A sharp interface limit of a nonlocal variational model for pattern formation in biomembranes

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Abstract. Pattern formation in biological membranes is often explained by a coupling of the local curvature of the thin elastic membrane to its local composition. This ansatz introduces nonlocal terms to Canham-Helfrich type models for membranes. This paper complements a study in [22]. The main result is the derivation of a Γ-limit in a parameter regime in which complex pattern formation is not expected. We consider the full nonlocal model as well as a local approximation and prove that the Γ-limit in both cases is of perimeter-type. A main step in the proof is a general strategy to include Neumann boundary conditions in the construction of a recovery sequence.

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1. Introduction

This paper complements studies in [22, 13], in which a Γ-limit of nonlocal second order energy functionals has been studied. The latter arise from a model that has been introduced in the physics literature to describe phase separation of a diblock copolymer in a membrane allowing out of plane (bending) distortions (see [33, 45, 35, 43]). The model builds on the classical continuum theory for membranes (see, e.g., [9, 30]), and incorporates the ansatz to link the local composition to the local curvature of the membrane (see e.g. [43] for details). This modelling approach has been identified as possible explanation for the formation of microdomains such as *lipid rafts* in biological membranes (see e.g. [3, 40]) although there are still debates in the literature on the origins and structures of these patterns (see e.g. [20, 36, 44, 36]). For completeness, we recall briefly the derivation from [22]. Precisely, the starting point is the functional

$$\mathcal{E}[\phi, h] = \int_D \left(f(\phi) + \frac{1}{2} b|\nabla \phi|^2 + \frac{1}{2}\sigma|\nabla h|^2 + \frac{1}{2}\kappa|\Delta h|^2 + \Lambda\phi\Delta h \right) d\bar{x},$$

in terms of the height profile $h : D \to \mathbb{R}$ of a membrane and the order parameter $\phi : D \to \mathbb{R}$ modelling the local composition. The function f is a non-negative multi-well potential whose minima represent the pure components. In this note, we shall restrict to the case of two components, i.e., double-well potentials f. The parameters b, σ and κ stand for the line tension between regions of different composition, the surface tension, and the bending rigidity, respectively. We assume that the membrane is almost flat so that its curvature is well approximated in terms of the Laplacian Δh. The parameter $\Lambda > 0$ introduces a coupling between the local curvature and the membrane and the local composition, and renders the problem nonlocal. This modelling ansatz is motivated by experimental findings (see e.g. [3, 40]). After optimizing in h and setting

$$\varepsilon := \sqrt{\frac{\kappa}{L^2\sigma}}, \qquad q := 1 - \frac{b\sigma}{\Lambda^2}, \qquad W(u) := \frac{2\kappa}{\Lambda^2} f(u), \qquad \mathcal{F}^*_\varepsilon := \frac{1}{\varepsilon}\frac{2\kappa}{\Lambda^2}\mathcal{E},$$

one ends up with the functional $\mathcal{F}^*_\varepsilon : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, given by

$$\mathcal{F}^*_\varepsilon[u] := \begin{cases} \frac{1}{\varepsilon}\int_\Omega \left(W(u) - u^2 + (1-q)\varepsilon^2|\nabla u|^2 + u\left(\mathbf{1} - \varepsilon^2\Delta\right)^{-1} u \right) dx, & \text{if } u \in W^{1,2}(\Omega); \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus W^{1,2}(\Omega). \end{cases} \tag{1.1}$$

Throughout the text, we will restrict to the case that $\Omega \subset \mathbb{R}^d$ is an open bounded set with $\mathcal{C}^4$-boundary, W is a double-well potential with minima at $\{\pm 1\}$ (see Assumptions 1.1 and 1.2 below for the precise assumptions), and the term $(\mathbf{1} - \varepsilon^2\Delta)$ is a differential operator subject to Neumann boundary conditions, see Section 2 for a detailed discussion. We shall mainly focus on the analysis of (1.1) but also consider a local approximation. Precisely, doing a long-wavelength approximation as presented in [22, Appendix] the functional further simplifies to

$$\mathcal{F}^*_{\varepsilon,ap}[u] = \int_\Omega \frac{1}{\varepsilon} W(u) - \varepsilon q|\nabla u|^2 + \varepsilon^3(\Delta u)^2 dx. \tag{1.2}$$

Note that $\mathcal{F}^*_{\varepsilon,ap}$ is in fact a local functional. The functionals (1.1) and (1.2) involve the parameters ε and q. We will deal with the case $\varepsilon \to 0$. For fixed line tension and composition-curvature coupling, the parameter q varies with the surface tension. It turns out that the biologically relevant values for q fall into the interval $(-1.1, 1)$ (see [22], based on the data from [34]). The case $q \geq 0$ and $\varepsilon \to 0$ has been treated in [22], and Γ-convergence to a perimeter functional has been proven, which corresponds to results for the local approximation (1.2) derived independently in [10] and [11]. The asymptotic analysis presented in the latter two references has been generalized to a large class of singularly perturbed second order functionals (see [2]). Qualitative properties of local minimizers of (1.2) have been studied extensively to explain the formation of periodic layered structures (see e.g. [4, 12, 37, 41]).
Nonlocal functionals similar to (1.1) have gained large attention in the numerical and analytical literature in the last years (see e.g. [28]). Related models for (almost) spherical membranes have been studied e.g. in [16, 17, 18, 19], for flat domains see [15] and the references therein. A sharp interface variant of (1.1) has recently been analyzed in a similar paramteter regime via autocorrelation functions (see [13]; we refer to [13, Section 2] for a comparison between the two models).

In this paper, we focus on the case $q \leq 0$. We note that a main advantage of the case $q \leq 0$ is that the functionals (1.1) and (1.2) are non-negative. For the local functional (1.2), a special case with $q < 0$ with a double-well potential related to the Fisher-Kolmogorov equation has been treated in [31], see also [8] for a related result. The case $q = 0$ and the Laplacian replaced by the full Hessian has been studied in [24].

Summarizing, the purpose of this note is threefold: First, we derive the Γ-limits of (1.1) and (1.2) as $\varepsilon \to 0$. For (1.1), we show how for $q < 0$, arguments from [22] simplify, and obtain analogous results under slightly weaker conditions on the potential W. In particular, we present a detailed proof of the limsup inequality, i.e., the construction of recovery sequences which also fixes a gap in the argument from [22] in the case $q \in (0, q_*)$, see Remark 3.4 and the beginning of Section 3.3. The main difficulty here lies in the Neumann boundary condition for the solution operator $(\mathbf{1} - \varepsilon^2\Delta)^{-1}$. We point out that our construction also works for $q > 0$ and therefore completes the proof of [22]. Second, for $d = 1$, we consider a much larger class of potentials, and prove refined results in the spirit of [10]. Finally, for the local functional, we consider the case $q = 0$ with and without Neumann boundary conditions, relying on the construction for the nonlocal case. We outline the main results in more detail below.

1.1. Main results

In the following, we will discuss our main results for the nonlocal and the local functional separately.

1.2. The nonlocal functional (1.1)

We need to introduce some notation. We start with the assumptions on the double-well potential.

Assumption 1.1. *For general $d \geq 1$, we assume that W satisfies the following conditions:*

(H1) $W \in C^2(\mathbb{R})$, $W(s) = 0$ if and only if $s \in \{\pm 1\}$, and $W''(\pm 1) > 0$.

(H2) There exist λ_1, $\lambda_2 > 0$ and $R > 1$ such that

$$\lambda_1 |s|^2 \leq W(s) \leq \lambda_2 |s|^2 \qquad \text{for all } s \in \mathbb{R} \text{ with } |s| \geq R.$$

We shall see that for $d = 1$, the assumptions on W can be relaxed. Precisely, we introduce the following.

Assumption 1.2. *For $d = 1$, we assume that W satisfies the following conditions:*

(H1') $W : \mathbb{R} \to [0, \infty)$ is locally Lipschitz continuous, $W(s) = 0$ if and only if $s \in \{\pm 1\}$.

(H2') There exist $L > 0$ and $R > 0$ such that

$$W(s) > L|s| \qquad \text{for all } s \in \mathbb{R} \text{ with } |s| \geq R.$$

Note that any function W satisfying Assumption 1.1 also satisfies Assumption 1.2. As in [22] we use the following notation to describe the limiting problem. Given a unit vector $\nu \in \mathbb{S}^{d-1}$, let $\{\nu_1, \cdots, \nu_{d-1}, \nu\}$ form an orthonormal basis of $\mathbb{R}^d$ and define

$$Q_\nu := Q_\nu(0,1) := \{x \in \mathbb{R}^d : |x \cdot \nu| < 1/2, \quad |x \cdot \nu_i| < 1/2, \quad i = 1, \ldots, d-1\}, \tag{1.3}$$

$$\begin{aligned}\mathcal{A}_\nu \ := \big\{ & v \in W^{2,2}_{\text{loc}}(\mathbb{R}^d) : v = -1 \text{ in a neighborhood of } x \cdot \nu = -1/2, \\ & v = 1 \text{ in a neighborhood of } x \cdot \nu = 1/2, \quad v(x) = v(x + \nu_i) \text{ for all } x \in \mathbb{R}^d, i = 1, \ldots, d-1 \big\},\end{aligned} \tag{1.4}$$

and

$$\begin{aligned} m_d \ &:= \ \inf \left\{ \int_{Q_\nu} \left(\frac{1}{\varepsilon} W(u) - \varepsilon q |\nabla u|^2 + \varepsilon^3 (\Delta v)^2 + \varepsilon^5 |\nabla \Delta v|^2 \right) dx : 0 < \varepsilon \leq 1, -\varepsilon^2 \Delta v + v = u, v \in \mathcal{A}_\nu \right\} \\ &= \ \inf \left\{ \frac{1}{T^{d-1}} \int_{T \cdot Q_\nu} \left(W(\tilde{u}) - q |\nabla \tilde{u}|^2 + (\Delta \tilde{v})^2 + |\nabla \Delta \tilde{v}|^2 \right) dx : 1 \leq T, -\Delta \tilde{v} + \tilde{v} = \tilde{u}, \tilde{v}(T \cdot) \in \mathcal{A}_\nu \right\}, \end{aligned}$$

where the second line follows by a change of variables with $\tilde{v}(\cdot) = v(\varepsilon \cdot)$, $\tilde{u}(\cdot) = u(\varepsilon \cdot)$, and $T := 1/\varepsilon$. Further, by changing coordinates one can see that m_d does not depend on ν, and we define $Q := Q_{e_d}$, $\mathcal{A} := \mathcal{A}_{e_d}$, where $\{e_1, \ldots, e_d\}$ denotes the standard basis of $\mathbb{R}^d$.

Our main result is the Γ-convergence of the sequence of functionals defined in (1.1) as $\varepsilon \to 0^+$ to

$$\mathcal{F}^*[u] := \begin{cases} m_d \operatorname{Per}(\{u = 1\}), & \text{if } u \in BV(\Omega; \{\pm 1\}), \\ +\infty, & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{\pm 1\}). \end{cases} \tag{1.5}$$

Precisely, we show the following theorem. Recall that we always assume that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with a C^4-boundary.

Theorem 1.3. *If $d = 1$ suppose that Assumption 1.2 holds; if $d > 1$, then suppose that Assumption 1.1 holds. Let $(\varepsilon_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$, $\varepsilon_n \to 0$. Then*

1. *Liminf Inequality: For every sequence $(u_n)_{n \in \mathbb{N}} \subset W^{1,2}(\Omega)$ with $u_n \to u$ in $L^2(\Omega)$ ($L^1(\Omega)$ if $d = 1$),*
$$\liminf_{n \to \infty} \mathcal{F}^*_{\varepsilon_n}[u_n] \geq \mathcal{F}^*[u]. \tag{1.6}$$
2. *Limsup Inequality: For every $u \in L^2(\Omega)$ there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset W^{1,2}(\Omega)$ such that $u_n \to u$ in $L^2(\Omega)$ ($L^1(\Omega)$ if $d = 1$) and*
$$\limsup_{n \to \infty} \mathcal{F}^*_{\varepsilon_n}[u_n] \leq \mathcal{F}^*[u]. \tag{1.7}$$

3. Compactness: If $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ is such that

$$\sup_n \mathcal{F}^*_{\varepsilon_n}[u_n] = M < \infty, \tag{1.8}$$

then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and $u \in BV(\Omega;\{\pm 1\})$ such that

$$u_{n_k} \to u, \quad \text{and} \quad v_{n_k} \to u \quad \text{in } L^2(\Omega) \quad (\text{resp. } L^1(\Omega) \text{ if } d=1),$$

where $(\mathbf{1} - \varepsilon_{n_k}^2 \Delta) v_{n_k} = u_{n_k}$ in Ω and $\frac{\partial v_{n_k}}{\partial \hat{n}} = 0$ on $\partial\Omega$.

Remark 1.4. *The assumption on Ω to be a C^4-domain is only needed in the construction of a recovery sequence, i.e., in the proof of the limsup-inquality. For compactness and the liminf-inequality, a piecewise C^2-boundary is sufficient.*

We note that in spite of the non-locality of the functionals (1.1), the Γ-limit functional turns out to be of the same structure as the one obtained for the local approximation (see [10, 11]), and also similar as for small but positive $q > 0$ (see [22]). By the general theory of Γ-convergence (see e.g. [5, 14]) Theorem 1.3 implies that sequences of (almost) minimizers of $\mathcal{F}^*_\varepsilon$ converge in the strong L^2- (resp. L^1-)topology to minimizers of $\mathcal{F}^*$. Note that the limit functional $\mathcal{F}^*$ is minimized by phase functions that minimize the area of interfaces, and therefore, the formation of microdomains is not expected in the parameter regime under consideration. We point out that the construction of a recovery sequence we present here in detail, also directly generalizes to the case $q > 0$.

1.2.1. Local functional (1.2)**.** We also deal with several variants of the local functional (1.2). Precisely, we consider $\mathcal{F}^*_{\varepsilon,ap} : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$, given by

$$\mathcal{F}^*_{\varepsilon,ap}[u] = \begin{cases} \int_\Omega \frac{W(u)}{\varepsilon} + \varepsilon^3 |\Delta u|^2 \, dx, & \text{if } u \in X, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.9}$$

The most natural choice for the set of admissible functions is $X = \{u \in L^2(\Omega) : \ \Delta u \in L^2(\Omega)\}$, but we can also deal with the case of additional Neumann boundary conditions. Since for every function $u \in L^2(\Omega)$ with $\Delta u \in L^2(\Omega)$ and Neumann boundary conditions, we have $u \in W^{2,2}(\Omega)$ (see e.g. [22, Proposition 2.10]), this results in the choice $X = \{u \in W^{2,2}(\Omega) : \ \partial_\nu u = 0 \text{ on } \partial\Omega\}$. Precisely, we have the following results.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded, Lipschitz domain and let $W : \mathbb{R} \to [0, +\infty)$ be continuous such that there exist $R > 0$ and $\lambda_1 > 0$ such that*

($\tilde{H}1$) *$W(u) = 0$ if and only if $u \in \{\pm 1\}$;*

($\tilde{H}2$) *$W(u) \geq \lambda_1 u^2$ whenever $|u| > R$.*

Let $\{\mathcal{F}^*_{\varepsilon,ap}\}_{\varepsilon>0}$ be as defined in (1.9) with

$$X = \{u \in L^2(\Omega) : \ \Delta u \in L^2(\Omega)\}. \tag{1.10}$$

Then the following holds

(i) The $\Gamma-$limit of $\{\mathcal{F}^*_{\varepsilon,ap}\}_{\varepsilon>0}$ as $\varepsilon \to 0$ with respect to the strong topology on $L^2(\Omega)$ is given by

$$\mathcal{F}^*_{ap}[u] := \begin{cases} \mathbf{m} Per(\{u = 1\}) & \text{if } u \in BV(\Omega; \{\pm 1\}) \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$\mathbf{m} = \min \left\{ \int_\mathbb{R} W(f) + |f''|^2 dt : f \in W^{2,2}_{loc}(\mathbb{R}), \lim_{t\to+\infty} f(t) = +1, \lim_{t\to-\infty} f(t) = -1 \right\}.$$

(ii) Compactness: Let $\{u_\varepsilon\}_{\varepsilon>0} \subseteq X$ be such that $\liminf_{\varepsilon\to 0^+} \mathcal{F}^*_{\varepsilon,ap}[u_\varepsilon] < +\infty$. Then there exists a subsequence $(u_{\varepsilon_n})_{n\in\mathbb{N}} \subseteq \{u_\varepsilon\}_{\varepsilon>0}$ and a function $u \in BV(\Omega; \{\pm 1\})$ such that $u_{\varepsilon_n} \to u$ in $L^2(\Omega)$ as $n \to \infty$.

We note that the Γ-limiting functional involves a one-dimensional optimal profile problem, and agrees with the one derived in [24]. There, functionals of the form (1.9) are considered, with the Laplacian replaced by the full Hessian.

1.2.2. Outline. The remainder of the paper is organized as follows. We briefly set the notation in Section 2. The Γ-limit in arbitrary space dimension for the nonlocal functional (1.1) is proven in Section 3, and subsequently, in Section 4, the Γ-limit in the case $d = 1$ is derived under weaker assumptions. In Section 4.1, the associated optimal profile problem is discussed. Finally, in Section 5 we deal with the local functionals (1.2). This section relies on one of the authors' Master's thesis [42], where also some more details can be found.

2. Notation and preliminary results

Throughout the text, C stands for a generic constant that may change from expression to expression. For a measurable set $E \subseteq \mathbb{R}^d$, we denote by $\mathcal{L}^d(E)$ or $|E|$ its d-dimensional Lebesgue measure, by $\mathcal{H}^{d-1}(E)$ its $(d-1)$ - dimensional Hausdorff measure, and by $\partial^* E$ its essential boundary. Except for section 5, we always assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set with $\mathcal{C}^4$-boundary.
Given $\Psi \in C_c^\infty(\mathbb{R}^d)$, with $\mathrm{supp}(\Psi) \subset B(0,1)$ and $\int_{\mathbb{R}^d} \Psi(x)\,dx = 1$, we define for $\varepsilon > 0$ the standard mollifier

$$\Psi_\varepsilon(x) := \frac{1}{\varepsilon^d}\Psi\left(\frac{x}{\varepsilon}\right). \tag{2.1}$$

If $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ then $\varphi_\varepsilon := \varphi * \Psi_\varepsilon \in C^\infty(\mathbb{R}^d)$. If additionally $\varphi \in L^\infty(\mathbb{R}^d)$, then $\varphi_\varepsilon \to \varphi$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for every $p \in [1,\infty)$,

$$\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)}, \quad \|\nabla\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\varepsilon^{-1}, \|\nabla^2\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\varepsilon^{-2} \quad \text{and} \quad \|\nabla^3\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq C\varepsilon^{-3}, \tag{2.2}$$

where the constants C depend on φ but not on ε. We will use the following special cases of the Gagliardo-Nirenberg interpolation inequalities from [26, 39]. For every open, bounded connected domain $E \subset \mathbb{R}^d$, there are constants C_1, $C_2 > 0$ such that for every $u : E \to \mathbb{R}$,

$$\|\nabla u\|_{L^2(E)} \leq C_1\|\nabla^2 u\|_{L^2(E)}^{1/2}\|u\|_{L^2(E)}^{1/2} + C_2\|u\|_{L^2(E)} \leq \frac{C_1}{2}\|\nabla^2 u\|_{L^2(E)} + \left(\frac{C_1}{2} + C_2\right)\|u\|_{L^2(E)}, \tag{2.3}$$

and

$$\|\nabla u\|_{L^{4/3}(E)} \leq C_1\|\nabla^2 u\|_{L^2(E)}^{1/2}\|u\|_{L^1(E)}^{1/2} + C_2\|u\|_{L^1(E)}. \tag{2.4}$$

In several places throughout the text we will need to make use of equiintegrability of an energy bounded sequence. Therefore we separate this implication in the lemma below.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ open and bounded. Let $W : \mathbb{R} \to [0,\infty)$ be such that there exists $R > 0$ and $\lambda_1 > 0$ with*

$$\lambda_1|s|^2 \leq W(s) \quad \text{for all } s \in \mathbb{R} \text{ with } |s| \geq R.$$

Let $\varepsilon_n \to 0$ and $(u_n) \subset L^1(\Omega)$ such that

$$\sup_n \int_\Omega \frac{W(u_n)}{\varepsilon_n} dx = M$$

for some constant $M \in \mathbb{R}$. Then the family $(|u_n|^2)_{n\in\mathbb{N}}$ is equiintegrable.

Proof. Fix $\epsilon > 0$ arbitrary and let $E \subset \Omega$ be a measurable set. Then it holds

$$\begin{aligned}\int_E |u_n|^2 dx &= \int_{E\cap\{|u_n|<R\}} |u_n|^2 dx + \int_{E\cap\{|u_n|\geq R\}} |u_n|^2 dx \\ &\leq \mathcal{L}^d(E)R^2 + \frac{1}{\lambda_1}\int_{E\cap\{|u_n|\geq R\}} W(u_n) dx \\ &\leq \mathcal{L}^d(E)R^2 + \frac{1}{\lambda_1}\varepsilon_n M.\end{aligned}$$

Since $\varepsilon_n \to 0$, there exists $N \in \mathbb{N}$ such that $\varepsilon_n < \frac{\lambda_1}{2M}\epsilon$ for all $n \geq N$. Consequently, if we choose $\tilde{\delta} := \varepsilon/(2R^2)$ then for all $n \geq N$ and all sets with $\mathcal{L}^d(E) \leq \delta$ we have $\|u_n\|^2_{L^2(E)} \leq \epsilon$. Now choose $\delta \leq \tilde{\delta}$ small enough such that $\mathcal{L}^d(E) \leq \delta$ implies $\|u_n\|^2_{L^2(E)} \leq \varepsilon$ also for the finitely many $u_1, \dots, u_{N-1}$. $\square$

It will be convenient to rewrite the functional $\mathcal{F}_\varepsilon$ in different ways, see also [22]. Given $u \in W^{1,2}(\Omega)$, we define $v \in W^{2,2}(\Omega)$ as the weak solution to

$$-\varepsilon^2 \Delta v + v = u \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial v}{\partial \hat{n}} = 0 \text{ on } \partial\Omega, \tag{2.5}$$

where $\hat{n}$ denotes the unit normal to $\partial\Omega$, and we use the abbreviatory notation $v := (\mathbf{1} - \varepsilon^2\Delta)^{-1}u$. If $\mathcal{F}^*_\varepsilon[u] < \infty$, then integration by parts and (2.5) allows to rewrite the functional as (see also [22] for the detailed computations)

$$\begin{aligned}
\mathcal{F}^*_\varepsilon[u] &= \int_\Omega \left(\frac{1}{\varepsilon} W(u) + (1-q)\varepsilon|\nabla u|^2 - \varepsilon|\nabla v|^2 - \varepsilon^3|\Delta v|^2 \right) dx \\
&= \int_\Omega \left(\frac{1}{\varepsilon} W(u) - \varepsilon q|\nabla u|^2 + \varepsilon^3(\Delta v)^2 + \varepsilon^5|\nabla\Delta v|^2 \right) dx \\
&= \int_\Omega \left(\frac{1}{\varepsilon} W(u) - \varepsilon q|\nabla v|^2 + (1-2q)\varepsilon^3(\Delta v)^2 + (1-q)\varepsilon^5|\nabla\Delta v|^2 \right) dx.
\end{aligned} \tag{2.6}$$

Due to (2.6) we may also view $\mathcal{F}^*_\varepsilon[u]$ as $\mathcal{F}_\varepsilon[v]$ with $\mathcal{F}_\varepsilon : W^{2,2}(\Omega) \to [0,\infty]$ and $u = -\varepsilon^2\Delta v + v$, i.e.,

$$\mathcal{F}_\varepsilon[v] = \begin{cases} \int_\Omega \left(\frac{1}{\varepsilon} W(-\varepsilon^2\Delta v + v) - \varepsilon q|\nabla v|^2 + (1-2q)\varepsilon^3(\Delta v)^2 + (1-q)\varepsilon^5|\nabla\Delta v|^2 \right) dx, & \text{if } \frac{\partial v}{\partial \hat{n}} = 0 \text{ on } \partial\Omega, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Here and throughout the text, we shall use the convention that $\mathcal{F}_\varepsilon[v] = +\infty$ if $\nabla\Delta v \notin L^2(\Omega)$. Note that for $q < 1/2$, $\mathcal{F}_\varepsilon[v] < \infty$ implies that $\Delta v \in W^{1,2}(\Omega)$. If we consider restrictions of the integral to smaller open domains $E \subset \Omega$, then we set

$$\mathcal{F}^*_\varepsilon[u; E] := \begin{cases} \frac{1}{\varepsilon} \int_E \left(W(u) - u^2 + (1-q)\varepsilon^2|\nabla u|^2 + u \left(\mathbf{1} - \varepsilon^2\Delta \right)^{-1} u \right) dx, & \text{if } u \in W^{1,2}(E); \\ +\infty, & \text{if } u \in L^2(E) \setminus W^{1,2}(E), \end{cases}$$

and similarly for $\mathcal{F}^*_{\varepsilon,ap}$ or $\mathcal{F}_\varepsilon$.

2.1. Some geometric preliminaries

In order to define the recovery sequence in Section 3.3 one needs to define diffeomorphisms of the tubular neighbourhood of the boundary of the domain. We sketch the construction here, for more details see e.g. [7]. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, C^4-domain. Then its boundary $\partial\Omega$ is a C^4-submanifold of $\mathbb{R}^d$ and thus there exists a normal bundle of the boundary that we denote $N(\partial\Omega) \subseteq \partial\Omega \times \mathbb{R}^d$ (see [7, Definition 11.1]) with the associated natural projection to the boundary

$$\begin{aligned} p &: N(\partial\Omega) \to \partial\Omega \\ p &: (x,v) \mapsto x. \end{aligned}$$

Note that the natural projection p is a mapping of class C^3 (see e.g. [25, Lemma]). Let us denote the subset of the normal bundle that contains only vectors of length less than 2δ by $N_{<2\delta}(\partial\Omega) := \{(x,v) \in N(\partial\Omega) : \|v\| < 2\delta\}$. Then, by the Tubular Neighbourhood Theorem (see e.g. [7, Def. 11.3 and Theorem 11.4] and compactness of $\partial\Omega$, there exists $\delta > 0$ such that

$$\begin{aligned} t &: N_{<2\delta}(\partial\Omega) \to V'_{2\delta} := \{x \in \mathbb{R}^d : \operatorname{dist}(x,\partial\Omega) < 2\delta\} \\ t &: (x,v) \mapsto x + v \end{aligned}$$

is a diffeomorphism, of class C^3. Let

$$(O_1, \phi_1), \dots, (O_M, \phi_M)$$

be an atlas on $\partial\Omega$, i.e., $\{O_1, \ldots, O_M\}$ is an open cover of $\partial\Omega$ and $\phi_1, \ldots, \phi_M$ are the corresponding local diffeomorphisms of class C^3 between O_i and subsets of $\mathbb{R}^{d-1}$. Using this atlas, we would like to define a new atlas on the inner neighbourhood of the boundary. Notice that $p^{-1}(O_i) \subset N(\partial\Omega)$ is actually diffeomorphic to $O_i \times \mathbb{R}^1$ and we may choose the diffeomorphism such that it preserves orientation. More precisely, for every $i = 1, \ldots, M$ we have the diffeomorphism

$$\begin{aligned} p^{-1}(O_i) &\to O_i \times \mathbb{R} \\ (x, v) &\mapsto (p(x, v), or_i(x, v)) . \end{aligned}$$

such that $or_i(x, v) > 0$ if $x + v \in \Omega$ and $or_i(x, v) < 0$ otherwise. In particular, for each $i = 1, \ldots, M$ there is the diffeomorphism

$$\begin{aligned} \Theta_i : p^{-1}(O_i) &\to O_i \times \mathbb{R} \\ (x, v) &\mapsto \left(p(x, v), \frac{or_i(x, v)}{|or_i(x, v)|} \|v\| \right) . \end{aligned}$$

Finally, we have an atlas on $N(\partial\Omega)$ given by $\{p^{-1}(O_i), (\phi_i \times id) \circ \Theta_i\}_{i=1}^M$.
Starting from this atlas, we would like to find diffeomorphisms that "straighten out" the inner neighbourhood of the boundary. Therefore we define the patches of the the 2δ-neighbourhood of the boundary via

$$U_i' := t\left(p^{-1}(O_i) \cap N_{<2\delta}(\partial\Omega)\right)$$

and diffeomorphisms on them via

$$\begin{aligned} \varphi_i :\ U_i' \xrightarrow{t^{-1}} p^{-1}(O_i) \cap N_{<2\delta}(\partial\Omega) &\overset{(\phi_i \times id)\circ\Theta_i}{\longrightarrow} \phi_i(O_i) \times (-2\delta, 2\delta) \\ x + v \mapsto (x, v) &\mapsto (\phi_i(x), \frac{or_i(x, v)}{|or_i(x, v)|} \|v\|). \end{aligned}$$

However, we are only interested in the part of the neighbourhood inside Ω (previously referred to as "inner neighbourhood"), so we define the inner patches as

$$U_i := U_i' \cap \Omega, \qquad i = 1, \ldots, M$$

and corresponding diffeomorphisms (of class C^3)

$$\begin{aligned} \varphi_i : U_i &\to \phi_i(O_i) \times (0, 2\delta) \\ x + v &\mapsto (\phi_i(x), \|v\|). \end{aligned}$$

Denote the whole inner neighbourhood by

$$V_{2\delta} := \bigcup_{i=1}^M U_i = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < 2\delta\}.$$

Then $\{U_i, \varphi_i\}_{i=1}^M$ is an atlas on $V_{2\delta}$.

We close this section by showing that for every $u \in BV(\Omega; \{\pm 1\})$ extending the trace values of u in normal direction into the tubular neighborhood $V_{2\delta}$ agrees with u up to a set of measure at most $o(\delta)$.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ with a C^4-boundary. For $u \in BV(\Omega; \{\pm 1\})$ define $\tilde{u} : V_{2\delta} \to \mathbb{R}$ by $\tilde{u}(x - t\nu(x)) = Tu(x)$ where $x \in \partial\Omega$, $t \in (0, 2\delta)$ and $\nu(x) \in S^{d-1}$ the outer normal to $\partial\Omega$. Then there exists $\delta_0 > 0$ and $C > 0$ such that for all $\delta \in (0, \delta_0)$ and $u \in BV(\Omega; \{\pm 1\})$ it holds*

$$\mathcal{L}^d\left(\{u(x) \neq \tilde{u}(x) : x \in V_{2\delta}\}\right) \leq C\delta |Du|(V_{2\delta}).$$

Proof. Note that by the discussion above, we obtain that there exists $\delta_0 > 0$ such that $V_{2\delta_0}$ can be diffeomorphically mapped to $\partial\Omega \times (0, 2\delta_0)$ via the orthogonal projection onto the boundary and the distance to the boundary. Moreover, the corresponding derivative and its inverse are bounded. We note that for $\delta \in (0, \delta_0)$ the same mapping maps $V_{2\delta}$ diffeomorphically to $\partial\Omega \times (0, 2\delta)$. Fix $\delta \in (0, \delta_0)$.

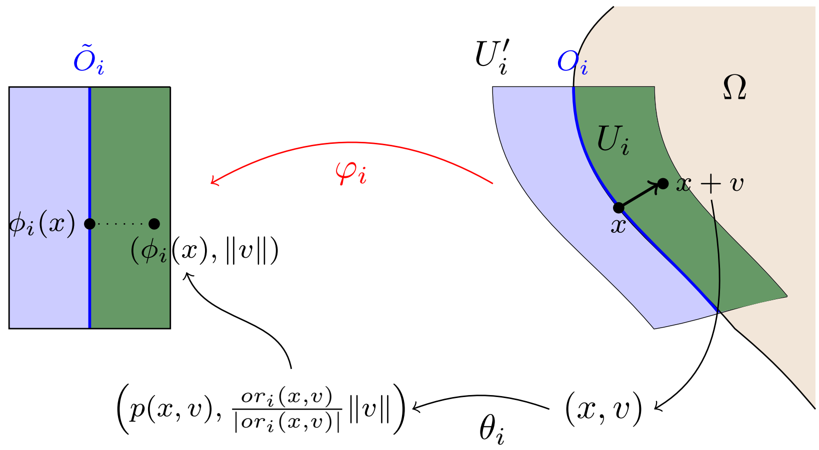

FIGURE 1. Local diffeomorphism on the tubular neighbourhood. The picture is oriented in such a way that e_d points to the right.

Now, let $v \in C^\infty(\overline{V_{2\delta}})$. Then we estimate

$$\int_{V_{2\delta}} |v(x) - \tilde{v}(x)|\,dx \leq C \int_{\partial\Omega} \int_0^{2\delta} |v(x - t\nu(x)) - v(x)|\,dt\,d\mathcal{H}^{d-1}(x) \tag{2.8}$$

$$\leq C \int_{\partial\Omega} \int_0^{2\delta} \int_0^{2\delta} |\partial_{\nu(x)} v(x - s\nu(x))|\,ds\,dt\,d\mathcal{H}^{d-1}(x) \tag{2.9}$$

$$\leq C\delta \int_{\partial\Omega} \int_0^{2\delta} |\nabla v(x - s\nu(x))|\,ds\,d\mathcal{H}^{d-1}(x) \tag{2.10}$$

$$\leq C\delta \int_{V_{2\delta}} |\nabla v(x)|\,dx. \tag{2.11}$$

Here, the constants in the first and and last inequality stem from the change of coordinates from $V_{2\delta}$ to $\partial\Omega \times (0, 2\delta)$ and vice versa. Note that these estimates are indeed uniform in δ by the uniform bounds on the corresponding derivatives.

Next, let $u \in BV(\Omega; \{\pm 1\})$. Then there exists a sequence $v_k \in C^\infty(\overline{V_{2\delta}})$ such that $v_k \to u$ in $L^1(V_{2\delta})$ and $\int_{V_{2\delta}} |\nabla v_k(x)|\,dx \to |Du|(V_{2\delta})$. Since the trace as a map from $BV(V_{2\delta})$ to $L^1(\partial\Omega)$ is continuous with respect to strict convergence, we obtain from (2.11)

$$\int_{V_{2\delta}} |u(x) - \tilde{u}(x)|\,dx \leq C\delta |Du|(V_{2\delta}).$$

Next, observe that since $u \in \{\pm 1\}$, the same follows for its trace Tu and consequently also for $\tilde{u}$. In particular, it holds $2|\{u(x) \neq \tilde{u}(x) : x \in V_{2\delta}\}| = \int_{V_{2\delta}} |u(x) - \tilde{u}(x)|\,dx$. Hence, the claim follows. □

3. Proof of Theorem 1.3 for $d \geq 1$

In this section, we will prove Theorem 1.3 in the case of general space dimension $d \geq 1$.

3.1. Compactness

To prove compactness (i.e., item 3. of Theorem 1.3) we will use an interpolation inequality from [11]. For that, we note that for any double-well potential W satisfying Assumption 1.1 there is a constant $c_W > 0$ such that $W(s) \geq c_W(s \mp 1)^2$ for all $s \geq 0$. Indeed, by Taylor's expansion

$$W(u) = \frac{W''(\pm 1)}{2}(u \mp 1)^2 + o((u \mp 1)^2),$$

and hence there is $\delta > 0$ such that (using that $W''(\pm 1) > 0$ by assumption), $W(u) \geq \frac{W''(\pm 1)}{2}(u \mp 1)^2$ for all $u \in (-1-\delta, -1+\delta) \cup (1-\delta, 1+\delta)$. Then, the lower bound on W in the compact set $[-R, -1-\delta] \cup [1+\delta, R]$ follows from continuity of W and the assumption that W has no zero in this set. Finally, for $|s| \geq R > 1$, the lower bound follows from the growth condition in Assumption 1.1. Hence, we can apply the following result to our setting.

Proposition 3.1. *Suppose that $W : \mathbb{R} \to [0, \infty)$ is continuous and that there is a constant $c_W > 0$ such that $W(s) \geq c_W(s \mp 1)^2$ for all $\pm s > 0$. Then for every bounded domain $\Omega \subset \mathbb{R}^d$, there exists a constant C, such that for all $v \in W^{2,2}(\Omega)$, and all $k > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$k \int_\Omega \varepsilon |\nabla v|^2 \, dx \leq \frac{1}{2} \int_\Omega \left(\frac{W(v)}{\varepsilon} + Ck^2 \varepsilon^3 |\nabla^2 v|^2 \right) dx. \tag{3.1}$$

Proof. This is shown in [11, Lemma 3.1, Remarks 3.2 and 3.5]. □

We now prove the compactness result for low-energy sequences, see the third item of Theorem 1.3.

Proof. Since $q \leq 0$, the form of the functional in (2.6) and the upper bound (1.8) imply that

$$\varepsilon_n^3 \|\Delta v_n\|^2_{L^2(\Omega)} \leq M. \tag{3.2}$$

Consider first the case $q < 0$. Then by the second line of (2.6),

$$\mathcal{F}^*_{\varepsilon_n}[u_n] \geq \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_n) - \varepsilon_n q |\nabla u_n|^2 \right) dx \geq \min\{1, -q\} \int_\Omega \left(\frac{1}{\varepsilon_n} W(u_n) + \varepsilon_n |\nabla u_n|^2 \right) dx,$$

and by the compactness result for the Modica-Mortola functional (see e.g. [38, Theorem I.]) we obtain a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in BV(\Omega; \{\pm 1\})$ such that $u_{n_k} \to u$ in $L^1(\Omega)$. In addition, by the hypotheses on W (see Assumption 1.1) and Lemma 2.1 ($|u_n|^2$) is equiintegrable and by Vitali's convergence theorem, we obtain L^2-convergence of $(u_{n_k})_{k\in\mathbb{N}}$ to u. Finally, since $v_n = \varepsilon_n^2 \Delta v_n + u_n$, the upper bound (3.2) implies that also $(v_{n_k})_{k\in\mathbb{N}}$ converges to u in L^2.
Consider now the case $q = 0$. We first prove L^2-convergence of $(v_{n_k})_{k\in\mathbb{N}}$ and deduce convergence of $(u_{n_k})_{k\in\mathbb{N}}$. Observe that by the hypotheses on W, since

$$\|u_n\|^2_{L^2(\Omega)} = \int_\Omega (v_n - \varepsilon_n \Delta v_n)^2 = \|v_n\|^2_{L^2(\Omega)} + 2\varepsilon_n \|\nabla v_n\|^2_{L^2(\Omega)} + \varepsilon_n^2 \|\Delta v_n\|^2_{L^2(\Omega)} \geq \|v_n\|^2_{L^2(\Omega)},$$

we have

$$\|v_n\|^2_{L^2(\Omega)} \leq \|u_n\|^2_{L^2(\Omega)} \leq R^2 \mathcal{L}^d(\Omega) + \frac{M\varepsilon_n}{\lambda_1} \leq C_1. \tag{3.3}$$

We define the modified potential $\tilde{W}$ as

$$\tilde{W}(x) := \begin{cases} W(x), & \text{if} \quad 0 \leq |x| \leq R, \\ S(x), & \text{if} \quad R \leq |x| \leq R+1, \\ \lambda_1 x^2, & \text{if} \quad |x| \geq R+1, \end{cases}$$

where S is an interpolating polynomial such that $\tilde{W} \in C^2(\mathbb{R})$. Note that S can be chosen such that $\tilde{W}$ satisfies Assumption 1.1, and additionally there is some constant $C_W > 0$ such that

$$\tilde{W} \leq CW, \qquad \text{and} \qquad \|\tilde{W}''\|_{L^\infty(\mathbb{R})} \leq C_W. \tag{3.4}$$

In particular, we may argue as before Proposition 3.1 to find that $\tilde{W}$ satisfies the assumptions of Proposition 3.1. Then clearly, by (1.8),

$$\int_\Omega \frac{1}{\varepsilon_n} \tilde{W}\left(-\varepsilon_n^2 \Delta v_n + v_n\right) dx \leq C\mathcal{F}^*_{\varepsilon_n}[u_n] \leq CM,$$

and

$$\int_\Omega \frac{1}{\varepsilon_n} \tilde{W}(v_n) \, dx \leq CM - \int_\Omega \frac{1}{\varepsilon_n} \left(\tilde{W}(-\varepsilon_n^2 \Delta v_n + v_n) - \tilde{W}(v_n) \right).$$

To bound the right-hand side, we employ Taylor's theorem, which yields

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon_n}\tilde{W}(v_n)\,dx &\leq CM + \left|\int_\Omega \varepsilon_n \tilde{W}'(v_n)\Delta v_n\,dx\right| + \frac{1}{2}\varepsilon_n^3\|\tilde{W}''\|_{L^\infty(\mathbb{R})}\|\Delta v_n\|^2_{L^2(\Omega)} \\ &\leq \varepsilon_n\|\tilde{W}''\|_{L^\infty(\mathbb{R})}\int_\Omega |\nabla v_n|^2\,dx + C(M,W), \end{aligned}\tag{3.5}$$

where we integrated by parts in the second term on the right-hand side and used (3.2) to bound the third term. Thus by Proposition 3.1 applied to $\tilde{W}$ with $k := \|\tilde{W}''\|_{L^\infty(\mathbb{R})}$ there is $\varepsilon_0 > 0$ such that for all $\varepsilon_n \in (0, \varepsilon_0)$

$$\|\tilde{W}''\|_{L^\infty(\mathbb{R})}\int_\Omega \varepsilon_n|\nabla v_n|^2\,dx \leq \frac{1}{2}\int_\Omega\left(\frac{\tilde{W}(v_n)}{\varepsilon_n} + K\varepsilon_n^3|\nabla^2 v_n|^2\right)dx,$$

where we denote the corresponding constant $C(k)$ by K. Since v_n satisfies Neumann boundary conditions, standard elliptic regularity estimates (see e.g. [32, 27, 29]) imply that

$$\|\nabla^2 v_n\|^2_{L^2(\Omega)} \leq C_e\left(\|v_n\|^2_{L^2(\Omega)} + \|\Delta v_n\|^2_{L^2(\Omega)}\right).$$

To close the estimate we observe using (3.5) that this yields

$$\int_\Omega\left(\frac{1}{\varepsilon_n}\tilde{W}(v_n) + K\varepsilon_n^3|\nabla^2 v_n|^2\right)dx \leq \int_\Omega\left(\frac{1}{2\varepsilon_n}\tilde{W}(v_n) + 2C_eK\varepsilon_n^3(\Delta v_n)^2\right)dx + C_eK\varepsilon_n^3\|v_n\|^2_{L^2(\Omega)}.$$

Absorbing the term involving $\tilde{W}$ from the right hand side on the left hand side and using (3.3) and (3.2) we obtain

$$\int_\Omega\left(\frac{1}{\varepsilon_n}\tilde{W}(v_n) + K\varepsilon_n^3|\nabla^2 v_n|^2\right)dx \leq 2C(M,W,K,C_e).$$

By the compactness result from [24, Proposition 3.1] applied to $(v_n)_{n\in\mathbb{N}}$, there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ and $u \in BV(\Omega;\{\pm1\})$ such that $v_{n_k} \to u$ in $L^1(\Omega)$. In addition, since $\tilde{W}$ satisfies the same growth condition as W, we have by Lemma 2.1 that $(|v_{n_k}|^2)_{k\in\mathbb{N}}$ is equi-integrable, and L^2-convergence of $(v_{n_k})_{k\in\mathbb{N}}$ follows from Vitali's Convergence Theorem. Since $u_n = -\varepsilon_n^2\Delta v_n + v_n$, the upper bound (3.2) yields the convergence of $(u_{n_k})_{k\in\mathbb{N}}$.

□

3.2. Liminf Inequality

We now turn to the proof of the liminf inequality (1.6). For that, we closely follow the lines of the proof presented in [31, 8] but, for the reader's convenience, present the full proof adjusted to our setting. Let $u \in BV(\Omega;\{\pm1\})$. Assume that $\varepsilon_n \to 0^+$, and that $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ satisfies (extracting a subsequence if needed)

$$\liminf_{n\to\infty}\mathcal{F}_{\varepsilon_n}[v_n;\Omega] = \lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}[v_n;\Omega] = M < \infty,\tag{3.6}$$

where $u_n \to u$ in $L^2(\Omega)$ and pointwise for $\mathcal{L}^d$-a.e. $x \in \Omega$, and $v_n = (\mathbf{1} - \varepsilon^2\Delta)^{-1}u_n$. Define $E_0 \subset \Omega$ by

$$u = \chi_{E_0} - \chi_{\Omega\setminus E_0},\tag{3.7}$$

and note that $\mathrm{Per}_\Omega(E_0) < \infty$ since $u \in BV(\Omega;\{\pm1\})$. Consider Q_ν as defined in (1.3), and for $x_0 \in \mathbb{R}^d$ and $r_0 > 0$ set

$$Q_\nu(x_0, r_0) := \left\{x \in \mathbb{R}^d : \frac{(x - x_0)}{r_0} \in Q_\nu\right\}.\tag{3.8}$$

In Appendix A, we use a blow-up argument (see, e.g., [10, 23]) to show that

$$\lim_{n\to\infty}\mathcal{F}_{\varepsilon_n}[v_n;\Omega] \geq \int_{\partial^* E_0}\lim_{k\to\infty}\mathcal{F}_{\varepsilon_k}[w_k;Q_\nu(0,1)]\,d\mathcal{H}^{n-1},$$

where $w_k \in W^{2,2}(Q_\nu(0,1))$ satisfies

$$\|w_k\|_{L^2(Q_\nu)} \leq \|v_{n_k}\|_{L^2(\Omega)} \quad \text{and} \quad \lim_{k\to\infty} \|w_k - u_0\|_{L^2(Q_\nu(0,1))} = 0$$

with

$$u_0(x) := \begin{cases} -1 & \text{if } x\cdot\nu < 0, \\ 1 & \text{if } x\cdot\nu > 0. \end{cases} \tag{3.9}$$

The lower bound (1.6) then follows from the following lower bound for $\mathcal{F}_{\varepsilon_n}[w_n; Q_\nu(0,1)]$.

Proposition 3.2. *Assume that $\varepsilon_n \to 0^+$ and that $(w_n)_{n\in\mathbb{N}} \subset W^{2,2}(Q_\nu(0,1))$ satisfies*

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu(0,1)] = M < \infty,$$

as well as $w_n \to u_0$ in $L^2(Q_\nu)$ and for $\mathcal{L}^d$-a.e. $x \in Q_\nu(0,1)$, where u_0 is defined in (3.9). Then

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu(0,1)] \geq m_d. \tag{3.10}$$

Proof. We will show that there exists a sequence $(\tilde{w}_n)_{n\in\mathbb{N}} \subset \mathcal{A}_\nu$ satisfying

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[\tilde{w}_n; Q_\nu(0,1)] \leq \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu(0,1)].$$

For that, we follow the lines of [22] (see also [10]), which rely on an appropriate slicing argument. Note that the argument simplifies since in our case the functional is non-negative. Let $(\Psi_{\varepsilon_n})_{n\in\mathbb{N}}$ be a sequence of mollifiers as defined in (2.1), and set $\varphi_n := u_0 * \Psi_{\varepsilon_n}$. Then, by construction, $\varphi_n \in \mathcal{A}_\nu$ for $\varepsilon_n > 0$ sufficiently small. We may assume that for all $n \in \mathbb{N}$ (recall that $Q_\nu = Q_\nu(0,1)$)

$$\mathcal{F}_{\varepsilon_n}[w_n; Q_\nu] \leq C < \infty,$$

and, consequently,

$$\|\nabla w_n\|_{L^2(Q_\nu)} \leq C\varepsilon_n^{-1/2}, \quad \|\Delta w_n\|_{L^2(Q_\nu)} \leq C\varepsilon_n^{-3/2}, \quad \text{and} \quad \|\nabla\Delta w_n\|_{L^2(Q_\nu)} \leq C\varepsilon_n^{-5/2}. \tag{3.11}$$

Further, by elliptic interior regularity estimates (see [27, 32]), we have that for any $\delta > 0$

$$\|\nabla^2 w_n\|_{L^2(Q_\nu^\delta)} \leq C_\delta(\|\Delta w_n\|_{L^2(Q_\nu)} + \|w_n\|_{L^2(Q_\nu)}) \leq C_\delta\varepsilon_n^{-3/2}, \tag{3.12}$$

where $Q_\nu^\delta := (1-\delta)Q_\nu$ and the constant C_δ depends on δ. We want to define $\tilde{w}_n$ to be equal to φ_n near the boundary of Q_ν and to be equal to w_n away from the boundary. More precisely, we fix $\delta \in (0,1/2)$ and $m \in \mathbb{N}$ such that $\delta < \frac{1}{2m}$. We first partition the set $\left\{x \in Q_\nu : \frac{1}{m} < \operatorname{dist}(x, \partial Q_\nu) \leq \frac{2}{m}\right\}$ into $\lfloor \varepsilon_n^{-1} \rfloor$ layers of width $\frac{1}{m\lfloor\varepsilon_n^{-1}\rfloor}$, i.e., for $i = 1, \ldots, \lfloor\varepsilon_n^{-1}\rfloor$ we set

$$Q_{m,n,i}^{\text{trans}} := \left\{x \in Q_\nu : \frac{1}{m} + \frac{i-1}{m\lfloor\varepsilon_n^{-1}\rfloor} < \operatorname{dist}(x, \partial Q_\nu) \leq \frac{1}{m} + \frac{i}{m\lfloor\varepsilon_n^{-1}\rfloor}\right\}.$$

For one of these layers, $Q_{m,n}^{\text{trans}} := Q_{m,n,i^*}^{\text{trans}}$ we have,

$$\begin{aligned} &\|w_n\|_{L^2(Q_{m,n}^{\text{trans}})}^2 \leq C\varepsilon_n\|w_n\|_{L^2(Q_\nu^\delta)}^2, \qquad \|\nabla w_n\|_{L^2(Q_{m,n}^{\text{trans}})}^2 \leq C\varepsilon_n\|\nabla w_n\|_{L^2(Q_\nu^\delta)}^2, \\ &\|\nabla^2 w_n\|_{L^2(Q_{m,n}^{\text{trans}})}^2 \leq C\varepsilon_n\|\nabla^2 w_n\|_{L^2(Q_\nu^\delta)}^2, \qquad \|w_n - \varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})}^2 \leq C\varepsilon_n\|w_n - \varphi_n\|_{L^2(Q_\nu^\delta)}^2, \\ &\text{and} \quad \mathcal{F}_{\varepsilon_n}[w_n; Q_{m,n}^{\text{trans}}] \leq C\varepsilon_n. \end{aligned} \tag{3.13}$$

We note that, since both $w_n \to u_0$ in $L^2(Q_\nu)$ and $\varphi_n \to u_0$ in $L^2(Q_\nu)$, we have

$$\varepsilon_n^{-1}\|w_n - \varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})}^2 \leq C\|w_n - \varphi_n\|_{L^2(Q_\nu)}^2 \to 0 \text{ as } n \to \infty. \tag{3.14}$$

We now define

$$\tilde{w}_{m,n}(x) := \eta_{m,n} w_n + (1 - \eta_{m,n})\varphi_n,$$

where $\eta_{m,n} \in C^\infty$ is such that

$$\eta_{m,n}(x) := \begin{cases} 0 & \text{if } x \in Q_{m,n}^{\text{out}} := \left\{ x \in Q_\nu : \operatorname{dist}(x, \partial Q_\nu) \le \frac{1}{m} + \frac{i^*-1}{m\lfloor \varepsilon_n^{-1} \rfloor} \right\}, \\ \in (0,1) & \text{if } x \in Q_{m,n}^{\text{trans}}, \\ 1 & \text{if } x \in Q_{m,n}^{\text{in}} := \left\{ x \in Q_\nu : \operatorname{dist}(x, \partial Q_\nu) > \frac{1}{m} + \frac{i^*}{m\lfloor \varepsilon_n^{-1} \rfloor} \right\}, \end{cases}$$

and

$$\|\nabla^k \eta_{m,n}\|_{L^\infty(Q_\nu)} \le C\left(\frac{m^k}{\varepsilon_n^k}\right), \quad k = 1, 2, 3. \tag{3.15}$$

Then $\tilde{w}_{m,n} \in \mathcal{A}_\nu$, and since $\tilde{w}_{m,n} = w_n$ in $Q_{m,n}^{\text{in}}$ and $\tilde{v}_{m,n} = \varphi_n$ in $Q_{m,n}^{\text{out}}$, we have

$$\begin{aligned} \mathcal{F}_{\varepsilon_n}[\tilde{w}_{m,n}; Q_\nu] &= \mathcal{F}_{\varepsilon_n}[w_n; Q_{m,n}^{\text{in}}] + \mathcal{F}_{\varepsilon_n}[\tilde{w}_{m,n}; Q_{m,n}^{\text{trans}}] + \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_{m,n}^{\text{out}}] \\ &\le \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu] + \mathcal{F}_{\varepsilon_n}[\tilde{w}_{m,n}; Q_{m,n}^{\text{trans}}] + \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_{m,n}^{\text{out}}]. \end{aligned} \tag{3.16}$$

To complete the proof, it remains to control $\mathcal{F}_{\varepsilon_n}[\tilde{w}_{m,n}; Q_{m,n}^{\text{trans}}]$ and $\mathcal{F}_{\varepsilon_n}[\varphi_n; Q_{m,n}^{\text{out}}]$. By continuity of W, the bounds on φ_n (see (2.2)) and using that $\nabla\varphi_n = 0$ on $Q_\nu \setminus \{|x \cdot \nu| \le \varepsilon_n\}$, we obtain

$$\begin{aligned} \mathcal{F}_{\varepsilon_n}[\varphi_n; Q_{m,n}^{\text{out}}] &= \int_{Q_{m,n}^{\text{out}}} \left(\frac{1}{\varepsilon_n} W(-\varepsilon_n^2 \Delta\varphi_n + \varphi_n) - \varepsilon_n q |\nabla(-\varepsilon_n^2 \Delta\varphi_n + \varphi_n)|^2 + \varepsilon_n^3 (\Delta\varphi_n)^2 + \varepsilon_n^5 |\nabla\Delta\varphi_n|^2 \right) dx \\ &\le \frac{C}{\varepsilon_n} \mathcal{L}^d(Q_{m,n}^{\text{out}} \cap \{x \in Q_{m,n}^{\text{out}} : |x \cdot \nu| < \varepsilon_n\}) \le \frac{C}{m} \end{aligned} \tag{3.17}$$

with a constant C independent of m. The contribution of $Q_{m,n}^{\text{trans}}$ can be bounded using the product rule, (3.11), (2.2), (3.15), (3.13), and (3.14). Then

$$\begin{aligned} &\varepsilon_n^{1/2} \|\nabla\tilde{w}_{m,n}\|_{L^2(Q_{m,n}^{\text{trans}})} \le C\varepsilon_n^{1/2} \Big(\|\nabla\eta_{m,n}\|_{L^\infty(Q_{m,n}^{\text{trans}})} \|w_n - \varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})} + \\ + \quad &\|\eta_{m,n}\|_{L^\infty(Q_{m,n}^{\text{trans}})} \|\nabla w_n\|_{L^2(Q_{m,n}^{\text{trans}})} + \\ + \quad &\|1 + \eta_{m,n}\|_{L^\infty(Q_{m,n}^{\text{trans}})} \|\nabla\varphi_n\|_{L^\infty(Q_{m,n}^{\text{trans}})} \left| Q_{m,n}^{\text{trans}} \cap \{x \in Q_{m,n}^{\text{trans}} : |x \cdot \nu| < \varepsilon_n\} \right|^{1/2} \Big) \\ \le \quad &C\varepsilon_n^{1/2} \left(\frac{m}{\varepsilon_n} \varepsilon_n^{1/2} \|w_n - \varphi_n\|_{L^2(Q_\nu)} + \varepsilon_n^{1/2} \|\nabla w_n\|_{L^2(Q_\nu^\delta)} + \frac{1}{\varepsilon_n} \frac{\varepsilon_n}{m^{1/2}} \right) \\ \to \quad &0, \text{ as } n \to \infty, \end{aligned} \tag{3.18}$$

and similarly recalling also (3.12), with a constant C_m depending on m,

$$\begin{aligned} &\varepsilon_n^{3/2} \|\nabla^2\tilde{w}_{m,n}\|_{L^2(Q_{m,n}^{\text{trans}})} \le C\varepsilon_n^{3/2} \Big(\|\nabla^2\eta_{m,n}\|_{L^\infty(Q_{m,n}^{\text{trans}})} \|w_n - \varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})} + \\ + \quad &\|\nabla\eta_{m,n}\|_{L^\infty(Q_{m,n}^{\text{trans}})} \left(\|\nabla w_n\|_{L^2(Q_{m,n}^{\text{trans}})} + \|\nabla\varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})} \right) + \|\nabla^2 w_n\|_{L^2(Q_{m,n}^{\text{trans}})} + \|\nabla^2\varphi_n\|_{L^2(Q_{m,n}^{\text{trans}})} \Big) \\ \le \quad &C_m \left(\|w_n - \varphi_n\|_{L^2(Q_\nu^\delta)} + \varepsilon_n \|\nabla w_n\|_{L^2(Q_\nu^\delta)} + \varepsilon_n^{1/2} \right) \to 0, \text{ as } n \to \infty, \end{aligned} \tag{3.19}$$

and

$$\varepsilon_n^{5/2} \|\nabla\Delta\tilde{w}_{m,n}\|_{L^2(Q_{m,n}^{\text{trans}})} \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

To bound the integral involving the potential W we proceed similarly as in the proof of compactness, and consider separately the regions in which $\left|-\varepsilon_n^2 \Delta\tilde{w}_{m,n} + \tilde{w}_{m,n}\right| \le R$ and $\left|-\varepsilon_n^2 \Delta\tilde{w}_{m,n} + \tilde{w}_{m,n}\right| > R$, respectively. Then by the growth condition on W and (3.19) we obtain

$$\begin{aligned} &\left| \frac{1}{\varepsilon_n} \int_{Q^{\text{trans}}_{m,n}} W(-\varepsilon_n^2 \Delta \tilde{w}_{m,n} + \tilde{w}_{m,n})\, dx \right| \leq \frac{\sup_{|s|\leq R} W(s)}{\varepsilon_n} \mathcal{L}^d(Q^{\text{trans}}_{m,n}) + \frac{\lambda_2}{\varepsilon_n} \int_{Q^{\text{trans}}_{m,n}} (-\varepsilon_n^2 \Delta \tilde{w}_{m,n} + \tilde{w}_{m,n})^2\, dx \\ \leq\ & \frac{C}{m} + 2\lambda_2 \int_{Q^{\text{trans}}_{m,n}} \varepsilon_n^3 (\Delta \tilde{w}_{m,n})^2\, dx + \frac{2\lambda_2}{\varepsilon_n} \int_{Q^{\text{trans}}_{m,n}} \left(\eta_{m,n}(w_n - \varphi_n) + \varphi_n\right)^2\, dx \\ \leq\ & \frac{C}{m} + 2\lambda_2 \int_{Q^{\text{trans}}_{m,n}} \varepsilon_n^3 (\Delta \tilde{w}_{m,n})^2\, dx + \frac{4\lambda_2}{\varepsilon_n} \mathcal{L}^d(Q^{\text{trans}}_{m,n}) + 4\lambda_2 \|w_n - \varphi_n\|^2_{L^2(Q^\delta_\nu)} \leq \frac{C}{m} \end{aligned} \tag{3.21}$$

with a constant C independent of m. Inserting (3.17)-(3.21) into (3.16), we find

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[\tilde{w}_{m,n}; Q_\nu] \leq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[w_n; Q_\nu] + \frac{C}{m},$$

and the assertion follows by taking a diagonal sequence. □

3.3. Limsup Inequality

We finally show the limsup inequality, i.e., the second item of Theorem 1.3. The proof follows the general lines of the respective proof for the case $q > 0$ in [22, Section 5]), which in turn is based on [10]. However, in the proof presented in [22, Section 5], there is a gap in the argument in Step 2. Precisely, if one sets $V_n = 0$ in a neighborhood of $\partial\Omega$, this induces additional surface energy on the boundary that is not reflected in the limit energy. Therefore, we present here a different construction in detail. The main difference will be in a region close to $\partial\Omega$ where the construction presented here will guarantee the Neumann boundary conditions by extending the (mollified) trace values of the limiting v constantly in normal direction. This will then be glued to the construction from [22] inside Ω which recovers the interfaces that are not too close to $\partial\Omega$. As a main tool to estimate corresponding error terms Lemma 2.2 is invoked. Although we consider the case $q < 0$, a careful inspection of the proof shows that the analogous construction works for $q \in (0, q_*)$.
We first note that it suffices to construct a recovery sequence for $\mathcal{F}_\varepsilon$ instead of $\mathcal{F}^*_\varepsilon$. Precisely, we will prove the following result.

Proposition 3.3. *Let $\varepsilon_n \to 0$ and let $\mathcal{F}_\varepsilon : W^{2,2}(\Omega) \to [0, \infty]$ be as in (2.7). Then for every $v \in L^2(\Omega)$ there exists a recovery sequence $(v_n)_{n\in\mathbb{N}} \subset W^{2,2}(\Omega)$, i.e. a sequence that satisfies $v_n \to v$ strongly in $L^2(\Omega)$ and*

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \leq \mathcal{F}^*[v].$$

We first show that Proposition 3.3 implies the limsup-inequality from Theorem 1.3. Indeed, suppose that Proposition 3.3 holds, and let $u \in L^2(\Omega)$ and $\varepsilon_n \to 0$. If $u \in L^2(\Omega) \setminus BV(\Omega; \{\pm 1\})$, then $\mathcal{F}^*[u] = \infty$ and there is nothing to prove. If $u \in BV(\Omega; \{\pm1\})$, let $(v_n)_{n\in\mathbb{N}} \subset W^{2,2}(\Omega)$ be the sequence from Proposition 3.3 for $v := u$. Since we may assume that $\mathcal{F}_{\varepsilon_n}[v_n] < \infty$, we have $\Delta v_n \in W^{1,2}(\Omega)$ and $\frac{\partial v_n}{\partial \hat{n}} = 0$ on $\partial\Omega$. We define the sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ as

$$u_n := -\varepsilon_n^2 \Delta v_n + v_n.$$

Then $\mathcal{F}^*_{\varepsilon_n}[u_n] = \mathcal{F}_{\varepsilon_n}[v_n]$, which by Proposition 3.3 yields

$$\limsup_{n\to\infty} \mathcal{F}^*_{\varepsilon_n}[u_n] \leq \mathcal{F}^*[u].$$

Further, since up to a subsequence

$$\sup_n \mathcal{F}^*_{\varepsilon_n}[u_n] < +\infty,$$

we obtain by the compactness result in Theorem 1.3 (item 3), that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same limit in $L^2(\Omega)$, i.e. $u_n \to u$ in $L^2(\Omega)$. Thus, $(u_n)_{n\in\mathbb{N}}$ is indeed a recovery sequence for u.
It remains to prove Proposition 3.3. The proof is divided into several steps depending on the shape

of the set $E := \{x \in \Omega : v(x) = -1\}$. The structure follows the proof of [22], the main novelty being *Step 1.* For the ease of notation we introduce the set

$$Y := \{v \in W^{2,2}(\Omega) : \frac{\partial v}{\partial \hat{n}} = 0 \text{ on } \partial\Omega \text{ and } \Delta v \in W^{1,2}(\Omega)\}. \tag{3.22}$$

Proof. (of Proposition 3.3) Suppose that $v \in BV(\Omega; \{\pm 1\})$, otherwise there is nothing to prove.

Step 1. Let us first assume that the function $v \in BV(\Omega; \{\pm 1\})$ has a flat jump set, i.e., it is of the form (possibly after a change of coordinates)

$$v(x) := \begin{cases} -1, & x_d < 0, \\ +1, & x_d \geq 0. \end{cases}$$

Let $\varepsilon_n \to 0$. Using the preliminary discussion and notation from the Section 2.1, let $V_{2\varepsilon_n}$ be the sequence of inner tubular neighbourhoods of $\partial\Omega$ and let $\{(U_i, \varphi_i)\}_{i=1}^M$ be an atlas on $V_{2\varepsilon_0}$.We get atlases on $V_{2\varepsilon_n}$ by respective restrictions. There exists a partition of unity of the full tubular neighbourhood $\xi_i^n : V'_{2\varepsilon_n} \to [0,1],\ i = 1, \dots, M$ such that

$$\begin{aligned} &\xi_i^n \equiv 0, \quad \text{on } V'_{2\varepsilon_n} \setminus U'_i, \\ &\sum_{i=1}^M \xi_i^n(x) = 1, \quad \text{for all } x \in V'_{2\varepsilon_n}, \text{ and} \\ &\|\nabla^k \xi_i^n\|_{L^\infty(V'_{2\varepsilon_n})} \leq C, \quad k = 0, 1, 2, 3. \end{aligned}$$

Since $v \in BV(\Omega; \{\pm 1\})$, there exists the trace $Tv \in L^1(\partial\Omega; \{\pm 1\})$. Denote by $\tilde{O}_i := \varphi_i(O_i) \subseteq \mathbb{R}^{d-1}$ (where $\{O_1, \dots, O_M\}$ form a corresponding covering of $\partial\Omega$ as in the Section 2.1). Now for each $i = 1, \dots, M$ consider the mappings

$$Tv \circ (\varphi_i)^{-1} \in L^1(\tilde{O}_i).$$

Let $(\psi_{\varepsilon_n})_{n \in \mathbb{N}}$ be the standard mollifiers on $\mathbb{R}^{d-1}$. Define

$$g_n^i := (Tv \circ (\varphi_i)^{-1}) * \psi_{\varepsilon_n}, \qquad i = 1, \dots, M.$$

Then $g_n^i \in C^\infty(\tilde{O}_i)$ and

$$\|\nabla^k g_n^i\|_{L^\infty(\tilde{O}_i)} \leq \frac{C}{\varepsilon_n^k}, \quad k = 0, 1, 2, 3.$$

Notice that we define g_n^i on the Euclidean space, so this is just the Euclidean gradient. Now let g_n, $n \in \mathbb{N}$ be smoothed boundary values, i.e.,

$$g_n : \partial\Omega \to \mathbb{R}, \qquad g_n(y) := \sum_{i=1}^M \xi_i^n(y) g_n^i\left(\varphi_i(y)\right).$$

Note that $g_n \in C^3(\partial\Omega)$, if $\xi_i^n \lfloor_{O_i} \in C^3(O_i), i = 1, \dots, M$. However, $\xi_i^n \lfloor_{O_i} = \xi_i^n \circ inc_i$ where $inc_i \in C^3(O_i; U_i)$ is an inclusion. Finally, we define

$$z_n : V_{2\varepsilon_n} \to \mathbb{R}, \qquad z_n := g_n \circ P_\nu,$$

where by P_ν we denote the normal projection from the tubular neighbourhood to the boundary. More precisely, using notation from the Section 2.1 we have

$$P_\nu = p \circ t^{-1} : U_i \to O_i, \qquad x + v \mapsto x.$$

Note that the normal projection P_ν and, consequently, the mappings z_n are of class C^3 if Ω is a C^4 domain. This follows from the regularity of the projection p and the mapping t which were discussed in Section 2.1. We define the auxiliary sequence $(w_n)_{n \in \mathbb{N}} \subseteq W^{2,2}(\Omega)$ using an almost

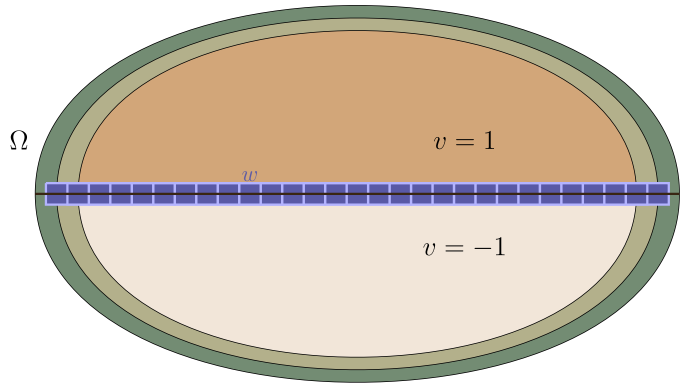

FIGURE 2. Sketch of the construction in Step 1: Close to the boundary the regularized boundary values are extended in normal direction into the domain (dark green area), the jump set is covered by cubes in which the (almost) optimal profile is used (purple squares). Eventually, the constructions are interpolated in the light green region.

optimal profile, similarly as in [22, Section 6, Step 1]. Precisely, for $\rho > 0$ there exist ε_0 and $w \in \mathcal{A}_\nu$ such that

$$\int_Q \Big(\frac{1}{\varepsilon_0} W(w - \varepsilon_0^2 \Delta w) - q\varepsilon_0 |\nabla w|^2 + (1-2q)\varepsilon_0^3 |\Delta w|^2 + (1-q)\varepsilon_0^5 |\nabla \Delta w|^2\Big)\, dx < m_d + \rho.$$

Using this almost-optimal profile we define the sequence $(w_n)_{n\in\mathbb{N}} \subset W^{2,2}(\Omega)$ with $(\Delta w_n)_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$ via

$$w_n(x) := \begin{cases} -1 & \text{if } x_d < -\frac{\varepsilon_n}{2\varepsilon_0}, \\ w(\frac{\varepsilon_0 x}{\varepsilon_n}) & \text{if } |x_d| \le \frac{\varepsilon_n}{2\varepsilon_0}, \\ +1 & \text{if } x_d > \frac{\varepsilon_n}{2\varepsilon_0}. \end{cases} \tag{3.23}$$

Now we choose a family of cut-off functions $(\mu_n)_{n\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that

$$\begin{aligned} &\mu_n \equiv 1 \quad \text{in } \Omega \setminus V_{2\varepsilon_n}, \\ &\mu_n \equiv 0 \quad \text{in } V_{\varepsilon_n}, \text{ and} \\ &\|\nabla^k \mu_n\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon_n^k}, \quad k = 0,1,2,3, \end{aligned} \tag{3.24}$$

and construct $(v_n)_{n\in\mathbb{N}}$ via

$$v_n(x) := \begin{cases} (1-\mu_n) z_n + \mu_n w_n & \text{if } x \in V_{2\varepsilon_n}, \\ w_n & \text{if } x \in \Omega \setminus V_{2\varepsilon_n}. \end{cases}$$

We claim that $(v_n)_{n\in\mathbb{N}}$ is a recovery sequence for v. First, note that by construction we have $(v_n)_{n\in\mathbb{N}} \subseteq Y$. Indeed, it holds $v_n \in W^{2,2}(\Omega)$, $\Delta v_n \in W^{1,2}(\Omega)$, and for each $n \in \mathbb{N}$, v_n satisfies Neumann boundary conditions since z_n satisfies them as well. Next, we show that $v_n \to v$ strongly in $L^2(\Omega)$. Indeed,

$$\begin{aligned} \|v_n - v\|_{L^2(\Omega)} &\le \|v_n - v\|_{L^2(V_{2\varepsilon_n})} + \|v_n - v\|_{L^2(\Omega\setminus V_{2\varepsilon_n})} \\ &= \|(1-\mu_n)(z_n - v) + \mu_n(w_n - v)\|_{L^2(V_{2\varepsilon_n})} + \|w_n - v\|_{L^2(\Omega\setminus V_{2\varepsilon_n})} \\ &\le \|z_n - v\|_{L^2(V_{2\varepsilon_n})} + C\|w_n - v\|_{L^2(\varepsilon_n)}, \end{aligned} \tag{3.25}$$

the first term in the previous expression satisfies

$$\|z_n - v\|_{L^2(V_{2\varepsilon_n})} \leq \|z_n - v\|_{L^\infty(V_{2\varepsilon_n})} \left(\mathcal{L}^d(V_{2\varepsilon_n})\right)^{1/2} \to 0, \text{ as } n \to \infty,$$

and for the second term it holds

$$\|w_n - v\|_{L^2(\Omega)} \leq \|w_n\|_{L^2(\{x\in\Omega:\ |x_d|\leq \frac{\varepsilon_n}{2\varepsilon_0}\})} + \|v\|_{L^2(\{x\in\Omega:\ |x_d|\leq \frac{\varepsilon_n}{2\varepsilon_0}\})} \overset{n\to\infty}{\longrightarrow} 0$$

exactly as in [22, Section 6, Step 1] (see also (3.26) below for a similar argument). It remains to show that

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \leq (m_d + \rho)\mathrm{Per}_\Omega\big(\{v = -1\}\big).$$

We use the estimate

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \leq \underbrace{\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; \Omega\setminus V_{2\varepsilon_n}]}_{I_1} + \underbrace{\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; V_{\varepsilon_n}]}_{I_2} + \underbrace{\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; V_{2\varepsilon_n}\setminus V_{\varepsilon_n}]}_{I_3},$$

and consider the contributions to the energy in the different domains separately. Let us start with I_1. We define for $k \in \mathbb{Z}^d$ the cube of sidelength ε as $Q(k\varepsilon, \varepsilon) := \varepsilon k + (-\varepsilon/2, \varepsilon/2)^d$. Moreover, for fixed $n \in \mathbb{N}$ denote the relevant region for the energy as

$$P_n := \{x \in \Omega \setminus V_{2\varepsilon_n} : v_n(x) \notin \{\pm 1\}\}.$$

Then it holds with $\varepsilon := \varepsilon_n/\varepsilon_0$

$$\begin{aligned}
I_1 &= \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; \Omega\setminus V_{2\varepsilon_n}] \\
&\leq \limsup_{n\to\infty} \int_{P_n} \Big(\frac{1}{\varepsilon_n} W(w_n - \varepsilon_n^2 \Delta w_n) - \varepsilon_n q|\nabla w_n|^2 + (1-2q)\varepsilon_n^3(\Delta w_n)^2 + (1-q)\varepsilon_n^5 |\nabla\Delta w_n|^2\Big)\, dx \\
&\leq \limsup_{n\to\infty} \sum_{k\in\mathbb{Z}^d} \int_{Q(k\varepsilon,\varepsilon)\cap P_n} \Big(\frac{1}{\varepsilon_n} W(w_n - \varepsilon_n^2 \Delta w_n) - \varepsilon_n q|\nabla w_n|^2 + (1-2q)\varepsilon_n^3(\Delta w_n)^2 + (1-q)\varepsilon_n^5 |\nabla\Delta w_n|^2\Big)\, dx \\
&\leq \mathcal{H}^{d-1}(\partial E\cap\Omega) \int_Q \left(\frac{1}{\varepsilon_0} W(w - \varepsilon_0^2\Delta w) - \varepsilon_0 q|\nabla w|^2 + (1-2q)\varepsilon_0^3(\Delta w)^2 + (1-q)\varepsilon_0^5|\nabla\Delta w|^2\right) dx \\
&\leq (m_d+\rho)\mathcal{H}^{d-1}(\partial E\cap\Omega) = (m_d+\rho)\mathrm{Per}_\Omega(\{v=-1\}).
\end{aligned} \tag{3.26}$$

We now turn to I_2, and set

$$R_n := \{x \in V_{\varepsilon_n} : v_n(x) \notin \{\pm 1\}\}.$$

Notice that by Lemma 2.2 $\mathcal{L}^d(R_n) \leq C\varepsilon_n(|Du|(V_{\varepsilon_n}) + \varepsilon_n)$, and $v_n - \varepsilon_n^2 \Delta v_n = v_n \in \{\pm 1\}$ in $V_{\varepsilon_n} \setminus \overline{R_n}$. Therefore, since $v_n = z_n$ in V_{ε_n},

$$
\begin{aligned}
I_2 &= \limsup_{n\to\infty} \int_{V_{\varepsilon_n}} \left(\frac{1}{\varepsilon_n} W(z_n - \varepsilon_n^2 \Delta z_n) + \varepsilon_n^3 (1-2q)|\Delta z_n|^2 - \varepsilon_n q |\nabla z_n|^2 + (1-q)\varepsilon_n^5 |\nabla \Delta z_n|^2 \right) dx \\
&= \limsup_{n\to\infty} \int_{R_n} \Big(\frac{1}{\varepsilon_n} W(g_n \circ P_\nu - \varepsilon_n^2 \Delta(g_n \circ P_\nu)) - \varepsilon_n q \left|\nabla (g_n \circ P_\nu)\right|^2 + \\
&\qquad + \varepsilon_n^3 (1-2q) \left|\Delta (g_n \circ P_\nu)\right|^2 + (1-q)\varepsilon_n^5 \left|\nabla \Delta (g_n \circ P_\nu)\right|^2 \Big) dx \\
&\leq C \limsup_{n\to\infty} (|Du|(V_{\varepsilon_n}) + \varepsilon_n) - q \underbrace{\limsup_{n\to\infty} \int_{R_n} \varepsilon_n |\nabla (g_n \circ P_\nu)|^2 dx}_{J_1} + \underbrace{\limsup_{n\to\infty} \int_{R_n} \varepsilon_n^3 (1-2q) |\Delta (g_n \circ P_\nu)|^2 dx}_{J_2} \\
&\qquad + \underbrace{\limsup_{n\to\infty} \int_{R_n} (1-q)\varepsilon_n^5 |\nabla \Delta (g_n \circ P_\nu)|^2 dx}_{J_3} \\
&= J_1 + J_2 + J_3.
\end{aligned}
\tag{3.27}
$$

Let us consider J_1 carefully.

$$
\begin{aligned}
J_1 &= |q| \limsup_{n\to\infty} \ \varepsilon_n \int_{R_n} |\nabla (g_n \circ P_\nu)|^2 dx \\
&= |q| \limsup_{n\to\infty} \ \varepsilon_n \int_{R_n} \left| \nabla \left(\sum_{i=1}^M \xi_i^n (g_n^i \circ \varphi_i) \circ P_\nu \right) \right|^2 dx \\
&\leq C|q| \limsup_{n\to\infty} \sum_{i=1}^M \varepsilon_n \int_{R_n} \left| \nabla \left(\xi_i^n (g_n^i \circ \varphi_i) \circ P_\nu \right) \right|^2 dx \\
&\leq C|q| \limsup_{n\to\infty} \ \varepsilon_n \sum_{i=1}^M \int_{R_n} \left| \nabla \left((\xi_i^n \circ P_\nu)(g_n^i \circ \varphi_i \circ P_\nu) \right) \chi_{U_i} \right|^2 dx \\
&\leq C|q| \limsup_{n\to\infty} \ \varepsilon_n \sum_{i=1}^M \int_{R_n \cap U_i} |\nabla (\xi_i^n \circ P_\nu) \cdot (g_n^i \circ \varphi_i \circ P_\nu)|^2 + |(\xi_i^n \circ P_\nu) \nabla (g_n^i \circ \varphi_i \circ P_\nu)|^2 dx
\end{aligned}
$$

Note at this point that it holds

$$
P_\nu(x) = (\varphi_i)^{-1} \left(P^e \left(\varphi_i(x) \right) \right), \quad \forall x \in V_{\varepsilon_n}
$$

where by P^e we denote the projection in Euclidean space

$$
\begin{aligned}
&P^e : \mathbb{R}^d \to \{x_d = 0\} \\
&P^e : (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1}, 0).
\end{aligned}
$$

This also implies that $P_\nu \in C^3(V_{\varepsilon_n}; \partial\Omega)$. Therefore we have

$$
J_1 \quad \leq \qquad C|q| \limsup_{n\to\infty} \ \varepsilon_n \sum_{i=1}^M \int_{R_n \cap U_i} \left(\left| \nabla (\xi_i^n \circ (\varphi_i)^{-1} \circ P^e \circ \varphi_i) \cdot (g_n^i \circ P^e \circ \varphi_i) \right) |^2 + \tag{3.28}
$$

$$
+ \left| (\xi_i^n \circ (\varphi_i)^{-1} \circ P^e \circ \varphi_i) \nabla (g_n^i \circ P^e \circ \varphi_i) \right|^2 \Big) dx \tag{3.29}
$$

$$
\leq \quad C|q| \limsup_{n\to\infty} \ \varepsilon_n \sum_{i=1}^M \int_{R_n \cap U_i} \left(C + C \left| \nabla (g_n^i \circ P^e \circ \varphi_i) \right|^2 \right) dx. \tag{3.30}
$$

It remains to consider the last term. Here the derivatives are taken with respect to the standard basis $(e_1, \dots, e_d)$. Let us instead consider another basis - fix $x \in \partial\Omega$, let $\nu = \nu(x)$ be the inward pointing unit normal to $\partial\Omega$ at x and $(a_1, \dots, a_{d-1}, \nu)$ the new local basis, where

$$a_j(x) := D_{\varphi_i(x)}(\varphi_i)^{-1}(e_j), \quad j = 1, \dots, d-1. \tag{3.31}$$

Here and in the sequel we use the notation $D_x f(v)$ to denote the derivative of a function f at a point x applied to v. Now for every $j = 1, \dots, d-1$ it holds

$$\begin{aligned}
\partial_{a_j}(g_n^i \circ P^e \circ \varphi_i)(x) &= D_x(g_n^i \circ P^e \circ \varphi_i)(a_j) \\
&= D_x(g_n^i \circ P^e \circ \varphi_i)(D_{\varphi_i(x)}(\varphi_i)^{-1}(e_j)) \\
&= D_{\varphi_i(x)}(g_n^i \circ P^e \circ \varphi_i \circ (\varphi_i)^{-1})(e_j) \\
&= D_{\varphi_i(x)}(g_n^i \circ P^e)(e_j) \\
&= (\partial_{e_j}(g_n^i \circ P^e))(\varphi_i(x)) \\
&= \sum_{k=1}^{d-1} \partial_{e_k} g_n^i(P^e(\varphi_i(x))) \cdot \partial_{e_j} P_k^e(\varphi_i(x)) \\
&= (\partial_{e_j} g_n^i)(P^e(\varphi_i(x)))
\end{aligned} \tag{3.32}$$

and $\partial_\nu(g_n^i \circ P^e \circ \varphi_i)(x) = 0$. Thus we have

$$\begin{aligned}
\int_{R_n \cap U_i} |\nabla(g_n^i \circ P^e \circ \varphi_i)|^2 dx &\le C \int_{R_n \cap U_i} |\nabla_{(a_1,\dots,a_{n-1},\nu)}(g_n^i \circ P^e \circ \varphi_i)|^2 dx \\
&\le C \int_{R_n \cap U_i} \sum_{j=1}^{d-1} |\partial_{e_j} g_n^i(P^e(\varphi_i(x)))|^2 dx \le \frac{C}{\varepsilon_n^2} \mathcal{L}^d(R_n \cap U_i).
\end{aligned} \tag{3.33}$$

Inserting this bound into (3.28), we obtain

$$J_1 \le C|q| \limsup_{n\to\infty} \sum_{i=1}^{M} \varepsilon_n \mathcal{L}^d(R_n \cap U_i) \left(C + \frac{C}{\varepsilon_n^2} \right) \to 0, \text{ as } n \to \infty.$$

Using similar ideas, we get that $J_2 = 0$ and $J_3 = 0$, and conclude (see (3.27))

$$I_2 = 0.$$

Finally, let us consider I_3.

$$M_n := \{x \in V_{2\varepsilon_n} \setminus V_{\varepsilon_n} : v_n(x) \notin \{\pm 1\}\}$$

and note that by Lemma 2.2 it holds $\mathcal{L}^d(M_n) \le C\varepsilon_n(|Dv|(V_{2\varepsilon_n}) + \varepsilon_n)$. Moreover, note that $\nabla v_n = \Delta v_n = \nabla\Delta v_n = 0$ almost everywhere outside M_n. Then we have

$$\begin{aligned}
I_3 &= \limsup_{n\to\infty} \int_{V_{2\varepsilon_n} \setminus V_{\varepsilon_n}} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n) + \varepsilon_n^3(1-2q)|\Delta v_n|^2 - \varepsilon_n q|\nabla v_n|^2 + \varepsilon_n^5(1-q)|\nabla\Delta v_n|^2 dx \\
&= \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n) + \varepsilon_n^3(1-2q)|\Delta v_n|^2 - q\varepsilon_n|\nabla v_n|^2 + (1-q)\varepsilon_n^5|\nabla\Delta v_n|^2 dx \\
&\le \underbrace{\limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n)}_{A_w} + \limsup_{n\to\infty} \int_{M_n} -\varepsilon_n q|\nabla((1-\mu_\delta)z_n + \mu_\delta w_n)|^2 dx \\
&\qquad + \limsup_{n\to\infty} \int_{M_n} \varepsilon_n^3(1-2q)|\Delta((1-\mu_\delta)z_n + \mu_\delta w_n)|^2 dx + \\
&\quad + \limsup_{n\to\infty} \int_{M_n} \varepsilon_n^5(1-q)|\nabla\Delta((1-\mu_\delta)z_n + \mu_\delta w_n)|^2 dx.
\end{aligned}$$

Let us only consider the first term in detail. Using the bounds for μ_n and z_n we estimate similarly to (3.26)

$$
\begin{aligned}
A_w &= \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n) dx \\
&= \limsup_{n\to\infty} \int_{M_n \cap \{|v_n - \varepsilon_n^2 \Delta v_n| < R\}} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n) dx + \int_{M_n \cap \{|v_n - \varepsilon_n^2 \Delta v_n| > R\}} \frac{1}{\varepsilon_n} W(v_n - \varepsilon_n^2 \Delta v_n) dx \\
&\le \limsup_{n\to\infty} \frac{C}{\varepsilon_n} \mathcal{L}^d (M_n) + \frac{1}{\varepsilon_n} \int_{M_n} \lambda_2 (v_n - \varepsilon_n^2 \Delta v_n)^2 dx \\
&\le C \limsup_{n\to\infty} \int_{M_n} \left(\frac{1}{\varepsilon_n} |v_n|^2 + \varepsilon_n^3 |\Delta v_n|^2 \right) dx \\
&\le C \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} |w_n|^2 + \varepsilon_n^3 \left(|\Delta w_n|^2 + |\nabla \mu_n|^2 |\nabla w_n|^2 + |D^2 \mu_n|^2 |w_n|^2 \right) dx \\
&\quad + C \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} |z_n|^2 + \varepsilon_n^3 \left(|\Delta z_n|^2 + |\nabla \mu_n|^2 |\nabla z_n|^2 + |D^2 \mu_n|^2 |z_n|^2 \right) dx \\
&\le C \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} |w_n|^2 + \varepsilon_n^3 |\Delta w_n|^2 + \varepsilon_n |\nabla w_n|^2 + \frac{1}{\varepsilon_n} |w_n|^2 dx \\
&\quad + C \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} dx \\
&\le C \limsup_{n\to\infty} \int_{M_n} \frac{1}{\varepsilon_n} |w_n|^2 + \varepsilon_n^3 |\Delta w_n|^2 + \varepsilon_n |\nabla w_n|^2 dx \quad + \limsup_{n\to\infty} C(|Dv|(V_{2\varepsilon_n}) + \varepsilon_n) \\
&\le \limsup_{n\to\infty} C \int_{M_n} \frac{1}{\varepsilon_n} \left| w\left(\frac{\varepsilon_0}{\varepsilon_n} x\right) \right|^2 + \frac{\varepsilon_0^4}{\varepsilon_n} \left| (\Delta w) \left(\frac{\varepsilon_0}{\varepsilon_n} x\right) \right|^2 + \frac{\varepsilon_0^2}{\varepsilon_n} \left| (\nabla w) \left(\frac{\varepsilon_0}{\varepsilon_n} x\right) \right|^2 dx \\
&\le \limsup_{n\to\infty} C |Dv|(V_{2\varepsilon_n}) \int_Q \frac{1}{\varepsilon_0} |w(y)|^2 + \varepsilon_0^3 |\Delta w(y)|^2 + \varepsilon_0 |\nabla w(y)|^2 dy \\
&= 0.
\end{aligned}
$$

We use similar computations as for J_1 to bound the other three terms, which yields

$$I_3 = 0.$$

This finally gives

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \le (m_d + \rho) \mathrm{Per}_\Omega(\{v = -1\}).$$

The result follows by letting $\rho \to 0$ and taking a diagonal sequence.

Step 2. Since $v \in BV(\Omega; \{\pm 1\})$, v can be written as $v = -\chi_E + \chi_{\Omega \setminus E}$ with E the set of finite perimeter in Ω. In this step, we assume that E has the form $E = \Omega \cap P$ for some polygonal set P. By a polygonal set we mean that there is $L \in \mathbb{N}$ such that $\partial P = H_1 \cup \dots H_L \cup F$ where H_i are pairwise disjoint convex and relatively open polyhedra of dimension $d-1$, $H_i \subset \{x \in \mathbb{R}^d : (x - x_i) \cdot \hat{n}_i = 0\}$ for some $x_i \in \mathbb{R}^d$ and $\hat{n}_i \in S^{d-1}$, $i = 1, \dots, L$, and F is the finite union of convex polyhedra of dimension $d-2$. The following construction is illustrated in Figure 3.3. For $0 < \delta < 1$ let

$$U_\delta := \{x \in \Omega : \mathrm{dist}(x, \Omega \cap F) \le \delta\}.$$

For each $i = 1, \dots, L$ consider $H_i' \subset H_i$ with C^∞-boundary such that

$$\left\{ x \in H_i \cap \Omega : \mathrm{dist}(x, \Omega \cap F) \ge \frac{\delta}{2} \right\} \subset H_i' \subset \overline{H_i'} \subset \Omega \cap H_i$$

and such that

$$\overline{H_i'} \cap U_{\frac{\delta}{4}} = \emptyset. \tag{3.34}$$

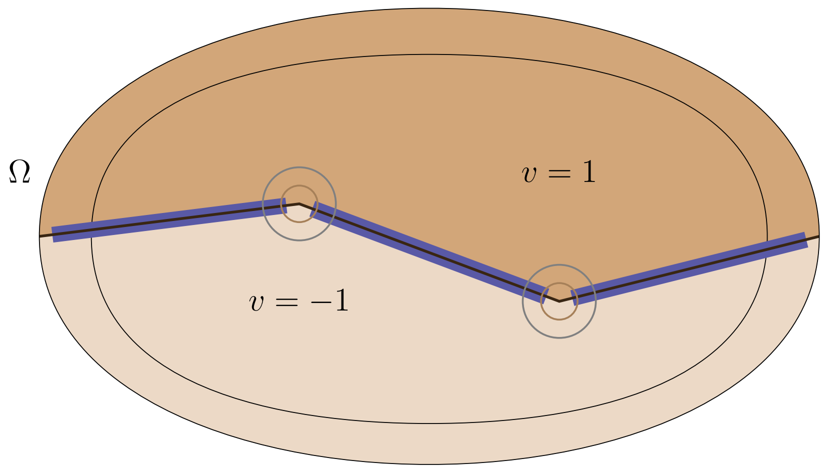

FIGURE 3. Sketch of the construction in Step 2

Fix $\eta \in (0, \frac{\delta}{2})$ and $\rho > 0$, and consider for $i = 1, \dots, L$ the η-neighbourhood of H_i', i.e. let

$$\Omega_i := \Big\{ x + t\hat{n}_i : x \in H_i', |t| < \eta \Big\}.$$

Without loss of generality we can assume that Ω_i are pairwise disjoint (one can choose η small). Next, we introduce some auxiliary sequences for our construction. First, similarly as in (3.23), for each Ω_i we can construct a sequence $(w_n^i)_{n\in\mathbb{N}} \subset W^{2,2}(\Omega)$ such that $w_n^i \to v$ strongly in $L^2(\Omega_i)$ and such that

$$\lim_{n\to+\infty} \mathcal{F}_{\varepsilon_n}[w_n^i; \Omega_i] \leq (m_d + \rho)\mathcal{H}^{n-1}(H_i \cap \Omega_i).$$

Next, we construct the mollifying sequence. Extend v to $\mathbb{R}^d$ such that $v = -\chi_P + \chi_{\mathbb{R}^d\setminus P}$. Let $(\varphi_n)_{n\in\mathbb{N}} \subseteq C^\infty(\mathbb{R}^d)$ with $\varphi_n := \varphi_{\varepsilon_n}$, be a mollifying sequence for v. Since $v \in L^\infty(\Omega)$, we have the bounds from (2.2). For fixed $0 < \delta < 1$ consider a cut-off function $\eta_\delta \in C_c^\infty(\mathbb{R}^d; [0,1])$ such that

$$\begin{aligned} &\eta_\delta \equiv 0 \text{ in } U_\delta, \\ &\eta_\delta \equiv 1 \text{ in } \mathbb{R}^d \setminus U_{2\delta}, \text{and} \\ &\|\nabla^k \eta_\delta\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\delta^k}, \quad k = 0,1,2,3. \end{aligned} \tag{3.35}$$

Finally, we define a sequence

$$w_n(x) := \begin{cases} \eta_\delta w_n^i + (1-\eta_\delta)\varphi_n, & x \in \overline{\Omega_i}, \quad i = 1, \dots, L; \\ \varphi_n, & x \in A := \Omega \setminus (\overline{\Omega_1} \cup \dots \cup \overline{\Omega_n}). \end{cases}$$

We construct the recovery sequence $(v_n)_{n\in\mathbb{N}} \subseteq Y$ using the interpolation procedure from the previous step and using the auxiliary sequences $(w_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$. Computations similar to the ones in the previous step and [22, Section 6, Step 2] yield that by a diagonal argument as $\delta \to 0$ a recovery sequence can be constructed in this way.

Step 3. Finally, suppose that $v \in BV(\Omega; \{\pm 1\})$, i.e. E is of finite perimeter. The recovery sequence is constructed using the polyhedral approximation from [6].

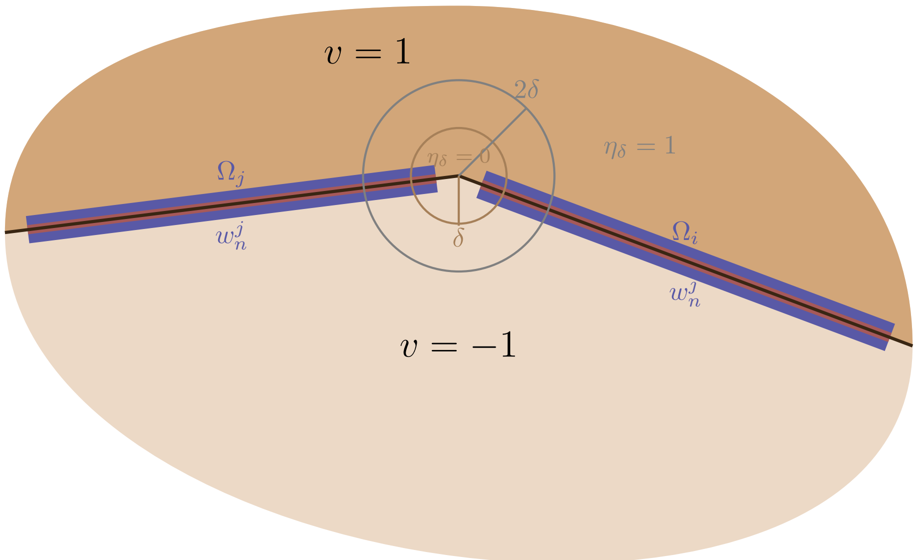

FIGURE 4. Sketch of the construction in Step 2 close to F

Notice that in particular it holds

$$v \in SBV(\Omega; \{\pm 1\}) \text{ and } |Du|(\Omega) = 2 \operatorname{Per}(E; \Omega) < \infty.$$

Thus, by [6, Theorem 2.1 & Corollary 2.4] there exists a sequence $(v_j)_{j\in\mathbb{N}} \subseteq SBV(\Omega; \{\pm 1\})$ such that

- $v_j \to v$ strongly in $L^1(\Omega)$ and thus also in $L^2(\Omega)$,
- for $E_j := \{v_j = -1\}$ there exists a finite number of $d-1$ dimensional simplexes $H_1, \ldots, H_M \subseteq \mathbb{R}^d$ such that

$$\mathcal{H}^{d-1}(\partial E_j \triangle \bigcup_{j=1}^{M} (H_j \cap \Omega) \,) = 0,$$

  where Δ denotes the symmetric difference, and
- $\operatorname{Per}(E_j; \Omega) \to \operatorname{Per}(E; \Omega)$.

Note that the sets E_j are in particular of polygonal type in the sense of Step 2. Thus, for each $j \in \mathbb{N}$ we can construct a sequence $(v_j^k)_{k\in\mathbb{N}} \subseteq Y$ such that

$$v_j^k \to v_j \text{ strongly in } L^2(\Omega) \text{ as } k \to \infty \text{ and}$$
$$\limsup_{k\to\infty} \mathcal{F}_{\varepsilon_k}[v_j^k] \leq m_d \operatorname{Per}(E_j; \Omega).$$

Finally, we obtain a recovery sequence for $v \in BV(\Omega; \{\pm 1\})$ by the standard diagonalization procedure.

$\square$

Remark 3.4. *We note at this point that the assumption $q \leq 0$ is not necessary for the proof of the* $\limsup$-*inequality. Following the steps from the proof above, one can construct a recovery sequence in the case $q > 0$ the same way as previously described. In this case, the only relevant change appears when showing the upper bound on the energies. However, all of the terms that were bounded by $C\delta$ in* *the setting above, are still bounded by the same value (one can bound the absolute value of those terms exactly as above). Thus, the same construction can be used to fill the gap in the construction from* [22, Section 6]*.*

4. One dimension $d = 1$

We consider now the special case $d = 1$ and show that we obtain a Γ-convergence result under the much weaker hypotheses on the double-well potential W stated in Assumption 1.2. Without loss of generality, we consider the normalized interval domain $I := (-1, 1)$. We make explicit use of the $1D$ structure, and first prove an estimate that allows us to obtain a lower bound on the energy in terms of the number of transition layers, following ideas from [24, Proposition 2.8 and Corollary 2.9]. We use the notation $u = -\varepsilon^2 v'' + v$ and consider for $(a, b) \subset I$,

$$\mathcal{F}_\varepsilon[v; (a, b)] := \int_a^b \left(\frac{1}{\varepsilon} W(u) - \varepsilon q |u'|^2 + \varepsilon^3 (v'')^2 + \varepsilon^5 (v''')^2 \right) dx. \tag{4.1}$$

Proposition 4.1. *Let $\varepsilon_n \to 0^+$, and let $(v_n)_{n\in\mathbb{N}} \subset W^{3,2}(I)$. Assume that there exists a partition*

$$-1 = x_0 < x_1 < \cdots < x_\nu = 1,$$

of I with the following property: Given $\eta > 0$, there exists $N_\eta \in \mathbb{N}$ such that for $n \geq N_\eta$ and $i = 1, \ldots, \nu - 1$, there exist $a_{i,n} \in (x_{i-1}, x_i)$, $b_{i,n} \in (x_i, x_{i+1})$ with $a_{i+1,n} > b_{i,n}$ such that

$$\begin{aligned} &|\varepsilon_n^k v_n^{(k)}(a_{i,n})| < \eta, \quad |\varepsilon_n^k v_n^{(k)}(b_{i,n})| < \eta, \quad k = 1, 2, 3, \\ &|v_n(a_{i,n}) + 1| < \eta, \quad |v_n(b_{i,n}) - 1| < \eta. \end{aligned} \tag{4.2}$$

Then we have

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \geq (\nu - 1) m_1.$$

Proof. Let $u_n = -\varepsilon_n^2 v_n'' + v$. Then, since the integrand in (4.1) is non-negative, and the intervals $(a_{i,n}, b_{i,n})$ are pairwise disjoint, we have

$$\begin{aligned} \mathcal{F}_{\varepsilon_n}[v_n] &= \int_{-1}^1 \left(\frac{1}{\varepsilon} W(u_n) - \varepsilon_n q |u_n'|^2 + \varepsilon_n^3 (v_n'')^2 + \varepsilon_n^5 (v_n''')^2 \right) dx \\ &\geq \sum_{i=1}^{\nu-1} \int_{a_{i,n}}^{b_{i,n}} \left(\frac{1}{\varepsilon_n} W(u_n) - \varepsilon_n q |u_n'|^2 + \varepsilon_n^3 (v_n'')^2 + \varepsilon_n^5 (v_n''')^2 \right) dx \\ &= \sum_{i=1}^{\nu-1} \mathcal{F}_{\varepsilon_n}[v_n; (a_{i,n}, b_{i,n})]. \end{aligned} \tag{4.3}$$

For $i = 1, \ldots, \nu - 1$, we consider the restriction of v_n to $(a_{i,n}, b_{i,n})$ and construct extensions $\tilde{v}_{i,n}$ to $(-1, 1)$ such that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; (a_{i,n}, b_{i,n})] \geq \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (-1, 1)] \geq m_1.$$

For the last inequality, we note that for $d = 1$, by a rescaling argument, we may replace the unit interval $Q_\nu = (-1/2, 1/2)$ and the set of admissible functions for the minimization in the definition of m_1, $\mathcal{A}_\nu$, by the interval $(-1, 1)$ and functions $v : (-1, 1) \to \mathbb{R}$ such that $v(\cdot/2) \in \mathcal{A}_\nu$. Then, set

$$\tilde{v}_{i,n}(x) := \begin{cases} -1 & \text{if } x \in [-1, a_{i,n} - \varepsilon_n), \\ \tilde{\psi}_{i,n}(x) & \text{if } x \in [a_{i,n} - \varepsilon_n, a_{i,n}], \\ v_n(x) & \text{if } x \in (a_{i,n}, b_{i,n}), \\ \tilde{\phi}_{i,n}(x) & \text{if } x \in [b_{i,n}, b_{i,n} + \varepsilon_n], \\ 1 & \text{if } x \in (b_{i,n} + \varepsilon_n, 1] \end{cases}$$

with

$$\tilde{\psi}_{i,n}(x) := \psi_n\left(\frac{x-a_{i,n}}{\varepsilon_n}\right), \qquad \text{where} \qquad \psi_n(s) := (s+1)^3 p_n(s) - 1, \tag{4.4}$$

with a quadratic polynomial

$$p_n(s) := p_{0,n} + p_{1,n}s + p_{2,n}s^2. \tag{4.5}$$

Then $\psi_n(-1) = -1$, $\psi_n'(-1) = 0$, $\psi_n''(-1) = 0$, and we choose the coefficients $p_{i,n}$ of p_n such that

$$\psi_n(0) = v_n(a_{i,n}), \quad \psi_n'(0) = \varepsilon_n v_n'(a_{i,n}), \quad \text{and} \quad \psi_n''(0) = \varepsilon_n^2 v_n''(a_{i,n}). \tag{4.6}$$

Then $\tilde{v}_{i,n}$ has a twice continuously differentiable representative on $[-1, b_{i,n})$. The extension to $[b_{i,n}, 1]$ can be chosen similarly. We note that the coefficients of p_n are bounded successively by

$$\begin{aligned}
|p_{0,n}| &\overset{(4.5)}{=} |p_n(0)| \overset{(4.4)}{=} |\psi_n(0)+1| \overset{(4.6)}{=} |v_n(a_{i,n})+1| \overset{(4.2)}{<} \eta, \qquad \text{and similarly} \\
|p_{1,n}| &= |3p_{0,n} + p_{1,n} - 3p_{0,n}| = |\psi_n'(0) - 3p_{0,n}| = |\varepsilon_n v_n'(a_{i,n}) - 3p_{0,n}| < 4\eta, \qquad \text{and} \\
|p_{2,n}| &= \frac{1}{2}|\psi_n''(0) - 6p_{0,n} - 6p_{1,n}| = \frac{1}{2}|\varepsilon_n^2 v_n''(a_{i,n}) - 6p_{0,n} - 6p_{1,n}| < 16\eta.
\end{aligned} \tag{4.7}$$

Fix $i = 1, \ldots, \nu - 1$ and consider the energy. We have

$$\begin{aligned}
\mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (-1,1)] &= \mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (-1, a_{i,n} - \varepsilon_n)] + \mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (a_{i,n} - \varepsilon_n, a_{i,n})] + \mathcal{F}_{\varepsilon_n}[v_n; (a_{i,n}, b_{i,n})] \\
&+ \mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (b_{i,n}, b_{i,n} + \varepsilon_n)] + \mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (b_{i,n} + \varepsilon_n, 1)].
\end{aligned}$$

We consider the terms on the right hand side separately. First, for $x \in (-1, a_{i,n} - \varepsilon_n)$, we have $\tilde{v}_n(x) \equiv -1$ and thus $\tilde{u}_\varepsilon(x) = -\varepsilon^2 \tilde{v}_n''(x) + \tilde{v}_n(x) = -1$. Hence, since $W(-1) = 0$,

$$\mathcal{F}_{\varepsilon_n}[\tilde{v}_n; (-1, a_{i,n} - \varepsilon_n)] = 0.$$

Next, for $a_{i,n} - \varepsilon_n < x < a_{i,n}$, set

$$z_{i,n} := \frac{x - a_{i,n}}{\varepsilon_n} \qquad \text{and note that} \qquad |z_{i,n}| \le 1.$$

We use the estimates (4.7) for the coefficients of p_n to estimate

$$|p_n(z_{i,n})| \le |p_{0,n}| + |p_{1,n}| + |p_{2,n}| < 21\eta,$$

and similarly for the derivatives. Hence, we find that there exists $C > 0$ independent of n and η such that for all $x \in (a_{i,n} - \varepsilon_n, a_{i,n})$, with $\tilde{u}_{i,n} := -\varepsilon_n^2 \tilde{v}_{i,n}'' + \tilde{v}_{i,n}$,

$$\begin{aligned}
|\tilde{v}_{i,n}(x) + 1| &= |\tilde{\psi}_{i,n}(x) + 1| = \left|\psi_n\left(\frac{x - a_{i,n}}{\varepsilon_n}\right) + 1\right| = |(z_{i,n}+1)^3 p_n(z_{i,n})| < C\eta, \\
|\tilde{v}_{i,n}'(x)| &= |\tilde{\psi}_{i,n}'(x)| = \frac{1}{\varepsilon_n}|3(z_{i,n}+1)^2 p_n(z_{i,n}) + (z_{i,n}+1)^3 p_n'(z_{i,n})| < \frac{C\eta}{\varepsilon_n}, \\
|\tilde{v}_{i,n}''(x)| &= |\tilde{\psi}_n''(x)| = \frac{1}{\varepsilon_n^2}|6(z_{i,n}+1)p_n(z_{i,n}) + 6(z_{i,n}+1)^2 p_n'(z_{i,n}) + (z_{i,n}+1)^3 p_n''(z_{i,n})| < \frac{C\eta}{\varepsilon_n^2}, \\
|\tilde{v}_{i,n}'''(x)| &= |\tilde{\psi}_{i,n}'''(x)| = \frac{1}{\varepsilon_n^3}|6p_n(z_{i,n}) + 18(z_{i,n}+1)p_n'(z_{i,n}) + 9(z_{i,n}+1)^2 p_n''(z_{i,n}) + (z_{i,n}+1)^3 p_n'''(z_{i,n})| \\
&< \frac{C\eta}{\varepsilon_n^3}, \\
|\tilde{u}_{i,n}'(x)| &= |-\varepsilon_n^2 \tilde{v}_n'''(x) + \tilde{v}_n'(x)| < \frac{C\eta}{\varepsilon_n}, \quad \text{and} \\
\frac{1}{\varepsilon_n}W(\tilde{u}_{i,n}(x)) &= \frac{1}{\varepsilon_n}W\left(-\varepsilon_n^2 \tilde{v}_{i,n}''(x) + \tilde{v}_{i,n}(x) + 1 - 1\right) \le \frac{C\eta}{\varepsilon_n},
\end{aligned}$$

where in the last estimate we used the Lipschitz continuity of W and $W(-1) = 0$. It follows that

$$\mathcal{F}_{\varepsilon_n}[\tilde{v}_{i,n}; (a_{i,n} - \varepsilon_n, a_{i,n})] = \int_{a_{i,n}-\varepsilon_n}^{a_{i,n}} \left(\frac{1}{\varepsilon_n}W(\tilde{u}_{i,n}) - \varepsilon_n q|\tilde{u}_{i,n}'|^2 + \varepsilon_n^3|\tilde{v}_{i,n}''|^2 + \varepsilon_n^5|\tilde{v}_{i,n}'''|^2\right) dx \le C\eta,$$

and hence, since $\eta > 0$ was arbitrary,

$$\limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}[\tilde v_{i,n}; (a_{i,n}-\varepsilon_n, a_{i,n})] = 0.$$

An analogous argument for the extension to $(b_{i,n}, 1)$ yields

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] \geq \sum_{i=1}^{\nu-1} \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; (a_{i,n}, b_{i,n})] \geq \sum_{i=1}^{\nu-1} \liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[\tilde v_{i,n}; (-1,1)] \geq (\nu-1)m_1,$$

which concludes the proof. □

4.1. Optimal Profile Problem

We now study the minimization problem defining m_1, where we follow the method in [24]. For simplicity of notation, we denote the class of admissible functions for m_1 by

$$\mathcal{A} := \{V \in W^{3,2}_{\text{loc}}(\mathbb{R}) : \text{ there exists } T>0, \text{ such that } V(z) = -1, \text{ if } z < -T, \quad V(z) = 1, \text{ if } z > T\}. \tag{4.8}$$

Proposition 4.2. *The constant m_1 is positive and*

$$m_1 = \min\left\{ \int_{\mathbb{R}} [W(U) - q|U'|^2 + |V''|^2 + |V'''|^2]\, dz : V \in W^{3,2}_{loc}(\mathbb{R}), \quad U = -V''+V, \quad V \to \pm 1 \text{ as } z \to \pm\infty \right\}.$$

Proof. We divide the proof into three steps, following the lines of the proof of [24, Lemma 2.1].
Step 1: $m_1 > 0$. Assume for the sake of a contradiction that $m_1 = 0$. Let $(V_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ (see (4.8)) be a minimizing sequence for m_1. Since $V_n \in W^{3,2}_{\text{loc}}(\mathbb{R})$ has a continuous representative and takes values -1 and 1 for $z < -T$ and $z > T$ respectively, there exists $z_n \in \mathbb{R}$ such that $V_n(z_n) = 0$. Without loss of generality we may assume $V_n(0) = 0$ (otherwise change variables $\tilde z = z - z_n$). Since $\mathcal{F}_\infty[V_n] \to 0$, we have $\|V_n'''\|_{L^2(\mathbb{R})} \to 0$ and $\|V_n''\|_{L^2(\mathbb{R})} \to 0$. Further, on any bounded $J \subset \mathbb{R}$ we have by (H2')

$$\begin{aligned}
\|V_n\|_{L^1(J)} &= \|U_n + V_n''\|_{L^1(J)} \leq \|U_n\|_{L^1(J)} + \|V_n''\|_{L^1(J)} \leq \|U_n\|_{L^1(J)} + |J|^{1/2}\|V_n''\|_{L^2(J)} \\
&\leq \int_{\{|U_n|<R\}} |U_n|\, dz + \int_{\{|U_n|\geq R\}} |U_n|\, dz + |J|^{1/2}\|V_n''\|_{L^2(J)} \\
&\leq |J|R + \frac{1}{L}\int_J W(U_n)\, dz + |J|^{1/2}\|V_n''\|_{L^2(J)} \leq C(J),
\end{aligned}$$

and by the interpolation inequality (2.4)

$$\|V_n'\|_{L^{4/3}} \leq C(J)(\|V_n\|^{1/2}_{L^1(J)}\|V_n''\|^{1/2}_{L^2(J)} + \|V_n\|_{L^1(J)}) \leq C(J),$$

where here and in the following $C(J)$ denotes a generic constant that depends on J but not on n. Consequently, by the Sobolev embedding, $(V_n)_{n\in\mathbb{N}}$ is bounded in $W^{3,2}(J)$ and we may extract a subsequence (not relabeled) converging in $W^{2,\infty}(J)$ to some affine function V, i.e.,

$$V(z) = az + b.$$

Since $U_n = -V_n'' + V_n \to 0 + V$ in $L^2(J)$, by Fatou's lemma, we have,

$$\begin{aligned}
m_1 &= \lim_{n\to\infty} \int_{\mathbb{R}} (W(U_n) - q|U_n'|^2 + |V_n''|^2 + |V_n'''|^2)\, dz \geq \lim_{n\to\infty} \int_J (W(U_n) - q|U_n'|^2 + |V_n''|^2 + |V_n'''|^2)\, dz \\
&\geq \int_J W(az+b)\, dz.
\end{aligned}$$

By the assumption $m_1 = 0$, this implies that $az + b = \pm 1$ for all $z \in J$. Hence, $a = 0$, $b = \pm 1$, which contradicts $V(0) = \lim V_n(0) = 0$. This shows that $m_1 > 0$.
Step 2: $\tilde m = m_1$, where

$$\tilde m := \inf\left\{ \int_{\mathbb{R}} [W(U) - q|U'|^2 + |V''|^2 + |V'''|^2]\, dz : V \in W^{3,2}_{loc}(\mathbb{R}), \quad U = -V''+V, \quad V \to \pm 1 \text{ as } z \to \pm\infty \right\}. \tag{4.9}$$

Clearly, $\tilde m \le m_1$. It remains to show that $m_1 \le \tilde m$. Fix $\delta > 0$ and let V be admissible for $\tilde m$ such that

$$\int_{\mathbb{R}} (W(U) - q|U'|^2 + |V''|^2 + |V'''|^2)\, dz \le \tilde m + \delta.$$

It follows that $V''' \in L^2(\mathbb{R})$ and $V'' \in L^2(\mathbb{R})$. Since V' is continuous by the Sobolev embedding, and $V \to \pm 1$ as $z \to \pm\infty$, there are sequences $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}} \subset \mathbb{R}$, such that $a_j \to -\infty$, $b_j \to \infty$, and

$$\lim_{j\to\infty} |V(a_j)+1| = \lim_{j\to\infty} |V(b_j)-1| = \lim_{j\to\infty} V^{(k)}(a_j) = \lim_{j\to\infty} V^{(k)}(b_j) = 0, \quad k = 1,2,3. \tag{4.10}$$

Define

$$\tilde V_i(z) := \begin{cases} -1, & \text{if } z \in (-\infty, a_i - 1), \\ \tilde\Psi_i(z), & \text{if } z \in [a_i - 1, a_i], \\ V(z), & \text{if } z \in (a_i, b_i), \\ \tilde\Phi_i(z), & \text{if } z \in [b_i, b_i + 1], \\ 1, & \text{if } z \in (b_i + 1, \infty), \end{cases}$$

where we want to choose the functions $\tilde\Phi_i$ and $\tilde\Psi_i$ such that for i sufficiently large

$$\mathcal{F}_\infty[V] \ge \mathcal{F}_\infty[\tilde V_i] - \delta \ge m_1 - \delta.$$

Since $\tilde m + \delta \ge \mathcal{F}_\infty[V]$ and $\delta > 0$ was arbitrary, this implies $\tilde m \ge m_1$. To construct $\tilde V_i$, we proceed as in the proof of Proposition 4.1 and choose

$$\tilde\Psi_i(z) := \psi_i(z - a_i), \qquad \text{where} \qquad \psi_i(s) := (s+1)^3 p_i(s) - 1 \qquad \text{with} \quad p_i(s) := p_{0,i} + p_{1,i}s + p_{2,i}s^2.$$

As before we have, $\psi_i(-1) = -1$, $\psi_i'(-1) = 0$, $\psi_i''(-1) = 0$, and we choose the coefficients $p_{0,i}, p_{1,i}, p_{2,i}$ such that

$$\psi_i(0) = V(a_i), \quad \psi_i'(0) = V'(a_i), \quad \psi_i''(0) = V''(a_i) \tag{4.11}$$

to guarantee that V is twice continuously differentiable. Similarly to the proof of Proposition 4.1, conditions (4.10) imply that for $i \to \infty$,

$$p_{0,i} = p_i(0) = \psi_i(0) + 1 = V(a_i) + 1 \to 0,$$

$$p_{1,i} = p_i'(0) = \psi_i'(0) - 3p_i(0) = V'(a_i) - 3p_{0,i} \to 0,$$

$$p_{2,i} = p_i''(0)/2 = \psi_i''(0)/2 - 3p_i'(0) - 3p_i(0) = V''(a_i)/2 - 3p_{1,i} - 3p_{0,i} \to 0.$$

The extension $\tilde\Phi_i$ to $[b_i, b_i+1]$ can be chosen similarly. Since for $z < a_i - 1$ and $z > b_i + 1$, $\tilde V(z) = \pm 1$ and $\tilde V$, V agree on (a_i, b_i), we have,

$$\begin{aligned} \mathcal{F}_\infty[V] &\ge \int_{a_i}^{b_i} (W(U) - q|U'|^2 + |V''|^2 + |V'''|^2)\, dz \\ &= \mathcal{F}_\infty[\tilde V] - \int_{b_i}^{b_i+1} (W(\tilde U) - q|\tilde U'|^2 + |\tilde V''|^2 + |\tilde V'''|^2)\, dz - \int_{a_i-1}^{a_i} (W(\tilde U) - q|\tilde U'|^2 + |\tilde V''|^2 + |\tilde V'''|^2)\, dz, \end{aligned}$$

where $\tilde U := -\tilde V'' + \tilde V$. To estimate the last two terms, note that for $a_i - 1 < z < a_i$, we have $|z - a_i| \le 1$, and hence, as in the proof of Proposition 4.1, for $i \to \infty$,

$$\begin{aligned} \|\tilde V_i(\cdot) + 1\|_{L^\infty(a_i-1,a_i)} &= \|\tilde\Psi_i(\cdot) + 1\|_{L^\infty(a_i-1,a_i)} = \|\psi_i(\cdot - a_i) + 1\|_{L^\infty(a_i-1,a_i)} \\ &= \|(\cdot - a_i + 1)^3 p_i(\cdot - a_i)\|_{L^\infty(a_i-1,a_i)} \to 0, \qquad \text{and} \end{aligned}$$

$$\|\tilde V_i'\|_{L^\infty(a_i-1,a_i)}, \|\tilde V_i''\|_{L^\infty(a_i-1,a_i)}, \|\tilde V_i'''\|_{L^\infty(a_i-1,a_i)} \to 0.$$

It follows that

$$\|\tilde U_i'\|_{L^\infty(a_i-1,a_i)} = \left\| -\tilde V_i''' + \tilde V_i' \right\|_{L^\infty(a_i-1,a_i)} \to 0,$$

and

$$\|W(\tilde{U}_i)\|_{L^\infty(a_i-1,a_i)} = \|W(-\tilde{V}_i'' + \tilde{V}_i)\|_{L^\infty(a_i-1,a_i)} \to 0,$$

which implies

$$\int_{a_i-1}^{a_i} (W(\tilde{U}_i) - q|\tilde{U}_i'|^2 + |\tilde{V}_i''|^2 + |\tilde{V}_i'''|^2)\, dz \to 0.$$

A similar argument yields that

$$\int_{b_i}^{b_i+1} (W(\tilde{U}_i) - q|\tilde{U}_i'|^2 + |\tilde{V}_i''|^2 + |\tilde{V}_i'''|^2)\, dz \to 0.$$

Hence, for i sufficiently large we obtain from (4.12),

$$\tilde{m} + \delta \geq \mathcal{F}_\infty[V] \geq \mathcal{F}_\infty[\tilde{V}_i] - \delta \geq m_1 - \delta,$$

which concludes the proof of the assertion $m_1 = \tilde{m}$.

Step 3: $\tilde{m}$ is attained. Let $(V_k)_{k\in\mathbb{N}}$ be a minimizing sequence for $\tilde{m}$. We can again assume without loss of generality that $V_k(0) = 0$ for all $k \in \mathbb{N}$ and that $(V_k)_{k\in\mathbb{N}}$ is equibounded in $W^{3,2}_{\mathrm{loc}}(\mathbb{R})$. Extracting a subsequence (not relabeled) we may assume that $V_k \to \overline{V}$ strongly in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ and weakly in $W^{3,2}_{\mathrm{loc}}(\mathbb{R})$. Hence if J is an arbitrary interval, using Fatou's Lemma and the lower semicontinuity of the L^p norms yields

$$\begin{aligned}\tilde{m} &= \lim_{k\to\infty} \int_{\mathbb{R}} (W(U_k) - q|U_k'|^2 + |V_k''|^2 + |V_k'''|^2)\, dz \geq \liminf_{k\to\infty} \int_J (W(U_k) - q|U_k'|^2 + |V_k''|^2 + |V_k'''|^2)\, dz \\ &\geq \int_J (W(\overline{U}) - q|\overline{U}'|^2 + |\overline{V}''|^2 + |\overline{V}'''|^2)\, dz.\end{aligned}$$

Since J was arbitrary, we have

$$\tilde{m} \geq \int_{\mathbb{R}} (W(\overline{U}) - q|\overline{U}'|^2 + |\overline{V}''|^2 + |\overline{V}'''|^2)\, dz,$$

and it remains to show that $\overline{V}$ is admissible for $\tilde{m}$. Let

$$L := \{l \in \mathbb{R} : l \text{ is a limit point of } \overline{V}(z) \text{ as } z \to \infty\}.$$

If $l \in L$ then, since $\overline{U} = -\overline{V}'' + \overline{V}$ and $\overline{V}'' \to 0$ as $z \to \infty$, l must be a limit point of $\overline{U}$. Since $\int_{\mathbb{R}} W(\overline{U})\, dz < \infty$, we have that $-1 \in L$ or $1 \in L$. Suppose that $1 \in L$ and assume for a contradiction that there exists some $1 \neq l \in L$. We may assume without loss of generality that $l \neq -1$ (otherwise, since $\overline{V} \in C^2(\mathbb{R})$, there exists another limit point $l \in (-1,1)$). By definition, there are two monotone sequences $(y_i)_{i\in\mathbb{N}}$ and $(z_i)_{i\in\mathbb{N}}$ such that $y_{i+1} - y_i \geq 3$, $z_i \in [y_i+1, y_{i+1}-1]$, $\overline{V}(y_i) \to 1$, and $\overline{V}(z_i) \to l$. For $0 < \delta < \min\{|l+1|, |l-1|\}$, define

$$\begin{aligned}h := \inf\Big\{ &\int_y^w (W(U) - q|U'|^2 + |V''|^2 + |V'''|^2)\, dz : \\ &w - y \geq 3, V \in W^{3,2}(y,w), U = -V'' + V, \exists z \in [y+1, w-1] \text{ s.t. } |V(z) - l| \leq \delta\Big\}.\end{aligned}$$

Since $|V(z_i) - l| \leq \delta$ for large $i \geq I$, we have

$$\int_{\mathbb{R}} (W(\overline{U}) - q|\overline{U}'|^2 + |\overline{V}''|^2 + |\overline{V}'''|^2)\, dz \geq \sum_{i=I}^\infty \int_{y_i}^{y_{i+1}} (W(\overline{U}) - q|\overline{U}'|^2 + |\overline{V}''|^2 + |\overline{V}'''|^2)\, dz \geq \sum_{i=I}^\infty h.$$

It follows that $h = 0$. We will show that this leads to a contradiction. Let $(V_n)_{n\in\mathbb{N}}$ be a minimizing sequence for h. Translating the intervals, we may assume without loss of generality that $z_n = 0$,

$y_n \leq -1$, $w_n \geq 1$, $V_n \to V = p_0 + p_1 z$ in $W^{2,\infty}([-1,1])$. It follows that

$$\begin{aligned} h &\geq \lim_{n\to\infty} \int_{y_n}^{w_n} (W(U_n) - q|U_n'|^2 + |V_n''|^2 + |V_n'''|^2)\,dz \\ &\geq \lim_{n\to\infty} \int_{-1}^{1} (W(U_n) - q|U_n'|^2 + |V_n''|^2 + |V_n'''|^2)\,dz \geq \int_{1}^{1} W(p_0 + p_1 z)\,dz \geq 0. \end{aligned}$$

If $h = 0$, we have $p_0 + p_1 z = \pm 1$ for all $z \in (-1,1)$, and so $V(z) = p_0 = \pm 1$ for all $z \in (-1,1)$. This contradicts $|V_n(0) - l| < \delta$ for all $n \in \mathbb{N}$. Hence, our assumption was wrong, and $\overline{V}(z) \to 1$ as $z \to \infty$. A similar argument shows that $\overline{V}(z) \to \pm 1$ as $z \to -\infty$.
Therefore, we have $\lim_{z\to\pm\infty} \overline{V}(z) \in \{\pm 1\}$. If $\lim_{z\to-\infty} \overline{V}(z) \neq \lim_{z\to\infty} \overline{V}(z)$ then either $\overline{V}(\cdot)$ or $\overline{V}(-\cdot)$ is admissible and hence a minimizer. Therefore, it remains to exclude the possibility $\overline{V}(z) \to -1$ as $z \to \pm\infty$. ($\overline{V}(z) \to 1$ as $z \to \pm\infty$ is analogous.) If this was the case, then there was $z_n \to \infty$ such that

$$|\overline{V}^{(k)}(z_n)| + |\overline{V}(z_n) + 1| + |\overline{U}(z_n) + 1| \to 0,$$

and, since $V_k \to V$ in $W^{2,\infty}_{\mathrm{loc}}$, up to a subsequence $V_{k_n}(z_n) \to -1$, $V_{k_n}^{'}(z_n), V_{k_n}^{''}(z_n) \to 0$, and

$$\begin{aligned} &\int_{\mathbb{R}} \left(W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2\right)\,dz = \int_{-\infty}^{z_n} \left(W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2\right)\,dz \\ + &\int_{z_n}^{\infty} \left(W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2\right)\,dz \geq \int_{-\infty}^{z_n} \left(W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2\right)\,dz + \tilde{m} \\ - &\int_{-\infty}^{z_n} \left(W(\tilde{U}_{k_n}) - q|\tilde{U}_{k_n}'|^2 + |\tilde{V}_{k_n}''|^2 + |\tilde{V}_{k_n}'''|^2\right)\,dz, \end{aligned}$$

where the extensions $\tilde{V}_{k_n}$ and $\tilde{U}_{k_n}$ of V_{k_n} and U_{k_n}, respectively, are defined as in (4.11). A similar argument as in Step 2. shows that

$$\lim_{n\to\infty} \int_{-\infty}^{z_n} (W(\tilde{U}_{k_n}) - q|\tilde{U}_{k_n}'|^2 + |\tilde{V}_{k_n}''|^2 + |\tilde{V}_{k_n}'''|^2)\,dz = 0.$$

Finally, since $V_{k_n} \to V$ in $W^{2,\infty}_{\mathrm{loc}}$, taking $n \to \infty$, and using Fatou's Lemma in the second line gives

$$\begin{aligned} \tilde{m} &= \lim_{n\to\infty} \int_{\mathbb{R}} (W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2)\,dz \\ &\geq \limsup_{n\to\infty} \int_{-\infty}^{z_n} (W(U_{k_n}) - q|U_{k_n}'|^2 + |V_{k_n}''|^2 + |V_{k_n}'''|^2)\,dz + \tilde{m} \geq \int_{-\infty}^{\infty} (W(\overline{U}) + |\overline{V}''|^2)\,dz + \tilde{m}, \end{aligned}$$

which implies $\overline{V}'' \equiv 0$ and hence $\overline{U} = -\overline{V}'' + \overline{V} = \overline{V}$ with $\overline{V}(z) = p_0 + p_1 z$. Again, $W(\overline{U}) \equiv 0$ implies $\overline{U} = \overline{V} = p_0 = -1$, which contradicts $V_n(0) = 0$ for all $n \in \mathbb{N}$. This concludes the proof. □

4.2. Compactness

We are now in the position to prove compactness of low-energy sequences i.e., item 3. of Theorem 1.3 in the case $d = 1$. Note that in the case $q < 0$, the result follows from

$$\mathcal{F}_{\varepsilon_n}[u_n] \geq \int_I \left(\frac{1}{\varepsilon} W(u_n) - \varepsilon_n q|u_n'|^2\right)\,dx$$

and the compactness result for the Modica-Mortola functional (see [38]). For the general case $q \leq 0$, we follow the proof of [24, Proposition 2.7], which requires to show some a priori estimates. We first note that for $q \leq 0$, (4.1) and (1.8) imply that

$$\|v_n''\|_{L^2(I)} \leq C\varepsilon_n^{-3/2}, \quad \text{and} \quad \|v_n'''\|_{L^2(I)} \leq C\varepsilon_n^{-5/2}, \tag{4.12}$$

and, consequently, $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are bounded in $L^1(I)$ since by (H2') and (1.8)

$$\begin{aligned} \|u_n\|_{L^1(I)} &\le \int_{\{|u_n|\le R\}} |u_n|\,dx + \int_{\{|u_n|>R\}} |u_n|\,dx \le R|I| + \frac{1}{L}\int_I W(u_n)\,dx \le C, \qquad \text{and} \\ \|v_n\|_{L^1(I)} &= \|u_n + \varepsilon_n^2 v_n''\|_{L^1(I)} \le \|u_n\|_{L^1(I)} + \varepsilon_n^2\|v_n''\|_{L^1(I)} \le \|u_n\|_{L^1(I)} + |I|^{1/2}\varepsilon_n^2\|v_n''\|_{L^2(I)} \le C. \end{aligned}$$

By (2.4) and (4.12), we have

$$\begin{aligned} \|u_n'\|_{L^{4/3}(I)} &= \left\|-\varepsilon_n^2 v_n''' + v_n'\right\|_{L^{4/3}(I)} \le \varepsilon_n^2\|v_n'''\|_{L^{4/3}(I)} + \|v_n'\|_{L^{4/3}(I)} \\ &\le C\varepsilon_n^2\|v_n'''\|_{L^2(I)} + \|v_n'\|_{L^{4/3}(I)} \\ &\le C\left(\varepsilon_n^2\|v_n'''\|_{L^2(I)} + \|v_n\|_{L^1(I)}^{1/2}\|v_n''\|_{L^2(I)}^{1/2} + \|v_n\|_{L^1(I)}\right) \le C\varepsilon_n^{-3/4}. \end{aligned} \tag{4.13}$$

The compactness result for $q = 0$ now follows from Proposition 4.1 and the proof of Proposition 2.7 in [24].

4.3. Liminf Inequality

The proof of the liminf inequality follows essentially from Proposition 4.1. Precisely, let $u \in BV(I;\{\pm 1\})$ and assume without loss of generality that there exists a partition

$$-1 = x_0 < x_1 < \cdots < x_\nu = 1,$$

with $u(x) = -1$ in (x_{2i-2}, x_{2i-1}) and $u(x) = 1$ in (x_{2i-1}, x_{2i}). Assume that $\varepsilon_n \to 0^+$ and that $(u_n)_{n\in\mathbb{N}} \subset W^{1,2}(I)$ satisfies (extracting a subsequence if needed)

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] = \lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n] = M < \infty,$$

and $u_n \to u$ in $L^1(I)$ and pointwise for $\mathcal{L}^1$-almost every $x \in I$. It follows from the compactness argument in the previous section that if $q \le 0$ and $k = 1, 2, 3$ then $\varepsilon_n^k v_n^{(k)} \to 0$. In addition, $v_n = u_n + \varepsilon_n^2 v_n'' \to u$ for $\mathcal{L}^1$-a.e. $x \in I$. Hence, the hypotheses of Proposition 4.1 are satisfied and the liminf inequality follows.

4.4. Limsup Inequality

To prove the limsup inequality we proceed as in *Step 1* in Subsection 3.3. Let $\delta > 0$ and $u \in BV(I;\{-1,1\})$ and $-1 = x_0 < x_1 < \cdots < x_\nu = 1$, where $x_1, \ldots, x_{\nu-1}$ are the jump points of u. Without loss of generality we may again assume $u = -1$ on $(-1, x_1)$. Pick an admissible $V \in \mathcal{A}$ (see (4.8)), such that with $U = -V'' + V$

$$\int_{\mathbb{R}} (W(U) - q|U'|^2 + |V''|^2 + |V'''|^2)\,dz \le m + \frac{\delta}{\nu}.$$

Consider a sequence $\varepsilon_n \to 0^+$. To construct a recovery sequence, we set for $i = 1, \ldots, \lfloor \frac{\nu}{2} \rfloor$

$$v_n(x) := \begin{cases} V(\frac{x - x_{2i-1}}{\varepsilon_n}) & \text{if } x \in I_{2i-1} := (\frac{x_{2i-1}+x_{2i-2}}{2}, \frac{x_{2i}+x_{2i-1}}{2}), \\ V(-\frac{x-x_{2i}}{\varepsilon_n}) & \text{if } x \in I_{2i} := (\frac{x_{2i}+x_{2i-1}}{2}, \frac{x_{2i+1}+x_{2i}}{2}), \\ u(x) & \text{otherwise.} \end{cases}$$

Then $v_n \to u$ in $L^1((-1,1))$, v_n is constant in $(-1, \frac{x_1-1}{2})$ and in $(\frac{x_{\nu-1}+1}{2}, 1)$ and hence satisfies Neumann boundary conditions. For n sufficiently large, we have $v_n \in W^{3,2}(-1,1)$ and with $u_n = v_n - \varepsilon_n^2 v_n''$, we have changing variables $y := \frac{1}{\varepsilon_n}(x - x_{2i-1})$

$$\begin{aligned} &\int_{I_{2i-1}} \left(\frac{1}{\varepsilon_n} W(u_n(x)) - \varepsilon_n q(u_n'(x))^2 + \varepsilon_n^3 (v_n''(x))^2 + \varepsilon_n^5 (v_n'''(x))^2\right) dx \\ \le\ &\int_{\mathbb{R}} \left(W(U(y)) - q(U'(y)^2 + (V''(y))^2 + (V'''(y))^2\right) dy \le m + \frac{\delta}{\nu}, \end{aligned}$$

and similarly

$$\int_{I_{2i}} \left(\frac{1}{\varepsilon_n} W(u_n(x)) - \varepsilon_n q (u_n'(x))^2 + \varepsilon_n^3 (v_n''(x))^2 + \varepsilon_n^5 (v_n'''(x))^2 \right) dx \le m + \frac{\delta}{\nu}.$$

Therefore, $\mathcal{F}_{\varepsilon_n}[v_n] \le (\nu - 1)m + \delta$, and since $\delta > 0$ was arbitrary, this concludes the proof of the limsup-inequality, see the discussion following Proposition 3.3.

5. Local approximation

We now turn to the local functional (1.2) and prove Theorem 1.5. Here we restrict ourselves to the case $q = 0$ (see the discussion in the introduction). Thus, we consider the family of functionals introduced in (1.9) with $X = \{u \in L^2(\Omega) : \Delta u \in L^2(\Omega)\}$. The proof of the Γ-convergence results follows from results in the literature, and therefore, we only sketch them here.

Proof. (of Theorem 1.5) (i) Let us first discuss the case $d = 1$. Note that by the Gagliardo-Nirenberg inequality in bounded domains (see e.g. [39]) in the case of a one-dimensional domain we have $X = \{u \in L^2(I) : u'' \in L^2(I)\} = W^{2,2}(I)$. Thus, the functionals $\mathcal{F}^*_{\varepsilon,ap}$ coincide with the functionals considered in [24], where, under weaker growth conditions on W, Γ-convergence and compactness is proven with respect to the strong L^1-topology. By Lemma 2.1 and Vitali's theorem, we deduce $\Gamma-$convergence and compactness with respect to the strong L^2-topology. Further, in [24], it is also shown that $\mathbf{m} > 0$ and

$$\mathbf{m} = \min\left\{ \int_{-\infty}^{+\infty} W(f(t)) + (f''(t))^2 dt : f \in W^{2,2}_{\text{loc}}(\mathbb{R}), \lim_{t\to+\infty} f(t) = 1, \lim_{t\to-\infty} f(t) = -1 \right\}.$$

Let us now discuss the case $d > 1$.

<u>$\liminf -$inequality</u> The liminf inequality follows very closely the lines of [31, 8] and we sketch it only for completeness. Let $\varepsilon_n \to 0$ and $(u_n)_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ with $u_n \to u$ in $L^2_{\text{loc}}(\Omega)$. We will show in part (ii) that $\liminf_{n\to\infty} \mathcal{F}^*_{\varepsilon_n,ap}[u_n] = +\infty$ if $u \notin BV(\Omega; \{\pm 1\})$, and hence, the liminf-inequality follows. Therefore we may assume that $u \in BV(\Omega; \{\pm 1\})$, and

$$\sup_{n\in\mathbb{N}} \mathcal{F}^*_{\varepsilon_n,ap}[u_n] < \infty.$$

We proceed by a blow-up argument, similarly as in Appendix A. For a Borel set $A \subseteq \Omega$, we set

$$\mu_{\varepsilon_n}(A) := \int_A \frac{W(u_n)}{\varepsilon_n} + \varepsilon^3 |\Delta u_n|^2 \, dx.$$

Then the sequence $(\mu_{\varepsilon_n})_{n\in\mathbb{N}}$ is equibounded, and there exists a subsequence (not relabeled) and a Radon measure μ such that such that

$$\mu_{\varepsilon_n} \overset{*}{\rightharpoonup} \mu. \tag{5.1}$$

By the Radon-Nikodym theorem we have the decomposition

$$\mu = \mu_a \mathcal{L}^d + \mu_J \mathcal{H}^{d-1} \lfloor J_u + \mu_s$$

where J_u denotes the jumpset of the function u. Suppose we had the inequality

$$\mu_J(x_0) \ge \mathbf{m}, \text{ for } \mathcal{H}^{d-1}\text{-almost every } x_0 \in J_u. \tag{5.2}$$

Let $(\varphi_k)_{k\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ be such that $0 \le \varphi_k \le 1$ and $\varphi_k(x) \nearrow 1$ on Ω. Then

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{F}^*_{\varepsilon_n,ap}[u_n] &\ge \liminf_{n\to\infty} \int_\Omega \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^3 |\Delta u_n|^2 \right) \varphi_k dx \\ &\overset{(5.1)}{=} \int_\Omega \varphi_k \, d\mu \\ &\ge \int_{J_u} \mu_J \varphi_k \, d\mathcal{H}^{d-1} \\ &\overset{(5.2)}{\ge} \mathbf{m} \int_{J_u} \varphi_k \, d\mathcal{H}^{d-1}.\end{aligned}$$

Taking the limit as $k \to \infty$, by the monotone convergence theorem we obtain the assertion

$$\liminf_{n\to\infty} \mathcal{F}^*_{\varepsilon_n,ap}[u_n] \ge \mathbf{m}\mathrm{Per}_\Omega(\{u=-1\}).$$

In order to prove (5.2), let $x_0 \in J_u$ be a Lebesgue point for μ with respect to $\mathcal{H}^{d-1}\lfloor J_u$, i.e.

$$\mu_J(x_0) = \lim_{\rho\to 0} \frac{\mu(Q_\nu(x_0,\rho))}{\mathcal{H}^{d-1}(Q_\nu(x_0,\rho)\cap J_u)} = \lim_{\rho\to 0} \frac{\mu(Q_\nu(x_0,\rho))}{\rho^{d-1}} \tag{5.3}$$

where $\nu := \nu_u(x_0)$ is the approximate normal and $Q_\nu(x_0,\rho)$ is the d-dimensional cube centered at x_0 with diameter ρ and with one of the sides normal to ν. By Besicovitch's differentiation theorem, (5.3) holds for $\mathcal{H}^{d-1}$-almost every point $x_0 \in J_u$. Since $x_0 \in J_u$, we can assume that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{(Q_\nu(x_0,\rho))^\pm} \left| u(x) - u^\pm(x_0) \right| dx = 0$$

where $(Q_\nu(x_0,\rho))^\pm := \{x \in Q_\nu(x_0,\rho) : \pm\langle x - x_0, \nu\rangle > 0\}$. Therefore we have

$$\begin{aligned}\mu_J(x_0) &= \lim_{\rho\to 0} \frac{1}{\rho^{d-1}} \int_{\overline{Q_\nu(x_0,\rho)}} d\mu \\ &\overset{(**)}{\ge} \lim_{\rho\to 0} \limsup_{n\to\infty} \frac{1}{\rho^{d-1}} \mu_{\varepsilon_n}(Q_\nu(x_0,\rho)) \\ &= \lim_{\rho\to 0} \limsup_{n\to\infty} \frac{1}{\rho^{d-1}} \int_{Q_\nu(x_0,\rho)} \left(\frac{W(u_n)}{\varepsilon_n} + \varepsilon_n^3 (\Delta u_n)^2 \right) dx \\ &= \lim_{\rho\to 0} \limsup_{n\to\infty} \rho \int_{Q_\nu} \left(\frac{W(u_n(x_0+\rho y))}{\varepsilon_n} + \varepsilon_n^3 (\Delta u_n(x_0+\rho y))^2 dy \right) \\ &= \lim_{\rho\to 0} \limsup_{n\to\infty} \int_{Q_\nu} \left(\frac{\rho}{\varepsilon_n} W(u_{n,\rho}(y)) + \frac{\varepsilon_n^3}{\rho^3} (\Delta u_{n,\rho}(y))^2 \right) dy \\ &\ge \lim_{\rho\to 0} \limsup_{n\to\infty} \mathcal{F}^*_{\frac{\varepsilon_n}{\rho},ap}[u_{n,\rho}; Q_\nu],\end{aligned}$$

where we denoted $u_{n,\rho}(y) := u_n(x_0+\rho y)$ and $(**)$ follows from [1, Proposition 1.62(a)]. Now by a standard diagonalization argument we can extract monotonically decreasing subsequences $(\varepsilon_{n_h})_{h\in\mathbb{N}}$ and $(\rho_h)_{h\in\mathbb{N}}$ such that

$$\sigma_h := \frac{\varepsilon_{n_h}}{\rho_h} \to 0, \qquad u_{\varepsilon_{n_h},\rho_h} \overset{L^2(Q_\nu)}{\longrightarrow} u \text{ as } h\to\infty, \qquad \text{and} \qquad \mu_J(x_0) \ge \lim_{h\to\infty} \mathcal{F}^*_{\sigma_h,ap}[u_{n_h,\rho_h}; Q_\nu].$$

By rotational symmetry, we may assume $\nu = e_n$ and set $Q(\delta) := (-\frac{1}{2}+\delta, \frac{1}{2}-\delta)^d$. Since $\mu_J(x_0) \ge \lim_{h\to\infty} \mathcal{F}^*_{\sigma_h,ap}[u_{n_h}; Q(\delta)]$ it suffices to show that $\lim_{h\to\infty} \mathcal{F}^*_{\sigma_h,ap}[u_{n_h}; Q(\delta)] \ge \mathbf{m}$. For simplicity of notation we for now drop the indices h and n_h. By density, we may assume that $u_{n_h} \in C^\infty(\overline{Q(\delta)})$.

$$\Delta_x u = \partial_{zz} u + \Delta_y u$$

with $z \in (-\frac{1}{2}+\delta, \frac{1}{2}-\delta)$ and $y \in Q^{'}(\delta) := (-\frac{1}{2}+\delta, \frac{1}{2}-\delta)^{d-1}$. Let $\eta \in C_0^\infty(Q(\delta))$ with $\|\eta\|_{L^\infty} = 1$, and note that

$$|\partial_{zz}u + \Delta_y u|^2\eta^2 = (\partial_{zz}u)^2\eta^2 + |\Delta_y u|^2\eta^2 + 2(\partial_{zz}u)\eta^2\Delta_y u. \tag{5.4}$$

We consider the last term, which is the one that does not have a sign, and note that by integration by parts (first in z and then in y) and Young's inequality

$$\int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\left(\int_{Q^{'}(\delta)}(\partial_{zz}u)\eta^2\Delta_y u\,dy\right)dz = -\int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\big((\partial_z u)\eta^2\Delta_y(\partial_z u) - 2(\partial_z u)\eta(\partial_z\eta)\Delta_y u\big)\,dydz$$
$$= \int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\left(|\nabla_y(\partial_z u)|^2\eta^2 - 2(\partial_z u)\eta\langle\nabla_y(\partial_z u), \nabla_y\eta\rangle_{\mathbb{R}^{d-1}} - 2(\partial_z u)\eta(\partial_z\eta)\Delta_y u\right)dydz$$
$$\geq \int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\left(|\nabla_y(\partial_z u)|^2\eta^2 - \big(|\nabla_y(\partial_z u)|^2\eta^2 + (\partial_z u)^2|\nabla_y\eta|^2\big) - \left(\frac{1}{2}|\Delta_y u|^2\eta^2 + 2(\partial_z u)^2(\partial_z\eta)^2\right)\right)dydz.$$

Therefore,

$$\int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}|\partial_{zz}u + \Delta_y u|^2\eta^2\,dydz \geq \int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\big((\partial_{zz}u)^2\eta^2 - 2(\partial_z u)^2|\nabla_y\eta|^2 - 4(\partial_z u)^2|\partial_z\eta|^2\big)\,dydz$$

Hence, with $c(\eta) := \max\left\{2|\nabla_y\eta|^2 + 4(\partial_z\eta)^2\right\}$, we have

$$\mathcal{F}^*_{\sigma,ap}[u, Q(\delta)] \geq \int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\left(\frac{1}{\sigma}W(u) + \sigma_h^3(\partial_{zz}u)^2\right)\eta^2\,dydz - \int_{Q(\delta)}c(\eta)\sigma^3(\partial_z u)^2\,dx. \tag{5.5}$$

By interior elliptic regularity (see e.g. [21, Section 6.3.1, Theorem 1]), there is a constant $C_R > 0$ depending only on δ such that for all $h \in \mathbb{N}$

$$\|\nabla^2 u_{n_h}\|_{L^2(Q(\delta))} \leq C_R\left(\|u_{n_h}\|_{L^2(Q_\nu)} + \|\Delta u_{n_h}\|_{L^2(Q_\nu)}\right),$$

and hence the (rescaled) Gagliardo-Nirenberg interpolation inequality (see (2.3)) yields, using the growth condition (H2) and the fact that $\mathcal{F}^*_{\sigma_h,ap}[u_{n_h}, Q_\nu] \leq C$,

$$\begin{aligned}
\sigma_h^3\|\partial_z u_{n_h}\|^2_{L^2(Q(\delta))} &\leq C\sigma_h^2\left(\frac{1}{\sigma_h}\int_{Q(\delta)}u_{n_h}^2\,dx + \sigma_h^3\|\nabla^2 u_{n_h}\|^2_{L^2(Q(\delta))}\right)\\
&\leq C(\delta)\sigma_h^2\left(\int_{Q(\delta)}\frac{W(u_{n_h})}{\sigma_h}\,dx + \sigma_h^3\|\Delta u_{n_h}\|^2_{L^2(Q(\delta))}\right) + C(\delta)\sigma_h.
\end{aligned} \tag{5.6}$$

Inserting the estimate (5.6) into (5.5), we obtain that for all $\delta > 0$ and all $\eta \in C_0^\infty(Q(\delta))$,

$$\lim_{h\to+\infty}\mathcal{F}^*_{\sigma_h,ap}[u_{n_h}, Q(\delta)] \geq \liminf_{h\to\infty}\int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\Big(\frac{W(u_{n_h})}{\sigma_h} + \sigma_h^3(\partial_{zz}u_{n_h})^2\Big)\eta^2\,dydz. \tag{5.7}$$

Now choose $\hat{\delta} > \delta$ and $\eta \in C_0^\infty(Q(\delta))$ such that $\eta \equiv 1$ on $Q(\hat{\delta}) \subset Q(\delta)$. Then, for such η we get

$$\begin{aligned}
\mu_J(x_0) &\geq \lim_{h\to\infty}\mathcal{F}^*_{\sigma_h,ap}[u_{n_h}, Q(\delta)]\\
&\overset{(5.7)}{\geq} \liminf_{h\to\infty}\int_{-\frac{1}{2}+\delta}^{\frac{1}{2}-\delta}\int_{Q^{'}(\delta)}\Big(\frac{W(u_{n_h})}{\sigma_h} + \sigma_h^3(u_{n_h})^2\Big)\eta^2\,dydz\\
&\geq \liminf_{h\to\infty}\int_{-\frac{1}{2}+\hat{\delta}}^{\frac{1}{2}-\hat{\delta}}\int_{Q^{'}(\hat{\delta})}\Big(\frac{W(u_{n_h})}{\sigma_h} + \sigma_h^3(\partial_{zz}u_{n_h})^2\Big)\,dydz\\
&\overset{\text{Fatou}}{\geq} \int_{Q^{'}(\hat{\delta})}\liminf_{h\to\infty}\int_{-\frac{1}{2}+\hat{\delta}}^{\frac{1}{2}-\hat{\delta}}\Big(\frac{W(u_{n_h})}{\sigma_h} + \sigma_h^3(\partial_{zz}u_{n_h})^2\Big)\,dzdy\\
&\geq \mathbf{m}\,\mathcal{H}^{d-1}(Q^{'}(\hat{\delta}))
\end{aligned}$$

where the last inequality follows from the one-dimensional case. Finally, by letting $\hat{\delta} \to 0$ we obtain (5.2).

$\underline{\limsup -\text{inequality}}$ A recovery sequence is constructed exactly as in [24]. Note that strong convergence in $L^2(\Omega)$ of the constructed sequence follows by Vitali convergence theorem (in particular, L^1 convergence implies convergence in measure and Lemma 2.1 yields uniform integrability).

(ii) Compactness: Consider $\{u_\varepsilon\}_\varepsilon \subseteq X$ such that $\liminf_{\varepsilon\to 0} \mathcal{F}^*_{\varepsilon,ap}[u_\varepsilon;\Omega] < \infty$. We use again that for every $v \in X$ and for every $U \subset\subset \Omega$ by [21, Section 6.3.1, Theorem 1] there holds

$$\|\nabla^2 v\|_{L^2(U)} \leq c(\Omega, U)\left(\|v\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right). \tag{5.8}$$

Thus we have for $\varepsilon > 0$ small enough, using the growth condition (H2),

$$\begin{aligned}
\mathcal{F}^*_{\varepsilon,ap}[u_n;\Omega] &= \int_\Omega \left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon^3(\Delta u_\varepsilon)^2\right) dx \\
&\geq \int_\Omega \frac{1}{\varepsilon}W(u_\varepsilon)dx + \varepsilon^3 C(\Omega,U)\int_U |\nabla^2 u_\varepsilon|^2 dx - \varepsilon^3 \int_\Omega u_n^2 dx \\
&= \varepsilon^3 C(\Omega,U)\int_U |\nabla^2 u_\varepsilon|^2 dx + \int_\Omega \left(\frac{1}{\varepsilon}W(u_\varepsilon) - \varepsilon^3 u_\varepsilon^2\right) dx \\
&\geq \varepsilon^3 C(\Omega,U)\int_U |\nabla^2 u_\varepsilon|^2 dx + \int_\Omega \left(\frac{1}{\varepsilon}W(u_\varepsilon) - c\varepsilon^3\left((W(u_\varepsilon) + R^2\right)\right) dx \\
&= \varepsilon^3 C(\Omega,U)\int_U |\nabla^2 u_\varepsilon|^2 dx + \int_\Omega \left(W(u_\varepsilon)\left(\frac{1}{\varepsilon} - c\varepsilon^3\right) - cR^2\varepsilon^3\right) dx \\
&\geq \int_U \frac{1}{2\varepsilon}W(u_\varepsilon) + C(\Omega,U)\varepsilon^3|\nabla^2 u_\varepsilon|^2 dx - cR^2\varepsilon_\varepsilon^3|\Omega| \\
&\geq \tilde{c}\int_U \left(\frac{1}{\varepsilon}W(u_\varepsilon) + \varepsilon^3|\nabla^2 u_\varepsilon|^2\right) dx - cR^2\varepsilon^3|\Omega|.
\end{aligned}$$

Now by [24, Proposition 3.1] there exists a subsequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ such that

$$u_{\varepsilon_n} \overset{L^1(U)}{\longrightarrow} u \in BV(U;\{\pm 1\}).$$

By a standard diagonal argument, we find a subsequence (not relabeled) such that $u_{\varepsilon_n} \longrightarrow u$ in $L^1_{loc}(\Omega)$. In particular, $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ converges in measure to u and is uniformly integrable (by Lemma 2.1). Thus Vitali's convergence theorem implies

$$u_{\varepsilon_n} \overset{L^2(\Omega)}{\longrightarrow} u \in BV(\Omega;\{\pm 1\}).$$

□

Appendix A

In this appendix, we briefly outline the blow-up argument to show that for a sequence $\{v_n\}_{n\in\mathbb{N}}$ with bounded energy, we have

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n;\Omega] \geq \int_{\partial^* E_0} \lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}[w_k; Q_\nu(0,1)]\, d\mathcal{H}^{d-1},$$

where $w_k \in W^{2,2}(Q_\nu(0,1))$ satisfies

$$\|w_k\|_{L^2(Q_\nu)} \leq \|v_{n_k}\|_{L^2(\Omega)} \quad \text{and} \quad \lim_{k\to\infty} \|w_k - u_0\|^2_{L^2(Q_\nu(0,1))} = 0$$

with

$$u_0(x) := \begin{cases} -1 & \text{if } x\cdot\nu < 0, \\ 1 & \text{if } x\cdot\nu > 0. \end{cases} \tag{5.9}$$

Let

$$Q_\nu(x_0, r_0) := \{x \in \mathbb{R}^d : (x - x_0)/r \in Q_\nu\} \tag{5.10}$$

Given a Borel set $A \subset \Omega$, let

$$\mu_n(A) = \mathcal{F}_{\varepsilon_n}[v_n; A]. \tag{5.11}$$

Then, there exists a signed Radon measure μ such that

$$\mu_n \overset{*}{\rightharpoonup} \mu. \tag{5.12}$$

Consider the nonnegative measure

$$\xi(A) := \mathcal{H}^{d-1}(A \cap \partial^* E_0) < \infty, \tag{5.13}$$

where $A \subset \Omega$ is a Borel set. Then, by Radon-Nykodym and Lebesgue Decomposition theorems (see [23, (1.180)]),

$$\mu = \mu_{ac} + \mu_s, \tag{5.14}$$

where $\mu_s \geq 0$ is a bounded Radon measure, $\mu_{ac} << \xi$ and

$$\lim_{n\to\infty} \mathcal{F}_{\varepsilon_n}[v_n; \Omega] \geq \mu(\Omega) \geq \int_{\Omega\cap\partial^* E_0} \frac{d\mu_{ac}}{d\mathcal{H}^{d-1}}(x) d\mathcal{H}^{d-1}. \tag{5.15}$$

In addition, for $x_0 \in \partial^* E \cap \Omega$,

$$\frac{d\mu_{ac}}{d\mathcal{H}^{d-1}}(x_0) = \lim_{r\to 0^+} \frac{\mu(Q_\nu(x_0, r))}{\mathcal{H}^{d-1}(Q_\nu(x_0, r) \cap \partial^* E_0)} = \lim_{r\to 0^+} \frac{\mu(Q_\nu(x_0, r))}{r^{d-1}} < \infty, \tag{5.16}$$

where ν is the outward unit normal to E_0 at x_0. Thus, choosing $(r_k)_{k\in\mathbb{N}}$ such that $\mu(\partial Q_v(x_0, r_k)) = 0$, yields for all $x_0 \in \Omega \cap \partial^* E_0$,

$$\begin{aligned}
\frac{d\mu_{ac}}{d\mathcal{H}^{d-1}}(x_0) &= \lim_{k\to\infty} \frac{1}{r_k^{d-1}} \lim_{n\to\infty} \int_{Q_\nu(x_0, r_k)} \left(\frac{1}{\varepsilon_n} W(u_n) - \varepsilon_n q |\nabla u_n|^2 + \varepsilon_n^3 (\Delta v_n)^2 + \varepsilon_n^5 |\nabla \Delta v_n|^2 \right) dx \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int_{Q_\nu(0,1)} \left(\frac{r_k}{\varepsilon_n} W(u_{n,k}(x)) - \frac{\varepsilon_n}{r_k} q |\nabla u_{n,k}|^2 + \frac{\varepsilon_n^3}{r_k^3} |\Delta v_{n,k}|^2 + \frac{\varepsilon_n^5}{r_k^5} |\nabla \Delta v_{n,k}|^2 \right) dx \\
&= \lim_{n\to\infty} \int_{Q_\nu(0,1)} \left(\frac{1}{\varepsilon_n} W(u_{n,n}(x)) - q\varepsilon_n |\nabla u_{n,n}|^2 + \varepsilon_n^3 |\Delta v_{n,n}|^2 + \varepsilon_n^5 |\nabla \Delta v_{n,n}|^2 \right) dx,
\end{aligned} \tag{5.17}$$

where we used change of variables $u_{n,k}(x) := u_n\left(\frac{x-x_0}{r_k}\right)$, $v_{n,k}(x) := v_n\left(\frac{x-x_0}{r_k}\right)$ in the second line and a diagonalization argument with $\varepsilon_n = \frac{\varepsilon_n}{r_n} \to 0$ in the third line. We remark that $-\varepsilon_n^2 \Delta v_{n,n} + v_{n,n} = u_{n,n}$ still holds in $Q_\nu := Q_\nu(0,1)$. In addition, since $v_n \to u$ in $L^2(\Omega)$,

$$\lim_{n\to\infty} \|v_{n,n} - u_0\|^2_{L^2(Q_\nu(0,1))} = 0, \tag{5.18}$$

where

$$u_0 := \begin{cases} -1, & \text{if } x \cdot \nu < 0, \\ 1, & \text{if } x \cdot \nu > 0, \end{cases} \tag{5.19}$$

since on the sets where $u\left(\frac{x-x_0}{r_k}\right)$ and u_0 differ,

$$\lim_{r\to 0} \frac{1}{r^d} \mathcal{L}^d(\{x \in Q_\nu(x_0, r) \setminus E_0 : x \cdot \nu < 0\}) = 0, \tag{5.20}$$

$$\lim_{r\to 0} \frac{1}{r^d} \mathcal{L}^d(\{x \in Q_\nu(x_0, r) \cap E_0 : x \cdot \nu > 0\}) = 0. \tag{5.21}$$

In addition, we may assume $Q_\nu(x_0, r_n) \subset \Omega$ and changing variables yields,

$$\|v_{n,n}\|_{L^2(Q_\nu)} \leq \|v_n\|_{L^2(\Omega)}. \tag{5.22}$$

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