RELAXATION AND OPTIMAL FINITENESS DOMAIN FOR DEGENERATE QUADRATIC FUNCTIONALS - ONE DIMENSIONAL CASE

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Abstract. The aim of this paper is the study, in the one-dimensional case, of the relaxation of a quadratic functional admitting a very degenerate weight w, which may not satisfy both the doubling condition and the classical Poincaré inequality. The main result deals with the relaxation on the greatest ambient space $L^0(\Omega)$ of measurable functions endowed with the topology of convergence in measure $\tilde{w}\,dx$. Here $\tilde{w}$ is an auxiliary weight fitting the degenerations of the original weight w. Also the relaxation w.r.t. the $L^2(\Omega, \tilde{w})$-convergence is studied. The crucial tool of the proof is a Poincaré type inequality, involving the weights w and $\tilde{w}$, on the greatest finiteness domain D_w of the relaxed functionals.

Contents

1. Introduction

This paper is devoted to the study in the one-dimensional framework of the integral representation of a functional obtained by relaxation of a quadratic weighted functional admitting a degenerate weight w. The main difficulty is that we do not require on w any additional assumption, as the doubling or Muckenhoupt condition (see Definitions

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2.7 and 2.8 below). We recall that, as proven in [13], in one dimension, the measures satisfying the doubling condition and the Poincaré inequality are precisely the Muckenhoupt A_2-weights. One of the main goals of the paper is to single out an appropriate ambient topological space containing the widest expected finiteness domain D_w of the relaxed functional (see (4)). Typically, this study has been carried out by prescribing a priori the ambient space.

More precisely, let us consider

$$F_X(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, w\, dx & \text{if } u \in C^1(\Omega) \\ +\infty & \text{if } u \in X \setminus C^1(\Omega), \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and X is an appropriate topological space composed of measurable functions. Let $\overline{F} := \mathrm{sc}^-(X) - F_X : X \to [0, +\infty]$ denote the relaxed functional (or lower semicontinuous envelope) of F w.r.t. the topology of X. Here w is a degenerate weight, i.e. we assume only that it is a nonnegative L^1_{loc} function, without any assumption on the function $\frac{1}{w}$. It is well-known that, if w is a Muckenhoupt weight in the A_2 class (this implies that $\frac{1}{w}$ belongs to L^1), then $X = L^2(\Omega, w)$ and the relaxed functional is finite in the Sobolev weighted space $W^{1,2}(\Omega, w)$ (for its definition see section 2) and it admits the following form

$$\overline{F}_X(u) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, w\, dx & \text{if } u \in W^{1,2}(\Omega, w) \\ +\infty & \text{if } u \in L^2(\Omega, w) \setminus W^{1,2}(\Omega, w). \end{cases}$$

If w is degenerate, the study of this relaxation problem is very complicated since it is unknown a priori what is the optimal natural ambient space where the finiteness domain

$$\mathrm{dom}(\overline{F}_X) = \{u \in X : \overline{F}_X(u) < +\infty\}$$

is contained. As well, a Meyers-Serrin type theorem needs in the weighted Sobolev space $W^{1,2}(\Omega, w)$, that is, whether $C^1(\Omega) \cap W^{1,2}(\Omega, w)$ is dense in $W^{1,2}(\Omega, w)$ (see [31]). Otherwise a Lavrentiev phenomenon may occur. The first space X considered in literature was the space $L^2(\Omega)$ (see [16, 23, 26, 27]). In particular a characterization of the relaxed functional w.r.t. the $L^2(\Omega)$ convergence is studied in [16]. Moreover, in [1, 2, 14, 15] the variational convergence of functionals of this type is considered. See also [11] (and the references therein) for the relation with the non occurrence of the Lavrentiev phenomenon.

On the other hand, another natural ambient space is the space $L^2(\Omega, w)$ firstly studied in the framework of the theory of Dirichlet forms (see [28]).

Recently, the theory of Sobolev spaces in metric measure spaces, initially developed in [17], has been extended to more general situations (see e.g. [3, 4, 5, 6, 7, 8, 9, 10, 25] and the references therein).

In all these theories, crucial tools are the doubling condition and the Poincaré inequality. We observe that we will consider very degenerate weights w, which may not satisfy these assumptions (see Remarks 4.12 and 5.2 below). Notice that our approach is different from the previous ones where the ambient space X is a priori fixed. For a comparison with these previous results see section 2.

Our investigation is confined to the relaxation of degenerate quadratic functionals in the simplest one-dimensional case, but with very general degenerations w. We are going to show that the space $L^2(\Omega)$ and $L^2(\Omega, w)$ are not always the appropriate ambient spaces for the relaxation of a quadratic functional with general degeneration w.

We consider a weight $w : \mathbb{R} \to \mathbb{R}$ satisfying

$$w \geq 0 \text{ a.e.}, \quad w \in L^1_{\text{loc}}(\mathbb{R}) . \tag{1}$$

Let $\Omega = (a, b)$ be a bounded open interval. Let $I_{\Omega,w}$ denote the set

$$I_{\Omega,w} := \Big\{ x \in \Omega : \exists\, \epsilon > 0 \text{ such that } \frac{1}{w} \in L^1\left((x - \epsilon, x + \epsilon)\right) \Big\}. \tag{2}$$

The set $I_{\Omega,w}$ is the biggest open set in Ω such that $\frac{1}{w}$ is locally summable. Without loss of generality we can assume that there exist two countable sets $\{a_i\}, \{b_i\}$ such that $a \leq a_i < b_i \leq b$, the intervals (a_i, b_i) are disjoint and

$$I_{\Omega,w} = \bigcup_{i=1}^{N_w} (a_i, b_i) , \tag{3}$$

with $N_w \in \mathbb{N} \cup \{+\infty\}$.

Definition 1.1. (i) If $I_{\Omega,w} = \emptyset$, we put $N_w := 0$.
(ii) If $1 \leq N_w < \infty$ we say that w is *finitely degenerate* in Ω.
(iii) If $N_w = \infty$ we say that w is *not finitely degenerate* in Ω.

Let

$$\begin{aligned} D_w := \Big\{ & u : \Omega \to \mathbb{R} : u \text{ (Lebesgue) measurable}, \\ & u \in W^{1,1}_{\text{loc}}(I_{\Omega,w}), \int_{I_{\Omega,w}} |u'|^2 \, w \, dx < +\infty \Big\} . \end{aligned} \tag{4}$$

The class D_w turns out to be the possible widest finiteness domain candidate for the relaxed functional $\overline{F}_X$ as soon as the convergence in X provides a mild pointwise convergence in $I_{\Omega,w}$ (see Lemma 4.5).

It is well-known (see Theorems 3.1 and 3.3 below) that when $X = L^2(\Omega)$

$$\text{dom}(\overline{F}_X) = D_w \cap L^2(\Omega).$$

On the other hand, it is easy to see that, for suitable w

$$D_w \nsubseteq L^2(\Omega)$$

(see Remark 5.3 below). Meanwhile, the same argument can be applied to the space $L^2(\Omega, w)$. This amounts that both $L^2(\Omega)$ and $L^2(\Omega, w)$ are not the appropriate spaces containing D_w.

The aim of our paper is to identify two ambient spaces which contain D_w and to provide a representation of the relaxed functional $\overline{F}_X$ in those spaces. The first ambient space is the greatest one $X = (L^0(\Omega), d_{\mathfrak{m}})$ or $(L^0(\Omega), d_{\widetilde{\mathfrak{m}}})$, where

$$L^0(\Omega) := \Big\{ u : \Omega \to \mathbb{R} : u \text{ is (Lebesgue) measurable} \Big\} , \tag{5}$$

$\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ are the measures on Ω

$$\mathfrak{m} = w\,dx \text{ and } \widetilde{\mathfrak{m}} = \tilde{w}\,dx\,, \tag{6}$$

$d_{\mathfrak{m}}$ and $d_{\widetilde{\mathfrak{m}}}$ are the distances defined according to (38) with $\mu = \mathfrak{m}$ and $\mu = \widetilde{\mathfrak{m}}$, respectively, which induce the convergence in measure (see (37) below). Here $\tilde{w}$ is an auxiliary new weight, associated to w, which fits the degeneration of w (see (25) for its definition) and it is is equal to 0 at the points where $\frac{1}{w}$ is not integrable. Then we deal with the relaxation on the ambient spaces $X = (L^0(\Omega), d_{\mathfrak{m}})$ and $(L^0(\Omega), d_{\widetilde{\mathfrak{m}}})$ and we study the lower semicontinuous envelopes w.r.t. the convergences in measure $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$, that is

$$\widehat{F^j} = \mathrm{sc}^-(d_{\mathfrak{m}}) - F^j_X, \quad \widetilde{F^j} = \mathrm{sc}^-(d_{\widetilde{\mathfrak{m}}}) - F^j_X \quad j = 1,2,3,4\,, \tag{7}$$

where F^j, $\;j = 1,2,3,4$ are defined in (13)–(16), and their finiteness domains

$$\widehat{D^j} := \{u \in L^0(\Omega) : \widehat{F^j}(u) < +\infty\,\}, \quad \widetilde{D^j} := \{u \in L^0(\Omega) : \widetilde{F^j}(u) < +\infty\,\}\,.$$

Our main result (see Theorem 4.18 (i)) states that

$$\widetilde{D^2} = D_w$$

and the following representation holds

$$\widetilde{F^2}(u) = \begin{cases} \displaystyle\int_{I_{\Omega,w}} |u'|^2\, w\,dx & \text{if } u \in D_w \\ +\infty & \text{if } u \in L^0(\Omega) \setminus D_w. \end{cases} \tag{8}$$

In particular, in the case when $w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$, we show that $\widehat{D^2} = \widetilde{D^2} = D_w$ and $\widehat{F^2} = \widetilde{F^2}$ on $L^0(\Omega)$ (see Theorem 4.18 (ii)). We also study the coincidence among the relaxed functionals $\widetilde{F^j}$ if $j = 1,2,3,4$ (see Corollary 4.20).
The second ambient space where we study the relaxation is $X = L^2(\Omega, \tilde{w})$, by considering the relaxed functionals

$$\overline{F^j} := \mathrm{sc}^-(L^2(\Omega,\tilde{w})) - F^j_X, \quad j = 1,2,3,4\,,$$

and their finiteness domains

$$D^j := \{u \in L^2(\Omega,\tilde{w}) : \overline{F^j}(u) < +\infty\,\}\,.$$

We are able to show that $D^2 = D_w \cap L^2(\Omega,\tilde{w})$ and $\overline{F^2} = \widetilde{F^2}$ on $L^2(\Omega,\tilde{w})$ (see Theorem 4.21). Note that, if the weight w is not finitely degenerate, it may happen that $D_w \nsubseteq L^2(\Omega,\tilde{w})$ (see Remark 5.3). However, if w is finitely degenerate, the same representation as in (8) holds for $\overline{F^2}$, that is, $D^2 = D_w$ and $\overline{F^2} = \widetilde{F^2}$ on $L^2(\Omega,\tilde{w})$ (see Corollary 4.22). We also study the coincidence among the relaxed functionals $\overline{F^j}$ if $j = 1,2,3,4$ (see Corollary 4.23).
A crucial tool of the proofs either of Theorems 4.18 and 4.21 is a Poincaré type inequality involving the two weights w and $\tilde{w}$ (see Theorem 4.11). Recall that, as proven in [29], an Hardy type inequality holds for the pair $(\tilde{w}, w)$ in the Muckenhoupt class, but unfortunately we need a Poincaré type inequality. The classical Poincaré inequality with

the usual rescaling does not work (see Remark 4.12). Anyway a Poincaré type inequality is true, but in a different form: for every $u \in D_w$

$$\sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 \tilde{w}(\eta)\, d\eta \le \int_a^b |u'(y)|^2 w(y)\, dy\,.$$

which does not seem to yield a Lipschitz approximation as in previous cases (see [21] and [30]).

2. Some previous results

In this section we will recall some previous results, where the relaxation of degenerate integral has been dealt with.

2.1. **Weighted L^2 and Sobolev spaces.** In order to introduce some definitions, according to the classical definitions of Sobolev spaces, let us fix a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary and a function $w : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$w \ge 0 \text{ a.e. in } \mathbb{R}^n,\ w \in L^1_{\text{loc}}(\mathbb{R}^n)\,.$$

If $\mathfrak{m}$ is a Radon measure on $\mathbb{R}^n$, let us define

$$L^2(\Omega, \mathfrak{m}) := \{u : \Omega \to \mathbb{R} : u \text{ Borel measurable}, \int_\Omega u^2\, d\mathfrak{m} < +\infty\}$$

and

$$L^2(\Omega, w) := L^2(\Omega, \mathfrak{m})$$

with $\mathfrak{m} = w\mathcal{L}^n$. If $w = 0$, then $L^2(\Omega, w) = \{0\}$, where we mean that for each function $u \in L^2(\Omega, w)$ we have $u(x) = 0$ for $w\mathcal{L}^n$ a.e. $x \in \Omega$.

If $X = L^p(\Omega)$ $(1 \le p < \infty)$, or $L^2(\Omega, w)$, we define the following type-Sobolev spaces:

$$W^1(\Omega, X, w) = \left\{u \in W^{1,1}_{\text{loc}}(\Omega) : (u, Du) \in X \times (L^2(\Omega, w))^n\right\}\,, \tag{9}$$

equipped with the norm

$$\|u\|_{X,w,\Omega} := \sqrt{\|u\|^2_X + \|Du\|^2_{L^2(\Omega,w)}}\,;$$

$$\begin{aligned} H^1(\Omega, X, w) :=& \text{the closure of } Lip(\Omega) \text{ in } W^1(\Omega, X, w) \\ & \text{endowed with the norm } \|\cdot\|_{X,w,\Omega}\,, \\ \tilde{H}^1(\Omega, X, w) :=& \Big\{u \in X : \exists (u_h)_h \subset Lip(\Omega), v \in (L^2(\Omega))^n, \\ & u_h \to u \text{ in } X, \sqrt{w}Du_h \to v \text{ in } (L^2(\Omega))^n\Big\}. \end{aligned}$$

We observe that

$$H^1(\Omega, X, w) \subseteq \tilde{H}^1(\Omega, X, w).$$

Remark 2.1. Since $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, in the definition of $\tilde{H}^1(\Omega, X, w)$, we may assume that $(u_h)_h \subset C^1(\bar{\Omega})$.

An explicit characterization of $\tilde{H}^1(\Omega, X, w)$ can be provided (see [16]).

Let
$$V \equiv V(\Omega, X, w)$$
denote the closure in $X \times (L^2(\Omega))^n$ of the linear subspace
$$\big\{(u, \sqrt{w}\nabla u) : u \in Lip(\Omega)\big\} \subset X \times (L^2(\Omega))^n\,,$$
and let Π_1 and Π_2 denote, respectively, the projections from $X \times (L^2(\Omega))^n$ into X and $(L^2(\Omega))^n$ respectively. Then it is easy to see that
$$\tilde{H}^1(\Omega, X, w) = \Pi_1(V(\Omega, X, w))\,.$$
For $u \in \Pi_1(V)$ let V_u denote the space
$$V_u := \big\{v \in (L^2(\Omega))^n : (u, v) \in V\big\}\,.$$

Remark 2.2. Since $V_u = \Pi_2\big((\{u\} \times (L^2(\Omega))^n) \cap V\big)$ and since Π_2 is an isomorphism from $\{u\} \times (L^2(\Omega))^n$ into $(L^2(\Omega))^n$, V_u is a closed affine subspace of $(L^2(\Omega))^n$ for each $u \in \Pi_1(V)$. In particular V_0 is a closed subspace of $(L^2(\Omega))^n$. For $(u, v) \in V$, we have that $V_u = v + V_0$.

If $u \in W^1(\Omega, X, w)$, we denote by Du the usual distributional gradient, that exists by definition (9). If w satisfies the additional property

$$\text{if } (\varphi_h)_h \subset Lip(\Omega), \|\varphi_h\|_X \to 0 \text{ and } \|\nabla\varphi_h - v\|_{L^2(\Omega,w)} \to 0 \quad \text{then } v = 0 \text{ a.e. in } \Omega\,, \tag{10}$$

then, if $u \in \tilde{H}^1(\Omega, X, w)$, V_u is a singleton and we are allowed to define the gradient $\nabla_{X,w}u$ in the following way: if $(\varphi_h)_h \subset Lip(\Omega)$ satisfies
$$\|\varphi_h - u\|_X \to 0 \text{ and } \|\nabla\varphi_h - v\|_{L^2(\Omega,w)} \to 0$$
then we set $\nabla_{X,w}u := v$.

Remark 2.3. In general the gradient of a function $u \in \tilde{H}^1(\Omega, X, w)$ does not need to be uniquely defined, that is the space V_u need not be a singleton. An example of this situation is given, for instance, in [20, Section 2.1].

Remark 2.4. An interesting case in which condition (10) occurs is when there exist a finite number of points $x_1, \dots, x_k$ in Ω such that $\frac{1}{w} \in L^1_{\text{loc}}(\Omega \setminus \{x_1, \dots, x_k\})$ (see [20, Section 2.1]). In this case it is easy to see that $\tilde{H}^1(\Omega, X, w) \subset W^{1,1}_{\text{loc}}(\Omega \setminus \{x_1, \dots, x_k\})$ and $\nabla_{X,w}u = Du$ a.e. in Ω for each $u \in \tilde{H}^1(\Omega, X, w)$. It is also interesting to observe that, even if $u \in \tilde{H}^1(\Omega, X, w)$ and it admits a distributional gradient, it may occur that $\nabla_{X,w}u \neq Du$ (see, for instance, [18, Example 2.1]). This means that, in general, $\tilde{H}^1(\Omega, X, w) \neq H^1(\Omega, X, w)$ and that $\big(W^1(\Omega, X, w), \|\cdot\|_{X,w,\Omega}\big)$ need not be complete.

If w satisfies the stronger assumption $\frac{1}{w} \in L^1(\Omega)$, it is well-known that
$$\big(W^1(\Omega, X, w), \|\cdot\|_{X,w,\Omega}\big)$$
is a Banach space and $\tilde{H}^1(\Omega, X, w) = H^1(\Omega, X, w) \subseteq W^1(\Omega, X, w)$. Moreover it is easy to see that
$$W^1(\Omega, L^2(\Omega, w), w) \subset W^1(\Omega, L^1(\Omega), w) \subset W^{1,1}(\Omega)\,.$$

In this case the agreement $H^1(\Omega, X, w) = W^1(\Omega, X, w)$ turns be out an important issue, which need not be true (see [18] and [10]).

Another characterization of $\tilde{H}^1(\Omega, X, w)$ by relaxation was provided in [16] in the case $X = L^p(\Omega)$.

Let $F : X \to [0, +\infty]$ denote the functional defined by

$$F(u) := \begin{cases} \int_\Omega |\nabla u|^2 \, w \, dx \text{ if } u \in Lip(\Omega) \\ +\infty \text{ otherwise} \end{cases}$$

and let $\overline{F} : X \to [0, +\infty]$ denote the relaxed functional (or lower semicontinuous envelope) of F w.r.t. the topology of X.

Theorem 2.5. *(*[16, Theorem 1.1]*) Let* $1 \le p < \infty$.

(i) $\tilde{H}^1(\Omega, L^p(\Omega), w) = \{u \in L^p(\Omega) : \overline{F}(u) < +\infty\}$.

(ii) *For* $u \in \tilde{H}^1(\Omega, L^p(\Omega), w)$ *and* $\overline{v} \in V_u$, *we have*

$$\overline{F}(u) = \min\left\{\int_\Omega |v|^2 \, dx : v \in V_u\right\} = \min\left\{\int_\Omega |\overline{v} + v|^2 \, dx : v \in V_0\right\}.$$

Corollary 2.6. *We consider the case* $X = L^p(\Omega)$. *Assume that* $\frac{1}{w} \in L^1_{\text{loc}}(\Omega \backslash \{x_1, \ldots, x_k\})$. *Then*

(i) $\tilde{H}^1(\Omega, X, w) \subset W^{1,1}_{\text{loc}}(\Omega \setminus \{x_1, \ldots, x_k\})$ *and* $\nabla_{X,w} u = Du$ *a.e. in* Ω*;*

(ii)

$$\overline{F}(u) = \int_\Omega |\nabla_{X,w} u|^2 \, w \, dx \quad \forall u \in \tilde{H}^1(\Omega, X, w).$$

Proof. This follows from Theorem 2.5 and previous arguments. □

When $X = L^2(\Omega, w)$, the space $\tilde{H}^1(\Omega, X, w)$ can be also characterized in the setting of metric measure Sobolev spaces (see, for instance, [4, 5, 17, 25]).

2.2. **Dirichlet forms approach.** In the setting of Dirichlet forms, property (10) can be understood saying that the form a defined by

$$D(a) := W^1(\Omega, L^2(\Omega, w), w) \subset H := L^2(\Omega, w)$$

$$a(u, v) := \int_\Omega DuDv \, w \, dx \quad u, v \in D(a)\,, \tag{11}$$

is closable (see [28, pg. 373], [2, 14, 15]). We recall some notions on the Dirichlet forms (for the general theory we refer to [22]). We fix a positive Radon measure μ on Ω, with $\operatorname{supp} \mu = \Omega$, which is called the "volume" measure on X. A *form* a in H is a non-negative definite symmetric bilinear form $a(u, v)$ defined on a linear subspace $D[a]$, called the domain of a, of the Hilbert space $H = L^2(X, \mu)$, equipped by the scalar product (u, v). It is possible to associate with $a[u, v]$ a quadratic functional

$$F(u) = a(u, u)$$

for every $u \in D[a]$. A form a is *closed* in $H = L^2(X, \mu)$ if its domain $D[a]$ is complete under the intrinsic inner product $a(u, v) + (u, v)$. The following characterization holds: a form a is closed in H if and only if the quadratic functional $F(u)$ is lower semicontinuous on H. Moreover a form a is *closable* in $H = L^2(X, \mu)$ if $(u_n) \subset D[a]$, $a(u_n - u_m, u_n - u_m) \to 0$, $(u_n, u_n) \to 0$, as $n, m \to +\infty$, imply $a(u_n, u_n) \to 0$, as $n \to +\infty$. We have that a form a is closable in $H = L^2(X, \mu)$ if and only if the completion of $D[a]$ under the intrinsic inner product $a(u, v) + (u, v)$ is injected in the space $H = L^2(X, \mu)$. The closure $\overline{a}(u, v)$ of a closable form a is a closed form and it coincides with the *relaxed form* defined by the relaxed functional $\overline{F}(u)$, by using the polarization identity

$$\overline{a}(u,v) = \frac{1}{2}\{\overline{a}(u+v, u+v) - \overline{a}(u,u) - \overline{a}(v,v)\} = \frac{1}{2}\{\overline{F}(u+v) - \overline{F}(u) - \overline{F}(v)\}.$$

Its domain is $D[\overline{a}] = \{u \in H : \overline{F}(u) < +\infty\}$. A form a in H is *Markovian* if for every $u \in D[a]$ the truncated function $v = \inf\{\sup\{u, 0\}, 1\}$ belongs to $D[a]$ and $a(v, v) \leq a(u, u)$. A *Dirichlet form* in H is a closed Markovian form in H. In [12] some suitable doubling condition and Poincaré inequality are considered. In this framework, a very particular case is a weighted Dirichlet form

$$a_w(u, v) = \int_\Omega Du\, Dv\, w dx$$

associated to the integral functional

$$F(u) = \int_\Omega |Du|^2 w dx.$$

It satisfies all the previous assumptions if the $\mu = w\mathcal{L}^n$ and w is a Muckenhoupt weight A_2 or a weight $w(x) = |\det F'|^{1-2/n}$ associated with a quasi-conformal transformation F in $\mathbb{R}^n$. Let us recall that, in the one-dimensional case, the following simple closability criterion was proved by Hamza (see [22, Th. 3.1.6] and [24]): the weighted form (11) is closable in $L^2(\Omega, w)$ if and only if the weight w satisfies the following so-called *Hamza's condition*, i.e.

$$\text{for a.e. } x \in \Omega = (a,b),\ w(x) > 0 \text{ implies that} \tag{12}$$
$$\exists\, \epsilon > 0 \text{ such that } \int_{x-\epsilon}^{x+\epsilon} \frac{1}{w(y)} dy < +\infty\,.$$

Eventually, for the reader's convenience, we recall the following definitions of doubling and A_2-weight.

Definition 2.7. We say that a weight $w \in L^1_{\text{loc}}(\Omega)$ is doubling on Ω if the measure $\mathfrak{m} := w\, dx$ is doubling, that is, there exists a constant $C > 0$ such that

$$\mathfrak{m}(B(x, 2r)) \leq C\mathfrak{m}(B(x, r))$$

for all $x \in \Omega$ and $r > 0$ such that $B(x, 2r) \subseteq \Omega$.

Definition 2.8. We say that a weight $w : \mathbb{R}^n \to [0, +\infty[$ is in the Muckenhoupt class A_2 if $w, \frac{1}{w} \in L^1_{\text{loc}}(\mathbb{R}^n)$ and there exists a constant $C > 0$ such that, for all balls B in $\mathbb{R}^n$, we have

$$\left(\frac{1}{|B|}\int_B w(x)\, dx\right)\left(\frac{1}{|B|}\int_B \frac{1}{w(x)}\, dx\right) \leq C,$$

where $|B|$ denotes the Lebesgue measure of B.

3. The one-dimensional case: previous results

Let w is a weight satisfying (1). Let $\Omega = (a,b)$ be a bounded open interval. We consider the following functionals defined on a topological space (X,τ), where X will be a suitable space of functions endowed with a topology τ.

$$F^1(u) \equiv F^1_X(u) := \begin{cases} \displaystyle\int_a^b |u'|^2\, w\, dx & \text{if } u \in C^1([a,b]) \\ +\infty & \text{if } u \in X \setminus C^1([a,b]) \end{cases} \tag{13}$$

$$F^2(u) \equiv F^2_X(u) := \begin{cases} \displaystyle\int_a^b |u'|^2\, w\, dx & \text{if } u \in \mathrm{Lip}([a,b]) \\ +\infty & \text{if } u \in X \setminus \mathrm{Lip}([a,b]) \end{cases} \tag{14}$$

$$F^3(u) \equiv F^3_X(u) := \begin{cases} \displaystyle\int_a^b |u'|^2\, w\, dx & \text{if } u \in H^1((a,b)) \\ +\infty & \text{if } u \in X \setminus H^1((a,b)) \end{cases} \tag{15}$$

$$F^4(u) \equiv F^4_X(u) := \begin{cases} \displaystyle\int_a^b |u'|^2\, w\, dx & \text{if } u \in AC([a,b]) = W^{1,1}((a,b)) \\ +\infty & \text{if } u \in X \setminus AC([a,b]) \end{cases} \tag{16}$$

and the corresponding lower semicontinuous envelopes w.r.t. the τ-convergence

$$\overline{F^j}(u) = \mathrm{sc}^-(\tau) - F_j(u) \qquad j = 1,2,3,4\,.$$

To our knowledge, in the one-dimensional case, integral representations for relaxed functionals was already provided in [26] and then in [16].

Theorem 3.1. *([26, Theorem 5)]) Let $X = H^1((a,b))$ endowed with the L^p topology with $1 \le p \le \infty$ and assume also that $0 \le w(x) \le c$ for a.e. $x \in (a,b)$ and for a suitable constant $c > 0$. Then*

$$\overline{F^3}(u) = \int_a^b |u'|^2\, \overline{w}\, dx \quad \forall u \in H^1((a,b))\,,$$

where

$$\overline{w}(x) = \lim_{\epsilon \to 0} 2\epsilon \Big[\int_{x-\epsilon}^{x+\epsilon} \frac{1}{w(y)} dy \Big]^{-1}.$$

Let $I \equiv I_{\Omega,w}$ denote the set in (2).

Remark 3.2. We point out the following two particular cases:
(i) If $I = \emptyset$, then $\frac{1}{w} \notin L^1((x-\epsilon, x+\epsilon))$ for every $x \in \Omega$ and for every $\epsilon > 0$. In this case $\overline{w} \equiv 0$ and $\overline{F^3}(u) = 0$ for every $u \in H^1(\Omega)$ and, by (57), even $\overline{F^4}(u) = 0$ for every $u \in H^1([a,b])$.
(ii) If $I = (a,b)$, then $\frac{1}{w} \in L^1_{\mathrm{loc}}((a,b))$; assume also that w satisfies the assumption of Theorem 3.1. We obtain that $w = \overline{w}$ a.e. and $\overline{F^3}(u) = F^3(u)$ for every $u \in H^1([a,b])$. Then, since $H^1([a,b]) \subset AC([a,b])$, as a consequence of Theorem 3.1,

$$\overline{F^4}(u) = F^4(u) \text{ for every } u \in H^1([a,b])\,. \tag{17}$$

We will prove (see Corollary 4.23) that (17) holds for each $u \in AC([a,b])$. In this case, we get the coincidence $w = w^* = \tilde{w}$.

In the one-dimensional case, the following improvement of Theorems 2.5 and 3.1 holds.

Theorem 3.3. *(*[16, Theorem 3.1]*) Let $X = L^p(\Omega)$ with $1 \le p < \infty$, endowed with the L^p- topology.*

(i) *I is the biggest open set in Ω such that $\frac{1}{w}$ is locally sommable;*

(ii)

$$\begin{aligned}\widetilde{H}^1(\Omega, L^p(\Omega), w) &:= \{u \in L^p(\Omega) : \overline{F}^2(u) < +\infty\} \\ &= \left\{u \in L^p(\Omega) \cap W^{1,1}_{\mathrm{loc}}(I) : \int_I |u'|^2\, w\, dx < +\infty\right\} \\ &= L^p(\Omega) \cap D_w\,;\end{aligned}$$

(iii)

$$\overline{F^2}(u) = \int_I |u'|^2\, w\, dx \quad \forall u \in \tilde{H}^1(\Omega, L^p(\Omega), w)\,.$$

Remark 3.4. Theorem 3.3 does not hold in higher dimensions, even though $\frac{1}{w} \in L^1(\Omega)$. Indeed in [18] it is has been showed that, if $n \ge 2$, there exists a weight w for which $\frac{1}{w} \in L^1(\Omega)$ and $\tilde{H}^1(\Omega, X, w) = H^1(\Omega, X, w) \subsetneq W^1(\Omega, X, w) \subset W^{1,1}(\Omega)$.

4. New result in the one-dimensional case

4.1. **Structure of the weight and optimal finiteness domain.** The set $I_{\Omega,w}$ defined in (2) is the biggest open set in (a,b) such that $\frac{1}{w}$ is locally sommable. Then it is well-known, being $I_{\Omega,w}$ an open set of the real line, $I_{\Omega,w}$ can be decomposed in the union of its open connected components, that is there exist a family of disjoint bounded open intervals (a_i, b_i) $i = 1, \ldots, N_w$, with N_w finite, i.e. $N_w \in \mathbb{N}$, or $N_w = \infty$, such that

$$I_{\Omega,w} = \bigcup_{i=1}^{N_w} (a_i, b_i)\,. \tag{18}$$

Notice also that the decomposition in (18) is unique and N_w is also uniquely defined. Moreover

$$\frac{1}{w} \in L^1_{\mathrm{loc}}(I_{\Omega,w})\,.$$

Let us stress the following simple characterization of weights satisfying Hamza's condition (12).

Proposition 4.1. *Let w be a weight on Ω. Then the following are equivalent:*

(i) *w satisfies Hamza's condition* (12)*;*

(ii) *$w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$.*

Moreover, if w is lower semicontinuous a.e. in $\Omega \setminus I_{\Omega,w}$ or Riemann integrable in Ω, then (ii) is satisfied.

Proof. The implication (i) ($\Rightarrow$) (ii) is immediate. Let us show the opposite implication. It is sufficient to show that

$$w(x) > 0 \text{ for a.e. } x \in I_{\Omega,w}\,. \tag{19}$$

By contradiction, assume there is a set $E \subset I_{\Omega,w}$ with $|E| > 0$ and $w(x) = 0$ for each $x \in E$. Then, there exists a point $x_0 \in E$ of density 1, that is

$$\lim_{r\to 0} \frac{|E \cap (x_0 - r, x_0 + r)|}{2r} = 1\,. \tag{20}$$

By (20), it follows that, there exists a small $r_0 > 0$, such that for each $r \in (0, r_0)$,

$$\infty = \int_{E\cap(x_0-r,x_0+r)} \frac{dx}{w} \le \int_{(x_0-r,x_0+r)} \frac{dx}{w}\,.$$

Thus a contradiction, since $x_0 \in I_{\Omega,w}$ and (19) follows. Assume now w is lower semicontinuous at $x \in \Omega \setminus I_{\Omega,w}$, and let prove that $w(x) = 0$. Indeed, by contradiction, if we assume that $w(x) > 0$, since

$$\liminf_{y\to x} w(y) \ge w(x),$$

then there exists $\epsilon > 0$ such that for every $y \in]x - \epsilon, x + \epsilon[$

$$w(y) > \frac{w(x)}{2} =: m.$$

This implies that

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1}{w(y)}\, dy < \frac{2\epsilon}{m} < \infty$$

and this is a contradiction. Moreover, if w is Riemann integrabile in $\Omega = (a, b)$, it is well-known that is continuous a.e. in $x \in (a, b)$, then $w(x) = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$. ☐

Remark 4.2. Note that a weight w in Ω may not satisfy the condition (ii) of Proposition 4.1, even though it is finitely degenerate. Indeed, there exist weights w in $(0, 1)$ with $I_{\Omega,w} = \emptyset$ and $w(x) > 0$ a.e. in $(0, 1)$ (see, for instance, [26, p. 212] or [20, p. 92]). Note that, if we extend such a weight as 1 in $(-1, 0]$, we obtain a finitely degenerate weight in $(-1, 1)$ which do not satisfy the condition (ii) of Proposition 4.1.

Remark 4.3. For each finite measure μ in Ω, if $N_w = \infty$, then $\lim_{i\to+\infty} \mu((a_i, b_i)) = 0$. Indeed, in this case, $\sum_{i=1}^{+\infty} \mu((a_i, b_i)) \le \mu((a, b)) < +\infty$.

If $I_{\Omega,w} \neq \emptyset$, let D_w denote the class defined in (4).
If $I_{\Omega,w} = \emptyset$ let us define $D_w := \{0\}$.

Remark 4.4. We note that, if $\frac{1}{w} \in L^1(\Omega)$, then, obviously, w is finitely degenerate in Ω with $N_w = 1$. In this case

$$D_w = \{u \in AC([a, b]) : \int_a^b |u'|^2\, w\, dx < +\infty\}$$

(since $I_{\Omega,w} = \Omega = (a, b)$ and $AC([a, b]) = W^{1,1}((a, b))$).

Theorems 3.1 and 3.3 (see also Remark 3.2) suggest that D_w contains the finiteness domain of a relaxed functional, when $X = L^2(\Omega, \mu)$ and μ is a finite Borel measure on Ω with its support sptμ containing $I_{\Omega,w}$. The lemma below confirms this suggestion.

Lemma 4.5 (Optimal finiteness domain)**.** *Let* $(u_h)_h \subset AC([a,b])$ *such that*

(a) $\sup_{h\in\mathbb{N}} \int_{I_{\Omega,w}} |u_h'|^2 w\,dx < +\infty$,

(b) *for every* $i = 1,\dots,N_w$ *there exists* c_i *such that* $a_i < c_i < b_i$ *and there exist finite the following limits*

$$\lim_{h\to+\infty} u_h(c_i) = d_i \in \mathbb{R}\,.$$

Then there exists a subsequence (u_{h_k}) *and a function* $u : I_{\Omega,w} \to \mathbb{R}$ *such that*

(i) $\lim_{k\to+\infty} u_{h_k}(x) = u(x)$ *for every* $x \in I_{\Omega,w}$,

(ii) $u \in D_w$,

(iii) $\int_{I_{\Omega,w}} |u'|^2\, w\,dx \le \liminf_{k\to+\infty} \int_{I_{\Omega,w}} |u_{h_k}'|^2\, w\,dx\,.$

Proof. Let us note that, by assumption (b), $I_{\Omega,w} \neq \emptyset$. By (a), there exist a subsequence $(u_{h_k})_k$ of $(u_h)_h$, and a function $v \in L^2(I_{\Omega,w}, w)$ such that

$$u_{h_k}' \to v \;\text{ weakly in } L^2(I_{\Omega,w}, w) \text{ as } k\to\infty\,. \tag{21}$$

Moreover, since $\frac{1}{w} \in L^1_{\text{loc}}(I_{\Omega,w})$ we have that

$$L^2_{\text{loc}}(I_{\Omega,w}, w) \subset L^1_{\text{loc}}(I_{\Omega,w})\,. \tag{22}$$

In particular, from (21) and (22), we get that $v \in L^1_{\text{loc}}(I_{\Omega,w})$ and

$$\int_\alpha^\beta u_{h_k}'\,dx \to \int_\alpha^\beta v\,dx \text{ as } k\to\infty\,, \tag{23}$$

for each $[\alpha,\beta] \subset I_{\Omega,w}$. Let us consider $u : \Omega \to \mathbb{R}$ defined in the following way: firstly for every $i = 1,\dots,N_w$

$$u^i(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus (a_i,b_i) \\ d_i + \displaystyle\int_{c_i}^x v(y)dy & \text{if } a_i < x < b_i. \end{cases}$$

Then we define

$$u(x) = \sum_{i=1}^{N_w} u^i(x)\chi_{(a_i,b_i)}(x).$$

By definition,

$$u \in W^{1,1}_{\text{loc}}(I_{\Omega,w}) \text{ and } u' = v \text{ a.e. in } I_{\Omega,w}\,.$$

For every $i = 1,\dots,N_w$,

$$u_{h_k}(x) = u_{h_k}(c_i) + \int_{c_i}^x {u'}_{h_k}(y)dy \quad \text{if } a_i < x < b_i\,.$$

By (b) and (23), taking the limit as $k\to\infty$ in the previous equality, condition (i) follows. Condition (ii) is immediate by the definition of u. Eventually, by (21) and the lower semicontinuity of the norm w.r.t. the weak convergence, (iii) is achieved. □

4.2. **Auxiliary weights.** Let $w : \Omega = (a,b) \to [0,\infty)$ be a weight, that is a function satisfying (1) and (18). Let $\tilde{w}, w^* : \Omega \to [0,+\infty[$ be defined as

$$w^*(x) := \begin{cases} \lim_{x\to a_i^+} \left(\int_x^{\frac{a_i+b_i}{2}} \frac{1}{w(y)}\, dy \right)^{-1} & \text{if } x = a_i \\ \left(\int_x^{\frac{a_i+b_i}{2}} \frac{1}{w(y)}\, dy \right)^{-1} & \text{if } a_i < x \le \frac{3a_i+b_i}{4} \\ \left(\int_{\frac{3a_i+b_i}{4}}^{\frac{a_i+3b_i}{4}} \frac{1}{w(y)}\, dy \right)^{-1} & \text{if } \frac{3a_i+b_i}{4} \le x \le \frac{a_i+3b_i}{4} \\ \left(\int_{\frac{a_i+b_i}{2}}^{x} \frac{1}{w(y)}\, dy \right)^{-1} & \text{if } \frac{a_i+3b_i}{4} \le x < b_i \\ \lim_{x\to b_i^-} \left(\int_{\frac{a_i+b_i}{2}}^{x} \frac{1}{w(y)}\, dy \right)^{-1} & \text{if } x = b_i \\ 0 & \text{if } x \in \Omega \setminus I_{\Omega,w}\,, \end{cases} \tag{24}$$

and

$$\tilde{w}(x) := \min\{w(x),\, w^*(x),\, 1\} \tag{25}$$

if $x \in (a,b)$ is a Lebesgue's point of w at x and 0 otherwise. Let us collect some properties of functions w^* and $\tilde{w}$ in the following proposition, whose proof is elementary taking the definitions into account.

Proposition 4.6 (Properties of w^* and $\tilde{w}$)**.**

(i) *If $\frac{1}{w}$ is not locally summable in Ω, i.e. $I_{\Omega,w} = \emptyset$, then $w^* = \tilde{w} \equiv 0$.*

(ii) *$\tilde{w} \in L^\infty(\Omega)$ and*

$$L^2(\Omega, w^*) \cup L^2(\Omega, w) \cup L^2(\Omega) \subset L^2(\Omega, \tilde{w})\,. \tag{26}$$

Moreover the inclusion of each space $L^2(\Omega,\mu)$ ($\mu = w^*\,dx, w\,dx,\, dx$) in $L^2(\Omega,\tilde{w})$ is continuous. In particular, the measure $\widetilde{\mathfrak{m}} = \tilde{w}\,dx$ is finite in Ω.

(iii) *For each $i = 1,\dots,N_w$, w^* is constant in $[\frac{3a_i+b_i}{4}, \frac{a_i+3b_i}{4}]$, increasing in $[a_i, \frac{3a_i+b_i}{4}]$, decreasing in $[\frac{a_i+3b_i}{4}, b_i]$ and absolutely continuous in each interval. In particular, it holds that*

$$0 < w^*(x) \le \sup_{y\in(a_i,b_i)} w^*(y) < \infty \quad \forall\, x \in (a_i,b_i)\,,$$

$$\inf_{x\in[\alpha,\beta]} w^*(x) > 0 \text{ for each } x \in [\alpha,\beta],\ a_i < \alpha < \beta < b_i,$$

and $w^*(a_i) = 0$ (respectively $w^*(b_i) = 0$) if and only if $\frac{1}{w} \notin L^1(a_i, \frac{a_i+b_i}{2})$ (respectively $\frac{1}{w} \notin L^1(\frac{a_i+b_i}{2}, b_i)$). Moreover

$$(w^*)' = \frac{(w^*)^2}{w} \quad \text{a.e. in } \left(a_i, \frac{3a_i+b_i}{4}\right) \cup \left(\frac{a_i+3b_i}{4}, b_i\right)\,.$$

(iv) *If $\frac{1}{w} \in L^1(\Omega)$, then there exists a constant $c > 0$ such that*

$$0 < \frac{1}{c} \le w^*(x) \le c \quad \text{a.e. } x \in \Omega\,.$$

(v) *If w is finitely degenerate in Ω, i.e.* (18) *holds with* $1 \le N_w < \infty$*, then there exists a constant* $c > 0$ *such that*

$$0 \le w^*(x) \le c \qquad a.e.\ x \in \Omega\,.$$

In particular, the measure $\mathfrak{m}^* := w^* dx$ *is finite in* Ω*.*

(vi) *If w is not finitely degenerate in Ω, i.e.* (18) *holds with* $N_w = \infty$*, then* $w^* \in L^\infty_{\rm loc}(I_{\Omega,w})$*. In particular, the measure* $\mathfrak{m}^* = w^* dx$ *is* σ*-finite in* Ω*.*

Example 4.7. If w is not finitely degenerate in Ω, then it can occur that $w^* \notin L^1(\Omega)$ as we will show later. On the contrary, $\tilde{w} \in L^\infty(\Omega)$ and the associated space $L^2(\Omega, \tilde{w})$ contains the main spaces of regular functions we will deal with, as AC, Lip, H^1 and C^1. Notice also that $\tilde{w}$ turns out to be a weight according to (1). Let us consider the following example. Let (a_i, b_i), $i : 1, \dots, \infty$, be a sequence of disjoint open intervals in $(0,1)$ and $(m_i)_i$ be a sequence of positive real numbers to be fixed later. Let $w : (0,1) \to [0, \infty[$ defined as follows

$$w(x) := \begin{cases} m_i(x-a_i)^\alpha & \text{if } a_i \le x \le \frac{a_i+b_i}{2} \\ m_i(b_i - x)^\alpha & \text{if } \frac{a_i+b_i}{2} \le x \le b_i \\ 0 & \text{outside,} \end{cases}$$

where $\alpha > 0$, $\alpha \ne 1$. It is immediate to see that w is not finitely degenerate if $\alpha > 1$, i.e. $N_w = \infty$, and $I_{\Omega,w} = \cup_{i=1}^{+\infty}(a_i, b_i)$. Let us fix $a_i \le x \le \frac{3a_i+b_i}{4}$, then, by definition of w^* we have

$$w^*(x) = \frac{(\alpha-1)m_i(x-a_i)^{\alpha-1}}{1 - \left(\frac{2(x-a_i)}{b_i-a_i}\right)^{\alpha-1}}.$$

Now, since

$$0 \le \frac{2(x-a_i)}{b_i - a_i} \le \frac{1}{2},$$

then

$$(\alpha-1)m_i(x-a_i)^{\alpha-1} \le w^*(x) \le \frac{(\alpha-1)m_i(x-a_i)^{\alpha-1}}{1-\left(\frac{1}{2}\right)^{\alpha-1}},$$

that is

$$w^*(x) \,\tilde{=}\, m_i(x-a_i)^{\alpha-1}, \qquad a_i \le x \le \frac{3a_i+b_i}{4}.$$

It is easy to see that

$$\int_{a_i}^{\frac{3a_i+b_i}{4}} w^*(x)\,dx \,\tilde{=}\, m_i(b_i-a_i)^\alpha$$

then, if we choose the sequence m_i such that

$$\sum_{i=1}^{+\infty} m_i(b_i-a_i)^\alpha = +\infty,$$

we can conclude that $w^* \notin L^1(\Omega)$.

Remark 4.8. We note that w^* is Lipschitz continuous in interval $[c,d] \subset (a_i, \frac{3a_i+b_i}{4})$ where it is nondecreasing and for every $x \in [c,d]$

$$|(w^*)'| \le \frac{(w^*(d))^2}{w(c)}\,.$$

The same condition holds for every $[c,d] \subset (\frac{a_i+3b_i}{4}, b_i)$ where w is nonincreasing.

4.3. **Poincaré-type inequalities.** Firstly, we prove some preliminary lemmas.

Proposition 4.9. *We fix $u \in D_w$ and $i = 1, \dots, N_w$. For every η, x such that $a_i < \eta \leq x \leq \frac{a_i+b_i}{2}$ we have:*

$$|u(x) - u(\eta)| \sqrt{w^*(\eta)} \leq \left(\int_\eta^x |u'(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} ; \tag{27}$$

$$|u(\eta)|^2 w^*(\eta) \leq 2|u(x)|^2 w^*(\eta) + 2 \int_{a_i}^x |u'(y)|^2 w(y)\, dy \,. \tag{28}$$

For every η, x such that $\frac{a_i+b_i}{2} \leq x \leq \eta < b_i$ we have:

$$|u(x) - u(\eta)| \sqrt{w^*(\eta)} \leq \left(\int_x^\eta |u'(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} ; \tag{29}$$

$$|u(\eta)|^2 w^*(\eta) \leq 2|u(x)|^2 w^*(\eta) + 2 \int_x^{b_i} |u'(y)|^2 w(y)\, dy \,. \tag{30}$$

Proof. Since $u \in AC_{\text{loc}}(a_i, b_i)$, for every $x \in]a_i, \frac{a_i+b_i}{2}]$ such that $a_i < \eta \leq x \leq \frac{a_i+b_i}{2}$ we have

$$\begin{aligned} |u(x) - u(\eta)| &= \left| \int_\eta^x u'(y)\, dy \right| \leq \left(\int_\eta^x |u'(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} \left(\int_\eta^x \frac{1}{w}(y)\, dy \right)^{\frac{1}{2}} \\ &\leq \left(\int_\eta^x |u'(y)|^2 w(y)\, dy \right)^{\frac{1}{2}} \left(\int_\eta^{\frac{a_i+b_i}{2}} \frac{1}{w}(y)\, dy \right)^{\frac{1}{2}} . \end{aligned} \tag{31}$$

Observe now that, if $a_i < \eta \leq \min\{\frac{3a_i+b_i}{4}, x\}$, then (27) follows by (31) and the definition of w^*; if $\frac{3a_i+b_i}{4} \leq \eta \leq x \leq \frac{a_i+b_i}{2}$, since

$$\left(\int_\eta^{\frac{a_i+b_i}{2}} \frac{1}{w}(y)\, dy \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{w^*(\eta)}} \,,$$

(27) still follows by (31) and the definition of w^*. Then, since

$$|u(\eta)|^2 \leq 2|u(x)|^2 + 2|u(\eta) - u(x)|^2 \,,$$

by (27), (28) follows. Similarly, (29) and (30) can be obtained. □

By Proposition 4.9, we can study the behaviour of functions in D_w near the end points a_i, b_i, $i = 1, \dots, N_w$.

Corollary 4.10. *Let $u \in D_w$ and fix $i = 1, \dots, N_w$.*

(i) $|u(\eta)|^2 w^*(\eta) \leq 2 \left| u\left(\frac{a_i+b_i}{2} \right) \right|^2 w^*(b_i) + 2 \int_{a_i}^{b_i} |u'(y)|^2 w(y)\, dy$, *for each $\eta \in (a_i, b_i)$. In particular $u \in L^2((a_i, b_i), w^*)$ and in the finitely degenerate case $u \in L^2(\Omega, w^*)$.*

(ii) *If* $\int_{a_i}^{\frac{a_i+b_i}{2}} \frac{1}{w} dx = +\infty$ *(respectively if* $\int_{\frac{a_i+b_i}{2}}^{b_i} \frac{1}{w} dx = +\infty$*) there exists* $\lim_{x \to a_i^+} u^2\, w^* = 0$ *(respectively* $\lim_{x \to b_i^-} u^2\, w^* = 0$*).*

(iii) *If* $\int_{a_i}^{\frac{a_i+b_i}{2}} \frac{1}{w} dx < \infty$ *(respectively if* $\int_{\frac{a_i+b_i}{2}}^{b_i} \frac{1}{w} dx < \infty$*), then*

$$u \in AC\Big(\big[a_i, \frac{a_i+b_i}{2}\big]\Big) \text{ (respectively } u \in AC\Big(\big[\frac{a_i+b_i}{2}, b_i\big]\Big).$$

Proof. (i) From (28) and (30) with $x = \frac{a_i+b_i}{2}$, we get that desired inequality.

(ii) Let $a_i < \eta \le x \le \frac{a_i+b_i}{2}$. By the hypothesis $\int_{a_i}^{\frac{a_i+b_i}{2}} \frac{1}{w} dx = +\infty$ and by definition of w^*, we have $\lim_{\eta \to a_i^+} w^*(\eta) = 0$. For fixed $x \in (a_i, \frac{a_i+b_i}{2})$ by (28) we have the following inequality

$$\limsup_{\eta \to a_i^+} |u(\eta)|^2 w^*(\eta) \le 2 \int_{a_i}^{x} |u'(y)|^2 w\, dy.$$

Taking the lim as $x \to a_i^+$ in the previous inequality, we get that

$$\lim_{\eta \to a_i^+} |u(\eta)|^2 w^*(\eta) = 0\,.$$

Respectively, if we assume $\int_{\frac{a_i+b_i}{2}}^{b_i} \frac{1}{w} dx = +\infty$, we have

$$\lim_{\eta \to b_i^-} |u(\eta)|^2 w^*(\eta) = 0\,.$$

(iii) Since $u \in AC([a_i + \delta, \frac{a_i+b_i}{2}])$, for each $\delta > 0$, in order to prove $u \in AC\big(\big[a_i, \frac{a_i+b_i}{2}\big]\big)$ it is sufficient to prove that there exists the following limit

$$\lim_{\eta \to a_i^+} u(\eta) \in \mathbb{R}. \tag{32}$$

Observe now that

$$u' \in L^1\left(a_i, \frac{a_i+b_i}{2}\right), \tag{33}$$

since

$$u' = u'\sqrt{w}\,\frac{1}{\sqrt{w}}$$

and $u'\sqrt{w}, \frac{1}{\sqrt{w}} \in L^2(a_i, \frac{a_i+b_i}{2})$.

Now, by the fundamental theorem of Calculus for every $\eta \in (a_i, \frac{a_i+b_i}{2}]$

$$u(\eta) = u\Big(\frac{a_i+b_i}{2}\Big) - \int_{\eta}^{\frac{a_i+b_i}{2}} u'(x)dx. \tag{34}$$

Thus, by (33) and (34), (32) follows. The other case is analogous. □

Theorem 4.11 (Poincaré type inequality on D_w)**.** *The following Poincaré type inequality holds: for every $u \in D_w$*

$$
\begin{aligned}
(35)\quad \sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 \tilde{w}(\eta)\, d\eta &\le \sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta \\
&\le \int_{I_{\Omega,w}} |u'(y)|^2 w(y)\, dy\,.
\end{aligned}
$$

Proof. The first inequality

$$\sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 \tilde{w}(\eta)\, d\eta \le \sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta$$

immediately follows since $\tilde{w} \le w^*$ on Ω. Let us show the second inequality. In (27) we take $x = \frac{a_i+b_i}{2}$, then

$$\left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta) \le \int_{a_i}^{\frac{a_i+b_i}{2}} |u'(y)|^2 w(y)\, dy.$$

By integrating w.r.t. to η we obtain

$$\int_{a_i}^{\frac{a_i+b_i}{2}} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta \le \frac{b_i-a_i}{2} \int_{a_i}^{\frac{a_i+b_i}{2}} |u'(y)|^2 w(y)\, dy.$$

Similarly we have

$$\int_{\frac{a_i+b_i}{2}}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta \le \frac{b_i-a_i}{2} \int_{\frac{a_i+b_i}{2}}^{b_i} |u'(y)|^2 w(y)\, dy.$$

Therefore

$$\int_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta \le (b_i-a_i) \int_{a_i}^{b_i} |u'(y)|^2 w(y)\, dy.$$

Hence

$$\fint_{a_i}^{b_i} \left| u(\eta) - u\left(\frac{a_i+b_i}{2}\right)\right|^2 w^*(\eta)\, d\eta \le \int_{a_i}^{b_i} |u'(y)|^2 w(y)\, dy.$$

The conclusion follows since $u \in D_w$ and so

$$\sum_{i=1}^{+\infty} \int_{a_i}^{b_i} |u'(y)|^2 w(y)\, dy \le \int_{I_{\Omega,w}} |u'(y)|^2 w(y)\, dy < +\infty.$$

□

Remark 4.12. Notice that, if $w(x) = |x|$, $\Omega = (-1, 1)$, then the doubling property holds for the measure $m = w\, dx$, but the Poincaré inequality does hold. Indeed there is an interesting characterization in [13] which provides that the Poincaré inequality holds if and only if w belongs to the Muckenhoupt class A_2, and it is well known that w is not in A_2.

4.4. **Convergence in measure.** We will consider two types of ambient spaces for the relaxation: the space $L^0(\Omega)$ endowed with the topology induced from the convergence in measure and the space $L^2(\Omega, \tilde{w})$.

Note that the measure $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$ in (6) are always finite on Ω , while $\mathfrak{m}^*$ is finite if w is a finitely degenerate and σ-finite in the general case (see Proposition 4.6). We are going to study the absolute continuity relationships bewteen $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$. It is easy to see that, in the general case $\mathfrak{m}$ may not be absolutely continuous w.r.t. $\widetilde{\mathfrak{m}}$, even though w is finitely degenerate (see Remark 4.2) . However if w satisfies Hamza's condition (12), then $\mathfrak{m}$ is absolutely continuous w.r.t. $\widetilde{\mathfrak{m}}$. The reverse relationship always turns out to be true.

Theorem 4.13. (i) $\widetilde{\mathfrak{m}} \ll \mathfrak{m}$ *in* Ω*;*
(ii) *if* $w = 0$ *a.e. in* $\Omega \setminus I_{\Omega,w}$*, then* $\mathfrak{m} \ll \widetilde{\mathfrak{m}}$ *in* Ω*.*

Proof. (i) It is immediate since, by definition of $\tilde{w}$ (see (25)), $\widetilde{\mathfrak{m}} \leq \mathfrak{m}$ on the class of measurable sets in Ω.
(ii) Let us show that $\mathfrak{m} \ll \widetilde{\mathfrak{m}}$ in Ω. Let $E \subset \Omega$ be mesaurable such $\widetilde{\mathfrak{m}}(E) = 0$. Then we can decompose E as

$$E = (E \cap (\Omega \setminus I_{\Omega,w})) \cup (E \cap I_{\Omega,w}) = E_1 \cup E_2 \,.$$

In particular, it follows that

$$\widetilde{\mathfrak{m}}(E_2) := \int_{E_2} \tilde{w}\, dx = 0\,. \tag{36}$$

From (19) and Proposition 4.6 (iii), it follows that $w(x) > 0$ and $w^*(x) > 0$, for a.e. $x \in I_{\Omega,w}$, respectively. Thus $\tilde{w}(x) > 0$, for a.e. $x \in I_{\Omega,w}$ and, by (36), we get $|E_2| = 0$, as well $\mathfrak{m}(E_2) = 0$. Therefore, since $w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$,

$$\mathfrak{m}(E) = \mathfrak{m}(E_1 \cup E_2) = \mathfrak{m}(E_1) + \mathfrak{m}(E_2) = 0\,,$$

and we are done. □

Let $L^0(\Omega)$ be the space defined in (5). Given a measure μ on Lebesgue measurable sets of Ω, we identify, as usual, two function $u, v \in L^0(\Omega)$ such that $u = v$ μ-a.e. in Ω. A natural convergence on $L^0(\Omega)$ is the *convergence in measure* μ. Let us recall that a sequence of functions $(u_h)_h \subset L^0(\Omega)$ is said to converge in measure μ to a function $u \in L^0(\Omega)$, written $u = \mu - \lim_{h\to\infty} u_h$ if

$$\lim_{h\to\infty} \mu\left(\{x \in \Omega : |u_h(x) - u(x)| > \epsilon\}\right) = 0 \quad \text{ for each } \epsilon > 0\,. \tag{37}$$

Let us collect in the following theorem some main properties of the convergence in measure we will need later.

Theorem 4.14. *Let* $(u_h)_h$ *and* u *be in* $L^0(\Omega)$*, and let* μ *be a measure on the* σ*-algebra of Lebesgue measurable subsets of* Ω*.*

(i) *If* μ *is finite and* $u_h \to u$ μ*-a.e. in* Ω *as* $h \to \infty$*, then* $u = \mu - \lim_{h\to\infty} u_h$*.*
(ii) *If* $u = \mu - \lim_{h\to\infty} u_h$*, there is a subsequence* $(u_{h_k})_k$ *such that* $u_{h_k} \to u$ μ*-a.e. in* Ω *as* $k \to \infty$*.*
(iii) *If* $(u_h)_h$ *and* u *are in* $L^p(\Omega, \mu)$*, with* $1 \leq p \leq \infty$*, and* $\lim_{h\to\infty} \|u_h - u\|_{L^p(\Omega,\mu)} = 0$*, then* $u = \mu - \lim_{h\to\infty} u_h$*.*

(iv) *Suppose that μ is finite and let*

$$d_\mu(u,v) := \int_\Omega \frac{|u-v|}{1+|u-v|}\, d\mu \text{ if } u,\, v \in L^0(\Omega)\,. \tag{38}$$

Then d_μ is a metric on $L^0(\Omega)$ and

$$\lim_{h\to+\infty} d_\mu(u_h,u) = 0 \iff u = \mu - \lim_{h\to\infty} u_h$$

Proof. See, for instance: (i) [19, Proposition 3.1.1]; (ii) [19, Proposition 3.1.2]; (iii) [19, Proposition 3.1.4]; (iv) [19, Chap. 3, Sect. 2, Exercise 5]. □

Let us now study the relationships between the convergence in measure $\mathfrak{m}$ and $\widetilde{\mathfrak{m}}$, as well as if they imply, up to a subsequence, the pointwise convergence in some points of $I_{\Omega,w}$.

Proposition 4.15. *Let $(u_h)_h$ and u be in $L^0(\Omega)$.*

(i) *Assume that $u = \mathfrak{m} - \lim_{h\to\infty} u_h$ (or $u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h$). Then there exists a subsequence $(u_{h_k})_k$ and a sequence of points $(c_i)_i$ such that*

$$c_i \in (a_i,b_i) \text{ and } \lim_{k\to\infty} u_{h_k}(c_i) = u(c_i) \text{ for every } i\,.$$

(ii) *Assume that $u = \mathfrak{m} - \lim_{h\to\infty} u_h$. Then it also holds that $u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h$.*

(iii) *Assume $w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$ and $u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h$. Then it also holds that $u = \mathfrak{m} - \lim_{h\to\infty} u_h$.*

Proof. (i) Suppose first that $u = \mathfrak{m} - \lim_{h\to\infty} u_h$. Then, from Theorem 4.14 (ii) with $\mu = \mathfrak{m}$, there exists a subsequence $(u_{h_k})_k$ and a $\mathfrak{m}$-null set $Z \subset \Omega$ such that

$$\lim_{k\to\infty} u_{h_k}(x) = u(x) \quad \forall\, x \in \Omega \setminus Z\,. \tag{39}$$

By contradiction, if $(a_i,b_i) \subset Z$ for some i, then $\mathfrak{m}((a_i,b_i)) = 0$. This would imply that $(a_i,b_i) \subset \Omega \setminus I_{\Omega,w}$ and then a contradiction. Thus

$$(a_i,b_i) \setminus Z \neq \emptyset \text{ for each } i = 1,2,\ldots, \tag{40}$$

and we get the desired conclusion. Suppose now that $u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h$. Then, still from Theorem 4.14 (ii) with $\mu = \widetilde{\mathfrak{m}}$, there is now $\widetilde{\mathfrak{m}}$- null set $Z \subset \Omega$ such that (39) holds. From Proposition 4.6 (ii), $\widetilde{\mathfrak{m}}((a_i,b_i)) > 0$ for each i. Therefore (40) holds. Thus we still get the desired conclusion.

(ii) From Theorem 4.13 (i), and since $\widetilde{\mathfrak{m}}$ is finite in Ω, by applying the Radon-Nikodym Theorem, there exists $f \in L^1(\Omega,\mathfrak{m}) = L^1(\Omega,w)$ such that

$$\widetilde{\mathfrak{m}}(E) = \int_E f\, d\mathfrak{m} \text{ for each measurable set } E \subset \Omega\,. \tag{41}$$

For given $\epsilon > 0$ let

$$E_h := \{x \in \Omega :\, |u(x) - u_h(x)| > \epsilon\}\ ,$$

then, since $\lim_{h\to\infty} \mathfrak{m}(E_h) = 0$, by (41) and the absolute continuity of the integral, we also get that $\lim_{h\to\infty} \widetilde{\mathfrak{m}}(E_h) = 0$.

(iii) From Theorem 4.13 (ii), and since $\widetilde{\mathfrak{m}}$ is finite in Ω, by applying the Radon-Nikodym Theorem, there exists $g : \Omega \to [0, \infty]$ such that

$$\mathfrak{m}(E) = \int_E g \, d\widetilde{\mathfrak{m}} \text{ for each measurable set } E \subset \Omega \,. \tag{42}$$

Since $\mathfrak{m}(\Omega) < \infty$, by (42), it follows that $g \in L^1(\Omega, \widetilde{\mathfrak{m}}) = L^1(\Omega, \tilde{w})$. Then, arguing as in (ii), we get the desired conclusion. $\square$

Remark 4.16. Note that, by assuming only that the weight w is finitely degenerate, the convergence in measure $\widetilde{\mathfrak{m}} = \tilde{w}\,dx$ does not imply the one in measure $\mathfrak{m} = w\,dx$. For instance, let $w : \Omega = (-1,1) \to [0,\infty]$ be the weight in Remark 4.2, $u_h := \begin{cases} 1 & \text{in } (-1,0] \\ h & \text{in } (0,1) \end{cases}$ $(h = 1, 2, \dots)$ and $u := \begin{cases} 1 & \text{in } (-1,0] \\ 0 & \text{in } (0,1) \end{cases}$. Then, it is easy to see that $u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h$, but the sequence $(u_h)_h$ cannot converge to u w.r.t. the convergence in measure $\mathfrak{m}$.

Remark 4.17. Note that each $L^p(\Omega, \mu)$, with $1 \le p \le \infty$, can be meant as a subspace of $L^0(\Omega)$. Indeed, if $u : \Omega \to \bar{\mathbb{R}}$ is a function in $L^p(\Omega, \mu)$ and $Z_u := \{x \in \Omega : |u(x)| = \infty\}$, then $|Z_u| = 0$. If $\tilde{u} : \Omega \to \mathbb{R}$ is defined as $\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \setminus Z_u \\ 0 & \text{if } x \in Z_u \end{cases}$, then $\tilde{u} \in L^0(\Omega)$. Moreover, if μ is finite, the map

$$(L^p(\Omega,\mu), \|\cdot\|_{L^p(\Omega,\mu)}) \ni u \mapsto \tilde{u} \in (L^0(\Omega), d_\mu)$$

is also continuous, by Theorem 4.14 (iii) and (iv).

4.5. **Relaxation results.** First we consider $X = (L^0(\Omega), d_{\widetilde{\mathfrak{m}}})$ and $(L^0(\Omega), d_{\mathfrak{m}})$ and the lower semicontinuous envelopes in (7).

Theorem 4.18. *Let w be a weight satisfying* (1)*.*

(i) *Then*

$$\widetilde{D^2} = D_w \tag{43}$$

and the representation (8) *holds for the relaxed functional $\widetilde{F^2}$.*

(ii) *If $w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$, then*

$$\widetilde{F^2} = \widehat{F^2} \text{ on } L^0(\Omega)\,.$$

Proof. (i) Firstly, we note that if $I_{\Omega,w} = \emptyset$, then $\tilde{w} \equiv 0$. This implies that $(L^0(\Omega), d_{\widetilde{\mathfrak{m}}}) = \{0\}$, $\widetilde{D^j} = \{0\}$ and $\widetilde{F^j}(u) = 0$ for each $u \in L^0(\Omega)$ j= 1, 2, 3, 4. Let us show (8). By Proposition 4.15 (i) and Lemma 4.5 , it follows that $\widetilde{D^2} \subseteq D_w$ and, by Proposition 4.9, we have that and for every $u \in \widetilde{D^2}$

$$u \in W^{1,1}_{\mathrm{loc}}(I_{\Omega,w}) \cap L^2(I_{\Omega,w}, w^*), \quad u^2 w^* \in L^\infty(I_{\Omega,w})\,.$$

Let us first show that for every $u \in L^0(\Omega)$

$$\int_{I_{\Omega,w}} |u'|^2 \, w \, dx \le \widetilde{F^2}(u).$$

Without loss of generality we can assume that $\widetilde{F^2}(u) < +\infty$. Therefore there exists a sequence $(u_h) \subset D_w$ such that $\lim_{h\to\infty} d_{\widetilde{\mathfrak{m}}}(u_h, u) = 0$ and

$$\widetilde{F^2}(u) = \lim_{h\to+\infty} F^2(u_h) = \lim_{h\to+\infty} \int_\Omega |u_h'|^2 \, w \, dx.$$

Again, we can apply Proposition 4.15 (i) and Lemma 4.5 and, up to a subsequence, we get

$$\int_{I_{\Omega,w}} |u'|^2 \, w \, dx \le \liminf_{h\to+\infty} \int_\Omega |u_h'|^2 \, w \, dx = \lim_{h\to+\infty} \int_\Omega |u_h'|^2 \, w \, dx = \widetilde{F^2}(u)$$

In order to complete the proof we have to prove that

$$\widetilde{F^2}(u) \le \int_{I_{\Omega,w}} |u'|^2 \, w \, dx \quad \forall u \in D_w \tag{44}$$

and so $D_w \subseteq \widetilde{D^2}$. Let us first prove that

$$\widetilde{F^2}(u) \le \int_{I_{\Omega,w}} |u'|^2 \, w \, dx \quad \forall u \in D_w \cap L^2(\Omega)\,. \tag{45}$$

By Theorem 3.3, for each $u \in D_w \cap L^2(\Omega)$, there exists $(u_h)_h \subset Lip(\Omega)$ such that

$$u_h \to u \quad \text{in } L^2(\Omega) \text{ as } h \to \infty\,, \tag{46}$$

and

$$\lim_{h\to\infty} F^2(u_h) = \int_{I_{\Omega,w}} |u'|^2 \, w \, dx\,. \tag{47}$$

By (26) and (46), it follows that

$$u_h \to u \quad \text{in } L^2(\Omega, \tilde{w}) \text{ as } h \to \infty\,. \tag{48}$$

Moreover, from Theorem 4.14 (iii) with $\mu = \tilde{w}\, dx$, (48) implies that

$$u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} u_h\,. \tag{49}$$

Thus, by (47), (49) and the definition of $\widetilde{F^2}$,

$$\widetilde{F^2}(u) \le \liminf_{h\to\infty} F^2(u_h) = \lim_{h\to\infty} F^2(u_h) = \int_{I_{\Omega,w}} |u'|^2 \, w \, dx\,,$$

and (45) follows. It is sufficient in order to complete the proof that, for each $u \in D_w$, there exists $(\tilde{u}_h)_h \subset D_w \cap L^2(\Omega)$ such that

$$u = \widetilde{\mathfrak{m}} - \lim_{h\to\infty} \tilde{u}_h\,, \tag{50}$$

and

$$\tilde{u}_h' \to u' \quad \text{in } L^2(I_{\Omega,w}, w) \text{ as } h \to \infty\,. \tag{51}$$

Indeed, from (45),(50) and (51) and the semicontinuity of $\widetilde{F^2}$, it will follow that

$$\widetilde{F^2}(u) \le \liminf_{h\to\infty} \widetilde{F^2}(\tilde{u}_h) \le \lim_{h\to\infty} \int_{I_{\Omega,w}} |\tilde{u}_h'|^2 \, w \, dx = \int_{I_{\Omega,w}} |u'|^2 \, w \, dx\,,$$

and we will get (44). Eventually let us show (51) and assume that $N_w = \infty$. The case $N_w < \infty$ follows by slight changes. Since $u' \in L^2(I_{\Omega,w}, w)$, by a classical result of measure theory, there exists a sequence of functions $(v_h)_h \subset C^0_c(I_{\Omega,w}) \subset L^2(I_{\Omega,w}, w)$ such that

$$\|v_h - u'\|^2_{L^2(I_{\Omega,w},w)} = \sum_{i=1}^{+\infty} \int_{a_i}^{b_i} |v_h - u'|^2 \, w \, dx \to 0 \text{ as } h \to +\infty \,. \tag{52}$$

Let us define, for given $h \in \mathbb{N}$, $\tilde{u}_h^{(i)} : (a_i, b_i) \to \mathbb{R}$, $i = 1, 2, \ldots, h$ as

$$\tilde{u}_h^{(i)}(x) := u\left(\frac{a_i + b_i}{2}\right) - \int_x^{\frac{a_i+b_i}{2}} v_h(y)\, dy \,, \qquad x \in (a_i, b_i). \tag{53}$$

and $\tilde{u}_h : (a, b) \to \mathbb{R}$ as

$$\tilde{u}_h := \sum_{i=1}^{h} \tilde{u}_h^{(i)} \, \chi_{(a_i,b_i)} \,. \tag{54}$$

Observe that $\tilde{u}_h^{(i)} \in C^1([a_i, b_i])$ for $i = 1, \ldots, h$, $(\tilde{u}_h)_h \subset D_w \cap C^1(I_{\Omega,w}) \cap L^2(\Omega)$ and

$$\tilde{u}_h\left(\frac{a_i + b_i}{2}\right) = u\left(\frac{a_i + b_i}{2}\right) \qquad \text{for each } i = 1, \ldots, h \,,$$

$$\tilde{u}'_h = v_h \text{ in } \cup_{i=1}^{h} (a_i, b_i) \text{ and } \tilde{u}'_h = 0 \text{ in } \cup_{i=h+1}^{\infty} (a_i, b_i) \,. \tag{55}$$

Thus, (51) follows. By Poincaré type inequality (35) with $\tilde{u}_h - u$ instead of u and since $\tilde{u}_h\left(\frac{a_i+b_i}{2}\right) = u\left(\frac{a_i+b_i}{2}\right)$, we have, for each $\epsilon > 0$,

$$\begin{aligned}
&\tilde{\mathfrak{m}}(\{x \in \Omega : |\tilde{u}_h - u| \geq \epsilon\}) \leq \frac{1}{\epsilon^2} \int_\Omega |\tilde{u}_h - u|^2 \, \tilde{w} \, dx \\
=&\frac{1}{\epsilon^2} \sum_{i=1}^{+\infty} \int_{a_i}^{b_i} |\tilde{u}_h - u|^2 \, \tilde{w} \, dx \\
\leq&\frac{b-a}{\epsilon^2} \sum_{i=1}^{+\infty} \fint_{a_i}^{b_i} |\tilde{u}_h - u|^2 \, \tilde{w} \, dx \\
\leq&\frac{b-a}{\epsilon^2} \int_{I_{\Omega,w}} |\tilde{u}'_h - u'|^2 \, w \, dx \\
=&\frac{b-a}{\epsilon^2} \left(\sum_{i=1}^{h} \int_{a_i}^{b_i} |v_h - u'|^2 \, w \, dx + \sum_{i=h+1}^{\infty} \int_{a_i}^{b_i} |u'|^2 \, w \, dx \right) \\
\leq&\frac{b-a}{\epsilon^2} \left(\int_{I_{\Omega,w}} |v_h - u'|^2 \, w \, dx + \sum_{i=h+1}^{\infty} \int_{a_i}^{b_i} |u'|^2 \, w \, dx \right) .
\end{aligned} \tag{56}$$

Since $u' \in L^2(I_{\Omega,w})$,

$$\lim_{h\to\infty} \sum_{i=h+1}^{\infty} \int_{a_i}^{b_i} |u'|^2 \, w \, dx = 0$$

as $h \to \infty$ in (56), by (52), (53) and (55) , (50) follows and we are done.
(ii) From Proposition 4.15 (ii) and (iii), the coincidence

$$\widetilde{F^2} = \widehat{F^2} \text{ on } L^0(\Omega)$$

immediately follows. $\square$

Remark 4.19. Under the assumptions of Theorem 4.18 (i), we do not know whether $\widetilde{F}^2 = \widehat{F}^2$ on $L^0(\Omega)$. Indeed, from Proposition 4.15 (ii), it follows that $\widetilde{F}^2 \le \widehat{F}^2$ on $L^0(\Omega)$, but the coincidence is not clear since the convergences w.r.t. measure $\mathfrak{m} = w\,dx$ and $\widetilde{\mathfrak{m}} = \tilde{w}\,dx$ in Ω are no longer equivalent (see Remark 4.16).

Corollary 4.20. *Let w be a weight satisfying* (1)*. For every $u \in L^0(\Omega)$ we have*

$$\widetilde{F^1}(u) = \widetilde{F^2}(u) = \widetilde{F^3}(u) = \widetilde{F^4}(u),$$

where $\widetilde{F^j}(u)$, $j = 1, 2, 3, 4$ are the functionals in (7)*.*

Proof. Since

$$F^4(u) \le F^3(u) \le F^2(u) \le F^1(u) \text{ for each } u \in L^0(\Omega)\,, \tag{57}$$

the inequalities

$$\widetilde{F^4}(u) \le \widetilde{F^3}(u) \le \widetilde{F^2}(u) \le \widetilde{F^1}(u) \text{ for each } u \in L^0(\Omega) \tag{58}$$

are trivial. Moreover, arguing as in the proof of Theorem 4.18, it follows that

$$\widetilde{D^j} \subseteq D_w \text{ and } \int_{I_{\Omega,w}} |u'|^2\, w\,dx \le \widetilde{F^j}(u) \text{ for each } u \in \widetilde{D^j},\ j = 1, 2, 3, 4\,. \tag{59}$$

Let us begin to prove that

$$\widetilde{F^2}(u) = \widetilde{F^3}(u) = \widetilde{F^4}(u) \text{ for each } u \in L^0(\Omega)\,. \tag{60}$$

By (43), (58) and (59) it follows that $D^j = D_w$ for each $j = 2, 3, 4$ and (60) follows. To conclude the proof we are going to show that the following inequality

$$\widetilde{F^1}(u) \le \widetilde{F^2}(u) \text{ for each } u \in L^0(\Omega)\,.$$

It suffices to apply the classical argument of approximation by convolution. We fix $u \in L^0(\Omega)$ and we can assume that $\widetilde{F^2}(u) < +\infty$; then there exists a sequence $(u_h)_h \subset Lip([a, b])$ such that $u_h \to u$ in $L^0(\Omega)$ and

$$\widetilde{F^2}(u) = \lim_{h\to+\infty} \int_a^b |u_h'|^2\, w\,dx < +\infty.$$

Let us extend u_h to the whole $\mathbb{R}$ by defining $u_h(x) = u_h(a)$ if $x \le a$ and $u_h(x) = u_h(b)$ if $x \ge b$. Let us consider $u_{h,\epsilon} := u_h * \rho_\epsilon$, where ρ_ϵ is a classical family of mollifiers on $\mathbb{R}$. Then, from the classical properties of the approximation by convolution, for given $\epsilon > 0$, $(u_{h,\epsilon})_h \subset C^\infty(\mathbb{R})$, $u_{h,\epsilon} \to u_h$ uniformly on $[a, b]$, as $\epsilon \to 0$, for a given h, $u'_{h,\epsilon} = u'_h * \rho_\epsilon$ and $u'_{h,\epsilon} \to u'_h$ in $L^p(\Omega)$ for every $p \in [1, \infty)$. Moreover

$$|u'_h * \rho_\epsilon|(x) \le \|u'_h\|_{L^\infty(\Omega)},\ x \in \Omega$$

for every $\epsilon > 0$. This implies that

$$F^1(u_{h,\epsilon}) = \int_a^b |u'_{h,\epsilon}|^2 \, w \, dx \to \int_a^b |u'_h|^2 \, w \, dx \,, \text{ as } \epsilon \to 0.$$

Therefore

$$\widetilde{F^1}(u_h) \leq \lim_{\epsilon \to 0^+} F^1(u_{h,\epsilon}) = \int_a^b |u'_h|^2 \, w \, dx.$$

Hence, we obtain

$$\widetilde{F^1}(u) \leq \liminf_{h \to +\infty} \widetilde{F^1}(u_h) \leq \liminf_{h \to +\infty} \int_a^b |u'_h|^2 \, w \, dx = \widetilde{F^2}(u).$$

□

Now we consider the relaxation w.r.t. the $L^2(\Omega, \tilde{w})$-topology, which is stronger than the convergence in measure $\tilde{\mathfrak{m}}$. By using the same strategy of the proof of Theorem 4.18, we show the two relaxed functional coincide. Indeed, let $X = L^2(\Omega, \tilde{w})$ were $\tilde{w}$ is the weight in (25) and the lower semicontinuous envelopes w.r.t. $L^2(\tilde{w})$-convergence, that is

$$\overline{F^j}(u) = \text{sc}^-(L^2(\tilde{w})) - F^j_X(u) \quad j = 1, 2, 3, 4 \tag{61}$$

and let

$$D^j = \{u \in L^2(\Omega, \tilde{w}) : \overline{F^j}(u) < +\infty\} \,.$$

We recall that, if $I_{\Omega,w} = \emptyset$, then $w^* \equiv 0$ (see Proposition 4.6 (i)) and so $\tilde{w} \equiv 0$, too. This implies that $L^2(\Omega, \tilde{w}) = \{0\}$, $D^j = \{0\}$ and $\overline{F^j}(u) = 0$, j= 1, 2, 3, 4.

Theorem 4.21. *Let w be a weight satisfying* (1)*. Then*

$$D^2 = D_w \cap L^2(\Omega, \tilde{w})$$

and the following representation holds for the relaxed functional

$$\overline{F^2}(u) = \begin{cases} \displaystyle\int_{I_{\Omega,w}} |u'|^2 \, w \, dx & \text{if } u \in D_w \cap L^2(\Omega, \tilde{w}) \\ +\infty & \text{if } u \in L^2(\Omega, \tilde{w}) \setminus D_w. \end{cases}$$

In particular

$$\widetilde{F^2} = \overline{F^2} \text{ on } D_w \cap L^2(\Omega, \tilde{w}) \,.$$

Proof. It is immediate that

$$\widetilde{F^2} \leq \overline{F^2} \text{ on } L^2(\Omega, \tilde{w}) \,.$$

In order to complete the proof we have only to prove that

$$\overline{F^2}(u) \leq \int_{I_{\Omega,w}} |u'|^2 \, w \, dx \quad \forall u \in D_w \,. \tag{62}$$

Let us first prove that

$$\overline{F^2}(u) \leq \int_{I_{\Omega,w}} |u'|^2 \, w \, dx \quad \forall u \in D_w \cap L^2(\Omega) \,. \tag{63}$$

As in the proof of Theorem 4.18, by Theorem 3.3, for each $u \in D_w \cap L^2(\Omega)$, there exists $(u_h)_h \subset Lip(\Omega)$ such that (48) and (47) hold. Thus, by (48) and the definition of $\overline{F^2}$,

$$\overline{F^2}(u) \leq \liminf_{h\to\infty} F^2(u_h) = \lim_{h\to\infty} F^2(u_h) = \int_{I_{\Omega,w}} |u'|^2\, w\, dx\,,$$

and (63) follows. It is sufficient in order to complete the proof that, for each $u \in D_w \cap L^2(\Omega, \tilde{w})$, there exists $(\tilde{u}_h)_h \subset D_w \cap L^2(\Omega)$ such that

$$\tilde{u}_h \to u \quad \text{in } L^2(\Omega, \tilde{w})\,, \tag{64}$$

and

$$\tilde{u}_h' \to u' \quad \text{in } L^2(I_{\Omega,w}, w) \text{ as } h \to \infty\,. \tag{65}$$

Indeed, from (63), (65) and the semicontinuity of $\overline{F^2}$, it will follow that

$$\overline{F^2}(u) \leq \liminf_{h\to\infty} \overline{F^2}(\tilde{u}_h) \leq \lim_{h\to\infty} \int_{I_{\Omega,w}} |\tilde{u}_h'|^2\, w\, dx = \int_{I_{\Omega,w}} |u'|^2\, w\, dx\,,$$

and we will get (62). Observe now that (64) and (65) can be proved by using the same sequence $(\tilde{u}_h)_h$ in (54). Indeed (51) immediately implies (65). Arguing as in (56), we get

$$\begin{aligned} &\int_\Omega |\tilde{u}_h - u|^2\, \tilde{w}\, dx \\ &\leq (b-a)\left(\int_{I_{\Omega,w}} |v_h - u'|^2\, w\, dx + \sum_{i=h+1}^{\infty} \int_{a_i}^{b_i} |u'|^2\, w\, dx\right). \end{aligned} \tag{66}$$

Since $u' \in L^2(I_{\Omega,w})$,

$$\lim_{h\to\infty} \sum_{i=h+1}^{\infty} \int_{a_i}^{b_i} |u'|^2\, w\, dx = 0\,.$$

Therefore, by (65) and (66) , (64) follows. □

If w is finitely degenerate, by Corollary 4.10 (i),

$$D_w \subset L^2(\Omega, w^*) \subset L^2(\Omega, \tilde{w})\,.$$

Thus, as an immediate consequence of Theorem 4.21, we get the characterization of relaxed functional $\overline{F^2}$ for finitely degenerate weights.

Corollary 4.22. *Let w be a finitely degenerate weight. Then*

$$D^2 = D_w$$

and the following representation holds for the relaxed functional

$$\overline{F^2}(u) = \begin{cases} \displaystyle\int_{I_{\Omega,w}} |u'|^2\, w\, dx & \text{if } u \in D_w \\ +\infty & \text{if } u \in L^2(\Omega, \tilde{w}) \setminus D_w. \end{cases}$$

In particular

$$\widetilde{F^2} = \overline{F^2} \text{ on } D_w\,.$$

Corollary 4.23. *Let w be a weight satisfying* (1)*. For every $u \in L^2(\Omega, \tilde{w})$ we have*

$$\overline{F^1}(u) = \overline{F^2}(u) = \overline{F^3}(u) = \overline{F^4}(u),$$

where $\overline{F^j}(u)$, $j = 1, 2, 3, 4$ are the functional in (61)*.*

Proof. The proof can be carried out as the one of Corollary 4.20 by replacing the role of the convergence in measure $\widetilde{\mathfrak{m}}$ with the one in $L^2(\Omega, \tilde{w})$ and the domain D_w with $D_w \cap L^2(\Omega, \tilde{w})$. □

5. Comparison between different Lebesgue weighted spaces

In this section we will present some examples in order to compare the different Lebesgue weighted spaces $L^2(\Omega, w)$ and $L^2(\Omega, w^*)$. Moreover we will show that space D_w may not be contained in $L^2(\Omega, w)$ and in $L^2(\Omega, w^*)$.

Example 5.1. We are going to study here the inclusion relationships between $L^2(\Omega, w^*)$ and $L^2(\Omega, w)$ by means of the behaviour of weight w. In particular we will prove they are independent. Namely we will show that all three cases

$$L^2(\Omega, w^*) = L^2(\Omega, w)\,, \tag{67}$$

$$L^2(\Omega, w^*) \subsetneq L^2(\Omega, w)\,, \tag{68}$$

$$L^2(\Omega, w^*) \supsetneq L^2(\Omega, w)\,, \tag{69}$$

can occur, even though w is finitely degenerate and $w = 0$ a.e. in $\Omega \setminus I_{\Omega,w}$. The same relationships holds by considering the corresponding spaces L^2_{loc}. Moreover we will see below that

$$L^2(\Omega, w^*) \not\subseteq L^2(\Omega, w) \tag{70}$$

and

$$L^2(\Omega, w) \not\subseteq L^2(\Omega, w^*)\,. \tag{71}$$

We will first consider the simple situation when the weight w is finitely degenerate with $N_w = 1$. More precisely, let $\Omega = (a, b) = (0, 1)$, $w : (0, 1) \to (0, \infty)$, $w \in L^1((0, 1))$ and $\frac{1}{w} \in L^1((\delta, 1))$ for each $\delta \in (0, 1)$. Under these assumptions, according to our notation, $I_{\Omega,w} = (a, b) = (a_1, b_1) = (0, 1)$ and the weight $w^* : (0, 1) \to (0, \infty)$ in (24) satisfies the following properties:

$$0 < \inf_{[1/2,1)} w^*(x) \le \sup_{[1/2,1)} w^*(x) < \infty\,, \tag{72}$$

$$w^* \in C^0((0, 1/2]) \text{ and } \exists \lim_{x \to 0^+} w^*(x) \in [0, \infty)\,. \tag{73}$$

(i) Assume that

$$\lim_{x \to 0^+} w^*(x) \in (0, \infty)\,. \tag{74}$$

Observe that (74) is equivalent to require that

$$\frac{1}{w} \in L^1((0,1))\,. \tag{75}$$

Then, from (72), (73) and (74), we can infer that

$$0 < \inf_{x\in(0,1)} w^*(x) \le \sup_{x\in(0,1)} w^*(x) < \infty\,,$$

and thus

$$L^2(\Omega, w^*) = L^2(\Omega)\,. \tag{76}$$

By choosing $w(x) = x^\alpha$ with $\alpha \in (-1,1)$, (74) is satisfied, since (75) holds. Therefore, by (76), we can conclude that, if $\alpha \in (0,1)$, since $w(x) < 1$ for each $x \in (0,1)$,

$$L^2(\Omega, w^*) = L^2(\Omega) \subsetneq L^2(\Omega, w)\,;$$

if $\alpha = 0$, since $w(x) = 1$ for each $x \in (0,1)$,

$$L^2(\Omega, w^*) = L^2(\Omega) = L^2(\Omega, w)\,;$$

if $\alpha \in (-1,0)$, since $w(x) > 1$ for each $x \in (0,1)$,

$$L^2(\Omega, w^*) = L^2(\Omega) \supsetneq L^2(\Omega, w)\,.$$

Therefore cases (67), (68) and (69) can occur.

(ii) Assume that

$$\lim_{x\to 0^+} w^*(x) = 0\,. \tag{77}$$

Observe that (77) is equivalent to require that

$$\frac{1}{w} \notin L^1((0,1))\,.$$

In particular, it holds true that

$$\limsup_{x\to 0^+} w(x) = 0 \text{ and } \lim_{x\to 0^+} \int_x^{1/2} \frac{1}{w}(y)\,dy = \infty\,.$$

Assume now that

$$\limsup_{x\to 0^+} \left(w(x) \int_x^{1/2} \frac{1}{w}(y)\,dy \right) < \infty\,. \tag{78}$$

Notice that (78) trivially holds if $w : (0,1/2) \to (0,\infty)$ is nondecreasing. From (78) and (72), we have that there is positive constant C such that

$$w(x) \le C\,w^*(x) \quad \forall\, x \in (0,1)\,,$$

which in turn implies (67) or (68).

The more interesting case is when (78) does not hold. For instance, when the weight w oscillates as $x \to 0^+$ and it is the case we are going to deal with. More precisely , let us denote

$$I_h^1 := \left(\frac{1}{h+1}, \frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h} \right) \right]\,,\ I_h^2 := \left(\frac{1}{2}\left(\frac{1}{h+1} + \frac{1}{h} \right), \frac{1}{h} \right]\,,$$

$$I^1 := \cup_{h=1}^\infty I_h^1, \quad I^2 := \cup_{h=1}^\infty I_h^2$$

and

$$I_h := I_h^1 \cup I_h^2 = \left(\frac{1}{h+1}, \frac{1}{h} \right] .$$

Let us define

$$\begin{aligned} w(x) &:= x^\gamma \, \chi_{I^1}(x) + x^3 \, \chi_{I^2}(x) \\ &= x^\gamma \sum_{h=1}^{\infty} \chi_{I_h^1}(x) + x^3 \sum_{h=1}^{\infty} \chi_{I_h^2}(x) \quad x \in (0,1) \end{aligned} \tag{79}$$

where $0 \leq \gamma < 1$ and χ_A denotes the characteristic function of a set A. Notice that

$$\frac{1}{w(x)} = \frac{1}{x^\gamma} \sum_{h=1}^{\infty} \chi_{I_h^1}(x) + \frac{1}{x^3} \sum_{h=1}^{\infty} \chi_{I_h^2}(x) \quad x \in (0,1) . \tag{80}$$

In this example, $I_{\Omega,w} = (0,1)$ and so $N_w = 1$, then it is finitely degenerate. Notice that $\frac{1}{w}$ is locally summable in $(0,1)$.

Let us prove that there exists a positive constant $c_1 > 0$ such that

$$\frac{1}{c_1} \, x^2 \leq w^*(x) \leq c_1 \, x^2 \quad \forall \, x \in (0, 1/4) . \tag{81}$$

From (79) and (81) it follows that the weights w and w^* are not comparable.

According to (24), by (80), if $x \in (0, 1/2)$,

$$\begin{aligned} \frac{1}{w^*(x)} &= \int_x^1 \frac{1}{w(y)} \, dy \\ &= \sum_{h=1}^{\infty} \int_{I_h^1 \cap [x,1]} y^{-\gamma} \, dy + \sum_{h=1}^{\infty} \int_{I_h^2 \cap [x,1]} y^{-3} \, dy \\ &= v_1(x) + v_2(x) . \end{aligned} \tag{82}$$

We are now going to estimate functions v_i $(i = 1, 2)$, from above and below. The estimate as far as v_1 is concerned is quite trivial. Indeed

$$\begin{aligned} 0 \leq v_1(x) &= \sum_{h=1}^{\infty} \int_{I_h^1 \cap [x,1]} y^{-\gamma} \, dy \leq \sum_{h=1}^{\infty} \int_{I_h \cap [x,1]} y^{-\gamma} \, dy \\ &= \int_x^1 y^{-\gamma} \, dy \leq \int_0^1 y^{-\gamma} \, dy \leq 1 \quad \forall \, x \in (0, 1/2) . \end{aligned} \tag{83}$$

Notice now that, if $N(x)$ denotes the integer part of $1/x$ with $x \in (0, 1/2)$, then

$$v_2(x) = \sum_{h=1}^{\infty} \int_{I_h^2 \cap [x,1]} y^{-3} \, dy = \sum_{h=1}^{N(x)-1} \int_{I_h^2} y^{-3} \, dy + \int_{I_{N(x)} \cap [x,1]} y^{-3} \, dy . \tag{84}$$

From (84), since for $1 \leq h \leq N(x) - 1$ we have

$$\frac{1}{2} \left(\frac{1}{h} + \frac{1}{h+1} \right) \geq x,$$

we can infer that

$$\sum_{h=1}^{N(x)-1}\int_{I_h^2} y^{-3}\,dy \le\ v_2(x) \le\ 2\int_x^1 y^{-3}\,dy = \frac{1}{x^2}-1 \quad \forall x\in(0,1/2)\,. \tag{85}$$

By a simple calculation, we get

$$\begin{aligned} v_2(x) \ge \sum_{h=1}^{N(x)-1}\int_{I_h^2} y^{-3}\,dy &\ge \sum_{h=1}^{N(x)-1} h^3\,\left|I_h^2\right| = \frac{1}{2}\sum_{h=1}^{N(x)-1}\frac{h^3}{h(h+1)} \\ &\ge \frac{1}{2}\sum_{h=1}^{N(x)-1} h = \frac{(N(x)-1)\,N(x)}{4} \\ &\ge \frac{1}{2}\left(\frac{1}{x}-2\right)\left(\frac{1}{x}-1\right) \qquad \forall x\in(0,1/2)\,. \end{aligned} \tag{86}$$

From (85) and (86), it follows that

$$\frac{1}{2}\left(\frac{1}{x}-2\right)\left(\frac{1}{x}-1\right) \le\ v_2(x) \le\ \frac{1}{x^2}-1 \quad \forall x\in(0,1/2)\,. \tag{87}$$

Therefore, by (82), (83) and (87), (81) follows. Eventually, by considering the weight w in (79), it is easy to see, because of (81), that (70) and (71) can occur.

Remark 5.2. The weight (79) is not a doubling weight. Indeed, let x_h, r_h such that $B(x_h, r_h) = (\frac{1}{2}(\frac{1}{h}+\frac{1}{h+1}), \frac{1}{h})$, then $r_h = \frac{1}{4}\frac{1}{h(h+1)}$. We obtain that

$$\mathfrak{m}(B(x_h, r_h)) = \int_{\frac{1}{2}(\frac{1}{h}+\frac{1}{h+1})}^{\frac{1}{h}} x^3\,dx \,\tilde{=}\, C_1\frac{1}{h^5} + o\big(\frac{1}{h^5}\big).$$

On the other hand, since

$$\left(\frac{1}{h}, \frac{1}{h}+\frac{1}{4h(h+1)}\right) \subseteq B(x_h, 2r_h),$$

we get

$$\mathfrak{m}(B(x_h, 2r_h)) \ge \int_{\frac{1}{h}}^{\frac{1}{h}+\frac{1}{4h(h+1)}} x^\gamma\,dx \,\tilde{=}\, C_2\frac{1}{h^{\gamma+2}} + o\big(\frac{1}{h^{\gamma+2}}\big).$$

We proceed by contradiction by assuming that $\mathfrak{m}$ is a doubling measure. Then there exists a constant C such that

$$C_2\frac{1}{h^{\gamma+2}} + o\big(\frac{1}{h^{\gamma+2}}\big) \le \mathfrak{m}(B(x_h, 2r_h)) \le C\mathfrak{m}(B(x_h, r_h)) \,\tilde{=}\, C_1\frac{1}{h^5} + o\big(\frac{1}{h^5}\big).$$

Thus we have a contradiction since $\gamma+2<5$.

Remark 5.3. If w is finitely degenerate, then, by Corollary 4.10 (i),

$$D_w \subseteq L^2(\Omega, w^*) \subseteq L^2(\Omega, \tilde{w})\,.$$

If w is not finitely degenerate, then $D_w \subseteq L^2_{\mathrm{loc}}(I_{\Omega,w}, w^*)$. We observe that $D_w \not\subseteq L^2(\Omega,\mu)$ for each finite measure μ on Ω such that $I_{\Omega,w} \subset spt(\mu)$. In fact, let $u(x) = \lambda_i$ on (a_i, b_i)

for every $i \in \mathbb{N}$; then $u \in D_w$, but $u \notin L^2(\Omega, \mu)$ if we choose

$$\lambda_i = \frac{1}{\mu((a_i, b_i))}.$$

Indeed,

$$\int_\Omega |u^2|\, d\mu = \sum_{i=1}^{+\infty} \int_{a_i}^{b_i} |u^2|\, d\mu = \sum_{i=1}^{+\infty} \frac{1}{\mu((a_i, b_i))}$$

does not converge, since $\mu((a_i, b_i)) \to 0$, as $i \to +\infty$, by Remark 4.3. In particular, this argument applies to measure $\mu = \tilde{w}\, dx$ since by (19) and Proposition 4.6, $I_{\Omega,w} \subset spt(\mu)$. Thus $D_w \not\subseteq L^2(\Omega, \mu) = L^2(\Omega, \tilde{w})$, if w is not finitely degenerate. This also implies that $D_w \not\subseteq L^2(\Omega, w)$, $D_w \not\subseteq L^2(\Omega, w^*)$ and $D_w \not\subseteq L^2(\Omega)$, if w is not finitely degenerate.

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