EXISTENCE OF MINIMIZERS FOR THE SDRI MODEL IN $\mathbb{R}^n$: WETTING AND DEWETTING REGIMES WITH MISMATCH STRAIN

SHOKHRUKH YU. KHOLMATOV AND PAOLO PIOVANO

ABSTRACT. The existence and the regularity results obtained in [37] for the variational model introduced in [36] to study the optimal shape of crystalline materials in the setting of stress-driven rearrangement instabilities (SDRI) are extended from two dimensions to any dimensions $n \geq 2$. The energy is the sum of the elastic and the surface energy contributions, which cannot be decoupled, and depend on configurational pairs consisting of a set and a function that model the region occupied by the crystal and the bulk displacement field, respectively. By following the physical literature, the "driving stress" due to the mismatch between the ideal free-standing equilibrium lattice of the crystal with respect to adjacent materials is included in the model by considering a discontinuous mismatch strain in the elastic energy. Since two-dimensional methods and the methods used in the previous literature where Dirichlet boundary conditions instead of the mismatch strain and only the wetting regime were considered, cannot be employed in this setting, we proceed differently, by including in the analysis the dewetting regime and carefully analyzing the fine properties of energy-equibounded sequences. This analysis allows to establish both a compactness property in the family of admissible configurations and the lower-semicontinuity of the energy with respect to the topology induced by the L^1-convergence of sets and a.e. convergence of displacement fields, so that the direct method can be applied. We also prove that our arguments work as well in the setting with Dirichlet boundary conditions.

1. INTRODUCTION

Elastic effects can strongly affect the structure of crystalline materials by inducing morphological destabilizations from the optimal free-standing crystalline equilibrium, that are often referred to as the family of *stress-driven rearrangement instabilities* (SDRI) [4, 19, 30, 34, 48]. In order to relieve the strain, atoms move from their crystalline order possibly inducing both bulk deformations and interface irregularities. The latter can be originated in various forms, such as the roughness of the exposed crystalline boundaries, the formation of internal cracks in the bulk, the nucleation of dislocations in the crystalline lattice, and the delamination at contact edges with adjacent materials. However, such corrugations and extra boundary interfaces are not favorable with respect to the surface energy, which would instead prescribe regular specific Wulff/Winterbottom-type shapes [44, 45, 50, 51]. Therefore, the surface energy competes against the destabilizing effect of the elastic energy with a regularizing effect: a delicate microscopical compromise between such opposite mechanisms must then be reached strongly affecting in a variety of ways the original crystalline-material macroscopical properties.

In the strive of capturing such interplay between elastic and (anisotropic) surface energy described by the physical literature [25, 35, 39, 46, 47, 49, 52], various mathematical models

Date: May 17, 2023.
2010 *Mathematics Subject Classification.* 49J45, 35R35, 74G65.
Key words and phrases. Minimal configurations, elastic energy, surface energy, mismatch strain, Dirichlet Boundary conditions, existence, regularity, compactness, lower-semicontinuity, density estimates, SDRI, interface instabilities, thin films, crystal cavities, fractures.

with a variational nature have been introduced in relation to the different settings relevant for the applications. A non-exhaustive list includes [6, 9, 20, 21, 27, 33, 38] for epitaxially-strained thin films deposited on supporting materials, [10, 11, 29] for fractures, [5, 40] for delamination, and, e.g., [28] for crystalline cavities. Establishing the existence of minimizers for such models even in dimension $n = 2$ is a challenging task especially due to compactness issues. Such issues were first solved in simplified settings, by working under the antiplane-shear assumption [8, 16], or by distinguishing the applications with *adhoc* geometric assumptions on the morphology of the crystalline materials, such as adopting graph-type and star-shapedness constraints on film profiles and crystal cavities, respectively. More recently, the development of several techniques related to GSBD-functions, a specific subclass of functions of bounded deformation [18], have been sucessfully applied to models related to the Griffith energy [11, 12, 13, 14, 18, 29]. Following this progress, there has been a growing effort [15, 17, 36, 37] to develop mathematical frameworks enabling the simultaneous treatment of the various mechanisms of mass rearrangement and boundary instabilities, which is of crucial importance, as often such phenomena concomitantly occur in applications.

The aim of this paper is to extend to dimension $n \geq 2$, and hence including the physical relevant case of $n = 3$, the existence and the regularity results obtained in [37] for $n = 2$ for the SDRI model introduced in [36]. In regard of the existence, such an extension was previously achieved in [17] for the *wetting regime*, i.e., the case for which it is more convenient for the crystal material to always cover the surface of a (supporting) adjacent material rather than letting it exposed, and the setting in which the stress driving effect characterizing SDRI is mathematically prescribed by introducing *boundary Dirichlet conditions.* Here we address also the dewetting regime and, as previously done by the authors in [36, 37] for $n = 2$, by following the physical literature [4, 19, 30, 34, 47, 48, 52] we avoid the use of any Dirichlet boundary conditions and we directly introduce a *mismatch strain* in the elastic energy. As suggested by its name, such strain is induced in the *free crystal*, i.e., the crystal of which we are studying the morphology, by the mismatch between its ideal free-standing equilibrium lattice and the lattice of adjacent materials. Since the approach used in [17] cannot be applied to this setting without boundary conditions as it is described below (see also [37]), we have developed an alternative strategy that allows us to tackle both the case with mismatch strain and the one with Dirichlet conditions (see Remark 2.10 for more details). Finally, the method of this paper extends (also to both the settings with and without Dirichlet conditions) the regularity results for the bulk displacements and the morphologies of the energy minimizing configurations obtained by the authors in [37] for $n = 2$ (besides extending the existence results of [37] to the presence of different adjacent materials and to Griffith-type models with mismatch strain and delamination).

To facilitate this generalization, we adopt the terminology introduced in [36, 37], by referring to the bounded region Ω in the space $\mathbb{R}^n$ where the free crystal is located as the *container* in analogy to capillarity problems, and to the region S occupied by adjacent materials outside the container, i.e., $S \subset \mathbb{R}^n \setminus \Omega$, as the *substrate* in analogy to the thin-film setting where S is the supporting material on which the film is being deposited. We notice that the *contact region* between the container and the substrate $\Sigma := \partial\Omega \cap \partial S$ is assumed to be a Lipschitz $(n-1)$-manifold and that S can be given by a finite number of different connected components possibly modeling different adjacent materials. The free crystals are then represented by configurational pairs of set-function type (A, u), where $A \subset \Omega$ is a set of finite perimeter denoting the region occupied by the free crystal and subject to the volume constraint $|A| = \mathtt{v}$ with $\mathtt{v} \in (0, |\Omega|]$, and u is a vector valued faction in $GSBD^2(\mathrm{Int(A \cup \Sigma \cup S)}) \cap \mathrm{H^1_{loc}(S)}$ denoting the displacement field of the free-crystal and substrate bulk materials with respect to their optimal equilibrium arrangements. The family of all such admissible configurational pairs (A, u) is denoted by $\mathcal{C}$.

The configurational energy of any free-crystal pair $(A, u) \in \mathcal{C}$ is defined by

$$\mathcal{F}(A, u) = \mathcal{W}(A, u) + \mathcal{S}(A, u), \tag{1.1}$$

where $\mathcal{S}$ and $\mathcal{W}$ represent the elastic and the surface energy, respectively. The elastic energy $\mathcal{W}$ in (1.1) is defined as in [27] by

$$\mathcal{W}(A, u) = \int_{A \cup S} \mathbb{C}(x)[\mathcal{E}u - \mathbf{M}_0] : [\mathcal{E}u - \mathbf{M}_0]\, \mathrm{d}x,$$

where $\mathbb{C}$ is a bounded measurable tensor-valued map $\mathbb{C}$ in $\Omega \cup S$ satisfying the coercivity assumption $\mathbb{C} \geq c\,\mathbb{I} > 0$ (in the sense of linear operators), where $\mathbb{I}$ is the identity tensor, $\mathcal{E}u$ is the approximate symmetric gradient of u (see (2.2)) and $\mathbf{M}_0$ is the (discontinuous) *mismatch strain* defined as

$$\mathbf{M}_0 = \begin{cases} \mathcal{E}u_0 & \text{in } \Omega, \\ 0 & \text{in } S \end{cases} \tag{1.2}$$

for some fixed $u_0 \in H^1(\mathbb{R}^n)$. In the special case in which the equilibrium lattice of the free crystal and of the substrate matches at Σ, we take $u_0 \equiv 0$. The surface energy $\mathcal{S}$ in (1.1) is defined as

$$\mathcal{S}(A, u) := \int_{\partial^* A \cup J_u} \psi(x, \nu(x)) \mathrm{d}\mathcal{H}^{n-1},$$

where $\partial^* A$ is the reduced boundary of A, J_u is the jump set of u, and the surface energy density $\psi(\cdot, \nu(\cdot))$ is given by

$$\psi(x, \nu(x)) := \begin{cases} \varphi(x, \nu_A(x)) & \text{if } x \in \Omega \cap \partial^* A, \\ 2\varphi(x, \nu_{J_u}(x)) & \text{if } x \in A^{(1)} \cap J_u, \\ \beta(x) & \text{if } x \in [\Sigma \cap \partial^* A] \setminus J_u, \\ \varphi(x, \nu_\Sigma(x)) & \text{if } x \in \Sigma \cap \partial^* A \cap J_u, \end{cases} \tag{1.3}$$

where $\nu_U(x)$ denotes the outward-pointing normal vector to U at $x \in \partial^* U$ for any set of finte perimeter $U \subset \mathbb{R}^n$, $\nu_\Sigma := \nu_S$, ν_{J_u} is the normal on J_u, $A^{(1)}$ is the set of points of density 1 for A, $\varphi \in C(\overline{\Omega} \times \mathbb{R}^n)$ is a a Finsler norm denoting the *anisotropic surface tension* of the free-crystal material, and $\beta \in L^\infty(\Sigma)$ represents the *relative adhesion coefficient* of Σ for which we assumed, as in capillarity theory (see, e.g., [24]), that

$$|\beta(x)| \leq \varphi(x, \nu_\Sigma) \quad \text{for a.e. } x \in \Sigma. \tag{1.4}$$

We notice that the weights in (1.3), which forbid to decouple the surface energy from the elastic energy making the energy $\mathcal{F}$ highly nonlocal, are consistent with the ones chosen in [17, 27, 28, 36, 37], where they were crucial to prove energy lower-semicontinuity-type properties. In particular, the anisotropy on *internal cracks* $A^{(1)} \cap J_u$ is weighted twice as much as the *free boundary* $\Omega \cap \partial^* A$ of the exposed boundary of the free crystal, because cracks can be approximated by "closing voids" as in [17, 27, 36]. The presence of the surface energy over $\Sigma \cap \partial^* A \cap J_u$ allows to consider a more general framework for thin films depositing on a substrate, in which cracks are allowed to appear not only inside the film material, but also along the surface of the substrate characterizing the *delamination region*, where debonding between the atoms of the two materials occurs, and as such, the corresponding surface tension in (1.3) is regarded as the same of the one on the free-crystal exposed boundary. Finally, on the complementary region to the delamination in $\Sigma \cap \partial^* A$ where the bulk displacement is continuous, the relative adhesion coefficient β is considered.

We observe that in the case of total wetting case, i.e., if $\beta(x) = -\varphi(x, \nu_\Sigma(x))$ for a.e. $x \in \Sigma$, we reduce to the setting of material voids considered in [17] (with the mismatch strain $\mathbf{M}_0$ replaced by a Dirichlet boundary condition). On the contrary, in the total dewetting case, i.e., if $\beta(x) = \varphi(x, \nu_\Sigma(x))$ for a.e. $x \in \Sigma$, then one can readily check that the energy $\mathcal{F}$ is minimized by configurational pairs with displacement $u \equiv u_0$ in Ω and null otherwise, and so characterized by having a zero elastic energy: the model reduces to the dewetted capillarity setting, or in other words, to the anisotropic isoperimetric problem in a container. Finally, in the case with $\mathtt{v} = |\Omega|$, we reduces to the Griffith model with the inclusion of possible delamination at the substrate boundary, which generalize also for $n = 2$ the setting considered by the authors in [36, 37] together with $S \neq \emptyset$.

We now present the two main results of the paper (see Section 2.2 for more detailed statements) and comment their proofs. We begin by observing that, since the values of the admissible displacement fields u in the void regions $\Omega \setminus A$ do not play any role in the energy of (A, u), as only a formal difference with respect to the previous presentation of the SDRI models introduced in [36, 37], for every $(A, u) \in \mathcal{C}$ we can redefine u in $\Omega \setminus A$ with a properly chosen constant such that $\Omega \cap \partial^* A \subset J_u$ (see Remark 2.1), and so without changing the value of $\mathcal{F}(A, u)$. We make use of this observation in the following.

Theorem 1.1 (**Existence of minimizing configurations**). *The minimum problem*

$$\min_{(A,u)\in\mathcal{C},\, |A|=\mathtt{v}} \mathcal{F}(A, u) \tag{1.5}$$

admits a solution.

We refer the Reader to Theorem 2.4 for a more detailed and comprehensive statement of the existence result of Theorem 1.1.

Theorem 1.1 is established by means of the *direct method of the calculus of variations* with respect to a properly chosen topology $\tau_{\mathcal{C}}$ with which we equip $\mathcal{C}$, and that is characterized by the convergence:

$$(A_k, u_k) \xrightarrow{\tau_{\mathcal{C}}} (A, u) \qquad \Longleftrightarrow \qquad \begin{cases} A_k \to A & \text{in } L^1(\mathbb{R}^n), \\ u_k \to u & \text{a.e. in } \Omega \cup S. \end{cases}$$

In order to establish the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{F}$ in Theorem 2.5 we consider the positive Radon measures μ_k and μ in $\mathbb{R}^n$ associated to the localized energy versions of $\mathcal{F}(A_k, u_k)$ and $\mathcal{F}(A, u)$, respectively, for which it holds that

$$\liminf_{k\to+\infty} \mathcal{F}(A_k, u_k) \geq \mathcal{F}(A, u) \qquad \Longleftrightarrow \qquad \liminf_{k\to+\infty} \mu_k(\mathbb{R}^n) \geq \mu(\mathbb{R}^n). \tag{1.6}$$

Then, we observe that, up to a subsequence, μ_k weakly* converges to some positive Radon measure μ_0, and that μ is absolutely continuous with respect to $\mathcal{H}^{n-1} \llcorner (\partial^* A \cup J_u \cup \Sigma) + \mathcal{L}^n \llcorner (\Omega \cup S)$, and we establish the following estimates for the Radon-Nikodym derivatives:

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner(\partial^* A \cup J_u \cup \Sigma)} \geq \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{H}^{n-1}\llcorner(\partial^* A \cup J_u \cup \Sigma)} \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial^* A \cup J_u \cup \Sigma, \tag{1.7}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{L}^n\llcorner(\Omega \cup S)} \geq \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n\llcorner(\Omega \cup S)} \quad \mathcal{L}^n\text{-a.e. in } \Omega \cup S, \tag{1.8}$$

which imply that $\lim \mu_k(\mathbb{R}^n) = \mu_0(\mathbb{R}^n) \geq \mu(\mathbb{R}^n)$ and, in view of (1.6), conclude the proof of the lower-semicontinuity. For the estimate (1.7) we need to distinguish between the estimate at the reduced boundary of A and at $\Sigma \setminus J_u$, where we can implement techniques developed in capillarity theory [1, 24], from the estimate at the (approximate) jump points of u, where we employ arguments based on the slicing properties of GSBD-functions as in the Griffith model [13, 14, 15], for which though extra care is needed: unless $\mathtt{v} = |\Omega|$, we cannot directly apply those arguments because at jump points we need to obtain different weights with respect to the ones at the reduced boundary of A. Rather, we replace J_{u_k} in small "holes" up to some error by means of Corollaries 3.3 and 3.5 in such a way that each slice intersects the boundary of those holes at least in two points (see the proof of Proposition 4.1), which in turns yields the desired estimate with weight 2 at such jump points (see Corollary 4.2). Finally, we prove (1.8) by using the convexity of $\mathcal{W}(A, \cdot)$ and by observing that the condition $u_k \to u$ a.e. in $\Omega \cup S$, together with the compactness result [14, Theorem 1.1], allows us to conclude that $\mathcal{E}u_k \rightharpoonup \mathcal{E}u$ in $L^2(\Omega \cup S)$. We recall that in [17] the authors prove the lower semicontinuity of an energy for crystalline voids via relaxation arguments. Namely, the authors start in the regular family of pair configurations given by voids with a Lipschitz boundary and Sobolev displacement fields, and then in the relaxation, the jump set appears as the void boundaries collapse, resulting in a coefficient 2 in front of the jump energy of $\mathcal{S}$. We are here actually arguing in the reverse direction: first we start in $\mathcal{C}$ with admissible pairs allowing displacements with jump sets, and then we carefully create an at most countable family of voids around them.

The $\tau_{\mathcal{C}}$-compactness of an energy-equibounded sequence $\{(A_k, u_k)\} \subset \mathcal{C}$ is established in Theorem 2.6. We easily get the uniform bounds on the perimeters of A_k, the $\mathcal{H}^{n-1}$- measure of the jumps J_{u_k}, and the L^2-norm of $\mathcal{E}u_k$ by the assumptions on the anisotropic surface tensions and the elasticity tensor (see Remark 2.3). Thus, we can directly deduce the convergence in $L^1(\mathbb{R}^n)$ up to a non-relabelled subsequence of A_k to some set $A \subset \Omega$ of finite perimeter. However, establishing the $\mathcal{L}^n$ a.e. convergence of the displacements u_k is delicate: by [14, Theorem 1.1] there could be an exceptional set E with $\mathcal{L}^n$ positive measure, in which $|u_k| \to +\infty$. The presence of such an exceptional set has been previously treated by prescribing Dirichlet boundary conditions [13, 14, 17]. For instance, in [17] the compactness issue is solved by considering in the proof an auxiliary more general class $GSBD^p_\infty$, $p > 1$, of displacements (which are allowed to attain the infinite value on a subset of their domain of also $\mathcal{L}^n$ positive measure) and then, by using the Dirichlet condition imposed on the displacements at the boundary, the authors are able to prove that the minimizing displacements belong to the original space $GSBD^p$. However, as in the setting with the mismatch strain (1.2), we cannot rely on any fixed boundary condition, one cannot even exclude the situation with $E = \Omega \cup S$ and hence, this issue unfortunately forbids the implementation of the strategy of [17] to our SDRI setting. The other option of excluding the presence of the exceptional set is based on the employment of Poincaré-Korn inequality for GSBD-functions citeCCF:2016 with small jump: the set Ω is partitioned into a Caccioppoli family $\{P_j\}$ of sets P_j in which a sequence $\{a_k^j\}$ of rigid displacements are defined in such a way that $u_k - a_k^j$ is convergent pointwise a.e. in P_j, so that one can conclude that the sequence

$$v_k := u_k - \sum_j a_k^j \chi_{P_j} \tag{1.9}$$

converges to some $u \in GSBD^p(\Omega)$ a.e. in Ω, $\mathcal{E}v_k \rightharpoonup \mathcal{E}u$ in $L^p(\Omega)$, and

$$\lim_{k\to+\infty} \mathcal{H}^{n-1}(J_{u_k}) \geq \mathcal{H}^{n-1}\Big(J_u \cup (\Omega \cap \bigcup_j \partial^* P_j)\Big)$$

(see [15, Theorem 1.1]). However, also this approach seems not implementable in our SDRI setting, since the functions v_k defined in (1.9) may admit extra jumps along the boundary of the partition phases P_j that should be counted with different weights in our setting with different surface tensions.

In view of these issues, in order to prove compactness we use a different strategy in this paper by directly partitioning the sets A and A_k (not only A!) into Caccioppoli families (that need to be created by starting from the connected components of the substrate) up to a controllable error (see Figures 3 and 5). Such strategy is a reminiscence of the ideas already used by the authors in [36, Theorem 2.7], of partitioning A_k by means of introducing extra circles closing the shrinking "necks", which though works only for $n = 2$ and under the constraint assumed in [36] on the number of boundary components for the admissible free-crystal regions. More precisely, we proceed here arguing as follows: First, by the classical Poincaré-Korn inequality we partition S in a family $\{S^i\}_{i\geq 1}$ of sets S^i such that for each $i \geq 1$ the set S^i is a union of connected components of S and there exists a sequence of rigid displacements $\{a^i_k\}$ such that, up to a subsequence, $u_k - a^i_k$ converges a.e. in S^i and $|a^i_k - a^j_k| \to +\infty$ a.e. in $\mathbb{R}^n$ for every $j \neq i$. Second, by applying [14, Theorem 1.1] with $u_k - a^i_k$ we construct a family $\{F^i\}_{i\geq 0}$ of pairwise disjoint Caccioppoli subsets of A, such that for $i \geq 1$ the sequence $u_k - a^i_k$ converges a.e. in $F^i \cup S^i$ and diverges to infinity otherwise, and $F^0 := A \setminus \bigcup_{i\geq 1} F^i$. Furthermore, since F^0 is the portion of the free crystal, so-called in the following "hanging phase" (see Figure 1), that does not "interact" with any substrate component, we can redefine the displacements in F^0 as u_0 (see (1.2)), which corresponds to providing a zero contribution to the overall elastic energy. Third, by using the $\mathcal{H}^{n-1}$-rectifiability of $\partial^* F^i$ and Propositions 4.1 and 5.2, we construct for any $\delta > 0$ a union $G^\delta_k \subset \Omega$ of open sets covering $\bigcup \partial^* F^i$ up to some error of order $O(\sqrt{\delta})$ and whose perimeter and volume are controlled, and we set

$$B^\delta_k := A_k \setminus G^\delta_k \quad \text{and} \quad v^\delta_k := u_0\chi_{F^0} + \sum_{i\geq 1}(u_k - a^i_k)\chi_{S^i\cup(F^i\setminus G^\delta_k)} + u_0\chi_{F^0}. \tag{1.10}$$

We notice that actually the definition of the v^δ_k in (1.10) is more involved (see (5.3)), as we need also to control the possible large jumps created along Σ, that though in the limit disappear (becoming wetting layer), by creating artificial small jumps in $A^\delta_k \setminus A$ and redefining v^δ_k in that set near Σ. The obtained configurations satisfy

$$\mathcal{F}(A_k, u_k) \geq \mathcal{F}(B^\delta_k, v^\delta_k) - c\sqrt{\delta}\Big(\mathcal{H}^{n-1}(\partial^* A_k) + \mathcal{H}^{n-1}(J_{u_k}) + \sum_{h=0}^{m} P(F^h)\Big) \tag{1.11}$$

for some constant $c > 0$ (see Proposition 5.1), from which Theorem 2.6 follows by a diagonal argument.

We also notice that in the case with Dirichlet boundary conditions, one see at most 2 elements in the partition, the hanging phase F^0 and a phase F^1 interacting with the substrate, since in this case we do not need to add any rigid displacements. Apart from this simplification, the methods used in the proof of Theorems 2.5 and 2.6 still work, even by relaxing the assumptions on the convex elastic energy densities, i.e., by allowing for a p-growth with respect to the strains (see Section 2.3). This allows us in particular to recover in Remark 2.10 the existence results for the model representing material voids in the framework with Dirichlet boundary conditions of [17] and the existence and regularity results for the Griffith fracture model with Dirichlet boundary conditions of [13].

The second main result of the paper relates to properties of partial regularity satisfied by the minimizers (A, u) of $\mathcal{F}$, such as the essential closedness of J_u and $\partial^* A$.

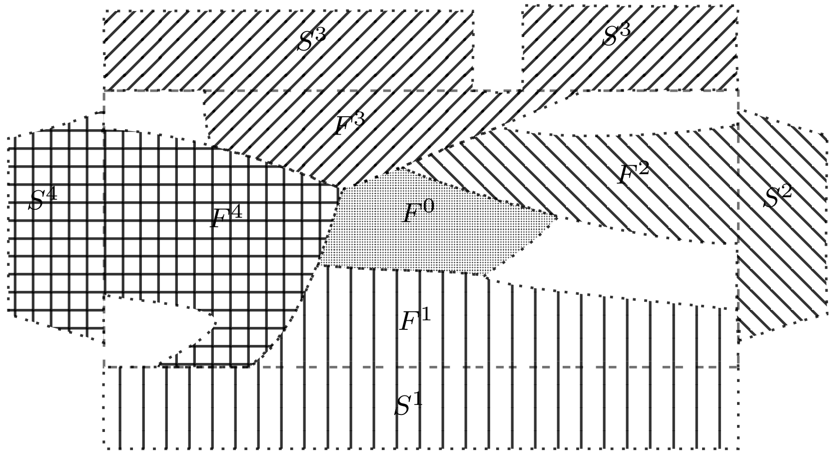

FIGURE 1. The partitions of the substrate and the free crystal into, respectively, the families $\{S^i\}_{i\geq 1}$ and $\{F^i\}_{i\geq 0}$ of Caccioppoli sets, which are used to prove the $\tau_{\mathcal{C}}$-compactness result, are depicted by representing the various phases of the free crystal that are interacting with the substrate with different line patterns and the remaining "hanging phase" F^0 with a point pattern.

Theorem 1.2 (Regularity results for minimizing configurations). *Let $(\widetilde{A}, \widetilde{u})$ be a solution of (1.5). Then the pair (A, u) defined by*

$$A := \mathrm{Int}(A^{(1)}) \qquad \text{and} \qquad u := \widetilde{u}\chi_{A\cup S} + \xi\chi_{\Omega\setminus A},$$

where $\xi \in \mathbb{R}^n$ is chosen such that $\Omega \cap \partial^* A \subset J_u$ (see Remark 2.1), is also a solution of (1.5). Furthermore, we have that

$$\mathcal{H}^{n-1}(\widetilde{A}^{(1)} \setminus A) < +\infty, \quad \mathcal{H}^{n-1}(J_u \setminus J_u^*) = 0, \qquad \text{and} \qquad \mathcal{H}^{n-1}(\overline{J_u^*} \setminus J_u^*) = 0,$$

where

$$J_u^* := \{x \in J_u : \ \theta(J_u, x) = 1\}$$

with $\theta(J_u, x)$ denoting the $(n-1)$-dimensional density of J_u at x. Finally, there exists a constant $c > 0$ such that if $E \subset A$ is a "hanging" component of A, i.e., if $\mathcal{H}^{n-1}([\partial^* E \cap \Sigma] \setminus J_u) = 0$, then $|E| \geq c$.

We refer the Reader to Theorem 2.7 for a more detailed statement of Theorem 1.2.

The proof of Theorem 1.2 is carried out by implementing in the SDRI setting the methods for the partial regularity of the minimizers of the Griffith model by means of the ideas already employed by the authors in [37] for $n = 2$: we introduce a localized version of $\mathcal{F}$ and establish uniform lower and upper $\mathcal{H}^{n-1}$ density estimates for the jump sets (see Section 6). by paying extra care to treat the presence of voids and of the different weights for the surface tension in the surface energy, which is a crucial difference from the Griffith model. We overcome such difficulties by means of the strategy employed in [43] and based on the *relative isoperimetric inequality* [3] to distinguish in the *Decay Lemma* the blows up "inside the free crystal" from the ones "in the voids", and by applying the approximation result of [12, Theorem 3].

The paper is organized as follows: In Section 2 we introduce the SDRI model, some preliminary results related to sets of finite perimeter and GSBD-functions, and state the main results. In Section 3 we provide some technical results which allows to replace a part of jump set with an open set without modifying too much the corresponding SDRI energy. Section 4 is devoted to the proof of the lower semicontinuity of $\mathcal{F}$. Section 5 contains the proof of the compactness for energy-equibounded sequences. In Section 6 we prove the decay estimates for $\mathcal{F}$ and the regularity results of Theorem 2.7. Finally, we conclude the paper with the Appendix containing the results related to the equivalence of the volume-constrained minimum problem with the volume-uncontrained penalized minimum problem, and to some properties of GSBD-functions.

2. Mathematical setting and formulation of the main results

Notation. Unless otherwise stated, all sets we consider are subsets of $\mathbb{R}^n$, in which the coordinates $(x_1,\dots,x_n)$ of $x\in\mathbb{R}^n$ are given with respect to the standard basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. The symbol $B_r(x)$ stands for the open ball in $\mathbb{R}^n$ centered at x and of radius $r>0$. The symbol $Q_r(x):=x+[-\frac{r}{2},\frac{r}{2}]^n$ stands for the standard n-dimensional (hyper) cube in $\mathbb{R}^n$ of sidelength r centered at x. We write $Q_r:=[-\frac{r}{2},\frac{r}{2}]^n$. Given $r>0$, $\nu\in\mathbb{S}^{n-1}$ and $x\in\mathbb{R}^n$ we denote by $Q_{r,\nu}(x)$ the cube of sidelength r centered at x whose sides are either parallel or perpendicular to ν. The characteristic function of a Lebesgue measurable set F is denoted by χ_F and its Lebesgue measure by $|F|$; we set also $\omega_n:=|B_1(0)|$. We denote by E^c the complement of E in $\mathbb{R}^n$. By $\mathcal{H}^{n-1}$ we denote by $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and we write $K=_{\mathcal{H}^{n-1}}L$ and $K\subset_{\mathcal{H}^{n-1}}L$ to mean $\mathcal{H}^{n-1}(K\Delta L)=0$ and $\mathcal{H}^{n-1}(K\setminus L)=0$.

Given an open set $U\subset\mathbb{R}^n$, the set of $L^1(U)$-functions having bounded total variation in U is denoted by $BV(U)$ and the elements of

$$BV(U;\{0,1\}):=\{E\subseteq U:\ \chi_E\in BV(U)\}$$

are called sets of finite perimeter in U. The standard references for BV-functions and sets of finite perimeter are for instance [3, 32, 41].

Given $E\in BV(U,\{0,1\})$ we denote

- by $P(E,U):=\int_U|D\chi_E|$ the perimeter of E in U;
- by ∂E the measure-theoretic boundary of E, i.e.,

$$\partial E:=\{x\in\mathbb{R}^n:\ 0<|B_\rho\cap E|<|B_\rho|\quad\forall\rho>0\};$$

- by ∂^*E the reduced boundary of E, i.e.,

$$\partial^*E:=\Big\{x\in\mathbb{R}^n:\ \exists\nu_E(x):=-\lim_{r\to0}\frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}\quad\text{and}\quad|\nu_E(x)|=1\Big\}.$$

- by ν_E the outer measure-theoretic unit normal to ∂^*E.

Given a Lebesgue measurable set $E\subseteq\mathbb{R}^n$ and $\alpha\in[0,1]$ we define

$$E^{(\alpha)}:=\left\{x\in\mathbb{R}^n:\ \lim_{\rho\to0^+}\frac{|B_\rho(x)\cap E|}{|B_\rho(x)|}=\alpha\right\}.$$

Given a set $K\subset\mathbb{R}^n$ and a point $x_0\in\mathbb{R}^n$, we denote by

$$\theta_*(K,x_0):=\liminf_{r\to0}\frac{\mathcal{H}^{n-1}(B_r(x_0)\cap K)}{\omega_{n-1}r^{n-1}}$$

and

$$\theta^*(K,x_0):=\limsup_{r\to0}\frac{\mathcal{H}^{n-1}(B_r(x_0)\cap K)}{\omega_{n-1}r^{n-1}}$$

the $(n-1)$-dimensional lower and upper density of K at x_0, respectively (see e.g., [3, page 78]). When these densities coincide, we denote their common value by $\theta(K, x_0)$. Recall that by [3, Theorem 2.63], K is $\mathcal{H}^{n-1}$-rectifiable if and only if $\theta(K, x) = 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in K$.

Given $x \in \mathbb{R}^n$ and $r > 0$, the blow-up map $\sigma_{x,r}$ is defined as

$$\sigma_{x,r}(y) = \frac{y - x}{r}. \tag{2.1}$$

Given an open set $U \subset \mathbb{R}^n$ and a metric space X we denote by $\mathrm{Lip}(U; X)$ the family of all Lipschitz functions $\psi : U \to X$. We denote by $\mathrm{Lip}(\psi)$ the Lipschitz constant of $\psi \in \mathrm{Lip}(U; X)$.

By $GSBD(U; \mathbb{R}^n)$ we denote the collection of all *generalized special functions of bounded deformation* (see [14, 18] for their definition and properties). Given $u \in GSBD(U; \mathbb{R}^n)$ we denote by $\mathcal{E}u \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ the *approximate symmetric gradient* and by J_u the jump set of u; we recall that by [18, Theorem 9.1]

$$\operatorname*{ap\,lim}_{y\to x} \frac{[u(y) - u(x) - \mathcal{E}u(x)(y - x)] \cdot (y - x)}{|y - x|^2} = 0 \qquad \text{for a.e. } x \in U \tag{2.2}$$

and by [18, Theorem 6.2] J_u is $\mathcal{H}^{n-1}$-rectifiable. Let us also define

$$GSBD^2(U) := \{u \in GSBD(U; \mathbb{R}^n) : \ \mathcal{E}u \in L^2(U; \mathbb{M}^{n\times n}_{\mathrm{sym}})\}.$$

Given a $\mathcal{H}^{n-1}$-rectifiable set $K \subset \overline{U}$, we consider a normal vector ν_K to its approximate tangent space and we denote by u^+_K and u^-_K the approximate limits of $u \in GSBD(U; \mathbb{R}^n)$ with respect to ν_K, i.e.,

$$u^+_K(x) := \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_K>0,\\ y\in U}} u(y) \quad \text{and} \quad u^-_K(x) := \operatorname*{ap\,lim}_{\substack{(y-x)\cdot\nu_K<0\\ y\in U}} u(y)$$

for every $x \in K$ whenever they exist [18, Definition 2.4]. We refer to u^+_K and u^-_K as the *two-sided traces* of u at K and we notice that they are uniquely determined up to a permutation when changing the sign of ν_K.

Let us recall some notation from [14] related to GSBD-functions. For $\xi \in \mathbb{S}^{n-1}$, $y \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$ and $v : B \to \mathbb{R}^n$ let

$$\Pi_\xi := \{x \in \mathbb{R}^n : \ x \cdot \xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R} : \ y + t\xi \in B\},$$

and

$$v^\xi_y(t) := v(y + t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t) \cdot \xi.$$

We denote by π_ξ the projection of $\mathbb{R}^n$ onto Π_ξ, i.e.,

$$\pi_\xi := x - (x \cdot \xi)\xi.$$

Recall that if $v \in GSBD^2(U)$ for an open set $U \subset \mathbb{R}^n$, then $\widehat{v}^\xi_y \in SBV^2_{\mathrm{loc}}(U^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi_\xi$. We denote by $\dot{u}^\xi_y$ the the absolutely continuous part of Du^ξ_y w.r.t. $\mathcal{L}^1$. Let us introduce

$$I^U_{y,\xi}(v) := \int_{U^\xi_y} |\dot{v}^\xi_y|^2 \mathrm{d}t$$

and

$$II^U_{y,\xi}(v) := |D[\tau(v \cdot \xi)]^\xi_y|(U^\xi_y),$$

where $\tau \in C^1(\mathbb{R}, (-\frac{1}{2}, \frac{1}{2}))$ and satisfies $0 \le \tau' \le 1$. By [14, Eq. 3.8]

$$\int_{\Pi_\xi} I^U_{y,\xi}(v) \mathrm{d}\mathcal{H}^{n-1}(y) = \int_U |\mathcal{E}v(x)\xi \cdot \xi|^2 \mathrm{d}x \le \int_U |\mathcal{E}v|^2 \mathrm{d}x \tag{2.3}$$

and by [14, Eq. 3.9] and obvious estimate $a \le 1 + a^2$

$$\int_{\Pi_\xi} II^U_{y,\xi}(v)\mathrm{d}\mathcal{H}^{n-1}(y) = |D_\xi[\tau(v\cdot\xi)]|(U) \le \int_U |\mathcal{E}v|\mathrm{d}x + \mathcal{H}^{n-1}(U\cap J_v)$$
$$\le |U| + \int_U |\mathcal{E}v|^2\mathrm{d}x + \mathcal{H}^{n-1}(U\cap J_v). \tag{2.4}$$

By the Fubini Theorem and the equality

$$\int_{\mathbb{S}^{n-1}} |\nu\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(\xi) = 2\omega_{n-1}, \qquad \nu\in\mathbb{S}^{n-1},$$

for any $\mathcal{H}^{n-1}$-rectifiable Borel set $L\subset\mathbb{R}^n$ and an open set $U\subset\mathbb{R}^n$ we have

$$\begin{aligned}\mathcal{H}^{n-1}(U\cap L) =& \frac{1}{2\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mathcal{H}^{n-1}(\xi)\int_{U\cap L}|\nu_L\cdot\xi|\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ =& \frac{1}{2\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\mathrm{d}\mathcal{H}^{n-1}(\xi)\int_{\Pi_\xi}\mathcal{H}^0(U^\xi_y\cap L^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y),\end{aligned} \tag{2.5}$$

where we applied the area formula with π_ξ in the second equality.

A linear function $a:\mathbb{R}^n\to\mathbb{R}^n$ satisfying $\nabla a = -(\nabla a)^T$ is called an (infinitesimal) *rigid displacement*.

2.1. **The SDRI model.** Given nonempty open sets $\Omega\subset\mathbb{R}^n$ and $S\subset\mathbb{R}^n\setminus\Omega$, we define the space of *admissible configurations* by

$$\mathcal{C} := \Big\{(A,u):\, A\in BV(\Omega;\{0,1\}),\ u\in GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma))\cap H^1_{\mathrm{loc}}(S)\Big\}$$

where $\Sigma := \partial S\cap\partial\Omega$.

The *energy* of admissible configurations is given by

$$\mathcal{F}:\mathcal{C}\to[-\infty,+\infty], \qquad \mathcal{F} := \mathcal{S}+\mathcal{W},$$

where $\mathcal{S}$ and $\mathcal{W}$ are the surface and elastic energies of the configuration, respectively. The surface energy of $(A,u)\in\mathcal{C}$ is defined as

$$\begin{aligned}\mathcal{S}(A,u) :=& \int_{\Omega\cap\partial^*A}\varphi(x,\nu_A(x))\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &+\int_{A^{(1)}\cap J_u}\big[\varphi(x,\nu_{J_u}(x))+\varphi(x,-\nu_{J_u}(x))\big]\,\mathrm{d}\mathcal{H}^{n-1}(x)\\ &+\int_{\Sigma\cap\partial^*A\setminus J_u}\beta(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)+\int_{\Sigma\cap\partial^*A\cap J_u}\varphi(x,-\nu_\Sigma(x))\,\mathrm{d}\mathcal{H}^{n-1}(x),\end{aligned}$$

where $\varphi:\overline{\Omega}\times\mathbb{S}^{n-1}\to(0,+\infty)$ and $\beta:\Sigma\to\mathbb{R}$ are Borel functions denoting the *anisotropy* of crystal and the *relative adhesion* coefficient of the substrate boundary, respectively, and $\nu_\Sigma := \nu_S$. In applications instead of $\varphi(x,\cdot)$ it is more convenient to use its positively one-homogeneous extension $|\xi|\varphi(x,\xi/|\xi|)$. With an abuse of notation we denote this extension also by φ.

The elastic energy of $(A,u)\in\mathcal{C}$ is defined as

$$\mathcal{W}(A,u) := \int_{A\cup S} W(x,\mathcal{E}u-\mathbf{M}_0)\mathrm{d}x,$$

where the elastic energy density W is a quadratic form

$$W(x,\mathbf{M}) := \mathbb{C}(x)\mathbf{M}:\mathbf{M},$$

determined by a tensor-valued measurable map $x \in \Omega \cup S \to \mathbb{C}(x)$, the so-called *stress-tensor*, in the Hilbert space $\mathbb{M}^{n\times n}_{\rm sym}$ of all $n \times n$-symmetric matrices with the natural inner product

$$\mathbf{M} : \mathbf{N} = \sum_{i,j=1}^{n} M_{ij} N_{ij}.$$

The *mismatch strain* $x \in \Omega \cup S \mapsto \mathbf{M}_0(x) \in \mathbb{M}^{n\times n}_{\rm sym}$ is given by

$$\mathbf{M}_0 := \begin{cases} \mathcal{E} u_0 & \text{in } \Omega, \\ 0 & \text{in } S, \end{cases}$$

for a fixed $u_0 \in H^1(\mathbb{R}^n)$.

Remark 2.1 (**Values of displacements outside a set**).

(i) The functional $\mathcal{F}(A, u)$ does not "see" the values of u in $\Omega \setminus A$, i.e.,

$$\mathcal{F}(A, u) = \mathcal{F}(A, u\chi_{A\cup S} + v\chi_{\Omega\setminus A}) \quad \text{for any } v \in GSBD^2(\Omega).$$

Thus, we can redefine u in $\Omega \setminus A$ arbitrarily without changing the energy of the configuration (A, u).

(ii) For any $(A, u) \in \mathcal{C}$ there exists an at most countable set $\Xi^{(A,u)} \subset \mathbb{R}^n$ such that for any $\xi \in \mathbb{R}^n \setminus \Xi^{(A,u)}$ the function

$$u^\xi := u\chi_{A\cup S} + \xi\chi_{\Omega\setminus A} \tag{2.6}$$

satisfies

$$J_{u^\xi} =_{\mathcal{H}^{n-1}} (\Omega \cap \partial^* A) \cup (\Sigma \cap J_u) \cup (A^{(1)} \cap J_u) \cup (\Sigma \setminus \partial^* A). \tag{2.7}$$

Indeed, for $\xi \in \mathbb{R}^n$ let $E^{(A,u)}_\xi := \{x \in \partial^* A \cup \Sigma : \ \mathrm{tr}_{A\cup S} u(x) = \xi\} \subset \Sigma \cup \partial^* A$ and let

$$\Xi^{(A,u)} := \{\xi \in \mathbb{R}^n : \ \mathcal{H}^{n-1}(E^{(A,u)}_\xi) > 0\}.$$

Since $\mathcal{H}^{n-1}(\partial^* A \cup \Sigma) < +\infty$ and $E^{(A,u)}_\xi \cap E^{(A,u)}_\eta = \emptyset$ for $\xi \neq \eta$, by slicing arguments (see e.g. [37, Proposition A.2]) the set $\Xi^{(A,u)}$ is at most countable. By the definition of jump, for any $\xi \in \mathbb{R}^n \setminus \Xi^{(A,u)}$ the function u^ξ satisfies (2.7).

(iii) For any countable set $\mathcal{U} \subset \mathcal{C}$ there exists an at most countable set $\Xi_{\mathcal{U}} \subset (0,1)^n$ such that for any $\xi \in (0,1)^n \setminus \Xi_{\mathcal{U}}$ and $(A, u) \in \mathcal{U}$ the function $\widetilde{u}^\xi$, defined as in (2.6), satisfies (2.7). Indeed, it is enough to define

$$\Xi_{\mathcal{U}} := \bigcup_{(A,u)\in\mathcal{U}} \Xi^{(A,u)}.$$

We introduce a topology in $\mathcal{C}$ as follows.

Definition 2.2. We say that a sequence $\{(A_k, u_k)\}$ converges to $(A, u) \in \mathcal{C}$ in the $\tau_{\mathcal{C}}$-topology (or shortly $\tau_{\mathcal{C}}$-converges) and denote as $(A_k, u_k) \xrightarrow{\tau_{\mathcal{C}}} (A, u)$ if

- $A_k \to A$ in $L^1(\mathbb{R}^n)$,
- $u_k \to u$ a.e. in $\Omega \cup S$.

2.2. **Main results.** Unless otherwise stated, throughout the paper the parameters Ω, S, φ, β, $\mathbb{C}$ of SDRI energy and volume constant $\mathtt{v}$ are assumed to satisfy the following:

(H0) Ω and S are bounded Lipschitz open sets, S has finitely many connected components, $\Sigma := \partial\Omega \cap \partial S$ is a Lipschitz $(n-1)$-manifold;

(H1) $\varphi \in C^0(\overline{\Omega} \times \mathbb{R}^n)$ and is a Finsler norm, i.e., there exist $b_2 \ge b_1 > 0$ such that for every $x \in \overline{\Omega}$, $\varphi(x, \cdot)$ is a norm in $\mathbb{R}^n$ satisfying

$$b_1|\xi| \le \varphi(x,\xi) \le b_2|\xi|, \qquad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n; \tag{2.8}$$

(H2) $\beta \in L^\infty(\Sigma)$ and satisfies

$$-\varphi(x,\nu_\Sigma(x)) \le \beta(x) \le \varphi(x,\nu_\Sigma(x)) \qquad \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma; \tag{2.9}$$

(H3) $\mathbb{C} \in L^\infty(\Omega \cup S) \cap C^0(\overline{\Omega})$ and there exists $b_4 \ge b_3 > 0$ such that

$$2b_3\, \mathbf{M} : \mathbf{M} \le \mathbb{C}(x)\mathbf{M} : \mathbf{M} \le 2b_4\, \mathbf{M} : \mathbf{M}, \qquad x \in \Omega \cup S, \quad \mathbf{M} \in \mathbb{M}^{n\times n}_{\mathrm{sym}}; \tag{2.10}$$

(H4) $\mathtt{v} \in (0, |\Omega|]$.

Remark 2.3 (**A priori bounds**). Hypotheses (H1)-(H3) are important to get a priori estimates for energy-equibounded countable families. Indeed, let $\mathcal{U} \subset \mathcal{C}$ be any at most countable family of $\mathcal{C}$ such that

$$M := \sup_{(A,u)\in\mathcal{U}} \mathcal{F}(A,u) < +\infty.$$

Then by (2.8) and (2.9)

$$\mathcal{S}(A,u) \le M \quad \text{and} \quad \mathcal{W}(A,u) \le M + \int_\Sigma |\beta| \mathrm{d}\mathcal{H}^{n-1} \le M + b_2\mathcal{H}^{n-1}(\Sigma).$$

Moreover:

(i) for any $(A,u) \in \mathcal{U}$

$$P(A) + \mathcal{H}^{n-1}(A^{(1)} \cap J_u) \le \frac{M + b_2\mathcal{H}^{n-1}(\Sigma)}{b_1} + P(\Omega)$$

and

$$\int_{A\cup S} |\mathcal{E}u|^2 \mathrm{d}x \le \frac{2M + 2b_2\mathcal{H}^{n-1}(\Sigma)}{b_3} + 3\int_\Omega |\mathcal{E}u_0|^2 \mathrm{d}x;$$

(ii) if $\mathcal{U} \ni (A_k,u_k) \overset{\tau_{\mathcal{C}}}{\to} (A,u)$ for some $(A,u) \in \mathcal{C}$, then[1]

$$\chi_{A_k\cup S}\mathcal{E}u_k \rightharpoonup \chi_{A\cup S}\mathcal{E}u \quad \text{in } L^2(\mathrm{Int}(\Omega \cup S \cup \Sigma)). \tag{2.11}$$

Now we formulate main results of the paper. First we deal with the existence of admissible configurations with minimal energy.

Theorem 2.4 (**Existence of minimizing configurations**). *The minimum problem*

$$\inf_{(A,u)\in\mathcal{C},\ |A|=\mathtt{v}} \mathcal{F}(A,u) \tag{2.12}$$

has a solution. Moreover, there exists $\lambda_0 > 0$ such that $(A,u) \in \mathcal{C}$ is a solution of (2.12) if and only if it solves

$$\inf_{(A,u)\in\mathcal{C}} \mathcal{F}^\lambda(A,u) \tag{2.13}$$

[1]Indeed, let $\Xi_{\mathcal{U}} \subset \mathbb{R}^n$ be the countable set, given by Remark 2.1 (c), and let $\xi \in (0,1)^n \setminus \Xi_{\mathcal{U}}$. Since the values of u_k are not important in $\Omega \setminus A_k$, we may assume $u = u^\xi$, where u^ξ is given as (2.6). Then $u_k \to u\chi_{A\cup S} + \xi\chi_{\Omega\setminus A}$ a.e. and hence, (2.11) follows from [14, Theorem 1.1].

for any $\lambda \ge \lambda_0$, *where*

$$\mathcal{F}^\lambda(A,u) := \mathcal{F}(A,u) + \lambda\big||A| - \mathrm{v}\big|.$$

To prove Theorem 2.4 we will apply direct methods of Calculus of Variations. To this aim we establish the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{F}$ and the τ-compactness of energy-equibounded sequences in $\mathcal{C}$.

Theorem 2.5 (**Lower semicontinuity**). *Assume that the sequence* $\{(A_k,u_k)\} \subset \mathcal{C}$ $\tau_{\mathcal{C}}$-*converges to* $(A,u) \in \mathcal{C}$. *Then*

$$\liminf_{k\to+\infty} \mathcal{F}(A_k,u_k) \ge \mathcal{F}(A,u). \tag{2.14}$$

Theorem 2.6 (**Compactness**). *Let* $\{(A_k,u_k)\} \in \mathcal{C}$ *be such that*

$$M := \sup_k \mathcal{F}(A_k,u_k) < +\infty.$$

Then there exists a subsequence $\{(A_{k_l},u_{k_l})\}$, *a sequence* $\{(B_l,v_l)\} \subset \mathcal{C}$ *and* $(A,u) \in \mathcal{C}$ *such that* $(B_l,v_l) \stackrel{\tau_{\mathcal{C}}}{\to} (A,u)$, $|A_{k_l}\Delta B_l| \to 0$ *and*

$$\liminf_{l\to+\infty} \mathcal{F}(A_{k_l},u_{k_l}) \ge \liminf_{l\to+\infty} \mathcal{F}(B_l,v_l) \ge \mathcal{F}(A,u).$$

Notice that our compactness result is analogous to those in [27, 36]. According to the proof, in general we have $|B_l| \le |A_{k_l}|$, i.e., the volume constraint may not be preserved. Rather, Theorems 2.5 and 2.6 allow to solve the unconstrained minimum problem (2.13), and then, as in [26, Theorem 1], using the equivalence of the minimum problems (2.12) and (2.13) (see Proposition A.1), we establish the existence of a volume-constraint minimizer.

It is worth to remark that in both Theorems 2.5 and 2.6 (and hence, in the existence) the assumption $\mathbb{C} \in C(\overline{\Omega})$ can be relaxed to $\mathbb{C} \in L^\infty(\Omega)$. The continuity of $\mathbb{C}$ is important in the (partial) regularity of minimizers of $\mathcal{F}$.

Theorem 2.7 (**Properties of minimizing configurations)**). *Let* $(\widetilde{A},\widetilde{u}) \in \mathcal{C}$ *be a solution of* (2.12) *and let*

$$A = \mathrm{Int}(\widetilde{A}^{(1)}) \quad \textit{and} \quad u = \widetilde{u}\chi_{A\cup S} + \xi\chi_{\Omega\setminus A},$$

where $\xi \in (0,1)^n$ *is chosen such that* $\Omega \cap \partial^* A \subset_{\mathcal{H}^{n-1}} J_u$ *(see Remark 2.1), and let*

$$J_u^* = \{x \in J_u :\ \theta(J_u,x) = 1\}.$$

Then:

(i) (A,u) *is a minimizer of* $\mathcal{F}$ *and*

$$\mathcal{H}^{n-1}(\widetilde{A}^{(1)} \setminus A) < +\infty, \quad \mathcal{H}^{n-1}(J_u \setminus J_u^*) = 0, \quad \mathcal{H}^{n-1}(\overline{J_u^*} \setminus J_u^*) = 0;$$

(ii) *for any* $x \in \Omega$ *and* $r \in (0, \min\{1, \mathrm{dist}(x,\partial\Omega)\})$

$$\frac{\mathcal{H}^{n-1}(Q_r(x) \cap J_u}{r^{n-1})} \le \frac{4nb_2 + \lambda_0}{b_1},$$

where λ_0 *is given by Theorem 2.4;*

(iii) *there exist* $\varsigma_0 = \varsigma_0(b_1,b_2,b_3,b_4) \in (0,1)$ *and* $R_0 = R_0(b_1,b_2,b_3,b_4) > 0$ *such that*

$$\frac{\mathcal{H}^{n-1}(Q_r(x) \cap J_u)}{r^{n-1}} \ge \varsigma_0$$

for all cubes $Q_r(x) \subset \Omega$ *centered at* $x \in \Omega \cap \overline{J_u^*}$ *with sidelength* $r \in (0,R_0)$;

(iv) *if* $E \subset A$ *is any connected component of* A *with* $\mathcal{H}^{n-1}([\partial^* E \cap \Sigma] \setminus J_u) = 0$, *then* $|E| \ge \omega_n\big(\frac{b_1 n}{\lambda_0}\big)^n$ *and* $u = u_0 + a$ *in* E *for some rigid displacement* a.

2.3. **Generalization and extra results related to Literature models.** In this section we discuss some models related to the SDRI model for which the proofs of the main results above can be adapted, by also recovering as a byproduct of our analysis some results already available in the Literature.

First we consider more general elastic energy densities.

Theorem 2.8 (**Elastic density with p-growth**). *For $p > 1$ let a measurable function $W_p : \mathrm{Int}(\Omega \cup S \cup \Sigma) \times \mathbb{M}^{n\times n}_{\rm sym} \to \mathbb{R}$ be such that*

(a1) *for any $x \in \mathrm{Int}(\Omega \cup S \cup \Sigma)$, $W_p(x,\cdot)$ is convex and there exist $c > 0$ and $f \in L^1(\mathrm{Int}(\Omega \cup S \cup \Sigma))$ such that*

$$W_p(x,\mathbf{M}) \ge c|\mathbf{M}|^p + f(x) \quad \text{for a.e. } x \in \mathrm{Int}(\Omega \cup S \cup \Sigma) \text{ and for all } \mathbf{M} \in \mathbb{M}^{n\times n}_{\rm sym}; \tag{2.15}$$

(a2) *for any $u \in GSBD^p(\mathrm{Int}(\Omega \cup S \cup \Sigma))$ the map $x \mapsto W_p(x,\mathcal{E}u(x))$ belongs to $L^1(\mathrm{Int}(\Omega \cup S \cup \Sigma))$.*

Let

$$\mathcal{C}_p := \{(A,u) :\ A \in BV(\Omega;\{0,1\}),\ u \in GSBD^p(\mathrm{Int}(\Omega \cup S \cup \Sigma))\}$$

be a class of admissible configurations and let

$$\mathcal{F}_p = \mathcal{S} + \mathcal{W}_p \qquad \text{in } \mathcal{C}_p,$$

where

$$\mathcal{W}_p(A,u) = \int_{A\cup S} W_p(x,\mathcal{E}u - \mathbf{M}_0)\,\mathrm{d}x.$$

Then for any $\mathtt{v} \in (0,|\Omega|]$ the minimum problem

$$\min_{(A,u)\in\mathcal{C}_p,\ |A|=\mathtt{v}} \mathcal{F}_p(A,u) \tag{2.16}$$

admits a solution. Moreover, there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ a configuration (A,u) is a solution to (2.16) if and only if it is a minimizer of

$$\mathcal{F}_p^\lambda(A,u) = \mathcal{F}(A,u) + \lambda\big||A| - \mathtt{v}\big|.$$

A standard example of W_p is

$$W_p(x,\mathbf{M}) = f(x)|\mathbf{M}|^p + g(x)$$

for some $f \in L^\infty(\mathrm{Int}(\Omega \cup S \cup \Sigma))$ with $f \ge c > 0$ a.e. and $g \in L^1(\mathrm{Int}(\Omega \cup S \cup \Sigma))$.

Now we study the existence of minimizers in models related to the SDRI setting, but with Dirichlet boundary conditions.

Theorem 2.9 (**Dirichlet case with a p-growth elastic density**). *For $p > 1$ let*

$$\mathcal{C}_{\rm Dir} := \{(A,u) :\ A \in BV(\Omega;\{0,1\}),\ u \in GSBD^p(\mathrm{Int}(\Omega \cup S \cup \Sigma)),\ u = u_0 \text{ in } S\},$$

where $u_0 \in H^1(\mathbb{R}^n)$ is fixed, and let

$$\mathcal{F}_{\rm Dir} := \mathcal{S} + \mathcal{W}_{\rm Dir} \qquad \text{in } \mathcal{C}_{\rm Dir},$$

where

$$\mathcal{W}_{\rm Dir}(A,u) := \int_A W_p(x,\mathcal{E}u)\,\mathrm{d}x$$

and the elastic energy density W_p satisfies all assumptions of Theorem 2.8. Then for any $\mathtt{v} \in (0,|\Omega|]$ the minimum problem

$$\min_{(A,u)\in\mathcal{C}_{\rm Dir},\ |A|=\mathtt{v}} \mathcal{F}_{\rm Dir}(A,u) \tag{2.17}$$

admits a solution. Moreover, there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ a configuration (A,u) is a solution to (2.17) and only if it is a minimizer of

$$\mathcal{F}^{\lambda}_{\mathrm{Dir}}(A,u) = \mathcal{F}_{\mathrm{Dir}}(A,u) + \lambda\big||A| - \mathtt{v}\big|.$$

Remark 2.10 (**Relation to some Literature results**)**.** As a consequence of Theorem 2.9 we have:

(i) Let $\beta(x) = -\varphi(x, \nu_\Sigma(x))$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ and let $W_p : \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfy
$$c'|\mathbf{M}|^p - c'' \le W_p(\mathbf{M}) \le c''(|\mathbf{M}|^p + 1)$$
for some $c'', c' > 0$. Then Theorem 2.9 coincides with the existence result [17, Proposition 5.8] in the setting of material voids.

(ii) Let $\beta = 0$ and W_p be as in (i). Then the minimizers of $\mathcal{F}_{\mathrm{Dir}}$ in $\mathcal{C}_{\mathrm{Dir}}$ with volume constraint $|\mathtt{v}| = |\Omega|$ (i.e., free-crystal regions have full $\mathcal{L}^n$-measure) coincide with the (strong) Griffith minimizers in [13] under Dirichlet boundary condition.

(iii) In the proof of Theorem 2.7 we work only in Ω, i.e., we study the regularity of $\partial^* A$ and J_u only in the points of Ω. Therefore, the assertion on the essential closedness of J_u and ∂^*A holds also for minimizers of $\mathcal{F}_{\mathrm{Dir}}$ with $W_p(x,\mathbf{M}) = \mathbb{C}(x)\mathbf{M} : \mathbf{M}$. In particular, this covers a partial regularity part of results in [13].

We anticipate here that we equip both $\mathcal{C}_p$ and $\mathcal{C}_{\mathrm{Dir}}$ with the same type of convergence introduced in $\mathcal{C}$, i.e.

$$(A_k,u_k) \xrightarrow{\tau} (A,u) \quad\Longleftrightarrow\quad A_k \xrightarrow{L^1(\mathbb{R}^n)} A \text{ and } u_k \to u \text{ a.e. in } \Omega\cup S. \tag{2.18}$$

3. Replacing cracks with voids

In this section we provide some technical results that allow to replace a portion of the jump set of the displacement fields with an open set without modifying too much the corresponding SDRI energy. These results will be used in both the lower-semicontinuity and the compactness results. We start with the following main ingredient of all crack-opening results.

Lemma 3.1. *Let $\delta \in (0,1/4)$, $Q := Q_{r,\nu}(x_0)$ be a cube, $\Gamma \subset Q$ is an $(n-1)$-dimensional Lipschitz graph and $K \subset Q$ be an $\mathcal{H}^{n-1}$-rectifiable set. Assume that*

(a1) *$x_0 \in \Gamma$, ν is the unit normal to Γ at x_0 and $|(x-x_0)\cdot\nu| \le r/2$ for all $x \in \Gamma$;*
(a2) *Γ separates Q into two open connected components G_1 and G_2;*
(a3) *$\theta(K,x_0) = \theta(K\cap\Gamma, x_0) = 1$, ν is the generalized unit normal to K at x_0, and*
$$(1-\delta)r^{n-1} \le \mathcal{H}^{n-1}(K\cap\Gamma) \le \mathcal{H}^{n-1}(\Gamma) \le (1+\delta)r^{n-1};$$
(a4) *$\mathcal{H}^{n-1}(K\setminus\Gamma) < \delta r^{n-1}$.*

Then there exist open sets $C, D \subset\subset Q$ of finite perimeter such that

(i) *$C \subset G_1$, and $\mathcal{H}^{n-1}(\partial C\setminus\partial^* C) = \mathcal{H}^{n-1}(\partial D\setminus\partial^*D) = 0$;*
(ii) *$\mathcal{H}^{n-1}(K\setminus\overline{C}) < 2\delta r^{n-1}$ and $\mathcal{H}^{n-1}(K\setminus D) < 2\delta r^{n-1}$;*
(iii) *$|C| < \delta r^n$ and $|D| < \delta r^n$;*
(iv) *$(1-2\delta)r^{n-1} \le \mathcal{H}^{n-1}(K\cap\partial C\cap\Gamma) \le \mathcal{H}^{n-1}(\partial C\cap\Gamma) < (1+\delta)r^{n-1}$;*
(v) *for any norm ϕ in $\mathbb{R}^n$ satisfying (4.1) one has*
$$\int_{\partial D}\phi(\nu_D)\mathrm{d}\mathcal{H}^{n-1} \le 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} + 5b_2\delta r^{n-1}. \tag{3.1}$$

$$\int_{\partial C} \phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} \le 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} + 5b_2\delta r^{n-1} \tag{3.2}$$

and

$$\int_{G_1\cap\partial C} \phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} \le \int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} + 3b_2\delta r^{n-1}, \tag{3.3}$$

Proof. Without loss of generality we assume that $\nu = \mathbf{e}_n$, $x_0 = 0$ and G_1 lies above Γ. Since Γ is a Lipschitz graph, $f \in \mathrm{Lip}(V)$ such that $\Gamma = \mathrm{graph}(f)$, where $V = [-\frac{r}{2}, \frac{r}{2}]^{n-1} \subset \mathbb{R}^{n-1}$. By (a1), $\|f\|_\infty \le r/2$, hence, Γ intersects only lateral sides of Q. Let

$$\epsilon := \frac{\delta}{4(1+\mathrm{Lip}(f))}.$$

Let $V'' \subset\subset V' \subset\subset V$ be any $(n-1)$-dimensional cubes in $\mathbb{R}^{n-1}$ such that

$$\mathcal{H}^{n-1}(V \setminus V'') < \epsilon r^{n-1}. \tag{3.4}$$

For $\gamma \in (0, \epsilon r)$ let $g \in \mathrm{Lip}_c(V; [0,\gamma])$ be such that $g \equiv \gamma$ in V'', $\mathrm{supp}\,(g) = \overline{V'}$ and $\|g\|_\infty \le 1$. Let C be the open set bounded between the graphs of f and $f+g$ and let D be the open set bounded between the graphs of $f+g$ and $f-g$. Since both ∂C and ∂D consists of two Lipschitz graphs, it is a set of finite perimeter.

We claim that C and D satisfy the assertion of the lemma.

(i) Since $\|f \pm g\|_\infty < 3r/4$ (by (a1) and choice of γ) and $g = 0$ on $V \setminus V'$, $C \subset G_1$ and $C, D \subset\subset Q_r$. Moreover, since V' is an $(n-1)$-dimensional hypercube, by the area formula

$$\mathcal{H}^{n-1}(\partial C \setminus \partial^* C) = \mathcal{H}^{n-1}(\partial D \setminus \partial^* D) \le (1+\mathrm{Lip}(f))\mathcal{H}^{n-1}(\overline{V'} \setminus V') = 0.$$

(i) By (a4)

$$\mathcal{H}^{n-1}(K \setminus \overline{C}) \le \mathcal{H}^{n-1}(\Gamma \cap K \setminus \overline{C}) + \mathcal{H}^{n-1}(K \setminus \Gamma) < \mathcal{H}^{n-1}(\Gamma \setminus \overline{C}) + \delta r^{n-1}.$$

Moreover, by contruction

$$\Gamma \setminus \overline{C} = \Gamma \setminus \partial C = \Gamma \setminus \overline{D} = f(V \setminus \overline{V'}),$$

and hence, by the area formula and (3.4)

$$\mathcal{H}^{n-1}(\Gamma \setminus \overline{C}) \le \int_{V\setminus V'} \sqrt{1+|\nabla f|^2}\mathrm{d}x' \le (1+\mathrm{Lip}(f))\mathcal{H}^{n-1}(V \setminus V') < \frac{\delta}{4}\, r^{n-1}. \tag{3.5}$$

Thus, $\mathcal{H}^{n-1}(K \setminus C) < \frac{5}{4}\,\delta r^{n-1}$. Similarly, $\mathcal{H}^{n-1}(\Gamma \setminus \overline{D}) = \mathcal{H}^{n-1}(\Gamma \setminus D) < \frac{1}{4}\delta r^{n-1}$.

(iii) By the Fubini's theorem, the choice of γ and also the area formula

$$|C| = \int_{V'} (f+g-f)\mathrm{d}x \le \gamma\mathcal{H}^{n-1}(V') < \epsilon r \int_V \sqrt{1+|\nabla f|^2}\mathrm{d}x' = \epsilon r\mathcal{H}^{n-1}(\Gamma)$$

and

$$|D| = \int_{V'} (f+g-(f-g))\mathrm{d}x \le 2\gamma\mathcal{H}^{n-1}(V') < 2\epsilon r \int_V \sqrt{1+|\nabla f|^2}\mathrm{d}x' = 2\epsilon r\mathcal{H}^{n-1}(\Gamma)$$

Hence, by (a3) $|C| < \frac{\delta(1+\delta)}{4} r^n$ and $|C| < \frac{\delta(1+\delta)}{2} r^n$.

(iv) By (a3)

$$\mathcal{H}^{n-1}(\partial C \cap \Gamma) < \mathcal{H}^{n-1}(\Gamma) \le (1+\delta) r^{n-1}.$$

Moreover, by (3.5)

$$\mathcal{H}^{n-1}(K\cap\Gamma) - \mathcal{H}^{n-1}(K\cap\partial C\cap\Gamma) = \mathcal{H}^{n-1}(K\cap\Gamma\setminus\partial C) \le \mathcal{H}^{n-1}(\Gamma\setminus\partial C) = \mathcal{H}^{n-1}(\Gamma\setminus C) < \frac{\delta}{4} r^{n-1}.$$

Hence, by (a3)

$$\mathcal{H}^{n-1}(K\cap\partial C\cap\Gamma)\geq\mathcal{H}^{n-1}(K\cap\Gamma)-\frac{\delta}{4}r^{n-1}>(1-\tfrac{5}{4}\delta)r^{n-1}.$$

(v) By the definition of C, the area formula, the convexity of ϕ, the definition of g, (4.1) and (3.4)

$$\begin{aligned}\int_{G_1\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}&=\int_{G_1\cap\mathrm{graph}(f+g)}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}=\int_{V'}\phi(-\nabla(f+g),1)\mathrm{d}\mathcal{H}^{n-1}\\&\leq\int_{V'}\phi(-\nabla f,1)\mathrm{d}\mathcal{H}^{n-1}+\int_{V'}\phi(-\nabla g,0)\mathrm{d}\mathcal{H}^{n-1}\\&\leq\int_{V}\phi(-\nabla f,1)\mathrm{d}\mathcal{H}^{n-1}+\int_{V'\setminus V''}\phi(-\nabla g,0)\mathrm{d}\mathcal{H}^{n-1}\\&\leq\int_{\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}+b_2\|g\|_\infty\mathcal{H}^{n-1}(V'\setminus V'')\\&\leq\int_{\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}+\frac{b_2\delta}{4}\,r^{n-1}.\end{aligned}$$

Moreover, by (a3)

$$\mathcal{H}^{n-1}(\Gamma\setminus K)=\mathcal{H}^{n-1}(\Gamma)-\mathcal{H}^{n-1}(\Gamma\cap K)\leq 2\delta r^{n-1},$$

and hence, by (4.1)

$$\int_\Gamma\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}\leq\int_{K\cap\Gamma}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+b_2\mathcal{H}^{n-1}(\Gamma\setminus K)\leq\int_K\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+\frac{9b_2}{4}\,\delta r^{n-1}.\quad(3.6)$$

Thus, (3.3) follows. Since $\partial C\cap\partial G_1=\Gamma$, the proof of 3.2 follows from (3.6) and (3.3). Similarly,

$$\begin{aligned}\int_{\partial D}\phi(\nu_D)\mathrm{d}\mathcal{H}^{n-1}=&\int_{V'}\Big[\phi(-\nabla(f+g),1)\mathrm{d}\mathcal{H}^{n-1}+\phi(-\nabla(f-g),1)\Big]\mathrm{d}\mathcal{H}^{n-1}\\\leq&2\int_{V'}\phi(-\nabla f,1)\mathrm{d}\mathcal{H}^{n-1}+2\int_{V'}\phi(-\nabla g,0)\mathrm{d}\mathcal{H}^{n-1}\\\leq&2\int_\Gamma\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}+\frac{b_2}{2}\,\delta r^{n-1}\\\leq&2\int_K\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+\frac{9b_2}{2}\,\delta r^{n-1}.\end{aligned}$$

□

The following result will be used in the proof of Proposition 4.1 with $K=A_k^{(1)}\cap J_{u_k}$ and allows to replace u_k with v_k, whose jump set is a reduced boundary of an open set of finite perimeter (see Corollary 3.3 below). Recall that this property is important to obtain the surface tension 2φ in the "interior" jump energy in the functional $\mathcal{S}$.

Lemma 3.2. *Let $U\subset\mathbb{R}^n$ be an open set, $K\subset U$ be a $\mathcal{H}^{n-1}$-rectifiable set and $\delta>0$. There exists an at most countable family $\{C_i\}_{i\geq1}$ of open sets of finite perimeter such that*

(i) $C_i\subset\subset U$ *and* $\mathcal{H}^{n-1}(\partial C_i\setminus\partial^*C_i)=0$;
(ii) $\mathcal{H}^{n-1}(K\setminus\bigcup_i C_i)<\delta$ *and* $|\bigcup_i C_i|<\delta$;

(iii) *for any norm ϕ in $\mathbb{R}^n$ satisfying* (4.1)

$$\sum_{i\ge1}\int_{\partial C_i}\phi(\nu_{C_i})\,\mathrm{d}\mathcal{H}^{n-1} < 2\int_K \phi(\nu_K)\,\mathrm{d}\mathcal{H}^{n-1}+\delta.$$

Proof. First we consider a special case.

Claim. Let $K=\mathrm{graph}(f)$ for some $f\in\mathrm{Lip}(V)$, where $V\subset\mathbb{R}^{n-1}$ is a bounded open set. Let $V''\subset\subset V'\subset\subset V$ be smooth open sets such that

$$\Big(1+\frac{1}{b_1}\Big)\int_{V\setminus V''}\phi(-\nabla f,1)\mathrm{d}x' + \mathcal{H}^{n-1}(V\setminus V'') < \frac{\delta}{2+2b_2}. \tag{3.7}$$

For $\gamma\in\big(0,\frac{\delta}{4[1+\mathcal{H}^{n-1}(V')]}\big)$ let $g\in\mathrm{Lip}(V;[0,\gamma])$ be such that $\mathrm{supp}\,(g)=\overline{V'}$, $g\equiv\gamma$ in V'' and $\|\nabla g\|_{L^\infty(V)}\le 1$. Then $g=0$ on $\partial V'$. Moreover, taking γ small enough we assume that the graphs of $f\pm g\big|_{V'}$ are compactly contained in U. Let C be the bounded open set whose boundary consists of the graphs of $f-g:V'\to\mathbb{R}$ and $f+g:V'\to\mathbb{R}$. Then $C\subset\subset U$ and by the area formula, triangle inequality for ϕ, (4.1), (3.7) and the inequality $\|\nabla g\|_\infty\le 1$

$$\begin{aligned}\int_{\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} =& \int_{V'}\Big(\phi(-\nabla(f+g),1)+\phi(-\nabla(f-g),1)\Big)\mathrm{d}x'\\ \le& 2\int_{V'}\phi(-\nabla f,1)\mathrm{d}x' + 2\int_{V'}\phi(-\nabla g,0)\mathrm{d}x'\\ \le& 2\int_{V''}\phi(-\nabla f,1)\mathrm{d}x' + 2\int_{V'\setminus V''}\phi(-\nabla f,1)\mathrm{d}x' + 2\int_{V'\setminus V''}\phi(\nabla g,0)\mathrm{d}x'\\ \le& 2\int_{V'}\phi(-\nabla f,1)\mathrm{d}x' + \delta = 2\int_K\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+\delta.\end{aligned}$$

Moreover, by (4.1) and (3.7)

$$\mathcal{H}^{n-1}(K\setminus C) = \int_{V\setminus V'}\sqrt{1+|\nabla f|^2}\mathrm{d}x' \le \frac{1}{b_1}\int_{V\setminus V''}\phi(-\nabla f,1)\mathrm{d}x' < \delta.$$

Finally since $0\le|g|\le\frac{\delta}{4[1+\mathcal{H}^{n-1}(V')]}$ it follows that

$$|C| = \int_{W'}[f+g-(f-g)]\mathrm{d}x' \le 2\|g\|_\infty\mathcal{H}^{n-1}(W')<\delta.$$

The equality $\mathcal{H}^{n-1}(\partial C\setminus\partial^* C)=0$ follows from the smoothness of V'.

Now we prove the lemma. By the countable $\mathcal{H}^{n-1}$-rectifiability of K there exists an at most countable family $\{\Gamma_i\}$ of Lipschitz graphs such that $\Gamma_i\subset U$, $\Gamma_i\cap\Gamma_j=\emptyset$ for $i\ne j$, and $\mathcal{H}^{n-1}(K\setminus\bigcup_i\Gamma_i)=0$. Since $\mathcal{H}^{n-1}\llcorner\Gamma_i$ is Radon, by the regularity of Radon measures for each i there exists a relatively open subset Γ_i' of Γ_i such that $\Gamma_i'\cap K\subset\Gamma_i\cap K$ and

$$\mathcal{H}^{n-1}(\Gamma_i'\setminus K) < \frac{\delta}{2^{i+2}(1+b_2)},\quad i\ge1. \tag{3.8}$$

For shortness, we assume $\Gamma_i=\Gamma_i'$. Then applying the claim above with $\delta:=\frac{\delta}{2^{i+1}(1+b_2)}$ and $\Gamma=\Gamma_i$ we find an open set $C_i\subset\subset U$ such that

$$|C_i|<\frac{\delta}{2^{i+1}(1+b_2)},\qquad \mathcal{H}^{n-1}(\Gamma_i\setminus C_i)<\frac{\delta}{2^{i+1}(1+b_2)} \tag{3.9}$$

and

$$\int_{\partial C_i} \phi(\nu_{C_i})\mathrm{d}\mathcal{H}^{n-1} \le 2\int_{\Gamma_i} \phi(\nu_{\Gamma_i})\mathrm{d}\mathcal{H}^{n-1} + \frac{\delta}{2^{i+1}(1+b_2)}. \tag{3.10}$$

Thus, by the pairwise disjointness of $\{\Gamma_i\}$

$$\begin{aligned}\mathcal{H}^{n-1}\Big(K\setminus\bigcup_j C_j\Big) \le& \mathcal{H}^{n-1}\Big(\bigcup_i\Big(\Gamma_i\setminus\bigcup_j C_j\Big)\Big) = \sum_i \mathcal{H}^{n-1}\Big(\Gamma_i\setminus\bigcup_j C_j\Big)\\ &\le \sum_i \mathcal{H}^{n-1}(\Gamma_i\setminus C_i) < \delta\end{aligned}$$

and by (3.8) and (3.10),

$$\begin{aligned}\sum_i\int_{\partial C_i}\phi(\nu_{C_i})\mathrm{d}\mathcal{H}^{n-1} \le& 2\sum_i\int_{\Gamma_i}\phi(\nu_{\Gamma_i})\mathrm{d}\mathcal{H}^{n-1}+\frac{\delta}{2}\\ \le& 2\sum_i\int_{\Gamma_i\cap K}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+2\sum_i\int_{\Gamma_i\setminus K}\phi(\nu_{\Gamma_i})\mathrm{d}\mathcal{H}^{n-1}+\frac{\delta}{2}\\ \le& 2\int_{\cup_i\Gamma_i\cap K}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+\sum_i\frac{2b_2\delta}{2^{i+2}(1+b_2)}+\frac{\delta}{2}\\ =& 2\int_K\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}+\delta.\end{aligned}$$

Finally, by the estimate for $|C_i|$ in (3.9)

$$\Big|\bigcup_i C_i\Big| \le \sum_i |C_i| < \delta.$$

□

Corollary 3.3. *Let $U \subset\subset \Omega$ be an open set, $(A,u)\in\mathcal{C}$ and $\delta>0$. Then there exists an open set $G\subset\subset U$ of finite perimeter such that*

(i) *the configuration (B,v) with $B:=A\setminus G$ and $v:=u\chi_{B\cup S}$ belongs to $\mathcal{C}$;*
(ii) $|G|<\delta$;
(iii) $\mathcal{H}^{n-1}(U\cap B^{(1)}\cap J_v)<\delta$;
(iv) *for any norm ϕ in $\mathbb{R}^n$ satisfying (4.1)*

$$\int_{U\cap\partial^*A}\phi(\nu_A)\mathrm{d}\mathcal{H}^{n-1}+2\int_{U\cap A^{(1)}\cap J_u}\phi(\nu_{J_u})\mathrm{d}\mathcal{H}^{n-1} \ge \int_{U\cap\partial^*B}\phi(\nu_B)\mathrm{d}\mathcal{H}^{n-1}-\delta.$$

Proof. Let $\epsilon := \frac{\delta}{8}$. Since $\mathcal{H}^{n-1}(U\cap A^{(1)}\cap J_u)<+\infty$, there exists an open set $U'\subset\subset U$ such that

$$\mathcal{H}^{n-1}((U\setminus U')\cap A^{(1)}\cap J_u)<\epsilon. \tag{3.11}$$

By Lemma 3.2 applied with U', $K:=U'\cap A^{(1)}\cap J_u$ and ϵ we find an at most countable family $\{C_i\}_{i\ge1}$ of open sets of finite perimeter such that

(a_1) $C_i\subset\subset U'$ and $\mathcal{H}^{n-1}(\partial C_i\setminus\partial^*C_i)=0$;
(a_2) $\mathcal{H}^{n-1}([U'\cap K]\setminus\bigcup_i C_i)<\epsilon$ and $|\bigcup_i C_i|<\epsilon$;
(a_3)

$$\sum_{i\ge1}\int_{\partial C_i}\phi(\nu_{C_i})\,\mathrm{d}\mathcal{H}^{n-1} < 2\int_{U'\cap K}\phi(\nu_K)\,\mathrm{d}\mathcal{H}^{n-1}+\epsilon.$$

Define

$$G := \bigcup_{i\ge1} C_i.$$

We claim that G satisfies the assertion of the lemma. Indeed, (i) is obvious and (ii) follows from (a_2). By construction, $B^{(1)}\cap J_v =_{\mathcal{H}^{n-1}} B^{(1)}\cap J_u$ and hence, by (3.11) and (a_2) we have

$$\mathcal{H}^{n-1}(U\cap B^{(1)}\cap J_v) \le \mathcal{H}^{n-1}((U\setminus U')\cap A^{(1)}\cap J_u) + \mathcal{H}^{n-1}(U'\cap A^{(1)}\cap J_u\setminus G) < 2\epsilon.$$

Finally, since $\partial^*B\setminus\partial^*A\subset\partial^*G$, by (a_3)

$$\begin{aligned}\int_{U\cap\partial^*B}\phi(\nu_B)\mathrm{d}\mathcal{H}^{n-1} &= \int_{U\cap\partial^*B\cap\partial^*A}\phi(\nu_B)\mathrm{d}\mathcal{H}^{n-1} + \int_{U\cap\partial^*B\setminus\partial^*A}\phi(\nu_B)\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{U\cap\partial^*A}\phi(\nu_A)\mathrm{d}\mathcal{H}^{n-1} + \sum_{i\ge1}\int_{\partial C_i}\phi(\nu_{C_i})\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{U\cap\partial^*A}\phi(\nu_A)\mathrm{d}\mathcal{H}^{n-1} + 2\int_{U'\cap K}\phi(\nu_{J_u})\,\mathrm{d}\mathcal{H}^{n-1} + \epsilon.\end{aligned}$$

□

The next lemma is a counterpart of Lemma 3.2 and relates to the "opening" of cracks along Σ. Notice that in this case the opening should not get out from Ω. Thus, we are replacing the jump of u only from one side (Corollary 3.5) and this is the reason for having φ (without factor 2) in the jump energy along Σ in the functional $\mathcal{S}$.

Lemma 3.4. *Let $U\subset\subset \mathrm{Int}(\Omega\cup S\cup\Sigma)$ be an open set, $\delta\in(0,1)$ and $K\subset U\cap\Sigma$ be any $\mathcal{H}^{n-1}$-measurable set. Then there exist an open set $C\subset U\cap\Omega$ of finite perimeter such that*

(i) *$C\subset\subset U$ and $\mathcal{H}^{n-1}(\partial C\setminus\partial^*C)=0$;*
(ii) *$\mathcal{H}^{n-1}(K\setminus\partial C)=\mathcal{H}^{n-1}(K\setminus\overline{C})<\delta$ and $|C|<\delta$;*
(iii) *$\mathcal{H}^{n-1}(U\cap\Sigma\cap\partial C\setminus K)<\delta$;*
(iv) *for any norm ϕ in $\mathbb{R}^n$ satisfying (4.1)*

$$\int_{\Omega\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}\le\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+\delta,$$

and

$$\int_{\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}\le 2\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+\delta.$$

Proof. Let

$$\epsilon := \frac{\delta}{8(1+b_2)(1+\mathcal{H}^{n-1}(\Sigma))}.$$

We divide the proof into two steps.

Step 1. Let $Q_r(x_0)\subset U$ be a cube centered at $x\in\Sigma$ such that $\Sigma\cap Q_r(x_0)=\mathrm{graph}(f)$ for some Lipschitz function $f:V\to\mathbb{R}$ and a cube $V\subset\mathbb{R}^{n-1}$, and assume that $S\cap Q_r(x_0)$ is a subgraph of f. Let $V''\subset\subset V'\subset\subset V$ be open sets such that

$$\mathcal{H}^{n-1}(V\setminus V'') < \frac{\mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)}{1+\mathrm{Lip}(f)}\,\epsilon$$

and for $\gamma\in(0,\frac{\mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)}{1+\mathcal{H}^{n-1}(V)}\,\epsilon)$ let $g\in\mathrm{Lip}_c(V;[0,\gamma])$ be such that $g\equiv1$ in V'', $\mathrm{supp}\,(g)=V'$ and $\mathrm{Lip}(g)<1$. We may assume that γ is so small that the set open set C, whose boundary lies on the graphs of f and $f+g$, is compactly contained in $Q_r(x_0)$ and $C\cap S=\emptyset$. Then

$$\begin{aligned}\int_{\Omega\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} &= \int_{V'}\phi(-\nabla(f+g),1)\mathrm{d}\mathcal{H}^{n-1} \le \int_{V'}\Big(\phi(-\nabla f,1)+\phi(-\nabla g,0)\Big)\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{V}\phi(-\nabla f,1)\mathrm{d}\mathcal{H}^{n-1} + b_2\mathrm{Lip}(f)\mathcal{H}^{n-1}(V'\setminus V'')\\ &< \int_{Q_r(x_0)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + b_2\mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)\epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} &= \int_{V'}[\phi(-\nabla(f+g),1)+\phi(-\nabla f,1)]\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{V'}\Big(2\phi(-\nabla f,1)+\phi(-\nabla g,0)\Big)\mathrm{d}\mathcal{H}^{n-1}\\ &< 2\int_{Q_r(x_0)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + b_2\mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)\epsilon.\end{aligned}$$

Also by the Fubini's theorem

$$|C| = \int_{V'} g\mathrm{d}x' \le \gamma\mathcal{H}^{n-1}(V') < \mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)\,\epsilon.$$

Finally,

$$\begin{aligned}\mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma\setminus\partial C) =& \mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma\setminus\overline{C}) = \int_{V\setminus V'}\sqrt{1+|\nabla f|^2}\mathrm{d}\mathcal{H}^{n-1}\\ \le& (1+\mathrm{Lip}(f))\mathcal{H}^{n-1}(V\setminus V') < \mathcal{H}^{n-1}(Q_r(x_0)\cap\Sigma)\,\epsilon.\end{aligned}$$

Step 2. Since Σ is Lipschitz and K is $\mathcal{H}^{n-1}$-rectifiable, we can find a finite family $Q_{r_1,\nu_1}(x_1),\ldots,Q_{r_m,\nu_m}(x_m)\subset U$ of pairwise disjoint cubes centered at K such that

- (a_1) for each j, $\Sigma\cap Q_{r_j,\nu_j}(x_j)$ is a graph of a Lipschitz function in ν_j direction;
- (a_2) $\theta(K,x_j)=\theta(\Sigma,x_j)=1$, and the unit normals $\nu_K(x_j)$ and $\nu_\Sigma(x_j)$ exist and coincide with ν_j;
- (a_3) $(1-\epsilon)r_j^{n-1} < \mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\cap K) \le \mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma) < (1+\epsilon)r_j^{n-1}$;
- (a_4) $\mathcal{H}^{n-1}\Big(K\setminus\bigcup_{j=1}^m Q_{r_j,\nu_j}(x_j)\Big) < \epsilon$.

Note that by (a_3)

$$\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\setminus K) < 2\epsilon r_j^{n-1} < \frac{2\epsilon}{1-\epsilon}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma). \tag{3.12}$$

By Step 1 for each j we can contruct an open set $C_j\subset\subset Q_{r_j,\nu_j}(x_j)$ with $C_j\cap S=\emptyset$ and

$$\int_{\Omega\cap\partial C_j}\phi(\nu_{C_j})\mathrm{d}\mathcal{H}^{n-1} < \int_{Q_{r_j,\nu_j}(x_j)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + b_2\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma)\epsilon \tag{3.13}$$

and

$$\int_{\partial C_j}\phi(\nu_{C_j})\mathrm{d}\mathcal{H}^{n-1} < 2\int_{Q_{r_j,\nu_j}(x_j)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + b_2\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma)\epsilon.$$

Moreover,

$$|C_j| < \mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma)\epsilon \tag{3.14}$$

and

$$\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\setminus\partial C_j)<\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma)\,\epsilon. \tag{3.15}$$

We claim that $C=\bigcup_{i=1}^m C_j$ satisfies all assertions of the lemma.

(i) By construction $C\subset\subset U$ and since each C_j is almost Lipschitz, $\mathcal{H}^{n-1}(\partial C_j\setminus\partial^* C_j)=0$. Hence, by the pairwise disjointness of $\overline{C_j}$, $\mathcal{H}^{n-1}(\partial C\setminus\partial^* C)=0$ and (i) follows.

(ii) By (a$_4$) and (3.15)

$$\begin{aligned}\mathcal{H}^{n-1}(K\setminus\partial C)\le&\mathcal{H}^{n-1}\Big(K\setminus\bigcup_{j=1}^m Q_{r_j,\nu_j}(x_j)\Big)+\sum_{j=1}^m\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\setminus\partial C_j)\\ <&\epsilon+\sum_{j=1}^m\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma)\,\epsilon\le(1+\mathcal{H}^{n-1}(\Sigma))\epsilon<\delta.\end{aligned}$$

Moreover, by (3.14)

$$|C|\le\sum_{j=1}^n|C_j|\le\mathcal{H}^{n-1}(\Sigma)\epsilon<\delta.$$

(iii) By (3.12)

$$\begin{aligned}\mathcal{H}^{n-1}(U\cap\Sigma\cap\partial C\setminus K)=&\sum_{j=1}^m\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\cap\partial C_j\setminus K)\\ \le&\sum_{j=1}^m\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\setminus K)<\frac{2\epsilon}{1-\epsilon}\mathcal{H}^{n-1}(\Sigma)<\delta.\end{aligned}$$

(iv) Since $\partial^*C\subset\cup_j\partial^*C_j$, by (3.13) we get

$$\int_{\Omega\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}\le\sum_{j=1}^m\int_{Q_{r_j,\nu_j}(x_j)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+b_2\mathcal{H}^{n-1}(\Sigma)\epsilon$$

Moreover, by (4.1)

$$\int_{Q_{r_j,\nu_j}(x_j)\cap\Sigma}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}\le\int_{Q_{r_j,\nu_j}(x_j)\cap K}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+b_2\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\setminus K),$$

and thus, by (3.12)

$$\int_{\Omega\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}\le\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+\Big(\tfrac{b_2}{1-\epsilon}+b_2\Big)\mathcal{H}^{n-1}(\Sigma)\epsilon<\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+\delta.$$

Finally, since $\Sigma\cap\partial C\subset K\cup(\Sigma\cap\partial C\setminus K)$,

$$\begin{aligned}\int_{\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}=&\int_{\Omega\cap\partial C}\phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1}+\int_{\Sigma\cap\partial C}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}\\ \le&2\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+3b_2\mathcal{H}^{n-1}(\Sigma)\epsilon+b_2\mathcal{H}^{n-1}(\Sigma\cap\partial C\setminus K)\\ <&2\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+7b_2\mathcal{H}^{n-1}(\Sigma)\epsilon<2\int_K\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}+\delta.\end{aligned}$$

□

Corollary 3.5. *Let $U \subset\subset \mathrm{Int}(\Omega \cup S \cup \Sigma)$ be an open set, $(A, u) \in \mathcal{C}$ and $\delta > 0$. Then there exists an open set $G \subset \Omega$ of finite perimeter such that*

(i) *$G \subset\subset U$ and $|G| < \delta$;;*
(ii) *the configuration (B, v) with $B := A \setminus G$ and $v := u\chi_{B\cup S}$ belongs to $\mathcal{C}$;*
(iii)
$$\mathcal{H}^{n-1}(\Sigma \cap \partial^* G \setminus (\partial^* A \cap J_u)) + \mathcal{H}^{n-1}(U \cap \Sigma \cap J_u \cap \partial^* A \setminus \partial^* G) < \delta$$
and
$$\mathcal{H}^{n-1}(U \cap B^{(1)} \cap J_v) < \delta + \mathcal{H}^{n-1}(U \cap \Sigma \cap J_v \cap \partial^* B) < \delta;$$
(iv) *for any norm ϕ in $\mathbb{R}^n$ satisfying (4.1)*
$$\int_{U\cap\Omega\cap\partial^* A} \phi(\nu_A) \mathrm{d}\mathcal{H}^{n-1} + 2\int_{U\cap A^{(1)}\cap J_u} \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} + \int_{U\cap\Sigma\cap\partial^* A\cap J_u} \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} \geq \int_{U\cap\Omega\cap\partial^* B} \phi(\nu_B) \mathrm{d}\mathcal{H}^{n-1} - \delta \geq \int_{U\cap\partial^* G} \phi(\nu_G) \mathrm{d}\mathcal{H}^{n-1} - \delta \tag{3.16}$$
and
$$\int_{U\cap\Omega\cap\partial^* A} \phi(\nu_A) \mathrm{d}\mathcal{H}^{n-1} + 2\int_{U\cap A^{(1)}\cap J_u} \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} + 2\int_{U\cap\Sigma\cap\partial^* A\cap J_u} \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} \geq \int_{U\cap\Omega\cap\partial^* B} \phi(\nu_B) \mathrm{d}\mathcal{H}^{n-1} + \int_{\Sigma\cap\partial^* G} \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} - \delta \geq \int_{\partial^* G} \phi(\nu_G) \mathrm{d}\mathcal{H}^{n-1} \tag{3.17}$$

Proof. The last inequalities in (3.16) and (3.17) follow from the definition of B.

Let $\epsilon := \frac{\delta}{16(1+b_2)}$. Let $U' \subset\subset \Omega \cap U$ be any open set such that
$$\mathcal{H}^{n-1}(\Omega \cap U \cap J_u \setminus U') < \epsilon.$$
By Corollary 3.3 applied with U', $(A, u) \in \mathcal{C}$ and ϵ we find an open set $D' \subset\subset U'$ of finte perimeter such that

(a_1) the configuration (B', v') with $B' := A \setminus D'$ and $v' := u\chi_{S\cup B'}$ belongs to $\mathcal{C}$;
(a_2) $|D'| < \epsilon$;
(a_3) $\mathcal{H}^{n-1}(U' \cap [B']^{(1)} \cap J_{v'}) < \epsilon$;
(a_4)
$$\int_{U'\cap\partial^* A} \phi(\nu_A) \mathrm{d}\mathcal{H}^{n-1} + 2\int_{U'\cap A^{(1)}\cap J_u} \phi(\nu_{J_u}) \mathrm{d}\mathcal{H}^{n-1} \geq \int_{U'\cap\partial^* B'} \phi(\nu_{B'}) \mathrm{d}\mathcal{H}^{n-1} - \epsilon.$$

Now choose another open set $U'' \subset\subset U$ such that $\overline{U'} \cap \overline{U''} = \emptyset$ and
$$\mathcal{H}^{n-1}((U \setminus U'') \cap \Sigma \cap \partial^* A \cap J_u) < \epsilon.$$
By Lemma 3.4 applied with U'', ϵ and $K := U'' \cap \Sigma \cap \partial^* A \cap J_u$ we find an open set $C' \subset U'' \cap \Omega$ of finite perimeter such that

(b_1) $C' \subset\subset U'$ and $\mathcal{H}^{n-1}(\partial C' \setminus \partial^* C') = 0$;
(b_2) $\mathcal{H}^{n-1}(K \setminus \partial C') = \mathcal{H}^{n-1}(K \setminus \overline{C'}) < \epsilon$ and $|C'| < \epsilon$;
(b_3) $\mathcal{H}^{n-1}(\Sigma \cap \partial C' \setminus K) < \epsilon$;
(b_4)
$$\int_{\Omega\cap\partial C'} \phi(\nu_{C'}) \mathrm{d}\mathcal{H}^{n-1} \leq \int_K \phi(\nu_\Sigma) \mathrm{d}\mathcal{H}^{n-1} + \epsilon,$$

and

$$\int_{\partial C'} \phi(\nu_{C'})\mathrm{d}\mathcal{H}^{n-1} \le 2\int_K \phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + \epsilon.$$

Define

$$G := C' \cup D'.$$

We claim that G satisfies the assertion of the lemma. Indeed, assertions (i)-(iii) follow from (a_1)-(a_3) and (b_1)-(b_3), whereas (iv) follows from the inclusion $\partial^* B \setminus \partial^* A \subset \Omega \cap \partial C' \cup \partial D'$ and conditions (a_4) and (b_4). □

4. $\tau_{\mathcal{C}}$-lower semicontinuity

In this section we prove Theorem 2.5 by following the arguments of [36, Proposition 4.1], and in particular by using density estimates for some Radon measures associated to $\mathcal{F}$. We start with the following lower bound for the localized surface energy.

Proposition 4.1. *Let $\delta \in (0,1)$, $Q_{r,\nu}(x_0) \subset\subset \mathrm{Int}(\Omega \cup \Sigma \cup S)$, $r > 0$, $\nu \in \mathbb{S}^{n-1}$, be a cube and $\Gamma \subset Q_{r,\nu}(x_0)$ be an $(n-1)$-dimensional Lipschitz graph separating $Q_{r,\nu}(x_0)$ into two connected components such that*

(a1) *$x_0 \in \Gamma$, $\nu_\Gamma(x_0) = \nu$ and*

$$|\nu_\Gamma(x) - \nu| < \delta \quad \text{and} \quad |(x - x_0)\cdot\nu| < \tfrac{\delta r}{2} \quad \text{for all } x \in \Gamma;$$

(a2) *$\mathcal{H}^{n-1}(Q_{r,\nu}(x_0) \cap \Gamma) < (1+\delta)r^{n-1}$.*

Assume that a sequence $\{(A_k, u_k)\} \subset \mathcal{C}$ and a configuration $(A, u) \in \mathcal{C}$ satisfy

(a3) *$u_k = \xi$ for some $\xi \in (0,1)^n \setminus \Xi_{\{(A_k,u_k)\}}$ (see Remark 2.1) and*

$$M := \sup_{k\ge1} \mathcal{F}(A_k, u_k) < +\infty;$$

(a4) *$A_k \to A$ in $L^1(\mathbb{R}^n)$;*

(a5) *$\mathcal{H}^{n-1}(Q_{r,\nu}(x_0) \cap \partial^*(A \cup S)) < \delta r^{n-1}$ and $|(A\cup S)\cap Q_{r,\nu}(x_0)| > (1-\delta)r^n$;*

(a6) *either*

$$u_k \to u \quad \text{a.e. in } Q_{r,\nu}(x_0)$$

and

$$K := Q_{r,\nu}(x_0) \cap J_u$$

or there exists a set of finite perimeter $E \subset Q_{r,\nu}(x_0)$ such that

$$u_k \to u \quad \text{a.e. in } Q_{r,\nu}(x_0)\setminus E \qquad \text{and} \qquad |u_k| \to +\infty \quad \text{a.e. in } Q_{r,\nu}(x_0)\cap E,$$

and

$$K := Q_{r,\nu}(x_0) \cap \partial^* E$$

(see Figure 2).

(a7) *the set K satisfies*

(a7.1) *$\nu_K(x_0) = \nu$ and $\theta(K, x_0) = \theta(\Gamma \cap K, x_0) = 1$;*

(a7.2) *$\mathcal{H}^{n-1}(K \cap \Gamma) > (1-\delta)r^{n-1}$;*

(a7.3) *$\mathcal{H}^{n-1}(K \setminus \Gamma) < \delta r^{n-1}$.*

We also denote by ϕ a norm in $\mathbb{R}^n$ satisfying

$$b_1 \le \phi(\nu) \le b_2, \quad \nu \in \mathbb{S}^{n-1}. \tag{4.1}$$

Let $C, D \subset\subset Q_{r,\nu}(x_0)$ be given by Lemma 3.1 applied with δ, Γ and K. Then there exist $c' = c'_{b_2} > 0$ and $k'_\delta := k'_\delta(b_2) > 0$ such that for any $k > k'_\delta$:

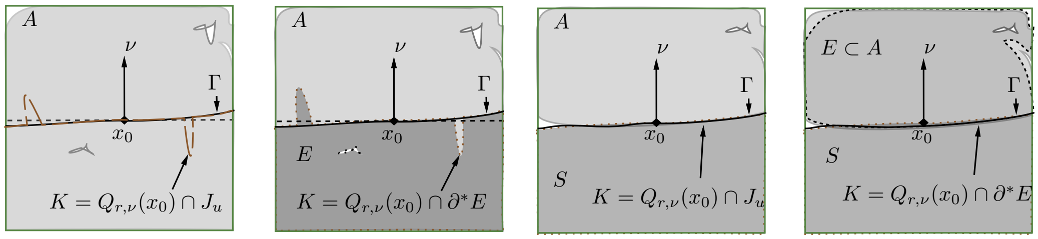

FIGURE 2. Set K in Proposition 4.1.

(i) *if* $Q_{r,\nu}(x_0) \subset\subset \Omega$, *then*

$$\int_{D\cap\partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{D\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\delta r^{n-1}$$
$$\geq \int_{\partial D} \phi(\nu_D)\mathrm{d}\mathcal{H}^{n-1} - (c' + 5b_2)\delta r^{n-1}; \qquad (4.2)$$

(ii) *if* $x_0 \in \Sigma$ *and* $\Gamma = Q_{r,\nu}(x_0) \cap \Sigma$, *then*

$$\int_{C\cap\partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{C\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\Sigma\cap\partial C\cap\partial^* A_k\cap J_{u_k}} \phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\delta r^{n-1}$$
$$\geq \int_{\partial C} \phi(\nu_C)\mathrm{d}\mathcal{H}^{n-1} - (c' + 5b_2)\delta r^{n-1}. \qquad (4.3)$$

The proof of this proposition is left after the proof of Theorem 2.5. In the proof of lower semicontinuity we only use the following corollary of Proposition 4.1; the assertions including sets C, D and E will used in the proof of compactness.

Corollary 4.2. *Under assumptions of Proposition 4.1, together with*

(a6) $u_k \to u$ *a.e. in* $Q_{r,\nu}(x_0)$ *and*

$$K := Q_{r,\nu}(x_0) \cap J_u,$$

there exist $c' = c'_{b_2} > 0$ *and* $k'_\delta := k'_\delta(b_2) > 0$ *such that for any* $k > k'_\delta$:

(i) *if* $Q_{r,\nu}(x_0) \subset\subset \Omega$, *then*

$$\int_{Q_{r,\nu}(x_0)\cap\partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{Q_{r,\nu}(x_0)\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}$$
$$\geq 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\delta r^{n-1};$$

(ii) *if* $x_0 \in \Sigma$ *and* $\Gamma = Q_{r,\nu}(x_0) \cap \Sigma$, *then*

$$\int_{Q_{r,\nu}(x_0)\cap\partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{Q_{r,\nu}(x_0)\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}$$

$$+ 2\int_{Q_{r,\nu}(x_0)\cap\Sigma\cap\partial^* A_k\cap J_{u_k}} \phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} \ge 2\int_K \phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\delta r^{n-1}.$$

Proof of Theorem 2.5. In view of Remark 2.1 we may assume that $u_k = \xi$ for some $\xi \in (0,1)^n \setminus \Xi_{\{(A_k,u_k)\}}$. Moreover, there is no loss of generality in assuming $liminf$ in (2.14) is a finite limit. Thus,

$$M := \sup_{k\ge1} \mathcal{F}(A_k,u_k) < +\infty.$$

In particular, $\{(A_k,u_k)\}$ satisfies the assumptions (a3) and (a4) of Proposition 4.1.

Let

$$\begin{aligned}\mu_k(B) :=& \int_{B\cap\Omega\cap\partial^* A_k} \varphi(x,\nu_{A_k})\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{B\cap A_k^{(1)}\cap J_{u_k}} \varphi(x,\nu_{J_{u_k}})\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ \int_{B\cap\Sigma\cap\partial^* A_k\setminus J_{u_k}} [\beta+\varphi(x,\nu_\Sigma)]\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{B\cap\Sigma\cap\partial^* A_k\cap J_{u_k}} \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ \int_{B\cap\Sigma\setminus\partial^* A_k} \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{B\cap(A\cup S)} W(x,\mathcal{E}u_k - \mathbf{M}_0)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\mu(B) :=& \int_{B\cap\Omega\cap\partial^* A} \varphi(x,\nu_{A})\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{B\cap A^{(1)}\cap J_{u}} \varphi(x,\nu_{J_{u}})\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ \int_{B\cap\Sigma\cap\partial^* A\setminus J_{u}} [\beta+\varphi(x,\nu_\Sigma)]\,\mathrm{d}\mathcal{H}^{n-1} + 2\int_{B\cap\Sigma\cap\partial^* A\cap J_{u}} \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}\\ &+ \int_{B\cap\Sigma\setminus\partial^* A} \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{B\cap(A\cup S)} W(x,\mathcal{E}u - \mathbf{M}_0)\mathrm{d}x\end{aligned}$$

be positive Radon measures in $\mathbb{R}^n$. Notice that

$$\mu_k(\mathbb{R}^n) = \mathcal{F}(A_k,u_k) + \int_\Sigma \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1} \tag{4.4}$$

and

$$\mu(\mathbb{R}^n) = \mathcal{F}(A,u) + \int_\Sigma \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.5}$$

In particular,

$$\sup_{k\ge1}\mu_k(\mathbb{R}^n) \le M + \int_\Sigma \varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1},$$

and thus, there exist a positive Radon measure μ_0 in $\mathbb{R}^n$ and a not relabelled subsequence $\{\mu_k\}$ such that $\mu_k \rightharpoonup^* \mu_0$. Let us show

$$\mu_0 \ge \mu. \tag{4.6}$$

Note that (2.14) directly follows from (4.6), (4.4), (4.5). By the nonnegativity of μ and μ_0, and the explicit form of the support of μ, to establish (4.6) it suffices to prove the following density estimates:

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner[\Omega\cap\partial^* A]}(x) \ge \varphi(x,\nu_A(x)) \quad \mathcal{H}^{n-1}\text{-a.e. } x\in(\Omega\cap\partial^* A)\cup(\Sigma\setminus\partial^* A), \tag{4.7a}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner[A^{(1)}\cap J_u]}(x) \ge 2\varphi(x,\nu_{J_u}(x)) \quad \mathcal{H}^{n-1}\text{-a.e. } x\in A^{(1)}\cap J_u, \tag{4.7b}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner[\Sigma\cap\partial^*A\cap J_u]}(x)\ge 2\varphi(x,\nu_\Sigma(x))\quad \mathcal{H}^{n-1}\text{-a.e. } x\in\Sigma\cap\partial^*A\cap J_u, \tag{4.7c}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner[\Sigma\cap\partial^*A]}(x)\ge \beta(x)+\varphi(x,\nu_\Sigma(x))\quad \mathcal{H}^{n-1}\text{-a.e. } x\in\Sigma\cap\partial^*A, \tag{4.7d}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner[\Sigma\setminus\partial^*A]}(x)\ge \varphi(x,\nu_\Sigma(x))\quad \mathcal{H}^{n-1}\text{-a.e. } x\in\Sigma\setminus\partial^*A, \tag{4.7e}$$

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{L}^{n}\llcorner[A\cup S]}(x)\ge W(x,\mathcal{E}u(x)-\mathbf{M}_0(x))\quad \mathcal{L}^{n}\text{-a.e. } x\in A\cup S. \tag{4.7f}$$

Proofs of (4.7a), (4.7d) *and* (4.7e). By assumptions (H1)-(H3), the capillary functional

$$\mathcal{C}(E;U)=\int_{U\cap\partial^*E}\varphi(x,\nu_E)\mathrm{d}\mathcal{H}^{n-1}+\int_{U\cap\Sigma\cap\partial^*E}[\beta+\varphi(x,\nu_\Sigma)]\,\mathrm{d}\mathcal{H}^{n-1}+\int_{U\cap\Sigma\setminus\partial^*E}\varphi(x,\nu_\Sigma)\,\mathrm{d}\mathcal{H}^{n-1}$$

is $L^1(U)$-lowersemicontinuous in any open set $U\subset\mathbb{R}^n$ (see e.g., [1, Theorem 3.4]). As $A_k\to A$ and $\mu_k\rightharpoonup^*\mu_0$, for any ball $B_r(x_0)$ with $\mu_0(\partial B_r(x_0))=0$ we have

$$\mu_0(B_r(x_0))=\lim_{k\to+\infty}\mu_k(B_r(x_0))\ge\liminf_{k\to+\infty}\mathcal{C}(A_k,B_r(x_0))\ge\mathcal{C}(A,B_r(x_0)).$$

This inequality and the Besicovitch differentiation theorem imply (4.7a), (4.7d) and (4.7e).

Proof of (4.7b). Fix $\epsilon\in(0,2^{-10})$ and let $K:=A^{(1)}\cap J_u$. By the $\mathcal{H}^{n-1}$-rectifiability of K, there exists an at most countable family $\{\Gamma_l\}$ of $(n-1)$-dimensional C^1-graphs such that

$$\mathcal{H}^{n-1}\Big(K\setminus\bigcup_{l\ge1}\Gamma_l\Big)=0.$$

Let $x_0\in L$ be such that

- (a$_1$) $x_0\in\Gamma_l$ for some $l\ge1$ so that the generalized unit normal $\nu_0:=\nu_K(x_0)$ to L at x_0 exists and equals to $\nu_{\Gamma_l}(x_0)$;
- (a$_2$) $\theta(K,x_0)=\theta(\Gamma_l\cap K,x_0)=1$;
- (a$_3$) $\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner K}(x_0)$ exists;
- (a$_4$) $\lim\limits_{r\to0}\frac{1}{r^{n-1}}\int_{Q_{r,\nu_0}(x_0)\cap K}\varphi(x_0,\nu_K)\mathrm{d}\mathcal{H}^{n-1}=\varphi(x_0,\nu_0)$.

By the $\mathcal{H}^{n-1}$-rectifiability of K, [3, Theorem 2.63] and Lebesgue-Besicovitch differentiation theorem, the set of $x_0\in K$ for which at least one of these conditions fails is $\mathcal{H}^{n-1}$-negligible. Since φ is uniformly continuous in $\overline{\Omega}$, there exists $r_{1,\epsilon}>0$ such that

$$|\varphi(x,\nu)-\varphi(y,\nu)|<\epsilon\quad\text{whenever } |x-y|<r_{1,\epsilon} \text{ and } \nu\in\mathbb{S}^{n-1}. \tag{4.8}$$

Decreasing $r_{1,\epsilon}$ if necessary, we assume that $Q_{r_{1,\epsilon},\nu_0}(x_0)\subset\subset\Omega$. Then for any $r\in(0,r_{1,\epsilon})$

$$\mu_k(Q_{r,\nu_0}(x_0))\ge\alpha_k(Q_{r,\nu_0}(x_0)) -\epsilon\Big(\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\partial^*A_k)+2\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap A_k^{(1)}\cap J_{u_k})\Big), \tag{4.9}$$

where

$$\alpha_k(U):=\int_{U\cap\partial^*A_k}\phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1}+2\int_{U\cap A_k^{(1)}\cap J_{u_k}}\phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}$$

and $\phi(\nu) := \varphi(x_0,\nu)$. By assumption (2.8) and the nonnegativity of the summands of μ_k we have an a priori bound

$$\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\partial^* A_k) + 2\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap A_k^{(1)}\cap J_{u_k}) \le \frac{\mu(Q_{r,\nu_0}(x_0))}{b_1},$$

and thus, inserting this in (4.9) we get

$$\Big(1+\frac{\epsilon}{b_1}\Big)\mu_k(Q_{r,\nu_0}(x_0)) \ge \alpha_k(Q_{r,\nu_0}(x_0)). \tag{4.10}$$

Now we estimate α_k from below using Corollary 4.2 (a). Since Γ_l is a C^1-graph, by (a_1) there exists $r_{2,\epsilon}\in(0,r_{1,\epsilon})$ such that

- Γ_l divides the cube $Q_{r_{2,\epsilon},\nu_0}(x_0)$ into two connected components;
- $|\nu_{\Gamma_l}(x)-\nu_0|<\epsilon$ for any $x\in Q_{r_{2,\epsilon},\nu_0}(x_0)\cap\Gamma_l$;
- $|(x-x_0)\cdot\nu_0|<\epsilon r/2$ for any $r\in(0,r_{2,\epsilon})$ and $x\in Q_{r,\nu_0}(x_0)\cap\Gamma_l$;
- $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\Gamma_l)<(1+\epsilon)r^{n-1}$ for all $r\in(0,r_{2,\epsilon})$.

In particular, for any $r\in(0,r_{2,\epsilon})$ the cube $Q_{r,\nu_0}(x_0)$ and the C^1-graph $\Gamma := Q_{r,\nu_0}(x_0)\cap\Gamma_l$ satisfy the assumptions (a1)-(a2) of Proposition 4.1. As we mentioned in the beginning of the proof, $\{(A_k,u_k)\}$ satisfies (a3)-(a4) of Proposition 4.1. Moreover, by assumptions $x_0\in A^{(1)}$ and (a_2) there exists $r_{3,\epsilon}\in(0,r_{2,\epsilon})$ such that

- $P(A,Q_{r,\nu_0}(x_0))<\epsilon r^{n-1}$ and $|A\cap Q_{r,\nu_0}(x_0)|>(1-\epsilon)r^{n-1}$ for all $r\in(0,r_{3,\epsilon})$;
- $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap K\cap\Gamma_l)>(1-\epsilon)r^{n-1}$ for any $r\in(0,r_{3,\epsilon})$;
- $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap K\setminus\Gamma_l)<\delta r^{n-1}$ for any $r\in(0,r_{3,\epsilon})$.

Thus, assumptions (a5)-(a7) of Proposition 4.1 also hold. Therefore, by Corollary 4.2 (i) there exists $k_\epsilon>0$ and $c'>0$ such that

$$\alpha_k(Q_{r,\nu_0}(x_0)) \ge 2\int_{Q_{r,\nu_0}(x_0)\cap K}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\epsilon r^{n-1}$$

for all $k>k_\epsilon$. This and (4.10) yield

$$\Big(1+\frac{\epsilon}{b_1}\Big)\mu_k(Q_{r,\nu_0}(x_0)) \ge 2\int_{Q_{r,\nu_0}(x_0)\cap K}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\epsilon r^{n-1}.$$

Now letting $k\to+\infty$ for a.e. $r\in(0,r_{3,\epsilon})$ we get

$$\Big(1+\frac{\epsilon}{b_1}\Big)\mu_0(Q_{r,\nu_0}(x_0)) \ge 2\int_{Q_{r,\nu_0}(x_0)\cap K}\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1} - c'\epsilon r^{n-1}.$$

Therefore, by (a_3) and (a_4)

$$\Big(1+\frac{\epsilon}{b_1}\Big)\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner K}(x_0) = \Big(1+\frac{\epsilon}{b_1}\Big)\lim_{r\to0^+}\frac{\mu_0(Q_{r,\nu}(x_0))}{r^{n-1}} \ge 2\varphi(x_0,\nu_0)-c'\epsilon.$$

Now letting $\epsilon\to0$ we obtain (4.7b).

Proof of (4.7c). Let $\epsilon\in(0,2^{-10})$ and let $L:=\Sigma\cap\partial^*A\cap J_u$. Since Σ is Lipschitz, L is $\mathcal{H}^{n-1}$-rectifiable.

Let $x_0\in L$ be such that

(b$_1$) $\nu_0:=\nu_\Sigma(x_0)$ exist and equals to $\nu_L(x_0)$;
(b$_2$) $\theta(L,x_0)=\theta(\Sigma,x_0)=\theta(\partial^*A,x_0)=1$;
(b$_3$) $\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner L}(x_0)$ exists.

(b$_4$) $\lim_{r\to 0}\frac{1}{r^{n-1}}\int_{Q_{r,\nu_0}(x_0)\cap L}\varphi(x_0,\nu_{J_u})\mathrm{d}\mathcal{H}^{n-1}=\varphi(x_0,\nu_0)$.

By the lipschitzianity of Σ, $\mathcal{H}^{n-1}$-rectifiablity of ∂^*A, [3, Theorem 2.63] and Besicovitch differentiation theorem, the set of $x_0\in L$ for which at least one of these conditions fails is $\mathcal{H}^{n-1}$-negligible.

Let $r_{1,\epsilon}>0$ be such that (4.8) holds and $Q_{r_{1,\epsilon},\nu_0}(x_0)\subset\subset \mathrm{Int}(\Omega\cup S\cup\Sigma)$. Then as in (4.10)

$$\Big(1+\frac{\epsilon}{b_1}\Big)\mu_k(Q_{r,\nu_0}(x_0))\ge\gamma_k(Q_{r,\nu_0}(x_0))$$

for any $r\in(0,r_{1,\epsilon})$, where

$$\gamma_k(U):=\int_{U\cap\Omega\cap\partial^*A_k}\phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1}+2\int_{U\cap A_k^{(1)}\cap J_{u_k}}\phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}+2\int_{U\cap\Sigma\cap\partial^*A_k\cap J_{u_k}}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}.$$

Since Σ is Lipschitz continuous, by (b$_1$) and (b$_2$) there exists $r_{2,\epsilon}\in(0,r_{1,\epsilon})$ such that

- Σ divides the cube $Q_{r_{2,\epsilon},\nu_0}(x_0)$ into two connected components;
- $|\nu_\Sigma(x)-\nu_\Sigma(x_0)|<\epsilon$ for any $x\in Q_{r_{2,\epsilon},\nu_0}(x_0)\cap\Sigma$;
- $|(x-x_0)\cdot\nu_0|<\epsilon r/2$ for any $r\in(0,r_{2,\epsilon})$ and $x\in Q_{r,\nu_0}(x_0)\cap\Sigma$;
- $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\Sigma)<(1+\epsilon)r^{n-1}$ for all $r\in(0,r_{2,\epsilon})$.

Moreover, since $x_0\in\Sigma\cap\partial^*A$ and $\theta(L,x_0)=\theta(\partial^*A,x_0)=1$, there exists $r_{3,\epsilon}\in(0,r_{2,\epsilon})$ such that

- $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\Sigma\cap\partial^*A)>(1-\epsilon)r^{n-1}$ and $\mathcal{H}^{n-1}(Q_{r,\nu_0}(x_0)\cap\partial^*A\setminus\Sigma)<\delta r^{n-1}$.

Thus, applying Corollary 4.2 (b) we find $k''_\epsilon>0$ and $c''>0$ such that

$$\gamma_k(Q_{r,\nu_0}(x_0))\ge 2\int_{Q_{r,\nu_0}(x_0)\cap L}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}-c''\delta r^{n-1}$$

for all $k>k''_\epsilon$. Therefore,

$$\Big(1+\frac{\epsilon}{b_1}\Big)\mu_k(Q_{r,\nu_0}(x_0))\ge 2\int_{Q_{r,\nu_0}(x_0)\cap L}\phi(\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}-c''\delta r^{n-1}$$

and hence, by (b$_3$) and (b$_4$)

$$\frac{\mathrm{d}\mu_0}{\mathrm{d}\mathcal{H}^{n-1}\llcorner L}(x_0)\ge 2\varphi(x_0,\nu_0).$$

Proof of (4.7f). By the nonnegativity of μ_k and our assumption $u_k=\xi$ on $\Omega\setminus A_k$

$$\begin{aligned}\mu_k(B_r(x))&\ge\int_{B_r(x)\cap(A_k\cup S)}W(y,\mathcal{E}u_k-\mathbf{M}_0)\mathrm{d}y\\&=\int_{B_r(x)\cap(\Omega\cup S)}W(y,\mathcal{E}u_k-\mathbf{M}_0)\mathrm{d}y-\int_{B_r(x)\cap(\Omega\setminus A_k)}W(y,-\mathbf{M}_0)\mathrm{d}y.\end{aligned}\tag{4.11}$$

Since $\mu_k\rightharpoonup^*\mu_0$, $\mathcal{E}u_k\rightharpoonup\mathcal{E}u$ in $L^2(\Omega\cup S)$ (see (2.11)) and $A_k\to A$ in $L^1(\mathbb{R}^n)$, letting $k\to+\infty$ in (4.11) for any ball $B_r(x)$ with $\mu_0(\partial B_r(x))=0$, we get

$$\begin{aligned}\mu_0(B_r(x))&=\lim_{k\to+\infty}\mu_k(B_r(x))\\&\ge\int_{B_r(x)\cap(\Omega\cup S)}W(y,\mathcal{E}u-\mathbf{M}_0)\mathrm{d}y-\int_{B_r(x)\cap(\Omega\setminus A)}W(y,-\mathbf{M}_0)\mathrm{d}y\\&=\int_{B_r(x)\cap(A\cup S)}W(y,\mathcal{E}u-\mathbf{M}_0)\mathrm{d}y,\end{aligned}$$

where in the equality we used $u = \xi$ in $\Omega \setminus A$. Now (4.7f) follows from the Besicovitch differentiation theorem. □

Remark 4.3. According to the proof of Theorem 2.5 both $\mathcal{S}$ and $\mathcal{W}$ are $\tau_{\mathcal{C}}$-lower semicontinuous in $\mathcal{C}$.

Now we prove bounds (4.2)-(4.3).

Proof of Proposition 4.1. We only prove (i). The last inequality in (4.2) directly follows from (3.1)-(3.3). Therefore, we establish only the first estimate. Without loss of generality, we assume $x_0 = 0$, $r = 1$ and $\nu = \mathbf{e}_n$. By (a1) $\Gamma \subset (-\frac{1}{2}, \frac{1}{2})^{n-1} \times (-\frac{\delta}{2}, \frac{\delta}{2})$, by (a3) and a priori estimates in Remark 2.3

$$M_1 := \sup_{k\ge1} \Big(\int_{\Omega\cup S} |\mathcal{E}u_k|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k}) \Big) < +\infty. \tag{4.12}$$

We prove (4.2) for $K = Q_1 \cap \partial^* E$ (i.e., in the case $|u_k| \to +\infty$ a.e. in $Q_1 \cap E$); the other case being similar. For any open set $G \subset Q_1$ define

$$\alpha_k(G) := \int_{G\cap\partial^* A_k} \phi(\nu_{A_k}) \mathrm{d}\mathcal{H}^{n-1} + 2\int_{G\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}}) \mathrm{d}\mathcal{H}^{n-1}.$$

Step 1. Let

$$\Upsilon := \{\xi \in \mathbb{S}^{n-1} :\ |\xi\cdot e_n| \ge 2\delta\}.$$

Then by (a1) for any $\xi \in \Upsilon$ and $x \in Q_1 \cap \Gamma$

$$|\xi \cdot \nu_\Gamma(x)| \ge |\xi\cdot \mathbf{e}_n| - |\xi \cdot (\nu_\Gamma(x) - \mathbf{e}_n)| > \delta,$$

and hence, $Q_1 \cap \Gamma$ is a graph also in ξ-direction, i.e., for any $y \in \Pi_\xi$ the line $\pi_\xi^{-1}(y)$ intersects $Q_1 \cap \Gamma$ at most at one point.

Step 2. Let D be given by Lemma 3.1 and let $U \subset\subset D$ be any open set such that $U \cap \Gamma \cap K \ne \emptyset$. Let also (B_k^U, v_k^U) be given by Corollary 3.3 applied with U, $\delta = \frac{|U|}{k}$ and (A_k, u_k). Then for all k :

- (a$_1$) $B_k^U \subset A_k$, $A_k \setminus B_k^U \subset\subset U$ and $|A_k \setminus B_k^U| < 1/k$;
- (a$_2$) $v_k^U = u_k$ in $B_k^U \cup S$;
- (a$_3$) $\mathcal{H}^{n-1}(U \cap [B_k^U]^{(1)} \cap J_{v_k^U}) < 1/k$;
- (a$_4$) $\alpha_k(U) + |U|/k \ge \Lambda_k(U)$, where

$$\Lambda_k(U) := \int_{U\cap\partial^* B_k^U} \phi(\nu_{B_k^U}) \mathrm{d}\mathcal{H}^{n-1}.$$

By (a$_1$) $B_k^U \to A$ in $L^1(\mathbb{R}^n)$, and by (a$_1$), (a$_2$) and also (a6)

$$v_k^U \to u \ \text{ a.e. in } U\setminus E \qquad \text{and} \qquad |v_k^U| \to +\infty \ \text{ a.e. in } U \cap E. \tag{4.13}$$

Moreover, by (4.12) and (a$_2$)

$$\sup_{k\ge1} \Big(\int_U |\mathcal{E}v_k^U|^2 \mathrm{d}x + \mathcal{H}^{n-1}(U \cap J_{v_k^U}) \Big) < +\infty.$$

We claim that

$$\liminf_{k\to+\infty} \Lambda_k(U) \ge \frac{2}{\phi^o(\xi)} \int_{U\cap\Gamma} |\nu_\Gamma\cdot\xi| \mathrm{d}\mathcal{H}^{n-1} - 2b_2 P(A, U) - 2b_2\mathcal{H}^{n-1}(U \cap [\Gamma\setminus \partial^* E]). \tag{4.14}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \Upsilon$.

To prove (4.14) we study some properties of one-dimensional slices $[\widehat{v}_k^U]_y^\xi$ of v_k^U. We closely follow the arguments of [14, pp. 11-13]; see also [15]. Let $k_j := k_j^U$ be such that

$$\liminf_{k\to+\infty}\int_{U\cap J_{v_k^U}}\phi(\nu_{J_{v_k^U}})\mathrm{d}\mathcal{H}^{n-1} = \lim_{j\to+\infty}\int_{U\cap J_{v_{k_j}^U}}\phi(\nu_{J_{v_{k_j}^U}})\mathrm{d}\mathcal{H}^{n-1}.$$

Applying (2.3) and (2.4) with $v = v_{k_j}^U$, (2.5) with $L = J_{v_{k_j}^U}$, and using (4.12) we find

$$\liminf_{j\to+\infty}\int_{\Pi_\xi}\Big[\mathcal{H}^0(J_{[\widehat{v}_{k_j}^U]_y^\xi}) + \kappa I_{y,\xi}^U(v_{k_j}^U) + \kappa II_{y,\xi}^U(v_{k_j}^U)\Big]\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty \tag{4.15}$$

for any $\kappa > 0$ and $\mathcal{H}^{n-1}$-a.e. $\xi\in\Upsilon$. Moreover, by [14, Lemma 2.7] and (4.13)

$$|v_k^U\cdot\xi| \to +\infty \quad \text{a.e. in } U\cap E \tag{4.16}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi\in\Upsilon$. Fix any $\xi\in\Upsilon$ satisfying (4.15) and (4.16) and consider the one-dimensional slices $[\widehat{v}_{k_j}^U]_y^\xi$ and $\widehat{u}_y^\xi$. In view of (4.15) and Fatou's lemma, for $\mathcal{H}^{n-1}$-a.e. $y\in\pi_\xi(U)$

$$\liminf_{k\to+\infty}\Big[\mathcal{H}^0(J_{[\widehat{v}_{k_j}^U]_y^\xi}) + \kappa I_{y,\xi}^U(v_{k_j}^U) + \kappa II_{y,\xi}^U(v_{k_j}^U)\Big] < +\infty.$$

Thus, for $\mathcal{H}^{n-1}$-a.e. $y\in\pi_\xi(U)$ there exists a subsequence $\{k_j^y\}\subset\{k_j\}$ (depending also on $\kappa>0$) such that

$$\begin{aligned}\liminf_{j\to+\infty}&\Big[\mathcal{H}^0(J_{[\widehat{v}_{k_j}^U]_y^\xi}) + \kappa I_{y,\xi}^U(v_{k_j}^U) + \kappa II_{y,\xi}^U(v_{k_j}^U)\Big]\\ &= \lim_{j\to+\infty}\Big[\mathcal{H}^0(J_{[\widehat{v}_{k_j^y}^U]_y^\xi}) + \kappa I_{y,\xi}^U(v_{k_j^y}^U) + \kappa II_{y,\xi}^U(v_{k_j^y}^U)\Big],\end{aligned} \tag{4.17}$$

and by (4.13) and (4.16)

$$[\widehat{v}_{k_j^y}^U]_y^\xi \to \widehat{u}_y^\xi \quad \mathcal{L}^1\text{-a.e. in } [U\setminus E]_y^\xi \quad\text{and}\quad \Big|[\widehat{v}_{k_j^y}^U]_y^\xi\Big| \to +\infty \quad \mathcal{L}^1\text{-a.e. in } [U\cap E]_y^\xi. \tag{4.18}$$

For $\tau(t) = \arctan(t)$, set $f_j := \tau\circ[\widehat{v}_{k_j^y}^U]_y^\xi$. Then $f_j\in SBV_{\rm loc}^2(U_y^\xi)$ and $J_{[\widehat{v}_{k_j^y}^U]_y^\xi} = J_{f_j}$. By (4.17), (4.18) and [2, Proposition 4.2] we find a not relabelled subsequence $\{v_{k_j^y}^U\}$ such that

$$f_j\to f_0 \quad \mathcal{L}^1\text{-a.e. in } U_y^\xi \text{ as } j\to+\infty.$$

By (4.18)

$$\begin{cases} f_0 = \tau\circ\widehat{u}_y^U & \text{in } [U\setminus E]_y^\xi,\\ |f_0| = \pi/2 & \text{in } [U\cap E]_y^\xi.\end{cases}$$

By [2, Proposition 4.2]

$$\liminf_{j\to+\infty}\mathcal{H}^0\Big(J_{[\widehat{v}_{k_j^y}^U]_y^\xi}\Big) = \liminf_{j\to+\infty}\mathcal{H}^0(J_{f_j}) \ge \mathcal{H}^0(J_{f_0}). \tag{4.19}$$

Thus, $\mathcal{H}^0(U_y^\xi\cap J_{f_0}) < +\infty$ and hence, $[U\cap E]_y^\xi$ consists of finitely many segments in each of which either $f_0\equiv\pi/2$ or $f_0\equiv-\pi/2$.

By (4.17) $\mathcal{H}^0(J_{f_j})$ is uniformly bounded and hence, there exists a further not relabelled subsequence and $N_y\in\mathbb{N}_0$ such that

$$\mathcal{H}^0(J_{f_j}) = N_y \quad\text{and}\quad J_{f_j} = \{t_j^1,\dots,t_j^{N_y}\}\subset U_y^\xi \quad \text{for all } j.$$

Then points of J_{f_j} converges to $M_y \le N_y$ points $t^1 < \ldots < t^{M_y}$. Since $II^U_{y,\xi}(v^U_{k^y_j})$ is uniformly bounded, the precise representatives of f_j uniformly bounded in $W^{1,1}_{\rm loc}(t^l, t^{l+1})$ so that $f_j \to f_0$ locally uniformly in (t^l, t^{l+1}) and $J_{f_0} \subset \{t^1, \ldots, t^{M_y}\}$. Repeating the arguments of [14, Section 1] we can show that $t^1 := U^\xi_y \cap [\partial^* E]^\xi_y \in J_{f_0}$.

Let us estimate the $\mathcal{H}^{n-1}$-measures of the sets

$$\begin{aligned} Y_0 :=& \{y \in \Pi_\xi \cap \pi_\xi(U \cap K) :\ N_y = 0\},\\ Y_1 :=& \{y \in \Pi_\xi \cap \pi_\xi(U \cap K) :\ N_y = 1\},\\ Y_2 :=& \{y \in \Pi_\xi \cap \pi_\xi(U \cap K) :\ N_y \ge 2\}. \end{aligned}$$

By (4.19) $\mathcal{H}^0(J_{f_0}) = 0$ for any $y \in Y_0$. Hence, $U \cap \pi_\xi^{-1}(y) \cap (\partial^* E \cup J_u) = \emptyset$, and therefore $Y_0 \subset \pi_\xi(U \cap \Gamma \setminus \partial^* E)$. Then by the 1-Lipschitz continuity of the projection π_ξ

$$\mathcal{H}^{n-1}(Y_0) \le \mathcal{H}^{n-1}(\pi_\xi(U \cap [\Gamma \setminus \partial^* E])) \le \mathcal{H}^{n-1}(U \cap [\Gamma \setminus \partial^* E]). \tag{4.20}$$

Now consider any $y \in Y_1$. By definition $\pi_\xi^{-1}(y)$ intersects $U \cap J_{v^U_{k^y_j}}$ just once and therefore, by the construction of (B^U_k, v^U_k) (see the proof of Corollary 3.3) either $y \in \pi_\xi\Big(U \cap [B^U_{k^y_j}]^{(1)} \cap J_{u_{k^y_j}} \cap J_{v^U_{k^y_j}}\Big)$ or $y \in \pi_\xi(U \setminus B^U_{k^y_j})$. If $y \in \pi_\xi(U \setminus B^U_{k^y_j})$, then t^1_j divides the line $U \cap \pi_\xi^{-1}(y)$ into two parts one is a subset of $U \cap B^U_{k^y_j}$ and the other is that of $U \setminus B^U_{k^y_j}$. Since $B^U_{k^y_j} \to A$ and $t^1 = U^\xi_y \cap [\partial^* E]^\xi_y \in J_{f_0}$, it follows that $t^1 \in \partial^* A$ and divides $U \cap \pi_\xi^{-1}(y)$ into two parts one belonging to $U \cap A$ other to $U \setminus A$. In particular, $y \in \pi_\xi(U \cap \partial^* A)$. Hence,

$$y \in \Big[U \cap [B^U_{k^y_j}]^{(1)} \cap J_{u_{k^y_j}} \cap J_{v^U_{k^y_j}})\Big]^\xi_y \cup \Big[U \cap \partial^* A\Big]^\xi_y$$

for all j. Thus,

$$\begin{aligned} \mathcal{H}^{n-1}(Y_1) =& \int_{Y_1} \mathcal{H}^0\left(\bigcap_j \Big(\Big[U \cap [B^U_{k^y_j}]^{(1)} \cap J_{u_{k^y_j}} \cap J_{v^U_{k^y_j}})\Big]^\xi_y \cup \Big[U \setminus B^U_{k^y_j}\Big]^\xi_y\Big)\right) \mathrm{d}\mathcal{H}^{n-1}(y)\\ \le& \int_{Y_1} \lim_{j\to+\infty} \mathcal{H}^0\Big(\Big[U \cap [B^U_{k^y_j}]^{(1)} \cap J_{u_{k^y_j}} \cap J_{v^U_{k^y_j}})\Big]^\xi_y\Big) \mathrm{d}\mathcal{H}^{n-1}(y)\\ &+ \int_{Y_1} \mathcal{H}^0\Big(\Big[U \cap \partial^* A\Big]^\xi_y\Big) \mathrm{d}\mathcal{H}^{n-1}(y). \end{aligned}$$

By the choice of $\{k^y_j\}$, the Fatou's lemma, the second equality in (2.5) and (a$_3$)

$$\begin{aligned} &\int_{Y_1} \lim_{j\to+\infty} \mathcal{H}^0\Big(\Big[U \cap [B^U_{k^y_j}]^{(1)} \cap J_{u_{k^y_j}} \cap J_{v^U_{k^y_j}})\Big]^\xi_y\Big) \mathrm{d}\mathcal{H}^{n-1}(y)\\ &\qquad\qquad \le \liminf_{k\to+\infty} \mathcal{H}^{n-1}\Big(U \cap [B^U_k]^{(1)} \cap J_{u_k} \cap J_{v^U_k})\Big) = 0. \end{aligned}$$

Similarly,

$$\int_{Y_1} \mathcal{H}^0\Big(\Big[U \cap \partial^* A\Big]^\xi_y\Big) \mathrm{d}\mathcal{H}^{n-1}(y) \le P(A, U).$$

Thus,

$$\mathcal{H}^{n-1}(Y_1) \le P(A, U). \tag{4.21}$$

Now using $\Pi_\xi\cap\pi_\xi(U) = Y_0\cup Y_1\cup Y_2$, from (4.20) and (4.21) we obtain

$$\mathcal{H}^{n-1}([\Pi_\xi\cap\pi_\xi(U)]\setminus Y_2) \le P(A,U) + \mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]).$$

Moreover, let

$$X := \{y\in\Pi_\xi\cap\pi_\xi(U):\ \pi_\xi^{-1}(y)\cap\Gamma\cap\partial^*E \text{ is a singleton}\}.$$

Then as above

$$\begin{aligned}\mathcal{H}^{n-1}(Y_2\setminus X) \le& \mathcal{H}^{n-1}([\Pi_\xi\cap\pi_\xi(U)]\setminus X) \le \mathcal{H}^{n-1}([U\cap(\Gamma\cup\partial^*A)]\setminus\partial^*E)\\ \le& \mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]) + P(A,U),\end{aligned}$$

and therefore,

$$\mathcal{H}^{n-1}([\Pi_\xi\cap\pi_\xi(U)]\setminus[Y_2\cap X]) \le 2P(A,U) + 2\mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]). \tag{4.22}$$

By the definition of X and Y_2 for any $y\in Y_2\cap X$ we have $\mathcal{H}^0(J_{f_0}) = 1$ and $N_y\ge 2$, therefore, we can improve (4.19) as

$$\lim_{j\to+\infty}\mathcal{H}^0\Big(J_{[\widehat{v}^U_{k^y_j}]^\xi_y}\Big) \ge 2 = 2\mathcal{H}^0([U\cap\Gamma]^\xi_y).$$

For such y from (4.17) we get

$$\liminf_{j\to+\infty}\Big[\mathcal{H}^0(J_{[\widehat{v}^U_{k_j}]^\xi_y}) + \kappa I^U_{y,\xi}(v^U_{k_j}) + \kappa II^U_{y,\xi}(v^U_{k_j})\Big] \ge 2\mathcal{H}^0([U\cap\Gamma]^\xi_y)$$

Now integrating over $X\cap Y_2$ and using (4.15) and the Fatou's lemma we get

$$\liminf_{k\to+\infty}\int_{\Pi_\xi}\Big[\mathcal{H}^0(J_{[\widehat{v}^U_k]^\xi_y}) + \kappa I^U_{y,\xi}(v^U_k) + \kappa II^U_{y,\xi}(v^U_k)\Big]\mathrm{d}\mathcal{H}^{n-1}(y) \ge 2\int_{X\cap Y_2}\mathcal{H}^0([U\cap\Gamma]^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y).$$

By the definition of Υ, $\mathcal{H}^0([U\cap\Gamma]^\xi_y) = 1$ for all $y\in\Pi_\xi\cap\pi_\xi(U)$ and therefore by (4.22)

$$\begin{aligned}\int_{X\cap Y_2}\mathcal{H}^0([U\cap\Gamma]^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y) \ge& \int_{\Pi_\xi\cap\pi_\xi(U)}\mathcal{H}^0([U\cap\Gamma]^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y)\\ &- P(A,U) - \mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]).\end{aligned}$$

Hence,

$$\begin{aligned}&\liminf_{k\to+\infty}\int_{\Pi_\xi}\Big[\mathcal{H}^0(J_{[\widehat{v}^U_k]^\xi_y}) + \kappa I^U_{y,\xi}(v^U_k) + \kappa II^U_{y,\xi}(v^U_k)\Big]\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\qquad\ge 2\int_{\Pi_\xi\cap\pi_\xi(U)}\mathcal{H}^0([U\cap\Gamma]^\xi_y)\mathrm{d}\mathcal{H}^{n-1}(y) - 2P(A,U) - 2\mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]).\end{aligned}$$

This, (2.5), (2.3), (2.4) as well as (4.12) yield

$$\begin{aligned}\liminf_{k\to+\infty}\int_{U\cap J_{v^U_k}}|\nu_{J_{v^U_k}}\cdot\xi|\mathrm{d}\mathcal{H}^{n-1} + (M_1+|U|)\kappa \ge& 2\int_{U\cap\Gamma}|\nu_\Gamma\cdot\xi|\mathrm{d}\mathcal{H}^{n-1}\\ &- 2P(A,U) - 2\mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E])\end{aligned}\tag{4.23}$$

Let ϕ^o be the dual norm to ϕ, i.e.,

$$\phi^o(\xi) = \sup_{\phi(\nu)=1}|\xi\cdot\nu|.$$

Then $|\xi\cdot\nu|\le\phi^o(\xi)\phi(\nu)$ and hence, by (4.23) and the arbitrariness of κ we get

$$\phi^o(\xi)\liminf_{k\to+\infty}\int_{U\cap J_{v_k^U}}\phi(\nu_{J_{v_k^U}})\mathrm{d}\mathcal{H}^{n-1}\ge 2\int_{U\cap\Gamma}|\nu_\Gamma\cdot\xi|\mathrm{d}\mathcal{H}^{n-1} - 2P(A,U)-2\mathcal{H}^{n-1}(U\cap[\Gamma\setminus\partial^*E]). \tag{4.24}$$

Now using $\phi^o(\xi)\ge 1/b_2$ from (4.24) we get (4.14).

Step 3. Now we prove (4.2).

Substep 3.1. Let

$$\mathbb{S}^{n-1}_{\phi^o}:=\{\xi\in\mathbb{R}^n:\ \phi^o(\xi)=1\}.$$

Since $\mathbb{S}^{n-1}_{\phi^o}$ is compact,

$$\phi(\eta)=\max_{i\ge1}\eta\cdot\xi_i$$

for any countable set $\{\xi_j\}_j\subset\mathbb{S}^{n-1}_{\phi^o}$ dense in $\mathbb{S}^{n-1}_{\phi^o}$.

Fix any such dense set $\{\xi_j\}_j\subset\mathbb{S}^{n-1}_{\phi^o}$ that if $\xi=\xi_j/|\xi_j|\in\Upsilon$, then (4.15) and (4.16) hold with ξ. By [23, Lemma 6] there exists a finite family $U_1,\dots,U_m$ of disjoint open set compactly contained in D such that

$$2\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}\le\sum_{j=1}^m 2\int_{U_j\cap\Gamma}|\nu_\Gamma\cdot\xi_j|\mathrm{d}\mathcal{H}^{n-1}+\delta. \tag{4.25}$$

Recalling the definition of (B_k^U,u_k^U) from Step 2, let us define

$$B_k=\bigcap_{j=1}^m B_k^{U_j}\quad\text{and}\quad v_k:=u_k\chi_{B_k\cup S}.$$

Then by (a$_2$) $B_k\subset A_k$, $A_k\setminus B_k\subset\subset D$ and

$$|A_k\setminus B_k|\le\sum_{j=1}^m|U_j\cap(A_k\setminus B_k^{U_j})|\le\sum_{j=1}^m\frac{|U_j|}{k}\le\frac{|D|}{k}.$$

Let $\Lambda_k(D)$ be defined as in (a$_4$) of Step 2 with (B_k,v_k) in place of (B_k^U,v_k^U). Then by the definition of (B_k,v_k), $\alpha_k(D)$ and (a_4)

$$\alpha_k(D)-\Lambda_k(D)=\sum_{j=1}^m\big(\alpha_k(U_j)-\Lambda_k(U_j)\big)\ge-\sum_{j=1}^m\frac{|U_j|}{k}\ge-\frac{|D|}{k}.$$

Thus,

$$\alpha_k(D)\ge\Lambda_k(D)-\frac{|D|}{k}. \tag{4.26}$$

Substep 3.2. Now we estimate $\Lambda_k(D)$ from below. Note that if $\xi_j/|\xi_j|\in\Upsilon$, then since $\phi^o(\xi_j)=1$, by (4.14)

$$2\int_{U_j\cap\Gamma}|\nu_\Gamma\cdot\xi_j|\mathrm{d}\mathcal{H}^{n-1}\le\liminf_{k\to+\infty}\Lambda_k(U_j)+2b_2P(A,U_j)+2b_2\mathcal{H}^{n-1}(U_j\cap[\Gamma\setminus\partial^*E]). \tag{4.27}$$

Now assume that $\xi:=\xi_j/|\xi_j|\notin\Upsilon$. Then by the definition of Υ and (a3)

$$|\nu_\Gamma(x)\cdot\xi|\le|(\nu_\Gamma(x)-\mathbf{e}_n)\cdot\xi|+|\mathbf{e}_n\cdot\xi|<3\delta$$

for any $x \in U_j \cap \Gamma$. Thus,

$$2\int_{U_j\cap\Gamma} |\nu_\Gamma\cdot\xi_j| \mathrm{d}\mathcal{H}^{n-1} \le 6\delta\mathcal{H}^{n-1}(U_j\cap\Gamma). \tag{4.28}$$

Now by (4.25), (4.27) and (4.28)

$$\begin{aligned}2\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} \le \delta + \sum_{j=1,\,j\in\Upsilon}^{m} \liminf_{k\to+\infty}\Lambda_k(U_j) + 6\delta\sum_{j=1,\,j\notin\Upsilon}^{m}\mathcal{H}^{n-1}(U_j\cap\Gamma)\\ + 2b_2\sum_{j=1}^{n}\Big[P(A,U_j) + 2\mathcal{H}^{n-1}(U_j\cap[\Gamma\setminus\partial^*E])\Big].\end{aligned}$$

Since set function $Q\mapsto\Lambda_k(Q)$ is additive and non-increasing and the family $\{U_j\}$ is pairwise disjoint,

$$\sum_{j=1,\,j\in\Upsilon}^{m}\liminf_{k\to+\infty}\Lambda_k(U_j) \le \liminf_{k\to+\infty}\Lambda_k(\cup_j U_j) \le \liminf_{k\to+\infty}\Lambda_k(D).$$

Moreover, by (a2)

$$\sum_{j=1,\,j\notin\Upsilon}^{m}\mathcal{H}^{n-1}(U_j\cap\Gamma) \le \mathcal{H}^{n-1}(Q_1\cap\Gamma) < 1+\delta,$$

and by (a2), (a5), (a7.2) and (a7.3)

$$\begin{aligned}&\sum_{j=1}^{n}\Big[P(A,U_j) + \mathcal{H}^{n-1}(Q_1\cap[\Gamma\setminus\partial^*E])\Big]\\ &\qquad\le P(A,Q_1) + \mathcal{H}^{n-1}(Q_1\cap\Gamma) - \mathcal{H}^{n-1}(Q_1\cap\Gamma\cap\partial^*E) \le 6\delta.\end{aligned}$$

Then

$$2\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} \le \delta + \liminf_{k\to+\infty}\Lambda_k(D) + 6\delta(1+\delta) + 6b_2\delta,$$

and hence,

$$\liminf_{k\to+\infty}\Lambda_k(D) \ge 2\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} - c_0\delta, \tag{4.29}$$

where

$$c_0 := 13 + 6b_2$$

depends only on b_2.

Substep 3.3. From (4.26) and (4.29) there exist $k_0 := k_0(\delta, b_2) > 0$ such that

$$\Lambda_k(D) \ge 2\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} - 2c_0\delta \tag{4.30}$$

for all $k > k_0$. Since $|D| < |Q| = 1$, one has $|D|/k < c_0\delta$ provided $k > \frac{1}{c_0\delta}$. Let

$$k'_\delta := \max\Big\{k_0, \frac{1}{c_0\delta}\Big\}.$$

Observe that

$$\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1} \ge \int_{D\cap\Gamma\cap\partial^*E}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}.$$

Moreover, by (a7.3)

$$\int_{D\cap\partial^*E\setminus\Gamma}\phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1}\le b_2\delta$$

and by Lemma 3.1 (ii)

$$\mathcal{H}^{n-1}(Q_1\cap\partial^*E\setminus D)<2\delta,$$

and therefore,

$$\int_{D\cap\Gamma}\phi(\nu_\Gamma)\mathrm{d}\mathcal{H}^{n-1}\ge\int_{Q_1\cap\partial^*E}\phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1}-3b_2\delta\qquad\text{for all } k>k'_\delta.$$

Combining these estimates with (4.26) and (4.30) we deduce

$$\alpha_k(D)\ge 2\int_K\phi(\nu_K)\mathrm{d}\mathcal{H}^{n-1}-(2c_0+6b_2)\delta.$$

Hence, $c':=c'_{b_2}=(2c_0+6b_2)$ satisfies the assertion. □

4.1. **Lower semicontinuity of $\mathcal{F}_p$ and $\mathcal{F}_{\rm Dir}$.** We conclude this section by showing that the functionals $\mathcal{F}_p$ and $\mathcal{F}_{\rm Dir}$ in Theorems 2.8 and 2.9, respectively, are lower semicontinuous with respect to the τ-convergence defined in (2.18). Indeed, the proof of the τ-lower semicontinuity of $\mathcal{S}$ in $\mathcal{C}_p$ and $\mathcal{C}_{\rm Dir}$ is exactly the same as the $\tau_{\mathcal{C}}$-lower semicontinuity of $\mathcal{S}$ in $\mathcal{C}$ (see the proof of Theorem 2.5). To prove the τ-lower semicontinuity of $\mathcal{W}_p$ and $\mathcal{W}_{\rm Dir}$ we notice that according to the proof of the density estimate (4.7f), we only need the convexity of $W_p(x,\cdot)$ and the weak convergence of $\mathcal{E}u_k$ to $\mathcal{E}u$ in $L^p(\mathrm{Int}(\Omega\cup S\cup\Sigma))$; the first condition is already stated in the assumption (a1) of W_p and the second condition follows from the lower bound in (a2) and the compactness result [14, Theorem 1.1].

5. Compactness in $\mathcal{C}$

In this section we prove Theorem 2.6. Note that if $\{(A_k,u_k)\}$ is an energy-equibounded sequence, then by a priori estimates (see Remark 2.3) we can find a set of finite perimeter $A\subset\Omega$ such that, up to a subsequence, $A_k\to A$ in $L^1(\mathbb{R}^n)$. Moreover, since each connected component S_i of S is Lipschitz, the convergence of u_k in S_i can be obtained by adding rigid displacements in S_i. However, since the rigid displacements for S_i may differ from those for S_j, $j\ne i$, we need to create extra jumps for the resulting displacement field. Hence, as in [36] we need to partition A_k to compensate those jumps. The following proposition provides such a partition up to some error.

Proposition 5.1. *Let $(A_k,u_k),(A,u)\in\mathcal{C}$ be admissible configurations, S^i for $i\in\{1,\dots,m\}$ be a nonempty union of some connected components of S such that $S^i\cap S^j=\emptyset$ and $S=\bigcup_{i=1}^m S^i$, $\{a_k^1\},\dots,\{a_k^m\}$ be sequences of rigid displacements, $u^1,\dots,u^m\in GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma))$ and $F^1,\dots,F^m\subset A$ be pairwise disjoint sets of finite perimeter. Assume that*

- *$\sup_k\mathcal{F}(A_k,u_k)<+\infty$ and $A_k\to A$ in $L^1(\mathbb{R}^n)$;*
- *for any $i\in\{1,\dots,m\}$ one has $u_k-a_k^i\to u^i$ a.e. in $S^i\cup F^i$ and $|u_k-a_k^i|\to+\infty$ a.e. $(S\setminus S^i)\cup(A\setminus F^i)$.*

Then for any $\delta\in(0,\frac18\min_{i\ne j}\{1,\mathrm{dist}(S^i,S^j)\})$ there exist a (not relabelled) subsequence $\{(A_k,u_k)\}$, $k_\delta>0$, $s_\delta\in(0,\delta)$ and a sequence $\{G_k^\delta\}\subset BV(\Omega;\{0,1\})$ such that

$$\mathcal{H}^{n-1}([A_k\setminus A]^{(1)}\cap\{\mathrm{dist}(\cdot,S)=s_\delta\})<c^*\delta,\tag{5.1a}$$

$$\mathcal{H}^{n-1}(\{\mathrm{dist}(\cdot,S)<s_\delta\}\cap\partial^*A)<c^*\delta, \tag{5.1b}$$

$$|G_k^\delta|<c^*\sqrt{\delta}\sum_{0\le i\le m}P(F^i), \tag{5.1c}$$

$$P(G_k^\delta)\le c^*\sum_{0\le i\le m}P(F^i), \tag{5.1d}$$

and the sequence $\{(B_k^\delta,v_k^\delta)\}$, *defined as*

$$B_k^\delta:=A_k\setminus G_k^\delta \tag{5.2}$$

and

$$v_k^\delta:=\begin{cases}u_k-a_k^i & \text{in } S^i\cup[F^i\setminus G_k^\delta]\cup[R_\delta^i\cap(B_k^\delta\setminus A)] \text{ for } i=1,\dots,m,\\ u_0 & \text{in } B_k^\delta\cap F^0,\\ \xi & \text{in } (\Omega\setminus B_k^\delta)\cup(B_k^\delta\setminus[A\cup\bigcup_{i=1}^m R_\delta^i]),\end{cases} \tag{5.3}$$

where $\xi\in(0,1)^n$,

$$R_\delta^i:=\{x\in\Omega:\ \mathrm{dist}(x,S^i)<s_\delta\},\qquad F^0:=A\setminus\bigcup_{i=1}^m F^i,$$

satisfies

$$\mathcal{S}(A_k,u_k)\ge\mathcal{S}(B_k^\delta,v_k^\delta)-c^*\sqrt{\delta}\Big[1+P(A_k)+\mathcal{H}^{n-1}(J_{u_k})+\sum_{i=0}^m\mathcal{H}^{n-1}(\partial^*F^i)\Big] \tag{5.4}$$

for all $k>k_\delta$. *Here constant* $c^*>0$ *depends only on* n, b_1 *and* b_2.

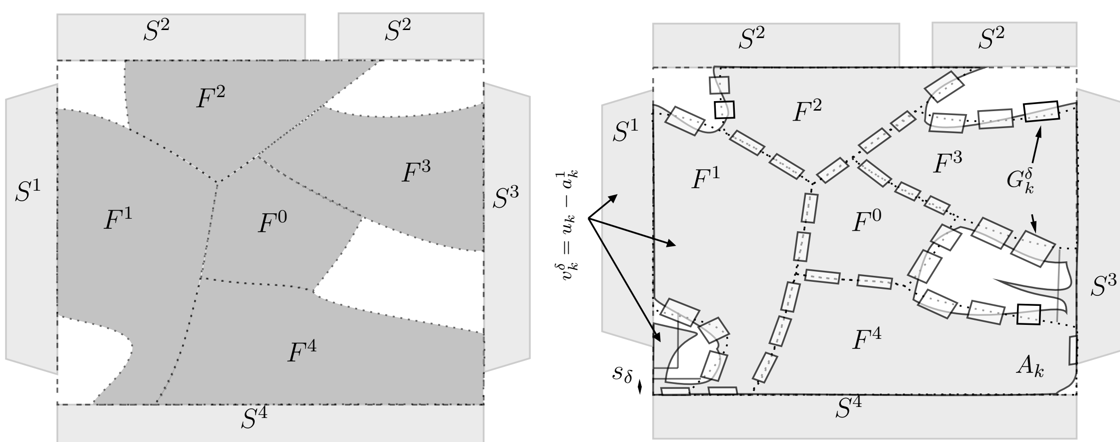

FIGURE 3. The partition of $A=\bigcup_{i\ge0}F^i$ and the construction of $B_k^\delta:=A_k\setminus G_k^\delta$ in Proposition 5.1. The set $G_{\delta,k}$ is a finite union of holes along the boundaries $F^i\cup\bigcup_{j\neq i}S^j$ in which $u_k-a_k^i$ converges. Note that the sets $\{F^i\setminus G_k^\delta\}_{i=0}^m$ partition B_k^δ. Since F^0 is a "hanging" component of A, i.e., not linked to the substrate, and hence, it is reasonable to assume that the elastic energy in F^0 is 0. Then we define the displacement fields v_k^δ as follows: in $S^i\cup(F^i\setminus G_k^\delta)$ for $i=1,\dots,m$ we set $v_k^\delta:=u_k-a_k^i$ and in $F^0\setminus G_k^\delta$ we write $v_k^\delta:=u_0$. Finally, since $A_k\setminus A$ may present large trace portions along ∂S on which v_k^δ forms a jump, we need to change the values of v_k^δ in $R_\delta^i\setminus A$ near S^i.

We postpone the proof of Proposition 5.1 after the proof of Theorem 2.6.

Proof of Theorem 2.6. Since S is Lipschitz open set with finitely many connected components, applying the Poincaré-Korn inequality and the Rellich-Kondrachov compactness theorem we find a not relabelled subsequence $\{(A_k, u_k)\}$, a partition $\{S^i\}_{i=1}^m$ of S and m sequences $\{a_k^1\}, \dots, \{a_k^m\}$ of rigid displacements such that

- (a_1) each S^i is the union of some connected components of S and $S = \bigcup_{i=1}^m S^i$;
- (a_2) for each $i \in \{1, \dots, m\}$ there exists $w^i \in H^1(S^i)$ such that $u_k - a_k^i$ converges to w^i weakly in $H^1(S^i)$ and a.e. in S^i;
- (a_3) if $i \neq j$, then $|a_k^i - a_k^j| \to +\infty$ a.e. in $\mathbb{R}^n$.

We may also assume $A_k \to A$ in $L^1(\mathbb{R}^n)$ for some $A \in BV(\Omega; \{0,1\})$. Since $\mathcal{E}v = \mathcal{E}(v + a)$ for any rigid displacement a, by Remark 2.3 we have

$$\sup_{k\ge1} \Big(P(A_k) + \mathcal{H}^{n-1}(J_{(u_k - a_k^i)\chi_{A_k \cup S}}) + \int_{A_k \cup S} |\mathcal{E}(u_k - a_k^i)|^2 \mathrm{d}x \Big) < +\infty$$

for any i. Hence, by [14, Theorem 1.1] there exist a not relabelled subsequence $\{(A_k, u_k)\}$ such that for each i the set

$$F_i := \{x \in \Omega : \limsup_{k\to+\infty} |(u_k(x) - a_k^i(x))\chi_{A_k}(x)| = +\infty\}$$

has finite perimeter and there exists a function $u^i \in GSBD^2(\mathrm{Int}(\Omega \cup S \cup \Sigma))$ such that

$$u_k - a_k^i \to u^i \quad \text{a.e. in } S^i \cup F^i,$$

where

$$F^i := A \setminus F_i.$$

By assumption (a_3) the sets $F^1, \dots, F^m$ are pairwise disjoint (see Figure 3).

Let $\delta_0 := 2^{-10} \min_{i\neq j}\{1, \mathrm{dist}(S^i, S^j)\}$ and consider any sequence $\delta_l \searrow 0$ with $\delta_1 < \delta_0$. By Proposition 5.1 for any $l \ge 1$ there exists a subsequence $\{(A_{k,l}, u_{k,l})\}_k \subset \{(A_{k,l-1}, u_{k,l-1})\}_k$, $k_{\delta_l} > 0$, $s_{\delta_l} \in (0, \delta_l)$ and a sequence $\{G_k^{\delta_l}\}_k$ of sets of finite perimeter satisfying (5.1a)-(5.1d) with $\delta = \delta_l$ such that the sequence $\{(B_k^{\delta_l}, v_k^{\delta_l})\}_k$, defined as (5.2)-(5.3), satisfies

$$\mathcal{S}(A_{k,l}, u_{k,l}) \ge \mathcal{S}(B_k^{\delta_l}, v_k^{\delta_l}) - c^* \sqrt{\delta_l} \Big[1 + P(A_{k,l}) + \mathcal{H}^{n-1}(J_{u_{k,l}}) + \sum_{i=0}^m \mathcal{H}^{n-1}(\partial^* F^i)\Big] \tag{5.5}$$

for all $k > k_{\delta_l}$. Here we set $(A_{k,0}, u_{k,0}) = (A_k, u_k)$. By (5.1d) we may also assume that $G_k^{\delta_l} \to G^{\delta_l}$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$, and therefore, $B_k^{\delta_l} \to A \setminus G^{\delta_l}$. Moreover, setting $v_k^{\delta_l} = \xi$ in $\Omega \setminus B_k^{\delta_l}$ and $B_k^{\delta_l} \setminus [\cup_i R_{\delta_l}^i \cup A]$ for some $\xi \in (0,1)^n \setminus \Xi_{\{B_k^{\delta_l}, u_k^{\delta_l}\}_{k,l}}$ (see Remark 2.1), by the choice of a_k^i we get $v_k^{\delta_l} \to v^{\delta_l}$ a.e. in $\Omega \cup S$, where

$$v^{\delta_l} := \sum_{i=1}^m u^i \chi_{S^i \cup (F^i \setminus G^{\delta_l})} + u_0 \chi_{F^0 \setminus G^{1/l}} + \xi \chi_{(\Omega\setminus A)\cup G^{1/l}}.$$

By (5.1c)-(5.1d)

$$|G^{\delta_l}| \le c^* \sqrt{\delta_l} \sum_{i=0}^m P(F^i), \qquad P(G^{\delta_l}) \le c^* \sum_{i=0}^m P(F^i),$$

and hence, $G^{\delta_l} \to \emptyset$ in $L^1(\mathbb{R}^n)$ as $l \to +\infty$. Therefore, $v^{\delta_l} \to u$ a.e. in $\Omega \cup S$ as $l \to +\infty$, where

$$u := \sum_{i=1}^m u^i \chi_{S^i \cup F^i} + u_0 \chi_{F^0} + \xi \chi_{\Omega \setminus A}.$$

By the nonnegativity and invariance w.r.t. rigid displacements of the elastic energy we have also

$$\mathcal{W}(A_{k,l}, u_{k,l}) \ge \mathcal{W}(B_k^{\delta_l}, v_k^{\delta_l}). \tag{5.6}$$

For each $l \ge 1$ let us choose $k_l > k_{\delta_l}$ and consider the sequences $\{(A_{k_l,l}, u_{k_l,l})\}_l$ and let $(B_l, v_l) := (B_{k_l}^l, u_{k_l}^l)$. We may also assume that $l \mapsto k_l$ is strictly increasing. By construction and the definition of u, one readily check that $(B_l, v_l) \xrightarrow{\tau_{\mathcal{C}}} (A, u)$. Moreover, by construction and (5.1c) $|A_{k_l,l}\Delta B_l| = |G_{k_l}^{\delta_l}| \to 0$. Finally, from (5.5) and (5.6) we immediately get

$$\liminf_{l\to+\infty} \mathcal{F}(A_{k_l,l}, u_{k_l,l}) \ge \liminf_{l\to+\infty} \mathcal{F}(B_l, u_l).$$

Thus, the subsequence $\{(A_{k_l,l}, u_{k_l,l})\}_l$, the sequence $\{(B_l, u_l)\}$ and the configuration (A, u) satisfy the assertions of Theorem 2.6. □

Note that by construction $|B_l| \le |A_{k_l}|$ and hence, in general our technique does not imply the compactness of energy-equibounded sequences $\{(A_k, u_k)\}$ satisfying a volume constraint.

5.1. **Proof of Proposition 5.1.** We start with the following estimates near the points of reduced boundary of A (in Proposition 5.1).

Proposition 5.2. *Let $\delta \in (0, 1/8)$, $U \subset \mathbb{R}^n$ be an open set, $E_k, E \in BV(U; \{0,1\})$, and $Q_{r,\nu}(x_0) \subset\subset U$, $r > 0$, $\nu \in \mathbb{S}^{n-1}$, be a cube such that*

(a1) $x_0 \in \partial^* E$, $\nu_E(x_0) = \nu$ *and*

$$1 - \delta < \frac{1}{\phi(\nu) r^{n-1}} \int_{Q_{r,\nu}(x_0)\cap\partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1} < 1 + \delta;$$

(a2)

$$\Big(\frac12 - \delta\Big) r^n < |E \cap Q_{r,\nu}^-(x_0)|, |E \cap Q_{r,\nu}^+(x_0)| < \Big(\frac12 + \delta\Big) r^n,$$

where $Q_{r,\nu}^{\pm}(x_0) = \{x \in Q_{r,\nu}(x_0) : \ (x - x_0)\cdot\nu \gtrless 0\}$;

(a3) $E_k \to E$ *in* $L^1(U)$.

We also denote by ϕ a norm in $\mathbb{R}^n$ satisfying (4.1). Then there exists $k_\delta > 0$ such that for any $k > k_\delta$ there is $t_k^\delta \in (\sqrt{\delta}, 2\sqrt{\delta})$ such that $\mathcal{H}^{n-1}(T_{t_k^\delta r} \cap \partial^* E_k) = 0$ and

$$\mathcal{H}^{n-1}(T_{t_k^\delta r} \cap E_k^{(1)}) + \mathcal{H}^{n-1}(T_{-t_k^\delta r} \cap (Q_1^- \setminus E^{(1)})) + \mathcal{H}^{n-1}(T_{-t_k^\delta r} \cap (E_k^{(1)}\Delta E^{(1)})) < 4\sqrt{\delta} r^{n-1},$$

where

$$T_t := \{x \in Q_{r,\nu}(x_0) : \ (x - x_0)\cdot\nu = t\}, \quad t \in (-r, r),$$

and the set

$$D_k^\delta := Q_{r,\nu}(x_0) \cap \{|(x - x_0)\cdot\nu| < t_k^\delta r\}$$

satisfies

$$\int_{D_k^\delta\cap\partial^* E_k} \phi(\nu_{E_k})\mathrm{d}\mathcal{H}^{n-1} \ge \phi(\nu)\mathcal{H}^{n-1}(T_{-t_k^\delta r}) - (4n + 12) b_2 \sqrt{\delta} r^{n-1}.$$

(see Figure 4).

In the proof of Proposition 5.1 we apply this proposition with $U = \Omega$, $E_k := A_k$ and $E = A$.

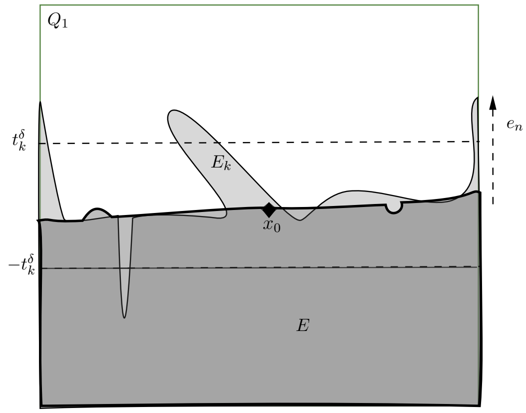

FIGURE 4. The sets E_k and E in Proposition 5.2.

Proof. Without loss of generality we assume that $x_0 = 0$, $\nu = \mathbf{e}_n$ and $r = 1$. By (a2)

$$|Q_1^+ \cap E| \le |E| - |E \cap Q_1^-| < 2\delta,$$

and hence, by (a3) there exists $k_\delta > 0$ such that

$$|Q_1^+ \cap E_k| < 2\delta \quad \text{and} \quad |E_k \Delta E| < \delta \quad \text{for all } k > k_\delta. \tag{5.7}$$

Also by (a2)

$$|Q_1^- \setminus E| \le |Q_1^-| - |Q_1^- \cap E| < \delta,$$

thus, by (5.7) and the coarea formula

$$4\delta > |Q_1^+ \cap E_k| + |Q_1^- \setminus E| + |E_k \Delta E| = \int_0^{1/2} \Big[\mathcal{H}^{n-1}(T_t \cap E_k^{(1)}) + \mathcal{H}^{n-1}(T_{-t} \cap [Q_1^- \setminus E^{(1)}]) + \mathcal{H}^{n-1}(T_t \cap [E_k^{(1)} \Delta E^{(1)}]) + \mathcal{H}^{n-1}(T_{-t} \cap [E_k^{(1)} \Delta E^{(1)}])\Big] \mathrm{d}t.$$

In particular there exists $t_k^\delta \in (\sqrt{\delta}, 2\sqrt{\delta})$ such that

$$\mathcal{H}^{n-1}(T_{t_k^\delta} \cap E_k^{(1)}) + \mathcal{H}^{n-1}(T_{-t_k^\delta} \cap (Q_1^- \setminus E^{(1)})) + \mathcal{H}^{n-1}(T_{-t_k^\delta} \cap (E_k^{(1)} \Delta E^{(1)})) < 4\sqrt{\delta}. \tag{5.8}$$

Define

$$D_k^\delta := \big(-1/2, 1/2\big)^{n-1} \times (-t_k^\delta, t_k^\delta)$$

(see Figure 4). Note that

$$\int_{D_k^\delta \cap \partial^* E_k} \phi(\nu_{E_k}) \mathrm{d}\mathcal{H}^{n-1} = \int_{\{x\cdot \mathbf{e}_n > -t_k^\delta\} \cap \partial^*(D_k^\delta \cap E_k)} \phi(\nu_{D_k^\delta \cap E_k}) \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial^* E_k \cap \overline{D_k^\delta} \cap \partial Q_1} \phi(\nu_{Q_1}) \mathcal{H}^{n-1} - \int_{E_k^{(1)} \cap T_{t_k^\delta}} \phi(\mathbf{e}_n) \mathcal{H}^{n-1}.$$

By the choice (5.8) of t_k^δ

$$\int_{E_k^{(1)} \cap T_{t_k^\delta}} \phi(\mathbf{e}_n) \mathcal{H}^{n-1} \le b_2 \mathcal{H}^{n-1}(E_k^{(1)} \cap T_{t_k^\delta}) < 4 b_2 \sqrt{\delta}$$

and

$$\int_{\partial^* E_k \cap \overline{D_k^\delta} \cap \partial Q_1} \phi(\nu_{Q_1}) \mathcal{H}^{n-1} \le b_2 \mathcal{H}^{n-1}(\partial D_k^\delta \cap \partial Q_1) < 4(n-1) b_2 \sqrt{\delta},$$

where $2(n-1)$ is the perimeter of $(-1/2,1/2)^{n-1}$. Moreover, by the anisotropic (local) minimality of half-spaces (see e.g. [7, Example 2.4])

$$\int_{\{x\cdot\mathbf{e}_n>-t_k^\delta\}\cap\partial^*(D_k\cap E_k)}\phi(\nu_{D_k\cap E_k})\mathrm{d}\mathcal{H}^{n-1}\ge\phi(\mathbf{e}_n)\mathcal{H}^{n-1}(E_k^{(1)}\cap T_{-t_k^\delta}),$$

and hence, by (5.8) (we can replace E_k with E)

$$\int_{\{x\cdot\mathbf{e}_n>-t_k^\delta\}\cap\partial^*(D_k\cap E_k)}\phi(\nu_{D_k\cap E_k})\mathrm{d}\mathcal{H}^{n-1}\ge\phi(\mathbf{e}_n)\mathcal{H}^{n-1}(E^{(1)}\cap T_{-t_k^\delta})-4b_2\sqrt{\delta}.$$

Again by (5.8)

$$\mathcal{H}^{n-1}(E^{(1)}\cap T_{-t_k^\delta})=\mathcal{H}^{n-1}(T_{-t_k^\delta})-\mathcal{H}^{n-1}((Q_1^-\setminus E^{(1)})\cap T_{-t_k^\delta})>\mathcal{H}^{n-1}(T_{-t_k^\delta})-4\sqrt{\delta}$$

and therefore,

$$\int_{D_k^\delta\cap\partial^*E_k}\phi(\nu_{E_k})\mathrm{d}\mathcal{H}^{n-1}\ge\phi(\mathbf{e}_n)\mathcal{H}^{n-1}(T_{-t_k^\delta})-4(n+3)b_2\sqrt{\delta}.$$

□

Now applying Proposition 4.1 and 5.2 we construct the set G_k^δ in Proposition 5.1.

Proof of Proposition 5.1. Without loss of generality we assume $u_k=\xi$ in $\Omega\setminus A_k$ for some $\xi\in(0,1)^n\setminus\Xi_{\{(A_k,u_k)\}\}}$ (see Remark 2.1).

By the uniform continuity of φ, there exists $r_\delta\in(0,1)$ such that

$$|\varphi(x,\nu)-\varphi(y,\nu)|<\delta\quad\text{for all } x,y\in\overline{\Omega}\text{ with } |x-y|<r_\delta. \tag{5.9}$$

Let

$$\begin{aligned}\widetilde{K}_1:=&\Sigma\cap\partial^*A\cap\bigcup_{i=1}^m\Big(\partial S^i\cap\bigcup_{j\ne i}\partial^*F^j\Big),\\ \widetilde{K}_2:=&\Omega\cap A^{(1)}\cap\bigcup_{i=0}^m\partial^*F^i,\\ \widetilde{K}_3:=&\Omega\cap\partial^*A\cap\bigcup_{i=0}^m\partial^*F^i.\end{aligned}$$

Since these sets are $\mathcal{H}^{n-1}$-rectifiable and pairwise disjoint, (by a simple covering argument) we can find open sets $U_1\subset\subset\mathrm{Int}(\Omega\cup S\cup\Sigma)$ and $U_2,U_3\subset\subset\Omega$ with disjoint closures such that

$$\sum_{i=1}^3\mathcal{H}^{n-1}\big(\widetilde{K}_i\setminus U_i\big)+\sum_{i=1}^3\mathcal{H}^{n-1}\Big(\widetilde{K}_i\cap\bigcup_{j\ne i}U_j\Big)<\delta. \tag{5.10}$$

Set

$$K_i:=U_i\cap\widetilde{K}_i,\quad i=1,2,3.$$

Note that around $\mathcal{H}^{n-1}$-a.e. point of $\cup_iK_i$ there exist $j\in\{1,\dots,m\}$ and a cube Q such that $\cup_iK_i$ "roughly divides" Q into two parts in one $u_k-a_k^j$ converges and in the other either u_k is constant or $|u_k-a_k^j|\to+\infty$. For convenience of the reader we divide the construction of G_k^δ into smaller steps.

Step 1. Using the $\mathcal{H}^{n-1}$-rectifiability of K_i, ∂^*A, ∂^*F^i, the lipschitzianity of Σ and the Borel regularity of corresponding unit normals we construct a fine cover of $\cup_i K_i$ as follows.

Substep 1.1: fine cover for K_1. For $\mathcal{H}^{n-1}$-a.e. $x\in K_1$ there exist $i_x,j_x\in\{1,\dots,m\}$ with $i_x\ne j_x$ and $r_x>0$ such that $x\in(\partial S^{i_x}\setminus\partial^*F^{i_x})\cap\partial^*F^{j_x}$ and:

- (a$_{1.1}$) $r_x<\frac14\min\{r_\delta,\mathrm{dist}(x,\partial U_1)\}$, where r_δ is defined in (5.9);
- (a$_{1.2}$) $\theta(\Sigma,x)=\theta(K_1,x)=\theta(\partial^*F^{j_x},x)=\theta(\partial^*A,x)=1$ and $\nu_\Sigma(x)$, $\nu_{K_1}(x)$, $\nu_{F^{j_x}}(x)$ and $\nu_A(x)$ exist and are parallel each other. For shortness, we set $\nu_x:=\nu_\Sigma(x)$;
- (a$_{1.3}$) $\Gamma_x:=Q_{r_x,\nu_x}(x)\cap\Sigma$ separates $Q_{r_x,\nu_x}(x)$ into two connected components;
- (a$_{1.4}$) for any $r\in(0,r_x)$

$$|\nu_{\Gamma_x}(y)-\nu_x|<\delta\quad\text{and}\quad|(y-x)\cdot\nu_x|<\tfrac{\delta r}{2}\quad\text{for all } y\in\Gamma_x, \tag{5.11a}$$

$$(1-\delta)r^{n-1}<\mathcal{H}^{n-1}(Q_r\cap\Gamma_x\cap\partial^*F^{j_x})\le\mathcal{H}^{n-1}(Q_r\cap\Gamma_x)<(1+\delta)r^{n-1}, \tag{5.11b}$$

$$\mathcal{H}^{n-1}\Big(\Big[Q_r\cap\bigcup_{j=0}^m\partial^*F^j\Big]\setminus\Gamma_x\Big)+\mathcal{H}^{n-1}(Q_r\cap[\partial^*F^{j_x}\Delta\Gamma_x])<\delta r^{n-1}, \tag{5.11c}$$

$$|(F^{j_x}\cup S)\cap Q_r|\ge(1-\delta)r^n, \tag{5.11d}$$

where $Q_r:=Q_{r,\nu_x}(x)$.

Removing an $\mathcal{H}^{n-1}$-negligible set from K_1 if necessary we assume that for all points $x\in K_1$ there exist r_x and i_x,j_x satisfying $(a_{1.1})$-$(a_{1.4})$.

Let us show that for any $x\in K_1$ and $r\in(0,r_x)$, the cube $Q_{r,\nu_x}(x)$, the sequence $\{(A_k,u_k-a_k^{j_x})\}$, the configuration (A,u^{j_x}), conditions (a$_{1.1}$)-(a$_{1.4}$), the sets $E:=Q_{r_x,\nu_x}(x)\setminus F^{j_x}$ and $K:=Q_{r_x,\nu_x}(x)\cap\partial^*F^{j_x}$ satisfy all assumptions of Proposition 4.1. Indeed, conditions for Γ follow from (a$_{1.3}$), (5.11a) and (5.11b), while conditions (a3)-(a4) for $\{(A_k,u_k)\}$ follows from our assumption in the beginning of the proof and the assumption of Proposition 5.1. The definition of F^{j_x} implies condition (a6) with $E:=Q_{r_x,\nu_x}(x)\setminus F^{j_x}$ and $K:=Q_{r_x,\nu_x}(x)\cap\partial^*F^{j_x}$. Finally, the estimates (5.11b) and (5.11c) together with (a$_{1.2}$) yield that $A\cup S$ and K satisfy conditions (a5) and (a7), respectively.

Substep 1.2: fine cover for K_2. For $\mathcal{H}^{n-1}$-a.e. $x\in K_3$ there exist $r_x>0$, $i_x,j_x\in\{0,\dots,m\}$ with $i_x\ne j_x$ and an $(n-1)$-dimensional C^1-graph Γ_x containing x such that

- (a$_{2.1}$) $r_x<\frac14\min\{r_\delta,\mathrm{dist}(x,\partial U_2)\}$.
- (a$_{2.2}$) $\theta(K_2,x)=\theta(\partial^*F^{i_x},x)=\theta(\partial^*F^{j_x},x)=\theta(K_2\cap\partial^*F^{i_x}\cap\partial^*F^{j_x}\cap\Gamma_x,x)=1$ and unit normals ν_{K_2}, $\nu_{F^{i_x}}(x)$ and $\nu_{F^{j_x}}(x)$ exist and is parallel to $\nu_x:=\nu_{\Gamma_x}(x)$;
- (a$_{2.3}$) Γ_x separates $Q_{r_x,\nu_x}(x)$ into two connected components;
- (a$_{2.4}$) for any $r\in(0,r_x)$

$$|\nu_{\Gamma_x}(y)-\nu_x|<\delta\quad\text{and}\quad|(y-x)\cdot\nu_x|<\tfrac{\delta r}{2}\quad\text{for all } y\in\Gamma_x\cap Q_r, \tag{5.12a}$$

$$\begin{aligned}(1-\delta)r^{n-1}&<\mathcal{H}^{n-1}(Q_r\cap\Gamma_x\cap K_2\cap\partial^*F^{i_x}\cap\partial^*F^{j_x})\\&\le\mathcal{H}^{n-1}(Q_r\cap\Gamma_x)<(1+\delta)r^{n-1},\end{aligned} \tag{5.12b}$$

$$\mathcal{H}^{n-1}(Q_r\cap[\Gamma_x\Delta(\partial^*F^{i_x}\cap\partial^*F^{j_x})])+\mathcal{H}^{n-1}\Big(\Big[Q_r\cap\bigcup_{j=0}^{N_2}\partial^*F^j\Big]\setminus\Gamma_x\Big)<\delta r^{n-1}, \tag{5.12c}$$

$$\Big(\tfrac12-\delta\Big)r^n\le|F^{i_x}\cap Q_r^-|,|F^{j_x}\cap Q_r^+|\le\Big(\tfrac12+\delta\Big)r^n, \tag{5.12d}$$

where $Q_r := Q_{r,\nu_x}(x)$ and $Q_r^\pm := \{y \in Q_r :\ (y-x)\cdot\nu_x \gtrless 0\}$. Here the volume density estimates follows from the definition of reduced boundary.

Removing an $\mathcal{H}^{n-1}$-negligible set from K_2 if necessary we assume that for all points $x \in K_2$ there exist r_x and i_x, j_x satisfying ($a_{2.1}$)-($a_{2.4}$). Then using $A = \cup_{j=0}^{N_2} F^j$ and $\partial^* A \subset \cup_{j=0}^{N_2}\partial^* F^j$ as in Substep 1.1. one can check that for any $x \in K_2$ and $r \in (0, r_x)$, the cube $Q_{r,\nu_x}(x)$, the sequence $\{(A_k, u_k - a_k^{i_x})\}$, the configuration (A, u^{i_x}) and the sets $E := Q_{r_x,\nu_x}(x) \setminus F^{i_x}$ and $K = Q_{r_x,\nu_x}(x) \cap \partial^* F^{i_x}$ satisfy all conditions of Proposition 4.1.

Substep 1.3: fine cover for K_3. For $\mathcal{H}^{n-1}$-a.e. $x \in K_3$ there exist $r_x > 0$, $i_x \in \{0, \dots, m\}$ and an $(n-1)$-dimensional C^1-graph Γ_x containing x such that

($a_{3.1}$) $r_x < \frac14 \min\{r_\delta, \operatorname{dist}(x, \partial U_4)\}$;
($a_{3.2}$) $\theta(K_3, x) = \theta(\partial^* F^{i_x}, x) = \theta(\partial^* A, x) = \theta(K_3 \cap \Gamma_x \cap \partial^* A \cap \partial^* F^{i_x}, x) = 1$ and the unit normals $\nu_{K_3}(x)$, $\nu_A(x)$ and $\nu_{F^{i_x}}(x)$ exist and coincide with $\nu_x := \nu_{\Gamma_x}(x)$;
($a_{3.3}$) Γ_x separates $Q_{r_x,\nu_x}(x)$ into two connected components;
($a_{3.4}$) for any $r \in (0, r_x)$

$$|\nu_{\Gamma_x}(y) - \nu_x| < \delta \quad\text{and}\quad |(y-x)\cdot\nu_x| < \tfrac{\delta r}{2} \quad \text{for all } y \in \Gamma_x \cap Q_r, \tag{5.13a}$$

$$(1-\delta) r^{n-1} < \mathcal{H}^{n-1}(Q_r \cap \Gamma_x \cap K_3 \cap \partial^* F^{i_x} \cap \partial^* A) \le \mathcal{H}^{n-1}(Q_r \cap \Gamma_x) < (1+\delta) r^{n-1}, \tag{5.13b}$$

$$\mathcal{H}^{n-1}(Q_r \cap [\Gamma_x \Delta(\partial^* F^{i_x} \cap \partial^* A)]) + \mathcal{H}^{n-1}\Big(\Big[Q_r \cap \bigcup_{j=0}^{N_3} \partial^* F^{i_x}\Big] \setminus \Gamma_x\Big) < \delta r^{n-1}, \tag{5.13c}$$

$$(1-\delta) r^{n-1} < \frac{1}{\varphi(x,\nu_x)} \int_{Q_r \cap \partial^* F^{i_x}} \varphi(x, \nu_{F^{i_x}}(y))\mathrm{d}\mathcal{H}^{n-1}(y) \tag{5.13d}$$

$$\le \frac{1}{\varphi(x,\nu_x)} \int_{Q_r \cap \partial^* A} \varphi(x, \nu_A(y))\mathrm{d}\mathcal{H}^{n-1}(y) < (1+\delta) r^{n-1}, \tag{5.13e}$$

$$\Big(\tfrac12 - \delta\Big) r^n < |Q_r^- \cap F^{i_x}| \le |Q_r^- \cap A| < \Big(\tfrac12 + \delta\Big) r^n, \tag{5.13f}$$

$$|Q_r^+ \cap A| < \delta r^n, \tag{5.13g}$$

where $Q_r := Q_{r,\nu_x}(x)$.

Removing an $\mathcal{H}^{n-1}$-negligible set from K_3 if necessary we assume that for all points $x \in K_3$ there exists $r_x > 0$ and i_x satisfying ($a_{3.1}$)-($a_{3.4}$). Then for any $x \in K_3$ and $r \in (0, r_x)$ the set $U = U_3$, the cube $Q_{r,\nu_x}(x)$, the sequence $E_k := Q_{r,\nu_x}(x) \cap A_k$, the set $E := Q_{r,\nu_x}(x) \cap A$ and conditions ($a_{3.1}$)-($a_{3.4}$) satisfy all assumptions of Proposition 5.2. Indeed, conditions (a1)-(a2) are given in (5.13e) and (5.13f), whereas (a3) follows from the assumption $A_K \to A$ in $L^1(\mathbb{R}^n)$ as $k \to +\infty$.

Step 2. Now we extract finitely many covering cubes still covering $\cup_i K_i$ up to some error of order $O(\sqrt{\delta})$, and create "holes" inside those cubes (i.e., the sets C_1^j, C_2^j and D_k^j in Figure 5). By Step 1, for each $i \in \{1,2,3\}$ the collection $\{\overline{Q_{r,\nu_x}(x)} :\ x \in K_i,\ r \in (0, r_x)\}$ of cubes provides a fine cover for K_i and hence, by the Vitali covering lemma we can extract an at most countable pairwise disjoint family $\{Q_{r_j^i,\nu_{x_j^i}}(x_j^i),\ x_j^i \in K_i\}$ such that

$$\mathcal{H}^{n-1}\Big(K_i \setminus \bigcup_j Q_{r_j^i,\nu_{x_j^i}}(x_j^i)\Big) = 0.$$

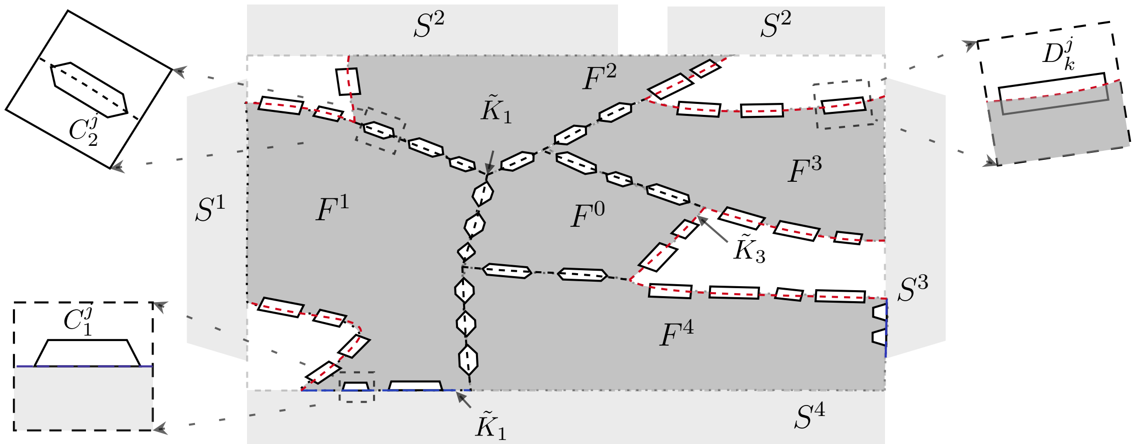

FIGURE 5. Construction of holes C_1^j, C_2^j and D_k^j.

Since $\mathcal{H}^{n-1}(K_i) < +\infty$, there exists $N_i \geq 1$ such that

$$\mathcal{H}^{n-1}\Big(K_i \setminus \bigcup_{j>N_i} Q_{r_j^i,\nu_{x_j^i}}(x_j^i)\Big) < \delta. \tag{5.14}$$

Moreover, decreasing r_j a bit necessary, we assume that $\overline{Q_{r_j^i,\nu_{x_j^i}}(x_j^i)} \cap \overline{Q_{r_{j'}^i,\nu_{x_{j'}^i}}(x_{j'}^i)} = \emptyset$ for all $1 \leq j < j' \leq N_i$. Since $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$, cubes belonging to the union of $\mathcal{G}_i := \{Q_{r_j^i,\nu_{x_j^i}}(x_j^i)\}_{j=1}^{N_i}$, $i = 1,2,3$, have disjoint closures. When no confusion arises, we drop the dependence of x_j^i and r_j^i on i.

Substep 2.1: definition of C_1^j. Let $Q_{r_j,x_j}(x_j) \in \mathcal{G}_1$ for some $j \in \{1,\dots,N_1\}$. By Substep 1.1 $x_j \in K_1 \cap \partial S^{l_j} \cap \partial^* F^{h_j}$ for some $l_j, h_j \in \{1,\dots,m\}$ with $l_j \neq h_j$. Applying Proposition 4.1 (ii) with $Q_{r_j,\nu_{x_j}}(x_j) \subset\subset \mathrm{Int}(\Omega \cup S \cup \Sigma)$, $\Gamma_{x_j} := Q_{r_j,\nu_{x_j}}(x_j) \cap \Sigma$, $\{(A_k, u_k - a_k^{h_j})\}$, (A, u^{h_j}), $E := Q_{r_j,\nu_{x_j}}(x_j) \setminus F^{h_j}$, $K := Q_{r_j,\nu_{x_j}}(x_j) \cap \partial^* F^{h_j}$ and $\phi(\cdot) = \varphi(x_j,\cdot)$ we find an open set $C_1^j \subset \Omega \cap Q_{r_j,\nu_{x_j}}(x_j)$ of finite perimeter (given by Lemma 3.1) and $k_\delta^{1,j} > 0$ such that

$$\begin{aligned}\int_{C_1^j\cap\partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{C_1^j\cap A_k^{(1)}\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\Sigma\cap\partial^* C_1^j\cap\partial^* A_k\cap J_{u_k}} \phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}\\ \geq 2\int_{Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j}} \phi(\nu_{F^{h_j}})\mathrm{d}\mathcal{H}^{n-1} - c'\delta r_j^{n-1}\\ \geq \int_{\partial^* C_1^j} \phi(\nu_{C_1^j})\mathrm{d}\mathcal{H}^{n-1} - (c' + 5b_2)\delta r_j^{n-1}\end{aligned} \tag{5.15}$$

for all $k > k_\delta^{1,j}$ and for some $c' > 0$ (depending only on b_2).

Let us estimate the perimeter and the volume of $\cup_j C_1^j$. By (5.11b)

$$r_j^{n-1} \leq \frac{1}{1-\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j} \cap \Sigma \cap \partial^* F^{h_j}) \tag{5.16}$$

and hence, by (2.8) and (3.2)

$$b_1\mathcal{H}^{n-1}(\partial^* C_1^j) \le \int_{\partial^* C_1^j} \phi(\nu_{C_1^j})\mathrm{d}\mathcal{H}^{n-1} \le 2\int_{Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j}} \phi(\nu_{F^{h_j}})\mathrm{d}\mathcal{H}^{n-1} + 5b_2\delta r_j^{n-1}$$
$$\le 2b_2\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j}) + 5b_2\delta r_j^{n-1}$$

so that

$$\mathcal{H}^{n-1}(\partial^* C_1^j) \le \frac{3b_2}{b_1}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j}). \tag{5.17}$$

Moreover,

$$|C_1^j| \le \delta r_j^n < \delta r_j^{n-1} \le 2\delta\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\Sigma\cap\partial^* F^{h_j})$$

and therefore,

$$\Big|\bigcup_{j=1}^{N_i} C_1^j\Big| \le 2\delta\sum_{h=1}^m \mathcal{H}^{n-1}(\partial^* F^h). \tag{5.18}$$

Let us estimate the error in covering K_1 by $\{C_1^j\}$. Fix some $j\in\{1,\dots,N_1\}$. Then by the definition of K_1, the error estimate (5.11c) and Lemma 3.1 (ii)

$$\mathcal{H}^{n-1}((Q_{r_j,\nu_{x_j}}(x_j)\cap K_1)\setminus\overline{C_1^j}) \le \mathcal{H}^{n-1}\Big(\Big[Q_{r_j,\nu_{x_j}}(x_j)\cap\bigcup_{j=0}^{N_1}\partial^* F^j\Big]\setminus\Gamma_{x_j}\Big) + \mathcal{H}^{n-1}(\Gamma_{x_j}\setminus\partial^* F^{h_j})$$
$$+\mathcal{H}^{n-1}([Q_{r_j,\nu_{x_j}}(x_j)\cap\partial^* F^{h_j}]\setminus\overline{C_1^j}) < 3\delta r_j^{n-1}$$

and thus, by (5.16) and the choice $\delta<1/8$

$$\mathcal{H}^{n-1}((Q_{r_j,\nu_{x_j}}(x_j)\cap K_1)\setminus\overline{C_1^j}) < 4\delta\,\mathcal{H}^{n-1}(Q_{r_j,\nu_{x_j}}(x_j)\cap\Sigma\cap\partial^* F^{h_j}). \tag{5.19}$$

From (5.14) and (5.19) it follows that

$$\mathcal{H}^{n-1}\Big(K_1\setminus\bigcup_{j=1}^{N_1}\overline{C_1^j}\Big) = \mathcal{H}^{n-1}\Big(K_1\setminus\bigcup_{j>N_1} Q_{r_j,\nu_j}(x_j)\Big) + \sum_{j=1}^{N_1}\mathcal{H}^{n-1}([Q_{r_j,\nu_{x_j}}(x_j)\cap K_1]\setminus\overline{C_1^j})$$
$$<\delta + 4\delta\sum_{j=1}^{N_1}\mathcal{H}^{n-1}(Q_{r_j,\nu_{x_j}}(x_j)\cap\Sigma\cap\partial^* F^{h_j})$$

so that by the disjointness of $\{F^h\}$

$$\mathcal{H}^{n-1}\Big(K_1\setminus\bigcup_{j=1}^{N_1}\overline{C_1^j}\Big) < \delta + 4\delta\sum_{h=1}^m\mathcal{H}^{n-1}(\partial^* F^h). \tag{5.20}$$

Substep 2.2: construction of C_2^j. Let $Q_{r_j,\nu_j}(x_j)\in\mathcal{G}_2$ for some $j\in\{1,\dots,N_2\}$ so that there exist $l_j,h_j\in\{0,\dots,m\}$ with $l_j\ne h_j\ne 0$ such that $x_j\in\partial^* F^{l_j}\cap\partial^* F^{h_j}$. As in Substep 2.1 applying Proposition 4.1 with $Q_{r_j,\nu_{x_j}}(x_j)\subset\subset\Omega$, Γ_{x_j}, $\{(A_k,u_k-a_k^{h_j})\}$, (A,u^{h_j}), $E:=Q_{r_j,\nu_j}(x_j)\setminus F^{h_j}$, $K:=Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j}$ and $\phi(\cdot)=\varphi(x_j,\cdot)$ we find an open set $C_2^j\subset\subset Q_{r_j,\nu_{x_j}}(x_j)$ of finite perimeter (given by Lemma 3.1) and $k_\delta^{2,j}>0$ such that

$$\int_{C_2^j\cap\partial^* A_k}\phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{C_2^j\cap A_k^{(1)}\cap J_{u_k}}\phi(\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1} \ge \int_{\partial C_2^j}\phi(\nu_{C_2^j})\mathrm{d}\mathcal{H}^{n-1} - c'\delta r_j^{n-1} \tag{5.21}$$

for all $k > k_\delta^{2,j}$, where c' depends only on b_2. As in Substep 2.1, by (5.12b)

$$r_j^{n-1} \le \frac{1}{1-\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{l_j} \cap \partial^* F^{h_j}) \tag{5.22}$$

by (2.8) and (3.1)

$$\mathcal{H}^{n-1}(\partial^* C_2^j) \le \frac{2b_2}{b_1}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{h_j}) + \frac{5b_2}{b_1}\,\delta r_j^{n-1} \le \frac{3b_2}{b_1}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{h_j}) \tag{5.23}$$

and

$$\Big|\bigcup_{j=1}^{N_2} C_2^j\Big| \le 2\delta \sum_{h=0}^m \mathcal{H}^{n-1}(\partial^* F^h). \tag{5.24}$$

Moreover,

$$\mathcal{H}^{n-1}\Big(K_2 \setminus \bigcup_{j=1}^{N_2} \overline{C_2^j}\Big) < \delta + 4\delta \sum_{h=0}^m \mathcal{H}^{n-1}(\partial^* F^h) \tag{5.25}$$

Substep 2.3: construction of D_k^j. Let $Q_{r_j,\nu_j}(x_j) \in \mathcal{G}_3$ for some $j \in \{1,\ldots,N_3\}$ and let $x_j \in \partial^* F^{h_j} \cap \partial^* A$ for some $h_j \in \{0,\ldots,m\}$. Using Proposition 5.2 applied with $U := Q_{r_j,\nu_{x_j}}(x_j)$, $E_k := Q_{r_j,\nu_{x_j}}(x_j) \cap A_k$, $E := Q_{r_j,\nu_{x_j}}(x_j) \cap A$ and $\phi(\cdot) = \varphi(x_j,\cdot)$ we find $k_\delta^{3,j} > 0$ such that for any $k > k_\delta^{3,j}$ there exists $t_{k,j}^\delta \in (\sqrt{\delta}, 2\sqrt{\delta})$ such that $\mathcal{H}^{n-1}(\partial^* A_k \cap T^j_{t^\delta_{k,j},r_j}) = 0$ and

$$\begin{aligned}\mathcal{H}^{n-1}(T^j_{t^\delta_{k,j}r_j} \cap A_k^{(1)}) + \mathcal{H}^{n-1}(T^j_{-t^\delta_{k,j}r_j} \cap [Q^-_{r_j,\nu_j}(x_j) \setminus A^{(1)}]) \\ + \mathcal{H}^{n-1}(T^j_{-t^\delta_{k,j}r_j} \cap [A_k^{(1)} \Delta A^{(1)}]) < 4\sqrt{\delta} r_j^{n-1},\end{aligned} \tag{5.26}$$

where

$$T_t^j := \{x \in Q_{r_j,\nu_j}(x_j) :\ (x - x_j)\cdot \nu_j = t\}, \quad t \in (-r_j, r_j),$$

and the set

$$D_k^j := \{x \in Q_{r_j,\nu_{x_j}}(x_j) :\ |(x-x_j)\cdot \nu_{x_j}| < t^\delta_{k,j} r_j\}$$

satisfy

$$\int_{D_k^j \cap \partial^* A_k} \phi(\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} \ge \phi(\nu_j)\mathcal{H}^{n-1}(T^j_{-t^\delta_{k,j}r_j}) - c'\sqrt{\delta} r_j^{n-1} \tag{5.27}$$

for some $c' > 0$ depending only on b_2 and n. Note that by (5.13b)

$$r_j^{n-1} \le \frac{1}{1-\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{h_j} \cap \partial^* A) \tag{5.28}$$

and hence by the choice of $t^\delta_{k,j}$ and (5.28)

$$|D_k^j| = 2t^\delta_{k,j} r^n \le \frac{4\sqrt{\delta}}{1-\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{h_j})$$

so that

$$\Big|\bigcup_{j=1}^{N_3} D_k^j\Big| \le 5\sqrt{\delta} \sum_{h=1}^m \mathcal{H}^{n-1}(\partial^* F^h). \tag{5.29}$$

Moreover, by the definition of D_k^j, (5.26), (5.28) and the equality $\mathcal{H}^{n-1}(T^j_{\pm t^\delta_{k,j}r_j}) = r_j^{n-1}$ we have

$$\mathcal{H}^{n-1}(\partial^* D_k^j) \le (2 + 4\sqrt{\delta}) r_j^{n-1} \le 4\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j) \cap \partial^* F^{h_j}). \tag{5.30}$$

Let us estimate the error in covering K_3 with $\{D_k^j\}$. Fox some $j\in\{1,\dots,N_3\}$. Recalling the definition of Γ_{x_j} in Substep 1.3 in view of (5.13a) we have $Q_{r_j,\nu_j}(x_j)\cap\Gamma_{x_j}\subset D_k^j$ and hence, by (5.13c) and (5.28)

$$\begin{aligned}\mathcal{H}^{n-1}&([K_3\cap Q_{r_j,\nu_j}(x_j)]\setminus D_k^j)\\ &\le\mathcal{H}^{n-1}\Big(\Big[Q_{r_j,\nu_j}(x_j)\cap\bigcup_{j=1}^{N_3}\Big]\setminus\Gamma_{x_j}\Big)+\mathcal{H}^{n-1}([Q_{r_j,\nu_j}(x_j)\cap\Gamma_{x_j}]\setminus[\partial^*A\cap\partial^*F^{h_j}])\\ &\le\delta r_j^{n-1}\le 2\delta\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\partial^*F^{h_j})\end{aligned}$$

and hence, by (5.14)

$$\mathcal{H}^{n-1}\Big(K_3\setminus\bigcup_{j=1}^{N_3}D_k^j\Big)=\mathcal{H}^{n-1}\Big(K_3\setminus\bigcup_{j>N_3}Q_{r_j,\nu_j}(x_j)\Big)+\sum_{j=1}^{N_3}\mathcal{H}^{n-1}\Big([K_3\cap Q_{r_j,\nu_j}(x_j)]\setminus D_k^j\Big)$$

so that

$$\mathcal{H}^{n-1}\Big(K_3\setminus\bigcup_{j=1}^{N_3}\overline{D_k^j}\Big)<\delta+2\delta\sum_{h=1}^{m}\mathcal{H}^{n-1}(\partial^*F^h).\tag{5.31}$$

Step 3: Definition of G_k^δ. Let $k_\delta^i:=\max_{j=1,\dots,N_i}k_\delta^{i,j}$, $i=1,2,3$, and for each $k>k_\delta:=\max\{k_\delta^1,k_\delta^2,k_\delta^3\}$ let us define

$$G_k^\delta:=\bigcup_{j=1}^{N_1}C_1^j\cup\bigcup_{j=1}^{N_2}C_2^j\cup\bigcup_{j=1}^{N_3}D_k^j,$$

obviously, G_k^δ is open. By (5.18), (5.24) and (5.29) as well as the inclusion $\partial^*A\subset\cup_j\partial^*F^j$ we get

$$|G_k^\delta|\le\Big|\bigcup_{j=1}^{N_1}C_1^j\Big|+\Big|\bigcup_{j=1}^{N_2}C_2^j\Big|+\Big|\bigcup_{j=1}^{N_3}D_k^j\Big|\le 8\sqrt{\delta}\sum_{h=0}^{m}\mathcal{H}^{n-1}(\partial^*F^h).$$

Moreover, summing the estimates (5.17), (5.23) and (5.30) and using the disjointness of the closures of C_1^j, C_2^j and D_k^j (because so are the containing cubes) we get

$$P(G_k^\delta)\le\sum_{j=1}^{N_1}P(C_1^j)+\sum_{j=1}^{N_2}P(C_2^j)+\sum_{j=1}^{N_3}P(D_k^j)\le\Big(4+\frac{3b_2}{b_1}\Big)\sum_{h=0}^{m}\mathcal{H}^{n-1}(\partial^*F^h).$$

Step 4: Definition of s_δ. Since $A_k\to A$ in $L^1(\mathbb{R}^n)$, by the coarea formula applied with the 1-Lipschitz function $f(x)=\mathrm{dist}(x,S)$

$$0=\lim_{k\to+\infty}|A_k\Delta A|=\int_0^\infty\mathcal{H}^{n-1}(\{x\in A_k\Delta A:\ \mathrm{dist}(x,S)=s\})\mathrm{d}s$$

and thus, passing to a not relabelled subsequence if necessary,

$$\lim_{k\to+\infty}\mathcal{H}^{n-1}(\{x\in A_k\Delta A:\ \mathrm{dist}(x,S)=s\})=0$$

for a.e. $s>0$. In particular, there exists $s_\delta\in(0,\delta)$ such that

$$\mathcal{H}^{n-1}([A_k\Delta A]\cap\{\mathrm{dist}(\cdot,S)=s_\delta\})<\delta\quad\text{and}\quad\mathcal{H}^{n-1}(\{0<\mathrm{dist}(\cdot,S)<s_\delta\}\cap\partial^*A)<\delta,$$

Step 5: Proof of (5.4). Let B_k^δ and v_k^δ be given by (5.2) and (5.3). As in the proof of lower semicontinuity, given $(B,v)\in\mathcal{C}$ and a Borel set $D\subset\mathbb{R}^n$, let us introduce

$$\begin{aligned}\mu_{B,v}(D) := &\int_{D\cap\partial^* B}\varphi(x,\nu_B)\mathrm{d}\mathcal{H}^{n-1} + 2\int_{D\cap B^{(1)}\cap J_v}\varphi(x,\nu_{J_v})\mathrm{d}\mathcal{H}^{n-1}\\ &+2\int_{D\cap\Sigma\cap\partial^*B\cap J_v}\varphi(x,\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1} + \int_{D\cap\Sigma\cap\partial^* B\setminus J_v}[\beta+\varphi(x,\nu_\Sigma)]\mathrm{d}\mathcal{H}^{n-1}\\ &+\int_{D\cap\Sigma\setminus\partial^*B}\varphi(x,\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

Since $\mu_{B,v}(\mathbb{R}^n) = \mathcal{S}(B,v) + \int_\Sigma\varphi(x,\nu_\Sigma)\mathrm{d}\mathcal{H}^{n-1}$, we have

$$\mathcal{S}(A_k,u_k) - \mathcal{S}(B_k^\delta,v_k^\delta) = \mu_{A_k,u_k}(\mathbb{R}^n) - \mu_{B_k^\delta,u_k^\delta}(\mathbb{R}^n).$$

By construction

$$\begin{aligned}&[\Omega\cap\partial^*A_k]\setminus\overline{G_k^\delta} = [\Omega\cap(\partial^*B_k^\delta]\setminus\overline{G_k^\delta},\quad [\Sigma\cap\partial^*A_k]\setminus\overline{G_k^\delta} = \Sigma\cap\partial^*B_k^\delta,\\ &[\Sigma\cap\partial^*A_k\cap J_{u_k}]\setminus\overline{G_k^\delta} = \Sigma\cap\partial^*B_k^\delta\cap J_{v_k^\delta},\quad \Sigma\setminus\Big(\partial^*A_k\cup\bigcup_{j=1}^{N_1}\partial^*C_1^j\Big) = \Sigma\setminus\partial^*B_k^\delta,\\ &[A^{(1)}\cap A_k^{(1)}\cap J_{u_k}]\setminus\overline{G_k^\delta} = A^{(1)}\cap B_k^{(1)}\cap J_{v_k^\delta},\\ &[A^{(1)}\cap J_{v_k^\delta}]\setminus J_{u_k} = \bigcup_{j=1}^m\partial^*F^j\setminus G_k^\delta,\quad J_{v_k^\delta}\cap\partial^*A\subset\bigcup_{j=0}^m\partial^*F^j\setminus G_k^\delta,\\ &[A_k^{(1)}\setminus A^{(1)}]\cap J_{v_k^\delta}\subseteq([R_\delta\setminus A]^{(1)}\cap J_{u_k})\cup([A_k\setminus A]^{(1)}\cap\partial R_\delta)\cup(R_\delta\cap\partial^*A),\end{aligned}$$

and hence,

$$\begin{aligned}\mathcal{S}(A_k,u_k)-\mathcal{S}(B_k^\delta,u_k^\delta)\ge{}&\mu_{A_k,u_k}(\overline{G_k^\delta}) - \mu_{B_k^\delta,u_k^\delta}(\overline{G_k^\delta}) - 2\int_{R_\delta\cap\partial^*A}\varphi(x,\nu_A)\mathrm{d}\mathcal{H}^{n-1}\\ &-2\int_{J_{v_k^\delta}\cap[B_k^\delta]^{(1)}\cap\bigcup_{i=0}^m\partial^*F^j}\varphi(x,\nu_{J_{v_k^\delta}})\mathrm{d}\mathcal{H}^{n-1} - 2\int_{[A_k\setminus A]^{(1)}\cap\partial R_\delta}\varphi(x,\nu_{R_\delta})\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}\tag{5.32}$$

By (2.8), the definition of $\widetilde{K}_j$ and U_j, the construction of C_1^j, C_2^j, D_k^j, the choice of s_δ and the error estimates (5.20), (5.25), (5.31) and (5.10) we have

$$\begin{aligned}&\int_{R_\delta\cap\partial^*A}\varphi(x,\nu_A)\mathrm{d}\mathcal{H}^{n-1} + \int_{J_{v_k^\delta}\cap[B_k^\delta]^{(1)}\cap\bigcup_{i=0}^m\partial^*F^j}\varphi(x,\nu_{J_{v_k^\delta}})\mathrm{d}\mathcal{H}^{n-1}\\ &\qquad+\int_{[A_k\setminus A]^{(1)}\cap\partial R_\delta}\varphi(x,\nu_{R_\delta})\mathrm{d}\mathcal{H}^{n-1}\le c_4^*\delta\Big(1+\sum_{h=0}^m\mathcal{H}^{n-1}(\partial^*F^h)\Big).\end{aligned}\tag{5.33}$$

Furthermore, from the additivity of the set-function $\alpha_{B,v}$ and disjointness of the closures of C_1^j, C_2^j and D_k^j we obtain

$$\begin{aligned}\mu_{A_k,u_k}(\overline{G_k^\delta}) - \mu_{B_k^\delta,u_k^\delta}(\overline{G_k^\delta}) ={}& \sum_{j=1}^{N_1}\Big[\mu_{A_k,u_k}(\overline{C_1^j}) - \mu_{B_k^\delta,u_k^\delta}(\overline{C_j^1})\Big] + \sum_{j=1}^{N_2}\Big[\mu_{A_k,u_k}(\overline{C_2^j}) - \mu_{B_k^\delta,u_k^\delta}(\overline{C_2^1})\Big]\\ &+\sum_{j=1}^{N_3}\Big[\mu_{A_k,u_k}(\overline{D_k^j}) - \mu_{B_k^\delta,u_k^\delta}(\overline{D_k^1})\Big] := I_1+I_2+I_3.\end{aligned}\tag{5.34}$$

Substep 5.1: A lower estimate for I_1. Let

$$\alpha_k^{1,j} := \int_{C_1^j\cap\partial^* A_k} \varphi(x,\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{C_1^j\cap A_k^{(1)}\cap J_{u_k}} \varphi(x,\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{\Sigma\cap\partial^* C_1^j\cap\partial^* A_k\cap J_{u_k}} \varphi(x,\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}.$$

By (5.9) and (5.15) we have

$$\alpha_k^{1,j} \ge \int_{\partial^* C_1^j} \varphi(x,\nu_{C_1^j})\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\overline{C_1^j}\cap[J_{u_k}\cup\partial^* A_k]) - \delta\mathcal{H}^{n-1}(\partial^* C_1^j) - c'\delta r_j^{n-1}. \tag{5.35}$$

Since $|\beta(x)| \le \phi(x,\nu_\Sigma)$ (see (2.9)), by the definition of μ_{A_k,u_k}, (B_k^δ, v_k^δ) and $\mu_{B_k^\delta,v_k^\delta}$ we have

$$\mu_{A_k,u_k}(\overline{C_j^1}) \ge \alpha_k^{1,j} \quad \text{and} \quad \int_{\partial^* C_1^j} \varphi(x,\nu_{C_1^j})\mathrm{d}\mathcal{H}^{n-1} = \mu_{B_k^\delta}(\overline{C_1^j}).$$

Therefore, from (5.35) and (5.16) we get

$$\mu_{A_k,u_k}(\overline{C_j^1}) - \mu_{B_k^\delta}(\overline{C_1^j}) \ge -\delta\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap[J_{u_k}\cup\partial^* A_k]) - \delta\mathcal{H}^{n-1}(\partial^* C_1^j) - \frac{c'\delta}{1-\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}\cap\Sigma\cap\partial^* F^{h_j}).$$

Summing these estimates in j and using the disjointness of $\{Q_{r_j,\nu_j}(x_j)\}$ and the perimeter estimate (5.17) of C_1^j we deduce

$$I_1 \ge -c_1^*\delta\Big(\mathcal{H}^{n-1}(J_{u_k}) + \mathcal{H}^{n-1}(\partial^* A_k) + \sum_{h=0}^m \mathcal{H}^{n-1}(\partial^* F^h)\Big) \tag{5.36}$$

for all $k > k_\delta^1 = \max_j k_\delta^{1,j}$ and for some c_1^* depending only on b_1 and b_2.

Substep 5.2: A lower estimate for I_2. Let

$$\alpha_k^{2,j} := \int_{C_2^j\cap\partial^* A_k} \varphi(x,\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1} + 2\int_{C_2^j\cap A_k^{(1)}\cap J_{u_k}} \varphi(x,\nu_{J_{u_k}})\mathrm{d}\mathcal{H}^{n-1}.$$

By (5.9) and (5.21)

$$\alpha_k^{2,j} \ge \int_{\partial^* C_2^j} \varphi(x,\nu_{C_2^j})\mathrm{d}\mathcal{H}^{n-1} - \delta\,\mathcal{H}^{n-1}(Q_{r_j,\nu_{x_j}}(x_j)\cap[\partial^* A_k\cup J_{u_k}] - \delta\mathcal{H}^{n-1}(\partial^* C_2^j) - c'\delta r_j^{n-1} \tag{5.37}$$

for all $k > k_\delta^{2,j}$ Since $\overline{C_2^j}\cap\Sigma = \emptyset$, from the definition of μ_{A_k,u_k}, (B_k^δ, v_k^δ) and $\mu_{B_k^\delta,v_k^\delta}$ we have

$$\mu_{A_k,u_k}(\overline{C_2^j}) = \alpha_k^{2,j} \quad \text{and} \quad \int_{\partial^* C_2^j} \varphi(x,\nu_{C_2^j})\mathrm{d}\mathcal{H}^{n-1} = \mu_{B_k^\delta,v_k^\delta}(\overline{C_2^j})$$

and thus, using (5.22) and (5.23) in (5.37) we obtain

$$\mu_{A_k,u_k}(\overline{C_2^j}) - \mu_{B_k^\delta,v_k^\delta}(\overline{C_2^j}) \ge -c_2^*\delta\Big(\mathcal{H}^{n-1}(Q_{r_j,\nu_{x_j}}(x_j)\cap[\partial^* A_k\cup J_{u_k}] + \mathcal{H}^{n-1}(Q_{r_j,\nu_{x_j}}(x_j)\cap\partial^* F^{l_j}\cap\partial^* F^{h_j})\Big)$$

for some constant $c_2^*>0$ depending only on b_1,b_2. Summing these estimates we get

$$I_2 \ge -c_2^*\delta\Big(\mathcal{H}^{n-1}(J_{u_k}) + \mathcal{H}^{n-1}(\partial^* A_k) + \sum_{h=0}^m \mathcal{H}^{n-1}(\partial^* F^h)\Big) \tag{5.38}$$

for all $k>k_\delta^2=\max_j k_\delta^{2,j}$.

Substep 5.3: A lower estimate for I_3. Let

$$\alpha_k^{3,j} := \int_{D_k^j\cap\partial^* A_k} \varphi(x,\nu_{A_k})\mathrm{d}\mathcal{H}^{n-1}.$$

Since $\mathcal{H}^{n-1}(T_{-t_{k,j}^\delta r_j})r_j^{n-1}$, using (5.9) and (5.27) we get

$$\alpha_k^{3,j} \ge \int_{Q_{r_j,\nu_{x_j}}(x_j)\cap T_{-t_{k,j}^\delta r}} \varphi(x,\nu_{x_j})\mathrm{d}\mathcal{H}^{n-1} - (\delta + c'\sqrt{\delta})r_j^{n-1} \tag{5.39}$$

for all $k>k_\delta^{3,j}$. Moreover, by the choice of $t_{k,j}^\delta$, (5.26) and (2.8)

$$\begin{aligned}
\mu_{B_k^\delta,u_k^\delta}(\overline{D_k^j}) \le & \int_{Q_{r_j,\nu_{x_j}}(x_j)\cap T_{-t_{k,j}^\delta r_j}} \varphi(x,\nu_{x_j})\mathrm{d}\mathcal{H}^{n-1} + \int_{Q_{r_j,\nu_{x_j}}(x_j)\cap A_k^{(1)}\cap T_{t_{k,j}^\delta r_j}} \varphi(x,\nu_{x_j})\mathrm{d}\mathcal{H}^{n-1}\\
& + \int_{\partial^* D_k^j\setminus[T_{-t_{k,j}^\delta r_j}\cup T_{t_{k,j}^\delta r_j}]} \varphi(x,\nu_{D_k^j})\mathrm{d}\mathcal{H}^{n-1}\\
\le & \int_{Q_{r_j,\nu_{x_j}}(x_j)\cap T_{-t_{k,j}^\delta r_j}} \varphi(x,\nu_{x_j})\mathrm{d}\mathcal{H}^{n-1} + 4b_2\sqrt{\delta}r_j^{n-1} + 2b_2 t_{k,j}^\delta r_j^{n-1}..
\end{aligned}$$

Now using $t_{k,j}^\delta \le 2\sqrt{\delta}$ and (5.28) in this estimate and combining with (5.39) and obvious inequality $\mu_{A_k,u_k}(\overline{D_k^j}) \ge \alpha_k^{3,j}$ (recall that $\overline{D_k^j}\cap\Sigma=\emptyset$) we get

$$\mu_{A_k,u_k}(\overline{D_k^j}) - \mu_{B_k^\delta,u_k^\delta}(\overline{D_k^j}) \ge -c_3^*\sqrt{\delta}\mathcal{H}^{n-1}(Q_{r_j,\nu_j}(x_j)\cap\partial^* F^{h_j})$$

for some c_3^* depending only on n and b_1,b_2. Summing these inequalities in j we get

$$I_3 \ge c_3^*\sqrt{\delta}\sum_{h=0}^m \mathcal{H}^{n-1}(\partial^* F^h) \tag{5.40}$$

for all $k>k_\delta^3=\max_j k_\delta^{3,j}$.

Including (5.36), (5.38) and (5.40) in (5.34) and using (5.33) in (5.32) we deduce

$$\mathcal{S}(A_k,u_k) - \mathcal{S}(B_k^\delta,u_k^\delta) \ge -c^*\sqrt{\delta}\Big(1 + \mathcal{H}^{n-1}(J_{u_k}) + \mathcal{H}^{n-1}(\partial^* A_k) + \sum_{i=0}^m \mathcal{H}^{n-1}(\partial^* F^i)\Big)$$

for all $k>k_\delta=\max\{k_\delta^1,k_\delta^2,k_\delta^3\}$. Finally, since the elastic energy density is nonnegative, and invariant w.r.t. to additive piecewise rigid displacements

$$\mathcal{W}(A_k,u_k) \ge \mathcal{W}(B_k^\delta,v_k^\delta)$$

and hence, (5.4) follows. □

From Theorems 2.5 and 2.6 together with Proposition A.1 implies that the minimum problem (2.12) is solvable.

Proof of Theorem 2.4. Fix any $\lambda > 0$ and let $\{(A_k, u_k)\} \subset \mathcal{C}$ be a minimizing sequence for $\mathcal{F}^\lambda$. Then $\sup_k \mathcal{F}(A_k, u_k) > +\infty$, and hence, by Theorem 2.6 there exists a not relabelled subsequence $\{(A_k, u_k)\}$, a sequence $\{(B_k, v_k)\} \subset \mathcal{C}$ and $(A, u) \in \mathcal{C}$ such that $(B_k, v_k) \overset{\tau_{\mathcal{C}}}{\to} (A, u)$, $|A_k \Delta B_k| \to 0$ and

$$\liminf_{k\to+\infty} \mathcal{F}(A_k, u_k) \ge \liminf_{k\to+\infty} \mathcal{F}(B_k, v_k) \ge \mathcal{F}(A, u). \tag{5.41}$$

Since the map $E \mapsto ||E| - \mathtt{v}|$ is $L^1(\mathbb{R}^n)$-continuous, from (5.41) it follows that

$$\liminf_{k\to+\infty} \mathcal{F}^\lambda(A_k, u_k) \ge \liminf_{k\to+\infty} \mathcal{F}^\lambda(B_k, v_k) \ge \mathcal{F}^\lambda(B, v).$$

Hence, (B, v) is a minimizer of $\mathcal{F}^\lambda$. By Proposition A.1 there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ every minimizer (A, u) of $\mathcal{F}^\lambda$ satisfies the volume constraint $|A| = \mathtt{v}$. Thus, (A, u) solves also the problem (2.12). Conversely, if (A, u) solves (2.12), then for $\lambda > \lambda_0$,

$$\begin{aligned}\min_{(B,v)\in\mathcal{C},\,|B|=\mathtt{v}} \mathcal{F}(B, v) =&\mathcal{F}(A, u) = \mathcal{F}^\lambda(A, u) \ge \min_{(B,v)\in\mathcal{C}} \mathcal{F}^\lambda(B, v)\\ &= \min_{(B,v)\in\mathcal{C},\,|B|=\mathtt{v}} \mathcal{F}^\lambda(B, v) = \min_{(B,v)\in\mathcal{C},\,|B|=\mathtt{v}} \mathcal{F}(B, v)\end{aligned}$$

and hence, (A, u) is a minimizer of $\mathcal{F}^\lambda$. □

5.2. **Compactness in $\mathcal{C}_p$ and $\mathcal{C}_{\mathrm{Dir}}$.** In this section we comment on the τ-compactness of energy-equibounded sequences in $\mathcal{C}_p$ and $\mathcal{C}_{\mathrm{Dir}}$; for the definition of τ-convergence see (2.18). Using (2.15) and the compactness result [14, Theorem 1.1] we have:

– if $\{(A_k, u_k)\} \subset \mathcal{C}_p$ is arbitrary sequence with $\sup_k \mathcal{F}_p(A_k, u_k) < +\infty$, then repeating the same arguments in the proof of Proposition 5.1 we construct a not relabelled subsequence, the set G_k^δ, numbers s_δ and k_δ satisfying (5.1a)-(5.1d) such that the configuration $(B_k^\delta, v_k^\delta) \in \mathcal{C}_p$, given by (5.2) and (5.3), satisfies

$$\mathcal{S}(A_k, u_k) - \mathcal{S}(B_k^\delta, u_k^\delta) \ge -c^*\sqrt{\delta}\Big(1 + \mathcal{H}^{n-1}(J_{u_k}) + \mathcal{H}^{n-1}(\partial^* A_k) + \sum_{i=0}^m \mathcal{H}^{n-1}(\partial^* F^i)\Big).$$

Then by (2.15)

$$\mathcal{W}(A_k, u_k) \ge \mathcal{W}(B_k^\delta, u_k) + \int_{G_k^\delta} W_p(x, \mathcal{E}v_k^\delta)\mathrm{d}x \ge \mathcal{W}(B_k^\delta, u_k) - \int_{G_k^\delta} |f|\mathrm{d}x.$$

Since $f \in L^1(\Omega \cup S)$, by (5.1c) and the absolute continuity of the Lebesgue integral we have

$$\mathcal{W}(A_k, u_k) \ge \mathcal{W}(B_k^\delta, u_k) + o_\delta, \tag{5.42}$$

where $o_\delta \to 0$ as $\delta \to 0$. Now the proof of the compactness in $\mathcal{C}_p$ runs exactly the same as Theorem 2.6 using (5.42) in place of (5.6);

– if $\{(A_k, u_k)\} \subset \mathcal{C}_p$ is arbitrary sequence with $\sup_k \mathcal{F}_{\mathrm{Dir}}(A_k, u_k) < +\infty$, then by [14, Theorem 1.1] in the proof of Theorem 2.6 we will have only two sets F^0 and F^1 partitioning A : the sequence u_k converges a.e. in F^1 (up to a subsequence) and $|u_k| \to +\infty$ a.e. in F^0. In particular, due to the Dirichlet condition for u_k in S, we do not need to add any rigid displacements, and then the proofs runs as in $\mathcal{C}_p$.

The τ-compactness in $\mathcal{C}_p$ (resp. $\mathcal{C}_{\mathrm{Dir}}$) and the τ-lower semicontinuity of $\mathcal{F}_p$ (resp. $\mathcal{F}_{\mathrm{Dir}}$) imply that for any $\lambda > 0$ there exists a minimizer of $\mathcal{F}_p^\lambda$ (resp. $\mathcal{F}_{\mathrm{Dir}}^\lambda$). Now observing that the proof of Proposition A.1 works also in $\mathcal{C}_p$ and $\mathcal{C}_{\mathrm{Dir}}$ (see Remark A.2) we conclude that both minimum problems (2.16) and (2.17) admit a solution.

6. Decay estimates

This section is devoted to the proof of the following density estimates for minimizers of $\mathcal{F}$.

Theorem 6.1 (**Density estimates**). *There exist $\varsigma_* = \varsigma_*(b_3, b_4) \in (0,1)$ and $R_* = R_*(b_1, b_2, b_3, b_4) > 0$, where b_i are given by (2.8) and (2.10), with the following property. Let $(A,u) \in \mathcal{C}$ be any minimizer of $\mathcal{F}$ in $\mathcal{C}$ such that $\Omega \cap \partial^* A \subset_{\mathcal{H}^{n-1}} J_u$ and $\int_{\Omega\setminus A} |\mathcal{E}u| \mathrm{d}x = 0$, and let*

$$J_u^* := \{x \in J_u : \ \theta(J_u, x) = 1\}. \tag{6.1}$$

Then for any $x \in \Omega$ and $r \in (0, \min\{1, \mathrm{dist}(x, \partial\Omega)\})$

$$\frac{\mathcal{H}^{n-1}(Q_r(x) \cap J_u)}{r^{n-1}} \le \frac{4nb_2 + \lambda_0}{b_1}. \tag{6.2}$$

Moreover, if $x \in \Omega \cap \overline{J_u^*}$ and $r \in (0, R_*)$ with $Q_r(x) \subset \Omega$, then

$$\frac{\mathcal{H}^{n-1}(Q_r(x) \cap J_u)}{r^{n-1}} \ge \varsigma_*. \tag{6.3}$$

In particular,

$$\mathcal{H}^{n-1}(\Omega \cap [\overline{J_u^*} \setminus J_u^*]) = 0. \tag{6.4}$$

Since J_u is $\mathcal{H}^{n-1}$-rectifiable, by the rectifiability criterion [3, Theorem 2.63] $\mathcal{H}^{n-1}(J_u \setminus J_u^*) = 0$. Thus, if we remove a $\mathcal{H}^{n-1}$-negligible set from J_u, then (6.4) implies that the jump set of u is essentially closed in Ω.

To prove Theorem 6.1 we follow the arguments of [37, Section 3]. First we introduce the local version $\mathcal{F}(\cdot; O) : \mathcal{C} \to \mathbb{R}$ of $\mathcal{F}$ in open sets $O \subset \Omega$ as

$$\mathcal{F}(A, u; O) := \mathcal{S}(A, u; O) + \mathcal{W}(A, u; O), \tag{6.5}$$

where $\mathcal{S}(\cdot; O)$ and $\mathcal{W}(\cdot; O)$ are the local versions of the surface and the elastic energy, i.e.,

$$\mathcal{S}(A, u; O) := \int_{O\cap\partial^* A} \varphi(y, \nu_A) d\mathcal{H}^{n-1} + 2\int_{O\cap A^{(1)}\cap J_u} \varphi(y, \nu_A)\, \mathrm{d}\mathcal{H}^{n-1}$$

and

$$\mathcal{W}(A, u; O) = \int_{O\cap A} \mathbb{C}(y)\mathcal{E}u : \mathcal{E}u \,\mathrm{d}y.$$

Next we introduce the notion of quasi-minimizers.

Definition 6.2 (**Θ-minimizers**). Given $\Theta \ge 0$, the configuration $(A, u) \in \mathcal{C}$ is a local Θ-*minimizer* of $\mathcal{F} : \mathcal{C} \to \mathbb{R}$ in O if

$$\mathcal{F}(A, u; O) \le \mathcal{F}(B, v; O) + \Theta|A\Delta B|$$

whenever $(B, v) \in \mathcal{C}$ with $A\Delta B \subset\subset O$ and $\mathrm{supp}\,(u - v) \subset\subset O$.

For any $(A, u) \in \mathcal{C}$ and any open set $O \subset\subset \Omega$ let

$$\Phi(A, u; O) := \inf\Big\{\mathcal{F}(B, v; O) : \ (B, v) \in \mathcal{C}, B\Delta A \subset\subset O, \ \mathrm{supp}\,(u - v) \subset\subset O\Big\}, \tag{6.6}$$

and let

$$\Psi(A, u; O) := \mathcal{F}(A, u; O) - \Phi(A, u; O) \tag{6.7}$$

be the *deviation of* (A, u) *from minimality* in O.

The following proposition is a generalization to our setting of [12, Theorem 4] established for the Griffith model.

Proposition 6.3. *Let $Q_R(x_0) \subset\subset \Omega$. Consider sequences of Finsler norms $\{\varphi_h\}$ and ellipticity tensors $\{\mathbb{C}_h\}$ such that $\{\mathbb{C}_h\}$ is equicontinuous in $\overline{Q_R(x_0)}$ and there exist $d_3, d_4, d_5 > 0$ with*

$$d_3 M : M \le \mathbb{C}_h(x) M : M \le d_4 M : M \quad \text{for all } (x, M) \in \overline{Q_R(x_0)} \times \mathbb{M}^{n\times n}_{\rm sym} \tag{6.8}$$

and

$$\inf_{(x,\nu)\in\overline{Q_R(x_0)}\times\mathbb{S}^{n-1}} \phi_h(x,\nu) \ge d_5 \sup_{(x,\nu)\in\overline{Q_R(x_0)}\times\mathbb{S}^{n-1}} \phi_h(x,\nu), \tag{6.9}$$

and define $\mathcal{F}_h$ and Ψ_h in $\mathcal{C}$ as in (6.5) and (6.7), respectively, with φ_h and $\mathbb{C}_h$ in places of φ and $\mathbb{C}$. Let $\{(A_h, u_h)\} \subset \mathcal{C}$ be such that

$$\int_{Q_R(x_0)\setminus A_h} |\mathcal{E}u_h| \mathrm{d}x = 0, \tag{6.10a}$$

$$M := \sup_{h\ge 1} \mathcal{F}_h(A_h, u_h; Q_R(x_0)) < \infty, \tag{6.10b}$$

$$\lim_{h\to\infty} \Psi_h(A_h, u_h; Q_R(x_0)) = 0, \tag{6.10c}$$

$$\lim_{h\to\infty} \mathcal{H}^{n-1}(Q_R(x_0) \cap J_{u_h}) = 0, \tag{6.10d}$$

$$Q_R(x_0) \cap \partial^* A_h \subset_{\mathcal{H}^{n-1}} J_{u_h}. \tag{6.10e}$$

Then there exist $u \in H^1(Q_R(x_0))$, an elasticity tensor $\mathbb{C} \in C^0(\overline{Q_R(x_0)}; \mathbb{M}^{n\times n}_{\rm sym})$ and sequences $\{a_j\}$ of rigid displacements and subsequences $\{(A_{h_j}, u_{h_j})\}$, $\{\varphi_{h_j}\}$ and $\{\mathbb{C}_{h_j}\}$ such that

(i) *$\mathbb{C}_{h_j} \to \mathbb{C}$ uniformly in $\overline{Q_R(x_0)}$ and*

$$w_j := u_{h_j} - a_j \to u \text{ a.e. in } Q_R(x_0) \quad \text{and} \quad \mathcal{E}w_j \rightharpoonup \mathcal{E}u \text{ in } L^2(Q_R(x_0))$$

as $j \to \infty$;

(ii) *for all $v \in u + H^1_0(Q_R(x_0))$*

$$\int_{Q_R(x_0)} \mathbb{C}(y)\mathcal{E}u : \mathcal{E}u \,\mathrm{d}y \le \int_{Q_R(x_0)} \mathbb{C}(y)\mathcal{E}v : \mathcal{E}v \,\mathrm{d}y; \tag{6.11}$$

(iii) *for any $r \in (0, R]$*

$$\lim_{j\to\infty} \mathcal{F}_h(A_{h_j}, u_{h_j}; Q_r(x_0)) = \int_{Q_r(x_0)} \mathbb{C}(x)\mathcal{E}u : \mathcal{E}u \,\mathrm{d}x. \tag{6.12}$$

Proof. Without loss of generality, we assume $R = 1$ and $x_0 = 0$. Also by (6.10d) we may assume $\mathcal{H}^{n-1}(Q_1 \cap J_{u_h}) < 1/4$ for any h. Let

$$b'_h := \inf_{(x,\nu)\in\overline{Q_R(x{:}0)}\times\mathbb{S}^{n-1}} \phi_h(x,\nu), \qquad b''_h := \sup_{(x,\nu)\in\overline{Q_R(x{:}0)}\times\mathbb{S}^{n-1}} \phi_h(x,\nu)$$

so that by (6.9)

$$d_5 b''_h \le b'_h \le b''_h \quad \text{for any } h. \tag{6.13}$$

By [11, Proposition 2] and (6.8), there exist a constant c_o (depending only on n and d_3) and sequences $\{\omega_h\}$ of a measurable subsets of Q_1 with $|\omega_h| \le c_o \mathcal{H}^{n-1}(Q_1 \cap J_{u_h})$ and $\{a_h\}$ of rigid displacements such that

$$\int_{Q_1\setminus\omega_h} |u_h - a_h|^2 \,\mathrm{d}x \le c_o \int_{Q_1} \mathbb{C}_h(x)\mathcal{E}u_h : \mathcal{E}u_h \,\mathrm{d}x. \tag{6.14}$$

By (6.10a) and (6.10b), $\|(u_h - a_h)\chi_{Q_1\setminus\omega_h}\|_{L^2(Q_1)} \le Mc_o$, and thus, there exist $u \in L^2(Q_1)$ and a not relablled subsequence such that $(u_h - a_h)\chi_{Q_1\setminus\omega_h} \rightharpoonup \widetilde{u}$ in $L^2(Q_1)$. Since $|\omega_h| \to 0$, the set

$$F := \{y \in Q_1 :\ \limsup_{h\to\infty} |u_h(y) - a_h(y)| = +\infty\}$$

satisfies $|F| = 0$. Furthermore, by (6.10a), (6.8) and (6.10b) as well as the equality $J_{u_h} = J_{u_h - a_h}$

$$\sup_{h\ge1} \int_{Q_1} |\mathcal{E}(u_h - a_h)|^2\,\mathrm{d}x + \mathcal{H}^{n-1}(Q_1 \cap J_{u_h-a_h}) < \frac{M}{d_3} + \frac{1}{4},$$

and hence, by [14, Theorem 1.1] there exist a not relabelled subsequence $\{u_h - a_h\}$ and $u \in GSBD^2(Q_1)$ such that

$$u_h - a_h \to u \qquad \text{a.e. in } Q_1 \tag{6.15}$$

$$\mathcal{E}(u_h - a_h) \rightharpoonup \mathcal{E}u \qquad \text{in } L^2(Q_1; \mathbb{M}^{n\times n}_{\mathrm{sym}}), \tag{6.16}$$

$$\mathcal{H}^{n-1}(Q_1 \cap J_u) \le \liminf_{h\to+\infty} \mathcal{H}^{n-1}(J_{u_h}) = 0. \tag{6.17}$$

Since the weak limit and the pointwise limit coincide (see e.g., [22, page 266]), $\widetilde{u} = u$ a.e. in Q_1. Moreover, (6.14), (6.15) and the Fatou's Lemma imply $u \in L^2(Q_1)$ and by (6.17) one has $\mathcal{H}^{n-1}(J_u) = 0$. Thus, by Lemma A.4 $u \in H^1(Q_1)$. Since our elastic energy is invariant under additive rigid displacements, without loss of generality further we assume $a_h = 0$ for any $h \ge 1$.

Next we prove (6.11). Let $v \in H^1(Q_1)$ be such that $\operatorname{supp}(u - v) \subset\subset Q_r$ for some $r \in (0,1)$. Fix $r'' < r' < r$ and let $\psi \in C^1_c(Q_r; [0,1])$ be a cut-off function with $\{0 < \psi < 1\} \subset \{u = v\} \cap Q_{r'}$ and $\operatorname{supp}(u - v) \subseteq \{\psi \equiv 1\} \subseteq Q_{r''}$. By (6.10d) and [12, Theorem 3] there exist a positive constant $c > 0$ (depending only on n, d_3 and d_4), a function $\widetilde{v}_h \in GSBD^2(Q_1)$, $r_h \in (r - \delta_h, r)$ with

$$\delta_h := \sqrt[2n]{\mathcal{H}^{n-1}(J_{u_h})}, \tag{6.18}$$

and a Lebesgue measurable set $\widetilde{\omega}_h \subset Q_{r_h}$ such that

- (a$_1$) $\widetilde{v}_h \in C^\infty(Q_{r-\delta_h})$, $\widetilde{v}_h = u_h$ in $Q_1 \setminus Q_{r_h}$, and
$$\mathcal{H}^{n-1}(J_{u_h} \cap \partial Q_{r_h}) = \mathcal{H}^{n-1}(J_{\widetilde{v}_h} \cap \partial Q_{r_h}) = 0;$$
- (a$_2$) $\mathcal{H}^{n-1}(J_{\widetilde{v}_h} \setminus J_{u_h}) < c\delta_h \mathcal{H}^{n-1}(J_{u_h} \cap (Q_r \setminus Q_{r-\delta_h}))$;
- (a$_3$) $|\widetilde{\omega}_h| \le c\delta_h^2 \mathcal{H}^{n-1}(Q_{r_h} \cap J_{u_h})$ and by (6.8),
$$\int_{Q_r\setminus\widetilde{\omega}_h} |\widetilde{v}_h - u_h|^2 \mathrm{d}x \le c\delta_h^4 \int_{Q_r} \mathbb{C}_h(x)\mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}x; \tag{6.19}$$
- (a$_4$) if $\eta \in \mathrm{Lip}(Q_1; [0,1])$, then
$$\int_{Q_r} \eta\mathbb{C}_h(x)\mathcal{E}\widetilde{v}_h : \mathcal{E}\widetilde{v}_h \mathrm{d}x \le \int_{Q_r} \eta\mathbb{C}_h(x)\mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}x + c\delta_h^s[1 + \mathrm{Lip}(\eta)] \int_{Q_r} \mathbb{C}_h(x)\mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}x \tag{6.20}$$

  for some $s \in (0,1)$ independent of h.

By (a$_1$) $\widetilde{v}_h \in H^1(Q_{r'})$ and $\operatorname{supp}(\widetilde{v}_h - u_h) \subset\subset Q_r$ for all sufficiently large h. By (6.15), (6.19) and the relation $\delta_h^{2n} = \mathcal{H}^{n-1}(Q_1 \cap J_{u_h}) \to 0$ it follows that $\widetilde{v}_h \to u$ a.e. in Q_1. Define

$$v_h := (1 - \psi)\widetilde{v}_h + \psi v. \tag{6.21}$$

Then (A_h, v_h) is an admissible configuration for $\Phi_h(A_h, u_h; Q_1)$ in (6.6). Therefore from (6.10c) and the definition of deviation it follows that

$$\mathcal{F}_h(A_h, u_h; Q_1) \le \mathcal{F}_h(A_h, v_h; Q_1) + o(1), \tag{6.22}$$

where $o(1) \to 0$ as $h \to \infty$. Note that by (a$_1$), (a$_2$), (6.13) and (6.10b)

$$\begin{aligned}\mathcal{S}(A_h, v_h; Q_1) - \mathcal{S}(A_h, u_h; Q_1) &= \int_{A_h^{(1)} \cap J_{\widetilde{v}_h}} \phi_h(x, \nu_{J_{\widetilde{v}_h}}) \mathrm{d}\mathcal{H}^{n-1} - \int_{A_h^{(1)} \cap J_{u_h}} \phi_h(x, \nu_{J_{u_h}}) \mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_{A_h^{(1)} \cap (J_{\widetilde{v}_h} \setminus J_{u_h}) \cap (Q_r \setminus Q_{r-\delta_h})} \phi_h(x, \nu_{J_{\widetilde{v}_h}}) \mathrm{d}\mathcal{H}^{n-1} \\ &\le b_h'' \mathcal{H}^{n-1}(J_{\widetilde{v}_h} \setminus J_{u_h}) \le c b_h'' \delta_h \mathcal{H}^{n-1}(J_{u_h} \cap (Q_r \setminus Q_{r-\delta_h})) \\ &\le \frac{c\delta_h}{d_5} \mathcal{S}(A_h, u_h; Q_1) \le \frac{Mc\delta_h}{d_5}.\end{aligned}$$

This estimate, (6.22) and the definition of localized elastic energy imply

$$\int_{A_h \cap Q_1} \mathbb{C}_h(x) \mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}\mathcal{H}^{n-1} \le \int_{A_h \cap Q_1} \mathbb{C}_h(x) \mathcal{E}v_h : \mathcal{E}v_h \mathrm{d}\mathcal{H}^{n-1} + o(1) \tag{6.23}$$

as $h \to +\infty$.

Next we estimate the integral in the right-hand-side of (6.23). By (6.21)

$$\mathcal{E}v_h = (1-\psi)\mathcal{E}\widetilde{v}_h + \psi \mathcal{E}v + \nabla\psi \odot (v - \widetilde{v}_h),$$

where $X \odot Y = (X \otimes Y + Y \otimes X)/2$. Since $\widetilde{v}_h \to u$ a.e. in Q_r and $u = v$ in $Q_r \setminus Q_{r'}$, one has $v_h \to v$ a.e. in Q_1.

We claim that $\widetilde{v}_h \to u$ strongly in $L^2_{\mathrm{loc}}(Q_r)$. Indeed, fix any $\rho \in (0, r)$. By (a$_1$) $\widetilde{v}_h \in H^1(Q_\rho)$. By (6.8), (6.10b) and (6.20) (applied with $\eta = 1$)

$$d_3 \int_{Q_\rho} |\mathcal{E}\widetilde{v}_h|^2 \mathrm{d}x \le d_3 \int_{Q_r} |\mathcal{E}\widetilde{v}_h|^2 \mathrm{d}x \le C \int_{Q_r} \mathbb{C}_h(x) \mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}x \le MC$$

for some constant $C > 0$ independent of h. Moreover, by the Poincaré-Korn inequality for each h there exist a rigid displacement e_h (possibly depending also on ρ) such that

$$\|\widetilde{v}_h - e_h\|_{H^1(Q_\rho)} \le \int_{Q_\rho} |\mathcal{E}\widetilde{v}_h|^2 \mathrm{d}x \le \frac{MCC'}{d_3}$$

and hence, the Rellich-Kondrachov Theorem implies the existence of $w \in H^1(Q_\rho)$ and not relabelled subsequence such that $\widetilde{v}_h - e_h \to w$ in $L^2(Q_\rho)$. Since $\widetilde{v}_h \to u$ a.e. in Q_1, $e_h \to w - u$ and hence, $e := u - w$ is also a rigid displacement. Then

$$\limsup_{h\to\infty} \|\widetilde{v}_h - u\|_{L^2(Q_\rho)} \le \limsup_{h\to\infty} \|\widetilde{v}_h - e_h - w\|_{L^2(Q_\rho)} + \limsup_{h\to\infty} \|e_h + (w-u)\|_{L^2(Q_\rho)} = 0,$$

and the claim follows.

Since $u = v$ out of $\{\psi = 1\}$, the claim implies $\widetilde{v}_h \to v$ strongly in $L^2(\{0 < \psi < 1\})$, and hence,

$$\lim_{h\to\infty} \int_{Q_r} |\nabla\psi \odot (v - \widetilde{v}_h)\big|_{A_h}|^2 \mathrm{d}x \le \liminf_{h\to\infty} \int_{\{0<\psi<1\}} |\nabla\psi \odot (v - \widetilde{v}_h)|^2 \mathrm{d}x = 0. \tag{6.24}$$

Thus, by definition (6.21) of v_h

$$\int_{Q_r \cap A_h} \mathbb{C}_h \mathcal{E}v_h : \mathcal{E}v_h \mathrm{d}x = \int_{Q_r \cap A_h} (1-\psi)^2 \mathbb{C}_h \mathcal{E}\widetilde{v}_h : \mathcal{E}\widetilde{v}_h \mathrm{d}x + \int_{Q_r \cap A_h} \psi^2 \mathbb{C}_h \mathcal{E}v : \mathcal{E}v \mathrm{d}x$$

$$
\begin{aligned}
&+\int_{Q_r\cap A_h}\mathbb{C}_h(\nabla\psi\odot(v-\widetilde{v}_h)):(\nabla\psi\odot(v-\widetilde{v}_h))\mathrm{d}x\\
&+\int_{Q_r\cap A_h}(1-\psi)\mathbb{C}_h\mathcal{E}\widetilde{v}_h:(\nabla\psi\odot(v-\widetilde{v}_h))\mathrm{d}x\\
&+\int_{Q_r}\psi\mathbb{C}_h\mathcal{E}v:(\nabla\psi\odot(v-\widetilde{v}_h))\mathrm{d}x\\
=&\int_{Q_r\cap A_h}(1-\psi)^2\mathbb{C}_h\mathcal{E}\widetilde{v}_h:\mathcal{E}\widetilde{v}_h\mathrm{d}x+\int_{Q_r\cap A_h}\psi^2\mathbb{C}_h\mathcal{E}v:\mathcal{E}v\mathrm{d}x+o(1)\\
\le&\int_{Q_r\cap A_h}(1-\psi)^2\mathbb{C}_h\mathcal{E}u_h:\mathcal{E}u_h\mathrm{d}x+\int_{Q_r\cap A_h}\psi^2\mathbb{C}_h\mathcal{E}v:\mathcal{E}v\mathrm{d}x+o(1),
\end{aligned}
\tag{6.25}
$$

where in the second equality we use (6.10b), (6.20) with $\eta\equiv 1$, (6.24), (6.8) and the Hölder inequality, while in the last inequality we use (6.20) with $\eta=(1-\psi)^2$ and (6.10d). Now combining (6.25) with (6.23) we get

$$
\int_{Q_r}(2\psi-\psi^2)\mathbb{C}_h\mathcal{E}u_h:\mathcal{E}u_h\mathrm{d}x\le\int_{Q_r}\psi^2\mathbb{C}_h\mathcal{E}v:\mathcal{E}v\mathrm{d}x+o(1).\tag{6.26}
$$

Since $\{\mathbb{C}_h\}$ is equibounded (see (6.8)) and equicontinuous, by the Arzela-Ascoli Theorem, there exist a (not relabelled) subsequence and an elasticity tensor $\mathbb{C}\in C^0(Q_1;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ such that $\mathbb{C}_h\to\mathbb{C}$ uniformly in Q_1. Hence, letting $h\to\infty$ in (6.26) and using (6.16) and the convexity of the elastic energy, we obtain

$$
\int_{Q_r}(2\psi-\psi^2)\mathbb{C}(y)\mathcal{E}u:\mathcal{E}u\,\mathrm{d}y\le\int_{Q_r}\psi^2\mathbb{C}(y)\mathcal{E}v:\mathcal{E}v\,\mathrm{d}y.\tag{6.27}
$$

By the choice of ψ, (6.27) implies

$$
\int_{Q_{r''}}\mathbb{C}(y)\mathcal{E}u:\mathcal{E}u\,\mathrm{d}y\le\int_{Q_r}\mathbb{C}(y)\mathcal{E}v:\mathcal{E}v\,\mathrm{d}y.\tag{6.28}
$$

Since r'' is arbitrary, letting $r''\nearrow r$ we deduce that (6.28) holds also with $r''=r$. Since $\mathrm{supp}\,(u-v)\subset\subset Q_r$, this implies (6.11).

It remains to prove (6.12). If we take $v=u$ in (6.26) and use $0\le\psi\le 1$ and $\psi=1$ in $Q_{r''}$ we get

$$
\begin{aligned}
\int_{Q_{r''}}\mathbb{C}\mathcal{E}u:\mathcal{E}u dx\le&\liminf_{h\to\infty}\int_{Q_{r''}}\mathbb{C}_h\mathcal{E}u_h:\mathcal{E}u_h\mathrm{d}x\\
&\le\limsup_{h\to\infty}\int_{Q_{r''}}\mathbb{C}_h\mathcal{E}u_h:\mathcal{E}u_h\mathrm{d}x\le\int_{Q_r}\mathbb{C}\mathcal{E}u:\mathcal{E}u\mathrm{d}x.
\end{aligned}
$$

Since r'' is arbitrary, letting $r''\nearrow r$ we deduce

$$
\lim_{h\to\infty}\int_{Q_r}\mathbb{C}_h\mathcal{E}u_h:\mathcal{E}u_h\mathrm{d}x=\int_{Q_r}\mathbb{C}\mathcal{E}u:\mathcal{E}u\mathrm{d}x.\tag{6.29}
$$

In view of (6.29) to prove (6.12) it suffices to establish

$$
\lim_{h\to\infty}\mathcal{S}_h(A_h;Q_r)=0\tag{6.30}
$$

for any $r\in(0,1)$. By (6.10e) $Q_1\cap\partial^*A_h\subset J_{u_h}$ up to an $\mathcal{H}^{n-1}$-negligible set. Thus, by (6.10d) and the relative isoperimetric inequality, up to a subsequence, either

$$
\lim_{h\to\infty}|Q_1\cap A_h|=0\tag{6.31}
$$

or

$$\lim_{h\to\infty} |Q_1 \setminus A_h| = 0. \tag{6.32}$$

We claim that there exists a not relabelled subsequence $\{A_h\}$ such that for a.e. $t \in (0,1)$

$$\lim_{h\to\infty} \int_{A_h\cap\partial Q_t} \phi_h(x,\nu_{Q_t})\mathrm{d}\mathcal{H}^{n-1} = 0 \tag{6.33}$$

if (6.31) holds, and

$$\lim_{h\to\infty} \int_{(Q_1\setminus A_h)\cap\partial Q_t} \phi_h(x,\nu_{Q_t})\mathrm{d}\mathcal{H}^{n-1} = 0 \tag{6.34}$$

if (6.32) holds.

We establish only (6.31), the proof of (6.34) being similar. By the coarea formula (applied with $f(x) = \max\{|x_1|,\dots,|x_n|\}$)

$$\lim_{h\to\infty} |Q_1 \cap A_h| = \lim_{h\to\infty} \int_0^1 \mathcal{H}^{n-1}(A_h\cap\partial Q_t)\mathrm{d}t = 0,$$

thus, passing to further not relabelled subsequence, $\lim\limits_{h\to\infty} \mathcal{H}^{n-1}(A_h\cap\partial Q_t) = 0$ for a.e. $t\in(0,1)$. In particular, if $\sup_h b''_h < +\infty$, then

$$\limsup_{h\to+\infty} \int_{A_h\cap\partial Q_t} \phi_h(x,\nu_{Q_t})\mathrm{d}\mathcal{H}^{n-1} \le \limsup_{h\to+\infty}\ b''_h\mathcal{H}^{n-1}(A_h\cap\partial Q_t) = 0. \tag{6.35}$$

On the other hand, if $b''_h \to +\infty$ (up to a subsequence), then by the coarea formula and the relative isoperimetric inequality in Q_1

$$b''_h \int_0^1 \mathcal{H}^{n-1}(A_h\cap\partial Q_t)\mathrm{d}t = b''_h|A_h\cap Q_1| \le b''_h c_n P(A_h,Q_1)^{\frac{n}{n-1}}, \tag{6.36}$$

where $c_n > 0$ is the relative isoperimetric inequality for cubes. By (6.9)

$$P(A_h,Q_1) \le \frac{1}{a_h}\,\mathcal{S}(A_h,Q_1) \le \frac{1}{d_5 b''_h}\,\mathcal{S}(A_h,Q_1) \le \frac{\mathcal{F}_h(A_h,u_h,Q_1)}{d_5 b''_h},$$

hence, by (6.36)

$$b''_h \int_0^1 \mathcal{H}^{n-1}(A_h\cap\partial Q_t)\mathrm{d}t \le c_n \left[\frac{M}{d_5}\right]^{\frac{n}{n-1}} [b''_h]^{-\frac{1}{n-1}}.$$

This and (6.10b) imply

$$\lim_{h\to+\infty} b''_h \int_0^1 \mathcal{H}^{n-1}(A_h\cap\partial Q_t)\mathrm{d}t = 0.$$

In particular,

$$\lim_{h\to+\infty}\ b''_h\mathcal{H}^{n-1}(A_h\cap\partial Q_t) = 0$$

for a.e. $t\in(0,1)$. Now the proof of (6.33) follows as in (6.35).

Now we prove (6.30) assuming (6.31). Given $t \in (r,1)$ for which (6.33) holds, define $E_h := A_h \setminus Q_t$. Then (E_h,u_h) is an admissible configuration in (6.6), and thus,

$$\mathcal{F}_h(A_h,u_h;Q_1) \le \Phi_h(A_h,u_h;Q_1) + o(1) \le \mathcal{F}_h(E_h,u_h;Q_1) + o(1), \tag{6.37}$$

where in the first inequality we use (6.10c) and in the second we use the definition of Φ_h. From the definition of E_h and (6.37) it follows that

$$\mathcal{S}_h(A_h;Q_t) \le \int_{A_h\cap\partial Q_t} \phi_h(x,\nu_{Q_t})\mathrm{d}\mathcal{H}^{n-1} + o(1).$$

This and (6.33) imply (6.30).

Now suppose that (6.32) holds. Let δ_h be defined as in (6.18), and let and ψ, and $r'' < r' < r$ and v_h be as in (6.21) with $v = u$. Fix any $t \in (r, 1)$ for which (6.34) holds and set $E_h := A_h \cup \overline{Q_t}$. Then for sufficiently large h that (E_h, v_h) is an admissible configuration for $\Phi_h(A_h, u_h; Q_1)$ in (6.6). Thus by (6.10c)

$$\mathcal{F}_h(A_h, u_h; Q_1) \le \mathcal{F}_h(E_h, v_h; Q_1) + o(1).$$

By the definition of $\mathcal{F}_h$, as in the proof of (6.26) we establish

$$\mathcal{S}_h(A_h; Q_t) + \int_{Q_r} (2\psi - \psi^2)\mathbb{C}_h\mathcal{E}u_h : \mathcal{E}u_h \mathrm{d}x \\ \le \int_{Q_r} \psi^2 \mathbb{C}_h\mathcal{E}u : \mathcal{E}u \mathrm{d}x + \int_{(Q_1\setminus A_h)\cap\partial Q_t} \phi_h(x, \nu_{Q_t})\mathrm{d}\mathcal{H}^{n-1} + o(1).$$

Thus, as in (6.27) letting $h \to \infty$ we obtain

$$\limsup_{h\to\infty} \mathcal{S}_h(E_h; Q_t) + \int_{Q_r} (2\psi - \psi^2)\mathbb{C}\mathcal{E}u : \mathcal{E}u dx \le \int_{Q_r} \psi^2 \mathbb{C}\mathcal{E}u : \mathcal{E}u dx. \tag{6.38}$$

Since $\psi = 1$ in $Q_{r''}$ and $|\psi| \le 1$, from (6.38) it follows that

$$\limsup_{h\to\infty} \mathcal{S}_h(A_h; Q_t) + \int_{Q_{r''}} \mathbb{C}\mathcal{E}u : \mathcal{E}u dx \le \int_{Q_r} \mathbb{C}\mathcal{E}u : \mathcal{E}u dx.$$

Now letting $r'' \to r$ we get (6.30). □

Recall that by [42, Theorem 6.2.1] if the elasticity tensor $\mathbb{C}$ is constant and satisfies (2.10), then there exists $C_{b_3,b_4} > 0$ such that every local minimizer $u \in H^1(Q_1(x_0))$ of the functional

$$v \in H^1(Q_1(x_0); \mathbb{R}^n) \mapsto \int_{Q_1(x_0)} \mathbb{C}\mathcal{E}v : \mathcal{E}v \mathrm{d}x \tag{6.39}$$

is analytic in $Q_1(x_0)$ and satisfies

$$\int_{Q_r(x_0)} \mathbb{C}\mathcal{E}u : \mathcal{E}u \, \mathrm{d}x \le C_{b_3,b_4}\, r^n \int_{Q_1(x)} \mathbb{C}\mathcal{E}u : \mathcal{E}u \, \mathrm{d}x \tag{6.40}$$

for any $r \in (0, 1/2)$. Let

$$\tau_0 := \tau_0(b_3, b_4) := (1 + C_{b_3,b_4})^{-2}. \tag{6.41}$$

Using Proposition 6.3 and repeating similar arguments in [13] we get the following decay property of the functional $\mathcal{F}$.

Proposition 6.4. *Assume (H1)-(H3). For any $\tau \in (0, \tau_0)$ there exist $\varsigma = \varsigma(\tau) \in (0, 1)$, $\vartheta := \vartheta(\tau) > 0$ and $R := R(\tau) > 0$ such that if $(A, u) \in \mathcal{C}$ satisfies*

$$Q_\rho(x) \cap \partial^* A \subseteq J_u,$$
$$\int_{Q_\rho(x)\setminus A} |\mathcal{E}u| \mathrm{d}x = 0,$$
$$\mathcal{H}^{n-1}(Q_\rho(x) \cap J_u) < \varsigma\rho^{n-1},$$
$$\mathcal{F}(A, u; Q_\rho(x)) \le (1 + \vartheta)\Phi(A, u; Q_\rho(x))$$

for some $Q_\rho(x) \subset\subset \Omega$ with $0 < \rho < R$, then

$$\mathcal{F}(A, u; Q_{\tau\rho}(x)) \le \tau^{n-1/2}\, \mathcal{F}(A, u; Q_\rho(x)).$$

Proof. Assume by contradiction that there exist $\tau \in (0,\tau_0)$, positive real numbers $\varsigma_h, \vartheta_h, \rho_h \to 0$, cubes $Q_{\rho_h}(x_h) \subset\subset \Omega$, and admissible configurations $(A_h,u_h) \in \mathcal{C}$ such that

$$Q_{\rho_h}(x_h) \cap \partial^* A_h \subseteq J_{u_h}, \tag{6.42a}$$

$$\int_{Q_{\rho_h}(x_h)\setminus A_h} |\mathcal{E}u_h| \mathrm{d}x = 0, \tag{6.42b}$$

$$\mathcal{H}^{n-1}(Q_{\rho_h}(x_h) \cap J_{u_h}) \le \varsigma_h \rho_h^{n-1}, \tag{6.42c}$$

$$\mathcal{F}(A_h,u_h;Q_{\rho_h}(x_h)) \le (1+\vartheta_h)\Phi(A_h,u_h;Q_{\rho_h}(x_h)), \tag{6.42d}$$

but

$$\mathcal{F}(A_h,u_h;Q_{\tau\rho_h}(x_h)) > \tau^{n-1/2}\mathcal{F}(A_h,u_h;Q_{\rho_h}(x_h)) \tag{6.43}$$

for any h. Note that $\mathcal{F}(A_h,u_h;Q_{\rho_h}(x_h)) > 0$. Let us define the rescaled energy $\mathcal{F}_h(\cdot;Q_1)$ as in (6.5) with

$$\phi_h(y,\nu) := \frac{(\rho_h/2)^{n-1}\varphi(x_h + \frac{1}{2}\rho_h y,\nu)}{\mathcal{F}(A_h,u_h;Q_{\rho_h}(x_h))}$$

in place of $\varphi(y,\nu)$ and

$$\mathbb{C}_h(y) := \mathbb{C}(x_h + \rho_h y)$$

in place of $\mathbb{C}(y)$, for $y \in Q_1$. In view of (6.42a)-(6.42d) for

$$E_h := \sigma_{x_h,\rho_h}(A_h)$$

(see definition of blow-up map $\sigma_{x,r}$ at (2.1)) and

$$v_h(y) := \frac{(\rho_h/2)^{\frac{n-2}{2}}\, u_h(x_h + \frac{1}{2}\rho_h y)}{\sqrt{\mathcal{F}(A_h,u_h;Q_{\rho_h}(x_h))}}$$

we have

$$\begin{aligned}
&\mathcal{F}_h(E_h,v_h;Q_1) = 1,\\
&Q_1 \cap \partial^* E_h \subset_{\mathcal{H}^{n-1}} J_{v_h},\\
&\int_{Q_1\setminus E_h} |\mathcal{E}v_h| \mathrm{d}x = 0,\\
&\mathcal{H}^{n-1}(Q_1 \cap \partial J_{v_h}) < 2^{n-1}\varsigma_h,\\
&\Psi_h(E_h,v_h;Q_1) \le \vartheta_h \Phi_h(E_h,v_h;Q_1) \le \vartheta_h \mathcal{F}_h(E_h,v_h;Q_1) = \vartheta_h,
\end{aligned}$$

where Φ_h and Ψ_h are defined as in (6.6) and (6.7) (with φ_h and $\mathbb{C}_h$ in places of φ and $\mathbb{C}$, respectively). By the boundednes of Ω, there exists $x_0 \in \overline{\Omega}$ such that, up to extracting a subsequence, $x_h \to x_0$ as $h \to +\infty$. In particular, $x_h + \rho_h y \to x_0$ for every $y \in \overline{Q_1}$. Then the uniform continuity of $\mathbb{C}$ implies that $\mathbb{C}_h \to \mathbb{C}_0 := \mathbb{C}(x_0)$ uniformly in $\overline{Q_1}$. Also by (2.8) ϕ_h satisfies (6.9) with $d_5 := b_1/b_2$. Thus, by Proposition 6.3 there exist $v \in H^1(Q_1)$ and infinitesimal rigid displacements a_h such that, up to a subsequence,

$$w_h := v_h - a_h \to v$$

pointwise a.e. in Q_1, $\mathcal{E}w_h \rightharpoonup \mathcal{E}v$ in $L^2(Q_1)$ as $h \to +\infty$, and

$$\lim_{h\to+\infty} \mathcal{F}_h(E_h,v_h;Q_r) = \lim_{h\to+\infty} \mathcal{F}_h(E_h,w_h;Q_r) = \int_{Q_r} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v \mathrm{d}x \tag{6.44}$$

for any $r \in (0,1]$. In particular, from (6.43) and (6.44) it follows that

$$\int_{Q_\tau} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v\mathrm{d}x = \lim_{h\to+\infty} \mathcal{F}(E_h, v_h; Q_\tau) \\ \ge \tau^{n-1/2} \lim_{h\to+\infty} \mathcal{F}(E_h, v_h; Q_1) \\ = \tau^{n-1/2} \int_{Q_1} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v\mathrm{d}x.$$

Since $\mathcal{F}_h(E_h, v_h; Q_1) = 1$, by (6.44) $\int_{Q_1} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}vdx = 1$. Moreover, as $\mathbb{C}_0$ is constant and v is a local minimizer of (6.39), applying (6.40) with $r := \tau$ and $R := 1$ we get

$$C_{b_3,b_4}\tau^n = C_{b_3,b_4}\tau^n \int_{Q_1} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v\mathrm{d}x \ge \int_{Q_\tau} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v\mathrm{d}x \\ \ge \tau^{n-1/2} \int_{Q_1} \mathbb{C}_0(x)\mathcal{E}v : \mathcal{E}v\mathrm{d}x = \tau^{n-1/2},$$

which contradicts to the assumption $\tau < \tau_0$. □

By employing the arguments of [43, Section 4.3] and using Proposition 6.4 we establish the following lower bound for $\mathcal{F}$.

Proposition 6.5. *Given $\tau \in (0,\tau_0)$, let $\varsigma := \varsigma(\tau) \in (0,1)$, $\vartheta = \vartheta(\tau) > 0$ and $R := R(\tau) > 0$ be as in Proposition 6.4 and let*

$$R_0 := R_0(\Theta, \tau, b_1) := \min\left\{R(\tau), \frac{b_1 n\omega_n^{1/n}\vartheta}{\Theta(2+\vartheta)}\right\}, \quad \Theta > 0.$$

Let $(A,u) \in \mathcal{C}$ be a Θ-minimizer of $\mathcal{F}$ in $Q_{r_0}(x_0)$ such that $\Omega \cap \partial^* A \subset_{\mathcal{H}^{n-1}} J_u$ and $\int_{\Omega\setminus A} |\mathcal{E}u|\mathrm{d}x = 0$. Then for any $x \in Q_{r_0}(x_0) \cap \overline{J_u^*}$, where J_u^* is given by (6.1), and any cube $Q_\rho(x) \subset Q_{r_0}(x_0)$ with $\rho \in (0, R_0)$ one has

$$\mathcal{F}(A, u; Q_\rho(x)) \ge b_1\varsigma\rho^{n-1}. \tag{6.45}$$

Proof. Let $(C,w), (D,v) \in \mathcal{C}$ and $O \subset \Omega$ be such that $C\Delta D \subset\subset O$. By the isoperimetric inequality, the inclusion $\partial^*(C\Delta D) \subset O \cap (\partial^* C \cup \partial^* D)$, (2.8), the definition of $\mathcal{S}(\cdot; O)$ and the nonnegativity of $\mathcal{W}(\cdot; O)$ one has

$$n\omega_n^{1/n} |C\Delta D|^{\frac{n-1}{n}} \le P(C\Delta D) \le P(C,O) + P(D,O) \\ \le \frac{\mathcal{S}(C,w,O) + \mathcal{S}(D,v,O)}{b_1} \le \frac{\mathcal{F}(C,w;O) + \mathcal{F}(D,v;O)}{b_1}, \tag{6.46}$$

From (6.46) and the Θ-minimality of (A,u) in $Q_{r_0}(x_0)$ we deduce

$$\mathcal{F}(A,u;Q_r(x)) \le \mathcal{F}(B,v;Q_r(x)) + \Theta|A\Delta B|^{\frac{1}{n}}|A\Delta B|^{\frac{n-1}{n}} \\ \le \mathcal{F}(B,v;Q_r(x)) + \frac{\Theta r}{b_1 n\omega_n^{1/n}}\Big(\mathcal{F}(A,u;Q_r(x)) + \mathcal{F}(B,v;Q_r(x))\Big) \tag{6.47}$$

for any $Q_r(x) \subset Q_{r_0}(x_0)$ and $(B,v) \in \mathcal{C}$ with $A\Delta B \subset\subset Q_r(x)$ and $\operatorname{supp}(u-v) \subset\subset Q_r(x)$, where in the last inequality we used the inequality $|A\Delta B| \le |Q_r| = r^n$. By the choice of R_0, if $r \in (0,R_0)$, then $\frac{\Theta r}{b_1 n\omega_n^{1/n}} \le \frac{\vartheta}{2+\vartheta}$, and thus, by (6.47)

$$\mathcal{F}(A,u;Q_r(x)) \le (1+\vartheta)\mathcal{F}(B,v;Q_r(x)).$$

By the arbitrariness of (B,v) this inequality is equivalent to

$$\mathcal{F}(A,u;Q_r(x)) \le (1+\vartheta)\Phi(A,u;Q_r(x)). \tag{6.48}$$

Now we prove (6.45). Fix any $x \in J_u^*$; for simplicity we suppose that $x=0$. By contradiction, assume that

$$\mathcal{F}(A,u;Q_\rho) < b_1\varsigma\rho^{n-1}$$

for some $Q_\rho \subset\subset Q_{r_0}(x_0)$ with $\rho \in (0,R_0)$. Then by the nonnegativity of the elastic energy and (2.8) one has

$$b_1\varsigma\rho^{n-1} > \mathcal{S}(A,u;Q_\rho) \ge b_1\mathcal{H}^{n-1}(Q_\rho \cap J_u)$$

so that

$$\mathcal{H}^{n-1}(Q_\rho \cap J_u) < \varsigma\rho^{n-1}.$$

By Proposition 6.4 and the definition (6.41) of τ_0

$$\mathcal{F}(A,u;Q_{\tau\rho}) \le \tau^{n-1/2}\mathcal{F}(A,u;Q_\rho) < b_1\varsigma(\tau\rho)^{n-1}$$

so that

$$\mathcal{H}^{n-1}(Q_{\tau\rho} \cap J_u) < \varsigma(\tau\rho)^{n-1}.$$

Then by induction,

$$\mathcal{H}^{n-1}(Q_{\tau^m\rho} \cap J_u) < \varsigma(\tau^m\rho)^{n-1} \quad \text{for any } m \ge 1.$$

However, by the definition of J_u^*

$$1 = \lim_{m\to+\infty} \frac{\mathcal{H}^{n-1}(Q_{\tau^m\rho} \cap J_u)}{(\tau^m\rho)^{n-1}} \le \frac{2b_1\varsigma}{2b_1} = \varsigma < 1,$$

a contradiction. Hence, (6.45) holds for any $x \in J_u^*$. Note that the map $\mathcal{F}(A,u;\cdot)$, defined for open sets $O \subset\subset Q_{r_0}(x_0)$ extends to a positive Borel measure in $Q_{r_0}(x_0)$, and therefore, by continuity of Borel measures, (6.45) extends also for $x \in Q_{r_0}(x_0) \cap \overline{J_u^*}$. □

Now we are ready to prove (6.2) and (6.3).

Proof of Theorem 6.1. Let (A,u) be a minimizer of $\mathcal{F}$ such that $\Omega \cap \partial^* A \subset J_u$ and $\int_{\Omega\setminus A} |Eu| \mathrm{d}x = 0$ and let $\lambda_0 > 0$ be given by Theorem 2.4. Since (A,u) is also a minimizer of $\mathcal{F}^{\lambda_0}$, for any open set $O \subset \Omega$ and $(B,v) \in \mathcal{C}$ with $A\Delta B \subset\subset O$ and $\operatorname{supp}(u-v) \subset\subset O$ we have

$$\mathcal{F}(A,u;O) \le \mathcal{F}(B,v;O) + \lambda_0\big||A|-|B|\big| \le \mathcal{F}(B,v;O) + \lambda_0\big|A\Delta B\big|.$$

Hence, (A,u) is λ_0-minimizer of $\mathcal{F}(\cdot;\Omega)$ in Ω.

Let us prove (6.2). Fix $x \in \Omega$ and let $r_x := \min\{1, \operatorname{dist}(x,\partial\Omega)\}$. Then by the λ_0-minimality of (A,u) for any $r \in (0,r_x)$ and $\rho \in (r,r_x)$

$$\mathcal{F}(A,u;Q_\rho(x)) \le \mathcal{F}(A \setminus \overline{Q_r}, u; Q_\rho(x)) + \lambda_0|Q_r(x) \cap A|, \tag{6.49}$$

where for shortness $Q_r := Q_r(x)$. Since $\mathcal{F}(A,u;Q_\rho(x) \setminus \overline{Q_r(x)}) = \mathcal{F}(A \setminus \overline{Q_r(x)}, u; Q_\rho(x) \setminus \overline{Q_r(x)})$, from (6.49) and the definition and nonnegativity of $\mathcal{F}$ we get

$$\mathcal{F}(A,u;Q_r(x)) \le \int_{\partial Q_r(x)} \varphi(x,\nu_{Q_r(x)}) d\mathcal{H}^{n-1} + \lambda_0 r^n.$$

By (2.8)

$$\int_{\partial Q_r(x)} \varphi(x,\nu_{Q_r(x)}) d\mathcal{H}^{n-1} \le b_2\mathcal{H}^{n-1}(\partial Q_r(x)) = 2nb_2r^n$$

thus, using $r \le 1$ we obtain

$$\mathcal{F}(A,u;\overline{Q_r(x)}) \le (2nb_2+\lambda_0)r^{n-1}. \tag{6.50}$$

Using the nonnegativity of $\mathcal{W}(A,u;Q_r(x))$, (2.8) and the equality $Q_r(x)\cap J_u = (Q_r(x)\cap \partial^* A)\cup (Q_r(x)\cap A^{(1)}\cap J_u)$ in (6.50) we get

$$\mathcal{F}(A,u;Q_r(x)) \ge \mathcal{S}(A,u;Q_r(x)) \ge b_1\mathcal{H}^{n-1}(Q_r(x)\cap J_u).$$

Therefore,

$$\mathcal{H}^{n-1}(Q_r(x)\cap J_u) \le \frac{2nb_2+\lambda_0}{b_1}\,r^{n-1}.$$

Next we prove (6.3). Fix $x\in \overline{J_u^*}$. For τ_0, given by (6.41), let $\varsigma_o=\varsigma(\tau_0/2)\in(0,1)$ and $R_o = R_0(\tau_0/2,b_1,b_2,\lambda_0)>0$ be as in Proposition 6.5. By (6.45)

$$\mathcal{F}(A,u;Q_{\gamma r}(x)) \ge b_1\varsigma_o(\gamma r)^{n-1} \tag{6.51}$$

for any $\gamma\in(0,1)$ and $r\in(0,R_o)$ with $Q_r(x)\subset\Omega$. Let

$$\varsigma_* := \varsigma(\tau_*),\quad \vartheta_* := \vartheta(\tau^*)\quad\text{and}\quad R_* := \min\{R(\tau_*),R_o\}$$

be given by Proposition 6.4 for

$$2\tau_* := \min\Big\{\frac{\tau_0}{2},\Big(\frac{b_1\varsigma_o}{2nb_2+\lambda_0}\Big)^2\Big\} \tag{6.52}$$

By contradiction, if $\mathcal{H}^{n-1}(Q_r(x)\cap J_u)<\varsigma_* r^{n-1}$, then applying (6.48) with $\tau=\tau_*$ we get

$$\mathcal{F}(A,u;Q_r(x)) \le (1+\vartheta_*)\Phi(A,u;Q_r(x)).$$

Hence, by Proposition 6.4

$$\mathcal{F}(A,u;Q_{\tau_* r}(x)) \le \tau_*^{n-1/2}\mathcal{F}(A,u;Q_r(x))$$

so that by (6.51) and (6.50)

$$\tau_*^{1/2} \ge \frac{b_1\varsigma_o}{2nb_2+\lambda_0},$$

which contradicts to (6.52).

Finally, (6.4) follows from the density estimates together with a covering argument. □

From Theorem 6.1 we get the partial regularity of minimizers of $\mathcal{F}$.

Proof of Theorem 2.7. (i)-(iii). Let $(\widetilde{A},\widetilde{u})\in\mathcal{C}$ be a minimizer of $\mathcal{F}$ and let

$$A' := \widetilde{A}^{(1)},\qquad u' := \widetilde{u}\chi_{A'\cup S}+\xi'\chi_{\Omega\setminus A'},$$

where $\xi'\in(0,1)^n$ is chosen such that $\Omega\cap\partial^*A'\subset J_{u'}$. By [41, Chapter 15], $\partial A' = \overline{\partial^* A'}$. Clearly, (A',u') is a minimizer of $\mathcal{F}$, and by Theorem 6.1 $\mathcal{H}^{n-1}(\overline{J_{u'}^*}\setminus J_{u'}^*)=0$. Since $J_{u'}$ is rectifiable, by [3, Theorem 2.63] $\mathcal{H}^{n-1}(J_{u'}\setminus J_{u'}^*)=0$ and hence observing $\Omega\cap\partial A' = \Omega\cap\overline{\partial^*A'}\subset\overline{J_{u'}}$ we observe

$$\mathcal{H}^{n-1}(A'\setminus \mathrm{Int}(A')) \le \mathcal{H}^{n-1}(\partial A') \le \mathcal{H}^{n-1}(\partial\Omega)+\mathcal{H}^{n-1}(J_{u'})<+\infty.$$

Now let

$$A := \mathrm{Int}(A')\quad\text{and}\quad u := \widetilde{u}\chi_{A\cup S}+\xi'\chi_{\Omega\setminus A}.$$

Since $|A\Delta A'|\le|\partial A'|=0$, $u=u'$ a.e. in $\Omega\cup S$ and hence, (A,u) is also a minimizer of $\mathcal{F}$. Moreover,

$$\mathcal{H}^{n-1}(\widetilde{A}^{(1)}\setminus A)\le\mathcal{H}^{n-1}(\partial A')<+\infty,\qquad \mathcal{H}^{n-1}(J_u\setminus J_u^*)=\mathcal{H}^{n-1}(J_{u'}\setminus J_{u'}^*)=0,$$

and

$$\mathcal{H}^{n-1}(\overline{J_u^*}\setminus J_u^*) = \mathcal{H}^{n-1}(\overline{J_{u'}^*}\setminus J_{u'}^*) = 0.$$

Thus, (i) follows. The assertions (ii) and (iii) directly follow from the minimality of (A,u) and Theorem 6.1.

(iv). Finally, if $E\subset A$ is a connected component of (the open set) A with $\mathcal{H}^{n-1}(\partial^*E\cap\Sigma\setminus J_u)=0$, then for $v := u\chi_{A\cup S\setminus E} + u_0\chi_E$ we have

$$\mathcal{S}(A,u)\ge\mathcal{S}(A,v)$$

and

$$\mathcal{W}(A,u)\ge\mathcal{W}(A,v). \tag{6.53}$$

In (6.53) the equality holds if and only of $u=u_0$ in E. Therefore, by the minimality of (A,u) it follows that $u=u_0$ in E (up to an additive rigid displacement). It remains to prove

$$|E|\ge\omega_n\Big(\frac{b_1 n}{\lambda_0}\Big)^n.$$

Consider the competitor $(A\setminus E,u)\in\mathcal{C}$. Since (A,u) solves (2.13), $\mathcal{F}^{\lambda_0}(A,u)\le\mathcal{F}^{\lambda_0}(A\setminus E,u)$ so that using $u=u_0$ in E and the additivity of the surface energy, we get

$$\int_{\partial^*E}\varphi(x,\nu_E)\mathrm{d}\mathcal{H}^{n-1}\le\lambda_0|E|.$$

Using (2.8) and the isoperimetric inequality in this estimate we obtain

$$\lambda_0|E|\ge b_1P(E)\ge b_1n\omega_n^{1/n}|E|^{\frac{n-1}{n}}.$$

Hence, $|E|\ge(b_1n\omega_n^{1/n}/\lambda_0)^n$ and (iv) follows. □

Appendix A.

A.1. **Equivalence of volume-constrained and uncontrained penalized minimum problems.** The following proposition can be seen an extension of [26, Theorem 1.1].

Proposition A.1. *Assume (H1)-(H3). There exists $\lambda_0>0$ (possibly depending on b_1, b_2 and Ω) with the following property: $(A,u)\in\mathcal{C}$ is a solution of (2.12) if and only if (A,u) is also a solution to (2.13) for all $\lambda\ge\lambda_0$.*

Proof. Note that any minimizer $(A,u)\in\mathcal{C}$ of $\mathcal{F}^\lambda$ with $|A|=\mathtt{v}$ is also minimizer of $\mathcal{F}$. Hence, it suffices to show that there exists $\lambda_0>0$ such that any minimizer (A,u) of $\mathcal{F}^\lambda$ for $\lambda>\lambda_0$ satisfies $|A|=\mathtt{v}$.

Assume by contradiction that there exist a sequence $\lambda_h\to\infty$ and a sequence $(A_h,u_h)\in\mathcal{C}$ minimizing $\mathcal{F}^{\lambda_h}$ such that $|A_h|\ne\mathtt{v}$. Take any $A_0\in BV(\Omega;\{0,1\})$ with $|A|=\mathtt{v}$. Then by minimality, $\mathcal{F}^{\lambda_h}(A_h,u_h)\le\mathcal{F}^{\lambda_h}(A_0,u_0)=\mathcal{F}(A_0,u_0)$ for all large h and hence, by (2.8) and (2.9),

$$\sup_{h\ge1}P(A_h)\le a:=\frac{\mathcal{F}(A_0,u_0)+b_2\mathcal{H}^{n-1}(\Sigma)+\mathcal{H}^{n-1}(\partial\Omega)}{b_1} \tag{A.1}$$

and

$$\sup_{h\ge1}\lambda_h||A_h|-\mathtt{v}|\le\mathcal{F}(A_0,u_0)+b_2\mathcal{H}^{n-1}(\Sigma).$$

This implies $|A_h|\to\mathtt{v}$ as $h\to\infty$. By compactness, there exists a finite perimeter set $A\subset\Omega$ and a not relabelled subsequence such that $\chi_{A_h}\to\chi_A$ a.e. in $\mathbb{R}^n$. In particular, $|A|=\mathtt{v}$.

Further we assume $|A_h| < \mathtt{v}$ for all h; the case $|A_h| > \mathtt{v}$ can be treated analogously. As in the proof of [26, Theorem 1.1] given $\epsilon \in (0, 2\epsilon_n)$, where $\epsilon_n > 0$ will be chosen later, there exist small $r > 0$ and $x_r \in \Omega$ such that $B_r(x) \subset\subset \Omega$ and

$$|A \cap B_{r/2}(x_r)| < \epsilon r^n, \qquad |A \cap B_r(x_r)| > \frac{\omega_n r^n}{2^{n+2}}.$$

For shortness, we suppose that $x_r = 0$ we write $B_r := B_r(x_r)$. Since $A_h \to A$ in $L^1(\mathbb{R}^n)$, for all large h,

$$|A_h \cap B_{r/2}| < \epsilon r^n, \qquad |A_h \cap B_r| > \frac{\omega_n r^n}{2^{n+2}}.$$

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be the bi-Lipschitz map which takes B_r into B_r defined as

$$\Phi(x) := \begin{cases} (1 - (2^n - 1)\sigma)x, & |x| < \frac{r}{2}, \\ x + \sigma\Big(1 - \frac{r^2}{|x|^2}\Big)x, & \frac{r}{2} \le x < r, \\ x, & |x| \ge r \end{cases}$$

for some $\sigma \in (0, \frac{1}{2^n})$. Recall from [26, pp. 420-422] that the Jacobian $J\Phi$ of Φ satisfies

$$J\Phi(y) \ge 1 + C_1(n)\,\sigma \qquad y \in B_r \setminus B_{r/2},$$

for some $C_1(n) > 0$, and

$$J\Phi(y) \le 1 + 2^n n\sigma \qquad y \in B_r.$$

Moreover, the tangential Jacobian $J_{n-1}T_x$ of Φ on the tangent space T_x of $\partial^* A_h$ satisfies

$$J_{n-1}T_x \le 1 + (1 + 2^n(n-1))\sigma, \qquad x \in B_r \cap \partial^* A_h. \tag{A.2}$$

Set

$$E_h := \Phi(A_h), \qquad v_h := u_h \chi_{A_h \setminus B_r} + u_0 \chi_{E_h \cap B_r}. \tag{A.3}$$

Note that $|E_h| < \mathtt{v}$ and $E_h \Delta A_h \subset \overline{B_r}$. Let us estimate

$$\begin{aligned} \mathcal{F}^{\lambda_h}(A_h, u_h) - \mathcal{F}^{\lambda_h}(E_h, v_h) = & \int_{\overline{B_r} \cap \partial^* A_h} \varphi(x, \nu_{A_h}) \mathrm{d}\mathcal{H}^{n-1} - \int_{\overline{B_r} \cap \partial^* E_h} \varphi(x, \nu_{E_h}) \mathrm{d}\mathcal{H}^{n-1} \\ & + 2\int_{\overline{B_r} \cap J_{u_h}} \varphi(x, \nu_{J_{u_h}}) \mathrm{d}\mathcal{H}^{n-1} - 2\int_{\overline{B_r} \cap J_{v_h}} \varphi(x, \nu_{J_{v_h}}) \mathrm{d}\mathcal{H}^{n-1} \\ & + \int_{B_r \cap A_h} W(x, \mathcal{E}u_h - \mathbf{M}_0) \mathrm{d}x - \int_{B_r \cap E_h} W(x, \mathcal{E}v_h - \mathbf{M}_0) \mathrm{d}x \\ & + \lambda_h \Big(|E_h| - |A_h|\Big) := I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{A.4}$$

By the definition of v_h and the nonnegativity of $\mathcal{W}$, $I_3 \ge 0$ and

$$I_2 \ge -2\int_{\partial B_r} \varphi(x, \nu_{J_{v_h}}) \mathrm{d}\mathcal{H}^{n-1} \ge -2b_2 n\omega_n r^{n-1}.$$

Moreover, by (A.2) and the area formula as well as from (2.8) and (A.1)

$$\begin{aligned} \int_{B_r \cap \partial^* E_h} \varphi(x, \nu_{E_h}) \mathrm{d}\mathcal{H}^{n-1} = & \int_{B_r \cap \partial^* A_h} \varphi(\Phi(y), \nu_{A_h}) J_{n-1}T_y \, d\mathcal{H}^{n-1}(y) \\ & \le 2b_2(1 + 2^n(n-1)\sigma)\mathcal{H}^{n-1}(B_r \cap \partial^* A_h) \le 2b_2(1 + (1 + 2^n(n-1))\sigma)a. \end{aligned}$$

Moreover, by (2.8)

$$\int_{\partial B_r \cap \partial^* E_h} \varphi(x, \nu_{E_h}) d\mathcal{H}^{n-1} \le 2b_2 \mathcal{H}^{n-1}(\partial B_r) \le 2n\omega_n b_2 r^{n-1},$$

thus,

$$I_1 \ge -2b_2(1+(1+2^n(n-1))\sigma)a - 2n\omega_n b_2 r^{n-1}.$$

Finally, repeating the same arguments of Step 4 in the proof of [26, Theorem 1.1], we obtain

$$I_4 \ge \lambda_h \sigma r^n \Big[C_1(n)\,\frac{\omega_n}{2^{n+1}} - C_1(n)\epsilon - (2^n-1)n\epsilon\Big],$$

thus,

$$\begin{aligned}\mathcal{F}^{\lambda_h}(A_h,u_h) - \mathcal{F}^{\lambda_h}(E_h,v_h) \ge & \lambda_h\sigma r^n\Big[C_1(n)\,\frac{\omega_n}{2^{n+1}} - C_1(n)\epsilon - (2^n-1)n\epsilon\Big] \\ & -2b_2(1+(1+2^n(n-1))\sigma)a - 2n\omega_n b_2 r^{n-1}. \qquad \text{(A.5)}\end{aligned}$$

Now if we define

$$\epsilon_n := \frac{C_1(n)\omega_n}{2^{n+2}[1+C_1(n)+(2^n-1)n]},$$

then from (A.5) applied with $\epsilon = \epsilon_n$ we deduce

$$\mathcal{F}^{\lambda_h}(A_h,u_h) - \mathcal{F}^{\lambda_h}(E_h,v_h) \ge \lambda_h\sigma\epsilon_n r^n - C$$

for some C independent of h. Thus, $\mathcal{F}^{\lambda_h}(A_h,u_h) > \mathcal{F}^{\lambda_h}(E_h,v_h)$ for all sufficiently large h, which contradicts to the minimality of (A_h,u_h). □

Remark A.2. The same proof of Proposition A.1 works also with $\mathcal{F}_p$ and $\mathcal{F}_{\mathrm{Dir}}$ in Theorems 2.8 and 2.9. Indeed, in case $\mathcal{F}_p$, for configuration (E_h,v_h), given by (A.3), the equality (A.4) is written as

$$\begin{aligned}\mathcal{F}_p^{\lambda_h}(A_h,u_h) - \mathcal{F}_p^{\lambda_h}(E_h,v_h) = & \int_{\overline{B_r}\cap\partial^* A_h} \varphi(x,\nu_{A_h})\mathrm{d}\mathcal{H}^{n-1} - \int_{\overline{B_r}\cap\partial^* E_h} \varphi(x,\nu_{E_h})\mathrm{d}\mathcal{H}^{n-1} \\ & +2\int_{\overline{B_r}\cap J_{u_h}} \varphi(x,\nu_{J_{u_h}})\mathrm{d}\mathcal{H}^{n-1} - 2\int_{\overline{B_r}\cap J_{v_h}} \varphi(x,\nu_{J_{v_h}})\mathrm{d}\mathcal{H}^{n-1} \\ & + \int_{B_r\cap A_h} W_p(x,\mathcal{E}u_h - \mathbf{M}_0)\mathrm{d}x - \int_{B_r\cap E_h} W_p(x,\mathcal{E}v_h - \mathbf{M}_0)\mathrm{d}x \\ & +\lambda_h\Big(|E_h| - |A_h|\Big) := I_1 + I_2 + I_3 + I_4.\end{aligned}$$

The estimates of I_1, I_2 and I_4 are the same, and by (2.15) for I_3 we have

$$I_3 \ge \int_{A_h\cap B_r} W_p(x,\mathcal{E}u_h - \mathbf{M}_0)\mathrm{d}x \ge -\int_{B_r} |f|\mathrm{d}x,$$

which is independent of h.

Similarly, in case of $\mathcal{F}_{\mathrm{Dir}}$ we define v_h in (A.3) as

$$v_h = u_h\chi_{A_h\setminus B_r}$$

and the proof runs as in the case of $\mathcal{F}_p$.

A.2. **Some properties of GSBD-functions.**

Lemma A.3. *Let U be an open set and $A \subset BV(U;\{0,1\})$. Assume that $u,v \in GSBD^2(U)$. Then $u\chi_A + v\chi_{U\setminus A} \in GSBD^2(U)$.*

Proof. Recall that by [18, Remark 9.3] if Since $w \in GSBD^2(U)$, then the Radon measure

$$\mu_w(B) := \mathcal{H}^{n-1}(B\cap J_w) + \int_B |\mathcal{E}w|\mathrm{d}x \quad \text{for all Borel sets } B \subset U,$$

can be used in [18, Definition 4.1]. Thus, $u\chi_A + v\chi_{U\setminus A}$ belongs to GSBD since, as A has finite perimeter in U, the measure

$$\lambda(B) = \mu_u(A\cap B) + \mu_v(B\setminus A) + \mathcal{H}^{n-1}(B\cap\partial^* A) \quad \text{for all Borel sets } B\subset U$$

can be used in Definition 4.1 of [18]. Since $\mathcal{E}u\chi_A + v\chi_{U\setminus A} = \mathcal{E}u\chi_A + \mathcal{E}v\chi_{U\setminus A}$, it follows that $u\chi_A + v\chi_{U\setminus A} \in GSBD^2(U)$. □

Note that this property does not hold for $GSBV$-functions, because the condition $u\chi_A + v\chi_{U\setminus A} \in GSBV(U)$ requires some regularity of the traces of u and v along $U\cap\partial^* A$. From Lemma A.3 we get $GSBD^2(\mathrm{Int}(\Omega\cup S\cup\Sigma)) = GSBD^2(\Omega\cup S)$.

Lemma A.4. *Let $n\ge 2$ and $D\subset\mathbb{R}^n$ be a connected bounded Lipschitz open set and let $u\in GSBD^2(D)$ be such that $\mathcal{H}^{n-1}(J_u)=0$. Then $u\in H^1(D)$ and there exists a rigid displacement a such that*

$$\|u-a\|_{H^1(D)} \le C_{n,D}\|\mathcal{E}u\|_{L^2(D)}$$

for some constant $C_{n,D}>0$ depending only on n and D.

Proof. Recall that by the Poincaré-Korn inequality for any connected Lipschitz set $U\subset\mathbb{R}^n$ there exists $C_{n,U}>0$ such that

$$\|v-a\|_{H^1(U)} \le C_{n,U}\|\mathcal{E}v\|_{L^2(U)} \tag{A.6}$$

for any $v\in H^1(U)$ and for some rigid displacement $a:\mathbb{R}^n\to\mathbb{R}^n$. Obviously, $C_{n,U}$ is independent of translation, and let us show

$$C_{n,\lambda U} \le C_{n,U} \quad \text{for any } \lambda\in(0,1]. \tag{A.7}$$

We may assume $0\in U$. Note that (A.6) is equivalent to

$$\min_{a \text{ rigid}} \|v-a\|_{H^1(U)} \le C_{n,U}\|\mathcal{E}v\|_{L^2(U)}, \quad v\in H^1(U). \tag{A.8}$$

Fix any $u\in H^1(\lambda U)$ and let $v_\lambda(x) := u(\lambda x)$. Then $v_\lambda\in H^1(U)$,

$$\int_U |v_\lambda(x)|^2\mathrm{d}x = \lambda^{-n}\int_{\lambda U}|u(y)|^2\mathrm{d}y$$

and

$$\int_U |\nabla v_\lambda(x)|^2\mathrm{d}x = \lambda^{2-n}\int_{\lambda U}|\nabla u(y)|^2\mathrm{d}y, \quad \int_U |\mathcal{E}v_\lambda(x)|^2\mathrm{d}x = \lambda^{2-n}\int_{\lambda U}|\mathcal{E}u(y)|^2\mathrm{d}y.$$

Then for any rigid displacement $a(x)=\mathbf{M}x+b$ we have

$$\|u-a\|^2_{H^1(\lambda U)} = \lambda^n\|v_\lambda - a_\lambda\|^2_{L^2(U)} + \lambda^{n-2}\|\nabla v_\lambda - \mathbf{M}\|^2_{L^2(U)} \le \lambda^{n-2}\|v_\lambda - a_\lambda\|^2_{H^1(U)},$$

where $a_\lambda(x) = \lambda\mathbf{M}x + b$. Now taking a_λ, satisfying (A.6) with $v=v_\lambda$, we have

$$\|u-a\|^2_{H^1(\lambda U)} \le \lambda^{n-2}\|v_\lambda - a_\lambda\|^2_{H^1(U)} \le C^2_{n,U}\lambda^{n-2}\|\mathcal{E}v_\lambda\|^2_{L^2(U)} = C^2_{n,U}\|\mathcal{E}u\|^2_{L^2(\lambda U)},$$

and thus, from (A.8) we get (A.7).

Now we prove the lemma. By [37, Proposition A.3] $u\in H^1_{\mathrm{loc}}(D)$ and hence, by (A.6) we just need to show $u\in H^1(D)$.

Step 1. First assume additionally that D is simply connected and 0 is in the interior of D. Consider the sequnce

$$D_i = (1-2^{-i})D, \quad i\in\mathbb{N},$$

of rescalings of D. Since $D_i \subset\subset D$ and $u \in H^1(D_i)$ by (A.6) and (A.7) there exists a rigid displacement $a_i : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|u - a_i\|_{H^1(D_i)} \le C_{n,D} \|\mathcal{E}u\|_{L^2(D_i)}. \tag{A.9}$$

Consider the sequence $\{a_i\}$. Since $D_1 \subset D_i \subset D$, by (A.9)

$$\|a_i - a_1\|_{H^1(D_1)} \le \|u - a_i\|_{H^1(D_i)} + \|u - a_1\|_{H^1(D_1)} \le C_{n,D} \|\mathcal{E}u\|_{L^2(D)}.$$

Thus, $\{a_i\}$ is uniformly bounded in $H^1(D_1)$. Since a_i are linear, up to a subsequence, $a_i \to a_0$ in $H^1_{\rm loc}(\mathbb{R}^n)$ and $a_i \to a_0$ a.e. in $\mathbb{R}^n$ for some rigid displacement a_0. Hence, by (A.9)

$$\|u - a_0\|_{H^1(D_i)} = \lim_{j\to+\infty} \|u - a_j\|_{H^1(D_i)} \le \limsup_{j\to+\infty} \|u - a_j\|_{H^1(D_j)} \le C_{n,D} \limsup_{j\to+\infty} \|\mathcal{E}u\|_{L^2(D_j)}$$

Since $D_j \nearrow D$ and $\mathcal{E}u \in L^2(D)$, by the monotone convergence theorem

$$\|u - a_0\|_{H^1(D_i)} \le C_{n,D} \|\mathcal{E}u\|_{L^2(D)}.$$

Letting $i \to +\infty$ in this inequality and using again the monotone convergence theorem we get $u - a_0 \in H^1(D)$, and thus, $u \in H^1(D)$.

Step 2. Now consider the general case. Since D is Lipschitz, for any $x \in \partial D$ there exists a cylinder R_x such that $D \cap R_x$ is a subgraph of a Lipchitz function. In particular, $D \cap R_x$ is Lipschitz and simply connected. For $x \in D$ let R_x be largest cube centered at x and contained in D. Then $\overline{D} \subseteq \bigcup_x R_x$ and hence, by the compactness of $\overline{D}$, there exists finitely many points $x_1, \dots, x_m$ such that $\overline{D} \subset \bigcup_{j=1}^m R_{x_j}$. Since $R_{x_j} \cap D$ is simply connected, by Step 1, $u \in H^1(R_{x_j} \cap D)$ and there exists a rigid displacement a_j such that

$$\|u - a_j\|_{H^1(R_{x_j}\cap D)} \le C_{n,R_{x_j}\cap D} \|\mathcal{E}u\|_{L^2(R_{x_j}\cap D)}.$$

Thus,

$$\|u\|^2_{H^1(D)} \le \sum_{j=1}^m \|u\|^2_{H^1(D\cap R_{x_j})} \le 2\sum_{j=1}^m \|u - a_j\|^2_{H^1(D\cap R_{x_j})} + 2\sum_{j=1}^m \|a_j\|^2_{H^1(D\cap R_{x_j})} < +\infty.$$

□

Acknowledgments. Sh. Kholmatov acknowledges support from the Austrian Science Fund (FWF) projects M2571-N32 and P33716. P. Piovano acknowledges support from the Austrian Science Fund (FWF) projects P 29681 and TAI 293, from the Vienna Science and Technology Fund (WWTF) together with the City of Vienna and Berndorf Privatstiftung through Project MA16-005, and from BMBWF through the OeAD-WTZ project HR 08/2020. Furthermore, P. Piovano is member of the Italian "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA) and has received funding from the INdAM - GNAMPA 2022 project CUP: E55F22000270001 and 2023 Project Codice CUP: E53C22001930001. Finally, P. Piovano is grateful for the support received as *Visiting Professor and Excellence Chair* at the Okinawa Institute of Science and Technology (OIST), Japan.

Data availability. The manuscript has no associated data.

References

[1] S. Almi, G. Dal Maso, R. Toader: A lower semicontinuity result for a free discontinuity functional with a boundary term. J. Math. Pures Appl. **108** (2017), 952–990.

[2] L. Ambrosio: A compactness theorem for a new class of functions of bounded variation. Boll. U. M. I. **3-B** (1989), 857–881.

[3] L. Ambrosio, N. Fusco, D. Pallara: Functions of Bounded Variation and Free Discontinuity problems. Oxford University Press, New York 2000.

[4] R. Asaro, W. Tiller: Interface morphology development during stress corrosion cracking: Part I. Via surface diffusion. Metall. Trans. **3** (1972), 1789–1796.

[5] J.-F. Babadjian, D. Henao: Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination. Interface Free Bound. **18** (2016), 545–578.

[6] P. Bella, M. Goldman, B. Zwicknagl: Study of island formation in epitaxially strained films on unbounded domains. Arch. Rational Mech. Anal. **218** (2015), 163–217.

[7] G. Bellettini, M. Novaga, Sh. Kholmatov: Minimizers of anisotropic perimeters with cylindrical norms. Comm. Pure Appl. Anal. **16** (2017), 1427–1454.

[8] M. Bonacini: Stability of equilibrium configurations for elastic films in two and three dimensions. Adv. Calc. Var. **8** (2015), 117–153.

[9] E. Bonnetier, A. Chambolle: Computing the equilibrium configuration of epitaxially strained crystalline films. SIAM J. Appl. Math. **62** (2002), 1093–1121.

[10] B. Bourdin, G. Francfort, J.-J. Marigo: The variational approach to fracture. Springer, Amsterdam, 2008.

[11] A. Chambolle, S. Conti, G. Francfort: Korn-Poincaré inequalities for functions with a small jump set. Indiana Univ. Math. J. **65** (2016), 1373–1399.

[12] A. Chambolle, S. Conti, F. Iurlano: Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. J. Math. Pures Appl. **128** (2019), 119–139.

[13] A. Chambolle, V. Crismale: Existence of strong solutions to the Dirichlet problem for the Griffith energy. Calc. Var. Partial Differential Equations **58** (2019), 136.

[14] A. Chambolle, V. Crismale: Compactness and lower semicontinuity in GSBD. J. Eur. Math. Soc. (JEMS) **23** (2021), 701–719.

[15] A. Chambolle, V. Crismale: Equilibrium configurations for nonhomogeneous linearly elastic materials with surface discontinuities. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (2022).

[16] A. Chambolle, M. Solci: Interaction of a bulk and a surface energy with a geometrical constraint. SIAM J. Math. Anal. **39**(2007), 77–102.

[17] V. Crismale, M. Friedrich: Equilibrium configurations for epitaxially strained films and material voids in three-dimensional linear elasticity. Arch. Rational Mech. Anal. **237** (2020), 1041–1098.

[18] G. Dal Maso: Generalised functions of bounded deformation. J. Eur. Math. Soc. **15** (2013), 1943–1997.

[19] A. Danescu: The Asaro-Tiller-Grinfeld instability revisited. Int. J. Solids Struct. **38** (2001), 4671–4684.

[20] E. Davoli, P. Piovano: Derivation of a heteroepitaxial thin-film model. Interface Free Bound. **22** (2020), 1–26.

[21] E. Davoli E., P. Piovano: Analytical validation of the Young-Dupré law for epitaxially-strained thin films. Math. Models Methods Appl. Sci., **29-12** (2019), 2183–2223.

[22] E. DiBenedetto: Real Analysis. Birkhäuser, Basel, 2002.

[23] E. De Giorgi, G. Buttazzo, G. Dal Maso: On the lower semicontinuity of certain integral functionals. Atti Acc. Naz. Lincei. Cl. Sc. Fis. Mat. Nat. Rend. **74** (1983), 274–282.

[24] G. De Philippis, F. Maggi: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. Arch. Rational Mech. Anal. **216** (2015), 473–568.

[25] X. Deng: Mechanics of debonding and delamination in composites: Asymptotic studies. Compos. Eng. **5** (1995), 1299–1315.

[26] L. Esposito, N. Fusco: A remark on a free interface problem with volume constraint. J. Convex Anal. **18** (2011), 417–426.

[27] I. Fonseca, N. Fusco, G. Leoni, M. Morini: Equilibrium configurations of epitaxially strained crystalline films: existence and regularity results. Arch. Rational Mech. Anal. **186** (2007), 477–537.

[28] I. Fonseca, N. Fusco, G.Leoni, V. Millot: Material voids in elastic solids with anisotropic surface energies. J. Math. Pures Appl. **96** (2011), 591–639.

[29] G. Francfort, A. Giacomini, O. Lopez-Pamies: Fracture with healing: a first step towards a new view of cavitation. Anal. PDE **12** (2019), 417–447.

[30] E. Fried, M. Gurtin: A unified treatment of evolving interfaces accounting for small deformations and atomic transport with emphasis on grain-boundaries and epitaxy. Adv. Appl. Mech. **40** (2004), 1–177.
[31] N. Fusco, V. Julin, M. Morini: The surface diffusion flow with elasticity in the plane. Commun. Math. Phys. **362** (2018), 571–607.
[32] E. Giusti: Minimal Surfaces and Functions of Bounded Variation. Birkhäuser, Boston, 1984.
[33] M. Goldman, B. Zwicknagl: Scaling law and reduced models for epitaxially strained crystalline films. SIAM J. Math. Anal. **46** (2014), 1–24.
[34] M.A. Grinfeld: The stress driven instabilities in crystals: mathematical models and physical manifestations. J. Nonlinear Sci. **3** (1993), 35–83.
[35] J. Hutchinson, Z. Suo: Mixed mode cracking in layered materials. Adv. Appl. Mech. **29** (1991), 63–191.
[36] Sh. Kholmatov, P. Piovano: A unified model for stress-driven rearrangement instabilities. Arch. Rational Mech. Anal. **238** (2020), 415–488.
[37] Sh. Kholmatov, P. Piovano: Existence of minimizers for the SDRI model in 2d: Wetting and dewetting regime with mismatch strain. Submitted, arXiv:2006.06096.
[38] L. Kreutz, P. Piovano: Microscopic validation of a variational model of epitaxially strained crystalline films. SIAM J. Math. Anal. **53-1** (2021), 453–490.
[39] A.A. León Baldelli, B. Bourdin, J.-J. Marigo, C. Maurini: Fracture and debonding of a thin film on a stiff substrate: analytical and numerical solutions of a one-dimensional variational model. Cont. Mech. Thermodyn. **25** (2013), 243–268.
[40] A.A. León Baldelli, J.-F. Babadjian, B. Bourdin, D. Henao, C. Maurini: A variational model for fracture and debonding of thin films under in-plane loadings. J. Mech. Phys. Solids **70** (2014), 320–348.
[41] F. Maggi: Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory. Cambridge University Press, Cambridge, 2012.
[42] C.B. Morrey Jr.: Multiple Integrals in Calculus of Variations. Classics in Mathematics, Springer, Berlin, 2008.
[43] P. Piovano: Evolution and Regularity Results for Epitaxially Strained Thin Films and Material Voids. Ph.D. Thesis - Carnegie Mellon University, 2012.
[44] P. Piovano, I. Velčić: Microscopical Justification of Solid-State Wetting and Dewetting. J Nonlinear Sci **32**, 32 (2022).
[45] P. Piovano, I. Velčić: Microscopical justification of the Winterbottom problem for well-separated lattices. Nonlinear Analysis textbf231, 113113 (2023).
[46] M. Siegel, M. Miksis, P. Voorhees: Evolution of material voids for highly anisotropic surface energy. J. Mech. Phys. Solids **52** (2004), 1319–1353.
[47] B. Spencer: Asymptotic derivation of the guled-wetting-layer model and the contact-angle condition for Stranski-Krastanov islands. J. Mech. Phys. Solids **52** (2004), 1319–1353.
[48] D.J. Srolovitz: On the stability of surfaces of stressed solids. Acta Metallurgica. 37-2 (1989), 621–625.
[49] H. Wang, Z. Li: The instability of the diffusion-controlled grain-boundary void in stressed solid. Acta Mech. Sinica **19** (2003), 330–339.
[50] W.L. Winterbottom: Equilibrium shape of a small particle in contact with a foreign substrate. Acta Metallurgica **15** (1967), 303–310.
[51] G. Wulff G.: Zur Frage der Geschwindigkeit des Wastums und der Auflösung der Kristallflachen. Krystallographie und Mineralogie. Z. Kristallner. **34** (1901), 449–530.
[52] Z.C. Xia, J.W. Hutchinson: Crack patterns in thin films. J. Mech. Phys. Solids **48** (2000), 1107–1131.

(Shokhrukh Yu. Kholmatov) Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern Platz 1, 1090 Wien (Austria)

E-mail address, Sh. Kholmatov: shokhrukh.kholmatov@univie.ac.at

(Paolo Piovano) Politecnico di Milano, Dipartimento di Matematica, P.zza Leonardo da Vinci 32, 20133 Milano, Italy, & WPI c/o Research Platform MMM "Mathematics-Magnetism-Materials", Fak. Mathematik Univ. Wien, A1090 Vienna

E-mail address, P. Piovano: paolo.piovano@polimi.it