \in T_0S$ such that $f(z_v) \neq 0$ and $t_v \neq 0$. Let us set $c := -\frac{f(z_v)}{t_v}$. We claim that

$$S = \{(z, t) \in \mathbb{H}^n \,:\, f(z) + ct = 0\} =: S_c.$$

Indeed, let $p = (z_p, t_p) \in S$. If $t_p = 0$, then $f(z_p) = 0$, and so $p \in S_c$. Assume then $t_p \neq 0$. Then, being S a t-graph, we infer that $f(z_p) \neq 0$.
Step 2. Let $v_1, \ldots, v_{2n-1}$ be a basis of HT_pS. Since S is ruled, then $p \cdot HT_pS \subseteq S$. We claim that there exists $j = 1, \ldots, 2n-1$ such that $Q(z_p, v_j) \neq 0$. Indeed, assume by contradiction that $Q(z_p, v_1) = \ldots = Q(z_p, v_{2n-1}) = 0$. In this case, recalling that S is ruled, it holds that

$$p \cdot HT_pS = \left\{ \left(z_p + \sum_{j=1}^{2n-1} \alpha_j v_j, t_p \right) \,:\, \alpha_1, \ldots, \alpha_{2n-1} \in \mathbb{R} \right\} \subseteq S. \tag{3.6}$$

We claim that

$$\operatorname{span}\{(v_1, 0), \ldots, (v_{2n-1}, 0)\} = \operatorname{span}\{Z_2|_0, \ldots, Z_n|_0, W_1|_0, \ldots, W_n|_0\}, \tag{3.7}$$

where $Z_2, \ldots, Z_n, W_1, \ldots, W_n$ are defined as (2.2). Indeed, if it was not the case, then (3.6) would imply the existence of $q = (z_q, t_p) \in S$ such that $z_q \in \pi(HT_0S)$. But since $t_p \neq 0$ and since $(z_q, 0) \in S$, we would contradict the fact that S is a t-graph. Notice that (3.7) implies that

$$Z_2|_0, \ldots, Z_n|_0, W_1|_0, \ldots, W_n|_0 \in H_p$$

and so, observing that

$$Z_j|_0 = a_j \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial x_j} = a_j X_1|_p - a_1 X_j|_p + (a_1 y_j - a_j y_1)T$$

for any $j = 2, \ldots, n$ and

$$W_j|_0 = b_j \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial y_j} = b_j X_1|_p - a_1 Y_j|_p + (-a_1 x_j - b_j y_1)T$$

for any $j = 1, \ldots, n$, we conclude that

$$z_p = \frac{y_1}{a_1}(-b_1, \ldots, -b_n, a_1, \ldots, a_n),$$

which implies in particular that $f(z_p) = 0$, a contradiction.
Step 3. Thanks to the previous step, we assume that there exists $j = 1, \ldots, 2n-1$ such

that $Q(z_p, v_j) \neq 0$. In this case, it holds that $p \cdot HT_pS \cap H_0 \cap S = p \cdot HT_pS \cap HT_0S \neq 0$. Therefore, let $q = (z_q, 0) \in p \cdot HT_pS \cap HT_0S$, and let $\lambda \in \mathbb{R}$ be such that

$$(z_q, 0) = (z_p + \lambda v_j, t_p + \lambda Q(z_p, v_j)), \tag{3.8}$$

or equivalently $\lambda = -\frac{t_p}{Q(z_p, v_j)}$. Arguing as in the proof of Proposition 3.2, we see that

$$P := \{(z_q, 0) + \alpha(z_q, 0) + \beta(v_j, Q(z_q, v_j)) \,:\, \alpha, \beta \in \mathbb{R}\} \subseteq S,$$

and so we conclude as above that $(v_j.Q(z_q, v_j)) \in T_0S$. This means that there exists $w \in \pi(HT_0S)$ and $\alpha \in \mathbb{R}$ such that

$$(v_j.Q(z_q, v_j)) = (w + \alpha z_v, \alpha t_v).$$

Therefore, recalling (3.8), we get that

$$(z_p, t_p) = (z_q, 0) - \lambda(v_j, Q(z_p, v_j)) = (z_q - \lambda w - \alpha\lambda z_v, -\alpha\lambda t_v)\,.$$

Therefore, since $z_q, w \in \pi(HT_0S)$ we can conclude that

$$f(z_p) + ct_p = -\alpha\lambda(f(z_v) + ct_v) = 0,$$

which implies that $p \in S_c$. Therefore we proved that $S \subseteq S_c$, and so, arguing as at the end of the proof of Proposition 3.2, we conclude that $S = S_c$.

Step 4. We conclude noticing that S_c is a non-vertical hyperplane, in such a case we already know that $S_0 \neq \emptyset$. A contradiction then follows. $\square$

Proof of Theorem 1.2. By Proposition 3.3, there exists $q \in S$ such that $T \in T_0\tau_{q^{-1}}(S)$. Moreover, by Proposition 2.6, $\tilde{S} := \tau_{q^{-1}}(S)$ is a non-characteristic ruled C^1-hypersurface. Notice that, by construction, $0 \in \tilde{S}$ and $T|_0 \in T_0\tilde{S}$. Therefore, thanks to Proposition 3.2, $\tilde{S}$ is a vertical hyperplane. To conclude, it suffices to notice that $S = \tau_q(\tilde{S})$ and that any left translation of a vertical hyperplane is itself a vertical hyperplane. $\square$

4. Ruled Intrinsic Cones

In this section we study ruled hypersurfaces among the class of hypersurfaces which are invariant under intrinsic dilations, that is the class of intrinsic cones. A set $C \subseteq \mathbb{H}^n$ is a *cone* if

$$\delta_\lambda(C) \subseteq C$$

for any $\lambda > 0$. It is easy to see that, if C is a cone, then $0 \in \overline{C}$, $\delta_\lambda(C) = C$ and $\delta_\lambda(\partial C) = \partial C$ for any $\lambda > 0$. We say that S is a *conical C^k-hypersurface*, for a given $k \geq 1$, if S is both a cone and a C^k-hypersurface. Notice that, in view of the aforementioned properties, if C is a cone with boundary of class C^k, then ∂C is a conical C^k-hypersurface. The simplest instance of non-characteristic conical C^1-hypersurfaces is given by vertical hyperplanes passing through the origin. Another simple instance is given by the horizontal plane H_0. In this case we already know that $(H_0)_0 = \{0\}$. Finally, if u is an homogeneous quadratic polynomial, then graph(u) is a conical smooth hypersurface. Moreover, in this last case, S_0 may be an infinite set. As an instance, let us consider the graph associated to $u(\bar{x}, \bar{y}) = \sum_{j=1}^n x_j y_j$. It is easy to see that

$$T_pS = \operatorname{span}\{X_1, \ldots, X_n, Y_1 + 2x_1T, \ldots, Y_n + 2x_nT\}$$

for any $p = (\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \in \operatorname{graph}(u)$. Therefore in this case we have that

$$S_0 = \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y})) \in \operatorname{graph}(u) \,:\, x_1 = \ldots = x_n = 0\}.$$

When a C^1-hyersurface is a cone, we can say more about the structure of S_0.

Proposition 4.1. *Let S be a conical C^1-hypersurface. Then S_0 is a cone.*

Proof. Let $p \in S_0$ and $\lambda > 0$. We prove that $q := \delta_\lambda(p) \in S_0$. If $p = 0$ the thesis is trivial. Assume that $p \neq 0$. We prove that $H_q = T_q S$. Since S is a cone, then $\delta_\lambda : S \longrightarrow S$ is a diffeomorphism, and consequently, recalling (2.3), $d\delta_\lambda|_p : HT_pS \longrightarrow HT_qS$ is an isomorphism. we conclude that $\dim(HT_pS) = \dim(HT_qS)$, which means that $q \in S_0$. □

Proposition 4.2. *Let S be a conical C^1-hypersurface. Then $S_0 \subseteq H_0$. Moreover, for any $p \in S_0$ there is a horizontal half line $\gamma : [0, +\infty) \longrightarrow S_0$ such that $\gamma(0) = 0$ and $\gamma(1) = p$.*

Proof. Let $p = (\bar{x}, \bar{y}, t) \in S_0 \setminus \{0\}$, and set $\gamma(0) = 0$ and $\gamma(\lambda) := \delta_\lambda(p)$. Then γ is a smooth curve with

$$\dot{\gamma}(\lambda) = (\bar{x}, \bar{y}, 2\lambda t) = \sum_{j=1}^{n} x_j X_j + \sum_{j=1}^{n} y_j Y_j + 2\lambda t T.$$

Moreover, thanks to Proposition 4.1, then $\gamma([0, +\infty)) \subseteq S_0$. Finally, since $\gamma(1) = p$, S is a cone and $p \in S_0$, then $\dot{\gamma}(1) \in T_pS = H_p$, and so $t = 0$. □

The shape of conical C^1-hypersurfaces strongly depends on the shape of the associated characteristic set. We begin recalling the following simple rigidity result for non-characteristic conical hypersurfaces, which can be obtained as an easy consequence of [FSSC, Theorem 4.1].

Theorem 4.3. *Let S be a conical C^1-hypersurface. Then $S_0 = \emptyset$ if and only if S is a vertical hyperplane. Moreover, it holds that*

$$S = \{(v, t) \,:\, (v, 0) \in HT_0S,\ t \in \mathbb{R}\}.$$

Thanks to Theorem 4.3, we know how to characterize non-characteristic conical C^1-hypersurfaces. Hence, in the rest of this section we assume that $S_0 \neq 0$. In this case, as the next proposition shows, it suffices to reduce to the analysis of t-graphs.

Proposition 4.4. *Let S be a conical C^1-hypersurface. If $S_0 \neq \emptyset$, then S is a t-graph.*

Proof. Since $S_0 \neq \emptyset$, then Proposition 4.1 implies that $0 \in S_0$. Therefore

$$H_0 = T_0S = \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial y_n}\right\}.$$

Hence there exists $r > 0$ and a function $\tilde{u} \in C^1(B^{2n}_{2r}(0))$ such that

$$S \cap B^{2n+1}_{2r}(0) = \{(\bar{x}, \bar{y}, \tilde{u}(\bar{x}, \bar{y}) \,:\, (\bar{x}, \bar{y}) \in B^{2n}_{2r}(0)\},$$

where here and throughout the paper we denote by $B^k_r(p)$ the Euclidean ball in $\mathbb{R}^k$ of radius r centered at p. Notice that, being S a cone, then

$$\tilde{u}(\lambda\bar{x}, \lambda\bar{y}) = \lambda^2 u(\bar{x}, \bar{y})$$

for any $(\bar{x}, \bar{y}) \in B^{2n}_{2r}(0)$ and for any $\lambda > 0$ such that $(\lambda\bar{x}, \lambda\bar{y}) \in B^{2n}_{2r}(0)$. Let us define $u : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ by

$$u(\bar{x}, \bar{y}) := \left(\frac{|(\bar{x}, \bar{y})|}{r}\right)^2 \tilde{u}\left(r\frac{(\bar{x}, \bar{y})}{|(\bar{x}, \bar{y})|}\right).$$

Then clearly $u \in C^1(\mathbb{R}^{2n})$ and $u \equiv \tilde{u}$ on $B^{2n}_{2r}(0)$. Moreover, by definition,

$$G := \{(\bar{x}, \bar{y}, u(\bar{x}, \bar{y}) \,:\, (\bar{x}, \bar{y}) \in \mathbb{R}^{2n}\}.$$

Let now $p = (\bar{x}, \bar{y}, t) \in S$. Then there exists $\lambda > 0$ such that $\delta_\lambda(p) \in B^{2n+1}_{2r}(0) \cap S$. Hence

$$\lambda^2 t = \tilde{u}(\lambda\bar{x}, \lambda\bar{y}) = u(\lambda\bar{x}, \lambda\bar{y}) = \lambda^2 u(\bar{x}, \bar{y}),$$

which allows to conclude that $t = u(\bar{x}, \bar{y})$ and $p \in G$. Therefore, being both S and G conical C^1-hypersurfaces, we conclude that $S = G$. □

In Section 2 we exhibited two examples of ruled conical smooth hypersurfaces, namely the horizontal hyperplane H_0 and the vertical hyperplanes passing through the origin. The aim of the rest of this section is to show that, in the class of conical C^2-hypersurfaces, these are the only possible examples. We begin with the following characterization, whose proof is inspired by the proof of [NGR, Lemma 4.4].

Theorem 4.5. *Let S be a conical C^1-hypersurface. Assume that $S_0 = \{0\}$. Then S is ruled if and only if S is the horizontal plane H_0.*

Proof. For sake of notational simplicity, we prove the statement when $n = 2$, being the other cases completely analogous. We already know that H_0 is ruled. Conversely, let S be ruled, and assume by contradiction that there exists $p = (z, t) \in S$ with $t \neq 0$. Then, thanks to Proposition 4.2, $p \in S \setminus S_0$. Moreover, $p \cdot HT_pS \cap S_0 = \emptyset$, since otherwise there would be an horizontal line joining p and 0, which contradicts the fact that horizontal lines passing through 0 lie in H_0. Therefore, being S ruled and thanks to Proposition 2.2, we infer that $p \cdot HT_pS \subseteq S$. It is well known (cf. for instance [CL]) that there exists an orthonormal basis u, v, w of HT_pS such that

$$J(u) = w \qquad \text{and} \qquad J(v) = \nu_S(p). \tag{4.1}$$

Let us set

$$M := \begin{bmatrix} u & v & J(u) & J(v) \end{bmatrix}^T.$$

Then, defining φ_R as in (2.5) and thanks to Proposition 2.9, we can assume that $u = X_1$, $v = X_2$ and $w = Y_1$. Let us define $\varphi : (0, +\infty) \times \mathbb{R}^3 \longrightarrow S$ by

$$\varphi(\lambda, \alpha, \beta, \gamma) := \delta_\lambda \left(p \cdot \left(\frac{\alpha}{\lambda} u + \frac{\beta}{\lambda} v + \frac{\gamma}{\lambda} w \right) \right).$$

Being S a ruled cone, the map φ is well-defined. Moreover, notice that

$$\begin{aligned} \varphi(\lambda, \alpha, \beta, \gamma) &= \delta_\lambda \left(z + \frac{\alpha u + \beta v + \gamma w}{\lambda}, t + \frac{\alpha Q(z,u) + \beta Q(z,v) + \gamma Q(z,w)}{\lambda} \right) \\ &= (\lambda z + \alpha u + \beta v + \gamma w, \lambda^2 t + \lambda\alpha Q(z,u) + \lambda\beta Q(z,v) + \lambda\gamma Q(z,w)) \\ &= (\lambda x_1 + \alpha, \lambda x_2 + \beta, \lambda y_1 + \gamma, \lambda y_2, \lambda^2 t + \lambda\alpha y_1 + \lambda\beta y_2 - \lambda\gamma x_1). \end{aligned}$$

Therefore, an easy computation shows that

$$D\varphi(\lambda, \alpha, \beta, \gamma) = \begin{bmatrix} x_1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 \\ y_1 & 0 & 0 & 1 \\ y_2 & 0 & 0 & 0 \\ 2\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1 & \lambda y_1 & \lambda y_2 & -\lambda x_1 \end{bmatrix}$$

We claim that $y_2 \neq 0$. Otherwise, recalling that $p \cdot HT_pS \subseteq S$, we would have that

$$(x_1, x_2, y_1, 0, t) \cdot (\alpha, \beta, \gamma, 0, 0) = (x_1 + \alpha, x_2 + \beta, y_1 + \gamma, 0, t + \alpha y_1 - \gamma x_1) \in S$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$. Therefore, choosing $\alpha = -x_1$, $\beta = -x_2$ and $\gamma = -y_1$, we conclude that $(0, 0, 0, 0, t) \in S$, which is a contradiction, since $0 \in S$ and S, thanks to Proposition 4.4, is a t-graph. Hence $y_2 \neq 0$, and so, since $\varphi(0, 0, 0, 0) = p$, there exists $r > 0$ such that $D\varphi$ has maximum rank in $B^4_r(0)$. In particular,

$$T_{\varphi(q)}S = \operatorname{span} \left\{ \frac{\partial \varphi}{\partial \lambda}(q), \frac{\partial \varphi}{\partial \alpha}(q), \frac{\partial \varphi}{\partial \beta}(q), \frac{\partial \varphi}{\partial \gamma}(q) \right\}$$

for any $q \in B^4_r(0)$. Notice that, if we define the 1-form ω by

$$\omega = dt - \sum_{j=1}^{n} y_j dx_j + \sum_{j=1}^{n} x_j dy_j,$$

then $v \in H$ if and only if $\omega(v) = 0$ for any $v \in T\mathbb{H}^n$. Fix $q = (\lambda, \alpha, \beta, \gamma) \in B^4_r(0)$. Then

$$\omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \lambda}(q) \right) = 2(\lambda t + \alpha y_1 + \beta y_2 - \gamma x_1),$$

and moreover

$$\omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \alpha}(q) \right) = -\gamma, \qquad \omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \beta}(q) \right) = 0, \qquad \omega|_{\varphi(q)} \left(\frac{\partial \varphi}{\partial \gamma}(q) \right) = \alpha.$$

Therefore, if we choose $\alpha = \gamma = 0$, we conclude that

$$\operatorname{span} \left\{ \frac{\partial \varphi}{\partial \alpha}(q), \frac{\partial \varphi}{\partial \beta}(q), \frac{\partial \varphi}{\partial \gamma}(q) \right\} \subseteq HT_{\varphi(q)}(S).$$

Moreover, since $y_2 \neq 0$ we can choose $\beta = -\frac{\lambda t}{y_2}$ to conclude that

$$\frac{\partial \varphi}{\partial \lambda}(q) \in HT_{\varphi(q)} S.$$

Since $\operatorname{rank}(D\varphi(q)) = 4$, we proved that

$$\varphi \left(\lambda, 0, -\frac{\lambda t}{y_2}, 0 \right) = \left(\lambda x_1, \lambda x_2 - \frac{\lambda t}{y_2}, \lambda y_1, \lambda y_2, 0 \right) \in S_0$$

for any $\lambda > 0$ small enough. Since $y_2 \neq 0$, we proved that there exists $\tilde{p} \neq 0$ such that $\tilde{p} \in S_0$. This is a contradiction with the assumption $S_0 = \{0\}$. $\square$

We are left with the analysis of ruled conical hypersurfaces S with infinite characteristic set. In this case we limit ourselves to consider conical C^2-hypersurfaces. Indeed, in this simpler situation, the next proposition shows that it suffices to consider graphs of quadratic polynomials.

Proposition 4.6. *Let S be a conical C^2-hypersurfaces. Assume that $S_0 \neq \emptyset$. Then $S = \operatorname{graph}(u)$ for some homogeneous quadratic polynomial u.*

Proof. We already know from Proposition 4.4 that $S = \operatorname{graph}(u)$, where $u \in C^1(\mathbb{R}^{2n})$. Moreover, since S is a C^2-hypersurface, then $u \in C^2(\mathbb{R}^{2n})$. Finally, since $0 \in S_0$, then $\nabla u(0) = 0$. Therefore, the second-order Taylor expansion of u in 0 reads as

$$u(p) = P_2(p) + o(|p|^2),$$

where P_2 is an homogeneous quadratic polynomial. We show that $u = P_2$. Let $p \in \mathbb{R}^{2n}$, and let $\alpha > 0$. Then it holds that

$$|u(p) - P_2(p)| = \frac{|u(\alpha p) - P_2(\alpha p)|}{\alpha^2} = |p|^2 \frac{o(\alpha^2 |p|^2)}{\alpha^2 |p|^2}$$

as $\alpha \to +\infty$. The thesis then follows letting $\alpha \to +\infty$. $\square$

Proof of Theorem 1.3. For sake of notational simplicity, we assume again that $n = 2$, being the other cases completely analogous. If $S_0 = \{0\}$ we already know from Proposition 4.5 that $S = H_0$. Let us assume that S_0 is infinite. We divide the proof into some steps.
Step 1. Thanks to Proposition 4.6, we assume that $S = \operatorname{graph}(u)$, where

$$u(\bar{x}, \bar{y}) = ax_1^2 + bx_2^2 + cy_1^2 + dy_2^2 + ex_1x_2 + fx_1y_1 + gx_1y_2 + hx_2y_1 + mx_2y_2 + py_1y_2,$$

for some $a, b, \dots, m, p \in \mathbb{R}$. Let us define $\varphi : \mathbb{R}^4 \longrightarrow \operatorname{graph}(u)$ by

$$\varphi(\bar{x}, \bar{y}) = (\bar{x}, \bar{y}, u(\bar{x}, \bar{y})).$$

Then φ is a global C^2 parametrization of S. Therefore, for any $p = (\bar{x}, \bar{y}) \in \mathbb{R}^4$, $T_{\varphi(p)}S$ is generated by

$$\frac{\partial \varphi}{\partial x_1}(p) = X_1 + (2ax_1 + ex_2 + (f-1)y_1 + gy_2)T,$$
$$\frac{\partial \varphi}{\partial x_2}(p) = X_2 + (ex_1 + 2bx_2 + hy_1 + (m-1)y_2)T,$$
$$\frac{\partial \varphi}{\partial x_2}(p) = Y_1 + ((f+1)x_1 + hx_2 + 2cy_1 + py_2)T$$

and

$$\frac{\partial \varphi}{\partial x_2}(p) = Y_2 + (gx_1 + (m+1)x_2 + py_1 + 2dy_2)T.$$

Let us define the 4×4 real-valued matrix M by

$$M = \begin{bmatrix} 2a & e & f-1 & g \\ e & 2b & h & m-1 \\ f+1 & h & 2c & p \\ g & m+1 & p & 2d \end{bmatrix},$$

and, for any $j = 1, \dots, 4$, we let v_j be the j-th row of M. Notice that $p = (z, t) \in S$ is a characteristic point of S if and only if $M \cdot z = 0$.

Step 2. We prove that $\operatorname{rank}(M) \in \{2, 3\}$. Since we are assuming that S_0 is infinite, then $\operatorname{rank}(M) \leq 3$, and so in particular S_0 is a linear subspace of $\mathbb{R}^4$ with $\dim(S_0) \geq 1$. Moreover, $\operatorname{rank}(M) \neq 0$, since otherwise we would have that $S = S_0 \subseteq H_0$, and so $S = S_0 = H_0$, which is impossible since 0 is the only characteristic point of H_0. Moreover, we claim that $\operatorname{rank}(M) \geq 2$. Otherwise, if $\operatorname{rank}(M) = 1$, then we can assume without loss of generality that $v_1 \neq 0$ and that there exist $A, B, C \in \mathbb{R}$ such that $v_2 = Av_1$, $v_3 = Bv_1$ and $v_4 = Cv_1$. Therefore, in particular, we have that $e = 2Aa$, $f = 2Ba - 1$ and $g = 2Ca$. Moreover, since $h = Be$ and $h = A(f-1)$, we infer that $0 = Be - A(f-1) = 2ABa - 2ABa + 2A = 2A$, and so $A = 0$. Moreover, since $p = Bg$ and $p = C(f-1)$, we conclude as above that $C = 0$. But this is impossible, since it would imply that $m - 1 = m + 1 = 0$. Therefore we conclude that $\operatorname{rank}(M) \in \{2, 3\}$.

Step 3. Let now $p = (z, p) \in S \setminus S_0$. Since then $M \cdot z \neq 0$, we can assume that $\langle v_1, z \rangle \neq 0$. Hence, there exists an open neighborhood $\tilde{U}$ of p such that $\langle v_1, z_q \rangle \neq 0$ for any $q = (z_q, t_q) \in \tilde{U}$. This implies in particular that $M \cdot z_q \neq 0$ for any $q \in \tilde{U}$, and so $\tilde{U} \cap S \subseteq S \setminus S_0$. Let now U be an open neighborhood of p such that $U \Subset \tilde{U}$. We are going to show that there exists an open neighborhood W of 0 such that

$$HT_pS \cap W \subseteq \{(\bar{x}, \bar{y}) \in \mathbb{R}^4 \, : \, u(\bar{x}, \bar{y}) = 0\} =: G. \tag{4.2}$$

Let us define

$$A = \frac{\langle v_2, z \rangle}{\langle v_1, z \rangle}, \qquad B = \frac{\langle v_3, z \rangle}{\langle v_1, z \rangle}, \qquad C = \frac{\langle v_4, z \rangle}{\langle v_1, z \rangle}.$$

Recalling the computations of the first step, it is clear that

$$HT_pS = \operatorname{span}\{X_2 - AX_1, Y_1 - BX_1, Y_2 - CX_1\}.$$

Therefore, being S ruled and $p \in S \setminus S_0$, it follows that

$$\begin{aligned}(x_1, x_2, y_1, y_2, u(\bar{x}, \bar{y})) \cdot (-\alpha A, -\beta B, -\gamma C, \alpha, \beta, \gamma, 0) \\ = (x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma, \\ u(\bar{x}, \bar{y}) - \alpha A y_1 - \beta B y_1 - \gamma C y_1 + \alpha y_2 - \beta x_1 - \gamma x_2) \in S\end{aligned}$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, noticing that

$$
\begin{aligned}
&u(x_1 - \alpha A - \beta B - \gamma C, x_2 + \alpha, y_1 + \beta, y_2 + \gamma) = \\
&\qquad ax_1^2 + a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 - 2a\alpha A x_1 - 2a\beta B x_1 - 2a\gamma C x_1 + 2a\alpha\beta AB \\
&\qquad + 2A\alpha\gamma AC + 2a\beta\gamma BC + bx_2^2 + 2b\alpha x_2 + b\alpha^2 + cy_1^2 + 2c\beta y_1 + c\beta^2 + dy_2^2 + 2d\gamma y_2 \\
&\qquad + d\gamma^2 + ex_1x_2 + e\alpha x_1 - e\alpha A x_2 - e\alpha^2 A - e\beta B x_2 - e\alpha\beta B - e\gamma C x_2 - e\alpha\gamma C \\
&\qquad + fx_1y_1 + f\beta x_1 - f\alpha A y_1 - f\alpha\beta A - f\beta B y_1 - f\beta^2 B - f\gamma C y_1 - f\beta\gamma C \\
&\qquad + gx_2y_2 + g\gamma x_1 - g\alpha A y_2 - g\alpha\gamma A - g\beta B y_2 - g\beta\gamma B - g\gamma C y_2 - g\gamma^2 C \\
&\qquad + hx_2y_1 + h\beta x_2 + h\alpha y_1 + h\alpha\beta + mx_2y_2 + m\gamma x_2 \\
&\qquad + m\alpha y_2 + m\alpha\beta + py_1y_2 + p\gamma y_1 + p\beta y_2 + p\beta\gamma,
\end{aligned}
$$

we infer that

$$
\begin{aligned}
&a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 - 2a\alpha A x_1 - 2a\beta B x_1 - 2a\gamma C x_1 + 2a\alpha\beta AB \\
&+ 2A\alpha\gamma AC + 2a\beta\gamma BC + 2b\alpha x_2 + b\alpha^2 + 2c\beta y_1 + c\beta^2 + 2d\gamma y_2 \\
&+ d\gamma^2 + e\alpha x_1 - e\alpha A x_2 - e\alpha^2 A - e\beta B x_2 - e\alpha\beta B - e\gamma C x_2 - e\alpha\gamma C \\
&+ (f+1)\beta x_1 - (f-1)\alpha A y_1 - f\alpha\beta A - (f-1)\beta B y_1 - f\beta^2 B - (f-1)\gamma C y_1 - f\beta\gamma C \\
&+ g\gamma x_1 - g\alpha A y_2 - g\alpha\gamma A - g\beta B y_2 - g\beta\gamma B - g\gamma C y_2 - g\gamma^2 C \\
&+ h\beta x_2 + h\alpha y_1 + h\alpha\beta + (m+1)\gamma x_2 \\
&+ (m-1)\alpha y_2 + m\alpha\beta + p\gamma y_1 + p\beta y_2 + p\beta\gamma = 0
\end{aligned}
$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough. Hence, recalling the definition of A, B and C, we conclude that

$$
\begin{aligned}
&+ a\alpha^2 A^2 + a\beta^2 B^2 + a\gamma^2 C^2 + 2a\alpha\beta AB + 2A\alpha\gamma AC + 2a\beta\gamma BC + b\alpha^2 \\
&+ c\beta^2 + d\gamma^2 - e\alpha^2 A - e\alpha\beta B - e\alpha\gamma C - f\alpha\beta A - f\beta^2 B - f\beta\gamma C \\
&- g\alpha\gamma A - g\beta\gamma B - g\gamma^2 C + h\alpha\beta + m\alpha\beta + p\beta\gamma = 0
\end{aligned}
$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$ small enough, which is equivalent to (4.2).

Step 4. Let us define

$$P_p := \operatorname{span}\{(-A, 1, 0, 0), (-B, 0, 1, 0), (-C, 0, 0, 1)\}.$$

Then (4.2) implies that $P_p \cap \pi(W) \subseteq G$. Moreover, it is easy to see that $N := (1, A, B, C)$ is the Euclidean normal to P_p in $\mathbb{R}^4$. Let us define $V = \pi(U)$. Since π is open, then V is an open neighborhood of z. Moreover, being S a t-graph, then $\pi|_S$ is invertible, $V = \pi(U \cap S) = \pi(U \cap (S \setminus S_0))$ and $U \cap S = \pi^{-1}(V)$. Therefore, if $\tilde{z} \in V$, we let $\tilde{z} = z_q$, where q is the unique point in $U \cap S$ such that $\pi(q) = z_q$. For any $z_q \in V$, we define

$$A_q = \frac{\langle v_2, z_q\rangle}{\langle v_1, z_q\rangle}, \qquad B_q = \frac{\langle v_3, z_q\rangle}{\langle v_1, z_Q\rangle}, \qquad C_q = \frac{\langle v_4, z_q\rangle}{\langle v_1, z_Q\rangle},$$

and we let

$$P_q := \operatorname{span}\{(-A_q, 1, 0, 0), (-B_q, 0, 1, 0), (-C_q, 0, 0, 1)\}.$$

Again, $N_q := (1, A_q, B_q, C_q)$ is the Euclidean normal to P_q in $\mathbb{R}^4$. Notice in particular that $A_p = A$, $B_p = B$, $C_p = C$ and $P_p = P$, and that, since $U \Subset \tilde{U} \subseteq S \setminus S_0$, W can be chosen in such a way that $P_q \cap \pi(W) \subseteq G$ for any $z_q \in V$. Moreover, thanks to the choice of U, A_q, B_q and C_q are smooth functions on V.

Step 5. We claim that one between A_q, B_q, C_q is not constant in any neighborhood of z. Indeed, let Z be a neighborhood of z, let $a_1,\dots,a_4,b_1,\dots,b_4\in\mathbb{R}$ be such that $b_1x'_1+b_2x'_2+b_3y'_1+b_4y'_2\neq 0$ for any $(\bar{x}',\bar{y}')\in Z$, and define

$$f(\bar{x}',\bar{y}') := \frac{a_1x'_1+a_2x'_2+a_3y'_1+a_4y'_2}{b_1x'_1+b_2x'_2+b_3y'_1+b_4y'_2}.$$

If f is constant on Z, then $\nabla f\equiv 0$ on Z. A simple computation shows that this is equivalent to

$$a_1b_2-a_2b_1=a_1b_3-a_3b_1=a_1b_4-a_4b_2=a_2b_3-a_3b_2=a_2b_4-a_4b_2=a_3b_4-a_4b_3=0.$$

This implies that the matrix

$$M=\begin{bmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & b_2 & b_3 & b_4\end{bmatrix}$$

has rank one. Therefore, if A_q, B_q and C_q were all constant functions on Z, then we would have that $\operatorname{rank}(M)\leq 1$, which contradicts the fact that $\operatorname{rank}(M)>1$. Therefore without loss of generality, we assume that A_q is not constant in any neighborhood of z.

Step 6. Since A_q is not constant in any neighborhood of z, there exists $s_1,s_2\in\mathbb{R}$ with $s_1<s_2$ such that $A\in(s_1,s_2)$ and for any $s\in(s_1,s_2)$ there exists $q_s\in U$ such that $N_{q_s}=(1,s,B_{q_s},C_{q_s})$. This implies that $\bigcup_{s\in(s_1,s_2)}P_{q_s}\cap\pi(W)$ has non-empty interior. But then, since $P_q\cap\pi(W)\subseteq G$ for any $q\in U$, G has non-empty interior. Being u a polynomial, the only possibility is that $u\equiv 0$, and thus $S=H_0$. □

Proof of Theorem 1.4. S is clearly a conical smooth hypersurface. Let $p\in S\setminus S_0$. It is well-known that

$$N(p)=\frac{1}{\sqrt{1+|\nabla u(z)|^2_{\mathbb{R}^{2n}}}}(\nabla u(z),-1)=\frac{1}{\sqrt{1+x_1^2+y_1^2}}(x_1,0,\dots,0,-y_1,0,\dots,0),$$

and so

$$\nu_{\mathbb{H}}(p)=\nu_{\mathbb{H}}(z)=\frac{1}{\sqrt{2(x_1-y_1)^2}}(x_1-y_1,0,\dots,0,x_1-y_1,0,\dots,0).$$

Since in this case $\nu_{\mathbb{H}}$ does not depend on t, an easy computation shows that

$$\operatorname{div}_{\mathbb{H}}\nu_{\mathbb{H}}(p)=\operatorname{div}_{\mathbb{R}^{2n}}\nu_{\mathbb{H}}(z)=0 \tag{4.3}$$

for any $p\in S\setminus S_0$. Since $n\geq 2$, (4.3) allows us to apply [CHY, Corollary F] and [BASCV, Theorem 5.15], which, together with [SC, Example 5.29], imply that S is minimal. We conclude noticing that, in view of Theorem 1.3, S is not ruled. □

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