A LAGRANGIAN APPROACH TO TOTALLY DISSIPATIVE EVOLUTIONS IN WASSERSTEIN SPACES

GIULIA CAVAGNARI, GIUSEPPE SAVARÉ, AND GIACOMO ENRICO SODINI

Dedicated to Ricardo H. Nochetto on the occasion of his 70^{th} *birthday.*

Abstract. We introduce and study the class of *totally dissipative* multivalued probability vector fields (MPVF) $\mathbf{F}$ on the Wasserstein space $(\mathcal{P}_2(\mathsf{X}), W_2)$ of Euclidean or Hilbertian probability measures. We show that such class of MPVFs is in one to one correspondence with law-invariant dissipative operators in a Hilbert space $L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ of random variables, preserving a natural maximality property. This allows us to import in the Wasserstein framework many of the powerful tools from the theory of maximal dissipative operators in Hilbert spaces, deriving existence, uniqueness, stability, and approximation results for the flow generated by a maximal totally dissipative MPVF and the equivalence of its Eulerian and Lagrangian characterizations.

We will show that demicontinuous single-valued probability vector fields satisfying a metric dissipativity condition as in [CSS23a] are in fact totally dissipative. Starting from a sufficiently rich set of discrete measures, we will also show how to recover a unique maximal totally dissipative version of a MPVF, proving that its flow provides a general mean field characterization of the asymptotic limits of the corresponding family of discrete particle systems. Such an approach also reveals new interesting structural properties for gradient flows of displacement convex functionals with a core of discrete measures dense in energy.

Contents

2020 *Mathematics Subject Classification.* Primary: 34A06, 47B44, 49Q22; Secondary: 34A12, 34A60, 28D05.
Key words and phrases. Measure differential equations/inclusions in Wasserstein spaces, probability vector fields, dissipative operators, measure-preserving isomorphisms, geodesically convex functionals, JKO scheme.

1. Introduction

The theory of evolutions of probability measures is experiencing an ever growing interest from the scientific community. On one side, this is justified by its numerous applications in modeling real-life dynamics: social dynamics, crowd dynamics for multi-agent systems, opinion formation, evolution of financial markets just to name a few. We refer the reader to the recent preprint [Pic23] for a more complete overview of the many applications of control theory for multi-agent systems, i.e. large systems of interacting particles/individuals. On the other side, dealing with mean-field evolutions, expecially in the framework of optimal control theory in Wasserstein spaces [For+19; Cav+22; CM22], provides interesting insights into mathematical research. We mention for instance the recent contributions [Ave22; BF21b; BF23] for the study of a well-posedness theory for differential inclusions in Wasserstein spaces, [AK22; BF21a; Pog16] for necessary conditions for optimality in the form of a Pontryagin maximum principle, the references [JMQ20; BF22a; CMP20] for the study of Hamilton-Jacobi-Bellman equations in this framework. Finally, other contributions devoted to the development of a viability theory for control problems in the space of probability measures are e.g. [AMQ21; BF22c; BF22b; CMQ21].
In addition to these studies, we have all the applications of the theory of gradient flows in Wasserstein spaces [AGS08] which are impossible to summarize here even briefly. In particular, in the case of geodesically convex (resp. λ-convex) functionals [McC97], the geometric viewpoint and the variational approach introduced by [Ott01; JKO98a] have been extremely powerful to construct a semigroup of contractions (resp. Lipschitz maps) [AGS08], which provides a robust background for various applications.

In the present paper, we continue the project, started in [CSS23a], to extend the theory beyond gradient flows. Our aim is to investigate the evolution semigroups generated by a λ-dissipative multivalued probability vector field (in short, MPVF) $\mathbf{F}$ in the Wasserstein space $(\mathcal{P}_2(\mathsf{X}), W_2)$. The space $\mathcal{P}_2(\mathsf{X})$ denotes the set of Borel probability measures with finite quadratic moment on a separable Hilbert space X. The geometric notion of dissipativity is intimately related to the L^2-Kantorovich-Rubinstein-Wasserstein distance W_2 between two measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$, which can be expressed by the solution of the Optimal Transport problem

$$W_2^2(\mu_0,\mu_1) := \min\Big\{ \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0,x_1) : \boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)\Big\}, \tag{1.1}$$

where $\Gamma(\mu_0,\mu_1)$ denotes the set of couplings $\boldsymbol{\mu} \in \mathcal{P}_2(\mathsf{X}\times\mathsf{X})$ with marginals μ_0 and μ_1. It is well known that the set $\Gamma_o(\mu_0,\mu_1)$ where the minimum in (1.1) is attained is a nonempty compact and convex subset of $\Gamma(\mu_0,\mu_1)$.
We refer to [CSS23a] for a detailed discussion of the various approaches to such kind of problems; let us only mention here the Cauchy-Lipschitz approach via vector fields [BF21b; BF23], the barycentric approach in [Pic19; Pic18; Cam+21] and the variational approach to characterize limit solutions of an Explicit Euler Scheme for evolution equations driven by dissipative MPVFs in [CSS23a].
Let us just recall here the main features of this approach. A MPVF $\mathbf{F}$ can be identified with a subset of the set of probability measures $\mathcal{P}_2(\mathsf{TX})$ on the space-velocity tangent bundle $\mathsf{TX} = \{(x,v) \in \mathsf{X}\times\mathsf{X}\}$, with proper domain $\mathrm{D}(\mathbf{F}) := \{\mathsf{x}_\sharp\Phi : \Phi \in \mathbf{F}\}$ and sections $\mathbf{F}[\mu] := \{\Phi \in \mathbf{F} :$

$\mathsf{x}_\sharp\Phi = \mu\}$, where $\mathsf{x}(x, v) := x$ is the projection on the first coordinate in TX. Since every element $\Phi \in \mathbf{F}$ has finite quadratic moment in the tangent bundle, the L^2-norm of the velocity marginal

$$|\Phi|_2^2 := \int |v|^2 \,\mathrm{d}\Phi(x, v) \quad \text{is finite.}$$

The disintegration $\{\Phi_x\}_{x\in\mathsf{X}}$ of $\Phi \in \mathbf{F}[\mu]$ with respect to μ provides a Borel field of probability measures on the space of velocity vectors, which can be interpreted as a probabilistic description of the velocity prescribed by $\mathbf{F}$ at every position/particle x, given the distribution μ. An important case, which is simpler to grasp, occurs when $\mathbf{F}$ is concentrated on maps and therefore $\Phi_x = \delta_{\boldsymbol{f}(x)}$ is a Dirac mass concentrated on the deterministic velocity $\boldsymbol{f}$ (in this case we say that $\mathbf{F}$ is deterministic): for every measure $\mu \in \mathrm{D}(\mathbf{F})$

$$\text{the elements } \Phi \in \mathbf{F}[\mu] \text{ have the form } (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f})_\sharp\mu \text{ for a vector field } \boldsymbol{f} \in L^2(\mathsf{X}, \mu; \mathsf{X}), \tag{1.2}$$

where $\boldsymbol{i}_\mathsf{X}$ denotes the identity map on X. In this case, $\mathbf{F}$ is *dissipative* if for every $\Phi_i = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}_i)_\sharp\mu_i \in \mathrm{D}(\mathbf{F})$, $i = 0, 1$,

$$\exists\, \boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1) \quad \text{optimal, such that} \quad \int \langle \boldsymbol{f}_0(x_0) - \boldsymbol{f}_1(x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le 0. \tag{1.3}$$

Notice however that, even in the deterministic case, the realization of $\mathbf{F}[\mu]$ as an element/subset of $\mathcal{P}_2(\mathsf{TX})$ is crucial to deal with varying base measures μ, since for different $\mu_0, \mu_1 \in \mathrm{D}(\mathbf{F})$ the representation (1.2) yields corresponding maps $\boldsymbol{f}_0, \boldsymbol{f}_1$ which belong to different L^2 spaces and therefore are not easy to be compared.
When $\mathbf{F}$ is not concentrated on maps, the dissipativity condition between two elements $\Phi_0 \in \mathbf{F}[\mu_0], \Phi_1 \in \mathbf{F}[\mu_1]$ guarantees the existence of a coupling $\boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1) \subset \mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$ such that the "space" marginal projection $(\mathsf{x}_0, \mathsf{x}_1)_\sharp\boldsymbol{\vartheta}$ is optimal, thus belongs to $\Gamma_o(\mu_0, \mu_1)$, and moreover

$$\int \langle v_1 - v_0, x_1 - x_0\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x_0, v_0; x_1, v_1) \le 0. \tag{1.4}$$

Such a property appears as a natural generalization of the corresponding condition introduced in [AGS08] for the Wasserstein subdifferentials of geodesically convex functionals.
The geometric interpretation of this condition becomes apparent by considering its equivalent characterization in terms of the first order expansion of the squared Wasserstein distance: in the case (1.2) it can be written as

$$W_2^2((\boldsymbol{i}_\mathsf{X} + h\boldsymbol{f}_0)_\sharp\mu_0, (\boldsymbol{i}_\mathsf{X} + h\boldsymbol{f}_1)_\sharp\mu_1) \le W_2^2(\mu_0, \mu_1) + o(h) \quad \text{as } h \downarrow 0.$$

In principle, one may interpret the flow generated by $\mathbf{F}$ in terms of absolutely continuous (w.r.t. the Wasserstein metric) curves $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$, $\mu(t) \equiv \mu_t$, in $\mathrm{D}(\mathbf{F})$ solving the continuity equation

$$\partial_t\mu_t + \nabla\cdot(\mu_t\, \boldsymbol{f}_t) = 0 \quad \text{in } (0, \infty) \times \mathsf{X}, \quad (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}_t)_\sharp\mu_t \in \mathbf{F},$$

and obeying a Cauchy condition $\mu_{|t=0} = \mu_0$. However, the derivation of such a precise formulation is not a simple task and, in general, it requires more restrictive assumptions on $\mathbf{F}$ as

$$\begin{gathered} \mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X}), \quad \mathbf{F}[\mu] = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}[\mu])_\sharp\mu \quad \text{(thus } \mathbf{F} \text{ is single-valued)}, \\ \mu_n \to \mu \quad \Longrightarrow \quad (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}[\mu_n])_\sharp\mu_n \to (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}[\mu])_\sharp\mu. \end{gathered} \tag{1.5}$$

We introduced in [CSS23a] the more flexible condition of EVI solutions, borrowed from the theory of gradient flows [AGS08] and from the Bénilan notion of integral solutions to dissipative evolutions in Hilbert/Banach spaces [Bén72]: a continuous curve $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$ with values in $\overline{\mathrm{D}(\mathbf{F})}$ is an EVI solution if it solves the system of Evolution Variational Inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) \le -[\Phi, \mu_t]_r \quad \text{for every } \Phi \in \mathbf{F}[\nu],\ \nu \in \mathrm{D}(\mathbf{F}) \quad \text{in } \mathscr{D}'((0, +\infty)), \tag{1.6}$$

where for every $\Phi = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp \nu \in \mathbf{F}$ the duality pairing $[\Phi, \mu]_r$ is defined by

$$[\Phi, \mu]_r := \min \Big\{ \int \langle \boldsymbol{f}(x_0), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\mu}(x_0, x_1) : \boldsymbol{\mu} \in \Gamma_o(\nu, \mu) \Big\}.$$

In [CSS23a], we studied the properties of the flow in $\mathcal{P}_2(\mathsf{X})$ generated by $\mathbf{F}$ by means of the *explicit Euler method* and we proved that, under suitable conditions, every family of discrete approximations obtained by the explicit Euler method converges to an EVI solution when the step size vanishes, also providing an optimal error estimate.
The use of the explicit Euler method is simple to implement and quite powerful when the domain of $\mathbf{F}$ coincides with the whole $\mathcal{P}_2(\mathsf{X})$ and $\mathbf{F}$ is locally bounded [CSS23a, Cor. 5.23], i.e. $|\Phi|_2$ remains uniformly bounded in a suitable neighborhood of every measure $\mu \in \mathcal{P}_2(\mathsf{X})$ (but much more general conditions are thoroughly discussed in [CSS23a]). Dealing with constrained evolutions or with operators which are not locally bounded requires a better understanding of the implicit Euler method.

***Maximal totally dissipative* MPVF*s*.** One of the starting points of the present investigation (see Sections 3.3 and 8) is the nontrivial fact that a large class of λ-dissipative MPVFs including the demicontinuous fields (1.5) satisfies a much stronger dissipativity condition, which we call *total λ-dissipativity*: in the simplest case $\lambda = 0$ when (1.2) holds and $\mathbf{F}$ is single-valued, such a property reads as

$$\int \langle \boldsymbol{f}[\mu_0](x_0) - \boldsymbol{f}[\mu_1](x_1), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le 0 \quad \text{for } \textit{every } \boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1) \tag{1.7}$$

and can be compared with the notion of L-monotonicity of [CD18, Definition 3.31]. Total dissipativity thus holds along arbitrary couplings between pairs of measures μ_0, μ_1 in the domain of $\mathbf{F}$, whereas the metric dissipativity condition (1.3) involves only optimal couplings. The relaxed version of (1.7) allowing for λ-dissipativity includes the class of Lipschitz probability vector fields $\boldsymbol{f}$ satisfying

$$\Big| \boldsymbol{f}[\mu_0](x_0) - \boldsymbol{f}[\mu_1](x_1) \Big| \le L \Big(|x_0 - x_1| + W_2(\mu_0, \mu_1) \Big) \quad \text{for every } x_i \in \mathsf{X},\ \mu_i \in \mathcal{P}_2(\mathsf{X})$$

for $\lambda = 2L$ (see Example 3.11).
Motivated by this remarkable property, it is natural to extend the notion of total dissipativity to a general MPVF $\mathbf{F}$. Here there are two possible approaches: the weakest one, modeled on the general definition of metric dissipativity (1.4), would require that for every $\Phi_0 \in \mathbf{F}[\mu_0], \Phi_1 \in \mathbf{F}[\mu_1]$ and *every* coupling $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$ ($\boldsymbol{\mu}$ is not optimal) there exists $\boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1)$ such that $(\mathsf{x}_0, \mathsf{x}_1)_\sharp \boldsymbol{\vartheta} = \boldsymbol{\mu}$ and (1.4) holds.
The strongest condition, which we will systematically investigate in this paper, requires that

$$\text{for every } \Phi_0, \Phi_1 \in \mathbf{F} \text{ and every } \boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1) \quad \int \langle v_1 - v_0, x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\vartheta}(x_0, v_0; x_1, v_1) \le 0. \tag{1.8}$$

It is clear that total dissipativity for arbitrary MPVFs is much stronger than the metric dissipativity condition (1.4). We address two main questions:

⟨Q.1⟩ What are the structural properties of totally dissipative MPVFs satisfying the stronger condition (1.8) and their relation with Lagrangian representations by dissipative operators in the Hilbert space
$$\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X}),$$
where $\mathbb{P}$ is a nonatomic probability measure on a standard Borel space $(\Omega, \mathcal{B})$, which provides the domain of the parametrization. A similar lifting approach has been used also in e.g. [Lio07; Car13; GT19; CD18; JMQ20], in particular for functions defined in $\mathcal{P}_2(\mathsf{X})$ and their Fréchet differential. This is the content of Section 3 and 4, with applications to the case of gradient flows in Section 5.

⟨Q.2⟩ Under which conditions a dissipative MPVF is totally dissipative and, more generally, it is possible to recover a unique maximal totally dissipative "version" of the initial MPVF starting from a sufficiently rich set of discrete measures. This is investigated first in Section 3.3 and more extensively in Section 8, starting from the results of Sections 6 and 7 on the geometry of discrete measures.

Lagrangian representations. Concerning the first question ⟨Q.1⟩, in Section 3.2 we will show that

> *there is a one-to-one correspondence between totally dissipative* MPVF*s and law invariant dissipative operators in the Hilbert space* $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$*; such a correspondence preserves maximality.*

This representation is very useful to import in the metric space $(\mathcal{P}_2(\mathsf{X}), W_2)$ all the powerful tools and results concerning semigroups of contractions generated by maximal dissipative operators in Hilbert spaces, see e.g. [Bré73]. This approach overcomes most of the technical limits of the explicit Euler method adopted in [CSS23a] and allows for a more general theory of existence, well posedness, and stability of solutions. In particular, even if the results are new and relevant also in the finite dimensional Euclidean case, the theory does not rely on any compactness argument and thus can be fully developed in a infinite dimensional separable Hilbert space X. We can in fact lift a totally dissipative MPVF $\mathbf{F}$ to a dissipative operator $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$, that we call *Lagrangian representation of* $\mathbf{F}$, defined by

$$(X, V) \in \boldsymbol{B} \quad \Longleftrightarrow \quad (X, V)_\sharp \mathbb{P} \in \mathbf{F}.$$

It turns out that $\boldsymbol{B}$ is law invariant (i.e. if $(X, V) \in \boldsymbol{B}$ and (X', V') has the same law as (X, V), then $(X', V') \in \boldsymbol{B}$ as well) and admits a maximal dissipative extension $\hat{\boldsymbol{B}}$ which is law invariant and corresponds to a maximal extension of $\mathbf{F}$ still preserving total dissipativity. In particular, $\mathbf{F}$ is maximal in the class of totally dissipative MPVFs if and only if $\boldsymbol{B}$ is a law invariant operator which is maximal dissipative.

Such a crucial result depends on two important properties: first of all, if the graph of $\boldsymbol{B}$ is strongly-weakly closed in $\mathcal{X} \times \mathcal{X}$ (in particular if $\boldsymbol{B}$ is maximal) then law invariance can also be characterized by invariance w.r.t. measure-preserving isomorphisms of Ω, i.e. essentially injective maps $g : \Omega \to \Omega$ such that $g_\sharp \mathbb{P} = \mathbb{P} = g^{-1}_\sharp \mathbb{P}$ (Theorem 3.4). The second property (Theorem 3.12) guarantees that every dissipative operator in $\mathcal{X}$ which is invariant by measure-preserving isomorphisms has a maximal dissipative extension enjoying the same invariance (and thus also law invariance). Such a result has been obtained in [CSS23b] and exploits remarkable results of [BW09; BW10] providing an explicit construction of a maximal extension of a monotone operator.

The equivalence between law-invariance and invariance by measure-preserving tranformations also plays a crucial role to prove that the resolvents of $\boldsymbol{B}$, its Yosida approximations, and the generated semigroup of contractions $(\boldsymbol{S}_t)_{t \geq 0}$ in $\mathcal{X}$ are still law invariant. The family $(\boldsymbol{S}_t)_t$ thus induces a projected semigroup of contractions in $\mathcal{P}_2(\mathsf{X})$ defined by

$$S_t(\mu_0) := (\boldsymbol{S}_t X_0)_\sharp \mathbb{P} \quad \text{whenever} \quad (X_0)_\sharp \mathbb{P} = \mu_0 \in \mathrm{D}(\mathbf{F}), \tag{1.9}$$

which is independent of the choice of X_0 parametrizing the initial law μ_0, which satisfies the EVI formulation (1.6) and the stability property (here for arbitrary $\lambda \in \mathbb{R}$)

$$|\boldsymbol{S}_t X_0 - \boldsymbol{S}_t Y_0|_{\mathcal{X}} \leq \mathrm{e}^{\lambda t} |X_0 - Y_0|_{\mathcal{X}}, \quad W_2(S_t(\mu_0), S_t(\nu_0)) \leq \mathrm{e}^{\lambda t} W_2(\mu_0, \nu_0). \tag{1.10}$$

Another crucial property of totally dissipative MPVFs concerns the *barycentric projection*, which can be obtained by taking the expected value of the disintegration $\{\Phi_x\}_{x \in \mathsf{X}}$ of an element $\Phi \in \mathbf{F}$ with respect to its first marginal $\mu = \mathsf{x}_\sharp \Phi$:

$$\boldsymbol{b}_\Phi(x) := \int v \, \mathrm{d}\Phi_x(v) \quad \text{for } \mu\text{-a.e. } x \in \mathsf{X}; \quad \boldsymbol{b}_\Phi \in L^2(\mathsf{X}, \mu; \mathsf{X}).$$

The barycenter $\boldsymbol{b}_\Phi$ also represents the conditional expectation $\mathbb{E}[V|X]$ of V given (the σ-algebra generated by) X, for every $(X,V)\in\mathbf{F}$ with $(X,V)_\sharp\mathbb{P}=\Phi$:

$$\mathbb{E}[V|X]=\boldsymbol{b}_\Phi\circ X\quad\text{in } L^2(\Omega,\sigma(X),\mathbb{P};\mathsf{X}).$$

It turns out that, if $\mathbf{F}$ is maximal totally dissipative (or, equivalently, its Lagrangian representation $\boldsymbol{B}$ is maximal dissipative), then $\mathbf{F}$ is invariant with respect to the barycentric projection:

$$(X,V)_\sharp\mathbb{P}=\Phi\in\mathbf{F}\quad\Longrightarrow\quad(\boldsymbol{i}_\mathsf{X},\boldsymbol{b}_\Phi)_\sharp\mu\in\mathbf{F},\quad(X,\mathbb{E}[V|X])\in\boldsymbol{B}. \tag{1.11}$$

Thanks to (1.11), for every $\mu_0\in \mathrm{D}(\mathbf{F})$, the solution μ_t expressed by the Lagrangian formula (1.9) can be characterized as a Lipschitz curve in $\mathcal{P}_2(\mathsf{X})$ satisfying the continuity equation

$$\partial_t\mu_t+\nabla\cdot(\mu_t\,\boldsymbol{v}_t)=0\quad\text{in } (0,+\infty)\times\mathsf{X} \tag{1.12}$$

for a Borel vector field $\boldsymbol{v}$ satisfying

$$t\mapsto\int|\boldsymbol{v}_t(x)|^2\,\mathrm{d}\mu_t(x)\quad\text{is locally integrable in } [0,+\infty),\quad(\boldsymbol{i}_\mathsf{X},\boldsymbol{v}_t)_\sharp\mu_t\in\mathbf{F}\text{ for a.e. } t>0. \tag{1.13}$$

We may also equivalently write (1.12) in a weaker form assuming that there exists a Borel family Φ_t, $t>0$, such that

$$\Phi_t\in\mathbf{F}[\mu_t]\quad\text{for a.e. } t>0,\quad t\mapsto\int|v|^2\,\mathrm{d}\Phi_t\quad\text{is locally integrable in } [0,+\infty),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int\zeta\,\mathrm{d}\mu_t=\int\langle v,\nabla\zeta(x)\rangle\,\mathrm{d}\Phi_t(x,v)\quad\text{for every } \zeta\in\mathrm{Cyl}(\mathsf{X})\text{ and a.e. } t>0.$$

A more precise formulation of (1.12) and (1.13) can be obtained by introducing the minimal selection $\boldsymbol{B}^\circ$ (i.e. the element of minimal norm) of $\boldsymbol{B}$: we will prove that for every $X\in\mathrm{D}(\boldsymbol{B})$ with $X_\sharp\mathbb{P}=\mu$, $\boldsymbol{B}^\circ$ is associated with a vector field $\boldsymbol{f}^\circ\in L^2(\mathsf{X},\mu;\mathsf{X})$ through the formula

$$V=\boldsymbol{B}^\circ X,\quad X_\sharp\mathbb{P}=\mu\quad\Longleftrightarrow\quad V=\boldsymbol{f}^\circ[\mu](X). \tag{1.14}$$

The measure $(\boldsymbol{i}_\mathsf{X},\boldsymbol{f}^\circ[\mu])_\sharp\mu$ can be characterized as the unique element $\Phi\in\mathbf{F}[\mu]$ minimizing $|\Phi|_2$ and the solution $\mu:[0,+\infty)\to\mathcal{P}_2(\mathsf{X})$ provided by (1.9) is also the unique Lipschitz curve satisfying the continuity equation

$$\partial_t\mu_t+\nabla\cdot(\mu_t\,\boldsymbol{f}^\circ[\mu_t])=0\quad\text{in } (0,+\infty)\times\mathsf{X} \tag{1.15}$$

with initial datum $\mu_0\in\mathrm{D}(\mathbf{F})$. It is remarkable that a maximal totally dissipative MPVF always admits a minimal selection which is concentrated on a map.

It turns out that the evolution driven by $\mathbf{F}$ preserves the class of discrete measures with finite support; if moreover $\mu_0=\frac{1}{N}\sum_{n=1}^N\delta_{x_n}\in\mathrm{D}(\mathbf{F})$ (or also in $\overline{\mathrm{D}(\mathbf{F})}$ if X has finite dimension) then the unique solution of (1.15) can be expressed in the form $\mu_t=\frac{1}{N}\sum_{n=1}^N\delta_{\mathsf{x}_n(t)}$ where $t\mapsto\mathsf{x}_n(t)$ are locally Lipschitz curves satisfying the system of ODEs

$$\dot{\mathsf{x}}_n(t)=\boldsymbol{f}^\circ[\mu_t](\mathsf{x}_n(t))\quad\text{a.e. in } (0,+\infty),\quad\mathsf{x}_n(0)=x_n,\quad n=1,\cdots,N. \tag{1.16}$$

Thanks to (1.10), if a sequence of discrete initial measures $\mu_0^N=\frac{1}{N}\sum_{n=1}^N\delta_{x_n^N}$ converges to a limit μ_0 in $\mathcal{P}_2(\mathsf{X})$ as $N\to\infty$, then the corresponding evolving measures μ_t^N obtained by solving (1.16) starting from μ_0^N will converge to $\mu_t=S_t(\mu_0)$. As a general fact [Szn91], this correspond to the propagation of chaos for the sequence of symmetric particle systems (1.16).

Maximality also shows that EVI curves are unique; when they are differentiable (in particular when $\mu_0\in\mathrm{D}(\mathbf{F})$) we can recover the representation (1.15) and the Lagrangian representation (1.9), in an even more refined version involving characteristic curves. This representation immediately yields regularity, stability, perturbation, and approximation results thanks to the corresponding statements in the Hilbertian framework.

Among the possible applications, we just recall that we can also use *the Implicit Euler Method* (corresponding to the JKO scheme for gradient flows) to construct the flow (Corollary 4.7).

Starting from $M^0_\tau := \mu_0 \in D(\mathbf{F})$, for every step size $\tau > 0$ we can find a (unique) sequence $(M^n_\tau)_{n\in\mathbb{N}}$ in $D(\mathbf{F})$ which at each step $n \in \mathbb{N}$ solves

$$(\mathsf{x} - \tau\mathsf{v})_\sharp \Phi^{n+1}_\tau = M^n_\tau \quad \text{for some } \Phi^{n+1}_\tau \in \mathbf{F}[M^{n+1}_\tau]. \tag{1.17}$$

Selecting $\tau := t/N$, the sequence $N \mapsto M^N_{t/N}$ converges to $S_t(\mu_0)$ as $N \to \infty$ with the a-priori error estimate

$$W_2(S_t(\mu_0), M^N_{t/N}) \le \frac{2t}{\sqrt{N}} \|\boldsymbol{f}^\circ[\mu_0]\|_{L^2(\mathsf{X},\mu_0;\mathsf{X})}. \tag{1.18}$$

When $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$ and $\mathbf{F}$ is single-valued as in (1.5), it follows that maximality is equivalent to the following demicontinuity condition: for every sequence $(\mu_n)_{n\in\mathbb{N}}$ converging to μ in $\mathcal{P}_2(\mathsf{X})$ one has

$$\sup_{n\to\infty} \int |\boldsymbol{f}[\mu_n]|^2 \,\mathrm{d}\mu_n < \infty, \quad (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}[\mu_n])_\sharp\mu_n \to (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}[\mu])_\sharp\mu \quad \text{in } \mathcal{P}(\mathsf{X} \times \mathsf{X}^w), \tag{1.19}$$

where X^w denotes the Hilbert space endowed with its weak topology. Clearly, in this case the map $\boldsymbol{f}$ representing $\mathbf{F}$ coincides with $\boldsymbol{f}^\circ$. Notice that (1.19) surely holds if $\mathbf{F}$ is represented by a map $\boldsymbol{f} : \mathcal{P}_2(\mathsf{X}) \to \mathrm{Lip}(\mathsf{X};\mathsf{X})$ (see also the map F' in [CB18, Section 2.3]) satisfying the integrated Lipschitz-like condition along arbitrary couplings

$$\int \Big|\boldsymbol{f}[\mu_0](x_0) - \boldsymbol{f}[\mu_1](x_1)\Big|^2 \mathrm{d}\boldsymbol{\mu}(x_0,x_1) \le L^2 \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0,x_1) \quad \text{for every } \boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1). \tag{1.20}$$

On the other hand, this class of regular dissipative PVFs is sufficiently rich to approximate the minimal selection of any maximal totally dissipative MPVF $\mathbf{F}$: in fact, by using the Yosida approximation, it is possible to find a sequence of regular PVFs $\mathbf{F}_n$ associated to Lipschitz fields $\boldsymbol{f}_n$ according to (1.20) (w.r.t. a possibly diverging sequence of Lipschitz constant L_n) satisfying the dissipativity condition (1.7) and

$$\lim_{n\to\infty} \int \big|\boldsymbol{f}_n[\mu](x) - \boldsymbol{f}^\circ[\mu](x)\big|^2 \,\mathrm{d}\mu(x) = 0 \quad \text{for every } \mu \in D(\mathbf{F}).$$

So, the class of totally dissipative MPVFs arises as a natural closure of more regular PVFs concentrated on dissipative Lipschitz maps. This statement (Corollary 3.23) justifies a posteriori the choice of the strongest notion of total dissipativity given in (1.8).
We cannot develop here all the applications of the abstract Hilbertian theory (which we aim to study in a future review paper) in the measure-theoretic setting: an incomplete list contains:

- regularizing effects under suitable assumptions on $\mathbf{F}$,
- asymptotic behaviour,
- error estimates for the Yosida regularization and for time discretizations (see also [CSS23a]), Chernoff and Trotter formulas,
- stability and convergence of sequences of λ-contractive semigroups,
- discrete to continuous limit and chaos propagation,
- the case of time dependent MPVFs,
- applications to the stochastic gradient descent method.

We will however devote a particular effort to clarify some relations between totally dissipative and metrically dissipative MPVFs, showing that they are intimately connected with the possibility to construct a MPVF starting from discrete measures.

Construction of a maximal totally dissipative* MPVF *from a discrete core. We investigate the second issue ⟨Q.2⟩ in Section 8, i.e. how to recover a (unique) maximal totally dissipative "version" of a (totally or metrically) λ-dissipative MPVF $\mathbf{F}$ defined on a sufficiently rich *core* C of discrete measures. This corresponds to the derivation of a mean-field description from a compatible family of discrete particle systems.

Just to give an idea of a simple case of core, we consider a totally convex subset D of the set $\mathcal{P}_f(\mathsf{X})$ of discrete measures with finite support: total convexity here means that, whenever the marginals $\mathsf{x}^i_\sharp\boldsymbol{\mu}$, $i=0,1$, of $\boldsymbol{\mu}\in\mathcal{P}_f(\mathsf{X}\times\mathsf{X})$ belong to D, then also $((1-t)\mathsf{x}^0+t\mathsf{x}^1)_\sharp\boldsymbol{\mu}$ belong to D for every $t\in(0,1)$.
For every $N\in\mathbb{N}$ we consider the collection C_N of uniform discrete measures $\mu_{\boldsymbol{x}}=\frac{1}{N}\sum_{n=1}^N\delta_{x_n}$ belonging to D, where $\boldsymbol{x}=(x_1,\cdots,x_N)$ is a vector in X^N with distinct coordinates. The set C_N corresponds to a subset C_N of X^N which is invariant under the action of the group of permutations $\mathrm{Sym}(N)$ of the components,

$$\sigma\boldsymbol{x}:=(x_{\sigma(1)},\cdots,x_{\sigma(N)}),\quad\text{for every}\quad\sigma\in\mathrm{Sym}(N),\ \boldsymbol{x}=(x_1,\cdots,x_N)\in\mathsf{X}^N.$$

We will suppose that C_N is relatively open in X^N for every $N\in\mathbb{N}$. Examples of D are provided by the collection of all the discrete measures μ such that $\mathrm{supp}(\mu)$ is contained in a given convex open subset U of X. Another interesting case, assuming $0\in\mathsf{U}$, is given by all the discrete measures such that $\mathrm{supp}(\mu)-\mathrm{supp}(\mu)\subset\mathsf{U}$. The case of the whole set $\mathcal{P}_f(\mathsf{X})$ is still interesting.
Suppose that we have a deterministic single-valued PVF $\mathbf{F}$ defined in $\mathrm{C}=\bigcup_N\mathrm{C}_N$ (when $\mathbf{F}$ is not deterministic, the construction is more subtle). We can then represent $\mathbf{F}$ on each C_N by a vector field $\boldsymbol{f}^N:\mathsf{C}_N\to\mathsf{X}^N$ satisfying the invariance property $\boldsymbol{f}^N(\sigma\boldsymbol{x})=\sigma\boldsymbol{f}^N(\boldsymbol{x})$, so that

$$\mathbf{F}[\mu_{\boldsymbol{x}}]=\frac{1}{N}\sum_{n=1}^N\delta_{(x_n,\boldsymbol{f}^N_n(\boldsymbol{x}))}\quad\text{for every }\boldsymbol{x}\in\mathsf{C}_N,$$

and, at least for a short time, the evolution of discrete measures in C_N can be described by $\mu_t=\frac{1}{N}\sum_{n=1}^N\delta_{x_n(t)}=\mu_{\boldsymbol{x}(t)}$ where the vector $\boldsymbol{x}(t)=(x_1(t),\cdots x_N(t))\in\mathsf{C}_N$ solves the system

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{f}^N(\boldsymbol{x}(t)).\tag{1.21}$$

We assume the following λ-dissipativity conditions on the maps $\boldsymbol{f}^N$: for every pair of integers $M,N\in\mathbb{N}$ with $M\mid N$, if $\boldsymbol{x}\in\mathsf{C}_M$, $\boldsymbol{y}\in\mathsf{C}_N$ and θ is an optimal correspondence from $\{1,\cdots,N\}$ to $\{1,\cdots,M\}$ such that

$$\mu_{\boldsymbol{x}}=\frac{1}{N}\sum_{n=1}^N\delta_{x_{\theta(n)}}\quad\text{and}\quad\frac{1}{N}\sum_{n=1}^N|y_n-x_{\theta(n)}|^2=W_2^2(\mu_{\boldsymbol{x}},\nu_{\boldsymbol{y}}),$$

then

$$\sum_{n=1}^N\langle\boldsymbol{f}^N_n(\boldsymbol{y})-\boldsymbol{f}^M_{\theta(n)}(\boldsymbol{x}),y_n-x_{\theta(n)}\rangle\le\lambda\sum_{n=1}^N|y_n-x_{\theta(n)}|^2.$$

We will show that $\mathbf{F}$ is in fact totally λ-dissipative and admits a unique maximal extension $\hat{\mathbf{F}}$, whose flow can be interpreted as the unique mean-field limit of the particle systems driven by (1.21). This fact guarantees two interesting properties: the local in time evolution corresponding to (1.21) admits a unique global extension which induces a semigroup $(S^N_t)_{t\ge0}$ on $\overline{\mathrm{C}_N}$ which corresponds to the restriction to $\overline{\mathrm{C}_N}$ of the semigroup S_t generated by $\hat{\mathbf{F}}$ (and characterized e.g. by the continuity equation (1.15) and by (1.16)). Moreover, thanks to (1.10) for every $\mu_0\in\overline{\mathrm{C}}$ and every sequence $(\mu_0^N)_{N\in\mathbb{N}}$ with $\mu_0^N\in\mathrm{C}_N$ and converging to μ_0 as $N\to\infty$ we have $S^N_t(\mu^N_0)\to S_t(\mu_0)$ in $\mathcal{P}_2(\mathsf{X})$ locally uniformly w.r.t. $t\in[0,+\infty)$.
Thanks to the stability properties of the Lagrangian flow, Theorem 4.9 also shows that the trajectories of the discrete particle system uniformly converge in a measure-theoretic sense to the characteristics of the mean-field system.
As a byproduct, we obtain that when the domain of a totally dissipative MPVF $\mathbf{F}$ contains a dense core then its maximal extension is unique and can be characterized by a suitable explicit construction starting from the core itself and its flow has a natural mean-field interpretation.

Our result also provides interesting applications to geodesically convex functionals and their approximations (see Sections 5,9).
First of all, if the proper domain of a lower semicontinuous and geodesically convex functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ contains a discrete core C which is dense in energy, then ϕ is totally convex, i.e. it is convex along all the linear interpolations induced by arbitrary couplings. An important class is provided by continuous and everywhere defined geodesically convex functionals, which thus turn out to be totally convex.
The same property holds for any functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ which arises as Mosco-like limit of a sequence of continuous and geodesically convex functionals which are everywhere finite. In particular, such approximation is impossible for all the functionals which are not totally convex, as the relative entropy functionals w.r.t. log-concave measures.

Plan of the paper. After a quick review in **Section 2** of the main tools on Wasserstein spaces used in the sequel, we summarize in Subsection 2.2 the notation and the results concerning Multivalued Probability Vector Fields and EVI solutions.
In **Section 3**, we introduce the notion of *totally dissipative* MPVF and we study its consequences in terms of existence and description of Lagrangian solutions: in Subsection 3.1 we study the properties of the Yosida approximations, the resolvent operator and the minimal selection of law-invariant operators in the Hilbert space $\mathcal{X}$ of parametrizations, Subsection 3.2 deals with the relation between dissipativity for such law-invariant subsets of $\mathcal{X}$ and the corresponding total dissipativity for their law. These results are used in Subsection 3.3 to study the particular case of deterministic totally dissipative PVFs.
Section 4 contains the main existence, uniqueness, stability, and approximation results for the Lagrangian flow generated by a totally dissipative MPVF, together with its various equivalent characterizations.

In **Section 5**, we study the behaviour of functionals $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ which are convex along any coupling, proving the existence of gradient flows (equivalently, EVI solutions for the MPVF given by their subdifferential) still exploiting their representation in terms of a convex functional ψ defined in the parametrization space $\mathcal{X}$.

Section 6 is devoted to study the properties of couplings between discrete measures, in particular showing that such couplings are "piece-wise" optimal. This property is then exploited in **Section 7** where we show that a dissipative MPVF is totally dissipative along discrete couplings.

In **Section 8** we show that starting from a dissipative MPVF $\mathbf{F}$ definied on a sufficiently rich *core* C of discrete measures, it is possible to construct a maximal totally dissipative MPVF $\hat{\mathbf{F}}$, in a unique canonical way.
Section 9 is in the same spirit but in the case of a geodesically convex functional ϕ: under analogous approximation properties, it is possible to show that ϕ is actually totally convex and then satisfies the assumptions of Section 5.

Finally, **Appendix A** contains many useful results related to λ-dissipative operators in Hilbert spaces that are more commonly known for $\lambda = 0$ (the main reference is [Bré73]), while **Appendix B** lists some of the results of [CSS23b] related to Borel partitions and approximations of couplings that are used in the present work.

Acknowledgments. G.S. and G.E.S. gratefully acknowledge the support of the Institute for Advanced Study of the Technical University of Munich, funded by the German Excellence Initiative. G.C. and G.S. have been supported by the MIUR-PRIN 2017 project *Gradient flows, Optimal Transport and Metric Measure Structures*. G.C. also acknowledges the partial support of INDAM-GNAMPA project 2022 *Evoluzione e controllo ottimo in spazi di Wasserstein* (CUP_E55F22000270001) and the funds FSR Politecnico di Milano Prog.TDG3ATEN02. G.S. also thanks IMATI-CNR, Pavia.

2. Preliminaries

In the following table, we provide a list of the adopted symbology for the reader's convenience. We then recall the main notions and results of Optimal Transport theory and finally, in Subsection 2.2, we collect the fundamental objects and basic results taken from [CSS23a] needed to develop our analysis. As a general rule, we will use bold letters to denote maps (even multivalued) with values in the Hilbert space X or measures/sets of measures in product spaces as couplings in $\mathsf{X}\times\mathsf{X}$ or probability vector fields in TX.

$\boldsymbol{b}_\Phi$	the barycenter of $\Phi\in\mathcal{P}(\mathsf{TX})$ as in Definition 2.3;
$\boldsymbol{B}^\lambda$	the λ-transformation of a set $\boldsymbol{B}$ as in Remark A.1;
$\boldsymbol{B}_\tau$	Yosida approximation of a maximal dissipative $\boldsymbol{B}$, see Appendix A;
$\boldsymbol{B}^\circ$	minimal selection of a maximal dissipative $\boldsymbol{B}$, see Appendix A;
$\mathrm{cl}(\mathbf{F})$	the sequential closure of $\mathbf{F}$, see Proposition 2.20;
$\mathrm{co}(E), \overline{\mathrm{co}}\,(E)$	convex and closed and convex envelope of a set E in a Hilbert space;
$\mathrm{D}(\mathbf{F})$	the proper domain of a set-valued function as in Definition 2.14;
$\mathrm{D}(\phi)$	the proper domain of a functional ϕ;
$\boldsymbol{f}^\circ$	the map defined in Theorem 3.17;
$\mathbf{F}, \mathbf{F}[\mu]$	a multivalued probability vector field and its section at $\mu\in\mathcal{P}_2(\mathsf{X})$, see Definition 2.14;
$\mathbf{F}^\lambda$	the λ-transformation of $\mathbf{F}$ as in (2.18);
$\Gamma(\mu,\nu)$	the set of admissible couplings between μ,ν, see (2.1);
$\Gamma_o(\mu,\nu)$	the set of optimal couplings between μ,ν, see Definition 2.3;
$\iota, \iota^2, \iota_X, \iota^2_{X,Y}$	the maps as in the beginning of Section 3;
$\boldsymbol{i}_X$	the identity map on a set X;
$\boldsymbol{J}_\tau$	the resolvent operator of a maximal dissipative $\boldsymbol{B}$, see Appendix A;
$\mathsf{m}_2(\nu)$	the 2-nd moment of $\nu\in\mathcal{P}_2(X)$ as in Definition 2.3;
$\lvert\Phi\rvert_2$	the partial 2-nd moment of $\Phi\in\mathcal{P}_2(\mathsf{TX})$ as in (2.5);
$\mathfrak{N}$	a directed subset of $\mathbb{N}$ w.r.t. the order induced by $\prec$, see Appendix B;
$(\Omega,\mathcal{B},\mathbb{P})$	a standard Borel space endowed with a nonatomic probability measure, Def. B.1;
$(\Omega,\mathcal{B},\mathbb{P},(\mathfrak{P}_N)_{N\in\mathfrak{N}})$	a $\mathfrak{N}$-refined standard Borel probability space, see Definition B.3;
$\mathcal{P}(X)$	the set of Borel probability measures on the topological space X;
$\mathcal{P}_f(X)$, $\mathcal{P}_{f,N}(X)$	the sets defined in (3.19),(3.20);
$\mathcal{P}_{f,\mathfrak{N}}, \mathcal{P}_{\#\mathfrak{N}}(X)$	the sets in (8.2);
$\mathcal{P}_c(X), \mathcal{P}_2(X)$	measures in $\mathcal{P}(X)$ with compact support or finite quadratic moment, see (2.6);
$\mathcal{P}_2^{sw}(\mathsf{TX})$	the space $\mathcal{P}_2(\mathsf{TX})$ endowed with the strong-weak topology as in Definition 2.4;
$\pi^i, \pi^{i,j}, \pi^{i,j,k}, \pi^{i,j,k,l}$	projections from a product space to one or more factors as in (2.1);
$[\cdot,\cdot]_r$, $[\cdot,\cdot]_l$	the pseudo scalar products as in Definition 2.12;
$[\Phi,\boldsymbol{\vartheta}]_{r,t}$, $[\Phi,\boldsymbol{\vartheta}]_{l,t}$	the duality pairings as in Definition 2.12;
$[\mathbf{F},\boldsymbol{\mu}]_{r,t}$, $[\mathbf{F},\boldsymbol{\mu}]_{l,t}$	the duality pairings as in Definition 2.18;
$\mathrm{S}(\Omega)=\mathrm{S}(\Omega,\mathcal{B},\mathbb{P})$	measure-preserving isomorphisms on $(\Omega,\mathcal{B},\mathbb{P})$, see Appendix B;
$\mathrm{S}_N(\Omega)$	subset of $\mathrm{S}(\Omega,\mathcal{B},\mathbb{P})$ of maps that are $\mathcal{B}_N-\mathcal{B}_N$ measurable;
$S_t, \boldsymbol{s}_t$	Eulerian and Lagrangian semigroups, Def. 4.1;
$\boldsymbol{S}_t$	semigroup generated by a maximal dissipative $\boldsymbol{B}$, see Appendix A;
$\mathcal{S}\,(\mathsf{X},D)\,,\,\mathcal{S}\,(\mathsf{X})$	the subsets of $\mathsf{X}\times\mathcal{P}_2(\mathsf{X})$ as in (2.15);
$W_2(\mu,\nu)$	the L^2-Wasserstein distance between μ and ν, see Definition 2.3;
X	a separable Hilbert space;
$\mathcal{X}, \mathcal{X}_N, \mathcal{X}_\infty$	the Hilbert spaces $L^2(\Omega,\mathcal{B},\mathbb{P};\mathsf{X})$, $L^2(\Omega,\mathcal{B}_N,\mathbb{P};\mathsf{X})$ and the union of the $\mathcal{X}_N$ respectively;
$\mathsf{X}^s, \mathsf{X}^w$	a separable Hilbert space endowed with its strong and weak topologies;
TX	the tangent bundle to X, usually endowed with the strong-weak topology;
$\mathsf{x}, \mathsf{x}^i, \mathsf{v}, \mathsf{v}^i$	the projection maps defined in (2.2) and in Section 2.2;
x^t	the evaluation map defined in (2.4).

In this first section of general measure theory preliminaries, we consider X, Y to be Lusin completely regular topological spaces. We recall that a topological space X is *completely regular* if it is Hausdorff and for every closed set C and point $x\in X\setminus C$ there exists $f: X\to[0,1]$

continuous function s.t. $f(C) = \{1\}$ and $f(x) = 0$. This general setting is convenient to be adapted to our analysis which deals with Borel probability measures defined in (subsets of) a separable Hilbert space X, which could be endowed with the strong or the weak topology.
We denote by $\mathcal{P}(X)$ the set of Borel probability measures on X endowed with the weak/narrow topology induced by the duality with the space of real valued continuous and bounded functions $\mathrm{C}_b(X)$. Thus, given a directed set $\mathbb{A}$, we say that a net $(\mu_\alpha)_{\alpha\in\mathbb{A}} \subset \mathcal{P}(X)$ converges narrowly to $\mu \in \mathcal{P}(X)$, and we write $\mu_\alpha \to \mu$ in $\mathcal{P}(X)$, if

$$\lim_\alpha \int_X \varphi \,\mathrm{d}\mu_\alpha = \int_X \varphi \,\mathrm{d}\mu \quad \forall \varphi \in \mathrm{C}_b(X).$$

Given $\mu \in \mathcal{P}(X)$ and a Borel function $f : X \to Y$, we define the *push-forward* $f_\sharp\mu \in \mathcal{P}(Y)$ of μ through f by

$$\int_Y \varphi \,\mathrm{d}(f_\sharp\mu) = \int_X \varphi \circ f \,\mathrm{d}\mu$$

for every $\varphi : Y \to \mathbb{R}$ bounded (or nonnegative) Borel function.
We recall the so-called *disintegration theorem* (see e.g. [AGS08, Theorem 5.3.1]).

Theorem 2.1. *Let $\mathbb{X}, X$ be Lusin completely regular topological spaces, $\boldsymbol{\mu} \in \mathcal{P}(\mathbb{X})$ and $r : \mathbb{X} \to X$ a Borel map. Denote with $\mu = r_\sharp\boldsymbol{\mu} \in \mathcal{P}(X)$. Then there exists a μ-a.e. uniquely determined Borel family of probability measures $\{\boldsymbol{\mu}_x\}_{x\in X} \subset \mathcal{P}(\mathbb{X})$ such that $\boldsymbol{\mu}_x(\mathbb{X} \setminus r^{-1}(x)) = 0$ for μ-a.e. $x \in X$, and*

$$\int_{\mathbb{X}} \varphi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{\mu}(\boldsymbol{x}) = \int_X \left(\int_{r^{-1}(x)} \varphi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{\mu}_x(\boldsymbol{x}) \right) \mathrm{d}\mu(x)$$

for every bounded Borel map $\varphi : \mathbb{X} \to \mathbb{R}$.

Remark 2.2. When $\mathbb{X} = X_1 \times X_2$ and r is the projection π^1 on the first component, we can canonically identify the disintegration $\{\boldsymbol{\mu}_x\}_{x\in X_1} \subset \mathcal{P}(\mathbb{X})$ of $\boldsymbol{\mu} \in \mathcal{P}(X_1 \times X_2)$ w.r.t. $\mu = \pi^1_\sharp\boldsymbol{\mu}$ with a family of probability measures $\{\mu_{x_1}\}_{x_1\in X_1} \subset \mathcal{P}(X_2)$. We write $\boldsymbol{\mu} = \int_{X_1} \mu_{x_1} \,\mathrm{d}\mu(x_1)$.

Given $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, we define the set of admissible transport plans

$$\Gamma(\mu,\nu) := \left\{ \boldsymbol{\gamma} \in \mathcal{P}(X \times Y) \mid \pi^1_\sharp\boldsymbol{\gamma} = \mu\,,\, \pi^2_\sharp\boldsymbol{\gamma} = \nu \right\}, \tag{2.1}$$

where we denoted by π^i, $i = 1, 2$, the projection on the i-th component and we call $\pi^i_\sharp\boldsymbol{\gamma}$ the i-th marginal of $\boldsymbol{\gamma}$.

2.1. **Wasserstein distance in Hilbert spaces and strong-weak topology**

From now on, we denote by X a separable (possibly infinite dimensional) Hilbert space with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. When it is necessary to specify it, we denote by X^s (resp. X^w) the Hilbert space X endowed with its strong (resp. weak) topology. We remark that X^w is a Lusin completely regular space and that X^s and X^w share the same class of Borel sets and thus of Borel probability measures. Therefore, we are allowed to adopt the simpler notation $\mathcal{P}(\mathsf{X})$ and to use the heavier $\mathcal{P}(\mathsf{X}^s)$ and $\mathcal{P}(\mathsf{X}^w)$ only when we will refer to the correspondent topology.
We adopt the notation TX for the tangent bundle to X, which is identified with the cartesian product $\mathsf{X} \times \mathsf{X}$ with the induced norm $|(x,v)| := \left(|x|^2 + |v|^2\right)^{1/2}$ and the strong-weak topology of $\mathsf{X}^s \times \mathsf{X}^w$(i.e. the product of the strong topology on the first component and the weak topology on the second one). The set $\mathcal{P}(\mathsf{TX})$ is defined thanks to the identification of TX with $\mathsf{X} \times \mathsf{X}$ and it is endowed with the narrow topology induced by the strong-weak topology in TX.
We will denote by $\mathsf{x}, \mathsf{v} : \mathsf{TX} \to \mathsf{X}$ the projection maps defined by

$$\mathsf{x}(x,v) := x, \quad \mathsf{v}(x,v) = v. \tag{2.2}$$

When dealing with the product space X^2 we use the notation

$$\mathsf{s} : \mathsf{X}^2 \to \mathsf{X}^2, \quad \mathsf{s}(x_0, x_1) := (x_1, x_0), \tag{2.3}$$

$$\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}, \quad \mathsf{x}^t(x_0, x_1) := (1-t)x_0 + t x_1, \quad t \in [0,1]. \tag{2.4}$$

Definition 2.3. *Given $\mu \in \mathcal{P}(\mathsf{X})$ and $\Phi \in \mathcal{P}(\mathsf{TX})$ we define*

$$\mathsf{m}_2^2(\mu) := \int_{\mathsf{X}} |x|^2 \,\mathrm{d}\mu(x), \quad |\Phi|_2^2 := \int_{\mathsf{TX}} |v|^2 \,\mathrm{d}\Phi(x,v) \tag{2.5}$$

and the spaces

$$\mathcal{P}_2(\mathsf{X}) := \{\mu \in \mathcal{P}(\mathsf{X}) \mid \mathsf{m}_2(\mu) < +\infty\}, \quad \mathcal{P}_2(\mathsf{TX}|\mu) := \Big\{\Psi \in \mathcal{P}(\mathsf{TX}) : \mathsf{x}_\sharp \Psi = \mu,\ |\Psi|_2 < \infty\Big\}. \tag{2.6}$$

Given $\Psi \in \mathcal{P}_2(\mathsf{TX}|\mu)$, the barycenter of Ψ *is the function $\boldsymbol{b}_\Psi \in L^2(\mathsf{X}, \mu; \mathsf{X})$ defined by*

$$\boldsymbol{b}_\Psi(x) := \int_{\mathsf{X}} v \,\mathrm{d}\Psi_x(v) \quad \text{for } \mu\text{-a.e. } x \in \mathsf{X}, \tag{2.7}$$

where $\{\Psi_x\}_{x\in\mathsf{X}} \subset \mathcal{P}_2(\mathsf{X})$ is the disintegration of Ψ w.r.t. μ. We set $\mathrm{bar}\,(\Psi) := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{b}_\Psi)_\sharp \mu$. *We say that Ψ is* concentrated on a map *(or it is* deterministic*) if $\Psi = \mathrm{bar}\,(\Psi)$.*

For the following recalls on Wasserstein spaces we refer e.g. to [AGS08, §7]. On $\mathcal{P}_2(\mathsf{X})$ we define the L^2-Wasserstein distance W_2 by

$$W_2^2(\mu, \mu') := \inf\left\{ \int_{\mathsf{X}\times\mathsf{X}} |x-y|^2 \,\mathrm{d}\boldsymbol{\gamma}(x,y) \mid \boldsymbol{\gamma} \in \Gamma(\mu, \mu') \right\}. \tag{2.8}$$

For the sequel, the set $\Gamma_o(\mu, \mu')$ denotes the subset of admissible plans in $\Gamma(\mu, \mu')$ realizing the infimum in (2.8). We say that a measure $\boldsymbol{\gamma} \in \mathcal{P}_2(\mathsf{X}\times\mathsf{X})$ is optimal if $\boldsymbol{\gamma} \in \Gamma_o(\pi^1_\sharp \boldsymbol{\gamma}, \pi^2_\sharp \boldsymbol{\gamma})$. We recall that $\boldsymbol{\gamma} \in \mathcal{P}_2(\mathsf{X}\times\mathsf{X})$ is optimal if and only if its support is cyclically monotone i.e.

$$\begin{gathered} \text{for every } N \in \mathbb{N} \text{ and } \{(x_n, y_n)\}_{n=1}^N \subset \operatorname{supp} \boldsymbol{\gamma} \text{ with } x_0 := x_N \text{ we have} \\ \sum_{n=1}^N \langle y_n, x_n - x_{n-1} \rangle \geq 0. \end{gathered} \tag{2.9}$$

We recall that the metric space $(\mathcal{P}_2(\mathsf{X}), W_2)$ is a complete and separable metric space and the W_2-convergence (sometimes denoted with $\xrightarrow{W_2}$) is stronger than the narrow convergence. More precisely, if $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{X})$ and $\mu \in \mathcal{P}_2(\mathsf{X})$, the following holds (see [AGS08, Remark 7.1.11])

$$\mu_n \xrightarrow{W_2} \mu, \text{ as } n \to +\infty \quad \Longleftrightarrow \quad \begin{cases} \mu_n \to \mu \text{ in } \mathcal{P}(\mathsf{X}^s), \\ \mathsf{m}_2(\mu_n) \to \mathsf{m}_2(\mu), \end{cases} \quad \text{as } n \to +\infty.$$

In the following Definition 2.4 and Proposition 2.5, we recall the topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$ (see [NS21; CSS23a]).

Definition 2.4 (Strong-weak topology in $\mathcal{P}_2(\mathsf{TX})$)**.** *We denote by $\mathcal{P}_2^{sw}(\mathsf{TX})$ the space $\mathcal{P}_2(\mathsf{TX})$ endowed with the coarsest topology which makes the following functions continuous*

$$\Phi \mapsto \int \zeta(x,v) \,\mathrm{d}\Phi(x,v), \quad \zeta \in \mathrm{C}_2^{sw}(\mathsf{TX}),$$

where $\mathrm{C}_2^{sw}(\mathsf{TX})$ is the Banach space of test functions $\zeta : \mathsf{TX} \to \mathbb{R}$ such that

$$\zeta \text{ is sequentially continuous in } \mathsf{TX} = \mathsf{X}^s \times \mathsf{X}^w,$$

$$\forall\, \varepsilon > 0\ \exists A_\varepsilon \geq 0 : |\zeta(x,v)| \leq A_\varepsilon(1 + |x|^2) + \varepsilon |v|^2 \quad \text{for every } (x,v) \in \mathsf{TX}.$$

The following Proposition (whose proof can be found in [NS21]) summarizes some of the properties of the topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$.

Proposition 2.5.

(1) *If* $(\Phi_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{TX})$ *is a sequence and* $\Phi \in \mathcal{P}_2(\mathsf{TX})$, *then* $\Phi_n \to \Phi$ *in* $\mathcal{P}_2^{sw}(\mathsf{TX})$ *as* $n \to \infty$ *if and only if*
   (a) $\Phi_n \to \Phi$ *in* $\mathcal{P}(\mathsf{TX}) = \mathcal{P}(\mathsf{X}^s \times \mathsf{X}^w)$,
   (b) $\lim_{n\to+\infty} \int |x|^2 \,\mathrm{d}\Phi_n(x,v) = \int |x|^2 \,\mathrm{d}\Phi(x,v)$,
   (c) $\sup_n \int |v|^2 \,\mathrm{d}\Phi_n(x,v) < \infty$.

(2) *For every compact set* $\mathcal{K} \subset \mathcal{P}_2(\mathsf{X}^s)$ *and every constant* $c < \infty$ *the sets*

$$\mathcal{K}_c := \Big\{ \Phi \in \mathcal{P}_2(\mathsf{TX}) : \mathsf{x}_\sharp \Phi \in \mathcal{K}, \quad \int |v|^2 \,\mathrm{d}\Phi(x,v) \le c \Big\}$$

are sequentially compact in $\mathcal{P}_2^{sw}(\mathsf{TX})$.

For the sequel, we recall the concept and main properties of geodesics in $\mathcal{P}_2(\mathsf{X})$. We denote equivalently by $\mu(t)$ or μ_t the evaluation at time $t \in \mathcal{I} \subset \mathbb{R}$ of a curve $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$.

Definition 2.6 (Geodesics)**.** *A curve* $\mu : [0,1] \to \mathcal{P}_2(\mathsf{X})$ *is said to be a* (constant speed) geodesic *if for all* $0 \le s \le t \le 1$ *we have*

$$W_2(\mu_s, \mu_t) = (t-s) W_2(\mu_0, \mu_1).$$

We also say that μ *is a* geodesic from μ_0 to μ_1.

Definition 2.7 (Geodesic and total convexity)**.** *We say that* $A \subset \mathcal{P}_2(\mathsf{X})$ *is a* geodesically convex set *if for any pair* $\mu_0, \mu_1 \in A$ *there exists a geodesic* $\mu : [0,1] \to \mathcal{P}_2(\mathsf{X})$ *from* μ_0 *to* μ_1 *such that* $\mu_t \in A$ *for all* $t \in [0,1]$.
We say that $A \subset \mathcal{P}_2(\mathsf{X})$ *is* totally convex *if for any pair* $\mu_0, \mu_1 \in A$ *and any coupling* $\boldsymbol{\gamma} \in \Gamma(\mu_0, \mu_1)$, *we have that* $(\mathsf{x}^t)_\sharp \boldsymbol{\gamma} \in A$ *for any* $t \in [0,1]$.

Remark 2.8. Since total convexity will play a crucial role in the present paper, let us recall a few examples of totally convex sets in $\mathcal{P}_2(\mathsf{X})$, which are induced by a lower semicontinuous and convex function $P : \mathsf{X} \to (-\infty, +\infty]$ and a real number c: it is sufficient to consider the set of $\mu \in \mathcal{P}_2(\mathsf{X})$ satisfying one of the following conditions:

$$P\Big(\int x \,\mathrm{d}\mu(x)\Big) \le c, \qquad \int P(x) \,\mathrm{d}\mu(x) \le c, \qquad \int P(x-y) \,\mathrm{d}\mu \otimes \mu(x,y) \le c.$$

Clearly, one can replace large with strict inequalities in the previous formulae. Choosing P as the indicator function of a convex set $\mathsf{U} \subset \mathsf{X}$ (i.e. $P(x) = 0$ if $x \in \mathsf{U}$, $P(x) = +\infty$ otherwise), one obtains conditions confining the barycenter, $\operatorname{supp}\mu$, or $\operatorname{supp}\mu - \operatorname{supp}\mu$ to a given set U.

The following useful result (see [AGS08, Theorem 7.2.1, Theorem 7.2.2]) on geodesics also points out that total convexity is stronger than geodesic convexity.

Theorem 2.9 (Properties of geodesics)**.** *Let* $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ *and* $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$. *Then* $\mu : [0,1] \to \mathcal{P}_2(\mathsf{X})$ *defined by*

$$\mu_t := (\mathsf{x}^t)_\sharp \boldsymbol{\mu}, \quad t \in [0,1], \tag{2.10}$$

is a (constant speed) geodesic from μ_0 *to* μ_1. *Conversely, any (constant speed) geodesic* μ *from* μ_0 *to* μ_1 *admits the representation* (2.10) *for a suitable plan* $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$.
Finally, if μ *is a geodesic connecting* μ_0 *to* μ_1, *then for every* $t \in (0,1)$ *there exists a unique optimal plan* $\boldsymbol{\mu}_{t0}$ *between* μ_t *and* μ_0 *(resp.* $\boldsymbol{\mu}_{t1}$ *between* μ_t *and* μ_1*) and it is concentrated on a map w.r.t.* μ_t, *meaning that there exist Borel maps* $\boldsymbol{r}_t, \boldsymbol{r}'_t : \mathsf{X} \to \mathsf{X}$ *such that*

$$\boldsymbol{\mu}_{t0} = (\boldsymbol{i}_X, \boldsymbol{r}_t)_\sharp \mu_t, \quad \boldsymbol{\mu}_{t1} = (\boldsymbol{i}_X, \boldsymbol{r}'_t)_\sharp \mu_t.$$

The following defines the counterpart of $\mathrm{C}_c^\infty(\mathbb{R}^d)$ when $\mathbb{R}^d$ is replaced by X.

Definition 2.10 (The space $\mathrm{Cyl}(\mathsf{X})$ of cylindrical functions)**.** *We denote by* $\Pi_d(\mathsf{X})$ *the space of linear maps* $\pi : \mathsf{X} \to \mathbb{R}^d$ *of the form* $\pi(x) = (\langle x, e_1\rangle, \cdots, \langle x, e_d\rangle)$ *for an orthonormal set* $\{e_1, \cdots, e_d\}$ *of* X*. A function* $\varphi : \mathsf{X} \to \mathbb{R}$ *belongs to the space of cylindrical functions on* X*,* $\mathrm{Cyl}(\mathsf{X})$*, if it is of the form*

$$\varphi = \psi \circ \pi$$

where $\pi \in \Pi_d(\mathsf{X})$ *and* $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$*.*

Given $\nu \in \mathcal{P}_2(\mathsf{X})$ we define the tangent bundle to $\mathcal{P}_2(\mathsf{X})$ at ν by

$$\mathrm{Tan}_\nu \mathcal{P}_2(X) := \overline{\{\nabla\varphi \mid \varphi \in \mathrm{Cyl}(\mathsf{X})\}}^{L^2(X,\nu;X)}.$$

If $\mathcal{I} \subset \mathbb{R}$ is an open interval and $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ is a locally absolutely continuous curve, we define the *metric velocity* of μ at $t \in \mathcal{I}$ as

$$|\dot{\mu}_t|^2 := \lim_{h\to 0} \frac{W_2^2(\mu_{t+h}, \mu_t)}{h^2},$$

which exists for a.e. $t \in \mathcal{I}$.
The following result (see [AGS08, Theorem 8.3.1, Proposition 8.4.5 and Proposition 8.4.6]) characterizes locally absolutely continuous curves in $\mathcal{P}_2(\mathsf{X})$.

Theorem 2.11 (Wasserstein velocity field)**.** *Let* $\mu : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ *be a locally absolutely continuous curve defined in an open interval* $\mathcal{I} \subset \mathbb{R}$*. There exists a Borel vector field* $\boldsymbol{v} : \mathcal{I} \times \mathsf{X} \to \mathsf{X}$ *and a set* $A(\mu) \subset \mathcal{I}$ *with* $\mathscr{L}(\mathcal{I} \setminus A(\mu)) = 0$ *such that for every* $t \in A(\mu)$ *the following hold*

(1) $\boldsymbol{v}_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathsf{X})$*;*
(2) $\int_{\mathsf{X}} |\boldsymbol{v}_t|^2 \,\mathrm{d}\mu_t = |\dot{\mu}_t|^2$*;*
(3) the continuity equation $\partial_t\mu_t + \nabla \cdot (\boldsymbol{v}_t\mu_t) = 0$ *holds in the sense of distributions in* $\mathcal{I} \times \mathsf{X}$*.*

Moreover, $\boldsymbol{v}_t$ *is uniquely determined in* $L^2(\mathsf{X}, \mu_t; \mathsf{X})$ *for* $t \in A(\mu)$ *and*

$$\lim_{h\to 0} \frac{W_2((\boldsymbol{i}_{\mathsf{X}} + h\boldsymbol{v}_t)_\sharp\mu_t, \mu_{t+h})}{|h|} = 0 \quad \textit{for every } t \in A(\mu).$$

2.2. Duality pairings

In this subsection we collect the main objects involving duality pairings between measures in $\mathcal{P}_2(\mathsf{TX})$. We report here a summary of the results needed in the sequel and we refer to [CSS23a] for a wider discussion on this matter.
As usual, we denote by $\mathsf{x}^0, \mathsf{v}^0, \mathsf{x}^1 : \mathsf{TX} \times \mathsf{X} \to \mathsf{X}$ the projection maps of a point (x_0, v_0, x_1) into x_0, v_0 or x_1, respectively (and similarly with $\mathsf{x}^0, \mathsf{v}^0, \mathsf{x}^1, \mathsf{v}^1$ when they are defined in $\mathsf{TX} \times \mathsf{TX}$).

Definition 2.12 (Metric-duality pairings)**.** *For every* $\Phi_0, \Phi_1 \in \mathcal{P}_2(\mathsf{TX})$*,* $\mu_1 \in \mathcal{P}_2(\mathsf{X})$*,* $\boldsymbol{\vartheta} \in \mathcal{P}_2(\mathsf{X} \times \mathsf{X})$*,* $t \in [0, 1]$ *and* $\Psi \in \mathcal{P}_2(\mathsf{TX}|\mathsf{x}^t_\sharp\boldsymbol{\vartheta})$*, we set*

$$\begin{aligned}
\Lambda(\Phi_0, \mu_1) &:= \left\{\boldsymbol{\sigma} \in \Gamma(\Phi_0, \mu_1) \mid (\mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\sigma} \in \Gamma_o(\mathsf{x}_\sharp\Phi_0, \mu_1)\right\},\\
\Lambda(\Phi_0, \Phi_1) &:= \left\{\boldsymbol{\Theta} \in \Gamma(\Phi_0, \Phi_1) \mid (\mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\Theta} \in \Gamma_o(\mathsf{x}_\sharp\Phi_0, \mathsf{x}_\sharp\Phi_1)\right\},\\
\Gamma_t(\Psi, \boldsymbol{\vartheta}) &:= \left\{\boldsymbol{\sigma} \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{X}) \mid (\mathsf{x}^0, \mathsf{x}^1)_\sharp\boldsymbol{\sigma} = \boldsymbol{\vartheta}, \quad (\mathsf{x}^t \circ (\mathsf{x}^0, \mathsf{x}^1), \mathsf{v}^0)_\sharp\boldsymbol{\sigma} = \Psi\right\}.
\end{aligned}$$

We set

$$[\Phi_0, \mu_1]_r := \min \left\{ \int_{\mathsf{TX}\times\mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \Lambda(\Phi_0, \mu_1) \right\},$$

$$[\Phi_0, \mu_1]_l := \max \left\{ \int_{\mathsf{TX}\times\mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \Lambda(\Phi_0, \mu_1) \right\},$$

$$[\Phi_0, \Phi_1]_r := \min \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \mid \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\},$$

$$[\Phi_0, \Phi_1]_l := \max \left\{ \int_{\mathsf{TX}\times\mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \mid \boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1) \right\},$$

$$[\Psi, \boldsymbol{\vartheta}]_{r,t} := \min \left\{ \int \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma}(x_0, v_0, x_1) \mid \boldsymbol{\sigma} \in \Gamma_t(\Psi, \boldsymbol{\vartheta}) \right\},$$

$$[\Psi, \boldsymbol{\vartheta}]_{l,t} := \max \left\{ \int \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma}(x_0, v_0, x_1) \mid \boldsymbol{\sigma} \in \Gamma_t(\Psi, \boldsymbol{\vartheta}) \right\}.$$

The following Theorem summarizes some of the properties of duality pairings analyzed in [CSS23a].

Theorem 2.13. *The following properties hold.*

(1) (Inversion) *For every $\boldsymbol{\vartheta} \in \mathcal{P}_2(\mathsf{X}^2)$, $t \in [0,1]$, $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mathsf{x}^t_\sharp \boldsymbol{\vartheta})$ it holds*

$$[\Phi, \boldsymbol{\vartheta}]_{r,t} = -[\Phi, \mathsf{s}_\sharp \boldsymbol{\vartheta}]_{l,1-t},$$

where s *is as in* (2.3).

(2) (Comparison) *For every $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ and every $\Phi_0 \in \mathcal{P}_2(\mathsf{TX}|\mu_0)$, $\Phi_1 \in \mathcal{P}_2(\mathsf{TX}|\mu_1)$, it holds*

$$[\Phi_0, \mu_1]_r = \inf_{\boldsymbol{\vartheta} \in \Gamma_o(\mu_0,\mu_1)} [\Phi_0, \boldsymbol{\vartheta}]_{r,0}, \qquad [\Phi_0, \mu_1]_l = \sup_{\boldsymbol{\vartheta} \in \Gamma_o(\mu_0,\mu_1)} [\Phi_0, \boldsymbol{\vartheta}]_{l,0},$$

$$[\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_r \le [\Phi_0, \Phi_1]_r, \qquad [\Phi_0, \mu_1]_l + [\Phi_1, \mu_0]_l \ge [\Phi_0, \Phi_1]_l,$$

and

$$[\Phi_0, \Phi_1]_r \le [\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_l \le [\Phi_0, \Phi_1]_l.$$

(3) (Restriction) *For every $\boldsymbol{\vartheta} \in \mathcal{P}_2(\mathsf{X}^2)$, every $0 \le s < t \le 1$ and every $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mathsf{x}^s_\sharp \boldsymbol{\vartheta})$, $\Psi \in \mathcal{P}_2(\mathsf{TX}|\mathsf{x}^t_\sharp \boldsymbol{\vartheta})$ we have*

$$[\Phi, \boldsymbol{\vartheta}]_{r,s} = \frac{1}{t-s} [\Phi, (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\vartheta}]_{r,0}, \quad [\Psi, \boldsymbol{\vartheta}]_{l,t} = \frac{1}{t-s} [\Psi, (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\vartheta}]_{l,1}. \tag{2.11}$$

(4) (Trivialization) *If $\boldsymbol{\vartheta} \in \mathcal{P}_2(\mathsf{X}^2)$, $t \in [0,1]$, $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mathsf{x}^t_\sharp \boldsymbol{\vartheta})$ and $\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}$ is $\boldsymbol{\vartheta}$-essentially injective or Φ is concentrated on a map, then $\Gamma_t(\Phi, \boldsymbol{\vartheta})$ contains a unique element and*

$$[\Phi, \boldsymbol{\vartheta}]_{r,t} = [\Phi, \boldsymbol{\vartheta}]_{l,t} = \int \langle \boldsymbol{b}_\Phi \left(\mathsf{x}^t(x_0, x_1) \right), x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\vartheta}(x_0, x_1), \tag{2.12}$$

with $\boldsymbol{b}_\Phi$ the barycenter of Φ as in Definition 2.3.

(5) (Semicontinuity) *Let $(\Phi^i_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{TX})$ be converging to Φ^i in $\mathcal{P}^{sw}_2(\mathsf{TX})$, $i = 0, 1$, let $(\boldsymbol{\vartheta}_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{X}^2)$ be converging to $\boldsymbol{\vartheta}$ in $\mathcal{P}_2(\mathsf{X}^2)$, let $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{P}_2(\mathsf{X})$ be converging to ν in $\mathcal{P}_2(\mathsf{X})$ and let $t \in [0,1]$. Then*

$$\liminf_{n\to\infty} \left[\Phi^0_n, \nu_n\right]_r \ge \left[\Phi^0, \nu\right]_r, \qquad \limsup_{n\to\infty} \left[\Phi^0_n, \nu_n\right]_l \le \left[\Phi^0, \nu\right]_l,$$

$$\liminf_{n\to\infty} \left[\Phi^0_n, \Phi^1_n\right]_r \ge \left[\Phi^0, \Phi^1\right]_r, \qquad \limsup_{n\to\infty} \left[\Phi^0_n, \Phi^1_n\right]_l \le \left[\Phi^0, \Phi^1\right]_l,$$

$$\liminf_{n\to\infty} [\Phi^0_n, \boldsymbol{\vartheta}_n]_{r,t} \ge [\Phi^0, \boldsymbol{\vartheta}]_{r,t}, \qquad \limsup_{n\to\infty} [\Phi^0_n, \boldsymbol{\vartheta}_n]_{l,t} \le [\Phi^0, \boldsymbol{\vartheta}]_{l,t}.$$

(6) *Let $\mathcal{I} \subset \mathbb{R}$ be an open interval, let $\mu^1, \mu^2 : \mathcal{I} \to \mathcal{P}_2(\mathsf{X})$ be locally absolutely continuous curves and let $\boldsymbol{v}^1, \boldsymbol{v}^2 : \mathcal{I} \times \mathsf{X} \to \mathsf{X}$ be Borel vector fields such that $\|\boldsymbol{v}^i_t\|_{L^2(\mathsf{X},\mu^i_t;\mathsf{X})} \in L^1_{loc}(\mathcal{I})$, $i = 1, 2$, and such that*

$$\partial_t \mu^i_t + \nabla \cdot (\boldsymbol{v}^i_t \mu^i_t) = 0$$

holds in the sense of distributions in $\mathcal{I} \times \mathsf{X}$, $i = 1, 2$. Let $A(\mu^1), A(\mu^2) \subset \mathcal{I}$ be as in Theorem 2.11. Then

(a) *for every $\nu \in \mathcal{P}_2(\mathsf{X})$ and every $t \in A(\mu^i)$, $i = 1, 2$, it holds*

$$\lim_{h \downarrow 0} \frac{W_2^2(\mu^i_{t+h}, \nu) - W_2^2(\mu^i_t, \nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^i_t)_\sharp \mu^i_t, \nu \right]_r,$$

$$\lim_{h \uparrow 0} \frac{W_2^2(\mu^i_{t+h}, \nu) - W_2^2(\mu^i_t, \nu)}{2h} = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^i_t)_\sharp \mu^i_t, \nu \right]_l;$$

(b) *there exists a subset $A \subset A(\mu^1) \cap A(\mu^2)$ of full Lebesgue measure such that $s \mapsto W_2^2(\mu^1_s, \mu^2_s)$ is differentiable in A and for every $t \in A$ it holds*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu^1_t, \mu^2_t) = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp \mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp \mu^2_t \right]_r = \left[(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^1_t)_\sharp \mu^1_t, (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}^2_t)_\sharp \mu^2_t \right]_l.$$

Proof. We give a few references for the proofs. Property (1) is [CSS23a, (3.27)]. Property (2) comes from the definition and [CSS23a, Corollary 3.7]. We sketch the proof only for the last property in (2): take $\boldsymbol{\sigma} \in \Lambda(\Phi_0, \mu_1)$ such that

$$[\Phi_0, \mu_1]_r = \int_{\mathsf{TX} \times \mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma},$$

and consider $\boldsymbol{\Theta} \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{TX})$ such that $(\mathsf{x}^0, \mathsf{v}^0, \mathsf{x}^1)_\sharp \boldsymbol{\Theta} = \boldsymbol{\sigma}$ and $(\mathsf{x}^1, \mathsf{v}^1)_\sharp \boldsymbol{\Theta} = \Phi_1$. Then $\boldsymbol{\Theta} \in \Lambda(\Phi_0, \Phi_1)$, so that

$$\begin{aligned}
[\Phi_0, \Phi_1]_r &\le \int_{\mathsf{TX} \times \mathsf{TX}} \langle x_0 - x_1, v_0 - v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \\
&= \int_{\mathsf{TX} \times \mathsf{X}} \langle x_0 - x_1, v_0 \rangle \, \mathrm{d}\boldsymbol{\sigma} + \int_{\mathsf{TX} \times \mathsf{TX}} \langle x_1 - x_0, v_1 \rangle \, \mathrm{d}\boldsymbol{\Theta} \\
&\le [\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_l.
\end{aligned}$$

The strategy for proving the remaining inequality in (2) is identical.
Assertion (3) follows from the fact that, if we define $T : \mathsf{TX} \times \mathsf{X} \to \mathsf{TX} \times \mathsf{X}$ and $\mathcal{L} : \mathcal{P}_2(\mathsf{TX} \times \mathsf{X}) \to \mathbb{R}$ as

$$T(x_0, v_0, x_1) := (\mathsf{x}^s(x_0, x_1), v_0, \mathsf{x}^t(x_0, x_1)), \qquad \mathcal{L}(\boldsymbol{\sigma}) := \int_{\mathsf{TX} \times \mathsf{X}} \langle v_0, x_0 - x_1 \rangle \, \mathrm{d}\boldsymbol{\sigma}(x_0, v_0, x_1),$$

it is clear that

$$[\Phi, \boldsymbol{\mu}]_{r,s} = \inf \left\{ \mathcal{L}(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \Gamma_s(\Phi, \boldsymbol{\mu}) \right\}, \quad [\Phi, (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\mu}]_{r,0} = \inf \left\{ \mathcal{L}(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \Gamma_0(\Phi, (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\mu}) \right\}.$$

Then, the first equality in the statement follows noting that $T_\sharp(\Gamma_s(\Phi, \boldsymbol{\mu})) = \Gamma_0(\Phi, (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\mu})$ and that $\mathcal{L}(T_\sharp \boldsymbol{\sigma}) = (t - s)\mathcal{L}(\boldsymbol{\sigma})$ for every $\boldsymbol{\sigma} \in \mathcal{P}_2(\mathsf{TX} \times \mathsf{X})$. The second equality follows from the first one and (1). Item (4) is [CSS23a, Remark 3.19]. Item (5) easily follows by [CSS23a, Lemma 3.15]. Finally, item (6) is provided by [CSS23a, Theorem 3.11, Theorem 3.14, Remark 3.12]. ☐

2.3. **Multivalued probability vector fields, metric dissipativity and EVI solutions**

We recall now the main definition of Multivalued Probability Vector Field and of metric dissipativity.

Definition 2.14 (Multivalued Probability Vector Field - MPVF)**.** *A* multivalued probability vector field $\mathbf{F}$ *is a nonempty subset of* $\mathcal{P}_2(\mathsf{TX})$ *with* $\mathrm{D}(\mathbf{F}) := \mathsf{x}_\sharp(\mathbf{F}) = \{\mathsf{x}_\sharp\Phi : \Phi \in \mathbf{F}\}$*. Given any* $\mu \in \mathcal{P}_2(\mathsf{X})$*, we define the* section $\mathbf{F}[\mu]$ *of* $\mathbf{F}$ *as*

$$\mathbf{F}[\mu] := \{\Phi \in \mathbf{F} \mid \mathsf{x}_\sharp\Phi = \mu\}.$$

We say that $\mathbf{F}$ *is a* Probability Vector Field (PVF) *if* $\mathsf{x}_\sharp$ *is injective in* $\mathbf{F}$*, i.e.* $\mathbf{F}[\mu]$ *contains a unique element for every* $\mu \in \mathrm{D}(\mathbf{F})$*.*
A selection $\mathbf{F}'$ *of a* MPVF $\mathbf{F}$ *is a* PVF *such that* $\mathbf{F}' \subset \mathbf{F}$ *and* $\mathrm{D}(\mathbf{F}') = \mathrm{D}(\mathbf{F})$*.*
A MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is* deterministic *or* concentrated on maps *if every* $\Phi \in \mathbf{F}$ *is deterministic (see Definition 2.3).*

Starting from a MPVF $\mathbf{F}$, the barycentric projection (2.7) induces a deterministic PVF which we call $\mathrm{bar}\,(\mathbf{F})$: it is defined by

$$\mathrm{bar}\,(\mathbf{F})\,[\mu] := \{\mathrm{bar}\,(\Phi) = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{b}_\Phi)_\sharp\mu : \Phi \in \mathbf{F}[\mu]\}, \quad \mu \in \mathrm{D}(\mathbf{F}). \tag{2.13}$$

We will also use the notation

$$\mathrm{map}\,(\mathbf{F})\,[\mu] := \big\{\boldsymbol{f} \in L^2(\mathsf{X}, \mu; \mathsf{X}) : (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f})_\sharp\mu \in \mathbf{F}[\mu]\big\}, \quad \mu \in \mathrm{D}(\mathbf{F}), \tag{2.14}$$

to extract the deterministic part of a MPVF $\mathbf{F}$: notice that a MPVF $\mathbf{F}$ is deterministic if and only if $\mathbf{F} = \mathrm{bar}\,(\mathbf{F}) = \{(\boldsymbol{i}_\mathsf{X}, \boldsymbol{f})_\sharp\mu : \boldsymbol{f} \in \mathrm{map}\,(\mathbf{F})\,[\mu],\ \mu \in \mathrm{D}(\mathbf{F})\}$.
Conversely, for a given set $D \subset \mathcal{P}_2(\mathsf{X})$ let us consider a continuous map $\boldsymbol{f} : \mathcal{S}\,(\mathsf{X}, D) \to \mathsf{X}$ where

$$\mathcal{S}\,(\mathsf{X}, D) := \{(x, \mu) \in \mathsf{X} \times D \mid x \in \mathrm{supp}(\mu)\}, \quad \text{with } \mathcal{S}\,(\mathsf{X}) := \mathcal{S}\,(\mathsf{X}, \mathcal{P}_2(\mathsf{X})). \tag{2.15}$$

If, for every $\mu \in D$, the integral $\displaystyle\int |\boldsymbol{f}(x,\mu)|^2 \,\mathrm{d}\mu(x)$ is finite, then $\boldsymbol{f}$ induces a PVF $\mathbf{F}$ defined by

$$\mathbf{F} = \big\{(\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}(\cdot,\mu))_\sharp\mu : \mu \in D\big\}, \quad \mathrm{D}(\mathbf{F}) = D.$$

We often adopt the convention to write $\boldsymbol{f}[\mu]$ for the function

$$\boldsymbol{f}[\mu](x) := \boldsymbol{f}(x, \mu), \quad x \in \mathrm{supp}(\mu),$$

in particular when $\boldsymbol{f}[\mu]$ is just an element of $L^2(\mathsf{X}, \mu; \mathsf{X})$.

Definition 2.15 (Metrically λ-dissipative MPVF)**.** *A* MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is (metrically)* λ-dissipative*,* $\lambda \in \mathbb{R}$*, if*

$$[\Phi_0, \Phi_1]_r \le \lambda W_2^2(\mu_0, \mu_1) \quad \forall\, \Phi_0, \Phi_1 \in \mathbf{F},\ \mu_0 = \mathsf{x}_\sharp\Phi_0,\ \mu_1 = \mathsf{x}_\sharp\Phi_1. \tag{2.16}$$

In case $\lambda = 0$*, we simply say that* $\mathbf{F}$ *is dissipative.*

Remark 2.16. Thanks to Theorem 2.13(2), (2.16) implies the weaker condition

$$[\Phi_0, \mu_1]_r + [\Phi_1, \mu_0]_r \le \lambda W_2^2(\mu_0, \mu_1), \quad \forall\, \Phi_0, \Phi_1 \in \mathbf{F},\ \mu_0 = \mathsf{x}_\sharp\Phi_0,\ \mu_1 = \mathsf{x}_\sharp\Phi_1. \tag{2.17}$$

Given a MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$, we define its λ-trasformation, $\mathbf{F}^\lambda$, and its opposite, $-\mathbf{F}$, as

$$\mathbf{F}^\lambda := L^\lambda_\sharp \mathbf{F} = \Big\{L^\lambda_\sharp\Phi \,:\, \Phi \in \mathbf{F}\Big\}, \tag{2.18}$$

$$-\mathbf{F} := \{(\mathsf{x}, -\mathsf{v})_\sharp\Phi : \Phi \in \mathbf{F}\}, \tag{2.19}$$

where $L^\lambda : \mathsf{TX} \to \mathsf{TX}$ is the bijective map defined by

$$L^\lambda := (\mathsf{x}, \mathsf{v} - \lambda\mathsf{x}).$$

Similar to Remark A.1 for the case of operators in Hilbert spaces, we recall the following result (cf. [CSS23a, Lemma 4.6])

Lemma 2.17. $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is a* λ*-dissipative* MPVF *(resp. satisfies* (2.17)*) if and only if* $\mathbf{F}^\lambda$ *is dissipative, i.e.* 0*-dissipative (resp. satisfies* (2.17) *with* $\lambda = 0$*).*

Definition 2.18. *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$, $\mu_0, \mu_1 \in \mathrm{D}(\mathbf{F})$. *We define the set*

$$\Gamma(\mu_0, \mu_1|\mathbf{F}) := \left\{ \boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1) \mid \mathsf{x}^t_\sharp \boldsymbol{\mu} \in \mathrm{D}(\mathbf{F}) \text{ for every } t \in [0,1] \right\}.$$

If $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1|\mathbf{F})$ *and* $t \in [0,1]$, *we define*

$$[\mathbf{F}, \boldsymbol{\mu}]_{r,t} := \sup \left\{ [\Phi, \boldsymbol{\mu}]_{r,t} \mid \Phi \in \mathbf{F}[\mu_t] \right\}, \qquad [\mathbf{F}, \boldsymbol{\mu}]_{l,t} := \inf \left\{ [\Phi, \boldsymbol{\mu}]_{l,t} \mid \Phi \in \mathbf{F}[\mu_t] \right\}.$$

In the following Theorem we discuss the behaviour of duality pairings with $\mathbf{F}$ along geodesics.

Theorem 2.19. *Let* $\mathbf{F}$ *be a* MPVF, *let* $\mu_0, \mu_1 \in \mathrm{D}(\mathbf{F})$, *let* $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1|\mathbf{F}) \cap \Gamma_o(\mu_0, \mu_1)$ *and let* $W^2 := W_2^2(\mu_0, \mu_1)$. *If* $\mathbf{F}$ *satisfies* (2.17), *then the following properties hold.*

(1) $[\mathbf{F}, \boldsymbol{\mu}]_{l,t} \le [\mathbf{F}, \boldsymbol{\mu}]_{r,t}$ *for every* $t \in (0,1)$;
(2) $[\mathbf{F}, \boldsymbol{\mu}]_{r,s} \le [\mathbf{F}, \boldsymbol{\mu}]_{l,t} + \lambda W^2 (t-s)$ *for every* $0 \le s < t \le 1$;
(3) $t \mapsto [\mathbf{F}, \boldsymbol{\mu}]_{r,t} + \lambda W^2 t$ *and* $t \mapsto [\mathbf{F}, \boldsymbol{\mu}]_{l,t} + \lambda W^2 t$ *are increasing respectively in* $[0,1)$ *and in* $(0,1]$;
(4) $[\mathbf{F}, \boldsymbol{\mu}]_{l,t} = [\mathbf{F}, \boldsymbol{\mu}]_{r,t}$ *at every point* $t \in (0,1)$ *where one of them is continuous and thus coincide outside a countable set.*

Proof. Item (1) immediately follows from the definition. Item (2) is proven in [CSS23a, Theorem 4.9], while (3) and (4) follow from (2). □

Proposition 2.20. *If* $\mathbf{F}$ *is a* λ*-dissipative* MPVF *then its sequential closure*

$$\mathrm{cl}(\mathbf{F}) := \Big\{ \Phi \in \mathcal{P}_2(\mathsf{TX}) : \exists\, \Phi_n \in \mathbf{F} : \Phi_n \to \Phi \text{ in } \mathcal{P}_2^{sw}(\mathsf{TX}) \Big\}. \tag{2.20}$$

is λ*-dissipative as well.*

Proof. It follows from Theorem 2.13(5). See also [CSS23a, Proposition 4.15]. □

We recall the definition of λ-EVI solution for a MPVF.

Definition 2.21 (λ-Evolution Variational Inequality)**.** *Let* $\mathbf{F}$ *be a* MPVF *and let* $\lambda \in \mathbb{R}$. *We say that a continuous curve* $\mu : \mathcal{I} \to \overline{\mathrm{D}(\mathbf{F})}$ *is a* λ-EVI solution *for the* MPVF $\mathbf{F}$ *if*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu_t, \mathsf{x}_\sharp \Phi) \le \lambda W_2^2(\mu_t, \mathsf{x}_\sharp \Phi) - [\Phi, \mu_t]_r \ \text{ in } \mathscr{D}'(\mathrm{int}\,(\mathcal{I})) \text{ for every } \Phi \in \mathbf{F},$$

where the writing $\mathscr{D}'(\mathrm{int}\,(\mathcal{I}))$ *means that the expression has to be unerstood in the distributional sense in* $\mathrm{int}\,(\mathcal{I})$.

Remark 2.22. In light of Theorem 2.13(6a) and recalling [CSS23a, Remark 5.2], an absolutely continuous curve $\mu : \mathcal{I} \to \overline{\mathrm{D}(\mathbf{F})}$ is a λ-EVI *solution* for the MPVF $\mathbf{F}$ if and only if

$$\lim_{h \downarrow 0} \frac{W_2^2(\mu_{t+h}, \nu) - W_2^2(\mu_t, \nu)}{2h} \le \lambda W_2^2(\mu_t, \mathsf{x}_\sharp \Phi) - [\Phi, \mu_t]_r \qquad \text{for every } t \in A(\mu) \text{ and every } \Phi \in \mathbf{F},$$

where $A(\mu) \subset \mathcal{I}$ is as in Theorem 2.11.

3. Invariant dissipative operators in Hilbert spaces and totally dissipative MPVFs

From now on, X will denote a separable Hilbert space; we will also consider a standard Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$ (see Appendix B and in particular Definition B.1) and the Hilbert space $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$. We will use capital letters $X, Y, V, \ldots$ to denote elements of $\mathcal{X}$ (i.e. X-valued random variables).
We denote by $\iota : \mathcal{X} \to \mathcal{P}_2(\mathsf{X})$ and $\iota^2 : \mathcal{X} \times \mathcal{X} \to \mathcal{P}_2(\mathsf{X} \times \mathsf{X}) \equiv \mathcal{P}_2(\mathsf{TX})$ the push-forward operators

$$\iota(X) := X_\sharp \mathbb{P}, \qquad \iota^2(X, V) := (X, V)_\sharp \mathbb{P}. \tag{3.1}$$

We frequently use the notations $\iota_X = \iota(X)$ and $\iota^2_{X,V} = \iota^2(X, V)$.

Definition 3.1 (Measure-preserving isomorphisms). *We denote by* $\mathrm{S}(\Omega)$ *the class of* $\mathcal{B}$-$\mathcal{B}$-*measurable maps* $g : \Omega \to \Omega$ *which are essentially injective and measure preserving, meaning that there exists a full* $\mathbb{P}$-*measure set* $\Omega_0 \in \mathcal{B}$ *such that* g *is injective on* Ω_0 *and* $g_\sharp\mathbb{P} = \mathbb{P}$. *Every* $g \in \mathrm{S}(\Omega)$ *has an inverse* $g^{-1} \in \mathrm{S}(\Omega)$ *(defined up to a* $\mathbb{P}$-*negligible set) such that* $g^{-1} \circ g = g \circ g^{-1} = \boldsymbol{i}_\Omega$ $\mathbb{P}$-*a.e. in* Ω.

In Section 3.1 we report some properties (see [CSS23b] for details and proofs) of the resolvent operator, the Yosida approximation and the minimal selection of a maximal λ-dissipative operator $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ which is invariant by measure-preserving isomorphisms. In Section 3.2 we study the relation between λ-dissipativity for an invariant subset $\boldsymbol{B}$ of $\mathcal{X} \times \mathcal{X}$, and correspondent total λ-dissipativity of the image/law $\mathbf{F}$ of $\boldsymbol{B}$ in $\mathcal{P}_2(\mathsf{TX})$. The particular case of deterministic MPVFs is considered in section 3.3. These results are then used, in Section 4, to analyze well-posedness of the Eulerian flow for $\mathbf{F}$ generated by the corresponding Lagrangian one for $\boldsymbol{B}$ and the generation of λ-EVI solutions in $\mathcal{P}_2(\mathsf{X})$.

3.1. Law invariant dissipative operators

Given a set $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ (as usual, we will identify subsets of $\mathcal{X} \times \mathcal{X}$ with multivalued operators), we define $\boldsymbol{B}X := \{V \in \mathcal{X} : (X, V) \in \boldsymbol{B}\}$ and the domain $\mathrm{D}(\boldsymbol{B}) := \{X \in \mathcal{X} : \boldsymbol{B}X \neq \emptyset\}$.
When $\boldsymbol{B}$ is maximal λ-dissipative, $\boldsymbol{J}_\tau$, $\boldsymbol{B}_\tau$ and $\boldsymbol{B}^\circ$ denote respectively the resolvent operator, the Yosida approximation and the minimal selection of $\boldsymbol{B}$ (we refer to Appendix A). Here, we just recall that $\boldsymbol{J}_\tau := (\boldsymbol{i}_\mathcal{X} - \tau\boldsymbol{B})^{-1}$ is a $(1 - \lambda\tau)^{-1}$-Lipschitz map defined on the whole $\mathcal{X}$ for every $0 < \tau < 1/\lambda^+$, where we set $\lambda^+ := \lambda \vee 0$ and $1/\lambda^+ = +\infty$ if $\lambda^+ = 0$. The minimal selection $\boldsymbol{B}^\circ : \mathrm{D}(\boldsymbol{B}) \to \mathcal{X}$ of $\boldsymbol{B}$ is also characterized by

$$\boldsymbol{B}^\circ X = \lim_{\tau \downarrow 0} \frac{\boldsymbol{J}_\tau X - X}{\tau}.$$

The Yosida approximation of $\boldsymbol{B}$ is defined by $\boldsymbol{B}_\tau := \frac{\boldsymbol{J}_\tau - I}{\tau}$. For every $0 < \tau < 1/\lambda^+$, $\boldsymbol{B}_\tau$ is maximal $\lambda/(1 - \lambda\tau)$-dissipative and $\frac{2-\lambda\tau}{\tau(1-\lambda\tau)}$-Lipschitz continuous.

If $\boldsymbol{B}$ is a maximal λ-dissipative operator, then there exists (cf. Theorems A.5,A.6 in Appendix A) a semigroup of $e^{\lambda t}$-Lipschitz transformations $(\boldsymbol{S}_t)_{t\geq 0}$ with $\boldsymbol{S}_t : \overline{\mathrm{D}(\boldsymbol{B})} \to \overline{\mathrm{D}(\boldsymbol{B})}$ s.t. for every $X_0 \in \mathrm{D}(\boldsymbol{B})$ the curve $t \mapsto \boldsymbol{S}_t X_0$ is included in $\mathrm{D}(\boldsymbol{B})$ and it is the unique locally Lipschitz continuous solution of the differential inclusion

$$\begin{cases} \dot{X}_t \in \boldsymbol{B}X_t & \text{a.e. } t > 0, \\ X|_{t=0} = X_0. \end{cases}$$

By Theorem A.5(3), we also have

$$\lim_{h \downarrow 0} \frac{\boldsymbol{S}_{t+h}X_0 - \boldsymbol{S}_t X_0}{h} = \boldsymbol{B}^\circ(\boldsymbol{S}_t X_0), \quad \text{for every } X_0 \in \mathrm{D}(\boldsymbol{B}) \text{ and every } t \geq 0.$$

Let us now consider the particular classes of operators which are invariant by measure-preserving isomorphisms or law-invariant.

Definition 3.2 (Invariant operators). *We say that a set (or a multivalued operator)* $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *is* invariant by measure-preserving isomorphisms *if for every* $g \in \mathrm{S}(\Omega)$ *it holds*

$$(X, V) \in \boldsymbol{B} \;\Rightarrow\; (X \circ g, V \circ g) \in \boldsymbol{B}.$$

A set $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *is* law invariant *if it holds*

$$(X, V) \in \boldsymbol{B},\; X', V' \in \mathcal{X},\; (X, V)_\sharp\mathbb{P} = (X', V')_\sharp\mathbb{P} \;\Rightarrow\; (X', V') \in \boldsymbol{B}.$$

An operator $\boldsymbol{A} : \mathcal{X} \supset D(\boldsymbol{A}) \to \mathcal{X}$, *is invariant by measure-preserving isomorphisms (resp. law invariant) if its graph is invariant by measure-preserving isomorphisms (resp. law invariant).*

We recall the notation in (2.15) and that $\iota\big(D(\boldsymbol{B})\big) = \Big\{X_\sharp\mathbb{P} : X \in \mathrm{D}(\boldsymbol{B})\Big\}$ is the image in $\mathcal{P}_2(\mathsf{X})$ of the domain of $\boldsymbol{B}$. The results in the following Lemma 3.3 and Theorem 3.4 are presented in [CSS23b, Section 4] to which we refer for the proofs.

Lemma 3.3 (Closed invariant sets)**.** *Let $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ be a closed set. Then $\boldsymbol{B}$ is invariant by measure-preserving isomorphisms if and only if it is law invariant.*

Theorem 3.4 (Representation of resolvents, Yosida approximations, and semigroups)**.** *Let $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ be a maximal λ-dissipative operator which is invariant by measure-preserving isomorphisms. Then for every $0 < \tau < 1/\lambda^+$, $t \geq 0$ the operators $\boldsymbol{B}, \boldsymbol{B}_\tau, \boldsymbol{J}_\tau, \boldsymbol{S}_t, \boldsymbol{B}^\circ$ are law invariant. Moreover there exist (uniquely defined) continuous maps $\boldsymbol{j}_\tau : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$, $\boldsymbol{b}_\tau : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$, and $\boldsymbol{s}_t : \mathcal{S}\left(\mathsf{X}, \overline{\iota\big(D(\boldsymbol{B})\big)}\right) \to \mathsf{X}$ such that*

(1) for every $\mu \in \mathcal{P}_2(\mathsf{X})$, the map $\boldsymbol{j}_\tau(\cdot,\mu) : \operatorname{supp}(\mu) \to \mathsf{X}$ is $(1-\lambda\tau)^{-1}$-Lipschitz continuous, for $0 < \tau < 1/\lambda^+$;

(2) for every $\mu \in \mathcal{P}_2(\mathsf{X})$, the map $\boldsymbol{b}_\tau(\cdot,\mu) : \operatorname{supp}(\mu) \to \mathsf{X}$ is $\frac{2-\lambda\tau}{\tau(1-\lambda\tau)}$-Lipschitz continuous, for $0 < \tau < 1/\lambda^+$;

(3) for every $\mu \in \overline{\iota\big(D(\boldsymbol{B})\big)}$, the map $\boldsymbol{s}_t(\cdot,\mu) : \operatorname{supp}(\mu) \to \mathsf{X}$ is $e^{\lambda t}$-Lipschitz continuous,

and

$$\text{for every } X \in \mathcal{X},\ \boldsymbol{J}_\tau X(\omega) = \boldsymbol{j}_\tau(X(\omega), X_\sharp\mathbb{P}) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{3.2}$$

$$\text{for every } X \in \mathcal{X},\ \boldsymbol{B}_\tau X(\omega) = \boldsymbol{b}_\tau(X(\omega), X_\sharp\mathbb{P}) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{3.3}$$

$$\text{for every } X \in \overline{\mathrm{D}(\boldsymbol{B})},\ \boldsymbol{S}_t X(\omega) = \boldsymbol{s}_t(X(\omega), X_\sharp\mathbb{P}) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{3.4}$$

together with the invariance and semigroup properties

$$\begin{gathered}\mu \in \overline{\iota\big(D(\boldsymbol{B})\big)} \quad \Rightarrow \quad \boldsymbol{s}_t(\cdot,\mu)_\sharp\mu \in \overline{\iota\big(D(\boldsymbol{B})\big)}; \quad \mu \in \iota\big(D(\boldsymbol{B})\big) \quad \Rightarrow \quad \boldsymbol{s}_t(\cdot,\mu)_\sharp\mu \in \iota\big(D(\boldsymbol{B})\big),\\ \boldsymbol{s}_{t+h}(x,\mu) = \boldsymbol{s}_h(\boldsymbol{s}_t(x,\mu), \boldsymbol{s}_t(\cdot,\mu)_\sharp\mu) \quad \text{for every } (x,\mu) \in \mathcal{S}\left(\mathsf{X}, \overline{\iota\big(D(\boldsymbol{B})\big)}\right),\ t,h \geq 0.\end{gathered} \tag{3.5}$$

Finally, for every $\mu \in \iota\big(D(\boldsymbol{B})\big)$, there exists a map $\boldsymbol{b}^\circ[\mu] \in L^2(\mathsf{X},\mu;\mathsf{X})$ such that for every $X \in \mathcal{X}$

$$\text{if } X_\sharp\mathbb{P} = \mu \text{ then } X \in \mathrm{D}(\boldsymbol{B}),\ \boldsymbol{B}^\circ X(\omega) = \boldsymbol{b}^\circ[\mu](X(\omega)) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{3.6}$$

For every $\mu \in \iota\big(D(\boldsymbol{B})\big)$, the map $\boldsymbol{b}^\circ[\mu]$ is λ-dissipative in a set $\mathsf{X}_0 \subset \mathsf{X}$ of full μ-measure and satisfies

$$\lim_{h\downarrow 0} \int \left| \frac{1}{h}(\boldsymbol{s}_{t+h}(x,\mu) - \boldsymbol{s}_t(x,\mu)) - \boldsymbol{b}^\circ[\boldsymbol{s}_t(\cdot,\mu)_\sharp\mu](\boldsymbol{s}_t(x,\mu)) \right|^2 \mathrm{d}\mu(x) = 0, \quad t \geq 0. \tag{3.7}$$

Notice that when $\mu \in \iota(\mathrm{D}(\boldsymbol{B}))$, (3.5) and (3.7) yield

$$\lim_{h\downarrow 0} \int \left| \frac{1}{h}(\boldsymbol{s}_h(x,\mu) - x) - \boldsymbol{b}^\circ[\mu](x) \right|^2 \mathrm{d}\mu(x) = 0. \tag{3.8}$$

Remark 3.5. By Theorem A.3(1) and Lemma 3.3, a maximal λ-dissipative operator $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$, $\lambda \in \mathbb{R}$, is law invariant if and only if it is invariant by measure-preserving isomorphisms. Hence, in this case, we will simply use the word *invariant.* Notice moreover that if $\boldsymbol{B}$ is law invariant, then also $\mathrm{D}(\boldsymbol{B})$ is *law invariant* in the sense that if $X \in \mathrm{D}(\boldsymbol{B})$ and $Y_\sharp\mathbb{P} = X_\sharp\mathbb{P}$ then also Y belongs to $\mathrm{D}(\boldsymbol{B})$. It is an immediate consequence of (3.6).

3.2. **Totally dissipative** MPVFs

The aim of this section is to study the properties of MPVFs enjoying a strong dissipativity property that we call total dissipativity.

Definition 3.6 (Total dissipativity)**.** *We say that a* MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is* totally λ-dissipative, $\lambda \in \mathbb{R}$, *if for every* $\Phi_0, \Phi_1 \in \mathbf{F}$ *and every* $\boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1)$ *we have*

$$\int \langle v_1 - v_0, x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\vartheta}(x_0, v_0, x_1, v_1) \le \lambda \int |x_1 - x_0|^2 \mathrm{d}\boldsymbol{\vartheta}. \tag{3.9}$$

We say that $\mathbf{F}$ *is* maximal totally λ-dissipative *if it is maximal in the class of totally λ-dissipative* MPVF*s: if* $\mathbf{F}' \supset \mathbf{F}$ *and* $\mathbf{F}'$ *is totally λ-dissipative, then* $\mathbf{F}' = \mathbf{F}$.

Of course, total λ-dissipativity implies λ-dissipativity (see Definition 2.15).

Remark 3.7. Notice that for a deterministic MPVF (recall Definition 2.14) total λ-dissipativity is equivalent to the following condition (when $\lambda = 0$ see the analogous notion of L-monotonicity of [CD18, Def. 3.31]): for every $\mu_i \in \mathrm{D}(\mathbf{F})$ and $\boldsymbol{f}_i \in \mathrm{map}\,(\mathbf{F}[\mu_i])$, $i = 0, 1$, and every $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$ it holds

$$\int \langle \boldsymbol{f}_1(x_1, \mu_1) - \boldsymbol{f}_0(x_0, \mu_0), x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le \lambda \int |x_1 - x_0|^2 \, \mathrm{d}\boldsymbol{\mu}(x_0, x_1). \tag{3.10}$$

We introduce now the natural notion of Lagrangian representation of a MPVF, based on the maps ι, ι^2 introduced in (3.1).

Definition 3.8 (Lagrangian representations and Eulerian images)**.** *Given* $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *and* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$, *we say that* $\boldsymbol{B}$ *is the* Lagrangian representation *of* $\mathbf{F}$ *if*

$$\boldsymbol{B} = (\iota^2)^{-1}(\mathbf{F}) = \Big\{ (X, V) \in \mathcal{X} \times \mathcal{X} : (X, V)_\sharp \mathbb{P} \in \mathbf{F} \Big\}.$$

Conversely, if $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *we say that* $\mathbf{F}$ *is the* Eulerian image *of* $\boldsymbol{B}$ *if*

$$\mathbf{F} = \iota^2(\boldsymbol{B}) = \Big\{ (X, V)_\sharp \mathbb{P} : (X, V) \in \boldsymbol{B} \Big\}.$$

Clearly, the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$ is law invariant, moreover $\boldsymbol{B}$ is the Lagrangian representation of $\mathbf{F}$ if and only if $\mathbf{F}$ is the Eulerian image of $\boldsymbol{B}$ and $\boldsymbol{B}$ is law invariant.
Similarly to Remark A.1 concerning operators in Hilbert spaces, we highlight the following result which allows a reduction of many arguments to the dissipative case $\lambda = 0$.

Lemma 3.9. *The following hold:*

(1) $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is totally λ-dissipative if and only if* $\mathbf{F}^\lambda$ *(cf. (2.18)) is totally* 0*-dissipative;*
(2) $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is maximal totally λ-dissipative if and only if* $\mathbf{F}^\lambda$ *is maximal totally* 0*-dissipative;*
(3) $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *is invariant by measure-preserving isomorphisms (resp. law invariant) if and only if* $\boldsymbol{B}^\lambda := \boldsymbol{B} - \lambda \boldsymbol{i}_{\mathcal{X}}$ *is invariant by measure-preserving isomorphisms (resp. law invariant);*
(4) $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ *is the Lagrangian representation of* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *if and only if* $\boldsymbol{B}^\lambda$ *is the Lagrangian representation of* $\mathbf{F}^\lambda$.

Proof. The proof of claim (1) is similar to [CSS23a, Lemma 4.6] and it is based on the bijectivity of the map $L^\lambda := (\mathsf{x}, \mathsf{v} - \lambda \mathsf{x}) : \mathsf{TX} \to \mathsf{TX}$. Hence, if $\Phi_i \in \mathbf{F}$ and $\Phi_i^\lambda := L^\lambda_\sharp \Phi_i \in \mathbf{F}^\lambda$, $i = 1, 2$, then $\boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1)$ if and only if $\boldsymbol{\vartheta}^\lambda \in \Gamma(\Phi_0^\lambda, \Phi_1^\lambda)$, with $\boldsymbol{\vartheta}^\lambda = (\mathsf{x}^0, \mathsf{v}^0 - \lambda \mathsf{x}^0, \mathsf{x}^1, \mathsf{v}^1 - \lambda \mathsf{x}^1)_\sharp \boldsymbol{\vartheta}$. We can thus prove only the left-to-right implication, the other will follow from the same procedure. We have

$$\begin{aligned}
\int \langle v_1 - v_0, x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\vartheta}^\lambda(x_0, v_0, x_1, v_1) &= \int \langle v_1 - v_0 - \lambda(x_1 - x_0), x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\vartheta}(x_0, v_0, x_1, v_1) \\
&= \int \langle v_1 - v_0, x_1 - x_0 \rangle \, \mathrm{d}\boldsymbol{\vartheta} - \lambda \int |x_1 - x_0|^2 \mathrm{d}\boldsymbol{\vartheta} \\
&\le 0,
\end{aligned}$$

by total λ-dissipativity of $\mathbf{F}$.
Items (2), (3) and (4) are straightforward. □

A first basic fact is stated by the following Proposition.

Proposition 3.10. *Let $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ be the Lagrangian representation of $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ according to Definition 3.8. Then $\mathbf{F}$ is totally λ-dissipative if and only if $\boldsymbol{B}$ is λ-dissipative.*

Proof. By Lemma 3.9 and Remark A.1, it is sufficient to prove the result in case $\lambda = 0$. Let us first assume that $\mathbf{F}$ is totally dissipative. Let $(X_0, V_0), (X_1, V_1) \in \boldsymbol{B}$. Since $\Phi_0 = (X_0, V_0)_\sharp \mathbb{P} \in \mathbf{F}$, $\Phi_1 = (X_1, V_1)_\sharp \mathbb{P} \in \mathbf{F}$ and $\boldsymbol{\vartheta} := (X_0, V_0, X_1, V_1)_\sharp \mathbb{P} \in \Gamma(\Phi_0, \Phi_1)$, (3.9) yields

$$\int \langle V_1 - V_0, X_1 - X_0 \rangle \,\mathrm{d}\mathbb{P} = \int \langle v_1 - v_0, x_1 - x_0 \rangle \,\mathrm{d}\boldsymbol{\vartheta} \le 0.$$

In order to prove the converse implication, let us assume that $\boldsymbol{B}$ is dissipative and take $\Phi_0, \Phi_1 \in \mathbf{F}$, $\boldsymbol{\vartheta} \in \Gamma(\Phi_0, \Phi_1)$ and $(X_0, V_0, X_1, V_1) \in \mathcal{X}^4$ such that $(X_0, V_0, X_1, V_1)_\sharp \mathbb{P} = \boldsymbol{\vartheta}$. Since $\Phi_0, \Phi_1 \in \mathbf{F}$, there exist $(X_0', V_0') \in \boldsymbol{B}$ and $(X_1', V_1') \in \boldsymbol{B}$ such that

$$(X_0', V_0')_\sharp \mathbb{P} = \Phi_0 = (X_0, V_0)_\sharp \mathbb{P}, \qquad (X_1', V_1')_\sharp \mathbb{P} = \Phi_1 = (X_1, V_1)_\sharp \mathbb{P}.$$

By the law invariance of $\boldsymbol{B}$, we have that $(X_0, V_0), (Y_0, W_0) \in \boldsymbol{B}$, so that

$$\int \langle v_1 - v_0, x_1 - x_0 \rangle \,\mathrm{d}\boldsymbol{\vartheta} = \langle V_1 - V_0, X_1 - X_0 \rangle_{\mathcal{X}} \le 0$$

by the dissipativity of $\boldsymbol{B}$. □

Example 3.11. Let us consider a map $\boldsymbol{f} : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$ (recall (2.15)) such that there exists $L > 0$ for which we have

$$|\boldsymbol{f}(x_1, \mu_1) - \boldsymbol{f}(x_0, \mu_0)| \le L \left(W_2(\mu_0, \mu_1) + |x_0 - x_1| \right) \qquad \text{for every } (x_0, \mu_0),\ (x_1, \mu_1) \in \mathcal{S}(\mathsf{X}).$$

We can also identify $\boldsymbol{f}$ with the map sending $\mu \mapsto f(\cdot, \mu) \in \mathrm{Lip}(\mathsf{X}; \mathsf{X})$ (compare with the framework analyzed by Bonnet and Frankowska in [BF21b; BF23] and with the hypoteses in [Cav+22; Amb+21]). Let us define the map $\boldsymbol{B} : \mathcal{X} \to \mathcal{X}$ and the (single-valued, deterministic) PVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ as

$$\begin{aligned} \boldsymbol{B}(X)(\omega) &:= \boldsymbol{f}(X(\omega), \iota_X), && X \in \mathsf{X},\ \omega \in \Omega, \\ \mathbf{F}[\mu] &:= (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}(\cdot, \mu))_\sharp \mu, && \mu \in \mathcal{P}_2(\mathsf{X}). \end{aligned}$$

It is not difficult to check that $\boldsymbol{B}$ is $2L$-Lipschitz and that $\mathbf{F}$ is maximal $2L$-totally dissipative. Indeed, for every $X, Y \in \mathcal{X}$, we have

$$\begin{aligned} |\boldsymbol{B}X - \boldsymbol{B}Y|_{\mathcal{X}} &= \left(\int_\Omega |\boldsymbol{B}X(\omega) - \boldsymbol{B}Y(\omega)|^2 \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/2} \\ &= \left(\int_\Omega |\boldsymbol{f}(X(\omega), \iota_X)) - \boldsymbol{f}(Y(\omega), \iota_Y)|^2 \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/2} \\ &\le L \left(\int_\Omega \left(W_2(\iota_X, \iota_Y) + |X(\omega) - Y(\omega)| \right)^2 \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/2} \\ &\le L \left(\left(\int_\Omega W_2^2(\iota_X, \iota_Y) \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/2} + \left(\int_\Omega |X(\omega) - Y(\omega)|^2 \,\mathrm{d}\mathbb{P}(\omega) \right)^{1/2} \right) \\ &\le 2L|X - Y|_{\mathcal{X}} \end{aligned}$$

so that $\boldsymbol{B}$ is $2L$-dissipative and therefore $\mathbf{F}$ is $2L$-totally dissipative as well by Proposition 3.10. Maximality follows by the maximality of $\boldsymbol{B}$ and the next Theorem.

Theorem 3.12 (Maximal dissipativity)**.**

(1) *Every λ-dissipative operator $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ which is invariant by measure-preserving isomorphisms has a maximal λ-dissipative extension with domain included in $\overline{\mathrm{co}}\,(\mathrm{D}(\boldsymbol{B}))$ which is invariant by measure-preserving isomorphisms (and therefore also law invariant).*

(2) *Let us suppose that $\boldsymbol{B} \subset \mathcal{X} \times \mathcal{X}$ is the λ-dissipative Lagrangian representation of the totally λ-dissipative* MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$*. Then $\boldsymbol{B}$ is maximal λ-dissipative if and only if $\mathbf{F}$ is maximal totally λ-dissipative.*

(3) *If $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ is a totally λ-dissipative* MPVF *with domain included in a closed and totally convex set* C*, then there exists a maximal totally λ-dissipative extension of $\mathbf{F}$ with domain included in* C.

Proof. By Lemma 3.9 and Remark A.1, it is sufficient to prove the result in case $\lambda = 0$. Claim (1) is [CSS23b, Theorem 4.6]. Notice that, being maximal λ-dissipative and invariant by measure-preserving isomorphisms, a maximal λ-dissipative extension of $\boldsymbol{B}$ is also law invariant by Lemma 3.3.
Claim (2) follows by the equivalence result of Proposition 3.10 and by Claim 1. In fact, if $\boldsymbol{B}$ is maximal dissipative it is clear that $\mathbf{F}$ is maximal. Conversely, suppose that $\mathbf{F}$ is maximal and $\boldsymbol{B}$ is its Lagrangian representation. By contradiction, if $\boldsymbol{B}$ is not maximal, Claim 1 shows that there exists a maximal and proper extension $\hat{\boldsymbol{B}}$ of $\boldsymbol{B}$ which is law invariant. Therefore, $\hat{\boldsymbol{B}}$ induces a strict extension of $\mathbf{F}$ which is totally dissipative.
Claim (3) is a consequence of Claim 1 and Claim 2. □

Remark 3.13. Notice that if $\boldsymbol{B}$ is the Lagrangian representation of a maximal totally λ-dissipative MPVF $\mathbf{F}$, then $\iota^{-1}\big(\overline{D(\mathbf{F})}\big) = \overline{D(\boldsymbol{B})}$. In fact, it is sufficient to prove that if $\iota(X) = \mu \in \overline{D(\mathbf{F})}$ then $X \in \overline{D(\boldsymbol{B})}$, since the converse inclusion is trivial. We can thus find a sequence $\mu_n \in D(\mathbf{F})$ converging to μ in $\mathcal{P}_2(\mathsf{X})$. Applying the last statement of Theorem B.5 we can then find a sequence $X_n \in \mathcal{X}$ such that $\iota(X_n) = \mu_n$ and $\lim_{n\to\infty} |X_n - X|_{\mathcal{X}} = 0$. We deduce that $X_n \in D(\boldsymbol{B})$ by Remark 3.5 and therefore $X \in \overline{D(\boldsymbol{B})}$.

We now apply Theorem 3.12 to get useful insights on the structure of totally dissipative MPVFs. The first result concerns the existence of a solution to the resolvent equation, which provides an equivalent characterization of maximality and will be the crucial tool to implement the Implicit Euler method, see Corollary 4.7.

Theorem 3.14 (Solution to the resolvent equation)**.** *A totally λ-dissipative* MPVF $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is maximal λ-dissipative if and only if for every $\mu \in \mathcal{P}_2(\mathsf{X})$ and every $0 < \tau < 1/\lambda^+$ there exists $\Phi \in \mathbf{F}$ such that $(\mathsf{x} - \tau\mathsf{v})_\sharp\Phi = \mu$.*

Proof. Let $\boldsymbol{B}$ be the Lagrangian representation of $\mathbf{F}$ that is λ-dissipative by Proposition 3.10. If $\mathbf{F}$ is maximal λ-dissipative, then $\boldsymbol{B}$ is maximal λ-dissipative as well by Theorem 3.12(3), so that for every $Y \in \mathcal{X}$ with $Y_\sharp\mathbb{P} = \mu$ and $0 < \tau < 1/\lambda^+$ there exists $(X, V) \in \boldsymbol{B}$ such that $X - \tau V = Y$ (cf. Theorem A.2(1)) so that $\Phi := (X, V)_\sharp\mathbb{P} \in \mathbf{F}$ satisfies $(\mathsf{x} - \tau\mathsf{v})_\sharp\Phi = \mu$.
Conversely, let us now suppose that $\mathbf{F}$ is not maximal λ-dissipative, so that $\boldsymbol{B}$ is not maximal λ-dissipative and it admits a proper maximal λ-dissipative law invariant extension $\hat{\boldsymbol{B}}$ by Theorem 3.12. Let $(\tilde{X}, \tilde{V}) \in \hat{\boldsymbol{B}} \setminus \boldsymbol{B}$, $0 < \tau < 1/\lambda^+$, $\tilde{Y} := \tilde{X} - \tau\tilde{V}$, and $\mu := \tilde{Y}_\sharp\mathbb{P}$. We claim that the equation $\Phi \in \mathbf{F}$, $(\mathsf{x} - \tau\mathsf{v})_\sharp\Phi = \mu$ has no solution. We argue by contradiction, and we suppose that $\Phi \in \mathbf{F}$ is a solution: we could find $(X, V) \in \boldsymbol{B}$ such that setting $(X, V)_\sharp\mathbb{P} = \Phi$ and setting $Y := X - \tau V$ we have $Y_\sharp\mathbb{P} = \mu$.
We use the maximal λ-dissipativity of $\hat{\boldsymbol{B}}$ and we denote by $\hat{\boldsymbol{J}}_\tau$ the resolvent associated to $\hat{\boldsymbol{B}}$, by $\hat{\boldsymbol{j}}_\tau$ the map induced by Theorem 3.4 as in (3.2), and we set $\hat{\boldsymbol{b}}_\tau(x) := \frac{1}{\tau}(\hat{\boldsymbol{j}}_\tau(x, \mu) - x)$, $x \in \mathrm{supp}(\mu)$. We have $\tilde{X} = \hat{\boldsymbol{J}}_\tau\tilde{Y} = \hat{\boldsymbol{j}}_\tau(\tilde{Y}, \mu)$, $X = \hat{\boldsymbol{J}}_\tau Y = \hat{\boldsymbol{j}}_\tau(Y, \mu)$, $\tilde{V} = \frac{1}{\tau}(\tilde{X} - \tilde{Y}) = \hat{\boldsymbol{b}}_\tau(\tilde{Y})$ and $V = \frac{1}{\tau}(X - Y) = \hat{\boldsymbol{b}}_\tau(Y)$. It follows that $(\tilde{X}, \tilde{V})_\sharp\mathbb{P} = (\hat{\boldsymbol{j}}_\tau(\cdot, \mu), \hat{\boldsymbol{b}}_\tau)_\sharp\mu = (X, V)_\sharp\mathbb{P} = \Phi \in \mathbf{F}$ so that $(\tilde{X}, \tilde{V})$ has the same law of (X, V) and therefore belongs to $\boldsymbol{B}$, a contradiction. □

We now show that a maximal totally λ-dissipative MPVF is sequentially closed in the strong-weak topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$, recall (2.20).

Proposition 3.15 (Strong-weak closure)**.** *The sequential strong-weak closure* $\mathrm{cl}(\mathbf{F})$ *of a totally λ-dissipative* MPVF $\mathbf{F}$ *is totally λ-dissipative as well. In particular, if* $\mathbf{F}$ *is maximal, then* $\mathrm{cl}(\mathbf{F}) = \mathbf{F}$.

Proof. As usual, it is sufficient to check the property for $\lambda = 0$. Let $\Phi', \Phi'' \in \mathrm{cl}(\mathbf{F})$ and $\boldsymbol{\vartheta} \in \Gamma(\Phi', \Phi'')$. Denoting by $\{e_i\}_{i\in\mathbb{N}}$ an orthonotmal system for X, we introduce on X and on TX respectively the distances

$$\mathsf{d}^w(v_1, v_2) := \sum_{i=1}^{\infty} 2^{-i}\big(|\langle v_1 - v_2, e_i\rangle| \wedge 1\big), \quad \mathsf{d}^{sw}((x_1, v_1), (x_2, v_2)) := \Big(|x_1 - x_2|_{\mathsf{X}}^2 + \mathsf{d}^w(v_1, v_2)^2\Big)\Big)^{1/2}$$

whose induced topologies are weaker than the weak (resp. the strong-weak) topology of X (resp. TX), see also the proof of [NS21, Proposition 3.4]. Denoting by W_2^{sw} the 2-Wasserstein distance on $\mathcal{P}_2(\mathsf{TX})$ induced by d^{sw}, we have

$$\Phi_n \to \Phi \quad \text{in } \mathcal{P}_2^{sw}(\mathsf{TX}) \quad \Rightarrow \quad W_2^{sw}(\Phi_n, \Phi) \to 0.$$

By definition of $\mathrm{cl}(\mathbf{F})$ we can find two sequences $(\Phi_n')_{n\in\mathbb{N}}, (\Phi_n'')_{n\in\mathbb{N}}$ in $\mathbf{F}$ respectively converging to Φ' and Φ'' in $\mathcal{P}_2^{sw}(\mathsf{TX})$. We denote by $\boldsymbol{\gamma}_n' \in \Gamma_o^{sw}(\Phi_n', \Phi')$ and $\boldsymbol{\gamma}_n'' \in \Gamma_o^{sw}(\Phi'', \Phi_n'')$ the corresponding optimal plans for W_2^{sw}.
Denoting the elements of TX^4 by $(x_1', v_1', x_1, v_1, x_2, v_2, x_2'', v_2'')$ and using the gluing Lemma we can find a plan $\boldsymbol{\sigma}_n \in \mathcal{P}_2(\mathsf{TX}^4)$ such that $(\mathsf{x}_1', \mathsf{v}_1', \mathsf{x}_1, \mathsf{v}_1)_\sharp \boldsymbol{\sigma}_n = \boldsymbol{\gamma}_n'$, $(\mathsf{x}_1, \mathsf{v}_1, \mathsf{x}_2, \mathsf{v}_2)_\sharp \boldsymbol{\sigma}_n = \boldsymbol{\vartheta}$, $(\mathsf{x}_2, \mathsf{v}_2, \mathsf{x}_2'', \mathsf{v}_2'')_\sharp \boldsymbol{\sigma}_n = \boldsymbol{\gamma}_n''$. We also have

$$\lim_{n\to\infty} \int |x_1' - x_1|^2 + |x_2 - x_2''|^2 + d^w(v_1', v_1)^2 + d^w(v_2'', v_2)^2 \,\mathrm{d}\boldsymbol{\sigma}_n = 0,$$
$$\sup_{n\in\mathbb{N}} \int \Big(|v_1'|^2 + |v_1|^2 + |v_2|^2 + |v_2''|^2\Big) \,\mathrm{d}\boldsymbol{\sigma}_n < \infty,$$

so that setting $\tilde{\boldsymbol{\sigma}}_n := (\mathsf{x}_1', \mathsf{x}_2'', \mathsf{v}_1', \mathsf{v}_2'')_\sharp \boldsymbol{\sigma}_n$ we have

$$\tilde{\boldsymbol{\sigma}}_n \to (\mathsf{x}_1, \mathsf{x}_2, \mathsf{v}_1, \mathsf{v}_2)_\sharp \boldsymbol{\vartheta} \quad \text{in } \mathcal{P}_2^{sw}(\mathsf{X}^2 \times \mathsf{X}^2).$$

Since $(\mathsf{x}_1', \mathsf{v}_1', \mathsf{x}_2'', \mathsf{v}_2'')_\sharp \boldsymbol{\sigma}_n \in \Gamma(\Phi_n', \Phi_n'')$, the total dissipativity of $\mathbf{F}$ yields

$$\int \langle \mathsf{v}_1 - \mathsf{v}_2, \mathsf{x}_1 - \mathsf{x}_2\rangle \,\mathrm{d}\tilde{\boldsymbol{\sigma}}_n = \int \langle \mathsf{v}_1' - \mathsf{v}_2'', \mathsf{x}_1' - \mathsf{x}_2''\rangle \,\mathrm{d}\boldsymbol{\sigma}_n \le 0 \quad \text{for every } n \in \mathbb{N}. \tag{3.11}$$

Since the function $\zeta(x_1, x_2; v_1, v_2) := \langle \mathsf{v}_1 - \mathsf{v}_2, \mathsf{x}_1 - \mathsf{x}_2\rangle$ belongs to $\mathrm{C}_2^{sw}(\mathsf{X}^2 \times \mathsf{X}^2)$, the convergence in $\mathcal{P}_2^{sw}(\mathsf{X}^2 \times \mathsf{X}^2)$ is sufficient to pass to the limit in (3.11) and thus get

$$\int \langle \mathsf{v}_1 - \mathsf{v}_2, \mathsf{x}_1 - \mathsf{x}_2\rangle \,\mathrm{d}\boldsymbol{\vartheta} \le 0. \qquad \square$$

We can also prove that the sections $\mathbf{F}[\mu]$ of a maximal totally dissipative MPVF are (conditionally) totally convex. In the following statement we consider the space $\mathsf{X} \times \mathsf{X}^N$ whose variables are denoted by $(x, v_1, \cdots, v_N)$ and the corresponding projections are $\mathsf{x}(x, v_1, \cdots, v_N) := x$, $\mathsf{v}_i(x, v_1, \cdots, v_N) := v_i$.

Proposition 3.16 (Total convexity of sections of maximal totally dissipative MPVF)**.** *If* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is a maximal totally λ-dissipative* MPVF, *then for every* $\mu \in D(\mathbf{F})$ *the section* $\mathbf{F}[\mu]$

satisfies the following total convexity property:

$$\text{if } \Lambda \in \mathcal{P}_2(\mathsf{X}\times\mathsf{X}^N) \text{ satisfies } (\mathsf{x},\mathsf{v}_i)_\sharp\Lambda \in \mathbf{F}[\mu] \text{ and } \alpha_i \ge 0,\ i=1,\cdots,N \text{ with } \sum_i \alpha_i = 1, \text{ then}$$
$$(\mathsf{x}, \sum_i \alpha_i \mathsf{v}_i)_\sharp \Lambda \in \mathbf{F}[\mu]. \tag{3.12}$$

Proof. Since $\mathbf{F}$ is maximal totally λ-dissipative, by Theorem 3.12, its Lagrangian representation $\boldsymbol{B} \subset \mathcal{X}\times\mathcal{X}$ is maximal λ-dissipative.
We can find $(X, V_1, V_2, \cdots V_N) \in \mathcal{X}\times\mathcal{X}^N$ such that $(X, V_1, V_2, \cdots V_N)_\sharp\mathbb{P} = \Lambda$. We deduce that $(X, V_i) \in \boldsymbol{B}$ since $(X, V_i)_\sharp\mathbb{P} \in \mathbf{F}$. since the sections of $\boldsymbol{B}$ are convex, we deduce that $(X, \sum_i \alpha_i V_i) \in \boldsymbol{B}$ as well, so that

$$(\mathsf{x}, \sum_i \alpha_i \mathsf{v}_i)_\sharp\Lambda = (X, \sum_i \alpha_i V_i)_\sharp\mathbb{P} \in \mathbf{F}. \qquad \square$$

We can now derive a remarkable information on the structure of a totally dissipative MPVF, which involves the barycentric projection introduced in (2.13).

Theorem 3.17 (Barycentric projection)**.** *Let* $\mathbf{F}$ *be a* MPVF *and* $\mu \in \mathrm{D}(\mathbf{F})$ *such that* $\mathbf{F}[\mu]$ *is closed in* $\mathcal{P}_2(\mathsf{TX})$ *and satisfies the total convexity property* (3.12)*. Then* $\mathrm{bar}\,(\mathbf{F})\,[\mu] \subset \mathbf{F}[\mu]$*. In particular, if* $\mathbf{F}$ *is a maximal totally* λ*-dissipative* MPVF*, then* $\mathrm{bar}\,(\mathbf{F}) \subset \mathbf{F}$*.*

Proof. We use an argument which is clearly inspired by the Law of Large Numbers.
Let $\{\Phi_x\}_{x\in\mathsf{X}}$ be the disintegration of $\Phi \in \mathbf{F}$ w.r.t. its first marginal $\mu \in \mathrm{D}(\mathbf{F})$. For a given integer N and every $x \in \mathsf{X}$ we define the product measure $\Phi_x^N := (\Phi_x)^{\otimes N} \in \mathcal{P}_2(\mathsf{X}^N)$ and the corresponding plan

$$\Lambda^N := \int \delta_x \otimes \Phi_x^N \,\mathrm{d}\mu(x) \in \mathcal{P}_2(\mathsf{X}\times\mathsf{X}^N).$$

It is clear that Λ^N satisfies the condition of Proposition 3.16: choosing $\alpha_i := 1/N$ we deduce that $\Psi^N := (\mathsf{x}, \frac{1}{N}\sum_i \mathsf{v}_i)_\sharp\Lambda^N \in \mathbf{F}[\mu]$.
Let now $\Psi := (\boldsymbol{i}_\mathsf{X}, \boldsymbol{b}_\Phi)_\sharp\mu$. We can easily estimate the squared Wasserstein distance between Ψ and Ψ^N by

$$W_2^2(\Psi^N, \Psi) \le \int \Big|\frac{1}{N}\sum_i v_i - \boldsymbol{b}_\Phi(x)\Big|^2 \mathrm{d}\Lambda^N = \frac{1}{N}\int \Big|v - \boldsymbol{b}_\Phi(x)\Big|^2 \mathrm{d}\Phi$$

where we used the following orthogonality for $i \neq j$

$$\int \langle v_i - \boldsymbol{b}_\Phi(x), v_j - \boldsymbol{b}_\Phi(x)\rangle \,\mathrm{d}\Lambda^N = \int \Big(\int \langle v_i - \boldsymbol{b}_\Phi(x), v_j - \boldsymbol{b}_\Phi(x)\rangle \,\mathrm{d}\Phi_x(v_i)\otimes\Phi_x(v_j)\Big)\,\mathrm{d}\mu(x) = 0$$

and the fact that

$$\int |v_i - \boldsymbol{b}_\Phi(x)|^2 \,\mathrm{d}\Lambda^N = \int \Big(\int |v_i - \boldsymbol{b}_\Phi(x)|^2 \,\mathrm{d}\Phi_x(v_i)\Big)\,\mathrm{d}\mu(x) = \int \Big|v - \boldsymbol{b}_\Phi(x)\Big|^2 \mathrm{d}\Phi.$$

We deduce that $\Psi^N \to \Psi$ in $\mathcal{P}_2(\mathsf{TX})$ as $N \to +\infty$, so that $\Psi \in \mathbf{F}[\mu]$ as well. $\square$

Corollary 3.18. *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *be a totally* λ*-dissipative* MPVF*. Then the extended* MPVF $\tilde{\mathbf{F}}$ *defined by*

$$\tilde{\mathbf{F}} := \mathbf{F} \cup \mathrm{bar}\,(\mathbf{F})\,,$$

with $\mathrm{bar}\,(\mathbf{F})$ *as in* (2.13)*, is totally* λ*-dissipative. In particular, for every* $\Phi_i \in \mathbf{F}[\mu_i]$*,* $i = 1, 2$*, and every* $\boldsymbol{\mu} \in \Gamma(\mu_1, \mu_2)$*,*

$$\int \langle \boldsymbol{b}_{\Phi_1}(x_1) - \boldsymbol{b}_{\Phi_2}(x_2), x_1 - x_2\rangle \,\mathrm{d}\boldsymbol{\mu}(x_1,x_2) \le \lambda \int |x_1 - x_2|^2 \,\mathrm{d}\boldsymbol{\mu}(x_1,x_2).$$

Proof. It is sufficient to consider an arbitrary maximal totally λ-dissipative extension $\hat{\mathbf{F}}$ of $\mathbf{F}$: by the previous Theorem 3.17 clearly $\hat{\mathbf{F}} \supset \tilde{\mathbf{F}}$. ☐

Theorem 3.19 (The minimal selection). *Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF*.*

(1) For every $\mu \in \mathrm{D}(\mathbf{F})$ there exists a unique vector field $\boldsymbol{f}^\circ[\mu] \in L^2(\mathsf{X}, \mu; \mathsf{X})$ such that

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}^\circ[\mu])_\sharp \mu \in \mathbf{F}[\mu], \quad \int |\boldsymbol{f}^\circ[\mu]|^2 \,\mathrm{d}\mu \le \int |v|^2 \,\mathrm{d}\Phi \quad \text{for every } \Phi \in \mathbf{F}[\mu]. \tag{3.13}$$

We denote the minimal selection of $\mathbf{F}$ *at μ by*

$$\mathbf{F}^\circ[\mu] := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}^\circ[\mu])_\sharp \mu. \tag{3.14}$$

(2) If $\boldsymbol{B}$ is the Lagrangian representation of $\mathbf{F}$, then for every $\mu \in \mathrm{D}(\mathbf{F})$, we have

$$\boldsymbol{f}^\circ[\mu] = \boldsymbol{b}^\circ[\mu] \quad \mu\text{-a.e.},$$

where $\boldsymbol{b}^\circ$ has been defined by (3.6) and, if $0 < \tau < 1/\lambda^+$, the following hold

$$\int \left|\boldsymbol{b}_\tau(x,\mu) - \boldsymbol{f}^\circ[\mu](x)\right|^2 \mathrm{d}\mu \le \int \left|\boldsymbol{f}^\circ[\mu]\right|^2 \mathrm{d}\mu - (1-2\lambda\tau) \int \left|\boldsymbol{b}_\tau(x,\mu)\right|^2 \mathrm{d}\mu, \tag{3.15}$$

$$(1-\lambda\tau)^2 \int \left|\boldsymbol{b}_\tau(x,\mu)\right|^2 \mathrm{d}\mu \uparrow \int \left|\boldsymbol{f}^\circ[\mu]\right|^2 \mathrm{d}\mu \quad \text{as } \tau \downarrow 0 \tag{3.16}$$

with $\boldsymbol{b}_\tau$ as in (3.3).

(3) The map $|\mathbf{F}|_2 : \mathcal{P}_2(\mathsf{X}) \to [0, +\infty]$ defined by

$$|\mathbf{F}|_2(\mu) := \begin{cases} \int |\boldsymbol{f}^\circ[\mu]|^2 \,\mathrm{d}\mu & \text{if } \mu \in \mathrm{D}(\mathbf{F}), \\ +\infty & \text{if } \mu \notin \mathrm{D}(\mathbf{F}) \end{cases} \tag{3.17}$$

is lower semicontinuous.

(4) Finally, if Y is a Polish space, $\boldsymbol{\mu} \in \mathcal{P}(\mathsf{X} \times \mathsf{Y})$ with marginal $\nu = \pi^2_\sharp \boldsymbol{\mu}$ and the disintegration $\{\mu_y\}_{y \in \mathsf{Y}}$ of $\boldsymbol{\mu}$ w.r.t. ν satisfies

$$\int_{\mathsf{X}\times\mathsf{Y}} |x|^2 \,\mathrm{d}\boldsymbol{\mu}(x,y) + \int_{\mathsf{Y}} |\mathbf{F}|_2(\mu_y) \,\mathrm{d}\nu(y) < +\infty, \tag{3.18}$$

then the map $\boldsymbol{f}(x,y) := \boldsymbol{f}^\circ[\mu_y](x)$ belongs to $L^2(\mathsf{X} \times \mathsf{Y}, \boldsymbol{\mu}; \mathsf{X})$ (in particular it is uniquely defined up to a $\boldsymbol{\mu}$-negligible set and it is $\boldsymbol{\mu}$-measurable).

Proof. Claim (1) is an immediate consequence of the closure of $\mathbf{F}$ in $\mathcal{P}_2^{sw}(\mathsf{TX})$ (so that the map $\Phi \mapsto |\Phi|_2$ has compact sublevels in the set $\mathcal{P}_2(\mathsf{TX})$ with fixed first marginal equal to μ) and of the previous Theorem 3.17.
To prove the second claim, it is enough to notice that, trivially, $\boldsymbol{b}^\circ(\cdot, \mu)$ satisfies (3.13). Estimates (3.15) and (3.16) follow by Theorem A.3(5).
The third claim still follows immediately by the closure of $\mathbf{F}$ in $\mathcal{P}_2^{sw}(\mathsf{TX})$ and the fact that the map $\Phi \mapsto |\Phi|_2^2$ defined by (2.5) is lower semicontinuous w.r.t. the topology of $\mathcal{P}_2^{sw}(\mathsf{TX})$.
Let us now prove claim (4). We first notice that (3.18) yields $\mu_y \in D(\mathbf{F})$ for ν-a.e. $y \in \mathsf{Y}$. Let us now prove that the map $\boldsymbol{b}_\tau(x,y) := \boldsymbol{b}_\tau(x, \mu_y)$ is $\boldsymbol{\mu}$-measurable.
Recall that the set

$$\mathcal{S}_0 := \left\{(x,\mu) \in \mathsf{X} \times \mathcal{P}(\mathsf{X}) : x \in \mathrm{supp}(\mu)\right\}$$

is a G_δ (thus Borel, cf. [FSS22, Formula (4.3)]) subset of $\mathsf{X} \times \mathcal{P}(\mathsf{X})$. Since the inclusion map of $\mathsf{X} \times \mathcal{P}_2(\mathsf{X})$ in $\mathsf{X} \times \mathcal{P}(\mathsf{X})$ is continuous, we deduce that

$$\mathcal{S} := \mathcal{S}_0 \cap (\mathsf{X} \times \mathcal{P}_2(\mathsf{X}))$$

is a G_δ set in $\mathsf{X} \times \mathcal{P}_2(\mathsf{X})$.

Since the map $j(x,y) := (x,\mu_y)$ is Borel from $\mathsf{X}\times\mathsf{Y}$ to $\mathsf{X}\times\mathcal{P}(\mathsf{Y})$, we deduce that the set $\mathcal{S}' := j^{-1}(\mathcal{S}) = (\{(x,y)\in\mathsf{X}\times\mathsf{Y} : x\in\operatorname{supp}(\mu_y)\}$ is Borel in $\mathsf{X}\times\mathsf{Y}$ and it is immediate to check that $\boldsymbol{\mu}$ is concentrated on $\mathcal{S}'$. Since the map $(x,\mu)\mapsto \boldsymbol{b}_\tau(x,\mu)$ is continuous in $\mathcal{S}$ (cf. Theorem 3.4) then its composition with j (which is the map $(x,y)\mapsto \boldsymbol{b}_\tau(x,\mu_y)$) is $\boldsymbol{\mu}$-measurable. Passing to the limit as $\tau\downarrow 0$ and using (3.15) and (3.16) we conclude that $\boldsymbol{b}_\tau\to\boldsymbol{f}$ in $L^2(\mathsf{X}\times\mathsf{Y},\boldsymbol{\mu};\mathsf{X})$ so that also $\boldsymbol{f}$ is $\boldsymbol{\mu}$-measurable. □

We now show that discrete measures are sufficient to reconstruct a maximal totally λ-dissipative MPVF. For a general Polish space X, we set

$$\mathcal{P}_f(X) := \Big\{\mu\in\mathcal{P}(X) \,:\, \operatorname{supp}(\mu) \text{ is finite}\Big\}, \tag{3.19}$$

$$\mathcal{P}_{f,N}(X) := \Big\{\mu\in\mathcal{P}_f(X) : N\mu(A)\in\mathbb{N} \quad \text{for every } A\subset X\Big\},\quad \mathcal{P}_{f,\infty}(X) := \bigcup_{N\in\mathbb{N}}\mathcal{P}_{f,N}(X). \tag{3.20}$$

Corollary 3.20. *Let $\mathbf{F}\subset\mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF *and let*

$$\mathrm{D}_{f,\infty}(\mathbf{F}) := \mathcal{P}_{f,\infty}(\mathsf{X})\cap\mathrm{D}(\mathbf{F}).$$

Then for every $\mu\in\mathrm{D}(\mathbf{F})$ there exists a sequence $\mu_n\in\mathrm{D}_{f,\infty}(\mathbf{F})$ such that $\mathbf{F}^\circ[\mu_n]\to\mathbf{F}^\circ[\mu]$ in $\mathcal{P}_2(\mathsf{TX})$ as $n\to\infty$, where $\mathbf{F}^\circ$ has been defined in (3.14). Moreover, a measure $\Phi\in\mathcal{P}_2(\mathsf{TX})$ with $\mathsf{x}_\sharp\Phi\in\overline{\mathrm{D}(\mathbf{F})}$ belongs to $\mathbf{F}$ if and only if for every $\mu\in\mathrm{D}_{f,\infty}(\mathbf{F})$ and every $\boldsymbol{\gamma}\in\Gamma(\Phi,\mu)$ we have

$$\int\langle v-\boldsymbol{f}^\circ(y,\mu), x-y\rangle\mathrm{d}\boldsymbol{\gamma}(x,v,y)\le\lambda\int|x-y|^2\,\mathrm{d}\boldsymbol{\gamma}(x,v,y).$$

Proof. We denote by $\boldsymbol{B}\subset\mathcal{X}\times\mathcal{X}$ the Lagrangian representation of $\mathbf{F}$ and we set $D := \iota^{-1}(\mathcal{P}_{f,\infty}(\mathsf{X}))$. Since $\mathcal{P}_{f,\infty}(\mathsf{X})$ is dense in $\mathcal{P}_2(\mathsf{X})$, by e.g. the last part of Theorem B.5 we have that D is dense in $\mathcal{X}$ and by Theorem 3.4 (see in particular (3.2)) it satisfies $\boldsymbol{J}_\tau(D)\subset D$. We can thus apply Corollary A.16. □

3.3. Totally dissipative PVFs concentrated on maps

We devote this section to the study of the important case of single-valued and everywhere defined PVFs. Recall that for a deterministic PVF, total λ-dissipativity can be equivalently stated as in Remark 3.7.

Definition 3.21 (Demicontinuity)**.** *A single-valued* PVF $\mathbf{F}$ *is* demicontinuous *if the map $\mu\mapsto\mathbf{F}[\mu]$ satisfies*

$$\mu_n\to\mu\quad\text{in } \mathcal{P}_2(\mathsf{X})\quad\Rightarrow\quad\mathbf{F}[\mu_n]\to\mathbf{F}[\mu]\quad\text{in } \mathcal{P}_2^{sw}(\mathsf{TX}).$$

A single-valued PVF $\mathbf{F}$ *is* hemicontinuous *if its domain is totally convex and, for every $\boldsymbol{\gamma}\in\mathcal{P}_2(\mathsf{X}\times\mathsf{X})$ with marginals in $\mathrm{D}(\mathbf{F})$, the restriction of $\mathbf{F}$ to the set $\{\mathsf{x}^t_\sharp\boldsymbol{\gamma} : t\in[0,1]\}$ is demicontinuous.*

Theorem 3.22 (Characterization of deterministic totally dissipative PVF)**.** *Let $\mathbf{F}$ be a* single-valued *totally λ-dissipative* PVF*.*

(1) If $\mathbf{F}$ is maximal, then it is deterministic and $\mathbf{F}[\mu]=(\boldsymbol{i}_\mathsf{X},\boldsymbol{f}^\circ[\mu])_\sharp\mu$ for every $\mu\in D(\mathbf{F})$, where $\boldsymbol{f}^\circ$ is the minimal selection of $\mathbf{F}$ as in Theorem 3.19.

(2) If $\mathrm{D}(\mathbf{F})=\mathcal{P}_2(\mathsf{X})$, then $\mathbf{F}$ is maximal if and only if it is deterministic and demicontinuous (or, equivalently, deterministic and hemicontinuous)

(3) If $\mathrm{D}(\mathbf{F})=\mathcal{P}_2(\mathsf{X})$ and $\mathbf{F}[\mu]=(\boldsymbol{i}_\mathsf{X},\boldsymbol{f}[\mu])_\sharp\mu$ for every $\mu\in\mathcal{P}_2(\mathsf{X})$, then $\mathbf{F}$ is maximal if and only if for every $\zeta\in\mathrm{C}_2^{sw}(\mathsf{TX})$ and for every sequence $\mu_n\to\mu$ in $\mathcal{P}_2(\mathsf{TX})$

$$\lim_{n\to\infty}\int\zeta(x,\boldsymbol{f}[\mu_n](x))\,\mathrm{d}\mu_n(x)=\int\zeta(x,\boldsymbol{f}[\mu](x))\,\mathrm{d}\mu(x).$$

Proof. Claim (1) is an obvious consequence of Theorem 3.17.
Claim (2): let us first assume that $\mathbf{F}$ is maximal and let $\boldsymbol{B}$ be its Lagrangian representation. Since $D(\boldsymbol{B}) = \mathcal{X}$, $\boldsymbol{B}$ is locally bounded (see Theorem A.3(3)) so that if a sequence μ_n is converging to μ in $\mathcal{P}_2(\mathsf{X})$ and $\Phi_n = \mathbf{F}[\mu_n]$, we can assume that there exists a constant $C > 0$ such that

$$\int |v|^2 \,\mathrm{d}\Phi_n(x,v) \le C \quad \text{for every } n \in \mathbb{N}.$$

The compactness criterium of Proposition 2.5 shows that $(\Phi_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{P}_2^{sw}(\mathsf{TX})$. On the other hand, since $\mathbf{F} = \mathrm{cl}(\mathbf{F})$ by Proposition 3.15, we know that any accumulation point of Φ_n belongs to $\mathbf{F}$ and therefore it should coincide with $\mathbf{F}[\mu]$.
In order to prove the converse implication, it is sufficient to consider the case $\lambda = 0$ and $\mathbf{F}$ deterministic and hemicontinuous; we try to reproduce the argument of [Bré73] in the measure theoretic framework.
We first observe that the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$ is everywhere defined and single-valued, since $\iota(X) = \mu$, and $\iota^2(X,V) = \mathbf{F}[\mu] = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f})_\sharp\mu$ yield $V = \boldsymbol{f}\circ X$.
Let $(Y,W) \in \mathcal{X}\times\mathcal{X}$ satisfying

$$\int \langle \boldsymbol{B}X - W, X - Y\rangle_\mathcal{X} \,\mathrm{d}\mathbb{P} \le 0 \quad \text{for every } X \in \mathcal{X}.$$

Replacing X with $Y_t := (1-t)Y + tX$, $t \in (0,1)$ and setting $\mu_t := \iota(Y_t)$, $\boldsymbol{f}_t := \boldsymbol{f}[\mu_t]$, $V_t := \boldsymbol{f}_t \circ Y_t = \boldsymbol{B}(Y_t)$ we get

$$\int \langle \boldsymbol{f}_t(Y_t) - W, Y_t - Y\rangle_\mathcal{X} \,\mathrm{d}\mathbb{P} = \frac{t}{1-t}\int \langle \boldsymbol{f}_t(Y_t) - W, X - Y_t\rangle_\mathcal{X} \,\mathrm{d}\mathbb{P} \le 0 \quad \text{for every } X \in \mathcal{X},$$

so that

$$\int \langle V_t - W, X - Y_t\rangle_\mathcal{X} \,\mathrm{d}\mathbb{P} \le 0 \quad \text{for every } X \in \mathcal{X}. \tag{3.21}$$

Let us now set $\boldsymbol{\vartheta}_t := (X, Y_t, V_t)_\sharp\mathbb{P} \in \mathcal{P}_2(\mathsf{X}^2\times\mathsf{X})$. Denoting by $\mathsf{x},\mathsf{y},\mathsf{v}$ the projections of the points of X^3 to their components, since $(\mathsf{y},\mathsf{v})_\sharp\boldsymbol{\vartheta}_t = \mathbf{F}[\mu_t]$, by hemicontinuity assumption we know that

$$(\mathsf{y},\mathsf{v})_\sharp\boldsymbol{\vartheta}_t \to (Y, \boldsymbol{f}_0\circ Y)_\sharp\mathbb{P} = \mathbf{F}[\mu_0], \quad \text{in } \mathcal{P}_2^{sw}(\mathsf{X}\times\mathsf{X}) \text{ as } t\downarrow 0.$$

On the other hand, $(\mathsf{x},\mathsf{y})_\sharp\boldsymbol{\vartheta}_t = \iota^2(X,Y_t)$ converges to $\iota^2(X,Y)$ in $\mathcal{P}_2(\mathsf{X}^2)$ so that by compactness, we can also find a sequence $n \mapsto t(n) \downarrow 0$ such that $\boldsymbol{\vartheta}_{t(n)} \to \boldsymbol{\vartheta}$ in $\mathcal{P}_2^{sw}(\mathsf{X}^2\times\mathsf{X})$. Clearly $(\mathsf{y},\mathsf{v})_\sharp\boldsymbol{\vartheta} = (\boldsymbol{i}_\mathsf{X}, \boldsymbol{f}_0)_\sharp\mu_0$ is concentrated on a graph, so that $\boldsymbol{\vartheta} = (X, Y, \boldsymbol{f}_0\circ Y)_\sharp\mathbb{P}$.
Since

$$\int \langle \boldsymbol{f}_t(Y_t), X - Y_t\rangle \,\mathrm{d}\mathbb{P} = \int \langle v, x - y\rangle \,\mathrm{d}\boldsymbol{\vartheta}_t$$

and the function $\zeta(x,y,v) := \langle v, x-y\rangle$ belongs to $\mathrm{C}_2^{sw}(\mathsf{X}^2\times\mathsf{X})$ we deduce that

$$\lim_{n\to\infty}\int \langle \boldsymbol{f}_{t(n)}(Y_{t(n)}), X - Y_{t(n)}\rangle \,\mathrm{d}\mathbb{P} = \int \langle v, x-y\rangle \,\mathrm{d}\boldsymbol{\vartheta} = \int \langle \boldsymbol{f}_0(Y), X - Y\rangle \,\mathrm{d}\mathbb{P}.$$

Thus, we can pass to the limit in (3.21) obtaining

$$\int \langle \boldsymbol{f}_0(Y) - W, X - Y\rangle_\mathcal{X} \,\mathrm{d}\mathbb{P} \le 0 \quad \text{for every } X \in \mathcal{X},$$

in particular it holds for $X = \boldsymbol{f}_0(Y) - W + Y$. We deduce that $W = \boldsymbol{f}_0\circ Y = \boldsymbol{B}Y$ so that $\boldsymbol{B}$ is maximal and $\mathbf{F}$ is maximal as well.
Claim (3) is just the equivalent way to express the demicontinuity of $\mathbf{F}$, recalling Definition 2.4. □

An important example of single-valued, everywhere defined demicontinuous PVF is provided by the Yosida approximation: starting from a maximal totally λ-dissipative MPVF $\mathbf{F}$ and its Lagrangian representation $\boldsymbol{B}$, for every $\tau \in (0, 1/\lambda^+)$ we consider its Yosida approximation $\boldsymbol{B}_\tau$ and define the corresponding (single-valued) PVF

$$\mathbf{F}_\tau := \iota^2(\boldsymbol{B}_\tau). \tag{3.22}$$

Notice that $\mathbf{F}_\tau$ is maximal totally $\lambda/(1-\lambda\tau)$-dissipative (see Theorem A.3). Moreover, by Theorem 3.22(1), (3.3) and (3.22) we get that

$$\mathbf{F}_\tau[\mu] = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}_\tau[\mu])_\sharp \mu, \quad \text{for all } \mu \in \mathcal{P}_2(\mathsf{X}),$$

where $\boldsymbol{f}_\tau : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$ are given by $\boldsymbol{f}_\tau[\mu](\cdot) := \boldsymbol{b}_\tau(\cdot, \mu)$ with $\boldsymbol{b}_\tau$ as in (3.3); notice $\boldsymbol{f}_\tau$ admits a continuous version defined in $\mathcal{S}(\mathsf{X})$ and $\boldsymbol{f}_\tau(\cdot, \mu)$ belongs to $\mathrm{Lip}(\mathrm{supp}(\mu); \mathsf{X})$ for every $\mu \in \mathcal{P}_2(\mathsf{X})$ and clearly admits a Lipschitz extension to X (see Theorem 3.4). Setting $L_\tau := \frac{1}{\tau}(2-\lambda\tau)/(1-\lambda\tau)$, by L_τ-Lipschitz continuity of $\boldsymbol{B}_\tau$ and the representation (3.3), we get the following Lipschitz condition

$$\int \Big|\boldsymbol{f}_\tau(x_0, \mu_0) - \boldsymbol{f}_\tau(x_1, \mu_1)\Big|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le L_\tau^2 \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \quad \text{for every } \boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1), \tag{3.23}$$

which clearly implies demicontinuity of $\mathbf{F}_\tau$. We have thus proved the following result, recalling also Theorem 3.19(2).

Corollary 3.23. *Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF*. There exist sequences $\lambda_n, L_n \in \mathbb{R}$ and a sequence of maps $\boldsymbol{f}_n : \mathcal{P}_2(\mathsf{X}) \to \mathrm{Lip}(\mathsf{X}, \mathsf{X})$ satisfying the Lipschitz condition (3.23) with L_n in place of L_τ inducing a sequence of single-valued maximal totally λ_n-dissipative* PVF*s $\mathbf{F}_n$, and satisfying*

$$\lim_{n\to\infty} \int \Big|\boldsymbol{f}_n[\mu](x) - \boldsymbol{f}^\circ[\mu](x)\Big|^2 \,\mathrm{d}\mu(x) = 0 \quad \textit{for every } \mu \in D(\mathbf{F}),$$

where $\boldsymbol{f}^\circ$ is as in Theorem 3.19.

4. Lagrangian and Eulerian flow generated by a totally dissipative MPVF

In this section, making use of the results obtained in the previous Section 3, we study well-posedness for λ-EVI solutions driven by a maximal totally λ-dissipative MPVF $\mathbf{F}$. These curves are characterized (time by time) as the law of the unique semigroup of Lipschitz transformations $\boldsymbol{S}_t$ of the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$. As in the previous Section, we will consider a standard Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$ and the Hilbert space $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$.

Definition 4.1 (Lagrangian flow)**.** *Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF*. We call* Lagrangian flow *the family of maps $\boldsymbol{s}_t : \mathcal{S}(\mathsf{X}, \overline{\mathrm{D}(\mathbf{F})}) \to \mathsf{X}$ defined by Theorem 3.4 starting from the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$.*
The Lagrangian flow induces a semigroup of $(\mathcal{P}_2(\mathsf{X}), W_2)$-Lipschitz transformations $S_t : \overline{\mathrm{D}(\mathbf{F})} \to \overline{\mathrm{D}(\mathbf{F})}$ defined by $S_t(\mu_0) := \boldsymbol{s}_t(\cdot, \mu_0)_\sharp \mu_0$.
We say that the continuous curve $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$ is a Lagrangian solution *of the flow generated by $\mathbf{F}$ if $\mu_t = S_t(\mu_0) = \boldsymbol{s}_t(\cdot, \mu_0)_\sharp \mu_0$ for every $t \ge 0$.*

Notice that, if μ is a Lagrangian solution, the semigroup property (3.5) of the Lagrangian flow $\boldsymbol{s}_t$ yields in particular

$$\mu_t = \boldsymbol{s}_{t-s}(\cdot, \mu_s)_\sharp \mu_s \quad \text{for every } 0 \le s \le t.$$

In particular, to construct a Lagrangian solution starting from $\mu_0 \in \mathrm{D}(\mathbf{F})$ it is sufficient to choose an arbitrary map $X_0 \in \mathcal{X}$ satisfying $(X_0)_\sharp \mathbb{P} = \mu_0$ and set $\mu_t := (X_t)_\sharp \mu_0$ for the (unique) locally

Lipschitz solution $X \in \mathrm{Lip}_{\mathrm{loc}}([0,\infty);\mathcal{X})$ of

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t = \boldsymbol{B}^\circ X_t \quad \text{a.e. in } (0,+\infty), \quad X_{|t=0} = X_0.$$

An immediate consequence of Theorem 3.4 is the following result.

Theorem 4.2 (Existence of Lagrangian solutions)**.** *If $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ is a maximal totally λ-dissipative* MPVF *then for every $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ there exists a unique Lagrangian solution $\mu : [0,+\infty) \to \mathcal{P}_2(\mathsf{X})$ starting from μ_0.*
If $\mu_0 \in \mathrm{D}(\mathbf{F})$, then $\mu_t \in \mathrm{D}(\mathbf{F})$ for every $t \geq 0$, the curve $\mu : [0,+\infty) \to \mathcal{P}_2(\mathsf{X})$ is locally Lipschitz continuous, and

$$\int |\boldsymbol{f}^\circ(x,\mu_t)|^2 \,\mathrm{d}\mu_t(x) \leq e^{\lambda t} \int |\boldsymbol{f}^\circ(x,\mu_0)|^2 \,\mathrm{d}\mu_0(x) \quad \textit{for every } t \geq 0, \tag{4.1}$$

where $\boldsymbol{f}^\circ$ is defined in Theorem 3.19 and induces a map $(x,t) \mapsto \boldsymbol{f}^\circ(x,\mu_t)$ which is $\boldsymbol{\mu}$-measurable with respect to $\boldsymbol{\mu} = \int \mu_t \,\mathrm{d}t$ in every set $\mathsf{X} \times (0,T)$, $T > 0$.
Moreover, μ is the unique Eulerian solution *of the flow generated by $\mathbf{F}$ in the following sense: $\mu : [0,+\infty) \to \mathcal{P}_2(\mathsf{X})$ is the unique distributional solution of*

$$\partial_t \mu_t + \nabla \cdot (\mu_t \,\boldsymbol{f}^\circ(\cdot,\mu_t)) = 0 \quad \textit{in } (0,+\infty) \times \mathsf{X} \tag{4.2}$$

among the class of locally absolutely continuous curves satisfying $\mu_{t=0} = \mu_0 \in \mathrm{D}(\mathbf{F})$ and

$$\int_0^T \int |\boldsymbol{f}^\circ(x,\mu_t)|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t < \infty \quad \textit{for every } T > 0. \tag{4.3}$$

Finally, for every $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ and $t > 0$ we have

(1) if $\mathrm{supp}(\mu_0)$ is finite, then $\mathrm{supp}(\mu_t)$ is finite and its cardinality is nonincreasing w.r.t. t. In particular, if $\mu_0 \in \mathcal{P}_{f,N}(\mathsf{X})$ for some $N \in \mathbb{N}$ (recall (3.20)) then $\mu_t \in \mathcal{P}_{f,N}(\mathsf{X})$ for every $t \geq 0$;
(2) if $\mathrm{supp}(\mu_0)$ is compact, then $\mathrm{supp}(\mu_t)$ is compact;
(3) if $\mathrm{supp}(\mu_0)$ is bounded, then $\mathrm{supp}(\mu_t)$ is bounded and $\mathrm{diam}(\mathrm{supp}(\mu_t)) \leq \mathrm{e}^{\lambda t}\,\mathrm{diam}(\mathrm{supp}(\mu_0))$;
(4) if $\int_{\mathsf{X}} |x|^p \,\mathrm{d}\mu_0(x) < +\infty$ for some $p \geq 1$, then $\int_{\mathsf{X}} |x|^p \,\mathrm{d}\mu_t(x) < +\infty$ and

$$\int |x-y|^p \,\mathrm{d}\mu_t \otimes \mu_t \leq \mathrm{e}^{p\lambda t} \int |x-y|^p \,\mathrm{d}\mu_0 \otimes \mu_0.$$

Proof. The existence and the regularity properties of Lagrangian solutions follow by Theorem 3.4, while (4.1) follows by Theorem A.5(4).
Property (3.7) clearly implies (4.2).
Concerning uniqueness of solutions to (4.2) satisfying (4.3), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} W_2^2(\mu_t^1,\mu_t^2) &\leq 2\int \langle \boldsymbol{f}^\circ(x_1,\mu_t^1) - \boldsymbol{f}^\circ(x_2,\mu_t^2), x_1 - x_2\rangle \,\mathrm{d}\boldsymbol{\mu}_t \\ &\leq 2\lambda \int |x_1 - x_2|^2 \mathrm{d}\boldsymbol{\mu}_t \\ &= 2\lambda W_2^2(\mu_t^1,\mu_t^2)\end{aligned}$$

for a.e. $t \geq 0$ and every $\boldsymbol{\mu}_t \in \Gamma_o(\mu_t^1,\mu_t^2)$, by Theorem 2.13(6b) thanks to (4.3), the total λ-dissipativity of $\mathbf{F}$ and (3.13). Hence, by Grönwall inequality, we get

$$W_2(\mu_t^1,\mu_t^2) \leq e^{\lambda t} W_2(\mu_0^1,\mu_0^2).$$

The $\boldsymbol{\mu}$-measurability of the map $(x,t) \mapsto \boldsymbol{f}^\circ(x,\mu_t)$ follows by continuity of $t \mapsto \mu_t$ together with Theorem 3.19(4) with $\mathsf{Y} = [0,T]$. Indeed, (3.18) holds thanks to (4.1).
The last assertions (1-4) come from the fact that $\mu_t = \boldsymbol{s}_t(\cdot,\mu_0)_\sharp\mu_0$ and this map is $e^{\lambda t}$-Lipschitz continuous (cf. Theorem 3.4(3)). □

Remark 4.3 (A sticky-particle interpretation). We may interpret property (1) of the previous Theorem 4.2 by saying that the flows of totally dissipative MPVFs model sticky particle evolutions, (see also [NS09]). This fact reflects at a dynamic level the barycentric projection property stated in Theorem 3.17. In contrast, we immediately see that the example of $\frac{1}{2}$-dissipative PVF, with $\mathsf{X} = \mathbb{R}$, analysed in [Pic19, Section 7.1], [Cam+21, Section 6] and later discussed in [CSS23a, Section 7.5, Example 7.11], cannot be maximally total $\frac{1}{2}$-dissipative since it produces a $\frac{1}{2}$-EVI solution which splits the mass for positive times if e.g. $\mu_0 = \delta_0$. Notice indeed that, as highlighted in the following Theorem 4.4, if $\mathbf{F}$ is maximal totally dissipative then Lagrangian and EVI solutions coincide.

It is remarkable that the Lagrangian flow $\boldsymbol{s}_t$ provides an explicit representation of the flow of Lipschitz transformations generated by the unique λ-EVI solution, see [CSS23a, Definition 5.21] and Definition 2.21.

Theorem 4.4 (EVI solutions and contraction)**.** *If* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *is a maximal totally* λ*-dissipative* MPVF*, then for every* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$*, the curve* $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$*,* $\mu_t := S_t(\mu_0)$*, is the unique* λ*-EVI solution starting from* μ_0 *and* S_t *is a semigroup of* $e^{\lambda t}$*-Lipschitz transformations satisfying*

$$W_2(S_t(\mu_0'), S_t(\mu_0'')) \le e^{\lambda t} W_2(\mu_0', \mu_0'') \quad \textit{for every } \mu_0', \mu_0'' \in \overline{\mathrm{D}(\mathbf{F})},\ t \ge 0.$$

Proof. The proof is an immediate consequence of [CSS23a, Theorem 5.22(e)] and Theorem 4.2. Indeed notice that [CSS23a, Theorem 5.22(e)] can be applied even if the absolutely continuous curve μ satisfies the differential inclusion

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t \in \mathbf{F}[\mu_t] \tag{4.4}$$

w.r.t. to any Borel vector field $\boldsymbol{v}_t$ s.t. $(\mu, \boldsymbol{v})$ solves the continuity equation and $t \mapsto |\boldsymbol{v}_t|_{L^2(\mathsf{X},\mu_t;\mathsf{X})} \in L^1_{loc}(0, +\infty)$. For instance it holds for the vector field $\boldsymbol{f}^\circ$. Indeed, the proof of [CSS23a, Theorem 5.22(e)] relies on [CSS23a, Theorem 5.17(2)] which holds even if the differential inclusion (4.4), with $\boldsymbol{v}$ the Wasserstein vector field, is replaced by a general velocity field $\boldsymbol{v}$ as above. See also [CSS23a, Remark 3.12]. □

As a further consequence, in the case of maximal λ-totally dissipative MPVF all the various definitions of solutions coincide.

Theorem 4.5. *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *be a maximal totally* λ*-dissipative* MPVF*, let* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$ *and let* $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$ *be a continuous curve starting from* μ_0*. The following properties are equivalent:*

(1) μ *is a Lagrangian solution.*
(2) μ *is a* λ-EVI *solution.*

If moreover $\mu_0 \in \mathrm{D}(\mathbf{F})$ *or there exists a sequence* $t_n \downarrow 0$ *for which* $\mu(t_n) \in \mathrm{D}(\mathbf{F})$*, the above conditions are also equivalent to the following ones:*

(3) there exists a Borel vector field $\boldsymbol{w}_t$ *satisfying*

$$t \mapsto \int |\boldsymbol{w}_t(x)|^2 \,\mathrm{d}\mu_t(x) \quad \textit{is locally integrable in } (0, +\infty), \quad (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w}_t)_\sharp \mu_t \in \mathbf{F} \quad \textit{for a.e. } t > 0 \tag{4.5}$$

and the pair $(\mu, \boldsymbol{w})$ *satisfies the continuity equation*

$$\partial_t \mu_t + \nabla \cdot (\mu_t \boldsymbol{w}_t) = 0 \quad \textit{in } (0, +\infty) \times \mathsf{X}; \tag{4.6}$$

(4) there exists a Borel family Φ_t, $t > 0$, such that

$$\Phi_t \in \mathbf{F}[\mu_t] \quad \textit{for a.e. } t > 0, \quad t \mapsto \int |v|^2 \,\mathrm{d}\Phi_t \quad \textit{is locally integrable in } (0, +\infty), \tag{4.7}$$

$$\int_0^\infty \Big(\int \partial_t \zeta(t,x) \,\mathrm{d}\mu_t + \int \langle v, \nabla \zeta(t,x) \rangle \,\mathrm{d}\Phi_t(x,v) \Big) \,\mathrm{d}t = 0 \quad \textit{for every } \zeta \in \mathrm{Cyl}((0,+\infty) \times \mathsf{X}); \tag{4.8}$$

(5) $\mu_t \in \mathrm{D}(\mathbf{F})$ for every $t > 0$, μ is locally Lipschitz in $(0, +\infty)$ and it satisfies

$$t \mapsto \int |\boldsymbol{f}^\circ(x, \mu_t)|^2 \,\mathrm{d}\mu_t \quad \textit{is locally bounded in } (0, +\infty),$$

and (4.2).

Proof. The equivalence between (1) and (2) is a consequence of Theorem 4.4.
We can now consider the case when $\mu_0 \in \mathrm{D}(\mathbf{F})$ (the argument for the case $\mu(t_n) \in \mathrm{D}(\mathbf{F})$ along an infinitesimal sequence t_n is completely analogous). Theorem 4.2 clearly yields (1) $\Rightarrow$ (5). The implication (5) $\Rightarrow$ (3) is obvious. Theorem 3.17 shows that (3) and (4) are equivalent. Indeed (3) implies (4) by choosing $\Phi_t := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{w}_t)_\sharp \mu_t$ and (4) implies (3) by choosing $\boldsymbol{w}_t := \boldsymbol{b}_{\Phi_t}$. The implication (3) $\Rightarrow$ (2) follows by Theorem 5.4(1) of [CSS23a]. □

In the case when $\mu_0 \in \mathcal{P}_f(\mathsf{X})$ has finite support (recall (3.19), (3.20)), we can obtain a more refined characterization, which also yields a regularization effect when X has finite dimension and recovers the characterization (1.16) anticipated in the Introduction. Recall that by Theorem 4.2(1) any Lagrangian solution starting from $\mu_0 \in \mathcal{P}_{f,N}(\mathsf{X})$ must stay in $\mathcal{P}_{f,N}(\mathsf{X})$ for every time $t \geq 0$.

Corollary 4.6 (Regularization effect and Wasserstein velocity field for discrete measures)**.** *Let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF*, let $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})} \cap \mathcal{P}_{f,N}(\mathsf{X})$ for some $N \in \mathbb{N}$ and let $\mu : [0, +\infty) \to \mathcal{P}_{f,N}(\mathsf{X})$ be a continuous curve starting from μ_0. Assume moreover that at least one of the following properties holds:*

(a) $\mu_0 \in \mathrm{D}(\mathbf{F})$,
(b) $\mathrm{D}(\mathbf{F}) \cap \mathcal{P}_{f,N}(\mathsf{X})$ has non empty relative interior in $\mathcal{P}_{f,N}(\mathsf{X})$,
(c) X has finite dimension.

Then conditions (1),…,(5) of Theorem 4.5 are equivalent and, in this case, the minimal selection $\boldsymbol{f}^\circ$ of $\mathbf{F}$ (cf. Theorem 3.19) coincides with the Wasserstein velocity field $\boldsymbol{v}$ of μ (cf. Theorem 2.11) and μ also satisfies the right-differentiability property

$$\boldsymbol{v}_t = \lim_{h \downarrow 0} \frac{1}{h} \Big(\boldsymbol{t}_t^{t+h} - \boldsymbol{i}_{\mathsf{X}} \Big) = \boldsymbol{f}^\circ[\mu_t] \quad \textit{in } L^2(\mathsf{X}, \mu_t; \mathsf{X}) \quad \textit{for every } t > 0, \tag{4.9}$$

where $\boldsymbol{t}_t^{t+h}$ is the optimal transport map pushing μ_t into μ_{t+h}.
Finally, μ is a Lagrangian solution for $\mathbf{F}$ starting from μ_0 if and only if there are curves $\mathsf{x}_n \in \mathrm{C}([0, +\infty); \mathsf{X})$, $n = 1, \cdots, N$ which are locally Lipschitz in $(0, +\infty)$ such that $\mu_t = \frac{1}{N} \sum_{n=1}^N \delta_{\mathsf{x}_n(t)}$ for every $t \geq 0$ and the curves $(\mathsf{x}_n(t))_{n=1}^N$ solve the system of ODEs

$$\dot{\mathsf{x}}_n(t) = \boldsymbol{f}^\circ(\mathsf{x}_n, \mu_t) \quad \textit{a.e. in } (0, +\infty). \tag{4.10}$$

Proof. Case (a) is part of Theorem 4.5. In order to prove the first equivalence statement in cases (b) and (c), we briefly anticipate an argument that we will develop more extensively in Section 8: we introduce the standard Borel space $\Omega := [0, 1)$ endowed with the Lebesgue measure (still denoted by $\mathbb{P}$), the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$, and we consider the closed subspace $\mathcal{X}_N \subset \mathcal{X}$ of maps $X : \Omega \to \mathsf{X}$ which are constant on each interval $[(k-1)/N, k/N)$, $k = 1, \cdots, N$. Thanks to Theorem 3.4, $\mathcal{X}_N$ is invariant with respect to the action of the resolvent map $\boldsymbol{J}_\tau$. We can thus apply Proposition A.8 obtaining that the operator $\boldsymbol{B}_N := \boldsymbol{B} \cap (\mathcal{X}_N \times \mathcal{X}_N)$ is maximal

λ-dissipative in $\mathcal{X}_N$ and, if we select a Lagrangian parametrization $X_0 \in \overline{\mathrm{D}(\boldsymbol{B}_N)}$ of μ_0, still by Proposition A.8(ii), we get that $\boldsymbol{S}_t X_0$, the semigroup generated by $\boldsymbol{B}$, coincides with $\boldsymbol{S}^N_t X_0$, the semigroup generated by $\boldsymbol{B}_N$ and, under any of the conditions (b) and (c), $\boldsymbol{S}^N$ has a regularizing effect (see Theorem A.7, Corollary A.9 and notice that, in case (c), $\mathcal{X}_N$ has finite dimension) so that $\boldsymbol{S}^N_t X_0 \in \mathrm{D}(\boldsymbol{B}_N) \subset \mathrm{D}(\boldsymbol{B})$ for every $t > 0$. We immediately obtain that the conditions (1), ..., (5) of Theorem 4.5 are equivalent.
In order to check (4.9) we can use (3.8) observing that, for sufficiently small h, $(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{s}_h)_\sharp \mu_t$ is an optimal coupling between μ_t and μ_{t+h}, being $\mu_t \in \mathcal{P}_{f,N}(\mathsf{X})$, see the next Lemma 6.1.
Finally, in order to check the last representation formula, it is sufficient to write μ_0 as $\frac{1}{N}\sum_{n=1}^N x_n$ for suitable points $x_n \in \mathsf{X}$ and to set $\mathsf{x}_n(t) := \boldsymbol{s}_t(x_n, \mu_0)$. □

A further application concerns the convergence of the Implicite Euler Scheme (also called JKO method in the framework of gradient flows, see Proposition 5.2). We just recall here the main Crandall-Liggett estimate, referring to [NSV00; NS06] for more refined a-priori and a-posteriori error estimates.

Corollary 4.7 (Implicit Euler Scheme)**.** *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *be a maximal totally* λ*-dissipative* MPVF*. For every* $\mu \in \mathcal{P}_2(\mathsf{X})$ *and every* $0 < \tau < 1/\lambda^+$ *there exists a unique* $\Phi \in \mathbf{F}$ *such that*

$$(\mathsf{x} - \tau \mathsf{v})_\sharp \Phi = \mu. \tag{4.11}$$

Moreover $M_\tau := \mathsf{x}_\sharp \Phi = \boldsymbol{j}_\tau(\cdot, \mu)_\sharp \mu$*, where* $\boldsymbol{j}_\tau$ *is as in Theorem 3.4 applied to the Lagrangian representation of* $\mathbf{F}$*. If* $\mu_0 \in \overline{\mathrm{D}(\mathbf{F})}$*, then setting* $M^0_\tau := \mu_0$*,* $M^{n+1}_\tau := \boldsymbol{j}_\tau(\cdot, M^n_\tau)_\sharp M^n_\tau$*,* $n \in \mathbb{N}$*, we have*

$$\lim_{N\to\infty} M^N_{t/N} = \mu_t \quad \text{for every } t \geq 0, \tag{4.12}$$

where $\mu_t = S_t(\mu_0)$ *with* S_t *as in Definition 4.1. Moreover, for every* $T \geq 0$ *there exist* $N(\lambda, T) \in \mathbb{N}$ *and* $C(\lambda, T) > 0$ *(with* $C(0,T) = 2T$*) such that*

$$W_2(M^N_{t/N}, \mu_t) \leq \frac{C(\lambda, T)}{\sqrt{N}} \left| \boldsymbol{f}^\circ[\mu_0] \right|_{L^2(\mathsf{X},\mu_0;\mathsf{X})}, \tag{4.13}$$

for every $0 \leq t \leq T$*,* $n \geq N(\lambda, T)$ *and every* $\mu_0 \in \mathrm{D}(\mathbf{F})$*, where* $\boldsymbol{f}^\circ$ *is as in Theorem 4.2.*

Proof. The existence of Φ satisfying (4.11) follows by Theorem 3.14. Uniqueness follows by the well posedness of $\boldsymbol{J}_\tau$ and its invariance by law, stated in Theorem 3.4. The approximation in (4.12) follows by the Lagrangian one

$$\boldsymbol{S}_t X = \lim_{N\to\infty} (\boldsymbol{J}_{t/N})^N (X)$$

holding for any $X \in \overline{\mathrm{D}(\boldsymbol{B})}$ (see Theorem A.6), $\boldsymbol{B}$ the Lagrangian representation of $\mathbf{F}$.
Finally, (4.13) follows by Theorem A.6. □

We conclude this section with two results concerning the uniqueness and the stabiility of the characteristic system representing the solution of (4.5) and (4.6).
Using the notation of Theorem 3.4, we preliminary observe that choosing $\mu_0 \in \mathrm{D}(\mathbf{F})$ the Lipschitz maps $\boldsymbol{s}_t(x) := \boldsymbol{s}_t(x, \mu_0)$ belong to $\mathrm{Lip}(\mathrm{supp}(\mu_0); \mathsf{X})$ and the curve $t \mapsto \boldsymbol{s}_t$ is Lipschitz in $L^2(\mathsf{X}, \mu_0; \mathsf{X})$ with derivative $\boldsymbol{b}^\circ_t(\boldsymbol{s}_t)$ where $\boldsymbol{b}^\circ_t(\cdot) := \boldsymbol{b}^\circ(\cdot, (\boldsymbol{s}_t)_\sharp \mu_0)$. It follows that for every $T > 0$ and for μ_0-a.e. x the curve $t \mapsto \boldsymbol{s}_t(x)$ belongs to $H^1(0,T;\mathsf{X})$ and satisfies $\dot{\boldsymbol{s}}_t(x) = \boldsymbol{b}^\circ_t(\boldsymbol{s}_t(x))$. We can thus associate to $(\boldsymbol{s}_t)_{t\geq 0}$ a μ_0-measurable map

$$\mathrm{s} : \mathsf{X} \to H^1(0,T;\mathsf{X}), \quad \mathrm{s}[x](t) := \boldsymbol{s}_t(x, \mu_0). \tag{4.14}$$

In a similar way, if $X_0 \in \mathcal{X}$ with $\iota(X_0) = \mu_0$, we can define

$$\mathrm{X}(\omega, t) := \boldsymbol{s}_t(X_0(\omega), \mu_0), \quad \mathrm{X}[\omega] := \mathrm{s} \circ X_0, \tag{4.15}$$

obtaining a distiguished Caratheodory representative of $\boldsymbol{S}_t X_0$ which satisfies

$$\mathrm{X}(\omega,t) = (\boldsymbol{S}_t X_0)(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega\in\Omega, \text{ for every } t>0 \tag{4.16}$$

and

$$\mathrm{X}(\omega,\cdot)\in H^1(0,T;\mathsf{X}) \quad \text{for } \mathbb{P}\text{-a.e. } \omega, \qquad \int\Big(\int_0^T |\partial_t \mathrm{X}(\omega,t)|^2\,\mathrm{d}t\Big)\,\mathrm{d}\mathbb{P}(\omega) \le Te^{2\lambda+T}|\boldsymbol{B}^\circ X_0|^2_{\mathcal{X}}, \tag{4.17}$$

since

$$\begin{aligned}\int\Big(\int_0^T |\partial_t \mathrm{X}(\omega,t)|^2\,\mathrm{d}t\Big)\,\mathrm{d}\mathbb{P}(\omega) &= \int\Big(\int_0^T |\boldsymbol{b}_t^\circ(\mathrm{X}(\omega,t))|^2\,\mathrm{d}t\Big)\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_0^T |\boldsymbol{B}^\circ \mathrm{X}(\cdot,t)|^2_{\mathcal{X}}\,\mathrm{d}t\\ &\le Te^{2\lambda+T}|\boldsymbol{B}^\circ X_0|^2_{\mathcal{X}},\end{aligned}$$

where we have used Theorem A.5(4). It follows that X can be identified with a $\mathbb{P}$-measurable map $\omega\mapsto \mathrm{X}[\omega]$ which belongs to $L^2(\Omega,\mathcal{B},\mathbb{P};H^1(0,T;\mathsf{X}))$.

Theorem 4.8 (Uniqueness of the characteristic fields)**.** *Let $\mathbf{F}\subset \mathcal{P}_2(\mathsf{TX})$ be a maximal totally λ-dissipative* MPVF*, let us fix $T>0$ and let us suppose that $(\mu,\boldsymbol{v})$ is a solution to (4.5) and (4.6) in the interval $[0,T]$ starting from $\mu_0\in \mathrm{D}(\mathbf{F})$. Let $\boldsymbol{\eta}\in\mathcal{P}(\mathrm{C}([0,T];\mathsf{X}))$ be a probability measure concentrated on absolutely continuous curves and satisfying the following properties:*

(1) $(\mathsf{e}_t)_\sharp\boldsymbol{\eta}=\mu_t$ for every $t\in[0,T]$, where $\mathsf{e}_t(\gamma):=\gamma(t)$ for every $\gamma\in\mathrm{C}([0,T];\mathsf{X})$;
(2) $\boldsymbol{\eta}$-a.e. γ is an integral solution of the differential equation $\dot\gamma(t)=\boldsymbol{v}_t(\gamma(t))$ a.e. in $[0,T]$.

Then $\boldsymbol{\eta}=\mathrm{s}_\sharp\mu_0$, where s *is defined as in (4.14). In particular $\boldsymbol{\eta}$ is unique and $\boldsymbol{v}_t(x)=\boldsymbol{b}_t^\circ(x)$ μ_t-a.e. in X.*

Proof. We can find a Borel map $\mathrm{Z}:\Omega\to\mathrm{C}([0,T];\mathsf{X})$ such that $\mathrm{Z}_\sharp\mathbb{P}=\boldsymbol{\eta}$. Let $\boldsymbol{B}$ be the Lagrangian representation of $\mathbf{F}$. We can then define $X_t:=\mathsf{e}_t\circ\mathrm{Z}$. Since $(X_t)_\sharp\mathbb{P}=\mu_t\in\mathrm{D}(\mathbf{F})$ by Theorem 4.5(5), recalling Remark 3.5 we see that $X_t\in D(\boldsymbol{B})\subset\mathcal{X}$. It is also clear that for $\mathbb{P}$-a.e. ω we have

$$X_{t+h}(\omega)-X_t(\omega)=\int_t^{t+h}\boldsymbol{v}_s(X_s(\omega))\,\mathrm{d}s$$

and therefore $|X_{t+h}-X_t|_{\mathcal{X}}\le\int_t^{t+h}\|\boldsymbol{v}_s\|_{L^2(\mathsf{X},\mu_s;\mathsf{X})}\,\mathrm{d}s$ so that $t\mapsto X_t$ belongs to $H^1(0,T;\mathcal{X})$. At every differentiability point we have $\dot X_t=\boldsymbol{v}_t(X_t)$ so that $(X_t,\dot X_t)_\sharp\mathbb{P}=(\boldsymbol{i}_{\mathsf{X}},\boldsymbol{v}_t)_\sharp\mu_t\in\mathbf{F}[\mu_t]$ and eventually $\dot X_t\in\boldsymbol{B}X_t$. We conclude that $X_t(\omega)=\boldsymbol{s}_t(X_0(\omega))$ and therefore $\boldsymbol{\eta}=\mathrm{s}_\sharp\mu_0$. □

Theorem 4.9 (Stability of the Lagrangian flows)**.** *Under the same conditions of the previous Theorem 4.8, let $(\mu_0^n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{D}(\mathbf{F})$ satisfying the following properties:*

(1) $(\mu_0^n)_{n\in\mathbb{N}}$ converges to μ_0 in $\mathcal{P}_2(\mathsf{X})$, as $n\to\infty$;
(2) $\sup_n|\mathbf{F}|_2(\mu_0^n)<\infty$, where $|\mathbf{F}|_2(\cdot)$ is defined in (3.17).

If $\mathrm{s}^n,\mathrm{s}:\mathsf{X}\to\mathrm{C}([0,T];\mathsf{X})$ are the Lagrangian maps defined as in (4.14) starting from μ_0^n and μ_0 respectively, then $(\boldsymbol{i}_{\mathsf{X}},\mathrm{s}^n)_\sharp\mu_0^n\to(\boldsymbol{i}_{\mathsf{X}},\mathrm{s})_\sharp\mu_0$ in $\mathcal{P}_2(\mathsf{X}\times\mathrm{C}([0,T];\mathsf{X}))$ as $n\to\infty$.

Proof. By the last part of Theorem B.5, we can select a sequence $(X_0^n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ strongly converging to X_0 such that $\iota(X_0^n)=\mu_0^n$ and $\iota(X_0)=\mu_0$. We now consider the family of $\mathbb{P}$-measurable maps $\mathrm{X}^n:\Omega\to H^1(0,T;\mathsf{X})\subset\mathrm{C}([0,T];\mathsf{X})$ defined as in (4.15) starting from X_0^n and the corresponding X defined starting from X_0. Our thesis follows if we prove that $\mathrm{X}^n\to\mathrm{X}$ in $L^2(\Omega,\mathcal{B},\mathbb{P};\mathrm{C}([0,T];\mathsf{X}))$.

The equivalence (4.16) and the contraction estimates on $\boldsymbol{S}_t$ (cf. (A.8)) show that

$$\begin{aligned}\|\mathrm{X}^n - \mathrm{X}\|^2_{L^2(\Omega;L^2(0,T;\mathsf{X}))} &= \int \Big(\int_0^T |\mathrm{X}^n(\omega,t) - \mathrm{X}(\omega,t)|^2 \,\mathrm{d}t \Big) \,\mathrm{d}\mathbb{P}(\omega) \\ &= \int_0^T \Big(\int |\mathrm{X}^n(\omega,t) - \mathrm{X}(\omega,t)|^2 \,\mathrm{d}\mathbb{P}(\omega) \Big) \,\mathrm{d}t \\ &= \int_0^T |\boldsymbol{S}_t X_0^n - \boldsymbol{S}_t X_0|^2_{\mathcal{X}} \,\mathrm{d}t \\ &\le T\mathrm{e}^{2\lambda+T} |X_0^n - X_0|^2_{\mathcal{X}} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Moreover, recalling (4.17), we have

$$\sup_n \|(\mathrm{X}^n)'\|^2_{L^2(\Omega;L^2(0,T;\mathsf{X}))} \le Te^{2\lambda+T} \sup_n |\boldsymbol{B}^\circ X_0^n|^2_{\mathcal{X}} < \infty \quad \text{for every } n \in \mathbb{N},$$

so that X^n is uniformly bounded in $L^2(\Omega, \mathcal{B}, \mathbb{P}; H^1(0,T;\mathsf{X}))$ by some finite constant $S > 0$. The interpolation inequality (cf. [Bre10, p.233 (iii)])

$$\|Y\|^2_{\mathrm{C}([0,T];\mathsf{X})} \le C\|Y\|_{L^2(0,T;\mathsf{X})}\|Y\|_{H^1(0,T;\mathsf{X})} \quad \text{for every } Y \in H^1(0,T;\mathsf{X}),$$

gives that the sequence X^n strongly converges to X in $L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathrm{C}([0,T];\mathsf{X}))$, since

$$\begin{aligned}\|\mathrm{X}^n - \mathrm{X}\|^2_{L^2(\Omega,\mathcal{B},\mathbb{P};\mathrm{C}([0,T];\mathsf{X}))} &= \int \|\mathrm{X}^n[\omega] - \mathrm{X}[\omega]\|^2_{\mathrm{C}([0,T];\mathsf{X})} \,\mathrm{d}\mathbb{P} \\ &\le C \int \|\mathrm{X}^n[\omega] - \mathrm{X}[\omega]\|_{L^2(0,T;\mathsf{X})} \|\mathrm{X}^n[\omega] - \mathrm{X}[\omega]\|_{H^1(0,T;\mathsf{X})} \,\mathrm{d}\mathbb{P} \\ &\le C\Big(\int \|\mathrm{X}^n[\omega] - \mathrm{X}[\omega]\|^2_{L^2(0,T;\mathsf{X})} \,\mathrm{d}\mathbb{P} \Big)^{1/2} \Big(\int \|\mathrm{X}^n[\omega] - \mathrm{X}[\omega]\|^2_{H^1(0,T;\mathsf{X})} \,\mathrm{d}\mathbb{P} \Big)^{1/2} \\ &\le C(S + \|\mathrm{X}\|_{L^2(\Omega,\mathcal{B},\mathbb{P};H^1(0,T;\mathsf{X}))}) \|\mathrm{X}^n - \mathrm{X}\|_{L^2(\Omega;L^2(0,T;\mathsf{X}))}.\end{aligned}$$

□

5. Totally convex functionals in $\mathcal{P}_2(\mathsf{X})$

In this section we analyze the case of a proper and lower semicontinuous functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ which is *totally* $(-\lambda)$*-convex*, $\lambda \in \mathbb{R}$, i.e. it is $(-\lambda)$-convex along any coupling:

$$\phi(\mathsf{x}^t_\sharp \boldsymbol{\mu}) \le (1-t)\phi(\mathsf{x}^0_\sharp \boldsymbol{\mu}) + t\phi(\mathsf{x}^1_\sharp \boldsymbol{\mu}) + \frac{\lambda}{2} t(1-t) \int_{\mathsf{X}\times\mathsf{X}} |x-y|^2 \,\mathrm{d}\boldsymbol{\mu}(x,y)$$

for every $\boldsymbol{\mu} \in \mathcal{P}_2(\mathsf{X} \times \mathsf{X})$, $t \in [0,1]$. Notice that, in particular, ϕ is $(-\lambda)$-convex along generalized geodesics [AGS08, Definition 9.2.4] and thus also geodesically $(-\lambda)$-convex. It is also easy to check that ϕ is totally $(-\lambda)$-convex if and only if

$$\phi^\lambda(\mu) := \phi(\mu) + \frac{\lambda}{2} \int |x|^2 \,\mathrm{d}\mu \quad \text{is totally convex.}$$

We recall that the Wasserstein subdifferential $\boldsymbol{\partial}\phi \subset \mathcal{P}_2(\mathsf{TX})$ of ϕ is defined as the set of $\Psi \in \mathcal{P}_2(\mathsf{TX})$ such that

$$\mathsf{x}_\sharp \Psi = \mu \in \mathrm{D}(\phi), \quad \phi(\nu) - \phi(\mu) \ge -[\Psi, \nu]_l + o\,(W_2(\mu,\nu)) \quad \text{as } \nu \to \mu \text{ in } \mathcal{P}_2(\mathsf{X}).$$

When ϕ is geodesically $(-\lambda)$-convex, then it is possibile to show that Ψ belongs to $\boldsymbol{\partial}\phi$ if and only if Ψ and $\mu = \mathsf{x}_\sharp \Psi \in D(\phi)$ satisfy

$$\phi(\nu) - \phi(\mu) \ge -[\Psi, \nu]_l - \frac{\lambda}{2} W_2^2(\mu,\nu) \quad \text{for every } \nu \in \mathcal{P}_2(\mathsf{X}). \tag{5.1}$$

It is easy to check that $-\boldsymbol{\partial}\phi$ (cf.(2.19)) is a λ-dissipative MPVF (see also [CSS23a, Section 7.1]), but in general not totally λ-dissipative.
Let us now consider a totally λ-convex, proper and lower semicontinuous functional ϕ. We fix a standard Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$, with $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ and we consider the Lagrangian parametrization of ϕ given by

$$\psi : \mathcal{X} \to (-\infty, +\infty] \quad \text{defined as} \quad \psi(X) := \phi(\iota_X) \quad \text{for every } X \in \mathcal{X}. \tag{5.2}$$

Clearly, ψ is proper, l.s.c. and $(-\lambda)$-convex, i.e. $X \mapsto \psi(X) + \frac{\lambda}{2}|X|^2$ is convex.

As a preliminary result, we study the (opposite of the) subdifferential of ψ, showing in particular that it is an invariant maximal λ-dissipative operator. This allows to consider its resolvent operator $\boldsymbol{J}_\tau$ and compare, in Proposition 5.2, the scheme generated by $\boldsymbol{J}_\tau$ with the Wasserstein JKO scheme ([JKO98b]) for the functional ϕ in $\mathcal{P}_2(\mathsf{X})$. We then show relations between $-\partial\psi$ and $-\boldsymbol{\partial}\phi$, dealing in particular with the respective elements of minimal norm. Finally, in Theorem 5.4, we show that the Lagrangian solution to the flow generated by the maximal totally λ-dissipative MPVF $\iota^2(-\partial\psi)$ is the unique Wasserstein gradient flow for ϕ and the unique λ-EVI solution for $-\boldsymbol{\partial}\phi$. Analogously to Theorem 4.4, this Wasserstein semigroup can be characterized as the law of the semigroup of Lipschitz transformations $\boldsymbol{S}_t$ of $-\partial\psi$.

Proposition 5.1 (Total subdifferential)**.** *Let $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ be a proper, lower semicontinuous and totally $(-\lambda)$-convex functional and let ψ be as in (5.2).*

(1) The opposite of the subdifferential of ψ, $-\partial\psi$, is an invariant maximal λ-dissipative operator in $\mathcal{X} \times \mathcal{X}$.

(2) The total subdifferential $-\boldsymbol{\partial}_{\mathrm{t}}\phi := \iota^2(-\partial\psi)$ *is maximal totally λ-dissipative.*

(3) An element $\Psi \in \mathcal{P}_2(\mathsf{TX})$ satisfying $\mu = \mathsf{x}_\sharp\Psi \in D(\phi)$ belongs to $-\boldsymbol{\partial}_{\mathrm{t}}\phi$ if and only if for every $\nu \in D(\phi)$ and every plan $\boldsymbol{\vartheta} \in \Gamma(\Psi, \nu)$ we have

$$\phi(\nu) - \phi(\mu) \geq \int \Big(\langle v, x - y \rangle - \frac{\lambda}{2}|x - y|^2 \Big) \,\mathrm{d}\boldsymbol{\vartheta}(x, v, y). \tag{5.3}$$

In particular $\boldsymbol{\partial}_{\mathrm{t}}\phi \subset \boldsymbol{\partial}\phi$.

Proof. As usual it is sufficient to check the case $\lambda = 0$.
Claim (1): by maximality of the λ-dissipative operator $-\partial\psi$ in $\mathcal{X} \times \mathcal{X}$ (cf. Theorem A.3(1) and Corollary A.4) and thanks to Theorem 3.4, it is enough to prove that $-\partial\psi$ is invariant by measure-preserving isomorphisms.
Let $(X, V) \in -\partial\psi$ and let $g \in \mathrm{S}(\Omega)$. We have

$$\psi(Y) - \psi(X) \geq \langle V, X - Y \rangle_{\mathcal{X}} \quad \text{for every } Y \in \mathcal{X}.$$

For every $Z \in \mathcal{X}$, choosing $Y := Z \circ g^{-1}$ we get

$$\begin{aligned} \psi(Z) - \psi(X \circ g) = \psi(Z \circ g^{-1}) - \psi(X) &\geq \langle V, X - Z \circ g^{-1} \rangle_{\mathcal{X}} \\ &= \langle V \circ g, X \circ g - Z \rangle_{\mathcal{X}}. \end{aligned}$$

This shows that $(X \circ g, V \circ g) \in -\partial\psi$.
Claim (2) follows immediately by Theorem 3.12(3).
Claim (3): let us first show that an element Ψ satisfying (5.3) belongs to $-\boldsymbol{\partial}_{\mathrm{t}}\phi$: it is sufficient to take a pair $(X, V) \in \mathcal{X} \times \mathcal{X}$ such that $\iota^2_{X,V} = \Psi$. For every $Y \in D(\psi)$, setting $\nu := \iota_Y \in D(\phi)$ and $\boldsymbol{\vartheta} := (X, V, Y)_\sharp\mathbb{P}$, we get

$$\psi(Y) - \psi(X) = \phi(\nu) - \phi(\mu) \geq \int \langle v, x - y \rangle \,\mathrm{d}\boldsymbol{\vartheta}(x, v, y) = \langle V, X - Y \rangle_{\mathcal{X}},$$

which shows that $V \in -\partial\psi(X)$ and therefore $\Psi \in \iota^2(-\partial\psi) = -\boldsymbol{\partial}_{\mathrm{t}}\phi$.
In order to prove the converse implication, we just take $\Psi = \iota^2(X', Y') \in \iota^2(-\partial\psi)$ for some $(X', Y') \in -\partial\psi$, $\nu \in D(\phi)$, and $\boldsymbol{\vartheta} \in \Gamma(\Psi, \nu)$. We can find elements $X, V, Y \in \mathcal{X}$ such that

$(X,V,Y)_\sharp\mathbb{P} = \boldsymbol{\vartheta}$. In particular $Y_\sharp\mathbb{P} = \nu$ so that $\psi(Y) = \phi(\nu)$ and $(X,V)_\sharp\mathbb{P} = \Psi$ so that $(X,V) \in -\partial\psi$, since $-\partial\psi$ is law invariant and the law of (X,V) coincides with the law of (X',Y'). Since $\psi(X) = \phi(\iota_X) = \phi(\mu)$ and $(X,V) \in -\partial\psi$, we get (5.3)

$$\phi(\nu) - \phi(\mu) = \psi(Y) - \psi(X) \geq \langle V, X - Y\rangle_{\mathcal{X}} = \int \langle v, x - y\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x,v,y). \qquad \square$$

In view of the invariance and the maximal λ-dissipativity of $-\partial\psi$, by Theorem 3.19(1,2) we have that the subdifferential of ψ contains elements concentrated on maps, in the sense that for every $X \in \mathrm{D}(\partial\psi)$ there exist $\boldsymbol{f} \in L^2(\mathsf{X}, \iota_X; \mathsf{X})$ such that $\boldsymbol{f} \circ X \in -\partial\psi(X)$. An analogous result has been obtained in [GT19, Theorem 3.19(iii)] for real-valued functionals when X has finite dimension (cf. also [JMQ20, Lemma 8, Proposition 5]).

The next result gives a correspondence between the minimal selection and the resolvent operators of $-\partial\psi$ and ϕ. It is remarkable that the minimal selection $\boldsymbol{\partial}^\circ\phi$ of $\boldsymbol{\partial}\phi$ is an element of the smaller set $\boldsymbol{\partial}_{\mathrm{t}}\phi$ and therefore coincides with $\boldsymbol{\partial}^\circ_{\mathrm{t}}\phi$. This fact guarantees that the "Eulerian-Wasserstein" approach to the gradient flow of ϕ coincides with the "Lagrangian-Hilbertian" construction.

In the following, $\boldsymbol{J}_\tau$ denotes the resolvent of the invariant maximal λ-dissipative operator $-\partial\psi$ for $0 < \tau < 1/\lambda^+$ with the corresponding map $\boldsymbol{j}_\tau$ introduced in Theorem 3.4.

Proposition 5.2 (JKO scheme, Wasserstein and total subdifferential)**.** *Let $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ be a proper, lower semicontinuous and totally $(-\lambda)$-convex functional and let ψ be as in (5.2). Then:*

(1) For every $\mu \in \mathcal{P}_2(\mathsf{X})$ and $0 < \tau < 1/\lambda^+$ the measure $\mu_\tau := \boldsymbol{j}_\tau(\cdot,\mu)_\sharp\mu$ is the unique solution of the JKO scheme for ϕ starting from μ, i.e. μ_τ is the unique minimizer of

$$\nu \mapsto \frac{1}{2\tau} W_2^2(\mu,\nu) + \phi(\nu). \tag{5.4}$$

Equivalently, if $\mu = \iota(X)$ for some $X \in \mathcal{X}$, then $\mu_\tau = \iota(\boldsymbol{J}_\tau X)$.

(2) For every $\mu = \iota_X \in \mathrm{D}(\boldsymbol{\partial}_{\mathrm{t}}\phi)$, the element of minimal norm $\boldsymbol{\partial}^\circ_{\mathrm{t}}\phi[\mu]$ (equivalently, the law of the element of minimal norm of $\partial\psi(X)$) is the element of minimal norm of $\boldsymbol{\partial}\phi[\mu]$.

(3) We have that $\iota(\mathrm{D}(\partial\psi)) = \mathrm{D}(\boldsymbol{\partial}_{\mathrm{t}}\phi) = \mathrm{D}(\boldsymbol{\partial}\phi)$ and the minimal selection $-\boldsymbol{\partial}^\circ\phi$ of $-\boldsymbol{\partial}\phi$ is concentrated on a map and it is totally λ-dissipative.

(4) The MPVF *$\iota^2(-\partial\psi)$ is the unique maximal totally λ-dissipative extension of $-\boldsymbol{\partial}^\circ\phi$ with domain included in $\overline{\mathrm{D}(\phi)}$.*

Proof. By Theorem 5.1 and Theorem 3.4, we have that μ_τ does not depend on the choice of $X \in \mathcal{X}$ such that $\iota_X = \mu$; if $\nu \in \mathcal{P}_2(\mathsf{X})$, $\nu \neq \mu_\tau$, we can thus find $(X', Y) \in \mathcal{X}^2$ such that $\iota^2_{X',Y} = (X',Y)_\sharp\mathbb{P} \in \Gamma_o(\mu,\nu)$, $\mu_\tau = (\boldsymbol{J}_\tau X')_\sharp\mathbb{P}$, and $Y \neq \boldsymbol{J}_\tau X'$, since $Y_\sharp\mathbb{P} = \nu \neq \mu_\tau = \boldsymbol{J}_\tau X'$. By the properties of the resolvent operator $\boldsymbol{J}_\tau$ (cf. Corollary A.4), we have that

$$\phi(\mu_\tau) + \frac{1}{2\tau} W_2^2(\mu_\tau,\mu) \leq \psi(\boldsymbol{J}_\tau X') + \frac{1}{2\tau}|\boldsymbol{J}_\tau X' - X'|^2_{\mathcal{X}} < \psi(Y) + \frac{1}{2\tau}|Y - X'|^2_{\mathcal{X}} = \phi(\nu) + \frac{1}{2\tau} W_2^2(\mu,\nu),$$

which shows that μ_τ is a strict minimizer of (5.4).

To prove (2), first of all notice that, thanks to [AGS08, Lemma 10.3.8], ϕ is a *regular functional* according to [AGS08, Definition 10.3.9]. Let $-\partial^\circ\psi(X)$ be the element of minimal norm in $-\partial\psi(X)$ and let us denote by $\Phi_\mu := (X, -\partial^\circ\psi(X))_\sharp\mathbb{P} \in -\boldsymbol{\partial}\phi[\iota_X]$ by Proposition 5.1. Denoting $\mu := \iota_X$, we have

$$|\Phi_\mu|_2^2 = |-\partial^\circ\psi(X)|^2_{\mathcal{X}} = \lim_{\tau\downarrow 0} \frac{\psi(X) - \psi(\boldsymbol{J}_\tau X)}{\tau} = \lim_{\tau\downarrow 0} \frac{\phi(\mu) - \phi(\mu_\tau)}{\tau} = |-\boldsymbol{\partial}^\circ\phi(\mu)|_2^2,$$

where $-\boldsymbol{\partial}^\circ\phi(\mu)$ denotes the unique element of minimal norm in $-\boldsymbol{\partial}\phi[\mu]$ (cf. [AGS08, Theorem 10.3.11]), the last equality comes from [AGS08, Remark 10.3.14] and the second equality comes from Corollary A.4. This proves (2).

Also (3) follows by Corollary A.4, while the fact that $-\boldsymbol{\partial}^\circ\phi = -\boldsymbol{\partial}_{\rm t}^\circ\phi[\mu]$ is concentrated on a map follows by Theorem 3.19(1) being $-\boldsymbol{\partial}_{\rm t}\phi[\mu]$ maximal totally λ-dissipative by Proposition 5.1(2). To prove (4) it is enough to notice that, if $\mathbf{G}$ is a maximal totally λ-dissipative extension of $-\boldsymbol{\partial}^\circ\phi$ with domain included in $\overline{\mathrm{D}(\phi)}$, then its Lagrangian representation $\boldsymbol{B}$ has domain included in $\overline{\mathrm{D}(\psi)}$ and it is λ-dissipative with every element of the minimal selection of $-\partial\psi$ (cf. Theorem 3.12). By (A.3) we thus get that $\boldsymbol{B} \subset -\partial\psi$ and thus, being both maximal λ-dissipative, they coincide. □

Remark 5.3 (Comparison with similar notions of subdifferentiability). Part of Proposition 5.2 can be compared with the deep results obtained by [GT19] for the Fréchet subdifferential of general (not necessarily λ-convex) real-valued functionals when X has finite dimension. Using our notation, [GT19] restricts the analysis to elements of the Wasserstein-Fréchet subdifferential $\boldsymbol{\partial}\phi$ of ϕ which can be expressed by maps; it is proven in [GT19, Theorem 3.21, Corollary 3.22] that such a subset of $\boldsymbol{\partial}\phi(\mu)$ is nonempty if and only if the Fréchet subdifferential of ψ at X with $\mu = \iota_X$ is nonemtpty. Moreover in [GT19, Theorem 3.14] it is proven that, given $\mu \in \mathrm{D}(\phi)$, all the maps $\boldsymbol{f}$ belonging to $\mathrm{Tan}_\mu\,\mathcal{P}_2(\mathsf{X})$ for which $(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp\mu$ belongs to $\boldsymbol{\partial}\phi(\mu)$ correspond to elements $\boldsymbol{f} \circ X$ in $\partial\psi(X)$; in particular [GT19, Corollary 3.22] shows that the element of minimal norm of the Fréchet subdifferential of ψ at X can be written as $\boldsymbol{f}^\circ \circ X$, where $\boldsymbol{f}^\circ$ is the element of minimal norm of the Fréchet subdifferential of ϕ at ι_X (compare in particular with items (2),(3) in Proposition 5.2). On the other hand, working with general MPVFs and elements in $\partial\psi(X)$ which not necessarily have the form $\boldsymbol{f} \circ X$ allows to prove the law invariance of $\partial\psi$ and to work with functions ϕ whose proper domain $\mathrm{D}(\phi)$ is strictly contained in $\mathcal{P}_2(\mathsf{X})$.
We also mention that the lifting technique we are using here is of fundamental relevance for the concept of L-derivative considered in [CD18, Definition 5.22], [Car13, Definition 6.1], and inspired by [Lio07]. Using our notation, in [CD18; Car13] a function $\phi : \mathcal{P}_2(\mathsf{X}) \to \mathbb{R}$ is said to be L-differentiable at $\mu = \iota_X \in \mathcal{P}_2(\mathsf{X})$, for $X \in \mathcal{X}$, if the lifted function $\psi : \mathcal{X} \to \mathbb{R}$ is Fréchet differentible at X. The notion of L-differentiability can also be used to define a notion of convexity (called L-convexity) for functionals $\phi : \mathcal{P}_2(\mathsf{X}) \to \mathbb{R}$ which are continuously differentiable: we refer the interested reader to [CD18, Section 5.5.1, Definition 5.70] and we only mention that for such a class of regular functionals this definition is equivalent to total convexity. If $\dim \mathsf{X} \geq 2$, we will also show (see Remark 9.2) that for continuous functionals taking real values total convexity is in fact equivalent to geodesic convexity.

Theorem 5.4 (Gradient flows of totally convex functionals). *Let $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ be a proper, lower semicontinuous and totally $(-\lambda)$-convex functional and let ψ be as in (5.2). For every $\mu_0 \in \overline{\mathrm{D}(\phi)}$, let us denote by $(S_t)_{t\geq 0}$ the family of semigroups in $\mathcal{P}_2(\mathsf{X})$ induced by the Lagrangian flow associated to the maximal total λ-dissipative* MPVF $-\boldsymbol{\partial}_{\rm t}\phi = \iota^2(-\partial\psi)$ *(cf. Definition 4.1). Then the locally Lipschitz curve $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$, $\mu_t := S_t(\mu_0)$, is the unique* gradient flow *for ϕ starting from μ_0, in the sense that*

$$(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp\mu_t = -\boldsymbol{\partial}^\circ\phi[\mu_t] = -\boldsymbol{\partial}_{\rm t}^\circ\phi[\mu_t] \quad \textit{for a.e. } t > 0,$$

where $\boldsymbol{v}$ is the Wasserstein velocity field of μ coming from Theorem 2.11 and therefore satisfies all the properties of [AGS08, Thm. 11.2.1].
*Moreover, $t \mapsto S_t(\mu_0)$ is also the unique $(-\lambda)$-*EVI *solution for the* MPVF $-\boldsymbol{\partial}\phi$ *starting from $\mu_0 \in \overline{\mathrm{D}(\phi)}$ and S_t is a semigroup of $e^{\lambda t}$-Lipschitz transformations satisfying*

$$W_2(S_t(\mu_0), S_t(\mu_1)) \leq e^{\lambda t} W_2(\mu_0, \mu_1) \quad \textit{for any } \mu_0, \mu_1 \in \overline{\mathrm{D}(\phi)}.$$

Proof. Since ϕ is lower semicontinuous and $(-\lambda)$-convex along generalized geodesics, in particular it is coercive thanks to [NS21, Theorem 4.3]: we can apply [AGS08, Theorem 11.2.1] to get that there exists a unique gradient flow $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$ for ϕ starting from μ_0. By [CSS23a, Theorem 5.22(e)] this also shows that μ is the unique $(-\lambda)$-EVI solution for $-\boldsymbol{\partial}\phi$ starting from μ_0.

Since $\boldsymbol{\partial}^\circ\phi = \boldsymbol{\partial}^\circ_{\mathrm{t}}\phi$ by Proposition 5.2, we can apply Theorem 4.2 and Theorem 4.4 to show that μ coincides with $S_t(\mu_0)$, first for every $\mu_0 \in D(\boldsymbol{\partial}\phi)$ and then also in its closure, thanks to the regularization effect. □

We conclude the section with a pivotal example of a functional ϕ to which the results of this section can be applied.

Example 5.5. Let $P, W : \mathsf{X} \to (-\infty, +\infty]$ be proper, lower semicontinuous and $(-\lambda)$-convex functions, with W even. We define the functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ as

$$\phi(\mu) := \int_{\mathsf{X}} P \,\mathrm{d}\mu + \frac{1}{2}\int_{\mathsf{X}\times\mathsf{X}} W(x-y)\,\mathrm{d}(\mu\otimes\mu)(x,y), \quad \mu \in \mathcal{P}_2(\mathsf{X}).$$

Notice that $W(0)$ is finite so that, if $x_0 \in \mathrm{D}(P)$, then $\phi(\delta_{x_0}) = P(x_0) + \frac{1}{2}W(0) < +\infty$, so that ϕ is proper. Moreover, by [AGS08, Propositions 9.3.2 and 9.3.5], we have that ϕ is lower semicontinuous and totally $(-\lambda \wedge 0)$-convex.

6. Local optimality and injectivity of couplings

In this section we study the local optimality and the injectivity of a few classes of couplings. We first start with arbitrary couplings between discrete measures.

6.1. **Local optimality of couplings between discrete measures**

We want to show that the linear interpolations induced by arbitrary couplings between discrete measures can be decomposed in a finite union of geodesics.
The main quantitative information is contained in the following lemma.

Lemma 6.1. *Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$, $\boldsymbol{\gamma} \in \Gamma(\mu_0, \mu_1)$. If μ_0 has finite support $S = \{\bar{x}_1, \cdots, \bar{x}_M\}$ with $\delta := \min\big\{|\bar{x}_i - \bar{x}_j| : i, j \in \{1, \cdots, M\},\ i \neq j\big\} > 0$ and*

$$\sup\Big\{|y-x| : (x,y) \in \operatorname{supp}\boldsymbol{\gamma}\Big\} \le \delta/2$$

then $\boldsymbol{\gamma} \in \Gamma_o(\mu_0, \mu_1)$ and $W_2^2(\mu_0, \mu_1) = \int |y-x|^2\,\mathrm{d}\boldsymbol{\gamma}$.

Proof. It is sufficient to prove that the support of γ satisfies the cyclical monotonicity condition (2.9).
If $\{(x_n, y_n)\}_{n=1}^N$ are points in $\operatorname{supp}\boldsymbol{\gamma}$ with $x_0 := x_N$ and $x_n \neq x_{n-1}$ then

$$\begin{aligned}\langle y_n, x_n - x_{n-1}\rangle &= \langle y_n - x_n, x_n - x_{n-1}\rangle + \langle x_n, x_n - x_{n-1}\rangle \\ &\ge -\frac{\delta}{2}|x_n - x_{n-1}| + \frac{1}{2}|x_n - x_{n-1}|^2 + \frac{1}{2}|x_n|^2 - \frac{1}{2}|x_{n-1}|^2 \\ &\ge \frac{1}{2}|x_n|^2 - \frac{1}{2}|x_{n-1}|^2\end{aligned}$$

since $|y_n - x_n| \le \delta/2$ and $|x_n - x_{n-1}| \ge \delta$. If $x_n = x_{n-1}$ we trivially have $\langle y_n, x_n - x_{n-1}\rangle = \frac{1}{2}|x_n|^2 - \frac{1}{2}|x_{n-1}|^2$, so that

$$\sum_{n=1}^N \langle y_n, x_n - x_{n-1}\rangle \ge \sum_{n=1}^N \frac{1}{2}|x_n|^2 - \frac{1}{2}|x_{n-1}|^2 = \frac{1}{2}|x_N|^2 - \frac{1}{2}|x_0|^2 = 0.$$

□

As a consequence we obtain the following result.

Theorem 6.2 (Local optimality of discrete interpolations)**.** *Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ be two measures with finite support, $\boldsymbol{\gamma} \in \Gamma(\mu_0, \mu_1)$ and $\mu_t := (\mathsf{x}^t)_\sharp\boldsymbol{\gamma}$, $t \in [0,1]$. Then the following properties hold.*

(1) For every $s \in [0,1]$ there exists $\delta > 0$ such that for every $t \in [0,1]$ with $|t-s| \le \delta$ $\boldsymbol{\gamma}_{s,t} := (\mathsf{x}^s, \mathsf{x}^t)_\sharp \boldsymbol{\gamma}$ is an optimal plan between μ_s and μ_t, so that

$$W_2^2(\mu_s, \mu_t) = \int |y-x|^2 \,\mathrm{d}\boldsymbol{\gamma}_{s,t} = |t-s|^2 \int |y-x|^2 \,\mathrm{d}\boldsymbol{\gamma}(x,y).$$

(2) There exist a finite number of points $t_0 = 0 < t_1 < t_2 < \cdots < t_K = 1$ such that for every $k = 1, \cdots, K$, $\mu|_{[t_{k-1}, t_k]}$ is a minimal constant speed geodesic and

$$W_2^2(\mu_{t'}, \mu_{t''}) = |t''-t'|^2 \int |y-x|^2 \,\mathrm{d}\boldsymbol{\gamma}(x,y) \quad \text{for every } t', t'' \in [t_{k-1}, t_k].$$

(3) The length of the curve $t \mapsto \mu_t$ coincides with $\Big(\int |y-x|^2 \,\mathrm{d}\boldsymbol{\gamma}\Big)^{1/2}$.

Proof. The first statement follows by Lemma 6.1, since every measure μ_s has finite support and for every $t \in [0,1]$

$$\begin{aligned}\sup\big\{|y-x| : (x,y) \in \operatorname{supp} \boldsymbol{\gamma}_{s,t}\big\} &= |t-s| \sup\big\{|y-x| : (x,y) \in \operatorname{supp} \boldsymbol{\gamma}\big\} \\ &\le |t-s| \max\{|y-x| : x \in \operatorname{supp} \mu_0,\ y \in \operatorname{supp} \mu_1\big\}.\end{aligned}$$

In order to prove the second claim, we define an increasing sequence $(t_n)_{n=0}^\infty \subset [0,1]$ by induction as follows:

- $t_0 := 0$;
- if $t_n < 1$ then $t_{n+1} := \sup\Big\{t \in (t_n, 1] : W_2^2(\mu_{t_n}, \mu_t) = |t-t_n|^2 \int |y-x|^2 \,\mathrm{d}\boldsymbol{\gamma}\Big\}$;
- if $t_n = 1$ then $t_{n+1} = 1$.

The sequence is well defined thanks to the first claim. It is easy to see that there exists $K \in \mathbb{N}$ such that $t_K = 1$. If not, t_n would be strictly increasing with limit $t_\infty \le 1$ as $n \to \infty$. By the first claim, there exists $r > 0$ such that the restriction of μ to $[t_\infty - r, t_\infty]$ is a minimal geodesic, so that whenever $t_n \ge t_\infty - r$ we should get $t_{n+1} = t_\infty$, a contradiction.
Claim (3) follows immediately by (2). □

6.2. **Injectivity of interpolation maps**

Given two pairs of points (a', b') and (a'', b'') in X^2 it is easy to check that

$$(1-t)a' + tb' \ne (1-t)a'' + tb'' \quad \text{for every } t \in (0,1) \quad \Leftrightarrow \quad b'' - b' \notin \Big\{-s(a''-a') : s > 0\Big\}. \tag{6.1}$$

In particular, given a set $A \subset \mathsf{X}$ we consider the set of directions

$$\operatorname{dir}(A) := \Big\{s(a'-a'') : s \in \mathbb{R},\ a', a'' \in A\Big\} = \bigcup_{s \in \mathbb{R}} s\big(A - A\big). \tag{6.2}$$

Definition 6.3. *Given $A, B \subset \mathsf{X}$ we say that* the chords of B are not aligned with the directions of A *if*

$$(B - B) \cap \operatorname{dir}(A) = \{0\}. \tag{6.3}$$

In this case, for every $t \in (0,1)$ the map $\mathsf{x}^t : \mathsf{X}^2 \to \mathsf{X}$ is injective on $A \times B$.

When X has at least dimension 2, it is remarkable that in the discrete setting, it is always possible to perturb the elements of a finite set B in order to satisfy condition (6.3) with respect to a fixed finite set A. In particular, we can always find a suitable small perturbation of the points in B, so that the chords of the perturbed set are not aligned with the directions of the fixed set A.

Proposition 6.4 (Injectivity by small perturbations)**.** *Assume that* $\dim \mathsf{X} \ge 2$ *and $A \subset \mathsf{X}$ be a finite set. For every finite set of distinct points $B = \{b_n\}_{n=1}^N \subset \mathsf{X}$ there exists a finite set $B' := \{b'_n\}_{n=1}^N$ of distinct points with $|b'_n - b_n| < 1$ such that, setting*

$$b_n(s) := (1-s)b_n + sb'_n, \quad B(s) := \{b_n(s)\}_{n=1}^N, \tag{6.4}$$

we have that $\#B(s) = N$ *for all* $s \in [0,1]$ *and*

$$(B(s) - B(s)) \cap \mathrm{dir}(A) = \{0\} \quad \text{for every } s \in (0,1]. \tag{6.5}$$

In particular, for every $t \in (0,1)$ *the restriction of the map* x^t *to* $A \times B(s)$ *is injective for every* $s \in (0,1]$.

Proof. We split the proof of the Proposition in two steps.
Claim 1: *there exists a finite set of distinct points* $B'' := \{b''_n\}_{n=1}^N$ *with* $|b''_n - b_n| < 1$ *satisfying*

$$(B'' - B'') \cap \mathrm{dir}(A) = \{0\}. \tag{6.6}$$

We can argue by induction with respect to the cardinality N of the set B. The statement is obvious in case $N = 1$ (it is sufficient to choose $b''_1 := b_1$).
Let us assume that the property holds for all the sets of cardinality $N-1 \geq 1$. We can thus find a finite set of distinct points $B''_{N-1} = \{b''_n\}_{n=1}^{N-1}$ satisfying $(B''_{N-1} - B''_{N-1}) \cap \mathrm{dir}(A) = \{0\}$. We look for a point $b''_N \in U \setminus B''_{N-1}$, where $U := \{x \in \mathsf{X} : |x - b_N| < 1\}$, such that $B''_N := B''_{N-1} \cup \{b''_N\}$ satisfies (6.6). b''_N should therefore satisfy

$$b''_N \in U, \quad b''_N - b''_n \notin \mathrm{dir}(A) \quad \text{for every } n \in \{1, \cdots, N-1\}.$$

Such a point surely exists, since $\mathrm{dir}(A)$ is a closed set with empty interior (here we use the fact that the dimension of X is at least 2) and the union $\bigcup_{n=1}^{N-1} \left(b''_n + \mathrm{dir}(A)\right)$ has empty interior as well, so that it cannot contain the open set U.
Claim 2: *If* B'' *satisfies the properties of the previous claim, then there exists* $\delta \in (0,1]$ *such that setting*

$$b'_n := (1-\delta) b_n + \delta b''_n, \tag{6.7}$$

the set $B' = \{b'_n\}_{n=1}^N$ *satisfies the thesis.*
We denote by a the cardinality $\#A$ of A and we first make a simple remark: for every $z, z'' \in \mathsf{X}$

$$\#\{s \in [0,1] : z(s) := (1-s)z + s z'' \in \mathrm{dir}(A)\} > \mathsf{a}^2 \quad \Rightarrow \quad z, z'' \in \mathrm{dir}(A). \tag{6.8}$$

Indeed, the set $A - A$ contains at most a^2 distinct elements, so that if the left hand side of (6.8) is true, then there are at least two distinct values $s_1, s_2 \in [0,1]$, $r_1, r_2 \in \mathbb{R}$ and a vector $w \in A - A$ such that $(1-s_1)z + s_1 z'' = r_1 w$, $(1-s_2)z + s_2 z'' = r_2 w$. We then get

$$z(s) = z(s_1) + \frac{s - s_1}{s_2 - s_1}(z(s_2) - z(s_1)) = r_1 w + \frac{(s-s_1)(r_2 - r_1)}{s_2 - s_1} w \in \mathrm{dir}(A) \quad \text{for every } s \in [0,1],$$

hence (6.8). As a particular consequence of (6.8) we get that if z'' does not belong to $\mathrm{dir}(A)$, then the set $\{s \in (0,1] : z(s) := (1-s)z + s z'' \in \mathrm{dir}(A)\}$ is finite, so that

$$\forall\, z, z'' \in \mathsf{X} : \; z'' \notin \mathrm{dir}(A) \quad \Rightarrow \quad \exists\, \delta > 0 : \; (1-s)z + s z'' \notin \mathrm{dir}(A) \quad \text{for every } s \in (0, \delta]. \tag{6.9}$$

Let us now apply property (6.9) to all the pairs (z, z'') of the form $z = b_n - b_m$, $z'' = b''_n - b''_m$, $n, m \in \{1, \cdots, N\}$, with $n \neq m$. Since $b''_n - b''_m \notin \mathrm{dir}(A)$ we deduce that there exists $\delta_{n,m} > 0$ such that

$$(1-s)(b_n - b_m) + s(b''_n - b''_m) \notin \mathrm{dir}(A) \quad \text{for every } s \in (0, \delta_{n,m}]. \tag{6.10}$$

Setting

$$\tilde\delta := \min\{|b_n - b_m| \,:\, n, m \in \{1, \ldots, N\}, \, n \neq m\} > 0$$

and choosing $\delta := \min_{n,m}\{\delta_{n,m}, \, \tilde\delta/3\} > 0$, then it is not difficult to check that B' satisfies the thesis, with b'_n as in (6.7). Indeed, $|b_n - b'_n| = \delta |b_n - b''_n| < 1$, and for every $s \in [0,1]$ and n we get

$$b_n(s) := (1-s) b_n + s b'_n = (1-s) b_n + s(1-\delta) b_n + s\delta b''_n = (1 - \delta s) b_n + \delta s b''_n$$

so that

$$b_n(s) - b_m(s) = (1 - \delta s)(b_n - b_m) + \delta s (b''_n - b''_m) \notin \mathrm{dir}(A)$$

thanks to (6.10) and the fact that $s\delta \leq \delta_{n,m}$. □

7. Total dissipativity of MPVFs along discrete measures

We will consider the following subsets of the space $\mathcal{P}_f(X)$ of probability measures with finite support in a general Polish space X: for every $N \in \mathbb{N}$

$$\begin{aligned} \mathcal{P}_{f,N}(X) &:= \Big\{\mu \in \mathcal{P}_f(X) : N\mu(A) \in \mathbb{N}\ \forall\, A \subset X\Big\}, \\ \mathcal{P}_{\#N}(X) &:= \Big\{\mu \in \mathcal{P}_f(X) : N\mu(A) \in \{0,1\}\ \forall\, A \subset X\Big\} \\ &= \Big\{\mu \in \mathcal{P}_{f,N}(X) : \#\operatorname{supp}(\mu) = N\Big\}. \end{aligned} \tag{7.1}$$

Notice that every measure $\mu \in \mathcal{P}_{f,N}(X)$ can be expressed in the form

$$\mu = \frac{1}{N}\sum_{n=1}^{N}\delta_{x_n} \quad \text{for some points } x_1, \cdots, x_N \in X.$$

The measure μ belongs to $\mathcal{P}_{\#N}(X)$ if the points $x_1, \cdots, x_n$ are distinct.
If $\mathbf{F}$ is a MPVF, $\mu_0, \mu_1 \in \mathcal{P}(\mathsf{X})$, we correspondingly set

$$\mathrm{D}_\star(\mathbf{F}) := \mathrm{D}(\mathbf{F}) \cap \mathcal{P}_\star(\mathsf{X}), \quad \Gamma_\star(\mu_0,\mu_1) := \Gamma(\mu_0,\mu_1) \cap \mathcal{P}_\star(\mathsf{X}\times\mathsf{X}), \tag{7.2}$$

where $\star$ is replaced by one of the symbols f, c, b, (f,N), $\#N$ above.
For every $\mu_0, \mu_1 \in \mathcal{P}_f(\mathsf{X})$ we introduce the L^∞-Wasserstein distance by

$$W_\infty(\mu_0,\mu_1) := \min\Big\{\big|\mathsf{x}^0 - \mathsf{x}^1\big|_{L^\infty(\mathsf{X}\times\mathsf{X},\boldsymbol{\mu};\mathsf{X})} : \boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)\Big\}. \tag{7.3}$$

In the following, we investigate the results recalled in Theorem 2.19 in the case of marginals μ_0, μ_1 with finite support, but removing the optimality requirement over the coupling $\boldsymbol{\mu}$. Recall that the set $\Gamma(\mu_0,\mu_1|\mathbf{F})$ has been introduced in Definition 2.18.

Lemma 7.1. *Let $\mathbf{F}$ be a MPVF satisfying (2.17) and let $\mu_0, \mu_1 \in \mathrm{D}_f(\mathbf{F})$ with $\boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1|\mathbf{F})$ satisfy at least one of the following conditions:*

(1) for every $t \in (0,1)$, x^t is $\boldsymbol{\mu}$-essentially injective;
(2) for every $t \in (0,1)$, there exists an element $\Phi_t \in \mathbf{F}[\mathsf{x}^t_\sharp\boldsymbol{\mu}]$ which is concentrated on a map.

Then

$$[\mathbf{F},\boldsymbol{\mu}]_{r,s} - [\mathbf{F},\boldsymbol{\mu}]_{l,t} \le \lambda(t-s)W^2, \quad W^2 := \int |x_0 - x_1|^2\,\mathrm{d}\boldsymbol{\mu}, \quad \text{for every } 0 \le s < t \le 1. \tag{7.4}$$

In particular, $t \mapsto [\mathbf{F},\boldsymbol{\mu}]_{r,t} + \lambda W^2 t$ and $t \mapsto [\mathbf{F},\boldsymbol{\mu}]_{l,t} + \lambda W^2 t$ are increasing respectively in $[0,1)$ and in $(0,1]$, $[\mathbf{F},\boldsymbol{\mu}]_{l,t} = [\mathbf{F},\boldsymbol{\mu}]_{r,t}$ at every $t \in (0,1)$ where one of them is continuous, hence they coincide outside a countable set of discontinuities.

Proof. By Theorem 2.19 it is not restrictive to assume $\lambda = 0$; we can also assume $s = 0$ and $t = 1$ thanks to (2.11). We set $\mu_t := \mathsf{x}^t_\sharp\boldsymbol{\mu}$ and we select an element $\Phi_t \in \mathbf{F}[\mu_t]$ (in case (2) we can also suppose that Φ_t is concentrated on a map).
Applying Theorem 6.2, we can find points $t_0 = 0 < t_1 < \cdots < t_K = 1$ such that

$$\boldsymbol{\mu}^k := (\mathsf{x}^{t_{k-1}}, \mathsf{x}^{t_k})_\sharp\boldsymbol{\mu} \in \Gamma(\mu_{t_{k-1}}, \mu_{t_k}|\mathbf{F}) \cap \Gamma_o(\mu_{t_{k-1}}, \mu_{t_k}) \quad \text{for every } k = 1, \cdots, K.$$

In particular, from (2.11) and Theorem 2.19(2), we get

$$[\Phi_{t_{k-1}},\boldsymbol{\mu}]_{r,t_{k-1}} = \frac{1}{t_k - t_{k-1}}[\Phi_{t_{k-1}},\boldsymbol{\mu}^k]_{r,0} \le \frac{1}{t_k - t_{k-1}}[\Phi_{t_k},\boldsymbol{\mu}^k]_{l,1} = [\Phi_{t_k},\boldsymbol{\mu}]_{l,t_k}.$$

Since, for $1 \le k < K$, x^{t_k} is $\boldsymbol{\mu}$-essentially injective (if assumption (1) holds) or Φ_{t_k} is concentrated on its barycenter (if assumption (2) holds), Theorem 2.13(4) yields $[\Phi_{t_k},\boldsymbol{\mu}]_{l,t_k} = [\Phi_{t_k},\boldsymbol{\mu}]_{r,t_k}$ so that

$$[\Phi_0,\boldsymbol{\mu}]_{r,0} \le [\Phi_1,\boldsymbol{\mu}]_{l,1}.$$

Taking the supremum w.r.t. $\Phi_0 \in \mathbf{F}[\mu_0]$ and the infimum w.r.t. $\Phi_1 \in \mathbf{F}[\mu_1]$ we obtain (7.4). The last part of the statement follows as in the proof of Theorem 2.19. $\square$

Theorem 7.2 (Self-improving dissipativity along discrete couplings)**.** *Assume that* $\dim \mathsf{X} \geq 2$. *Let* $\mathbf{F}$ *be a* MPVF *satisfying* (2.17)*,* $N \in \mathbb{N}$*, let* $\mu_0, \mu_1 \in \mathrm{D}_f(\mathbf{F})$*,* $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$ *and let* $\mu_t = \mathsf{x}^t_\sharp \boldsymbol{\mu}$*,* $t \in [0,1]$*. Assume that one of the following conditions is satisfied:*

(1) $\boldsymbol{\mu} \in \mathcal{P}_{f,N}(\mathsf{X} \times \mathsf{X})$ *and for every* $t \in (0,1)$ μ_t *belongs to the relative interior of* $\mathrm{D}_{f,N}(\mathbf{F})$ *in* $\mathcal{P}_{f,N}(\mathsf{X})$*;*
(2) for every $t \in (0,1)$ μ_t *belongs to the interior of* $\mathrm{D}_f(\mathbf{F})$ *in the metric space* $(\mathcal{P}_f(\mathsf{X}), W_\infty)$*.*

Then

$$[\mathbf{F}, \boldsymbol{\mu}]_{r,s} - [\mathbf{F}, \boldsymbol{\mu}]_{l,t} \leq \lambda(t-s) W^2, \quad W^2 := \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}, \quad \textit{for every } 0 \leq s < t \leq 1. \tag{7.5}$$

Proof. We carry out the proof in case (1), the proof in case (2) is analogous. By Theorem 2.19 it is not restrictive to assume $\lambda = 0$; we can also assume $s = 0$ and $t = 1$ thanks to (2.11). By Theorem 6.2 we can find $0 < \delta < 1/2$ and $\tau \in (\delta, 1-\delta)$ s.t. $\mathsf{x}^\delta, \mathsf{x}^\tau$ and $\mathsf{x}^{1-\delta}$ are $\boldsymbol{\mu}$-essentially injective and $(\mathsf{x}^0, \mathsf{x}^\delta)_\sharp \boldsymbol{\mu}$, $(\mathsf{x}^{1-\delta}, \mathsf{x}^1)_\sharp \boldsymbol{\mu}$ are optimal. In this way, since by Theorem 2.19 the relation (7.5) is true both for the case $s = 0, t = \delta$ and $s = 1-\delta, t = 1$, we only need to prove it for $s = \delta$ and $t = 1-\delta$.
We set $A = \operatorname{supp}(\mu_\delta) \cup \operatorname{supp}(\mu_{1-\delta})$ and $B = \operatorname{supp}(\mu_\tau)$. By compactness, we can find $\varepsilon > 0$ such that every measure in $\mathcal{P}_{f,N}(\mathsf{X})$ in the W_2-neighborhood of radious $\varepsilon > 0$ around μ_t is contained in $\mathrm{D}(\mathbf{F})$ for every $\delta \leq t \leq 1-\delta$.
Applying Proposition 6.4 we can find a map $\boldsymbol{b} : B \to \mathsf{X}$ with values in the open ball of radious ε centered at 0 such that setting $\boldsymbol{b}^s(x) := x + s\boldsymbol{b}(x)$ for every $s \in [0,1]$ and $x \in B$, the set $B^s := \boldsymbol{b}^s(B)$ satisfies $(B^s - B^s) \cap \operatorname{dir}(A) = \{0\}$ and $\# B^s = \# \operatorname{supp}(\mu_\tau)$ for every $s \in (0,1]$. Considering the measures $\nu_s := (\boldsymbol{b}^s)_\sharp \mu_\tau$, we can pick $\Psi_s \in \mathbf{F}[\nu_s]$ with barycenter $\boldsymbol{v}_s : B^s \to \mathsf{X}$, i.e.

$$\boldsymbol{v}_s(y) := \int v \,\mathrm{d}\Psi_s(y, v).$$

Now for every $(x_0, x_1) \in \operatorname{supp}((\mathsf{x}^\delta, \mathsf{x}^{1-\delta})_\sharp \boldsymbol{\mu})$ we set

$$x_a := \mathsf{x}^a(x_0, x_1), \quad \boldsymbol{b}^{s,\tau} := \boldsymbol{b}^s(x_a), \quad \boldsymbol{v}^{s,\tau} := \boldsymbol{v}^s(\boldsymbol{b}^{s,\tau}),$$

where $a = \frac{\tau - \delta}{1 - 2\delta}$. Notice that $x_a \in B = \operatorname{supp}(\mu_\tau)$, so the above definitions are well-posed. Let us consider $\Phi_\delta \in \mathbf{F}[\mu_\delta]$, $\Phi_{1-\delta} \in \mathbf{F}[\mu_{1-\delta}]$ and $\boldsymbol{\sigma} \in \mathcal{P}(\mathsf{TX} \times \mathsf{TX})$ s.t. $(\mathsf{x}^0, \mathsf{x}^1)_\sharp \boldsymbol{\sigma} = (\mathsf{x}^\delta, \mathsf{x}^{1-\delta})_\sharp \boldsymbol{\mu}$,

$(\mathsf{x}^0, \mathsf{v}^0)_\sharp \boldsymbol{\sigma} = \Phi_\delta$ and $(\mathsf{x}^1, \mathsf{v}^1)_\sharp \boldsymbol{\sigma} = \Phi_{1-\delta}$. For every $(x_0, v_0, x_1, v_1) \in \mathrm{supp}(\boldsymbol{\sigma})$ we have

$$\begin{aligned}
&\langle v_0 - v_1, x_0 - x_1\rangle = \langle v_0 - \boldsymbol{v}^{s,\tau}, x_0 - x_1\rangle + \langle v_1 - \boldsymbol{v}^{s,\tau}, x_1 - x_0\rangle \\
&\quad= \frac{1}{a}\langle v_0 - \boldsymbol{v}^{s,\tau}, x_0 - x_a\rangle + \frac{1}{1-a}\langle v_1 - \boldsymbol{v}^{s,\tau}, x_1 - x_a\rangle \\
&\quad= \frac{1}{a}\langle v_0 - \boldsymbol{v}^{s,\tau}, x_0 - \boldsymbol{b}^{s,\tau}\rangle + \frac{1}{1-a}\langle v_1 - \boldsymbol{v}^{s,\tau}, x_1 - \boldsymbol{b}^{s,\tau}\rangle \\
&\qquad + \frac{1}{a}\langle v_0 - \boldsymbol{v}^{s,\tau}, \boldsymbol{b}^{s,\tau} - x_a\rangle + \frac{1}{1-a}\langle v_1 - \boldsymbol{v}^{s,\tau}, \boldsymbol{b}^{s,\tau} - x_a\rangle \\
&\quad= \frac{1}{a}\langle v_0 - \boldsymbol{v}^{s,\tau}, x_0 - \boldsymbol{b}^{s,\tau}\rangle + \frac{1}{1-a}\langle v_1 - \boldsymbol{v}^{s,\tau}, x_1 - \boldsymbol{b}^{s,\tau}\rangle \\
&\qquad + \frac{1}{a(1-a)}\langle \boldsymbol{v}^{1,\tau} - \boldsymbol{v}^{s,\tau}, \boldsymbol{b}^{s,\tau} - x_a\rangle + \frac{1}{a(1-a)}\langle (1-a)v_0 + av_1 - \boldsymbol{v}^{1,\tau}, \boldsymbol{b}^{s,\tau} - x_a\rangle \\
&\quad= \frac{1}{a}\langle v_0 - \boldsymbol{v}^{s,\tau}, x_0 - \boldsymbol{b}^{s,\tau}\rangle + \frac{1}{1-a}\langle v_1 - \boldsymbol{v}^{s,\tau}, x_1 - \boldsymbol{b}^{s,\tau}\rangle \\
&\qquad + \frac{s}{(1-s)a(1-a)}\langle \boldsymbol{v}^{1,\tau} - \boldsymbol{v}^{s,\tau}, \boldsymbol{b}^{1,\tau} - \boldsymbol{b}^{s,\tau}\rangle + \frac{s}{a(1-a)}\langle (1-a)v_0 + av_1 - \boldsymbol{v}^{1,\tau}, \boldsymbol{b}^{1,\tau} - x_a\rangle.
\end{aligned} \tag{7.6}$$

We have that

$$\begin{aligned}
\int \langle v_0 - \boldsymbol{v}^{s,\tau}(x_0,x_1), x_0 - \boldsymbol{b}^{s,\tau}(x_0,x_1)\rangle \,\mathrm{d}\boldsymbol{\sigma} &= [\Phi_\delta, \boldsymbol{\mu}^{s,\tau}]_{r,0} - [\Psi_s, \boldsymbol{\mu}^{s,\tau}]_{l,1}, \\
\int \langle v_1 - \boldsymbol{v}^{s,\tau}(x_0,x_1), x_1 - \boldsymbol{b}^{s,\tau}(x_0,x_1)\rangle \,\mathrm{d}\boldsymbol{\sigma} &= [\Phi_{1-\delta}, \tilde{\boldsymbol{\mu}}^{s,\tau}]_{r,0} - [\Psi_s, \tilde{\boldsymbol{\mu}}^{s,\tau}]_{l,1}, \\
\int \langle \boldsymbol{v}^{1,\tau}(x_0,x_1) - \boldsymbol{v}^{s,\tau}(x_0,x_1), \boldsymbol{b}^{1,\tau}(x_0,x_1) - \boldsymbol{b}^{s,\tau}(x_0,x_1)\rangle \,\mathrm{d}\boldsymbol{\sigma} &= [\Psi_1, \boldsymbol{\vartheta}^{s,\tau}]_{r,0} - [\Psi_s, \boldsymbol{\vartheta}^{s,\tau}]_{l,1},
\end{aligned} \tag{7.7}$$

where $\boldsymbol{\mu}^{s,\tau} = (\mathsf{x}^0, \boldsymbol{b}^{s,\tau})_\sharp \boldsymbol{\sigma}$, $\tilde{\boldsymbol{\mu}}^{s,\tau} = (\mathsf{x}^1, \boldsymbol{b}^{s,\tau})_\sharp \boldsymbol{\sigma}$, $\boldsymbol{\vartheta}^{s,\tau} = (\boldsymbol{b}^{1,\tau}, \boldsymbol{b}^{s,\tau})_\sharp \boldsymbol{\sigma}$ and the equalities with the pseudo scalar products come from the fact that all those plans are concentrated on a map w.r.t. their first marginal. Indeed, we can use Theorem 2.13(4) thanks to the $\boldsymbol{\mu}$-essential injectivity of $\mathsf{x}^\delta, \mathsf{x}^\tau$, $\mathsf{x}^{1-\delta}$, and use the fact that the cardinality of B^s is constant w.r.t. s. By construction, these plans satisfy the hypotheses of Lemma 7.1 so that all the expressions at the right-hand side of (7.7) are nonpositive. Combining this fact with (7.6), we end up with

$$\int \langle v_0 - v_1, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma} \le \frac{s}{a(1-a)} \int \langle (1-a)v_0 + av_1 - \boldsymbol{v}^{1,\tau}, \boldsymbol{b}^{1,\tau} - x_a\rangle \,\mathrm{d}\boldsymbol{\sigma}.$$

Passing to the limit as $s \downarrow 0$ we obtain

$$\int \langle v_0 - v_1, x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\sigma} \le 0.$$

Passing to the supremum w.r.t. $\Phi_\delta \in \mathbf{F}[\mu_\delta]$ and to the infimum w.r.t. $\Phi_{1-\delta} \in \mathbf{F}[\mu_{1-\delta}]$, we get

$$[\mathbf{F}, (\mathsf{x}^\delta, \mathsf{x}^{1-\delta})_\sharp \boldsymbol{\mu}]_{r,0} - [\mathbf{F}, (\mathsf{x}^\delta, \mathsf{x}^{1-\delta})_\sharp \boldsymbol{\mu}]_{l,1} \le 0,$$

which is (7.5) with $s = \delta$ and $t = 1-\delta$ thanks to Theorem 2.13(3). □

Remark 7.3. If $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ is a λ-dissipative MPVF with $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$, then $\mathbf{F}$ is λ-dissipative along discrete couplings thanks to Theorem 7.2 and Theorem 2.13, i.e.

$$[\Phi, \Psi]_r \le \lambda \int_{\mathsf{X}\times\mathsf{X}} |x-y|^2 \,\mathrm{d}\boldsymbol{\gamma}(x,y)$$

for every $\Phi, \Psi \in \mathbf{F}$ and any $\boldsymbol{\gamma} \in \Gamma(\mathsf{x}_\sharp\Phi, \mathsf{x}_\sharp\Psi)$ such that $\mathsf{x}_\sharp\Phi, \mathsf{x}_\sharp\Psi$ belong to $\mathcal{P}_f(\mathsf{X})$.

Let us introduce the notion of "collisionless couplings".

Definition 7.4 (Convexity along collisionless couplings)**.** *Let* $\mu_0, \mu_1 \in \mathcal{P}_f(\mathsf{X})$*. We say that* $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$ *is* collisionless *if* x^t *is* $\boldsymbol{\mu}$*- essentially injective for every* $t \in [0,1]$*.*
We say that a set $C \subset \mathcal{P}_f(\mathsf{X})$ *is* convex along collisionless couplings *if for every collisionless* $\boldsymbol{\nu} \in \mathcal{P}_f(\mathsf{X}^2)$*, with* $\mathsf{x}^0_\sharp\boldsymbol{\nu}, \mathsf{x}^1_\sharp\boldsymbol{\nu} \in C$*, and every* $t \in (0,1)$ *we have* $\mathsf{x}^t_\sharp\boldsymbol{\nu} \in C$*.*

Notice that if $\mu_0, \mu_1 \in \mathcal{P}_{\#N}(\mathsf{X})$ a coupling $\boldsymbol{\mu}$ in $\Gamma(\mu_0, \mu_1)$ is collisionless if and only if

$$\boldsymbol{\mu} \in \Gamma_{\#N}(\mathsf{X}^2), \quad \mathsf{x}^t_\sharp\boldsymbol{\mu} \in \mathcal{P}_{\#N}(\mathsf{X}) \quad \text{for every } t \in (0,1). \tag{7.8}$$

Theorem 7.5 (Selfimproving dissipativity along collisionless couplings)**.** *Assume that* $\dim \mathsf{X} \geq 2$*,* $N \in \mathbb{N}$*, let* $\mathbf{F}$ *be a* MPVF *satisfying* (2.17) *and such that* $\mathrm{D}_{\#N}(\mathbf{F})$ *is convex along collisionless couplings. If* μ_0, μ_1 *belong to the interior of* $\mathrm{D}_{\#N}(\mathbf{F})$ *in the metric space* $(\mathcal{P}_{\#N}(\mathsf{X}), W_2)$ *and* $\boldsymbol{\mu} \in \Gamma_{\#N}(\mu_0, \mu_1)$ *then*

$$[\mathbf{F}, \boldsymbol{\mu}]_{r,0} - [\mathbf{F}, \boldsymbol{\mu}]_{l,1} \leq \lambda W^2, \quad W^2 := \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}. \tag{7.9}$$

Proof. The proof is very similar to the one of Theorem 7.2, we keep the same notation.
Since $\boldsymbol{\mu} \in \mathcal{P}_{\#N}(\mathsf{X}^2)$, x^0, x^1 are $\boldsymbol{\mu}$-essentially injective, so that we can select $\delta = 0$. We can then choose $\tau \in (0,1)$ such that x^τ is $\boldsymbol{\mu}$-essentially injective and $\varepsilon > 0$ sufficiently small so that the ball of radious ε centered at μ_t is contained in $\mathrm{D}_{\#N}(\mathbf{F})$ for every $t \in [0,1]$. We can then proceed with the same perturbation argument of the previous proof. In the last part of the proof, we use the fact that $\mathrm{D}_{\#N}(\mathbf{F})$ is convex along collisionless couplings. □

The following result shows that in case of a deterministic demicontinuous PVF (recall Definition 3.21) λ-dissipativity yields total λ-dissipativity. Similarly, we can lift the Lipschitz continuity along optimal couplings to arbitrary couplings.

Theorem 7.6 (Deterministic demicontinuous dissipative PVFs are totally dissipative)**.** *Let* $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ *be a deterministic demicontinuous λ-dissipative* PVF *with* $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$*, of the form*

$$\mathbf{F}[\mu] := (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}(\cdot, \mu))_\sharp\mu, \quad \mu \in \mathcal{P}_2(\mathsf{X}), \tag{7.10}$$

for a map $\boldsymbol{f} : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$*, where* $\mathcal{S}(\mathsf{X})$ *is as in* (2.15)*. Then* $\mathbf{F}$ *is maximal totally λ-dissipative.*
If moreover there exists $L > 0$ *for which the following condition holds: for every* $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ *there exists* $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ *satisfying*

$$\int_{\mathsf{X}\times\mathsf{X}} |\boldsymbol{f}(x_1, \mu_1) - \boldsymbol{f}(x_0, \mu_0)|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \leq L^2 \int_{\mathsf{X}\times\mathsf{X}} |x_1 - x_0|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1), \tag{7.11}$$

then (7.11) *holds for every* $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$*.*

Proof. By Lemma 7.1(2) and the fact that $\mathbf{F}$ is single-valued and concentrated on a map $\boldsymbol{f} : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$, recalling Theorem 2.13(4) we know that $\mathbf{F}$ satisfies (3.9), or, equivalently, (1.7) for every $\mu_0, \mu_1 \in \mathcal{P}_f(\mathsf{X})$. We use an approximation procedure to get the general formulation for every $\mu_0, \mu_1 \in \mathcal{P}_2(\mathsf{X})$ and every $\boldsymbol{\mu} \in \Gamma(\mu_0, \mu_1)$: we take sequences $(\mu_0^n)_n, (\mu_1^n)_n \subset \mathcal{P}_f(\mathsf{X})$ such that $W_2(\mu_0^n, \mu_0) \to 0$ and $W_2(\mu_1^n, \mu_1) \to 0$ and optimal plans $\boldsymbol{\gamma}_0^n \in \Gamma_o(\mu_0^n, \mu_0)$ and $\boldsymbol{\gamma}_1^n \in \Gamma_o(\mu_1, \mu_1^n)$. Let $\boldsymbol{\sigma}_n \in \mathcal{P}(\mathsf{X}^4)$ be such that $\pi^{1,2}_\sharp\boldsymbol{\sigma}_n = \boldsymbol{\gamma}_0^n$, $\pi^{2,3}_\sharp\boldsymbol{\sigma}_n = \boldsymbol{\mu}$ and $\pi^{3,4}_\sharp\boldsymbol{\sigma}_n = \boldsymbol{\gamma}_1^n$. Notice that we also have that $\boldsymbol{\mu}_n := \pi^{1,4}_\sharp\boldsymbol{\sigma}_n$ belongs to $\Gamma(\mu_0^n, \mu_1^n)$ and converges to $\boldsymbol{\mu}$ in $\mathcal{P}_2(\mathsf{X}^2)$ as $n \to \infty$. Thanks to the demicontinuity of $\mathbf{F}$ and the fact that $\mathbf{F}$ is concentrated on $\boldsymbol{f}$, we obtain that $\boldsymbol{\vartheta}_n := (\boldsymbol{i}_{\mathsf{X}\times\mathsf{X}}, \boldsymbol{f}(x_0, \mu_0) \times \boldsymbol{f}(x_1, \mu_1))_\sharp\boldsymbol{\mu}_n$ converges to $\boldsymbol{\vartheta}_n := (\boldsymbol{i}_{\mathsf{X}\times\mathsf{X}}, \boldsymbol{f}(x_0, \mu_0) \times \boldsymbol{f}(x_1, \mu_1))_\sharp\boldsymbol{\mu}$ in $\mathcal{P}_2^{sw}(\mathsf{X}^2 \times \mathsf{X}^2)$. We can then pass to the limit in the inequality

$$\int \langle \boldsymbol{f}(x_1, \mu_1) - \boldsymbol{f}(x_0, \mu_0), x_1 - x_0 \rangle \,\mathrm{d}\boldsymbol{\mu}_n(x_0, x_1) = \int \langle v_1 - v_0, x_1 - x_0 \rangle \,\mathrm{d}\boldsymbol{\vartheta}_n(x_0, x_1, v_0, v_1) \leq 0$$

obtaining

$$\int \langle \boldsymbol{f}(x_1,\mu_1) - \boldsymbol{f}(x_0,\mu_0), x_1 - x_0\rangle \,\mathrm{d}\boldsymbol{\mu}(x_0,x_1) = \int \langle v_1 - v_0, x_1 - x_0\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x_0,x_1,v_0,v_1) \le 0.$$

We can eventually apply Theorem 3.22 to get the maximality of $\mathbf{F}$.
Concerning the second part of the Theorem, let us first show that the condition (7.11) holds for every $\mu_0,\mu_1 \in \mathcal{P}_f(\mathsf{X})$ and every $\boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)$: by Theorems 6.2 and 2.9 there exists some $K \in \mathbb{N}$ and points $0 = t_0 < t_1 < \cdots < t_{K-1} < t_K = 1$ such that $(\mathsf{x}^{t_{i-1}}, \mathsf{x}^{t_i})_\sharp\boldsymbol{\mu}$ is the unique element of $\Gamma_o(\mathsf{x}^{t_{i-1}}_\sharp\boldsymbol{\mu}, \mathsf{x}^{t_i}_\sharp\boldsymbol{\mu})$ for every $i = 1,\dots,K$. We thus have for every $i = 1,\dots,K$ that

$$\left(\int_{\mathsf{X}} \left|\boldsymbol{f}(\mathsf{x}^{t_i}, \mathsf{x}^{t_i}_\sharp\boldsymbol{\mu}) - \boldsymbol{f}(\mathsf{x}^{t_{i-1}}, \mathsf{x}^{t_{i-1}}_\sharp\boldsymbol{\mu})\right|^2 \mathrm{d}\boldsymbol{\mu}\right)^{1/2} \le L(t_i - t_{i-1}) \left(\int_{\mathsf{X}\times\mathsf{X}} |x_1 - x_0|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0,x_1)\right)^{1/2}.$$

Summing up these inequalities for $i = 1,\dots,K$ and using the triangular inequality in $L^2(\mathsf{X}\times\mathsf{X},\boldsymbol{\mu};\mathsf{X})$, we get that (7.11) holds for every $\mu_0,\mu_1 \in \mathcal{P}_f(\mathsf{X})$ and every $\boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)$.
By using the same approximation procedure (and the same notation) of the first part of this proof, we show that (7.11) holds for every $\mu_0,\mu_1 \in \mathcal{P}_2(\mathsf{X})$ and every $\boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)$: in fact we have the estimate

$$\begin{aligned}
\left(\int_{\mathsf{X}\times\mathsf{X}} |\boldsymbol{f}(x_1,\mu_1) - \boldsymbol{f}(x_0,\mu_0)|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0,x_1)\right)^{1/2} &= \|\boldsymbol{f}(\pi^3,\mu_1) - \boldsymbol{f}(\pi^2,\mu_0)\|_{L^2(\mathsf{X}^2,\boldsymbol{\sigma}_n;\mathsf{X})} \\
\le \|\boldsymbol{f}(\pi^3,\mu_1) - \boldsymbol{f}(\pi^4,\mu_1^n)\|_{L^2(\mathsf{X}^2,\boldsymbol{\sigma}_n;\mathsf{X})} &+ \|\boldsymbol{f}(\pi^4,\mu_1^n) - \boldsymbol{f}(\pi^1,\mu_0^n)\|_{L^2(\mathsf{X}^2,\boldsymbol{\sigma}_n;\mathsf{X})} \\
+ \|\boldsymbol{f}(\pi^1,\mu_0^n) - \boldsymbol{f}(\pi^2,\mu_0)\|_{L^2(\mathsf{X}^2,\boldsymbol{\sigma}_n;\mathsf{X})}& \\
\le L\Big(W_2(\mu_1^n,\mu_1) + W_2(\mu_0,\mu_0^n)\Big) + L &\left(\int_{\mathsf{X}^2} |x-y|^2 \,\mathrm{d}\boldsymbol{\mu}_n(x,y)\right)^{1/2}.
\end{aligned}$$

Passing to the limit as $n \to +\infty$, we get that (7.11) holds for every $\mu_0,\mu_1 \in \mathcal{P}_2(\mathsf{X})$ and every $\boldsymbol{\mu} \in \Gamma(\mu_0,\mu_1)$. □

8. Construction of a totally λ-dissipative MPVF from a discrete core

We have seen at the end of Section 3.2 (Corollary 3.20) that a maximal totally λ-dissipative MPVF is determined by its restriction to the set of uniform discrete measures.
In this section, we want to investigate a closely related question, which plays a crucial role in the construction of a maximal totally λ-dissipative MPVF: if we assign a MPVF $\mathbf{F}$ on a sufficiently rich subset of discrete measures, is it possible to uniquely construct a maximal extension of $\mathbf{F}$?
In the Hilbert setting, such kind of problems are well understood if the domain of the initial operator is open and convex (see in particular [Qi83], Proposition A.12 and Theorem A.13). However, dealing with open sets at the level of $\mathcal{P}_2(\mathsf{X})$ will prevent the use of discrete measures. We will circumvent this difficulty by a suitable localization of the open condition in each subset $\mathcal{P}_{\#N}(\mathsf{X})$, which relies on the notion of *discrete core.*
In order to allow for the greatest flexibility, we consider collections of discrete measures indexed by an unbounded directed subset $\mathfrak{N} \subset \mathbb{N}$ with respect to the partial order given by

$$m \prec n \quad \Leftrightarrow \quad m \mid n, \tag{8.1}$$

where $m \mid n$ means that $n/m \in \mathbb{N}$. We write $m \precnsim n$ if $m \prec n$ and $m \neq n$. Typical examples are the set of all natural integers $\mathfrak{N} := \mathbb{N}$ or the dyadic one $\mathfrak{N} := \{2^n : n \in \mathbb{N}\}$. We set

$$\mathcal{P}_{f,\mathfrak{N}}(\mathsf{X}) := \bigcup_{N\in\mathfrak{N}} \mathcal{P}_{f,N}(\mathsf{X}), \quad \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X}) := \bigcup_{N\in\mathfrak{N}} \mathcal{P}_{\#N}(\mathsf{X}), \tag{8.2}$$

observing that, for every $N \in \mathfrak{N}$, $\mathcal{P}_{f,N}(\mathsf{X})$ is closed in $\mathcal{P}_2(\mathsf{X})$ and $\mathcal{P}_{\#N}(\mathsf{X})$ is a relatively open and dense subset of $\mathcal{P}_{f,N}(\mathsf{X})$.

Definition 8.1 ($\mathfrak{N}$-core). *Let $\mathfrak{N}$ be an unbounded directed subset of $\mathbb{N}$ w.r.t. the order relation $\prec$ as in (8.1). A discrete $\mathfrak{N}$-core is a set $\mathrm{C} \subset \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$ such that $\overline{\mathrm{C}} \subset \mathcal{P}_2(\mathsf{X})$ is totally convex and the family $\mathrm{C}_N := \mathrm{C} \cap \mathcal{P}_{\#N}(\mathsf{X})$, $N \in \mathfrak{N}$, satisfies the following properties:*

(1) C_N is nonempty and relatively open in $\mathcal{P}_{\#N}(\mathsf{X})$ (or, equivalently, in $\mathcal{P}_{f,N}(\mathsf{X})$);
(2) C_N coincides with the relative interior in $\mathcal{P}_{f,N}(\mathsf{X})$ of $\overline{\mathrm{C}} \cap \mathcal{P}_{\#N}(\mathsf{X})$.

In the next result we will present several equivalent characterizations of $\mathfrak{N}$-cores, which we fully justify in the next section, adopting a Lagrangian viewpoint (see Lemma 8.12).

Lemma 8.2 (Equivalent characterizations of $\mathfrak{N}$-cores). *Let $\mathrm{C} \subset \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$; then the following properties are equivalent:*

(a) *C is a $\mathfrak{N}$-core;*
(b) *there exists a subset D of $\mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})$ such that, for every $N \in \mathfrak{N}$, the set $\mathrm{D}_N := \mathrm{D} \cap \mathcal{P}_{f,N}(\mathsf{X})$ satisfies the following two conditions*
   (1') *D_N is relatively open in $\mathcal{P}_{f,N}(\mathsf{X})$,*
   (2') *D_N is convex along couplings in $\mathcal{P}_{f,N}(\mathsf{X} \times \mathsf{X})$,*
   and $\mathrm{C} = \mathrm{D} \cap \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$;
(c) *there exists a totally convex and closed subset E of $\mathcal{P}_2(\mathsf{X})$ such that*
   (1") *for every $N \in \mathfrak{N}$ the sets*

$$\mathring{\mathrm{E}}_N := \textit{relative interior of } \big(\mathrm{E} \cap \mathcal{P}_{f,N}(\mathsf{X})\big) \textit{ in } \mathcal{P}_{f,N}(\mathsf{X})$$

   are not empty,
   (2") *$\mathrm{E} \cap \mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})$ is dense in E,*
   and $\mathrm{C} = \bigcup_{N\in\mathfrak{N}} \mathring{\mathrm{E}}_N \cap \mathcal{P}_{\#N}(\mathsf{X})$;
(d) *the family of sets $\mathrm{C}_N = \mathrm{C} \cap \mathcal{P}_{\#N}(\mathsf{X})$ satisfies*
   (1*) *C_N is relatively open in $\mathcal{P}_{\#N}(\mathsf{X})$ (or, equivalently, in $\mathcal{P}_{f,N}(\mathsf{X})$),*
   (2*) *C_N is convex along collisionless couplings (cf. Definition 7.4),*
   (3*) *if $M, N \in \mathfrak{N}$, $M \mid N$ then $\overline{\mathrm{C}_M} = \overline{\mathrm{C}_N} \cap \mathcal{P}_{f,M}(\mathsf{X})$,*
   (4*) *$\overline{\mathrm{C}_N}$ is convex along couplings in $\mathcal{P}_{f,N}(\mathsf{X} \times \mathsf{X})$.*

In the above cases the sets C_N, D_N, $\mathring{\mathrm{E}}_N$, C, D and E are linked by the following relations

$$\mathrm{C}_N = \mathrm{D}_N \cap \mathcal{P}_{\#N}(\mathsf{X}) = \mathring{\mathrm{E}}_N \cap \mathcal{P}_{\#N}(\mathsf{X}), \quad \mathrm{C} = \bigcup_{N\in\mathfrak{N}} \mathrm{C}_N,, \tag{8.3}$$

$$\mathrm{D}_N = \mathring{\mathrm{E}}_N = \textit{ relative interior of } \overline{\mathrm{C}_N} \textit{ in } \mathcal{P}_{f,N}(\mathsf{X}), \quad \mathrm{D} = \bigcup_{N\in\mathfrak{N}} \mathrm{D}_N = \bigcup_{N\in\mathfrak{N}} \mathring{\mathrm{E}}_N, \tag{8.4}$$

$$\overline{\mathrm{C}_N} = \overline{\mathrm{D}_N} = \mathrm{E} \cap \mathcal{P}_{f,N}, \tag{8.5}$$

$$\overline{\mathrm{C}} = \overline{\mathrm{D}} = \mathrm{E}. \tag{8.6}$$

Lemma 8.3. *Let $\mathrm{C} \subset \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$; if $\dim(\mathsf{X}) \geq 2$, then condition* (4*) in Lemma 8.2 follows by (1*)-(3*).

Our first result shows how to recover a totally λ-dissipative MPVF starting from a (metrically) λ-dissipative MPVF $\mathbf{F}$ whose domain is a $\mathfrak{N}$-core C.

Theorem 8.4 (From dissipativity to total dissipativity). *Let X be a separable Hilbert space, let $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ be a MPVF and let $\mathrm{C} \subset \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$ be a $\mathfrak{N}$-core. Let us assume either one of the the following hypotheses:*

(i) $\mathbf{F}$ is λ-dissipative, $\mathrm{D}(\mathbf{F}) = \mathrm{C}$ and $\dim(\mathsf{X}) \geq 2$;
(ii) $\mathbf{F}$ is totally λ-dissipative and $\mathrm{C} \subset \mathrm{D}(\mathbf{F}) \subset \overline{\mathrm{C}}$.

For every $N \in \mathfrak{N}$ consider the MPVF $\hat{\mathbf{F}}_N$ *defined by the following formula:* $\Phi \in \hat{\mathbf{F}}_N[\mu]$ *if and only if* $\Phi \in \mathcal{P}_{f,N}(\mathsf{TX})$, $\mu \in \overline{\mathrm{C}_N}$ *and for every* $\nu \in \mathrm{C}_N$, $\Psi \in \mathbf{F}[\nu]$, $\boldsymbol{\vartheta} \in \Gamma_{f,N}(\Phi, \nu)$ *we have*

$$\int \langle v_0 - \boldsymbol{b}_\Psi(x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x_0, v_0, x_1) \le \lambda \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\vartheta}(x_0, v_0, x_1). \tag{8.7}$$

We have the following properties:

(1) *For every $N \in \mathfrak{N}$, the elements of $\hat{\mathbf{F}}_N$ satisfy* (3.9) *for couplings $\boldsymbol{\vartheta} \in \mathcal{P}_{f,N}(\mathsf{TX} \times \mathsf{TX})$ and* $\mathrm{D}(\hat{\mathbf{F}}_N)$ *contains* C_N.

(2) *For every $\mu \in \overline{\mathrm{C}_N}$, $\boldsymbol{f} \in \operatorname{map}\left(\hat{\mathbf{F}}_N\right)[\mu]$ (cf.* (2.14)*) if and only if for every $\nu \in \mathrm{C}_N$, $\Psi \in \mathbf{F}[\nu]$, $\boldsymbol{\mu} \in \Gamma_{f,N}(\mu, \nu)$ we have*

$$\int \langle \boldsymbol{f}(x_0) - \boldsymbol{b}_\Psi(x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le \lambda \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1). \tag{8.8}$$

Moreover, in order for $\boldsymbol{f}$ to belong to $\operatorname{map}\left(\hat{\mathbf{F}}_N\right)[\mu]$, it is sufficient to check (8.8) *only for all the measures $\nu \in \mathrm{C}_N$ and all the couplings $\boldsymbol{\mu} \in \Gamma(\mu, \nu)$ such that $\boldsymbol{\mu}$ is the unique element of $\Gamma_o(\mu, \nu)$.*

(3) *$M \mid N$ implies $\mathrm{D}(\hat{\mathbf{F}}_M) \subset \mathrm{D}(\hat{\mathbf{F}}_N)$.*

(4) *The* MPVF

$$\hat{\mathbf{F}}_\infty[\mu] := \bigcup_{M \in \mathfrak{N}} \bigcap_{M \mid N} \hat{\mathbf{F}}_N[\mu] \quad \textit{with domain} \quad \mathrm{D}(\hat{\mathbf{F}}_\infty) = \bigcup_{M \in \mathbb{N}} \mathrm{D}(\hat{\mathbf{F}}_M) \supset \mathrm{C} \tag{8.9}$$

is totally λ-dissipative.

(5) *There exists a unique maximal totally λ-dissipative* MPVF $\hat{\mathbf{F}}$ *extending $\hat{\mathbf{F}}_\infty$ whose domain is contained in $\overline{\mathrm{C}}$. For every $\mu \in \overline{\mathrm{C}}$, $\hat{\mathbf{F}}[\mu]$ is characterized by all the measures $\Phi \in \mathcal{P}_2(\mathsf{TX}|\mu)$ satisfying*

$$\int \langle v - \boldsymbol{f}(y), x - y\rangle \,\mathrm{d}\boldsymbol{\vartheta}(x, v, y) \le \lambda \int |x - y|^2 \,\mathrm{d}\boldsymbol{\vartheta} \tag{8.10}$$

for every $\boldsymbol{\vartheta} \in \Gamma(\Phi, \nu)$ with $\nu \in \mathrm{D}(\hat{\mathbf{F}}_\infty)$ and $(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp \nu \in \hat{\mathbf{F}}_\infty$. The MPVF $\hat{\mathbf{F}}$ *also coincides with the strong closure of $\hat{\mathbf{F}}_\infty$ in $\mathcal{P}_2(\mathsf{TX})$. Finally, if $\mu \in \mathrm{C}$ then the minimal selection $\hat{\mathbf{F}}^\circ$ of $\hat{\mathbf{F}}$ satisfies*

$$\hat{\mathbf{F}}^\circ[\mu] \in \hat{\mathbf{F}}_\infty[\mu].$$

We discuss two particular cases in more detail: the first one occurs when $\mathbf{F}$ is a deterministic λ-dissipative MPVF: as in Theorem 7.6 we obtain that λ-dissipativity implies total λ-dissipativity; here however, we deal with a MPVF (not necessarily single-valued) defined in a much smaller domain.

Theorem 8.5 (Deterministic dissipative MPVFs on a core are totally dissipative)**.** *Let us suppose that $\dim \mathsf{X} \ge 2$ and $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$ is a deterministic λ-dissipative* MPVF *whose domain is a $\mathfrak{N}$-core* C. *Then $\mathbf{F}$ is totally λ-dissipative, $\hat{\mathbf{F}}_\infty$ (cf.* (8.9)*) is a totally λ-dissipative extension of $\mathbf{F}$ and, for every $\mu \in \bigcup_{N \in \mathfrak{N}} \overline{\mathrm{C}_N}$, $\boldsymbol{f} \in \operatorname{map}\left(\hat{\mathbf{F}}_\infty\right)[\mu]$ if and only if*

$$\int \langle \boldsymbol{f}(x_0) - \boldsymbol{g}(x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \le \lambda \int |x_0 - x_1|^2 \,\mathrm{d}\boldsymbol{\mu}(x_0, x_1) \tag{8.11}$$

for all the measures $\nu \in \mathrm{C}$, $\boldsymbol{g} \in \operatorname{map}(\mathbf{F})[\nu]$, and all the couplings $\boldsymbol{\mu} \in \Gamma(\mu, \nu)$ such that $\boldsymbol{\mu}$ is the unique element of $\Gamma_o(\mu, \nu)$. The MPVF $\hat{\mathbf{F}}$ *of Theorem 8.4(5) provides the unique maximal totally λ-dissipative extension of $\mathbf{F}$ with domain included in $\overline{\mathrm{C}}$. If moreover $\mathbf{F}$ is single-valued and the restriction of $\mathbf{F}$ to each set C_N, $N \in \mathfrak{N}$, is demicontinous, then the restrictions of $\hat{\mathbf{F}}_\infty$ and $\hat{\mathbf{F}}^\circ$ to* C *coincide with* $\mathbf{F}$.

A second case occurs when we know that $\mathbf{F}$ is totally λ-dissipative.

Theorem 8.6 (Unique maximal extension of totally dissipative MPVF). *If* $\mathbf{F}$ *is a totally* λ-*dissipative* MPVF *whose domain contains a dense* $\mathfrak{N}$*-core* C. *Then the* MPVF $\hat{\mathbf{F}}$ *constructed as in Theorem 8.4 provides the unique maximal totally* λ*-dissipative extension of* $\mathbf{F}$ *with domain included in* $\overline{\mathrm{C}}$.

We devote the remaining part of this section to the proof of the above main theorems. We adopt a Lagrangian viewpoint, lifting the MPVF $\mathbf{F}$ to the Hilbert space $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ and parametrizing probability measures by random variables in $\mathcal{X}$ as we did in Section 3.2.

8.1. **Lagrangian representations of $\mathfrak{N}$-cores**

For the whole section, we fix a standard Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$ (see Definition B.1).
Given $\mathfrak{N}$ an unbounded directed subset of $\mathbb{N}$ w.r.t. the order relation $\prec$ as in (8.1), we consider a $\mathfrak{N}$-segmentation of $(\Omega, \mathcal{B}, \mathbb{P})$ (see Definition B.3) that we denote by $(\mathfrak{P}_N)_{N\in\mathfrak{N}}$. We define $\mathcal{B}_N := \sigma(\mathfrak{P}_N)$, $N \in \mathfrak{N}$, and we denote by $(\Omega, \mathcal{B}, \mathbb{P}, (\mathfrak{P}_N)_{N\in\mathfrak{N}})$, with $\mathfrak{P}_N = \{\Omega_{N,n}\}_{n\in I_N}$ and $I_N := \{0, \dots, N-1\}$, the $\mathfrak{N}$-refined probability space as in Definition B.3 induced by $(\mathfrak{P}_N)_{N\in\mathfrak{N}}$ on $(\Omega, \mathcal{B}, \mathbb{P})$. We set

$$\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X}), \quad \mathcal{X}_N := L^2(\Omega, \mathcal{B}_N, \mathbb{P}; \mathsf{X}), \quad N \in \mathfrak{N}, \quad \mathcal{X}_\infty := \bigcup_{N\in\mathfrak{N}} \mathcal{X}_N,$$

and we recall that $\mathcal{X}_\infty$ is dense in $\mathcal{X}$ by Proposition B.4.
Even if the choice of a general standard Borel space allows for a great generality, it would not be restrictive to focus on the canonical example below, at least at a first reading.

Example 8.7. The canonical example of $\mathfrak{N}$-refined standard Borel probability space is

$$([0, 1), \mathcal{B}([0, 1)), \lambda, (\mathfrak{I}_N)_{N\in\mathfrak{N}}),$$

where λ is the one dimensional Lebesgue measure restricted to $[0, 1)$ and $\mathfrak{I}_N = (I_{N,k})_{k\in I_N}$ with $I_{N,k} := [k/N, (k+1)/N)$, $k \in I_N$ and $N \in \mathfrak{N}$. The space $\mathcal{X}_N$ can then be identified with the class of functions which are (essentially) constant in each subintervals $I_{N,k}$, $k \in I_N$, of the partition $\mathfrak{I}_{N,k}$.

As in Section 3, we parametrize measures in $\mathcal{P}(\mathsf{X})$ by random variables in $(\Omega, \mathcal{B}, \mathbb{P})$ and we use the notation $\iota : \mathcal{X} \to \mathcal{P}_2(\mathsf{X})$ for the map sending $X \in \mathcal{X}$ to $\iota(X) = X_\sharp\mathbb{P} = \iota_X \in \mathcal{P}_2(\mathsf{X})$. Recall that

$$W_2(\iota_X, \iota_Y) \le |X - Y|_{\mathcal{X}} \quad \text{for every } X, Y \in \mathcal{X}. \tag{8.12}$$

If $(X, V) \in \mathcal{X} \times \mathcal{X}$ recall the notation $\iota^2_{X,V} = (X, V)_\sharp\mathbb{P} \in \mathcal{P}_2(\mathsf{TX})$.
We can identify $\mathcal{X}_N$ with the space X^N of vectors $\boldsymbol{x} : I_N \to \mathsf{X}$ such that $X(\omega) = \boldsymbol{x}(n)$ whenever $\omega \in \Omega_{N,n}$. In this case we set $X = \mathscr{I}_N(\boldsymbol{x})$. Clearly $\iota(\mathcal{X}_N) = \mathcal{P}_{f,N}(\mathsf{X})$ and $\iota(\mathcal{X}_\infty) = \mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})$.
The isomorphism $\mathscr{I}_N$ preserves the scalar product on X^N

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathsf{X}^N} := N^{-1} \sum_{n=0}^{N-1} \langle \boldsymbol{x}(n), \boldsymbol{y}(n)\rangle = \mathbb{E}\big[\langle \mathscr{I}_N(\boldsymbol{x}), \mathscr{I}_N(\boldsymbol{y})\rangle\big] = \langle \mathscr{I}_N(\boldsymbol{x}), \mathscr{I}_N(\boldsymbol{y})\rangle_{\mathcal{X}} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathsf{X}^N.$$

The conditional expectation $\Pi_N = \mathbb{E}[\cdot|\mathcal{B}_N]$ provides the orthogonal projection of an arbitrary map $X \in \mathcal{X}$ onto $\mathcal{X}_N$:

$$\Pi_N[X](\omega) = N \int_{\Omega_{N,n}} X \,\mathrm{d}\mathbb{P} \quad \text{if } \omega \in \Omega_{N,n}.$$

Notice that

$$\text{if } M \mid N \text{ then } \mathcal{B}_M \subset \mathcal{B}_N \text{ and } \Pi_M = \Pi_M \circ \Pi_N.$$

For every $X = \mathscr{I}_N(\boldsymbol{x}) \in \mathcal{X}_N$ the probability measure $\iota_X = X_\sharp\mathbb{P}$ takes the form $\iota_X = N^{-1}\sum_{n=0}^{N-1}\delta_{\boldsymbol{x}(n)} \in \mathcal{P}_{f,N}(\mathsf{X})$.

We denote by $\mathsf{O}_N \subset \mathsf{X}^N$ the subset of the injective maps and by $\mathcal{O}_N := \mathscr{I}_N(\mathsf{O}_N) \subset \mathcal{X}_N$. Clearly, $\iota(\mathcal{O}_N) = \mathcal{P}_{\#N}(\mathsf{X})$. Since the complement of O_N is the union of a finite number of proper closed subspaces with empty interior $S_{ij} := \{\boldsymbol{x} \in \mathsf{X}^N : \boldsymbol{x}(i) = \boldsymbol{x}(j)\}$, $i \neq j$, of X^N, then O_N is open and dense in X^N.

Every permutation $\sigma \in \mathrm{Sym}(I_N)$ acts on X^N via $\sigma\boldsymbol{x}(n) := \boldsymbol{x}(\sigma(n))$ and can be thus extended to $\mathcal{X}_N$ via $\sigma(\mathscr{I}_N(\boldsymbol{x})) := \mathscr{I}_N(\sigma(\boldsymbol{x}))$. It is not difficult to see that, for every $X, Y \in \mathcal{X}_N$, $\iota_X = \iota_Y$ is equivalent to $Y = \sigma X$ for some $\sigma \in \mathrm{Sym}(I_N)$.
As in Section 3, we denote by $\mathrm{S}(\Omega)$ the class of $\mathcal{B}$-$\mathcal{B}$-measurable maps $g : \Omega \to \Omega$ which are essentially injective and measure-preserving, meaning that there exists a full $\mathbb{P}$-measure set $\Omega_0 \in \mathcal{B}$ such that g is injective on Ω_0 and $g_\sharp\mathbb{P} = \mathbb{P}$. Moreover, for every $N \in \mathfrak{N}$, we denote by $\mathrm{S}_N(\Omega) := \mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{B}_N)$ the subset of $\mathrm{S}(\Omega)$ of $\mathcal{B}_N$-$\mathcal{B}_N$ measurable maps.

Remark 8.8. Clearly, if $X = \mathscr{I}_N(\boldsymbol{x}) \in \mathcal{X}_N$ and $g \in \mathrm{S}_N(\Omega)$ then $X \circ g \in \mathcal{X}_N$ and there exists a unique permutation $\sigma = \sigma_g \in \mathrm{Sym}(I_N)$ such that $X \circ g = \sigma_g X = \mathscr{I}_N(\boldsymbol{x} \circ \sigma_g)$. Conversely, if $\sigma \in \mathrm{Sym}(I_N)$ there exists $g \in \mathrm{S}_N(\Omega)$ such that $\sigma = \sigma_g$, as shown in Lemma B.2. We set $G[\sigma] := \big\{g \in \mathrm{S}_N(\Omega) : \sigma_g = \sigma\big\}$.

There is an interesting relation between projections and permutations.

Lemma 8.9. *Let $N, M \in \mathfrak{N}$ be such that $M \mid N$. If $\mathcal{K}$ is a convex subset of $\mathcal{X}_N$ invariant by the action of* $\mathrm{Sym}(I_N)$*, then*

$$\mathcal{K} \cap \mathcal{X}_M = \Pi_M(\mathcal{K}). \tag{8.13}$$

Moreover,

$$\overline{\mathcal{K}} \cap \mathcal{X}_M = \overline{\mathcal{K} \cap \mathcal{X}_M} \tag{8.14}$$

and, denoting by $\mathring{\mathcal{K}}$ the relative interior of $\mathcal{K}$ in $\mathcal{X}_N$, if $\mathring{\mathcal{K}}$ is not empty then we have

$$\mathring{\mathcal{K}} \cap \mathcal{X}_M \text{ coincides with the relative interior of } \mathcal{K} \cap \mathcal{X}_M \text{ in } \mathcal{X}_M. \tag{8.15}$$

Proof. Let us first compute the explicit representation of the orthogonal projection $\Pi_M(X)$ for every $X \in \mathcal{X}_N$. If $K := N/M$ we consider the cyclic permutation $\sigma : I_N \to I_N$ defined by

$$\sigma(n) := \begin{cases} mK + k + 1 & \text{if } n = mK + k,\ m \in I_M,\ 0 \le k < K - 1, \\ mK & \text{if } n = mK + K - 1,\ m \in I_M, \end{cases}$$

and its powers σ^p, $p \in I_K$. It is not difficult to check that $\sigma^K = \sigma^0 = \boldsymbol{i}_{I_N}$ and for every $Y \in \mathcal{X}_M$ we have $\sigma^p Y = Y$ for every $p \in I_K$. Therefore for every $X \in \mathcal{X}_N$ we obtain the representation

$$\Pi_M(X) = \frac{1}{K}\sum_{p=0}^{K-1} \sigma^p X.$$

If $\mathcal{K}$ is a convex subset of $\mathcal{X}_N$ invariant by the action of $\mathrm{Sym}(I_N)$, we get $\Pi_M(X) \in \mathcal{K}$ for every $X \in \mathcal{K}$, so that $\Pi_M(\mathcal{K}) = \mathcal{K} \cap \mathcal{X}_M$, hence we proved (8.13).
In order to check (8.14), we observe that in general $\overline{\mathcal{K} \cap \mathcal{X}_M} \subset \overline{\mathcal{K}} \cap \mathcal{X}_M$; on the other hand $\overline{\mathcal{K}} \cap \mathcal{X}_M = \Pi_M(\overline{\mathcal{K}}) \subset \overline{\Pi_M(\mathcal{K})} = \overline{\mathcal{K} \cap \mathcal{X}_M}$ by (8.13).
Similarly, if we denote by $\mathring{\mathcal{K}}_M$ the relative interior of $\mathcal{K} \cap \mathcal{X}_M$ in $\mathcal{X}_M$, as a general fact $\mathring{\mathcal{K}} \cap \mathcal{X}_M \subset \mathring{\mathcal{K}}_M$ so that $\mathring{\mathcal{K}}_M$ is not empty, since by (8.13) $\mathring{\mathcal{K}} \cap \mathcal{X}_M = \Pi_M(\mathring{\mathcal{K}})$ is not empty. On the other hand, by (8.14), $\overline{\mathring{\mathcal{K}} \cap \mathcal{X}_M} = \overline{\mathring{\mathcal{K}}} \cap \mathcal{X}_M = \overline{\mathcal{K}} \cap \mathcal{X}_M = \overline{\mathcal{K} \cap \mathcal{X}_M} = \overline{\mathring{\mathcal{K}}_M}$ so that the open convex sets $\mathring{\mathcal{K}} \cap \mathcal{X}_M$ and $\mathring{\mathcal{K}}_M$ have the same closure and therefore coincide. □

We will now show that if the sections $\mathcal{A} \cap \mathcal{X}_N$ of a set $\mathcal{A} \subset \mathcal{X}_\infty$ are invariant by the action of $\mathrm{Sym}(I_N)$, then $\overline{A}$ is law invariant.

Lemma 8.10. *Let $\mathcal{A} \subset \mathcal{X}_\infty$ be a set such that $\mathcal{A}_N := \mathcal{A} \cap \mathcal{X}_N$ are invariant w.r.t.* $\mathrm{Sym}(I_N)$ *for every $N \in \mathfrak{N}$. Then $\overline{\mathcal{A}}$ is law invariant.*

Remark 8.11. The same statement applies to subsets of $\mathcal{X}_\infty \times \mathcal{X}_\infty$.

Proof. Since $\overline{\mathcal{A}}$ is a closed set, by Lemma 3.3, it is sufficient to prove that it is invariant by measure-preserving isomorphisms: for every $X \in \overline{\mathcal{A}}$ and $g \in \mathrm{S}(\Omega)$ we want to show that $X \circ g \in \overline{\mathcal{A}}$. It is enough to prove that there exist $Z_n \in \mathcal{A}$ s.t. $Z_n \to X \circ g$. Let X_n be a sequence in $\mathcal{A}$ such that $X_n \to X$; since $\mathcal{A} \subset \mathcal{X}_\infty$, for every $n \in \mathbb{N}$, there exists some $N_n \in \mathfrak{N}$ such that $X_n \in \mathcal{A}_{N_n}$. Let $(b_k)_k \subset \mathfrak{N}$ be the sequence given by Proposition B.4; by Theorem B.5(1) applied to $(\Omega, \mathcal{B}, \mathbb{P}, (\mathfrak{P}_{b_k})_{k\in\mathbb{N}})$ and $\boldsymbol{\gamma} := (\boldsymbol{i}_\Omega, g)_\sharp \mathbb{P}$, we can find a strictly increasing sequence $(M_j)_j \subset \mathbb{N}$ and maps $g_j \in \mathrm{S}_{b_{M_j}}(\Omega)$ such that

$$(U,W)_\sharp(\boldsymbol{i}_\Omega, g_j)_\sharp \mathbb{P} \to (U,W)_\sharp(\boldsymbol{i}_\Omega, g)_\sharp \mathbb{P} \ \text{ in } \mathcal{P}_2(\mathsf{X}^2)$$

for every $U, V \in \mathcal{X}$. Since M_j is strictly increasing and (B.1) holds, then we can find a strictly increasing sequence $n \mapsto j(n)$ such that $g_{j(n)} \in \mathrm{S}_{N_n}(\Omega)$. Thus setting $g'_n := g_{j(n)}$, $n \in \mathbb{N}$, by the invariance of $\mathcal{A}_{N_n}$, we get that $Z_n := X_n \circ g'_n \in \mathcal{A}_{N_n} \subset \mathcal{A}$ and of course we have

$$(U,W)_\sharp(\boldsymbol{i}_\Omega, g'_n)_\sharp \mathbb{P} \to (U,W)_\sharp(\boldsymbol{i}_\Omega, g)_\sharp \mathbb{P} \ \text{ in } \mathcal{P}_2(\mathsf{X}^2) \tag{8.16}$$

for every $U, V \in \mathcal{X}$. We are left to show that

$$Z_n \to X \circ g \text{ in } \mathcal{X}. \tag{8.17}$$

Since $|Z_n - X \circ g'_n|_{\mathcal{X}} = |X_n - X|_{\mathcal{X}}$, in order to get (8.17) it is enough to show that $X \circ g'_n \to X \circ g$ which, on the other hand, is implied by $X \circ g'_n \rightharpoonup X \circ g$, since $|X \circ g'_n|_{\mathcal{X}} = |X|_{\mathcal{X}} = |X \circ g|_{\mathcal{X}}$. Let $Y \in \mathcal{X}$ and let us take $U = Y, W = X$ in (8.16) so that

$$\langle X \circ g'_n, Y \rangle_{\mathcal{X}} = \int_{\mathsf{X}^2} \langle x, y \rangle \,\mathrm{d}((Y,X) \circ (\boldsymbol{i}_\Omega, g'_n))_\sharp \mathbb{P} \to \int_{\mathsf{X}^2} \langle x, y \rangle \,\mathrm{d}((Y,X) \circ (\boldsymbol{i}_\Omega, g))_\sharp \mathbb{P} = \langle X \circ g, Y \rangle_{\mathcal{X}},$$

since $\varphi(x,y) := \langle x, y \rangle$ is a real valued function on X^2 with less than quadratic growth (see e.g. [AGS08, Proposition 7.1.5, Lemma 5.1.7]). This shows that $X \circ g'_n \rightharpoonup X \circ g$ as desired, thus (8.17) and so $X \circ g \in \overline{\mathcal{A}}$. □

If C is a $\mathfrak{N}$-core and $N \in \mathfrak{N}$, we set

$$\begin{gathered} \mathcal{C}_N := \Big\{ X \in \mathcal{X}_N : \iota_X \in \mathrm{C}_N \Big\}, \quad \mathcal{C}_\infty := \Big\{ X \in \mathcal{X}_\infty : \iota_X \in \mathrm{C} \Big\} = \bigcup_{N \in \mathfrak{N}} \mathcal{C}_N \\ \mathcal{D}_N := \mathrm{co}(\mathcal{C}_N), \quad \mathcal{D}_\infty := \bigcup_{N \in \mathfrak{N}} \mathcal{D}_N, \quad \mathcal{E}_\infty := \overline{\mathcal{C}_\infty}. \end{gathered} \tag{8.18}$$

Notice that $\mathcal{C}_N$ is in fact a subset of $\mathcal{O}_N$ and $\mathcal{D}_N$ is a subset of $\mathcal{X}_N$.

Lemma 8.12. *Assume that $\mathrm{C} \subset \mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$ satisfies property (d) in Lemma 8.2. Then for every $N \in \mathfrak{N}$ it holds:*

(1) $\mathcal{C}_N$ and $\mathcal{D}_N$ are relatively open subsets of $\mathcal{X}_N$, invariant with respect to the action of permutations of $\mathrm{Sym}(I_N)$.
(2) The relative interior of $\overline{\mathcal{C}_N}$ in $\mathcal{X}_N$ coincides with $\mathcal{D}_N$, in particular $\mathcal{C}_N$ is dense in $\mathcal{D}_N$ and $\overline{\mathcal{C}_N} = \overline{\mathcal{D}_N}$.
(3) $\mathcal{D}_N \cap \mathcal{O}_N = \mathcal{C}_N$.
(4) If $M \in \mathfrak{N}$ and $M \mid N$ then $\mathcal{D}_M = \mathcal{D}_N \cap \mathcal{X}_M = \Pi_M(\mathcal{D}_N)$ and $\overline{\mathcal{D}_M} = \overline{\mathcal{D}_N} \cap \mathcal{X}_M = \Pi_M(\overline{\mathcal{D}_N})$.
(5) $\mathcal{C}_\infty \subset \mathcal{D}_\infty \subset \overline{\mathcal{D}_\infty} = \overline{\mathcal{C}_\infty} = \mathcal{E}_\infty$ and $\mathcal{E}_\infty$ is convex.
(6) $\mathcal{D}_N = \mathcal{D}_\infty \cap \mathcal{X}_N = \Pi_N(\mathcal{D}_\infty)$ and $\overline{\mathcal{D}_N} = \mathcal{E}_\infty \cap \mathcal{X}_N = \Pi_N(\mathcal{E}_\infty)$.
(7) $\mathcal{E}_\infty = \overline{\mathcal{D}_\infty} = \overline{\mathcal{C}_\infty}$ is law invariant.

Proof. (1) The set $\mathcal{C}_N$ is relatively open, since the map $X \mapsto \iota_X$ is Lipschitz from $\mathcal{X}_N$ to $\mathcal{P}_{f,N}(\mathsf{X})$, thanks to (8.12), and C_N is relatively open in $\mathcal{P}_{f,N}(\mathsf{X})$. $\mathcal{D}_N$ is relatively open in $\mathcal{X}_N$ since it is the convex hull of an open set.

(2) Since $\mathcal{D}_N = \mathrm{co}(\mathcal{C}_N)$ we immediately get $\mathcal{D}_N \subset \overline{\mathcal{C}_N}$. On the other hand, since $\mathcal{D}_N \supset \mathcal{C}_N$ we also have $\overline{\mathcal{D}_N} = \overline{\mathcal{C}_N}$. Being $\mathcal{D}_N$ open and convex, it coincides with the interior of its closure.

(3) It is clear that $\mathcal{C}_N \subset \mathcal{D}_N \cap \mathcal{O}_N$. Let now show that any element $X = \mathscr{I}_N(\boldsymbol{x}) \in \mathcal{D}_N \cap \mathcal{O}_N$ belongs to $\mathcal{C}_N$. If B_N is the open unit ball in X^N it is easy to see that there exists a sufficiently small $\varepsilon > 0$ such that the open set $\mathcal{B}_\varepsilon := \{(\mathscr{I}_N(\boldsymbol{x}+\varepsilon\boldsymbol{z}), \mathscr{I}_N(\boldsymbol{x}-\varepsilon\boldsymbol{z})) : \boldsymbol{z} \in \mathsf{B}_N\}$ is contained in $\big(\mathcal{D}_N \cap \mathcal{O}_N\big)^2$. Since $\mathcal{C}_N$ is relatively open and dense in $\mathcal{D}_N \cap \mathcal{O}_N$ the intersection of $\mathcal{B}_\varepsilon$ with $\big(\mathcal{C}_N\big)^2$ is not empty.
It follows that we can find $\boldsymbol{z} \in \mathsf{B}_N$ such that $\mathscr{I}_N(\boldsymbol{x}+\varepsilon\boldsymbol{z}), \mathscr{I}_N(\boldsymbol{x}-\varepsilon\boldsymbol{z}) \in \mathcal{C}_N$ and therefore $X \in \mathcal{C}_N$ since C_N is convex along collisionless couplings.

(4) If we apply Lemma 8.9 to the closure of $\mathcal{D}_N$ we obtain $\Pi_M(\overline{\mathcal{D}_N}) = \overline{\mathcal{D}_N} \cap \mathcal{X}_M$. On the other hand, using property (3*) in Lemma 8.2 and the density of $\mathcal{C}_N$ in $\mathcal{D}_N$ we have $\overline{\mathcal{D}_N} \cap \mathcal{X}_M = \overline{\mathcal{C}_N} \cap \mathcal{X}_M = \overline{\mathcal{C}_M} = \overline{\mathcal{D}_M}$.
Similarly, still using Lemma 8.9, we obtain $\Pi_M(\mathcal{D}_N) = \mathcal{D}_N \cap \mathcal{X}_M$. Applying (8.15) to $\overline{\mathcal{D}_N}$, since by (2) the relative interior of $\overline{\mathcal{D}_N}$ in $\mathcal{X}_N$ is $\mathcal{D}_N$, we get that $\mathcal{D}_N \cap \mathcal{X}_M$ coincides with the relative interior in $\mathcal{X}_M$ of $\overline{\mathcal{D}_N} \cap \mathcal{X}_M = \overline{\mathcal{D}_M}$ which is equal to $\mathcal{D}_M$, again by claim (2).

(5) Since $\overline{\mathcal{C}_N} = \overline{\mathcal{D}_N}$ for every $N \in \mathfrak{N}$ by claim (2), we have that $\overline{\mathcal{D}_\infty} \subset \overline{\mathcal{C}_\infty}$ so that equality follows by the trivial inclusion $\mathcal{C}_\infty \subset \mathcal{D}_\infty$. Since $\overline{\cup\overline{\mathcal{C}_N}} = \overline{\mathcal{C}_\infty}$, to prove that $\overline{\mathcal{C}_\infty}$ is convex, it is enough to show that $\cup\overline{\mathcal{C}_N}$ is convex. If $X, Y \in \cup\overline{\mathcal{C}_N}$, we can find $M, N \in \mathfrak{N}$ such that $X \in \overline{\mathcal{C}_N}$ and $Y \in \overline{\mathcal{C}_M}$, so that by claim (4), both X and Y belong to $\overline{\mathcal{C}_{MN}}$ which is convex by assumption.

(6) The first property follows by the identity $\mathcal{D}_N = \mathcal{D}_L \cap \mathcal{X}_N = \Pi_N(\mathcal{D}_L)$ for any $L \in \mathfrak{N}$ such that $N \mid L$ (cf. claim (4)) and the fact that $\mathcal{D}_\infty = \cup\Big\{\mathcal{D}_L : L \in \mathfrak{N},\ N \mid L\Big\}$, since $\mathfrak{N}$ is a directed set.
Setting $\mathcal{D}' := \cup_{N\in\mathfrak{N}}\overline{\mathcal{D}_N}$ and starting from the second identity of claim (4), the same argument shows that $\overline{\mathcal{D}_N} = \mathcal{D}' \cap \mathcal{X}_N = \Pi_N(\mathcal{D}')$. Since $\overline{\mathcal{D}_N}$ is closed, we get that $\Pi_N(\mathcal{D}')$ is closed, so that $\Pi_N(\overline{\mathcal{D}'}) \subset \overline{\Pi_N(\mathcal{D}')} = \Pi_N(\mathcal{D}')$. Since clearly $\Pi_N(\mathcal{D}') \subset \Pi_N(\overline{\mathcal{D}'})$, we get that $\overline{\mathcal{D}_N} = \mathcal{D}' \cap \mathcal{X}_N = \Pi_N(\mathcal{D}') = \Pi_N(\overline{\mathcal{D}'})$. We also have $\overline{\mathcal{D}'} \cap \mathcal{X}_N = \Pi_N(\overline{\mathcal{D}'} \cap \mathcal{X}_N) \subset \Pi_N(\overline{\mathcal{D}'}) = \mathcal{D}' \cap \mathcal{X}_N$, so that we get $\overline{\mathcal{D}'} \cap \mathcal{X}_N = \mathcal{D}' \cap \mathcal{X}_N$. The thesis then follows since $\overline{\mathcal{D}'} = \overline{\mathcal{D}_\infty} = \mathcal{E}_\infty$.

(7) The fact that $\mathcal{E}_\infty$ is law invariant follows from Lemma 8.10 and the previous claim, which shows that $\mathcal{E}_\infty \cap \mathcal{X}_M = \overline{\mathcal{D}_M}$ which is invariant w.r.t. $\mathrm{Sym}(I_M)$ by claim (1). □

As an immediate consequence of Lemma 8.12 we have the following result.

Corollary 8.13 (Cores are totally convex)**.** *If* C *is as in Lemma 8.12, then* $\overline{\mathrm{C}}$ *is totally convex.*

We can now justify the equivalent characterization of $\mathfrak{N}$-cores of Lemma 8.2.

Proof of Lemma 8.2.
Claim 1: (d) *implies* (a), (b) *and* (c).
The fact that (d) implies (b) and (c) follows by setting $\mathrm{D} := \iota(\mathcal{D}_\infty)$ defined in (8.18) and $\mathrm{E} := \overline{\mathrm{C}}$, as a consequence of Lemma 8.12 and Corollary 8.13. We prove that (d) implies (a): by Corollary 8.13, we have that $\overline{\mathrm{C}}$ is totally convex. Notice that the sets C_N are nonempty for every $N \in \mathfrak{N}$ thanks to (3*) and the fact that C is nonempty. Finally, by Lemma 8.12, we have that the relative interior in $\mathcal{P}_{f,N}(\mathsf{X})$ of $\overline{\mathrm{C}} \cap \mathcal{P}_{\#N}(\mathsf{X})$ is given by $\mathrm{D}_N \cap \mathcal{P}_{\#N}(\mathsf{X}) = \mathrm{C}_N$ (cf. Lemma 8.12(3)).
Claim 2: (b) *implies* (d).
If D is a subset of $\mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})$ satisfying conditions $(1'), (2')$ and $\mathrm{C} = \mathrm{D} \cap \mathcal{P}_{\#\mathfrak{N}}$, we see that $\mathrm{C}_N = \mathrm{D}_N \cap \mathcal{P}_{\#N}(\mathsf{X})$ for every $N \in \mathfrak{N}$. Clearly C_N is relatively open and convex along collisionless couplings in $\mathcal{P}_{f,N}(\mathsf{X})$. Also, since $\mathcal{P}_{\#N}(\mathsf{X})$ is obviously dense in $\mathcal{P}_{f,N}(\mathsf{X})$ and D_N

is open, we see that C_N is dense D_N i.e. $\overline{\mathrm{C}_N} = \overline{\mathrm{D}_N}$. It is also clear that $\overline{\mathrm{C}_N}$ is convex along couplings in $\mathcal{P}_{f,N}(\mathsf{X}\times\mathsf{X})$. Finally $\overline{\mathrm{D}_N} \cap \mathcal{P}_{f,M}(\mathsf{X}) = \overline{\mathrm{D}_M}$ thanks to the convexity of D_N and D_M, as an application of (8.15) to their Lagrangian representations.

Claim 3: (c) *implies* (b).

Let E be a totally convex and closed subset of $\mathcal{P}_2(\mathsf{X})$ satisfying conditions $(1''), (2'')$ and $\mathrm{C} = \cup_{N\in\mathfrak{N}}\mathring{\mathrm{E}}_N \cap \mathcal{P}_{\#N}(\mathsf{X})$. We define D_N and D as in (8.4). The only thing to check is that

$$\mathrm{D} \cap \mathcal{P}_{f,N}(\mathsf{X}) = \mathrm{D}_N. \tag{8.19}$$

Denote by $\mathcal{E}_\infty$ the Lagrangian parametrization of E (hence, law invariant) and denote by $\mathcal{E}_N := \mathcal{E}_\infty \cap \mathcal{X}_N$, which is closed and convex. The relative interior $\mathring{\mathcal{E}}_N$ of $\mathcal{E}_N$ in $\mathcal{X}_N$ provides a Lagrangian parametrization of $\mathring{\mathrm{E}}_N = \mathrm{D}_N$. Hence, proving (8.19) is equivalent to prove that $\mathcal{D}' \cap \mathcal{X}_N = \mathring{\mathcal{E}}_N$, where $\mathcal{D}' := \bigcup_{N\in\mathfrak{N}} \mathring{\mathcal{E}}_N$. Using (8.15), if $M \mid N$ we get $\mathring{\mathcal{E}}_N \cap \mathcal{X}_M = \mathring{\mathcal{E}}_M$, also observing that $\mathcal{E}_N$ is invariant by the action of $\mathrm{Sym}(I_N)$, as a consequence of the law invariance of $\mathcal{E}_\infty$. Therefore we deduce that $\mathcal{D}' \cap \mathcal{X}_M = \mathring{\mathcal{E}}_M$.

Claim 4: (a) *implies* (c).

It is clear that setting $\mathrm{E} := \overline{\mathrm{C}}$ we have that E it totally convex and closed. Moreover, since $\mathring{\mathrm{E}}_N$ contains the relative interior in $\mathcal{P}_{f,N}(\mathsf{X})$ of $E \cap \mathcal{P}_{\#N}(\mathsf{X})$ (coinciding with C_N), $\mathring{\mathrm{E}}_N$ is not empty. Since the intersection of $\mathring{\mathrm{E}}_N$ with $\mathcal{P}_{\#N}(\mathsf{X})$ is given by C_N, we immediately see that $\cup_N(\mathring{\mathrm{E}}_N \cap \mathcal{P}_{\#N}(\mathsf{X})) = \mathrm{C}$. Finally

$$\overline{\mathrm{E} \cap \mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})} = \overline{\cup_N \overline{\mathrm{E} \cap \mathcal{P}_{\#N}(\mathsf{X})}} = \overline{\cup_N \overline{\mathrm{C}_N}} = \overline{\mathrm{C}},$$

where we have used again that the intersection of $\mathring{\mathrm{E}}_N$ with $\mathcal{P}_{\#N}(\mathsf{X})$ is given by C_N and that the closure of $\mathrm{E} \cap \mathcal{P}_{\#N}(\mathsf{X})$ coincides with the closure of its (relative) interior. □

Proof of Lemma 8.3. Assume that (1*)-(3*) hold. We need to prove that $\overline{\mathrm{C}_N}$ is convex along couplings in $\mathcal{P}_{f,N}(\mathsf{X}\times\mathsf{X})$ for every $N\in\mathfrak{N}$. This is equivalent to prove the convexity of $\overline{\mathcal{C}_N}$ so that it is sufficient to show that, for every $X_0, X_1 \in \mathcal{C}_N$ and $t\in[0,1]$, their linear interpolation $X_t := (1-t)X_0 + tX_1$ belongs to $\overline{\mathcal{C}_N}$. By Proposition 6.4, we can find small perturbations $X_1(s)$ of X_1, $s\in[0,1]$, such that $X_1(s)\in\mathcal{C}_N$, $X_1(s)\to X_1$ as $s\downarrow 0$, and the perturbed interpolation $X_{s,t} := (1-t)X_0 + tX_1(s)$ belongs to $\mathcal{C}_N$ for every $t\in[0,1]$ and $s>0$. It follows that the coupling $\boldsymbol{\mu}_s = \iota^2_{X_0,X_1(s)}$ belongs to $\mathcal{P}_{\#N}(\mathsf{X}\times\mathsf{X})$ and it is collisionless for every $s>0$ and therefore $\mu_{s,t} = \mathsf{x}^t_\sharp\boldsymbol{\mu}_s$ belongs to C_N for every t. Since $\mu_{s,t} = \iota(X_{s,t})$ we have $X_{s,t}\in\mathcal{C}_N$. Passing to the limit as $s\downarrow 0$ we conclude that $X_t\in\overline{\mathcal{C}_N}$. □

8.2. Lagrangian representations of discrete MPVFs: construction of $\hat{\mathbf{F}}_N$

Let us now study in more detail the Lagrangian representations of a MPVF $\mathbf{F}\subset\mathcal{P}_2(\mathsf{TX})$ defined on a $\mathfrak{N}$-core. If $\Phi\in\mathbf{F}$ we can consider the (not empty) set of all the maps $(X,V)\in\mathcal{X}^2$ such that $(X,V)_\sharp\mathbb{P} = \Phi$. A particular case arises when the first marginal $\mu = \mathsf{x}_\sharp\Phi$ of Φ belongs to $\mathcal{P}_{f,N}(\mathsf{X})$. In this case, X has the form $X = \mathscr{I}_N(\boldsymbol{x})\in\mathcal{X}_N$, so that $\mu = X_\sharp\mathbb{P} = \frac{1}{N}\sum_{k\in I_N}\delta_{\boldsymbol{x}(k)}$, and we can construct V from the representation of Φ given by

$$\Phi = \frac{1}{N}\sum_{k\in I_N}\Phi_k, \quad \mathsf{x}_\sharp\Phi_k = \delta_{\boldsymbol{x}(k)},$$

for a family $\{\Phi_k\}_{k\in I_N}\subset\mathcal{P}(\mathsf{TX})$, by setting $V(\omega) := V_k(\omega)$ if $\omega\in\Omega_{N,k}$, where $V_k\in L^2(\Omega_{N,k},\mathbb{P}|_{\Omega_{N,k}};\mathsf{X})$ are maps such that $(V_k)_\sharp\mathbb{P}|_{\Omega_{N,k}} = \frac{1}{N}\mathsf{v}_\sharp\Phi_k$.

In the general case, when $\Phi\in\mathcal{P}_2(\mathsf{TX})$, it is easy to check that if $(X,V)_\sharp\mathbb{P} = \Phi$ and $Y\in\mathcal{X}$ then

$$[\Phi, \iota^2_{X,Y}]_{r,0} \le \langle V, X-Y\rangle_{\mathcal{X}}. \tag{8.20}$$

A particular important case arises when $X \in \mathcal{O}_N$ and $Y \in \mathcal{X}_N$: in this case Φ_k is uniquely determined by the disintegration of Φ w.r.t. μ, and $V|_{\Omega_{N,k}}$ coincides with V_k, with V_k as above, and

$$\langle V, X - Y\rangle_{\mathcal{X}} = \langle \Pi_N V, X - Y\rangle_{\mathcal{X}}, \quad \Pi_N V(\omega) = \boldsymbol{b}_\Phi(\boldsymbol{x}(k)) \quad \text{if } \omega \in \Omega_{N,k}, \tag{8.21}$$

where $\boldsymbol{b}_\Phi$ is the barycenter of Φ as in Definition 2.3. It is easy to check that

$$[\Phi, \iota^2_{X,Y}]_{r,0} = \langle V, X - Y\rangle_{\mathcal{X}} = \langle \Pi_N V, X - Y\rangle_{\mathcal{X}} \quad \text{if } (X,V)_\sharp \mathbb{P} = \Phi,\ X \in \mathcal{O}_N,\ Y \in \mathcal{X}_N, \tag{8.22}$$

since $\iota^2_{X,Y}$ is concentrated on a map.

Proposition 8.14. *Assume the same hypotheses of Theorem 8.4. Let us define the sets*

$$\boldsymbol{F} := \Big\{(X,V) \in \mathcal{C}_\infty \times \mathcal{X} : (X,V)_\sharp \mathbb{P} \in \mathbf{F}\Big\}, \quad \boldsymbol{F}_N := \Big\{(X, \Pi_N V) : X \in \mathcal{C}_N,\ (X,V) \in \boldsymbol{F}\Big\}, \tag{8.23}$$

where $N \in \mathfrak{N}$. Then $\boldsymbol{F}_N \subset \mathcal{X}_N \times \mathcal{X}_N$ is λ-dissipative, has open domain $\mathrm{D}(\boldsymbol{F}_N) = \mathcal{C}_N$, and it is invariant by permutations: if $(X,V) \in \boldsymbol{F}_N$ and $\sigma \in \mathrm{Sym}(I_N)$, then $(\sigma X, \sigma V) \in \boldsymbol{F}_N$.

Proof. In case (ii) of Theorem 8.4, the dissipativity of $\boldsymbol{F}_N$ immediately follows. In case (i) of Theorem 8.4, (8.22) and the λ-dissipativity of $\mathbf{F}$ along couplings in $\mathcal{P}_{\#N}(\mathsf{X} \times \mathsf{X})$, given by Theorem 7.5, yield

$$(X,V),\ (Y,W) \in \boldsymbol{F}_N \quad \Rightarrow \quad \langle V - W, X - Y\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}}, \tag{8.24}$$

so that $\boldsymbol{F}_N$ is λ-dissipative. In any of the cases (i) and (ii) of Theorem 8.4, if $(X,V) \in \boldsymbol{F}_N$ and $\sigma \in \mathrm{Sym}(I_N)$, then there exists $W \in \mathcal{X}$ such that $(X,W)_\sharp \mathbb{P} \in \mathbf{F}$ and $V = \Pi_N W$. By Lemma B.2, we can write $\sigma X = X \circ g \in \mathcal{C}_N$ for some $g \in G[\sigma]$ and $(X \circ g, W \circ g)_\sharp \mathbb{P} \in \mathbf{F}$. To conclude, it suffices to notice that $\Pi_N(W \circ g) = \sigma V$. □

Proposition 8.15. *Under the same assumptions of Theorem 8.4, for every $N \in \mathfrak{N}$ the λ-dissipative operator $\boldsymbol{F}_N$ admits a unique maximal λ-dissipative extension $\hat{\boldsymbol{F}}_N$ in $\mathcal{X}_N \times \mathcal{X}_N$ with $\mathcal{D}_N \subset \mathrm{D}(\hat{\boldsymbol{F}}_N) \subset \overline{\mathcal{D}_N}$. The operator $\hat{\boldsymbol{F}}_N$ can be equivalently characterized by*

$$(X,V) \in \hat{\boldsymbol{F}}_N \quad \Leftrightarrow \quad X \in \overline{\mathcal{D}_N},\ V \in \mathcal{X}_N,\ \langle V - W, X - Y\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \forall (Y,W) \in \boldsymbol{F}_N, \tag{8.25}$$

and, whenever $X \in \mathcal{D}_N$, $\hat{\boldsymbol{F}}_N X = \overline{\mathrm{co}}\left(\bar{\boldsymbol{F}}_N X\right)$, where

$$\bar{\boldsymbol{F}}_N X := \Big\{V \in \mathcal{X}_N : \exists\, (X_n, V_n) \in \boldsymbol{F}_N : X_n \to X,\ V_n \rightharpoonup V\Big\}. \tag{8.26}$$

$\hat{\boldsymbol{F}}_N$ is invariant with respect to permutations, i.e.

$$(X,V) \in \hat{\boldsymbol{F}}_N,\ \sigma \in \mathrm{Sym}(I_N) \quad \Rightarrow \quad (\sigma X, \sigma V) \in \hat{\boldsymbol{F}}_N \tag{8.27}$$

and for every $X, Y \in \mathcal{D}_N$, we have

$$V \in \hat{\boldsymbol{F}}_N X,\ \Psi \in \mathbf{F}[\iota_Y] \quad \Rightarrow \quad \langle V, X - Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \le \lambda |X - Y|^2_{\mathcal{X}}. \tag{8.28}$$

Finally, if $M \mid N = KM$, $X \in \overline{\mathcal{D}_M}$, and $(X,V) \in \hat{\boldsymbol{F}}_N$ then $\Pi_M V \in \hat{\boldsymbol{F}}_M X$. Conversely, if $X \in \mathcal{D}_M$ and $W \in \hat{\boldsymbol{F}}_M X$ then there exists $V \in \mathcal{X}_N$ such that

$$(X,V) \in \hat{\boldsymbol{F}}_N, \quad W = \Pi_M V. \tag{8.29}$$

Remark 8.16. It is worth noticing that the Eulerian image of $\hat{\boldsymbol{F}}_N$ is the MPVF $\hat{\mathbf{F}}_N$ defined in Theorem 8.4.

Proof. (8.25) and (8.26) follow by the fact that $\mathcal{D}_N$ is convex and open and the domain of $\boldsymbol{F}_N$ is dense in $\mathcal{D}_N$, see Lemma 8.12 and Theorem A.13 in the Appendix.
Using (8.25) it is immediate to check that $\hat{\boldsymbol{F}}_N$ satisfies (8.27), since for every $(X,V) \in \hat{\boldsymbol{F}}_N$ and $(Y,W) \in \boldsymbol{F}_N$

$$\langle \sigma V - W, \sigma X - Y\rangle_{\mathcal{X}} = \langle V - \sigma^{-1} W, X - \sigma^{-1} Y\rangle_{\mathcal{X}} \le \lambda |X - \sigma^{-1} Y|^2_{\mathcal{X}} = \lambda |\sigma X - Y|^2_{\mathcal{X}},$$

since $\boldsymbol{F}_N$ and the scalar product in $\mathcal{X}_N$ are invariant by the action of permutations in $\mathrm{Sym}(I_N)$. If $(X,V) \in \boldsymbol{F}_N$, (8.28) follows immediately since there exists $W \in \mathcal{X}$ such that $\Phi := (X,W)_\sharp\mathbb{P} \in \mathbf{F}$, $V = \Pi_N W$, and (8.22) yields $\langle V, X-Y\rangle_{\mathcal{X}} = [\Phi, \iota^2_{X,Y}]_{r,0}$ so that

$$\langle V, X-Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} = [\Phi, \iota^2_{X,Y}]_{r,0} + [\Psi, \iota^2_{Y,X}]_{r,0} \le \lambda |X-Y|^2_{\mathcal{X}}. \tag{8.30}$$

Notice that in case (i) of Theorem 8.4, the last inequality in (8.30) follows by (7.5), while this is obvious in case (ii) of Theorem 8.4.
If $X \in \mathcal{D}_N$ and $V \in \bar{\boldsymbol{F}}_N X$ according to (8.26), then there exist $(X_n, V_n) \in \boldsymbol{F}_N$, $X_n \in \mathcal{C}_N$, such that $X_n \to X$ and $V_n \rightharpoonup V$. We can pass to the limit in (8.30) written for (X_n, V_n) and using Theorem 2.13(5) we obtain that (X,V) satisfies (8.30) as well. Finally, since (8.30) holds for every $V \in \bar{\boldsymbol{F}}_N X$, it also holds for every $V \in \overline{\mathrm{co}}\left(\bar{\boldsymbol{F}}_N X\right)$, hence (8.28).
Let us now suppose that $M \mid N$, $(X,V) \in \hat{\boldsymbol{F}}_N$ and $X \in \mathcal{D}_M$. We want to show that $W := \Pi_M V$ belongs to $\hat{\boldsymbol{F}}_M X$ by using (8.25). If $(Y,U) \in \boldsymbol{F}_M$ with $Y \in \mathcal{C}_M$, we have $U = \Pi_M U'$ with $(Y,U')_\sharp\mathbb{P} =: \Phi \in \mathbf{F}$, so that (8.28) yields

$$\langle V, X-Y\rangle_{\mathcal{X}} + [\Phi, \iota^2_{Y,X}]_{r,0} \le \lambda |X-Y|^2_{\mathcal{X}}. \tag{8.31}$$

Since $Y \in \mathcal{O}_M$ and $X \in \mathcal{X}_M$, we have $[\Phi, \iota^2_{Y,X}]_{r,0} = \langle U, Y-X\rangle_{\mathcal{X}}$ by (8.22); since $X-Y \in \mathcal{X}_M$, we also have $\langle V, X-Y\rangle_{\mathcal{X}} = \langle \Pi_M V, X-Y\rangle_{\mathcal{X}}$ and we get

$$\langle W, X-Y\rangle_{\mathcal{X}} + \langle U, Y-X\rangle_{\mathcal{X}} = \langle V, X-Y\rangle_{\mathcal{X}} + [\Phi, \iota^2_{Y,X}]_{r,0} \le \lambda |X-Y|^2_{\mathcal{X}}. \tag{8.32}$$

Hence, by (8.25) $(X,W) \in \hat{\boldsymbol{F}}_M$. In particular, the above property shows that if $\boldsymbol{G} : \mathcal{D}_N \to \mathcal{X}_N$ is an arbitrary single-valued selection of $\hat{\boldsymbol{F}}_N$, the restriction of $\Pi_M \circ \boldsymbol{G}$ to $\mathcal{D}_M$ is a selection of $\hat{\boldsymbol{F}}_M$. We fix such a selection. To conclude we need to prove that the property holds also if $X \in \overline{\mathcal{D}_M}$. Recall that by Lemma 8.12(3), $\overline{\mathrm{D}(\boldsymbol{F}_M)} = \overline{\mathcal{C}_M} = \overline{\mathcal{D}_M}$. Then if $X \in \overline{\mathcal{D}_M}$, by Corollary A.14 we have that W belongs to $\hat{\boldsymbol{F}}_M X$ if and only if

$$\langle W - (\Pi_M \circ \boldsymbol{G})\,|_{\mathcal{D}_M}(Y), X-Y\rangle_{\mathcal{X}} \le \lambda |X-Y|^2_{\mathcal{X}} \quad \text{for every } Y \in \mathcal{D}_M,$$

i.e., if and only if

$$\langle W - \boldsymbol{G}Y, X-Y\rangle_{\mathcal{X}} \le \lambda |X-Y|^2_{\mathcal{X}} \quad \text{for every } Y \in \mathcal{D}_M. \tag{8.33}$$

If $V \in \hat{\boldsymbol{F}}_N X$, then using Corollary A.14 we have

$$\langle V - \boldsymbol{G}Y, X-Y\rangle_{\mathcal{X}} \le \lambda |X-Y|^2_{\mathcal{X}} \quad \text{for every } Y \in \mathcal{D}_N \supset \mathcal{D}_M,$$

hence (8.33) holds and we get $\Pi_M V \in \hat{\boldsymbol{F}}_M X$.
Let us now show the converse implication. If $X \in \mathcal{D}_M$ and $W \in \hat{\boldsymbol{F}}_M X$, we need to prove that $W \in \Pi_M \left(\hat{\boldsymbol{F}}_N X\right)$. Since $\overline{\mathrm{D}(\boldsymbol{G})} = \overline{\mathcal{D}_N}$, by Corollary A.14 and Theorem A.13 applied to $\boldsymbol{B} = \boldsymbol{G}$, we get $\Pi_M \left(\hat{\boldsymbol{F}}_N X\right) = \Pi_M \left(\hat{\boldsymbol{G}} X\right) = \Pi_M \left(\overline{\mathrm{co}}\left(\bar{\boldsymbol{G}} X\right)\right)$, where

$$\bar{\boldsymbol{G}} X := \left\{ Z \in \mathcal{X}_N : \exists\, X_n \in \mathcal{D}_N : X_n \to X,\ \boldsymbol{G} X_n \rightharpoonup Z \right\}.$$

Similarly, denoting by $\mathcal{G} := (\Pi_M \circ \boldsymbol{G})\,|_{\mathcal{D}_M}$, by Corollary A.14 and Theorem A.13 we get

$$\begin{aligned} \hat{\boldsymbol{F}}_M X = \hat{\mathcal{G}} X = \overline{\mathrm{co}}\left(\bar{\mathcal{G}} X\right) &= \overline{\mathrm{co}}\left(\{ Z \in \mathcal{X}_M : \exists\, X_n \in \mathcal{D}_M : X_n \to X,\ \mathcal{G} X_n \rightharpoonup Z \}\right) \\ &\subset \Pi_M \left(\overline{\mathrm{co}}\left(\bar{\boldsymbol{G}} X\right)\right), \end{aligned}$$

where the proof of the last equality can be pursued as follows. We first observe that

$$\begin{aligned} &\{ Z \in \mathcal{X}_M : \exists\, X_n \in \mathcal{D}_M : X_n \to X,\ \mathcal{G} X_n \rightharpoonup Z \} \\ &\subset \Pi_M(\{ W \in \mathcal{X}_N : \exists\, X_n \in \mathcal{D}_N : X_n \to X,\ \boldsymbol{G} X_n \rightharpoonup W \}) = \Pi_M(\bar{\boldsymbol{G}} X), \end{aligned}$$

by using the local boundedness of $\boldsymbol{G}$ as a selection of $\hat{\boldsymbol{G}}$ (see Theorem A.3(3)) and the fact that Π_M is a linear and continuous operator. Then we notice that

$$\overline{\mathrm{co}}\left(\Pi_M(\bar{\boldsymbol{G}}X)\right) = \overline{\Pi_M(\mathrm{co}(\bar{\boldsymbol{G}}X))} = \Pi_M(\overline{\mathrm{co}}\left(\bar{\boldsymbol{G}}X\right)),$$

where the first equality follows by linearity of Π_M and, for the second, we exploit again the local boundedness of $\bar{\boldsymbol{G}}$ as a selection of $\hat{\boldsymbol{G}}$ and the linearity and continuity of Π_M. Hence the conclusion. □

It is remarkable that $\hat{\boldsymbol{F}}_N$ can also be characterized by those $(X,V) \in \overline{\mathcal{D}_N} \times \mathcal{X}_N$ satisfying inequality (8.25) *restricted to those* $Y \in \mathcal{C}_N$ *for which* $\iota^2(X,Y)$ *is the unique optimal coupling between* $\iota(X)$ *and* $\iota(Y)$.

Proposition 8.17. *We assume the same hypothesis of Theorem 8.4. Let* $X \in \overline{\mathcal{D}_N}$ *and* $V \in \mathcal{X}_N$ *be satisfying*

$$\begin{gathered}\langle V-W, X-Y\rangle_{\mathcal{X}} \le \lambda|X-Y|^2_{\mathcal{X}}\\ \textit{for every } (Y,W)\in \boldsymbol{F}_N \textit{ s.t. } \iota^2(X,Y) \textit{ is the unique element of } \Gamma_o(\iota_X,\iota_Y).\end{gathered} \tag{8.34}$$

Then $(X,V) \in \hat{\boldsymbol{F}}_N$.

Proof. Let us consider an arbitrary element $(Y,W) \in \boldsymbol{F}_N$; in particular $Y \in \mathcal{C}_N$. Since $\mathcal{C}_N \subset \mathcal{D}_N$ and $\mathcal{D}_N$ coincides with the interior of the convex set $\overline{\mathcal{D}_N}$ (relatively to $\mathcal{X}_N$), we deduce that all the points $Y_t := (1-t)X + tY$ belong to $\mathcal{D}_N$ for $t \in (0,1]$.
Since for t in a neighborhood of 1 we have that $Y_t \in \mathcal{C}_N \subset \mathcal{O}_N$, we deduce that $Y_t \in \mathcal{O}_N$ with possible finite exceptions (observe that if two lines $t \mapsto (1-t)x_i + ty_i$, $i = 1,2$, in $\mathcal{X}$ coincide at two distinct values of t then they coincide everywhere). Therefore there exists $\varepsilon > 0$ such that $Y_t \in \mathcal{O}_N$ for every $t \in (0,\varepsilon)$. Since $\mathcal{D}_N \cap \mathcal{O}_N = \mathcal{C}_N$ (cf. Lemma 8.12), we deduce that $Y_t \in \mathcal{C}_N$ for every $t \in (0,\varepsilon)$.
By Theorem 6.2, we can thus find $\tau \in (0,\varepsilon)$ such that $Y_\tau \in \mathcal{C}_N$ and $\iota^2(X,Y_\tau)$ is the unique optimal coupling between ι_X and ι_{Y_τ}. Let $W_\tau \in \boldsymbol{F}_N(Y_\tau)$, then

$$\begin{aligned}\langle V-W, X-Y\rangle_{\mathcal{X}} &= \langle W_\tau - W, X-Y\rangle_{\mathcal{X}} + \langle V-W_\tau, X-Y\rangle_{\mathcal{X}}\\ &= \frac{1}{1-\tau}\langle W_\tau - W, Y_\tau - Y\rangle_{\mathcal{X}} + \frac{1}{\tau}\langle V-W_\tau, X-Y_\tau\rangle_{\mathcal{X}}\\ &\le \lambda|X-Y|^2_{\mathcal{X}},\end{aligned}$$

where we have used (8.34) and the λ-dissipativity of $\boldsymbol{F}_N$. Since (Y,W) is an arbitrary element of $\boldsymbol{F}_N$, we deduce that $(X,V) \in \hat{\boldsymbol{F}}_N$ by (8.25). □

Let us now show that, under the particular assumptions of Theorem 8.5, $\hat{\boldsymbol{F}}_N$ coincides with $\boldsymbol{F}$ on $\mathcal{C}_N$.

Corollary 8.18. *Under the assumptions of Theorem 8.4, assume also that the* MPVF $\mathbf{F}$ *is deterministic. Then* $\hat{\boldsymbol{F}}_N$ *is an extension of* $\boldsymbol{F}_N = \boldsymbol{F}$ *on* $\mathcal{C}_N$*, for every* $N \in \mathfrak{N}$*. Under the further assumptions that* $\mathbf{F}$ *is a single-valued* PVF *and demicontinuous on each* C_N*, then* $\boldsymbol{F}_N$ *coincides with* $\hat{\boldsymbol{F}}_N$ *on* $\mathcal{C}_N$.

Proof. The first statement is an immediate consequence of Proposition 8.15; the equality $\boldsymbol{F}_N = \boldsymbol{F}$ on $\mathcal{C}_N$ follows from the fact that $\mathbf{F}$ is a deterministic MPVF by assumption. Let us now assume that $\mathbf{F}$ is single-valued and its restriction to C_N is demicontinuous. Let X be an element of $\mathcal{C}_N$, $\mu = \iota(X)$; $\mathbf{F}[\mu]$ contains a unique element Φ which may be represented as $\mathrm{bar}(\Phi) = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{b}_\Phi)_\sharp\mu$ so that there is a unique element $V = \boldsymbol{b}_\Phi \circ X \in \mathcal{X}_N$ such that $(X,V)_\sharp\mathbb{P} = \Phi$. This shows that $\boldsymbol{F}X$ is single-valued. If $W \in \bar{\boldsymbol{F}}_N X$, we can find a sequence $(X_n, \boldsymbol{F}X_n) = (X_n, \boldsymbol{f}_n \circ X_n)$ converging in the strong-weak topology of $\mathcal{X} \times \mathcal{X}$ to (X,W), for maps $\boldsymbol{f}_n \in L^2(\mathsf{X}, \mu_n; \mathsf{X})$ with $\mu_n = \iota_{X_n}$. On the other hand, since $\mathbf{F}$ is demicontinuous and deterministic, we have that

$\mathbf{F}[\iota_{X_n}] = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}_n)_\sharp\mu_n \to \mathbf{F}[\iota_X] = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp\mu$ in $\mathcal{P}_2^{sw}(\mathsf{TX})$ for a map $\boldsymbol{f} \in L^2(\mathsf{X}, \mu; \mathsf{X})$. If $\psi \in \mathrm{C}_b(\mathsf{X}; \mathsf{X})$, we can test the convergence in $\mathcal{P}_2^{sw}(\mathsf{TX})$ against $\zeta(x, y) := \langle \psi(x), y\rangle$ so that

$$\langle \psi(X_n), \boldsymbol{f}_n \circ X_n\rangle_{\mathcal{X}} = \int_{\mathsf{X}} \zeta \, \mathrm{d}(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f}_n)_\sharp\mu_n \to \int_{\mathsf{X}} \zeta \, \mathrm{d}(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp\mu = \langle \psi(X), \boldsymbol{f} \circ X\rangle_{\mathcal{X}}.$$

On the other hand $\psi(X_n) \to \psi(X)$ and $\boldsymbol{f}_n \circ X_n \rightharpoonup W$ so that we deduce that

$$\langle \psi(X), \boldsymbol{f} \circ X\rangle_{\mathcal{X}} = \langle \psi(X), W\rangle_{\mathcal{X}} \quad \text{for every } \psi \in \mathrm{C}_b(\mathsf{X}; \mathsf{X}).$$

By arbitrariety of ψ we deduce that $W = \boldsymbol{f} \circ X = \boldsymbol{F}X$. We thus deduce that $\bar{\boldsymbol{F}}_N X$ coincides with $\boldsymbol{F}X$ and then it contains a unique element V, and therefore by (8.26) $\hat{\boldsymbol{F}}_N X = \mathrm{co}(\boldsymbol{F}_N X) = V$ as well. □

A similar result holds under the assumptions of Theorem 8.6. Let us first recall that, by Corollary 3.18, if $\mathbf{F}$ is totally λ-dissipative also $\tilde{\mathbf{F}} := \mathbf{F} \cup \mathrm{bar}\,(\mathbf{F})$ is totally λ-dissipative.

Corollary 8.19. *Under the assumptions of Theorem 8.6, let $\tilde{\boldsymbol{B}}$ be the Lagrangian representation of $\tilde{\mathbf{F}} = \mathbf{F} \cup \mathrm{bar}\,(\mathbf{F})$, and let $\boldsymbol{B}'$ be any λ-dissipative extension of $\tilde{\boldsymbol{B}}$. For every $N \in \mathfrak{N}$, $Y \in \overline{\mathcal{D}_N}$, $(Y, W) \in \boldsymbol{B}'$, we have $(Y, \Pi_N W) \in \hat{\boldsymbol{F}}_N$ and, in particular,*

$$\langle V - \Pi_N W, X - Y\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \textit{for every } (X, V) \in \hat{\boldsymbol{F}}_N,\ Y \in \overline{\mathcal{D}_N},\ (Y, W) \in \boldsymbol{B}', \tag{8.35}$$

where $\hat{\boldsymbol{F}}_N$ is constructed as in Proposition 8.15 starting from the restriction of $\mathbf{F}$ to C.

Proof. Observe that, by construction, $\boldsymbol{F}$ (constructed starting from the restriction of the MPVF $\mathbf{F}$ to C) and $\boldsymbol{F}_N$ are subsets of $\tilde{\boldsymbol{B}}$ hence of $\boldsymbol{B}'$; this implies that $\boldsymbol{F}_N$ is dissipative with $\boldsymbol{B}'$ in the sense that

$$\langle X - Y, V - W\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \text{for every } (X, V) \in \boldsymbol{F}_N,\ (Y, W) \in \boldsymbol{B}'. \tag{8.36}$$

Restricting (8.36) to $Y \in \overline{\mathcal{D}_N}$, the very definition of $\hat{\boldsymbol{F}}_N$ in (8.25) yields $(Y, \Pi_N W) \in \hat{\boldsymbol{F}}_N$; in particular, we get (8.35). □

In the general case, we can still improve the compatibility result obtained in (8.28) between $\hat{\boldsymbol{F}}_N$ and $\mathbf{F}$ with the following.

Lemma 8.20. *We keep the same assumptions of Theorem 8.4, and let $N \in \mathfrak{N}$. Then*

$$\langle V, X - Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \le \lambda |X - Y|^2_{\mathcal{X}} \tag{8.37}$$

for every $(X, V) \in \hat{\boldsymbol{F}}_N$, $Y \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ and every $\Psi \in \mathbf{F}[\iota_Y]$.

Proof. We start by proving (8.37) in case $X \in \mathcal{D}_N$. Let $Y_s := (1 - s)Y + sX \in \mathcal{D}_N$ for every $s \in (0, 1]$; then, by (8.28), we have

$$\langle V, X - Y_s\rangle_{\mathcal{X}} + [\mathbf{F}, \iota^2_{Y_s,X}]_{r,0} \le \lambda |X - Y_s|^2_{\mathcal{X}}.$$

Using (2.11) we can rewrite the above equation as

$$\langle V, X - Y_s\rangle_{\mathcal{X}} + (1 - s)[\mathbf{F}, \iota^2_{Y,X}]_{r,s} \le \lambda |X - Y_s|^2_{\mathcal{X}}.$$

Using (7.5) in case (i) of Theorem 8.4, or the total λ-dissipativity of $\mathbf{F}$ in case (ii) of Theorem 8.4, we get

$$\langle V, X - Y_s\rangle_{\mathcal{X}} + (1 - s)[\mathbf{F}, \iota^2_{Y,X}]_{r,0} - s(1 - s)\lambda |X - Y|^2_{\mathcal{X}} \le \lambda |X - Y_s|^2_{\mathcal{X}}.$$

Passing to the limit as $s \downarrow 0$, we obtain

$$\langle V, X - Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \forall X \in \mathcal{D}_N,\ V \in \hat{\boldsymbol{F}}_N X, Y \in \mathrm{D}(\hat{\boldsymbol{F}}_N), \Psi \in \mathbf{F}[\iota_Y]. \tag{8.38}$$

We come now to the general case; let $(X,V) \in \hat{\boldsymbol{F}}_N$, $Y \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ and $\Psi \in \mathbf{F}[\iota_Y]$. We define $Z = (X+Y)/2 \in \overline{\mathcal{D}_N}$ and, given $T \in \mathcal{D}_N$ and $V_T \in \hat{\boldsymbol{F}}_N T$, we set $Z_t := (1-t)Z + Tt \in \mathcal{D}_N$ for every $t \in (0,1]$; we take, for every $t \in (0,1]$, some $V_t \in \hat{\boldsymbol{F}}_N Z_t$. Clearly

$$(X,V), (Z_t, V_t), (T, V_T) \in \hat{\boldsymbol{F}}_N \quad \forall\, t \in (0,1].$$

We compute

$$\begin{aligned}
&\langle V, X-Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} = \\
&= \langle V - V_t, X - Y\rangle_{\mathcal{X}} - \langle V_t, Y - X\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \\
&= 2\langle V - V_t, X - Z\rangle_{\mathcal{X}} - 2\langle V_t, Y - Z\rangle_{\mathcal{X}} + 2[\Psi, \iota^2_{Y,Z}]_{r,0} \\
&= 2\langle V - V_t, X - Z_t\rangle_{\mathcal{X}} - 2\langle V_t, Y - Z_t\rangle_{\mathcal{X}} + 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} \\
&\quad + 2\langle V - V_t, Z_t - Z\rangle_{\mathcal{X}} - 2\langle V_t, Z_t - Z\rangle_{\mathcal{X}} - 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} + 2[\Psi, \iota^2_{Y,Z}]_{r,0} \\
&= 2\langle V - V_t, X - Z_t\rangle_{\mathcal{X}} - 2\langle V_t, Y - Z_t\rangle_{\mathcal{X}} + 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} \\
&\quad + 4\langle V_T - V_t, Z_t - Z\rangle_{\mathcal{X}} + 2\langle V - 2V_T, Z_t - Z\rangle_{\mathcal{X}} - 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} + 2[\Psi, \iota^2_{Y,Z}]_{r,0} \\
&= 2\langle V - V_t, X - Z_t\rangle_{\mathcal{X}} + 2\langle V_t, Z_t - Y\rangle_{\mathcal{X}} + 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} \\
&\quad + \frac{4t}{1-t}\langle V_T - V_t, T - Z_t\rangle_{\mathcal{X}} + 2t\langle V - 2V_T, T - Z\rangle_{\mathcal{X}} - 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} + 2[\Psi, \iota^2_{Y,Z}]_{r,0} \\
&\le 2t\langle V - 2V_T, T - Z\rangle_{\mathcal{X}} - 2[\Psi, \iota^2_{Y,Z_t}]_{r,0} + 2[\Psi, \iota^2_{Y,Z}]_{r,0} \\
&\quad + 2\lambda|X - Z_t|^2_{\mathcal{X}} + 2\lambda|Y - Z_t|^2_{\mathcal{X}} + \frac{4t}{1-t}\lambda|T - Z_t|^2_{\mathcal{X}},
\end{aligned}$$

where we have used again (2.11), the λ-dissipativity of $\hat{\boldsymbol{F}}_N$ and (8.38) applied to $Z_t \in \mathcal{D}_N$, $V_t \in \hat{\boldsymbol{F}}_N Z_t$. Passing to the lim sup as $t \downarrow 0$, we get

$$\langle V, X-Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \le 2[\Psi, \iota^2_{Y,Z}]_{r,0} - 2\liminf_{t\downarrow 0}[\Psi, \iota^2_{Y,Z_t}]_{r,0} + \lambda|X-Y|^2_{\mathcal{X}} \le \lambda|X-Y|^2_{\mathcal{X}}$$

by Theorem 2.13(5). □

8.3. **Lagrangian representation of the maximal extension**

The last and crucial step in the construction of $\hat{\mathbf{F}}$ of Theorem 8.4 exploits an important invariance property of the resolvents of $\hat{\boldsymbol{F}}_N$ with respect to N.

Proposition 8.21. *We keep the same assumptions of Theorem 8.4. For every $X \in \mathcal{X}_\infty$ and every $0 < \tau < 1/\lambda^+$ there exists a unique $X_\tau \in \mathcal{X}_\infty$ such that*

$$X \in \mathcal{X}_N \Rightarrow X_\tau \in \mathrm{D}(\hat{\boldsymbol{F}}_N) \subset \mathcal{X}_N \ \text{ and } \ X_\tau - X \in \tau\,\hat{\boldsymbol{F}}_N X_\tau. \tag{8.39}$$

Moreover

$$|X_\tau(\omega') - X_\tau(\omega'')| \le \frac{1}{1-\lambda\tau}|X(\omega') - X(\omega'')| \quad \text{for every } \omega', \omega'' \in \Omega. \tag{8.40}$$

Proof. Since $X \in \mathcal{X}_\infty$, there exits $N \in \mathfrak{N}$ such that $X \in \mathcal{X}_N$. Since $\hat{\boldsymbol{F}}_N$ is maximal λ-dissipative, recalling Theorem A.2(1), there exists a unique solution $X_{\tau,N} \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ of

$$X_{\tau,N} - X \in \tau\,\hat{\boldsymbol{F}}_N(X_{\tau,N}).$$

The invariance of $\hat{\boldsymbol{F}}_N$ by permutations, stated in (8.27), shows that $(\sigma X)_{\tau,N} = \sigma(X_{\tau,N})$ for every $\sigma \in \mathrm{Sym}(I_N)$. In particular, by λ-dissipativity of $\hat{\boldsymbol{F}}_N$ we have

$$\langle \sigma X_{\tau,N} - \sigma X - (X_{\tau,N} - X), \sigma X_{\tau,N} - X_{\tau,N}\rangle_{\mathcal{X}} \le \lambda\tau|\sigma X_{\tau,N} - X_{\tau,N}|^2_{\mathcal{X}}$$

so that

$$(1-\lambda\tau)\,|\sigma X_{\tau,N} - X_{\tau,N}|_{\mathcal{X}} \le |\sigma X - X|_{\mathcal{X}} \quad \text{for every } \sigma \in \mathrm{Sym}(I_N).$$

If $\omega' \in \Omega_{N,i}$, $\omega'' \in \Omega_{N,j}$, $i, j \in I_N$, and we choose as σ the transposition which shifts i with j, we get

$$\frac{2}{N}(1-\lambda\tau)^2|X_{\tau,N}(\omega') - X_{\tau,N}(\omega'')|^2 \le \frac{2}{N}|X(\omega') - X(\omega'')|^2$$

which yields (8.40).
Let us now suppose that $X \in \mathcal{X}_M$ with $M \mid N$. Then $X_{\tau,N}$ belongs to $\mathcal{X}_M$ by (8.40), so that $X_{\tau,N} \in \overline{\mathcal{D}_N} \cap \mathcal{X}_M = \overline{\mathcal{D}_M}$ by Lemma 8.12(4). By Proposition 8.15, for every $Y \in \mathcal{D}_M$ and $W \in \hat{\boldsymbol{F}}_M Y$ we can find $V \in \hat{\boldsymbol{F}}_N Y$ such that $W = \Pi_M V$, so that by λ-dissipativity of $\hat{\boldsymbol{F}}_N$ we have

$$\langle X_{\tau,N} - X - \tau V, X_{\tau,N} - Y\rangle_{\mathcal{X}} \le \lambda\tau|X_{\tau,N} - Y|^2_{\mathcal{X}}. \tag{8.41}$$

Since $X_{\tau,N} - Y \in \mathcal{X}_M$, we can replace V with $W = \Pi_M V$ in (8.41), thus obtaining that $X_{\tau,N} - X \in \tau\hat{\boldsymbol{F}}_M(X_{\tau,N})$ by Corollary A.14, i.e. $X_{\tau,N} = X_{\tau,M}$. If M, N are arbitrary and $X \in \mathcal{X}_M \cap \mathcal{X}_N$, then setting $R := MN$ the previous argument shows that $X_{\tau,M} = X_{\tau,R} = X_{\tau,N}$. □

Corollary 8.22. *We keep the same assumptions of Theorem 8.4, let $M \in \mathfrak{N}$ and let $X \in \mathrm{D}(\hat{\boldsymbol{F}}_M)$. Then*

(1) $X \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ for every $N \in \mathfrak{N}$ s.t. $M \mid N$.
(2) $\hat{\boldsymbol{F}}^\circ X := \lim_{\tau\downarrow 0} \frac{X_\tau - X}{\tau} \in \hat{\boldsymbol{F}}_M X$. In particular $\hat{\boldsymbol{F}}^\circ X \in \hat{\boldsymbol{F}}_N X$ for every $N \in \mathfrak{N}$ s.t. $M \mid N$.
(3) $|\hat{\boldsymbol{F}}^\circ X|_{\mathcal{X}} \le |V|_{\mathcal{X}}$ for every $V \in \hat{\boldsymbol{F}}_N X$ and for every $N \in \mathfrak{N}$ s.t. $M \mid N$.
(4) $(1-\lambda\tau)|X_\tau - X|_{\mathcal{X}} \le \tau|\hat{\boldsymbol{F}}^\circ X|_{\mathcal{X}}$ for every $0 < \tau < 1/\lambda^+$.

Moreover, for every $X, Y \in \bigcup_{N\in\mathfrak{N}} \mathrm{D}(\hat{\boldsymbol{F}}_N)$, we have

$$\langle \hat{\boldsymbol{F}}^\circ X - \hat{\boldsymbol{F}}^\circ Y, X - Y\rangle_{\mathcal{X}} \le \lambda|X - Y|^2_{\mathcal{X}}. \tag{8.42}$$

Proof. By Theorem A.3(5) there exists the limit

$$\lim_{\tau\downarrow 0} \frac{X_\tau - X}{\tau} = \hat{\boldsymbol{F}}^\circ X \in \hat{\boldsymbol{F}}_M X$$

and (4) holds. If $N \in \mathfrak{N}$ is s.t. $M \mid N$, then $X \in \mathrm{D}(\hat{\boldsymbol{F}}_M) \subset \overline{\mathcal{D}_M} \subset \overline{\mathcal{D}_N}$, by Lemma 8.12. Moreover by Proposition 8.21, we have that

$$\frac{X_\tau - X}{\tau} \in \hat{\boldsymbol{F}}_N X_\tau \quad \forall\, 0 < \tau < 1/\lambda^+.$$

In particular

$$\langle \frac{X_\tau - X}{\tau} - W, X_\tau - Y\rangle_{\mathcal{X}} \le \lambda|X_\tau - Y|^2_{\mathcal{X}} \quad \forall (Y, W) \in \boldsymbol{F}_N \quad \forall\, 0 < \tau < 1/\lambda^+,$$

so that, passing to the limit as $\tau \downarrow 0$, we get

$$\langle \hat{\boldsymbol{F}}^\circ X - W, X - Y\rangle_{\mathcal{X}} \le \lambda|X - Y|^2_{\mathcal{X}} \quad \forall (Y, W) \in \boldsymbol{F}_N,$$

since $X_\tau \to X$ as $\tau \downarrow 0$ by Theorem A.3(4). This proves that $(X, \hat{\boldsymbol{F}}^\circ X) \in \hat{\boldsymbol{F}}_N$ and, in particular, that $X \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$. This proves (1) and (2), while (3) immediately follows, also using Theorem A.3(5).
Finally, if $X, Y \in \bigcup_{N\in\mathfrak{N}} \mathrm{D}(\hat{\boldsymbol{F}}_N)$, then there exist $N, M \in \mathfrak{N}$ s.t. $X \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ and $Y \in \mathrm{D}(\hat{\boldsymbol{F}}_M)$ so that, taking $R := MN$, we have

$$(X, \hat{\boldsymbol{F}}^\circ X), (Y, \hat{\boldsymbol{F}}^\circ Y) \in \hat{\boldsymbol{F}}_R$$

by (2). The λ-dissipativity of $\hat{\boldsymbol{F}}_R$ gives (8.42). □

We can therefore define the operator $\hat{\boldsymbol{F}}_\infty \subset \mathcal{X} \times \mathcal{X}$

$$\hat{\boldsymbol{F}}_\infty := \Big\{(X,V) \in \mathcal{X}_\infty \times \mathcal{X}_\infty : \exists\, M \in \mathfrak{N} : (X,V) \in \hat{\boldsymbol{F}}_N \ \forall\, N \in \mathfrak{N},\ M \mid N\Big\}. \tag{8.43}$$

Equivalently, $\hat{\boldsymbol{F}}_\infty$ has domain $\mathrm{D}(\hat{\boldsymbol{F}}_\infty) = \bigcup_{N\in\mathfrak{N}} \mathrm{D}(\hat{\boldsymbol{F}}_N)$ and

$$\hat{\boldsymbol{F}}_\infty X = \bigcup_{M\in\mathfrak{N}} \bigcap_{M|N} \hat{\boldsymbol{F}}_N X \quad \text{for every } X \in \mathrm{D}(\hat{\boldsymbol{F}}_\infty). \tag{8.44}$$

Notice that $\hat{\boldsymbol{F}}_\infty$ is the Lagrangian representation of the MPVF $\hat{\mathbf{F}}_\infty$ defined by Theorem 8.4. We can summarize the previous results in the following statement.

Corollary 8.23. *We keep the same assumptions of Theorem 8.4. The operator $\hat{\boldsymbol{F}}_\infty$ defined by (8.43) or (8.44) satisfies the following properties:*

(1) $\hat{\boldsymbol{F}}_\infty$ is λ-dissipative with domain $\mathrm{D}(\hat{\boldsymbol{F}}_\infty) = \bigcup_{N\in\mathfrak{N}} \mathrm{D}(\hat{\boldsymbol{F}}_N)$ and $\mathcal{C}_\infty \subset \mathcal{D}_\infty \subset \mathrm{D}(\hat{\boldsymbol{F}}_\infty) \subset \overline{\mathrm{D}(\hat{\boldsymbol{F}}_\infty)} = \overline{\mathcal{C}_\infty} = \overline{\mathcal{D}_\infty}$.

(2) The map $\hat{\boldsymbol{F}}^\circ$ defined by Corollary 8.22 provides the minimal selection $(\hat{\boldsymbol{F}}_\infty)^\circ$.

(3) For every $X \in \mathcal{X}_\infty$ and every $0 < \tau < 1/\lambda^+$ there exists a unique $X_\tau \in \mathrm{D}(\hat{\boldsymbol{F}}_\infty)$ such that $X_\tau - X \in \tau\, \hat{\boldsymbol{F}}_\infty X_\tau$.

Proof. Claim (1) follows by Proposition 8.15 and Lemma 8.12. Claim (2) comes by (8.43) and Corollary 8.22. Claim (3) is a consequence of Proposition 8.21 and the λ-dissipativity of $\hat{\boldsymbol{F}}_\infty$. □

Corollary 8.24. *Under the assumptions of Theorem 8.4, there exists a unique maximal extension $\hat{\boldsymbol{F}}$ of $\hat{\boldsymbol{F}}_\infty$ and it satisfies the following:*

(1) $\mathrm{D}(\hat{\boldsymbol{F}}) \subset \overline{\mathrm{D}(\hat{\boldsymbol{F}}_\infty)} = \overline{\mathcal{C}_\infty}$,

$$\mathcal{X}_N \cap \mathrm{D}(\hat{\boldsymbol{F}}) = \mathrm{D}(\hat{\boldsymbol{F}}_N), \quad \mathcal{X}_\infty \cap \mathrm{D}(\hat{\boldsymbol{F}}) = \mathrm{D}(\hat{\boldsymbol{F}}_\infty), \tag{8.45}$$

and, if $X \in \mathcal{X}_\infty$ and $0 < \tau < 1/\lambda^+$, then

$$\boldsymbol{J}_\tau X = X_\tau, \tag{8.46}$$

where $\boldsymbol{J}_\tau$ is the resolvent operator of $\hat{\boldsymbol{F}}$ and X_τ is as in Proposition 8.21.

(2) When restricted to $\mathrm{D}(\hat{\boldsymbol{F}}_N)$ (resp. $\mathrm{D}(\hat{\boldsymbol{F}}_\infty)$), the minimal selection of $\hat{\boldsymbol{F}}$ coincides with $(\hat{\boldsymbol{F}}_N)^\circ$ (resp. $(\hat{\boldsymbol{F}}_\infty)^\circ = \hat{\boldsymbol{F}}^\circ$ as in Corollary 8.23(2)).

(3) The following characterization holds

$$(X,V) \in \hat{\boldsymbol{F}} \quad \Leftrightarrow \quad X \in \overline{\mathcal{C}_\infty},\ \langle V - W, X - Y\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \text{for every } (Y,W) \in \hat{\boldsymbol{F}}_\infty; \tag{8.47}$$

or, equivalently,

$$(X,V) \in \hat{\boldsymbol{F}} \quad \Leftrightarrow \quad X \in \overline{\mathcal{C}_\infty},\ \langle V - \hat{\boldsymbol{F}}^\circ Y, X - Y\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \text{for every } Y \in \mathrm{D}(\hat{\boldsymbol{F}}_\infty). \tag{8.48}$$

(4) $\hat{\boldsymbol{F}} = \overline{\hat{\boldsymbol{F}}_\infty}^{\mathcal{X}\times\mathcal{X}}$.

Proof. Thanks to Corollary 8.23, the existence and uniqueness of the maximal extension $\hat{\boldsymbol{F}}$ of $\hat{\boldsymbol{F}}_\infty$ with domain $\mathrm{D}(\hat{\boldsymbol{F}}) \subset \overline{\mathrm{D}(\hat{\boldsymbol{F}}_\infty)}$ and characterized by (8.47) follows by Lemma A.15, with $D = \mathcal{X}_\infty$.

Notice that (8.46) holds since, by Corollary 8.23(3), when $X \in \mathcal{X}_\infty$ then X_τ plays the role of the resolvent for $\hat{\boldsymbol{F}}_\infty$ and we just proved that $\hat{\boldsymbol{F}}$ is a maximal extension of $\hat{\boldsymbol{F}}_\infty$. We prove the equivalences in (8.45): let $X \in \mathcal{X}_N \cap \mathrm{D}(\hat{\boldsymbol{F}})$ and $0 < \tau < 1/\lambda^+$, then

$$\frac{\boldsymbol{J}_\tau X - X}{\tau}$$

belongs to $\hat{\boldsymbol{F}}_N X_\tau$ thanks to Proposition 8.21 and (8.46), moreover it is bounded being $X \in \mathrm{D}(\hat{\boldsymbol{F}})$ (cf. Theorem A.3(5)). By maximality of $\hat{\boldsymbol{F}}_N$ and applying again Theorem A.3(5), we deduce that $X \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$, hence $\mathcal{X}_N \cap \mathrm{D}(\hat{\boldsymbol{F}}) \subset \mathrm{D}(\hat{\boldsymbol{F}}_N)$. The reverse inclusion is trivial.
Claim (2) comes from Claim (1) and Theorem A.3(5). The assertion involving $\hat{\boldsymbol{F}}_\infty$ comes from Corollary 8.23(2) and the proof of Lemma A.15.
The characterization in (8.48) is a consequence of Corollary A.16, applied to $\boldsymbol{B} = \hat{\boldsymbol{F}}$ with $D = \mathcal{X}_\infty$, and of (8.45).
Finally, Claim (4) comes by Lemma A.15 and the density of $\mathcal{X}_\infty$ in $\mathcal{X}$. □

Remark 8.25. Notice that Corollary 8.24(2) makes the notation $\hat{\boldsymbol{F}}^\circ$, used in Corollary 8.22, coherent with the one used in Appendix A to denote the minimal selection of $\hat{\boldsymbol{F}}$.

Theorem 8.26. *Under the assumptions of Theorem 8.4, $\hat{\boldsymbol{F}}$ is a law invariant maximal λ-dissipative operator according to Definition 3.2 and the Eulerian images $\hat{\mathbf{F}}_N, \hat{\mathbf{F}}_\infty, \hat{\mathbf{F}}$ of $\hat{\boldsymbol{F}}_N, \hat{\boldsymbol{F}}_\infty, \hat{\boldsymbol{F}}$ respectively (cf. Definition 3.8) satisfy the properties stated in Theorem 8.4.*
Moreover, if $Y \in \mathrm{D}(\boldsymbol{F})$ and $\Psi \in \mathbf{F}[\iota_Y]$, we have

$$\langle V, X - Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X}]_{r,0} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \text{for every } (X, V) \in \hat{\boldsymbol{F}}. \tag{8.49}$$

Finally, if $X \in \mathcal{C}_N$ for some $N \in \mathfrak{N}$ and $\Phi \in \mathbf{F}[\iota_X]$, then

$$|\hat{\boldsymbol{F}}^\circ X|^2_{\mathcal{X}} \le \int_{\mathsf{X}} |\boldsymbol{b}_\Phi|^2 \,\mathrm{d}\iota_X, \tag{8.50}$$

where $\boldsymbol{b}_\Phi$ is the barycenter of Φ as in Definition 2.3.

Proof. We can apply Lemma 8.10 (see also Remark 8.11) to the set $\hat{\boldsymbol{F}}_\infty \subset \mathcal{X}_\infty \times \mathcal{X}_\infty$. By construction, if $(X, V) \in \hat{\boldsymbol{F}}_\infty \cap (\mathcal{X}_M \times \mathcal{X}_M)$ there exists some $M' \in \mathfrak{N}$ such that $(X, V) \in \hat{\boldsymbol{F}}_N$ for all N multiple of M'. In particular, choosing $M'' \in \mathfrak{N}$ so that $M \mid M''$ and $M' \mid M''$, $(X, V) \in \hat{\boldsymbol{F}}_N$ for all N multiple of M''. On the other hand, all the permutations $\sigma \in \mathrm{Sym}(I_M)$ induce admissible permutations of $\mathrm{Sym}(I_N)$; therefore, by (8.27), we have that $(\sigma X, \sigma Y)$ belongs to $\hat{\boldsymbol{F}}_N$ for every N multiple of M''. We deduce that $(\sigma X, \sigma Y) \in \hat{\boldsymbol{F}}_\infty$ so that $\hat{\boldsymbol{F}}_\infty \cap (\mathcal{X}_M \times \mathcal{X}_M)$ is invariant by $\mathrm{Sym}(I_M)$. Since $\hat{\boldsymbol{F}}$ is the closure of $\hat{\boldsymbol{F}}_\infty$ by Corollary 8.24(4), Lemma 8.10 yields that $\hat{\boldsymbol{F}}$ is law invariant.
Let us now consider the Eulerian images $\hat{\mathbf{F}}_N, \hat{\mathbf{F}}_\infty$ and $\hat{\mathbf{F}}$. Since $\hat{\boldsymbol{F}}$ is law-invariant and maximal λ-dissipative, by Proposition 3.10 and Theorem 3.12, we have that the MPVF $\hat{\mathbf{F}}$ is maximal totally λ-dissipative. Since $\hat{\mathbf{F}}$ is an extension of $\hat{\mathbf{F}}_\infty$, we deduce that $\hat{\mathbf{F}}_\infty$ is totally λ-dissipative as well. The remaining part of the statement of Theorem 8.4 follows by Proposition 8.15, Proposition 8.17, Corollary 8.22, Corollary 8.23, Corollary 8.24 and the definitions of $\hat{\boldsymbol{F}}_\infty$ and $\hat{\boldsymbol{F}}$.
We now prove (8.49). Let $Y \in \mathrm{D}(\boldsymbol{F})$ and let $N \in \mathfrak{N}$ be such that $Y \in \mathcal{C}_N$; let $((X_n, V_n))_n$ as before. If $\Psi \in \mathbf{F}[\iota_Y]$, then, for every $n \in \mathbb{N}$, we can find $M_n \in \mathfrak{N}$ such that

$$(X_n, V_n) \in \hat{\boldsymbol{F}}_N, Y \in \mathrm{D}(\hat{\boldsymbol{F}}_N) \quad \forall N \in \mathfrak{N},\, M_n \prec N.$$

By Lemma 8.20, we have

$$\langle V_n, X_n - Y\rangle_{\mathcal{X}} + [\Psi, \iota^2_{Y,X_n}]_{r,0} \le \lambda |X_n - Y|^2_{\mathcal{X}} \quad \forall n \in \mathbb{N}.$$

Passing to the liminf as $n \to +\infty$ and using Theorem 2.13(5) we obtain (8.49).

Let now $X \in \mathcal{C}_N \subset \mathcal{D}_N$ for some $N \in \mathfrak{N}$, and observe that, since $\mathcal{D}_N$ is open by Lemma 8.12, then $\boldsymbol{J}_\tau X \in \mathcal{D}_N$ for $0 < \tau < 1/\lambda^+$ sufficiently small, since $\boldsymbol{J}_\tau X \to X$ as $\tau \downarrow 0$, where $\boldsymbol{J}_\tau$ is the resolvent of $\hat{\boldsymbol{F}}$. We can thus apply (8.28) and get

$$\frac{1}{\tau}\langle \boldsymbol{J}_\tau X - X, \boldsymbol{J}_\tau X - X\rangle_{\mathcal{X}} + [\Phi, \iota^2_{X,\boldsymbol{J}_\tau X}]_{r,0} \le \lambda |X - J_\tau X|^2_{\mathcal{X}}.$$

Since we have shown that $\hat{\boldsymbol{F}}$ is an invariant maximal λ-dissipative operator, by Theorem 3.4, there exists a Lipschitz function f such that $\boldsymbol{J}_\tau X = f \circ X$; thus $\iota^2_{X,\boldsymbol{J}_\tau X}$ is concentrated on a map so that, by Theorem 2.13(4), we have

$$[\Phi, \iota^2_{X,\boldsymbol{J}_\tau X}]_{r,0} = \langle \boldsymbol{b}_\Phi, X - \boldsymbol{J}_\tau X\rangle_{\mathcal{X}}.$$

We hence get

$$\frac{1}{\tau}|\boldsymbol{J}_\tau X - X|^2_{\mathcal{X}} \le |X - \boldsymbol{J}_\tau X|_{\mathcal{X}} \Big(|\boldsymbol{b}_\Phi| + \lambda |X - \boldsymbol{J}_\tau X|_{\mathcal{X}}\Big);$$

dividing by $|X - \boldsymbol{J}_\tau X|_{\mathcal{X}}$ and passing to the limit as $\tau \downarrow 0$, we obtain (8.50) (cf. Theorem A.3(5)). □

We conclude this section with the proofs of Theorems 8.5. and 8.6

Proof of Theorem 8.5. Let us first check that $\mathbf{F} \subset \hat{\mathbf{F}}_\infty$. It is sufficient to prove that if $\mu \in \mathrm{C}_M$ and $M \mid N$, $M, N \in \mathfrak{N}$, then every element $\Phi = (\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{f})_\sharp\mu \in \mathbf{F}[\mu]$ belongs to $\hat{\mathbf{F}}_N[\mu]$. Adopting a Lagrangian viewpoint (thanks to Theorem 8.26), if $X \in \mathcal{C}_M$ we want to show that $V = \boldsymbol{f} \circ X$ belongs to $\hat{\boldsymbol{F}}_N X$. This follows easily by the fact that $\mathcal{C}_M \subset \overline{\mathcal{D}_N}$, the λ-dissipativity of $\mathbf{F}$ and Proposition 8.17. Being $\hat{\mathbf{F}}_\infty$ totally λ-dissipative, the inclusion $\mathbf{F} \subset \hat{\mathbf{F}}_\infty$ shows that $\mathbf{F}$ is totally λ-dissipative and $\hat{\mathbf{F}}_\infty$ is a totally λ-dissipative extension of $\mathbf{F}$. By construction, $\hat{\mathbf{F}}$ is a maximal totally λ-dissipative extension of $\mathbf{F}$ and its uniqueness follows as a particular case of Theorem 8.6. The characterization in (8.11) follows by definition of $\hat{\mathbf{F}}_\infty$ and Proposition 8.17. Let us now check the second statement, under the assumptions that $\mathbf{F}$ is also single-valued and demicontinuous in C_N. By Corollary 8.22, we know that, on each $\mathcal{C}_N$, the minimal selection $\hat{\boldsymbol{F}}^\circ$ is a subset of $\hat{\boldsymbol{F}}_N$ and therefore, by Corollary 8.18, $\hat{\boldsymbol{F}}^\circ X = \boldsymbol{F}X$ for every $X \in \mathcal{C}_\infty$. □

Proof of Theorem 8.6. Let $\hat{\boldsymbol{B}}$ be a law invariant maximal λ-dissipative extension of the Lagrangian representation $\boldsymbol{B}$ of $\mathbf{F}$ with domain included in the convex set $\overline{\mathcal{C}_\infty}$, whose existence is given by Theorem 3.12. Notice that $\iota^2(\hat{\boldsymbol{B}})$ is maximal totally λ-dissipative and contains $\mathbf{F}$ so that it also contains $\mathrm{bar}\,(\mathbf{F})$ by Theorem 3.17. We deduce that $\hat{\boldsymbol{B}}$ is the Lagrangian representation of a λ-dissipative extension of $\mathbf{F} \cup \mathrm{bar}\,(\mathbf{F})$.
We want to show that $\hat{\boldsymbol{B}} \subset \hat{\boldsymbol{F}}$ and we split the argument in a few steps.

Claim 1: *for every* $Y \in \mathrm{D}(\hat{\boldsymbol{B}}) \cap \left(\bigcup_{N\in\mathfrak{N}} \overline{\mathcal{D}_N}\right)$ *and* $W \in \hat{\boldsymbol{B}}Y$*, we have* $W \in \hat{\boldsymbol{F}}Y$.

Let Y and W be as above and let $X \in \mathrm{D}(\hat{\boldsymbol{F}}_\infty)$. We can find some $M, L \in \mathfrak{N}$ such that $Y \in \mathrm{D}(\hat{\boldsymbol{B}}) \cap \overline{\mathcal{D}_M}$ and $X \in \mathrm{D}(\hat{\boldsymbol{F}}_L)$. In particular $Y \in \mathrm{D}(\hat{\boldsymbol{B}}) \cap \overline{\mathcal{D}_N}$ and $X \in \mathrm{D}(\hat{\boldsymbol{F}}_N)$ for every $N \in \mathfrak{N}$ such that $ML \mid N$ (cf. Corollary 8.22 and Lemma 8.12). By (8.35) we have

$$\langle X - Y, \hat{\boldsymbol{F}}^\circ X - \Pi_N W\rangle_{\mathcal{X}} \le \lambda |X - Y|^2_{\mathcal{X}} \quad \text{for every } N \in \mathfrak{N} \text{ such that } ML \mid N. \tag{8.51}$$

Passing to the limit as $N \to \infty$ in $\mathfrak{N}$ and using (8.48) we deduce that $(Y, W) \in \hat{\boldsymbol{F}}$.

Claim 2: $\mathrm{D}(\hat{\boldsymbol{B}}) \cap \left(\bigcup_{N\in\mathfrak{N}} \overline{\mathcal{D}_N}\right) = \mathrm{D}(\hat{\boldsymbol{B}}) \cap \mathcal{X}_\infty$.

It is sufficient to prove that $\mathrm{D}(\hat{\boldsymbol{B}}) \cap \overline{\mathcal{D}_N} = \mathrm{D}(\hat{\boldsymbol{B}}) \cap \mathcal{X}_N$ for every $N \in \mathfrak{N}$ and since $\overline{\mathcal{D}_N} \subset \mathcal{X}_N$ it is sufficient to prove the inclusion

$$\mathrm{D}(\hat{\boldsymbol{B}}) \cap \mathcal{X}_N \subset \overline{\mathcal{D}_N}. \tag{8.52}$$

We first show that

$$\overline{\mathrm{D}(\hat{\boldsymbol{F}})} \cap \mathcal{X}_N \subset \overline{\mathcal{D}_N}. \tag{8.53}$$

Indeed, by Proposition 8.21 and Corollary 8.24, for every $X \in \overline{\mathrm{D}(\hat{\boldsymbol{F}})} \cap \mathcal{X}_N$ and $\tau > 0$, $\boldsymbol{J}_\tau X$ belongs to $\mathrm{D}(\hat{\boldsymbol{F}}_N) \subset \overline{\mathcal{D}_N}$: passing to the limit as $\tau \downarrow 0$, since $X \in \overline{\mathrm{D}(\hat{\boldsymbol{F}})}$, we conclude that

X belongs to $\overline{\mathcal{D}_N}$ as well, thus proving (8.53). Since $\mathrm{D}(\hat{\boldsymbol{B}}) \subset \overline{\mathcal{D}_\infty} = \overline{\mathrm{D}(\hat{\boldsymbol{F}})}$, by (8.53), we get $\mathrm{D}(\hat{\boldsymbol{B}}) \cap \mathcal{X}_N \subset \overline{\mathrm{D}(\hat{\boldsymbol{F}})} \cap \mathcal{X}_N \subset \overline{\mathcal{D}_N}$, which shows (8.52).
Claim 3: $\hat{\boldsymbol{B}} \subset \hat{\boldsymbol{F}}$.
Setting $\hat{\boldsymbol{B}}_0 := \hat{\boldsymbol{B}} \cap (\mathcal{X}_\infty \times \mathcal{X})$, the first two claims yield $\hat{\boldsymbol{B}}_0 \subset \hat{\boldsymbol{F}}$. On the other hand, the maximal λ-dissipativity and the law invariance of $\hat{\boldsymbol{B}}$ show (cf. Theorem 3.4) that $\mathcal{X}_\infty$ is invariant under the action of the resolvent of $\hat{\boldsymbol{B}}$; since $\mathcal{X}_\infty$ is also dense in $\mathcal{X}$, we can apply (A.24) of Lemma A.15 obtaining that $\hat{\boldsymbol{B}}$ coincides with the strong closure of $\hat{\boldsymbol{B}}_0$ in $\mathcal{X} \times \mathcal{X}$ which is also contained in $\hat{\boldsymbol{F}}$, since $\hat{\boldsymbol{F}}$ is maximal λ-dissipative.

□

8.4. **Examples and applications**

Let us suppose that $\mathbf{F}$ satisfies the assumptions of Theorem 8.4 and let $\hat{\mathbf{F}}$ be the maximal totally λ-dissipative MPVF induced by $\mathbf{F}$. Since C is dense in $\mathrm{D}(\hat{\mathbf{F}})$, if we characterize the Lagrangian solutions to the flow generated by $\hat{\mathbf{F}}$ starting from every measure of C, we can then obtain all the other evolutions by approximation.
We want to show that the evolution of every measure in the core C can be characterized in a metric way, involving only $\mathbf{F}$.

Theorem 8.27. *Under the assumptions of Theorem 8.4, let $\mu_0 \in \overline{\mathrm{C}_N}$ for some $N \in \mathfrak{N}$ and let $\mu : [0, +\infty) \to \mathcal{P}_2(\mathsf{X})$ be a continuous curve starting from μ_0. The following properties are equivalent:*

(1) μ is a Lagrangian solution of the flow generated by $\hat{\mathbf{F}}$ (cf. Definition 4.1);
(2) μ is locally absolutely continuous in $(0, +\infty)$, it takes values in $\overline{\mathrm{C}_N}$, in particular $\mu_t \in \mathcal{P}_{f,N}(\mathsf{X})$ for every $t \geq 0$, and μ is a λ-EVI solution for the restriction of $\mathbf{F}$ to C_N;
(3) μ is locally absolutely continuous in $[0, +\infty)$ and locally Lipschitz continuous in $(0, +\infty)$, there exists a constant $C > 0$ such that the Wasserstein velocity field $\boldsymbol{v}$ of μ (cf. Theorem 2.11) satisfies
$$I_\lambda(t)\Big(\int |\boldsymbol{v}_t|^2 \,\mathrm{d}\mu_t\Big)^{1/2} \leq C \quad \text{a.e. in } (0,1), \tag{8.54}$$
$\mu_t \in \mathrm{D}(\hat{\mathbf{F}}_N) \subset \mathrm{D}(\hat{\mathbf{F}})$ for every $t > 0$, and it holds
$$\boldsymbol{v}_t = \hat{\boldsymbol{f}}^\circ[\mu_t] \quad \text{for } \mathscr{L}^1\text{-a.e. } t > 0, \tag{8.55}$$
where $\hat{\boldsymbol{f}}^\circ$ is the minimal selection map induced by $(\hat{\mathbf{F}})^\circ$ as in Theorem 3.19 and $I_\lambda(t)$ is as in (A.11).

Proof. We split the proof in various steps.
Claim 1. *(1) ⇔ (3)*
To see that (3) implies (1), it is sufficient to notice that by (8.55) μ satisfies the inclusion $(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{v}_t)_\sharp \mu_t \in \hat{\mathbf{F}}[\mu_t]$ for a.e. $t > 0$, so that it is clearly a λ-EVI solution for $\hat{\mathbf{F}}$ (see also [CSS23a, Theorem 5.4(1)]); by Theorem 4.5 we get that μ is a Lagrangian solution of the flow generated by $\hat{\mathbf{F}}$. We are left to check that (1) implies (3).
Since $\mu_0 \in \overline{\mathrm{C}_N}$, we can represent μ_0 as $\iota(X_0)$ for some $X_0 \in \overline{\mathcal{C}_N} = \overline{\mathcal{D}_N} = \overline{\mathrm{D}(\hat{\boldsymbol{F}}_N)}$ (cf. Lemma 8.12 and Proposition 8.15); if $(\boldsymbol{S}_t)_{t \geq 0}$ is the semigroup generated by $\hat{\boldsymbol{F}}$ we have $\mu_t = \iota(X_t)$ where $X_t = \boldsymbol{S}_t X_0$.
By Corollary 8.24(1), the restriction of the resolvent $\boldsymbol{J}_\tau$ of $\hat{\boldsymbol{F}}$ to $\mathcal{X}_N$ coincides with the resolvent of $\hat{\boldsymbol{F}}_N$: using the exponential formula (cf. Theorem A.6), we obtain that the restriction of the semigroup $(\boldsymbol{S}_t)_{t \geq 0}$ to $\overline{\mathcal{D}_N}$ coincides with the semigroup generated by $\hat{\boldsymbol{F}}_N$. Since the interior of the domain of $\hat{\boldsymbol{F}}_N$ in $\mathcal{X}_N$ is not empty (cf. Proposition 8.15 and Lemma 8.12), we can apply Theorem A.7 obtaining that $\boldsymbol{S}_t X_0$ is locally absolutely continuous in $[0, +\infty)$, it is locally Lipschitz in

$(0,+\infty)$, it satisfies $I_\lambda(t)|\dot X_t|_{\mathcal X} \le C$ in $(0,1)$ for a suitable constant C (so that we get (8.54)), it belongs to $\mathrm D(\hat{\boldsymbol F}_N)$ for every $t>0$, and it solves the equation

$$\dot X_t = (\hat{\boldsymbol F}_N)^\circ X_t \quad \text{for } \mathscr L^1\text{-a.e. } t>0.$$

Corollary 8.24(2) then shows that $\dot X_t = (\hat{\boldsymbol F})^\circ X_t$ as well, so that we get (4.2), with $\hat{\boldsymbol f}^\circ$ in place of $\boldsymbol f^\circ$, and therefore (8.55): indeed the tangent space $\mathrm{Tan}_{\mu_t}\mathcal P_2(\mathsf X)$ (cf. Theorem 2.11 and [AGS08, Theorem 8.3.1, Propositions 8.4.5, 8.4.6]) coincides with $L^2(\mathsf X,\mu_t;\mathsf X)$ being $\mathrm{supp}(\mu_t)$ of finite cardinality.

Claim 2: *(3) ⇒ (2)*
We know that solves the continuity equation with velocity field $\boldsymbol v_t = \hat{\boldsymbol f}^\circ[\mu_t]$ so that, by Corollary 8.24(2), we have $(\boldsymbol i_{\mathsf X},\boldsymbol v_t)_\sharp\mu_t \in \hat{\mathbf F}_N$. Let $\Phi\in\mathbf F$ with $\nu := \mathsf x_\sharp\Phi \in \mathrm C_N$ and let $t\in A(\mu)\subset[0,+\infty)$, where $A(\mu)$ is the full $\mathscr L^1$-measure set given by Theorem 2.13(6a). By Theorem 8.4(2) we have that

$$\int_{\mathsf X\times\mathsf X}\langle \boldsymbol v_t(x), x-y\rangle\,\mathrm d\boldsymbol\mu_t(x,y) \le \int_{\mathsf X\times\mathsf X}\big(-\langle \boldsymbol b_\Phi(y), y-x\rangle + \lambda|x-y|^2\big)\,\mathrm d\boldsymbol\mu_t(x,y) \tag{8.56}$$

for every $\boldsymbol\mu_t\in\Gamma_{f,N}(\mu_t,\nu)$. Choosing $\boldsymbol\mu_t$ optimal, by Theorem 2.13(6a) we have that

$$\frac{\mathrm d}{\mathrm dt}\frac12 W_2^2(\mu_t,\nu) = [(\boldsymbol i_{\mathsf X},\boldsymbol v_t)_\sharp\mu_t,\nu]_r \le \int_{\mathsf X\times\mathsf X}\langle \boldsymbol v_t(x), x-y\rangle\,\mathrm d\boldsymbol\mu_t(x,y).$$

On the other hand, since $\boldsymbol\mu_t$ is concentrated on a map w.r.t. ν, (2.12) gives that

$$\int_{\mathsf X\times\mathsf X}\langle \boldsymbol b_\Phi(y), y-x\rangle\,\mathrm d\boldsymbol\mu_t(x,y) = [\Phi,\mathsf s_\sharp\boldsymbol\mu_t]_{r,0},$$

where $\mathsf s$ is as in (2.3). So that, using (8.56), we obtain that

$$\frac{\mathrm d}{\mathrm dt}\frac12 W_2^2(\mu_t,\nu) \le -[\Phi,\mathsf s_\sharp\boldsymbol\mu_t]_{r,0} + \lambda W_2^2(\mu_t,\nu).$$

By passing to the supremum w.r.t. $\boldsymbol\mu_t\in\Gamma_o(\mu_t,\nu)$ and recalling Theorem 2.13(2), we finally obtain

$$\frac{\mathrm d}{\mathrm dt}\frac12 W_2^2(\mu_t,\nu) \le -\,[\Phi,\mu_t]_r + \lambda W_2^2(\mu_t,\nu);$$

this implies that μ is a λ-EVI solution for the restriction of $\mathbf F$ to $\mathrm C_N$.

Claim 3. *(2) ⇒ (1)*
We apply [CSS23a, Lemma 5.3, (5.5a)] obtaining that for every t in a set $A(\mu)\subset[0,+\infty)$ of full $\mathscr L^1$-measure, every $\nu\in\mathrm C_N$ and $\Phi\in\mathbf F[\nu]$ we have

$$\big[(\boldsymbol i_{\mathsf X},\boldsymbol v_t)_\sharp\mu_t,\nu\big]_r + \big[\Phi,\mu_t\big]_r \le \lambda W_2^2(\mu_t,\nu), \tag{8.57}$$

where $\boldsymbol v_t$ is the Wasserstein velocity field of μ. Let $t\in A(\mu)$ be fixed; restricting (8.57) to all the measures ν for which $\Gamma_o(\mu_t,\nu)$ contains a unique element (denoted by $\boldsymbol\mu$), Theorem 2.13(4) yields

$$\big[(\boldsymbol i_{\mathsf X},\boldsymbol v_t)_\sharp\mu_t,\nu\big]_r = \int\langle \boldsymbol v_t(x_0), x_0-x_1\rangle\,\mathrm d\boldsymbol\mu(x_0,x_1),$$
$$\big[\Phi,\mu_t\big]_r = \int\langle \boldsymbol b_\Phi(x_1), x_1-x_0\rangle\,\mathrm d\boldsymbol\mu(x_0,x_1).$$

Proposition 8.17 and (8.57) then yield that $(\boldsymbol i_{\mathsf X},\boldsymbol v_t)_\sharp\mu_t\in\hat{\mathbf F}_N[\mu_t]$.
Let us now consider the Lagrangian solution $\tilde\mu_t := S_t(\mu_0)$ of the flow driven by $\hat{\mathbf F}$. By the first Claim, we know that $\tilde\mu$ is absolutely continuous, $\tilde\mu_t\in\mathrm D(\hat{\mathbf F}_N)\subset\mathrm D(\hat{\mathbf F})\cap\overline{\mathrm C_N}$ for $t>0$, and satisfies (8.55).

We can then compute the derivative of $W_2^2(\mu_t, \tilde\mu_t)$: for $\mathscr{L}^1$-a.e. $t > 0$, we can choose an arbitrary $\boldsymbol{\mu}_t \in \Gamma_o(\mu_t, \tilde\mu_t)$, in particular a coupling in $\mathcal{P}_{f,N}(\mathsf{X} \times \mathsf{X})$, obtaining, by Theorem 2.13(6b),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \tilde\mu_t) = \int \langle \boldsymbol{v}_t(x_0) - \hat{\boldsymbol{f}}^\circ[\tilde\mu_t](x_1), x_0 - x_1\rangle \,\mathrm{d}\boldsymbol{\mu}_t(x_0, x_1) \le \lambda W_2^2(\mu_t, \nu)$$

by λ-dissipativity of $\hat{\mathbf{F}}_N$, since $(\boldsymbol{i}_{\mathsf{X}}, \hat{\boldsymbol{f}}^\circ[\tilde\mu_t])_\sharp \tilde\mu_t \in \hat{\mathbf{F}}_N$ by Corollary 8.24(2). We thus have that $\mu_t = \tilde\mu_t$ for every $t \ge 0$ and $\boldsymbol{v}_t = \hat{\boldsymbol{f}}^\circ[\mu_t]$. □

Remark 8.28. The example of $\frac{1}{2}$-dissipative PVF $\mathbf{F}$, with $\mathsf{X} = \mathbb{R}$ discussed in Remark 4.3 provides also a counterexample to the validity of the above Theorem 8.27 in case $\dim(\mathsf{X}) = 1$ and $\mathbf{F}$ is not totally $\frac{1}{2}$-dissipative: the evolutions driven by $\mathbf{F}$ and $\hat{\mathbf{F}}$ should coincide by Theorem 8.27, but this is impossible since $\hat{\mathbf{F}}$ is maximal totally $\frac{1}{2}$-dissipative and the evolution driven by $\mathbf{F}$ splits mass, a contradiction with Theorem 4.2.

We can now fully justify the example given in the Introduction.

Example 8.29. Assume that $\dim \mathsf{X} \ge 2$ and that $\mathbf{F}$ is a λ-dissipative single-valued deterministic PVF induced by a map $\boldsymbol{f} : \mathcal{S}(\mathsf{X}, \mathrm{C}) \to \mathsf{X}$, where C is a core as in Definition 8.1. This means that $\boldsymbol{f}$ induces a vector field $\boldsymbol{f}^N : \mathsf{C}_N \to \mathsf{X}^N$ defined on $\mathsf{C}_N := \mathscr{I}_N^{-1}(\mathcal{C}_N)$ (where $\mathcal{C}_N$ is as in (8.18)), which is an open subset of X^N, whose vectors have distinct coordinates: for every $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathsf{C}_N$ we have

$$\boldsymbol{f}^N(\boldsymbol{x}) := (\boldsymbol{f}(x_n, \iota \circ \mathscr{I}_N(\boldsymbol{x})))_{n=1,\dots,N}.$$

Clearly $\boldsymbol{f}^N$ is invariant with respect to permutations, in the sense that $\boldsymbol{f}^N(\sigma\boldsymbol{x}) = \sigma\boldsymbol{f}^N(\boldsymbol{x})$, for every $\boldsymbol{x} \in \mathsf{C}_N$ and every $\sigma \in \mathrm{Sym}(I_N)$. If $\mathbf{F}$ is demicontinuous in C_N, $\boldsymbol{f}^N$ is demicontinuous (i.e. strongly-weakly continuous) in C_N.
The previous theorem shows that starting from $\mu^N = \frac{1}{N}\sum_{n=1}^N \delta_{x_n^N} \in \mathrm{C}_N$ the evolution $\mu_t^N = S_t(\mu^N)$, at least for a short time, has the form

$$\mu_t^N = \frac{1}{N}\sum_{n=1}^N \delta_{x_n^N(t)} \quad \text{where} \quad \dot x_n^N(t) = \boldsymbol{f}_n^N(\boldsymbol{x}^N(t)).$$

Such an evolution admits a unique extension (see Theorem 8.5) which in fact corresponds to the unique maximal (and invariant by permutation) extension of the λ-dissipative vector field $\boldsymbol{f}^N$ to $\overline{\mathsf{C}_N}$. It is then possible to follow the path of each single particle by using the Lagrangian flow starting from $\mu_0 \in \mathrm{C}$ and defining N locally Lipschitz curves $x_n^N(t) = \boldsymbol{s}_t(x_n^N, \mu_0)$. If now $\mu^N \to \mu_0$ as $N \to \infty$ with a uniform control of the initial velocities, i.e.

$$\sup_N \frac{1}{N}\sum_{n=1}^N |\boldsymbol{f}_n^N(\boldsymbol{x}^N)|^2 < \infty,$$

then the measures μ_t^N will converge to $\mu_t = S_t(\mu_0)$ for every $t \ge 0$ in $\mathcal{P}_2(\mathsf{X})$ and, by Theorem 4.9, the measures carried on the discrete trajectories $\frac{1}{N}\sum_{n=1}^N \delta_{x_n^N(\cdot)} \in \mathcal{P}_2(\mathrm{C}([0,T];\mathsf{X}))$ will converge to $\mathrm{s}_\sharp\mu_0$ where s is the Lagrangian map starting from μ_0 as in (4.14).

Example 8.30 (A kinetic model of collective motion). Consider in the phase space $\mathsf{X} := \mathbb{R}^d \times \mathbb{R}^d$ the evolution of N-particles characterized by position-velocity coordinates $(x_n, v_n) \in \mathsf{X}$, $n = 1, \dots, N$, satisfying the system [DOr+06; CCR11]

$$\begin{cases} \dot x_n(t) = v_n(t), \\ \dot v_n(t) = (\alpha - \beta|v_n(t)|^2)v_n(t) + \dfrac{1}{N}\displaystyle\sum_{m=1}^N \boldsymbol{h}(x_n(t) - x_m(t)), \end{cases} \tag{8.58}$$

with $\alpha \geq 0, \beta > 0$ and $\boldsymbol{h} : \mathbb{R}^d \to \mathbb{R}^d$ a given Lipschitz vector field. For a given $\mu \in \mathcal{P}_2(\mathsf{X})$ we can consider the lower semicontinuous and $(-\alpha)$-totally convex functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$

$$\phi(\mu) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big(\frac{\beta}{4}|v|^4 - \frac{\alpha}{2}|v|^2\Big) \,\mathrm{d}\mu(x,v), \tag{8.59}$$

whose proper domain is $\mathrm{D}(\phi) := \Big\{\mu \in \mathcal{P}_2(\mathsf{X}) : \displaystyle\int |v|^4 \,\mathrm{d}\mu(x,v) < \infty\Big\}$. The minimal selection of $-\boldsymbol{\partial}_{\mathrm{t}}\phi(\mu)$ is given by $(\boldsymbol{i}_{\mathsf{X}}, \boldsymbol{g})_\sharp\mu$ with

$$\boldsymbol{g}(x,v;\mu) := \Big(0, (\alpha - \beta|v|^2)v\Big) \tag{8.60}$$

with proper domain $\mathrm{D}(\boldsymbol{\partial}_{\mathrm{t}}\phi) = \Big\{\mu \in \mathcal{P}_2(\mathsf{X}) : \displaystyle\int |v|^6 \,\mathrm{d}\mu(x,v) < \infty\Big\}$.
We can also define the deterministic PVF induced as in (7.10) by $\boldsymbol{h} : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$

$$\boldsymbol{h}(x,v;\mu) := \Big(v, \int_{\mathbb{R}^d \times \mathbb{R}^d} \boldsymbol{h}(x-y) \,\mathrm{d}\mu(y,w)\Big). \tag{8.61}$$

It is easy to check that a collection of N particles $(x_n(t), v_n(t))$ satisfies (8.58) if and only if the measure $\mu_t = \frac{1}{N}\sum_{n=1}^N \delta_{(x_n(t),v_n(t))}$ is a Lagrangian solution of the system (4.10) associated with the deterministic PVF

$$\boldsymbol{f}(x,v;\mu) := \boldsymbol{g}(x,v;\mu) + \boldsymbol{h}(x,v;\mu), \quad \mu \in \mathrm{D}(\boldsymbol{\partial}_{\mathrm{t}}\phi). \tag{8.62}$$

Since the Lagrangian representation of $\boldsymbol{f}$ corresponds to the sum of a maximal α-dissipative operator (the subdifferential of $\psi = \phi \circ \iota$) and a Lipschitz operator, it is maximal α-dissipative thanks to [Bré73, Lemma 2.4, Chapter II], so that the deterministic PVF associated with (8.62) is totally α-dissipative and we can apply all the results of Section 4.

In the following we give an example of totally dissipative MPVF $\mathbf{F}$ having a core contained in its domain.

Example 8.31. Let $W : \mathsf{X} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, even and convex function and denote by $\mathrm{D}(W)$ its proper domain. Let $\boldsymbol{B} \subset \mathsf{X} \times \mathsf{X}$ be a maximal dissipative set (see Appendix A) and suppose that $0 \in \mathrm{int}\,(\mathrm{D}(W))$ and $\mathrm{int}\,(\mathrm{D}(\boldsymbol{B})) \neq \emptyset$. Possible examples of W and $\boldsymbol{B}$ are given by the indicator of a convex set in X (or a function diverging at the boundary of a convex set) and the gradient of a convex function in X (or its sum with a linear and antisymmetric function) respectively. Let $\boldsymbol{u}_W$ be an odd single-valued measurable selection of ∂W and let $\boldsymbol{v}_{\boldsymbol{B}}$ be an arbitrary single-valued selection of $\boldsymbol{B}$. We define the set

$$E := \{\mu \in \mathcal{P}_c(\mathsf{X}) \,:\, \mathrm{supp}\,\mu \subset \mathrm{int}\,(\mathrm{D}(\boldsymbol{B}))\,, \mathrm{supp}\,\mu - \mathrm{supp}\,\mu \subset \mathrm{int}\,(\mathrm{D}(W))\}\,,$$

where $\mathcal{P}_c(\mathsf{X})$ denotes the subset of measures in $\mathcal{P}(\mathsf{X})$ with compact support. We define the single-valued probability vector field $\mathbf{F}$ as follows:

$$\mathbf{F}[\mu] := \begin{cases} (\boldsymbol{i}_{\mathsf{X}}, -(\boldsymbol{u}_W * \mu) + \boldsymbol{v}_{\boldsymbol{B}})_\sharp\,\mu, & \text{if } \mu \in E \\ \emptyset & \text{otherwise} \end{cases}, \qquad \mu \in \mathcal{P}_2(\mathsf{X}).$$

Notice that the convolution between $\boldsymbol{u}_W$ and μ is well posed since μ is discrete; moreover $(\boldsymbol{u}_W * \mu) + \boldsymbol{v}_{\boldsymbol{B}} \in L^2(\mathsf{X}, \mu; \mathsf{X})$ if $\mu \in E$; indeed $\boldsymbol{v}_{\boldsymbol{B}}$ and $\boldsymbol{u}_W$ are both locally bounded in the interior of the respective domains (see Corollary A.4 and Theorem A.3(3) and recall that $\mathrm{int}\,(\mathrm{D}(\partial W)) = \mathrm{int}\,(\mathrm{D}(W))$), so that $\mathrm{D}(\mathbf{F}) = E$ and $\mathbf{F} \subset \mathcal{P}_2(\mathsf{TX})$. It is not difficult to check that

$\mathbf{F}$ is totally dissipative: for every $\boldsymbol{\gamma} \in \Gamma(\mu, \nu)$ and every $\mu, \nu \in E$,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathsf{X}\times\mathsf{X}} W(y_1 - y_2)\,\mathrm{d}(\nu\otimes\nu)(y_1,y_2) - \frac{1}{2}\int_{\mathsf{X}\times\mathsf{X}} W(x_1 - x_2)\,\mathrm{d}(\mu\otimes\mu)(x_1,x_2)\\
&\ge \frac{1}{2}\int_{\mathsf{X}^4} \langle \boldsymbol{u}_W(x_1 - x_2), (y_1 - y_2) - (x_1 - x_2)\rangle\,\mathrm{d}(\boldsymbol{\gamma}\otimes\boldsymbol{\gamma})(x_1,y_1,x_2,y_2)\\
&= \frac{1}{2}\int_{\mathsf{X}^3} \langle \boldsymbol{u}_W(x_1 - x_2), y_1 - x_1\rangle\,\mathrm{d}\mu(x_2)\,\mathrm{d}\boldsymbol{\gamma}(x_1,y_1)\\
&\quad + \frac{1}{2}\int_{\mathsf{X}^3} \langle \boldsymbol{u}_W(x_2 - x_1), y_2 - x_2\rangle\,\mathrm{d}\mu(x_1)\,\mathrm{d}\boldsymbol{\gamma}(x_2,y_2)\\
&= \int_{\mathsf{X}\times\mathsf{X}} \langle (\boldsymbol{u}_W * \mu)(x), y - x\rangle\,\mathrm{d}\boldsymbol{\gamma}(x,y),
\end{aligned}$$

where we have used Fubini's theorem and the fact that $\boldsymbol{u}_W$ is odd. This immediately gives that

$$\int_{\mathsf{X}\times\mathsf{X}} \langle (-\boldsymbol{u}_W * \mu)(x) + (\boldsymbol{u}_W * \nu)(y), x - y\rangle\,\mathrm{d}\boldsymbol{\gamma}(x,y) \le 0. \tag{8.63}$$

Thus

$$\begin{aligned}
&\int_{\mathsf{X}\times\mathsf{X}} \langle -(\boldsymbol{u}_W * \mu)(x) + \boldsymbol{v}_{\boldsymbol{B}}(x) + (\boldsymbol{u}_W * \nu)(y) - \boldsymbol{v}_{\boldsymbol{B}}(y), x - y\rangle\,\mathrm{d}\boldsymbol{\gamma}(x,y)\\
&= \int_{\mathsf{X}\times\mathsf{X}} \langle (-\boldsymbol{u}_W * \mu)(x) + (\boldsymbol{u}_W * \nu)(y), x - y\rangle\,\mathrm{d}\boldsymbol{\gamma}(x,y)\\
&\quad + \int_{\mathsf{X}\times\mathsf{X}} \langle \boldsymbol{v}_{\boldsymbol{B}}(x) - \boldsymbol{v}_{\boldsymbol{B}}(y), x - y\rangle\,\mathrm{d}\boldsymbol{\gamma}(x,y)\\
&\le 0,
\end{aligned}$$

where we have used (8.63) and the dissipativity of $\boldsymbol{B}$.
Given any unbounded directed subset $\mathfrak{N} \subset \mathbb{N}$, we can define D as

$$\mathrm{D} := \{\mu \in \mathcal{P}_{f,\mathfrak{N}}(\mathsf{X}) \,:\, \operatorname{supp}\mu \subset \operatorname{int}(\mathrm{D}(\boldsymbol{B}))\,,\ \operatorname{supp}\mu - \operatorname{supp}\mu \subset \operatorname{int}(\mathrm{D}(W))\}\,.$$

Trivially, being $\mathrm{D} \subset \mathcal{P}_c(\mathsf{X})$, then $\mathrm{D} \subset \mathrm{D}(\mathbf{F}) \cap \mathcal{P}_{f,\mathfrak{N}}(\mathsf{X})$. Moreover, for any $N \in \mathfrak{N}$, the set $\mathrm{D}\cap\mathcal{P}_{f,N}(\mathsf{X})$ is open in $\mathcal{P}_{f,N}(\mathsf{X})$ and convex along couplings in $\mathcal{P}_{f,N}(\mathsf{X}\times\mathsf{X})$, being both $\operatorname{int}(\mathrm{D}(\partial W))$ and $\operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ convex sets (see Corollary A.4 and Theorem A.3(3)). Thus, setting $\mathrm{C} := \mathrm{D}\cap\mathcal{P}_{\#\mathfrak{N}}(\mathsf{X})$ and recalling Lemma 8.2, then Definition 8.1 is satisfied for C.

Example 8.32. Assume $\dim\mathsf{X} \ge 2$. Let $\mathsf{U} \subset \mathsf{X}$ be an open convex subset of X containing 0 (e.g. an open ball of radious $r > 0$ centered at 0) and let A be the set of all measures $\mu \in \mathcal{P}_2(\mathsf{X})$ such that

$$\operatorname{supp}\mu - \int x\,\mathrm{d}\mu(x) \subset \mathsf{U}.$$

In the case U is an open ball, A imposes the constraint that the support of μ is contained in the ball with same radious as U centered at the barycenter of μ. We can then consider the set $\mathrm{D} := \bigcup_{N\in\mathbb{N}}(\mathrm{A}\cap\mathcal{P}_{f,N})$ and inducing a corresponding core C as in Lemma 8.2.
Let $\boldsymbol{f} : \mathcal{S}(\mathsf{X}) \to \mathsf{X}$ be a map as in Theorem 7.6 inducing a λ-dissipative demicontinuous PVF $\mathbf{F}$ by (7.10).
The restriction of $\boldsymbol{f}$ to $\mathcal{S}(\mathsf{X},\mathrm{C})$ induces a unique maximal totally λ-dissipative MPVF $\mathbf{F}'$, whose evolution corresponds to the evolution driven by $\boldsymbol{f}$ and constrained by A.

We conclude with an example of two probability vector fields $\mathbf{F}, \mathbf{G}$ generating the same evolution semigroup. The assumptions could be considerably refined: we just discuss a simple case, for ease of exposition.

Example 8.33 (Superposition of PVFs). Let $(\Theta, \mathcal{T}, \mathfrak{m})$ be a probability space and let $\boldsymbol{f} : \mathsf{X}\times\Theta \to \mathsf{X}$ be a $\mathcal{B}(\mathsf{X}) \otimes \mathcal{T}$-measurable map satisfying the properties

$$\boldsymbol{f}(\cdot,\theta) : \mathsf{X} \to \mathsf{X} \quad \text{is } \lambda\text{-dissipative and demicontinuous for } \mathfrak{m}\text{-a.e. } \theta \in \Theta,$$

$$\text{there exists } A > 0 \text{ such that } |\boldsymbol{f}(x,\theta)| \le A(1 + |x|^2) \text{ for every } x \in \mathsf{X} \text{ and } \mathfrak{m}\text{-a.e. } \theta \in \Theta.$$

We denote by $\pi^{\mathsf{X}} : \mathsf{X} \times \Theta \to \mathsf{X}$ the projection on the first component, $\pi^{\mathsf{X}}(x,\theta) := x$, and we set

$$\mathbf{F}[\mu] := (\pi^{\mathsf{X}}, \boldsymbol{f})_\sharp(\mu \otimes \mathfrak{m}), \quad \mu \in \mathcal{P}_2(\mathsf{X}). \tag{8.64}$$

Clearly

$$|\mathbf{F}[\mu]|_2^2 = \int_{\mathsf{X}} \Big(\int_\Theta |\boldsymbol{f}(x,\theta)|^2 \,\mathrm{d}\mathfrak{m}(\theta)\Big) \,\mathrm{d}\mu(x) \le A(1 + \mathsf{m}_2^2(\mu)) < \infty$$

so that $\mathrm{D}(\mathbf{F}) = \mathcal{P}_2(\mathsf{X})$. Using the plan $\Sigma := (\mathsf{x}^0, \boldsymbol{f}(\mathsf{x}^0, \boldsymbol{i}_\Theta), \mathsf{x}^1, \boldsymbol{f}(\mathsf{x}^1, \boldsymbol{i}_\Theta))_\sharp(\boldsymbol{\mu} \otimes \mathfrak{m})$ where $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$, we see that $\mathbf{F}$ is λ-dissipative. Its barycentric selection (cf. (2.13)) $\mathbf{G} := \mathrm{bar}\,(\mathbf{F})$ is a deterministic PVF induced by the demicontinuous map

$$\boldsymbol{g}(x) := \int_\Theta \boldsymbol{f}(x,\theta)\,\mathrm{d}\mathfrak{m}(\theta). \tag{8.65}$$

$\mathbf{G}$ is a maximal totally λ-dissipative PVF (cf. Theorem 3.22). Whenever $\boldsymbol{f}(\cdot,\theta)$ is not constant in a set $\Theta_0 \subset \Theta$ of positive $\mathfrak{m}$-measure (and therefore $\mathbf{F} \ne \mathbf{G}$), then $\mathbf{F}$ cannot be totally λ-dissipative since this would lead to a contradiction with the maximality of its barycentric projection $\mathbf{G}$. Applying [CSS23a, Corollary 5.23, Theorem 5.27], we know that $\mathbf{F}$ generates a unique λ-EVI flow whose trajectories have the barycentric property, and therefore coincide with the Lagrangian solutions of the flow generated by $\mathbf{G}$, i.e. $\mathbf{F}$ and $\mathbf{G}$ generates the same evolution semigroup. It would not be difficult to check that $\mathbf{G}$ coincides with the operator $\hat{\mathbf{F}}$ of Theorem 8.4 constructed from the restriction of $\mathbf{F}$ to the core of discrete measures.

9. Geodesically convex functionals with a core dense in energy are totally convex

In this section, we provide sufficient conditions for the total $(-\lambda)$-convexity property (cf. Section 5), $\lambda \in \mathbb{R}$, of a functional $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ which is proper, lower semicontinuous and geodesically $(-\lambda)$-convex (see [AGS08, Definition 9.1.1]) with proper domain $\mathrm{D}(\phi) := \{\mu \in \mathcal{P}_2(\mathsf{X}) : \phi(\mu) < +\infty\}$, where we assume $\dim(\mathsf{X}) \ge 2$. This ensures the applicability of the results of Section 5, in particular Theorem 5.4.
Recall that $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ is geodesically $(-\lambda)$-convex if for any μ_0, μ_1 in $\mathrm{D}(\phi)$ there exists $\boldsymbol{\mu} \in \Gamma_o(\mu_0, \mu_1)$ such that

$$\phi(\mu_t) \le (1-t)\phi(\mu_0) + t\phi(\mu_1) + \frac{\lambda}{2} t(1-t) W_2^2(\mu_0, \mu_1) \qquad \forall\, t \in [0,1],$$

where $\mu_t := \mathsf{x}^t_\sharp \boldsymbol{\mu}$.

Theorem 9.1 (Geodesic convexity vs total convexity). *Assume that* $\dim \mathsf{X} \ge 2$, $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ *is a proper l.s.c. geodesically* $(-\lambda)$*-convex functional such that* $\mathrm{D}(\phi)$ *contains a* $\mathfrak{N}$*-core* C *(see Definition 8.1) which is* dense in energy, *meaning that for every* $\mu \in \mathrm{D}(\phi)$ *there exists* $(\mu_n)_n \subset \mathrm{C}$ *such that*

$$\mu_n \to \mu \quad \text{and} \quad \phi(\mu_n) \to \phi(\mu).$$

Then ϕ is totally $(-\lambda)$-convex (cf. Section 5).

Proof. Notice that ϕ is geodesically (resp. totally) $(-\lambda)$-convex if and only if $\phi_\lambda := \phi + \frac{\lambda}{2}\mathsf{m}_2^2(\cdot)$ is geodesically (resp. totally) convex. Moreover the assumptions of the present Theorem hold for ϕ if and only if they hold for ϕ_λ. We can thus prove the Theorem only in case $\lambda = 0$. We proceed in a few steps, keeping the notation of Section 8.1. First of all, we introduce a standard

Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$ as in Definition B.1 and let $\mathcal{X} := L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$. We lift ϕ to the l.s.c. functional $\psi : \mathcal{X} \to (-\infty, +\infty]$ defined as

$$\psi(X) := \phi(\iota(X)) \quad \text{for every } X \in \mathcal{X}. \tag{9.1}$$

(1) *The restriction of ψ to $\mathcal{C}_N$ is continuous and locally convex.*
By construction the function ψ is finite and lower semicontinuous in $\mathcal{C}_N$. It is also clear, recalling Lemma 6.1, that for every $X \in \mathcal{C}_N$ there is an open ball $\mathcal{B}$ of $\mathcal{X}_N$ and centered at X such that $\mathcal{B} \subset \mathcal{C}_N$ and the restriction of ψ to $\mathcal{B}$ is convex. Since $\mathcal{B}$ is open, it follows that ψ is locally convex and continuous in $\mathcal{C}_N$: in particular it is convex along each segment contained in $\mathcal{C}_N$.

(2) *For every $X_0, X_1 \in \mathcal{C}_N$ we have*

$$\psi((1-t)X_0 + tX_1) \le (1-t)\psi(X_0) + t\psi(X_1). \tag{9.2}$$

Let $X_0, X_1 \in \mathcal{C}_N$; setting $A := \mathrm{supp}(\iota(X_0))$ and $B := \mathrm{supp}(\iota(X_1))$ we can apply Proposition 6.4 and use the fact that $\mathcal{C}_N$ is relatively open to find $X_1' \in \mathcal{C}_N$ such that $X_1(s) := (1-s)X_1 + sX_1' \in \mathcal{C}_N$ for every $s \in [0,1]$ and $X_{s,t} := (1-t)X_0 + tX_1(s)$ belongs to $\mathcal{O}_N$ for every $t \in [0,1]$ and $s \in (0,1]$. Since C_N is convex along collisionless couplings, we deduce that $X_{s,t} \in \mathcal{C}_N$ for every $s,t \in (0,1)$ and $\psi(X_{s,t}) \le (1-t)\psi(X_0) + t\psi(X_1(s))$. Passing to the limit as $s \downarrow 0$, using the the lower semicontinuity of ψ and its continuity in $\mathcal{C}_N$ we deduce (9.2).

(3) *Let $K \in \mathbb{N}$, $X_1, X_2, \cdots X_K \in \mathcal{C}_N$ and $\beta_1, \cdots, \beta_K \ge 0$ with $\sum_{k=1}^K \beta_k = 1$. For every $\varepsilon > 0$ there exist $X_k' \in \mathcal{C}_N$ with $|X_k - X_k'| < \varepsilon$, $k = 1, \cdots, K$, such that $\sum_{k=1}^K \beta_k X_k' \in \mathcal{C}_N$.*
It is sufficient to observe that the map $S_K : \mathcal{X}^K \to \mathcal{X}$, $S_K(X_1, \cdots, X_K) := \sum_{k=1}^K \beta_k X_k$ is linear, continuous, and surjective, in particular it is an open map. If $X_1, X_2, \cdots X_K \in \mathcal{C}_N$ and $\mathcal{B}_\varepsilon$ is an open ball of radious ε around the corresponding vector in $\mathcal{X}^K$ and contained in $(\mathcal{C}_N)^K$, $S_K\big(\mathcal{B}_\varepsilon\big)$ is open in $\mathcal{D}_N = \mathrm{co}(\mathcal{C}_N)$ so that its intersection with the open and dense subset $\mathcal{C}_N$ is not empty.

(4) *For every $K \in \mathbb{N}$, $X_1, X_2, \cdots X_K \in \mathcal{C}_N$ and $\alpha_1, \cdots, \alpha_K \ge 0$ with $\sum_{k=1}^K \alpha_k = 1$ we have*

$$\psi\Big(\sum_{k=1}^K \alpha_k X_k\Big) \le \sum_{k=1}^K \alpha_k \psi(X_k). \tag{9.3}$$

We argue by induction on the number K. By claim (2) the statement is true if $K = 2$. Let us assume that it is true for $K \in \mathbb{N}$ and let us consider $X_k \in \mathcal{C}_N$, $1 \le k \le K+1$ and corresponding coefficients α_k. It is not restrictive to assume $0 < \alpha_{K+1} < 1$ and we set $\beta_k := \alpha_k/(1-\alpha_{K+1})$, $1 \le k \le K$, so that $\beta_k \ge 0$ and $\sum_{k=1}^K \beta_k = 1$.
We can use the previous claim and for every $\varepsilon > 0$ we can find $X_k'(\varepsilon) \in \mathcal{C}_N$ with $|X_k'(\varepsilon) - X_k| < \varepsilon$ such that $X'(\varepsilon) := \sum_{k=1}^K \beta_k X_k'(\varepsilon) \in \mathcal{C}_N$.
Using claim (2) we get

$$\psi\big((1-\alpha_{K+1})X'(\varepsilon) + \alpha_{K+1}X_{K+1}\big) \le (1-\alpha_{K+1})\psi\big(X'(\varepsilon)\big) + \alpha_{K+1}\psi\big(X_{K+1}\big).$$

Using the induction step we also get

$$(1-\alpha_{K+1})\psi\big(X'(\varepsilon)\big) \le \sum_{k=1}^K \alpha_k \psi\big(X_k'(\varepsilon)\big).$$

Combining the two inequalities and passing to the limit as $\varepsilon \downarrow 0$ using the lower semicontinuity of ψ and its continuity in $\mathcal{C}_N$ we conclude.

(5) *ψ is convex in $\overline{\mathcal{D}_N}$.*
Let us consider the convex envelope of the restriction of ψ to $\mathcal{D}_N = \mathrm{co}(\mathcal{C}_N)$ defined by

$$\psi_N(X) := \inf\Big\{\sum_{k=1}^K \alpha_k \psi(X_k) : X_k \in \mathcal{C}_N,\ \alpha_k \ge 0,\ \sum_{k=1}^K \alpha_k = 1,\ \sum_{k=1}^K \alpha_k X_k = X,\ K \in \mathbb{N}\Big\}, \quad X \in \mathcal{D}_N.$$

By the previous claim $\psi(X) \le \psi_N(X)$ for every $X \in \mathcal{D}_N$. We then consider the lower semicontinuous envelope $\bar\psi_N : \overline{\mathcal{D}_N} \to (-\infty, +\infty]$ of ψ_N defined by

$$\bar\psi_N(X) := \inf\Big\{ \liminf_{n\to\infty} \psi_N(X_n) : X_n \in \mathcal{D}_N,\ X_n \to X \quad \text{as } n \to \infty \Big\}, \quad X \in \overline{\mathcal{D}_N}.$$

Since ψ is lower semicontinuous and ψ_N is continuous in $\mathcal{C}_N$, we have

$$\psi(X) \le \bar\psi_N(X) \quad \text{for every } X \in \overline{\mathcal{D}_N}, \quad \bar\psi_N(X) = \psi_N(X) = \psi(X) \quad \text{if } X \in \mathcal{C}_N. \tag{9.4}$$

We want to show that $\psi \equiv \bar\psi_N$ in $\overline{\mathcal{D}_N}$. Let us suppose that $X \in \overline{\mathcal{D}_N}$, with $\psi(X) < \infty$. We take $Y \in \mathcal{C}_N$, so that $X_t := (1-t)X + tY \in \mathcal{D}_N$ for every $t \in (0,1]$ (since $\overline{\mathcal{D}_N}$ is convex and its relative interior coincides with $\mathcal{D}_N$ by Lemma 8.12) and $X_t \in \mathcal{C}_N$ with possibly finite exceptions. Therefore, possibly replacing Y with X_{t_0} for a sufficiently small $t_0 > 0$, it is not restrictive to assume that $X_t \in \mathcal{C}_N$ for every $t \in (0,1]$ and $\iota^2(X,Y)$ is the unique optimal coupling between its marginals (see Lemma 6.2) , so that ψ is convex along $(X_t)_{t\in[0,1]}$ being ϕ geodesically convex. We deduce that

$$\bar\psi_N(X_t) = \psi(X_t) \le (1-t)\psi(X) + t\psi(Y) \quad \text{for every } t \in (0,1],$$

so that $\bar\psi_N(X) \le \liminf_{t\downarrow 0} \bar\psi_N(X_t) \le \psi(X)$.

(6) *ψ is convex.*

Let $X, Y \in \mathrm{D}(\psi)$, and let $\mu = \iota(X), \nu = \iota(Y) \in \mathcal{P}_2(\mathsf{X})$. We thus have that $\mu, \nu \in \mathrm{D}(\phi) \subset \overline{\mathrm{C}}$. By density, we can find sequences $(\mu_n)_n, (\nu_n) \subset \mathrm{C}$ such that $W_2(\mu_n, \mu) \to 0$, $W_2(\nu_n, \nu) \to 0$, $\phi(\mu_n) \to \phi(\mu)$ and $\phi(\nu_n) \to \phi(\nu)$ as $n \to +\infty$. By the last part of Theorem B.5, we can find sequences $(X_n)_n, (Y_n)_n \subset \mathcal{C}_\infty$ such that $\iota_{X_n} = \mu_n$, $\iota_{Y_n} = \nu_n$, $X_n \to X$ and $Y_n \to Y$. Since $X_n \in \mathcal{C}_{M(n)}$, $Y_n \in \mathcal{C}_{N(n)}$ for some $M(n), N(n) \in \mathfrak{N}$ and $\mathfrak{N}$ is a directed set, we can find $P(n) \in \mathfrak{N}$ such that $M(n) \mid P(n)$, $N(n) \mid P(n)$; so that $X_n, Y_n \in \overline{\mathcal{D}_{P(n)}}$. By claim (5), we we have that

$$\psi((1-t)X_n + tY_n) \le (1-t)\psi(X_n) + t\psi(Y_n), \quad \text{for any } n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$ and using the lower semicontinuity of ψ yield the sought convexity. □

Remark 9.2 (Geodesic convexity implies total convexity for continuous functionals). Let $\phi : \mathcal{P}_2(\mathsf{X}) \to \mathbb{R}$ be a lower semicontinuous and geodesically $(-\lambda)$-convex functional which is approximable by discrete measures, i.e. for every $\mu \in \mathcal{P}_2(\mathsf{X})$ there exists a sequence $\mu_n \in \mathcal{P}_{\#\mathbb{N}}(\mathsf{X})$ converging to μ such that $\phi(\mu_n) \to \phi(\mu)$ (e.g. ϕ is continuous). Then ϕ satisfies the assumptions of Theorem 9.1 with $\mathrm{C} = \mathcal{P}_{\#\mathbb{N}}(\mathsf{X})$. This in particular gives that such kind of functionals are totally $(-\lambda)$-convex and locally Lipschitz.

As a consequence, we notice that non totally $(-\lambda)$-convex functionals cannot be approximated in the Mosco sense by everywhere finite, continuous and geodesically $(-\lambda)$-convex functionals defined on $\mathcal{P}_2(\mathsf{X})$ (this is because total $(-\lambda)$-convexity is preserved by the Mosco limit).

As previously mentioned, thanks to Theorem 9.1 we are allowed to apply all the results obtained in Section 5 to the totally $(-\lambda)$-convex functional ϕ. In particular, we get existence and uniqueness of the λ-EVI solution for the MPVF $\mathbf{F} := -\boldsymbol{\partial}\phi$ starting from $\mu_0 \in \overline{\mathrm{D}(\phi)}$ and its Lagrangian characterization as the law of the semigroup generated by $-\partial\psi$, where ψ is defined as in (9.1). We conclude the section by showing that the total subdifferential $-\boldsymbol{\partial}_{\mathrm{t}}\phi := \iota^2(-\partial\psi)$ coincides with the operator $\hat{\mathbf{F}}$ obtained by the $\mathfrak{N}$-core construction of Theorem 8.4.

Proposition 9.3. *Let us suppose that* $\dim \mathsf{X} \ge 2$, $\phi : \mathcal{P}_2(\mathsf{X}) \to (-\infty, +\infty]$ *is a proper, l.s.c. geodesically $(-\lambda)$-convex functional such that* $\mathrm{D}(\boldsymbol{\partial}\phi)$ *contains a $\mathfrak{N}$-core* C *which is dense in energy in the sense that for every $\mu \in \mathrm{D}(\phi)$ there exists $(\mu_n)_n \subset \mathrm{C}$ s.t.*

$$\mu_n \to \mu, \quad \phi(\mu_n) \to \phi(\mu).$$

The maximal totally λ-dissipative MPVF $\hat{\mathbf{F}}$*, obtained by Theorem 8.4 starting from the minimal selection* $-\boldsymbol{\partial}^\circ\phi$ *restricted to* C*, coincides with* $-\boldsymbol{\partial}_{\rm t}\phi$ *defined as in Section 5. Equivalently, if* $\psi := \phi\circ\iota$ *and* $\hat{\boldsymbol{F}}$ *is the Lagrangian representation of* $\hat{\mathbf{F}}$*, then*

$$\hat{\boldsymbol{F}} = -\partial\psi.$$

Proof. By Theorem 9.1, we have that ϕ is totally $(-\lambda)$-convex so that we can apply the results of Section 5. By Propositions 5.1 and 5.2 we know that $\boldsymbol{\partial}^\circ\phi$ coincides with $\boldsymbol{\partial}^\circ_{\rm t}\phi$ and $\boldsymbol{\partial}^\circ_{\rm t}\phi$ is totally λ-dissipative.
Theorem 8.6 shows that $\hat{\mathbf{F}}$ provides the unique maximal totally λ-dissipative extension of the restriction of $\boldsymbol{\partial}^\circ_{\rm t}\phi$ to C with domain included in $\overline{\rm C}$. Therefore, $\hat{\mathbf{F}}$ must coincide with $\boldsymbol{\partial}_{\rm t}\phi$, being $\boldsymbol{\partial}_{\rm t}\phi$ maximal totally λ-dissipative as well (cf. Proposition 5.1) and observing that by Proposition 5.2(3) we have $\mathrm{D}(\boldsymbol{\partial}_{\rm t}\phi) = \mathrm{D}(\boldsymbol{\partial}\phi) \subset \overline{\rm C}$. □

Appendix A. Dissipative operators in Hilbert spaces and extensions

In this section, we recall useful definitions, properties and results on λ-dissipative operators in Hilbert spaces used in Sections 3 and 8, with $\lambda \in \mathbb{R}$. Our main reference is [Bré73].
Let $\mathcal{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $\langle\cdot,\cdot\rangle$. Given $E \subset \mathcal{H}$, we denote by $\mathrm{co}(E)$ the convex hull of E and by $\overline{\mathrm{co}}\,(E)$ its closure. Given an operator $\boldsymbol{B} \subset \mathcal{H}\times\mathcal{H}$ (which we identify with its graph) we define its sections $\boldsymbol{B}x \equiv \boldsymbol{B}(x) := \{v \in \mathcal{H} : (x,v) \in \boldsymbol{B}\}$ and its domain $\mathrm{D}(\boldsymbol{B}) := \{x \in \mathcal{H} : \boldsymbol{B}x \neq \emptyset\}$. An operator $\boldsymbol{B} \subset \mathcal{H}\times\mathcal{H}$ is λ-dissipative ($\lambda\in\mathbb{R}$) if

$$\langle v-w, x-y\rangle \le \lambda|x-y|^2 \quad \text{for every } (x,v),\ (y,w) \in \boldsymbol{B}. \tag{A.1}$$

A λ-dissipative operator $\boldsymbol{B}$ is maximal if it is maximal w.r.t. inclusion in the class of λ-dissipative operators or, equivalently, if [Bré73, Chap. II, Def. 2.2]

$$(x,v) \in \mathcal{H}\times\mathcal{H}, \quad \langle v-w, x-y\rangle \le \lambda|x-y|^2 \quad \text{for every } (y,w) \in \boldsymbol{B} \quad \Rightarrow \quad (x,v) \in \boldsymbol{B}. \tag{A.2}$$

Remark A.1 (Dissipativity, monotonicity). Let $\boldsymbol{B} \subset \mathcal{H}\times\mathcal{H}$; we define $-\boldsymbol{B} := \{(x,-v) : (x,v) \in \boldsymbol{B}\}$ and we say that $\boldsymbol{B}$ is λ-monotone if $-\boldsymbol{B}$ is $(-\lambda)$-dissipative. It is easy to check that $\boldsymbol{B}$ is λ-dissipative if and only if $\boldsymbol{B}^\lambda := \boldsymbol{B} - \lambda\boldsymbol{i}_{\mathcal{H}}$ is 0-dissipative (or simply, dissipative) if and only if $-\boldsymbol{B}^\lambda$ is 0-monotone (or simply, monotone). The same holds for maximal λ-dissipativity, maximal dissipativity and maximal monotonicity (with analogous definition). Observe also that $\mathrm{D}(\boldsymbol{B}) = \mathrm{D}(\boldsymbol{B}^\lambda) = \mathrm{D}(-\boldsymbol{B}^\lambda)$.

We list in the following theorems a few well known properties of λ-dissipative operators that have been extensively used in the previous sections. Since these results are more commonly known for $\lambda = 0$ (cf. [Bré73]), we prefer to state them here in the general case. For this reason, in the proofs, we point out only the changes that have to be made compared to the case $\lambda = 0$. Recall that $\lambda^+ := \lambda \vee 0$ and we set $1/\lambda^+ = +\infty$ if $\lambda^+ = 0$.

Theorem A.2. *Let* $\boldsymbol{B} \subset \mathcal{H}\times\mathcal{H}$ *be a* λ*-dissipative operator. Then:*

(1) $\boldsymbol{B}$ *is maximal if and only if the resolvent operator* $\boldsymbol{J}_\tau := (\boldsymbol{i}_{\mathcal{H}} - \tau\boldsymbol{B})^{-1}$ *is a* $(1-\lambda\tau)^{-1}$*-Lipschitz continuous map defined on the whole* $\mathcal{H}$ *for every* $0 < \tau < 1/\lambda^+$*;*

(2) there exists a maximal extension $\hat{\boldsymbol{B}}$ *of* $\boldsymbol{B}$ *(meaning that* $\boldsymbol{B} \subset \hat{\boldsymbol{B}}$ *and* $\hat{\boldsymbol{B}}$ *is maximal* λ*-dissipative) whose domain is included in* $\overline{\mathrm{co}}\,(\mathrm{D}(\boldsymbol{B}))$*.*

Proof. (1) We can use Remark A.1 and apply [Bré73, Proposition 2.2] to $-\boldsymbol{B}^\lambda$ and then obtain that $\boldsymbol{B}$ is maximal λ-dissipative if and only if $((1+\lambda\vartheta)\boldsymbol{i}_{\mathcal{H}} - \vartheta\boldsymbol{B})^{-1}$ is a contraction on $\mathcal{H}$ for every $\vartheta > 0$. Since $x \mapsto x/(1-\lambda x)$ is a bijection between $(0, 1/\lambda^+)$ and $(0,+\infty)$, this is equivalent to say that $((1-\lambda\tau)^{-1}(\boldsymbol{i}_{\mathcal{H}} - \tau\boldsymbol{B}))^{-1}$ is a contraction on $\mathcal{H}$ for every $0 < \tau < 1/\lambda^+$ which is to say that $\boldsymbol{J}_\tau$ is a $(1-\lambda\tau)^{-1}$-Lipschitz map defined on the whole $\mathcal{H}$.
(2) This follows immediately from Remark A.1 and [Bré73, Corollary 2.1]. □

Theorem A.3. *Let $\boldsymbol{B}$ be a maximal λ-dissipative operator. Then:*

(1) $\boldsymbol{B}$ is closed in the strong-weak (or the weak-strong) topology in $\mathcal{H}\times\mathcal{H}$;

(2) for every $x\in \mathrm{D}(\boldsymbol{B})$, the section $\boldsymbol{B}x$ is closed and convex so that it contains a unique element of minimal norm denoted by $\boldsymbol{B}^\circ x$;

(3) if $\operatorname{int}(\operatorname{co}(\mathrm{D}(\boldsymbol{B})))\neq\emptyset$, then $\operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ is convex, $\operatorname{int}(\mathrm{D}(\boldsymbol{B}))=\operatorname{int}\left(\overline{\mathrm{D}(\boldsymbol{B})}\right)\neq\emptyset$ and $\boldsymbol{B}$ is locally bounded in the interior of its domain;

(4) $\overline{\mathrm{D}(\boldsymbol{B})}$ is convex and for every $x\in\overline{\mathrm{D}(\boldsymbol{B})}$, $\boldsymbol{J}_\tau x\to x$ as $\tau\downarrow 0$;

(5) for every $0<\tau<1/\lambda^+$, the Moreau-Yosida approximation of $\boldsymbol{B}$, $\boldsymbol{B}_\tau:=\frac{\boldsymbol{J}_\tau-\boldsymbol{i}_\mathcal{H}}{\tau}$, is maximal $\frac{\lambda}{1-\lambda\tau}$-dissipative and $\frac{2-\lambda\tau}{\tau(1-\lambda\tau)}$-Lipschitz continuous. Moreover, for every $x\in\mathrm{D}(\boldsymbol{B})$,

$$(1-\lambda\tau)|\boldsymbol{B}_\tau x|\uparrow|\boldsymbol{B}^\circ x|,\quad \text{as } \tau\downarrow 0,$$
$$\boldsymbol{B}_\tau x\to\boldsymbol{B}^\circ x,\quad \text{as } \tau\downarrow 0,$$
$$|\boldsymbol{B}_\tau x-\boldsymbol{B}^\circ x|^2\le|\boldsymbol{B}^\circ x|^2-(1-2\lambda\tau)|\boldsymbol{B}_\tau x|^2,\quad \text{for } 0<\tau<1/\lambda^+.$$

If $x\notin\mathrm{D}(\boldsymbol{B})$, then $|\boldsymbol{B}_\tau x|\to+\infty$. Finally, $\boldsymbol{B}_\tau\to\boldsymbol{B}$ in the graph sense:

$$\text{for every } (x,v)\in\boldsymbol{B} \text{ there exists } (x_\tau)_{\tau>0}\subset\mathcal{H} \text{ such that } x_\tau\to x,\ \boldsymbol{B}_\tau x_\tau\to v,\ \text{as } \tau\downarrow 0.$$

(6) $\boldsymbol{B}^\circ$ is a principal selection *of $\boldsymbol{B}$ i.e.*

$$(x,v)\in\overline{\mathrm{D}(\boldsymbol{B})}\times\mathcal{H},\quad \langle v-\boldsymbol{B}^\circ y,x-y\rangle\le\lambda|x-y|^2\quad\text{for every } y\in\mathrm{D}(\boldsymbol{B})\quad\Rightarrow\quad(x,v)\in\boldsymbol{B}. \tag{A.3}$$

Proof. (1) and (2) follow immediately from (A.2).
(3) follows immediately by Remark A.1 and [Bré73, Proposition 2.9].
(4) follows by Remark A.1 and [Bré73, Theorem 2.2] observing that

$$\lim_{\tau\downarrow 0}\boldsymbol{J}_\tau x=\lim_{\vartheta\downarrow 0}(1+\lambda\vartheta)(\boldsymbol{i}_\mathcal{H}+\vartheta(-\boldsymbol{B}^\lambda))^{-1}x=x.$$

(5) The Lipschitz constant of $\boldsymbol{B}_\tau$ can be estimated by $\frac{1}{\tau}(L+1)$, where L is the Lipschitz constant of $\boldsymbol{J}_\tau$, so that the value of the constant follows by Theorem A.2(1). The fact that $\boldsymbol{B}_\tau$ is $\lambda/(1-\lambda\tau)$ dissipative is a consequence of the inequality

$$\langle\boldsymbol{B}_\tau x-\boldsymbol{B}_\tau y,x-y\rangle=\frac{1}{\tau}\langle\boldsymbol{J}_\tau x-\boldsymbol{J}_\tau y,x-y\rangle-\frac{1}{\tau}|x-y|^2\le\frac{\lambda}{1-\lambda\tau}|x-y|^2,$$

where we used the Lipschitz continuity of $\boldsymbol{J}_\tau$. Maximality of $\boldsymbol{B}_\tau$ follows by Remark A.1 and [Bré73, Proposition 2.6]. The fact that $(1-\lambda\tau)|\boldsymbol{B}_\tau x|$ is increasing and bounded from above by $|\boldsymbol{B}^\circ x|$ follows precisely as in the proof of [Bré73, Proposition 2.6]: exploiting the dissipativity inequality

$$\langle\boldsymbol{B}^\circ x-\boldsymbol{B}_\tau x,x-\boldsymbol{J}_\tau x\rangle\le\lambda|x-\boldsymbol{J}_\tau x|^2$$

one gets that $|\boldsymbol{B}_\tau x|^2(1-\lambda\tau)\le\langle\boldsymbol{B}^\circ x,\boldsymbol{B}_\tau x\rangle$ for every $x\in\mathrm{D}(\boldsymbol{B})$. Substituting to $\boldsymbol{B}$, in the same inequality, the $\lambda/(1-\lambda\eta)$-dissipative operator $\boldsymbol{B}_\eta$, we get that

$$|\boldsymbol{B}_{\eta+\tau}x|^2(1-\lambda(\tau+\eta))\le(1-\lambda\eta)\langle\boldsymbol{B}_\eta x,\boldsymbol{B}_{\eta+\tau}x\rangle\quad\text{for every } x\in\mathcal{H} \text{ and every } 0<\eta,\tau<1/\lambda^+.$$

This shows that the quantity $(1-\lambda\tau)|\boldsymbol{B}_\tau x|$ is nondecreasing as $\tau\downarrow 0$ for every $x\in\mathcal{H}$. This means in particular that there exists the limit $\ell:=\lim_{\tau\downarrow 0}|\boldsymbol{B}_\tau x|\in[0,+\infty]$. The above estimate also gives that

$$|\boldsymbol{B}_{\eta+\tau}x-\boldsymbol{B}_\eta x|^2\le|\boldsymbol{B}_\eta x|^2-\frac{1-\lambda(\eta+2\tau)}{1-\lambda\eta}|\boldsymbol{B}_{\eta+\tau}x|^2\quad\text{for every } x\in\mathcal{H}, \tag{A.4}$$

so that $(\boldsymbol{B}_\tau x)_\tau$ is Cauchy whenever it is bounded. Thus, if $x\in\mathrm{D}(\boldsymbol{B})$, then $(1-\lambda\tau)|\boldsymbol{B}_\tau x|\le|\boldsymbol{B}^\circ x|$ so that $\boldsymbol{B}_\tau x\to v$ for some $v\in\mathcal{H}$. By (1), $(x,v)\in\boldsymbol{B}$ and $|v|\le|\boldsymbol{B}^\circ x|$ which implies that $v=\boldsymbol{B}^\circ x$. On the other hand, if $x\notin\mathrm{D}(\boldsymbol{B})$, we have that $|\boldsymbol{B}_\tau x|\to+\infty$: indeed, if by contradiction $|\boldsymbol{B}_\tau x|$ is bounded, then we have shown that $\boldsymbol{B}_\tau x$ must converge to some $v\in\mathcal{H}$ so that we also have $\boldsymbol{J}_\tau x=\tau\boldsymbol{B}_\tau x+x\to x$. Since $(\boldsymbol{J}_\tau x,\boldsymbol{B}_\tau x)\in\boldsymbol{B}$ and $(\boldsymbol{J}_\tau x,\boldsymbol{B}_\tau x)\to(x,v)$, by (1) we deduce that

$(x,v) \in \boldsymbol{B}$, a contradiction. Observe that passing to the limit as $\eta \downarrow 0$ in (A.4), we get that $|\boldsymbol{B}_\tau x - \boldsymbol{B}^\circ x|^2 \le |\boldsymbol{B}^\circ x|^2 - (1-2\lambda\tau)|\boldsymbol{B}_\tau x|^2$. To conclude the proof of (5) we only need to show the graph convergence of $\boldsymbol{B}_\tau$ to $\boldsymbol{B}$. Let $(x,v) \in \boldsymbol{B}$ and let us define $x_\tau := x - \tau v$. Then $x_\tau \to x$ and $\boldsymbol{J}_\tau x_\tau = x$. Then $\boldsymbol{B}_\tau x_\tau = (x - x_\tau)/\tau = v$.
(6) Follows exactly as in [Bré73, Proposition 2.7]: performing similar computations, we get

$$\frac{1}{2}\langle y_1 - y_2, x_1 - x_2\rangle \le -\langle y_1 + y_2, x - \boldsymbol{J}_\tau x\rangle + \lambda(|\boldsymbol{J}_\tau x - x_1|^2 + |\boldsymbol{J}_\tau x - x_2|^2)$$

for every $(x_1,y_1),(x_2,y_2) \in M$, where

$$\boldsymbol{M} = \{(y,w) \in \overline{\mathrm{D}(\boldsymbol{B})} \times \mathcal{H} : \langle \boldsymbol{B}^\circ x - w, x - y\rangle \le \lambda|x-y|^2 \quad \text{for every } x \in \mathrm{D}(\boldsymbol{B})\},$$

and $x := (x_1+x_2)/2$. Passing to the limit as $\tau \downarrow 0$ we obtain that $\boldsymbol{M}$ is λ-dissipative so that, since $\boldsymbol{B} \subset \boldsymbol{M}$, we get that $\boldsymbol{M} = \boldsymbol{B}$. □

For the next result we recall that a proper functional $\psi : \mathcal{H} \to (-\infty, +\infty]$ is said to be λ-convex if the map $x \mapsto \psi(x) - \frac{\lambda}{2}|x|^2$ is convex. Its Fréchet subdifferential $\partial\psi$ is characterized by

$$(x,v) \in \partial\psi \quad \Leftrightarrow \quad x \in \mathrm{D}(\psi) \text{ and } \psi(y) - \psi(x) \ge \langle v, y-x\rangle + \frac{\lambda}{2}|x-y|^2 \quad \text{for every } y \in \mathcal{H}.$$

Corollary A.4. *Let $\psi : \mathcal{H} \to (-\infty,+\infty]$ be a proper, lower semicontinuous and $(-\lambda)$-convex function. Then $-\partial\psi$ is a maximal λ-dissipative operator. Moreover, denoting by $\boldsymbol{B} := -\partial\psi$, we have that*

$$\lim_{\tau\downarrow 0} \frac{\psi(x) - \psi(\boldsymbol{J}_\tau x)}{\tau} = |\boldsymbol{B}^\circ x|^2 \quad \textit{for every } x \in \mathrm{D}(\boldsymbol{B}),$$

$$\frac{1}{2\tau}|x - \boldsymbol{J}_\tau x|^2 + \psi(\boldsymbol{J}_\tau x) < \frac{1}{2\tau}|x-y|^2 + \psi(y) \quad \textit{for every } x,y \in \mathcal{H},\ y \ne \boldsymbol{J}_\tau x.$$

Proof. Notice that $\psi_\lambda := \psi + \frac{\lambda}{2}|\cdot|^2$ is convex and that $\partial\psi_\lambda = \partial\psi + \lambda \boldsymbol{i}_{\mathcal{X}}$ so that by [Bré73, Example 2.3.4] and Remark A.1, the operator $-\partial\psi_\lambda$ is maximal dissipative and thus $-\partial\psi$ is maximal λ-dissipative. By definition of subdifferential of a $(-\lambda)$-convex function, we have that for every $0 < \tau < 1/\lambda^+$ it holds

$$\psi(x) - \psi(\boldsymbol{J}_\tau x) \ge \langle \boldsymbol{B}_\tau x, \boldsymbol{J}_\tau x - x\rangle - \frac{\lambda}{2}|\boldsymbol{J}_\tau x - x|^2 = \tau|\boldsymbol{B}_\tau x|^2 - \frac{\lambda}{2}|\boldsymbol{J}_\tau x - x|^2,$$
$$\psi(\boldsymbol{J}_\tau x) - \psi(x) \ge \langle \boldsymbol{B}^\circ x, x - \boldsymbol{J}_\tau x\rangle - \frac{\lambda}{2}|\boldsymbol{J}_\tau x - x|^2 = -\tau\langle \boldsymbol{B}^\circ x, \boldsymbol{B}_\tau x\rangle - \frac{\lambda}{2}|\boldsymbol{J}_\tau x - x|^2.$$

Dividing the first (resp. the second) inequality by $\tau > 0$ (resp. $-\tau < 0$) and passing to the lim inf (resp. to the lim sup) as $\tau \downarrow 0$, gives the desired equality thanks to Theorem A.3(5). The fact that the limit diverges outside the domain of $\boldsymbol{B}$ follows again by Theorem A.3(5) and the first inequality above. The last assertion follows simply observing that $y \mapsto \Psi(\tau,x;y) := \frac{1}{2\tau}|x-y|^2 + \psi(y)$ is proper and strictly convex, so that z is a strict minimum point for $\Psi(\tau,x;\cdot)$ if and only if $0 \in \partial\Psi(\tau,x;z)$, which is satisfied if and only if $z = \boldsymbol{J}_\tau x$. □

Theorem A.5. *Let $\boldsymbol{B}$ be a maximal λ-dissipative operator and let $x_0 \in \mathrm{D}(\boldsymbol{B})$. There exists a unique locally Lipschitz function $x : [0,+\infty) \to \mathcal{H}$, with $x(0) = x_0$, such that:*

(1) $x(t) \in \mathrm{D}(\boldsymbol{B})$ for every $t > 0$;
(2) $\dot{x}(t) \in \boldsymbol{B}x(t)$ for a.e. $t > 0$;
(3) the map $t \mapsto \boldsymbol{B}^\circ x(t)$ is right continuous, $t \mapsto x(t)$ is right differentiable at every $t \ge 0$ and its right derivative at t coincides with $\boldsymbol{B}^\circ(x(t))$ for every $t \ge 0$;
(4) the function $t \mapsto e^{-\lambda t}|\boldsymbol{B}^\circ x(t)|$ is decreasing in $[0,+\infty)$.

Moreover, if $x,y : [0,+\infty) \to \mathcal{H}$ are solutions of the differential inclusion in (2), then

$$|x(t) - y(t)| \le e^{\lambda t}|x(0) - y(0)| \quad \textit{for every } t \ge 0.$$

Proof. The proof of the last assertion is trivial. The proof of the points (1),(2),(3) and (4) is completely analogous to the one of [Bré73, Theorem 3.1] with only few differences that we point out in case $\lambda \neq 0$. In what follows, we take $0 < \tau, \eta < 1/\lambda^+$. To prove existence one starts from the approximate problems

$$\dot{x}_\tau - \boldsymbol{B}_\tau x_\tau = 0$$

which have unique smooth solutions thanks to e.g. [Bré73, Theorem 1.6] together with the estimate

$$|\boldsymbol{B}_\tau x_\tau(t)| = |\dot{x}_\tau(t)| \le e^{\frac{\lambda t}{1-\lambda\tau}} |\boldsymbol{B}_\tau x_0| \le \frac{e^{\frac{\lambda t}{1-\lambda\tau}}}{1-\lambda\tau} |\boldsymbol{B}^\circ x_0| \quad \text{for every } t \ge 0, \tag{A.5}$$

still provided by [Bré73, Theorem 1.6] and Theorem A.3(5). Performing the same computations of the proof of [Bré73, Theorem 3.1], using λ-dissipativity instead of monotonicity, one obtains

$$|x_\tau(t) - x_\eta(t)| \le C(\lambda, t) |\boldsymbol{B}^\circ x_0| \sqrt{\tau + \eta} \quad \text{for every } t \ge 0,$$

where $C(\lambda, t)$ is a positive constant that depends in a continuous way only on λ and t. This proves that x_τ converges locally uniformly to x on $[0, +\infty)$ with the estimate

$$|x_\tau(t) - x(t)| \le C(\lambda, t) |\boldsymbol{B}^\circ x_0| \sqrt{\tau} \quad \text{for every } t \ge 0. \tag{A.6}$$

Since

$$|\boldsymbol{J}_\tau x_\tau - x_\tau| = \tau |\boldsymbol{B}_\tau x_\tau| \le \tau \frac{e^{\frac{\lambda t}{1-\lambda\tau}}}{1-\lambda\tau} |\boldsymbol{B}^\circ x_0|,$$

we also get that $\boldsymbol{J}_\tau x_\tau$ converges to x locally uniformly in $[0, +\infty)$ and this, together with the estimate (A.5) and Theorem A.3(1), shows that $x(t) \in \mathrm{D}(\boldsymbol{B})$ and $|\boldsymbol{B}^\circ x(t)| \le e^{\lambda t} |\boldsymbol{B}^\circ x_0|$ for every $t \ge 0$; in particular this proves (1). Since $|\dot{x}_\tau|$ is uniformly bounded on every interval $[0, T]$ by (A.5), it converges weakly* in $L^\infty([0, T]; \mathcal{H})$ (and thus also weakly in $L^2([0, T]; \mathcal{H})$) to a function $v \in L^\infty([0, T]; \mathcal{H})$ which turns out to be the almost everywhere derivative of x in $[0, T]$ (cf. [Bré73, Appendix]) so that, applying Theorem A.3(1) to the extension of $\boldsymbol{B}$ to $L^2([0, T]; \mathcal{H})$ (see [Bré73, Examples 2.1.3, 2.3.3] and Remark A.1), we obtain (2) and also the inequality

$$|\dot{x}(t)| \le e^{\lambda t} |\boldsymbol{B}^\circ x_0| \quad \text{for a.e. } t > 0. \tag{A.7}$$

Observing now that, for every $t_0 \ge 0$, $t \mapsto x(t + t_0)$ is a solution of (2) with initial datum $x(t_0)$, we get that $|\boldsymbol{B}^\circ x(t + t_0)| \le e^{\lambda t} |\boldsymbol{B}^\circ x(t_0)|$ which proves (4). It remains only to prove (3). The right continuity of $t \mapsto |\boldsymbol{B}^\circ x(t)|$ follows precisely as in [Bré73, Theorem 3.1]: it is enough to prove it at $t = 0$; if $0 < t_n < 1$ is such that $t_n \downarrow 0$, then $|\boldsymbol{B}^\circ(x(t_n))| \le e^{\lambda_+} |\boldsymbol{B}^\circ x_0|$ by (4), so that, up to a unrelabeled subsequence, $\boldsymbol{B}^\circ(x(t_n))$ converges weakly to some $v \in \mathcal{H}$. Since $x(t_n) \to x_0$ and thanks to Theorem A.3(1), v belongs to $\boldsymbol{B} x_0$. However $|v| \le |\boldsymbol{B}^\circ x_0|$ so that it must be $v = \boldsymbol{B}^\circ x_0$. The strong convergence follows observing that $\limsup |\boldsymbol{B}^\circ(x(t_n))| \le |v| = |\boldsymbol{B}^\circ x_0|$. Being the limit independent of the subsequence, we obtain convergence of the whole sequence. We still follow the proof of [Bré73, Theorem 3.1] to prove the right differentiability of x and the inclusion for its right derivative: for every $t_0, h > 0$ we have that

$$|x(t_0 + h) - x(t_0)| = \left| \int_{t_0}^{t_0 + h} \dot{x}(s) \, \mathrm{d}s \right| \le \frac{e^{\lambda h} - 1}{\lambda} |\boldsymbol{B}^\circ(x(t_0))|,$$

where we have applied (A.7) to $t \mapsto x(t + t_0)$. If t_0 is a point of differentiability for $x(t)$ such that $\dot{x}(t_0) \in \boldsymbol{B} x(t_0)$, dividing by h and passing to the limit as $h \downarrow 0$ in the above inequality, we get that $|\dot{x}(t_0)| \le |\boldsymbol{B}^\circ x(t_0)|$ so that $\dot{x}(t_0) = \boldsymbol{B}^\circ x(t_0)$. We can thus integrate this equality in $[t_0, t_0 + h]$ for every $t_0 \ge 0$ and every $0 < h < 1$ to obtain that

$$\lim_{h \downarrow 0} \frac{x(t_0 + h) - x(t_0)}{h} = \lim_{h \downarrow 0} \int_0^1 \boldsymbol{B}^\circ x(t_0 + sh) \, \mathrm{d}s = \boldsymbol{B}^\circ x(t_0),$$

where we used the right continuity of $t \mapsto \boldsymbol{B}^\circ(x(t))$ and the dominated convergence theorem that we can apply since $|\boldsymbol{B}^\circ x(t_0 + rh)| \le e^{\lambda_+} |\boldsymbol{B}^\circ x(t_0)|$ by (4). This concludes the proof of (3). □

Theorem A.6. *If $\boldsymbol{B}$ is maximal λ-dissipative, there exists a semigroup of Lipschitz transformations $\boldsymbol{S}_t : \overline{\mathrm{D}(\boldsymbol{B})} \to \overline{\mathrm{D}(\boldsymbol{B})}$ such that, for every $x \in \mathrm{D}(\boldsymbol{B})$, the curve $t \mapsto x(t) := \boldsymbol{S}_t x$ is the unique solution of the differential inclusion $\dot{x}(t) \in \boldsymbol{B}x(t)$, for a.e. $t > 0$, starting from x. Moreover, we have*

$$|\boldsymbol{S}_t x - \boldsymbol{S}_t y| \le e^{\lambda t}|x - y| \quad \text{for every } x, y \in \overline{\mathrm{D}(\boldsymbol{B})} \text{ and every } t \ge 0. \tag{A.8}$$

Finally, for every $x \in \overline{\mathrm{D}(\boldsymbol{B})}$ we have that

$$\boldsymbol{J}^n_{t/n} x \to \boldsymbol{S}_t x \quad \text{as } n \to +\infty$$

and for every $T \ge 0$ there exist $N(\lambda, T) \in \mathbb{N}$, $C(\lambda, T) > 0$ (with $C(0, T) = 2T$) such that

$$|\boldsymbol{J}^n_{t/n} x - \boldsymbol{S}_t x| \le C(\lambda, T)\frac{|\boldsymbol{B}^\circ x|}{\sqrt{n}} \quad \text{for every } 0 \le t \le T,\ n \ge N(\lambda, T),\ x \in \mathrm{D}(\boldsymbol{B}). \tag{A.9}$$

Proof. The first assertion follows by extending by continuity the semigroup (whose existence follows by Theorem A.5) from $\mathrm{D}(\boldsymbol{B})$ to the whole $\overline{\mathrm{D}(\boldsymbol{B})}$ (see also [Bré73, Remark 3.2]). The second assertion for $\lambda < 0$ follows immediately from [Bré73, Corollaries 4.3, 4.4] applied to $-\boldsymbol{B}$. We only prove the second assertion in case $\lambda > 0$ following the same strategy of [Bré73, Corollaries 4.3, 4.4]. We fix $x_0 \in \mathrm{D}(\boldsymbol{B})$ and we consider as in the proof of Theorem A.5 the approximated problems

$$\dot{x}_\tau(t) - \boldsymbol{B}_\tau x_\tau(t) = 0, \quad x_\tau(0) = x_0,$$

where we are assuming from now on that $0 < \tau < 1/\lambda$. By [Bré73, Theorem 1.7] we have that

$$\begin{aligned}|x_\tau(t) - \boldsymbol{J}^n_\tau x_0| &\le (1-\lambda\tau)^{-n} e^{\lambda t}|x_0 - \boldsymbol{J}_\tau x_0| \left(\left(n - \frac{t}{\tau(1-\lambda\tau)}\right)^2 + \frac{t}{\tau(1-\lambda\tau)}\right)^{1/2} \\ &\le |\boldsymbol{B}^\circ x_0|(1-\lambda\tau)^{-n-1} e^{\lambda t}\left(\left(\tau n - \frac{t}{1-\lambda\tau}\right)^2 + \frac{t\tau}{1-\lambda\tau}\right)^{1/2},\end{aligned}$$

where we have also used that $\boldsymbol{J}_\tau$ is $(1-\lambda\tau)^{-1}$-Lipschitz continuous (see Theorem A.2(1)) and Theorem A.3(5). Using this inequality together with (A.6) with $\tau = t/n$ we get that for every $T \ge 0$ we can find an integer $N(\lambda, T)$ and a positive constant $C(\lambda, T)$ such that

$$|\boldsymbol{J}_\tau x_0 - \boldsymbol{S}_t x_0| \le C(\lambda, T)\frac{|\boldsymbol{B}^\circ x_0|}{\sqrt{n}} \quad \text{for every } n \ge N(\lambda, T) \text{ and every } t \in [0, T].$$

This proves (A.9) and also the convergence of $\boldsymbol{J}^n_{t/n} x_0$ to $\boldsymbol{S}_t x_0$, whenever $x_0 \in \mathrm{D}(\boldsymbol{B})$. In case $y_0 \in \overline{\mathrm{D}(\boldsymbol{B})}$ and $x_0 \in \mathrm{D}(\boldsymbol{B})$ we can estimate

$$\begin{aligned}|\boldsymbol{J}^n_{t/n} y_0 - \boldsymbol{S}_t y_0| &\le |\boldsymbol{J}^n_{t/n} y_0 - \boldsymbol{J}^n_{t/n} x_0| + |\boldsymbol{S}_t y_0 - \boldsymbol{S}_t x_0| + |\boldsymbol{S}_t x_0 - \boldsymbol{J}^n_{t/n} x_0| \\ &\le |x_0 - y_0|\left((1-\lambda t/n)^{-n} + e^{\lambda t}\right) + |\boldsymbol{S}_t x_0 - \boldsymbol{J}^n_{t/n} x_0|,\end{aligned}$$

where we have used again Theorem A.2(1). Passing to the limit as $n \to +\infty$ gives that

$$\limsup_{n\to+\infty} |\boldsymbol{J}^n_{t/n} y_0 - \boldsymbol{S}_t y_0| \le 2e^{\lambda t}|x_0 - y_0|$$

ans passing to the inf w.r.t. $x_0 \in \mathrm{D}(\boldsymbol{B})$ gives the sought convergence. □

The following result corresponds to [Bré73, Theorem 3.3] and concerns the regularizing effect for the semigroup generated by maximal λ-dissipative operators whose domain has nonempty interior.

Theorem A.7. *Let $\boldsymbol{B}$ be a maximal λ-dissipative operator such that $\mathrm{int}\,(\mathrm{D}(\boldsymbol{B})) \neq \emptyset$ and let $x_0 \in \overline{\mathrm{D}(\boldsymbol{B})}$. Then the curve $x(t) := \boldsymbol{S}_t x_0$, $t \ge 0$ (cf. Theorem A.6) has the following properties:*

(1) x is locally absolutely continuous in $[0, +\infty)$ and locally Lipschitz in $(0, +\infty)$;

(2) $x(t) \in \mathrm{D}(\boldsymbol{B})$ *for every* $t > 0$;
(3) there exists a constant $C > 0$ *(depending solely on* x_0, λ *and* $\boldsymbol{B}$*) such that*

$$I_\lambda(t)|\dot{x}(t)| \le C \quad \text{for a.e. } t \in (0,1), \tag{A.10}$$

where

$$I_\lambda(t) := \int_0^t \mathrm{e}^{\lambda(s-t)}\,\mathrm{d}s = \begin{cases} \frac{1-\mathrm{e}^{-\lambda t}}{\lambda} & \text{if } \lambda \neq 0, \\ t & \text{if } \lambda = 0, \end{cases} \quad t \ge 0. \tag{A.11}$$

Proof. The proof closely follows the one of [Bré73, Theorem 3.3] and it is divided in several claims.

Claim 1. *For every* $y \in \operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ *there exist* $\varrho, M > 0$ *such that*

$$\varrho|v| \le \langle v, y-x\rangle + M(|x-y|+\varrho) + \lambda^+(|x-y|+\varrho)^2 \quad \text{for every } (x,v) \in \boldsymbol{B}.$$

Let $y \in \operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ and let $(x,v) \in \boldsymbol{B}$ be fixed. By Theorem A.3(3), there exist $\varrho, M > 0$ such that, for every $z \in \mathcal{H}$ with $|z| = 1$ and every $w \in \boldsymbol{B}(y-\varrho z)$, it holds $|w| \le M$. Testing the λ-dissipativity of $\boldsymbol{B}$ with $(x,v),(y-\varrho z, w) \in \boldsymbol{B}$, we get

$$\langle v-w, x-y+\varrho z\rangle \le \lambda|x-y+\varrho z|^2$$

so that

$$\begin{aligned}\varrho\langle v, z\rangle &\le \langle v, y-x\rangle + \lambda^+(|x-y|^2 + 2\varrho\langle x-y, z\rangle + \varrho^2|z|^2) + M(|x-y|+\varrho|z|) \\ &\le \langle v, y-x\rangle + M(|x-y|+\varrho) + \lambda^+(|x-y|+\varrho)^2.\end{aligned}$$

Passing to the supremum in $z \in \mathcal{H}$ with $|z| = 1$ proves the claim.

We consider, as in the proof of Theorem A.5, the approximated problems

$$\dot{x}_\tau(t) - \boldsymbol{B}_\tau x_\tau(t) = 0, \quad x_\tau(0) = x_0,$$

where we are assuming from now on that $0 < \tau < 1/\lambda^+$.

Claim 2. *For every* $T > 0$*, the curves* x_τ *and* $\boldsymbol{J}_\tau x_\tau$ *converge to* $t \mapsto \boldsymbol{S}_t x_0$ *uniformly in* $[0,T]$ *as* $\tau \downarrow 0$.

Let us first show that x_τ converges to $t \mapsto \boldsymbol{S}_t x_0$ uniformly in $[0,T]$: let us denote by $(\boldsymbol{S}_t^\tau)_{t\ge 0}$ the semigroup associated by Theorem A.6 to the maximal $\frac{\lambda}{1-\lambda\tau}$-dissipative operator $\boldsymbol{B}_\tau$ (cf. Theorem A.3(5)), so that in particular $x_\tau(t) = \boldsymbol{S}_t^\tau x_0$ for every $t \ge 0$. For every $y_0 \in \mathrm{D}(\boldsymbol{B})$ and $t \in [0,T]$, we estimate

$$\begin{aligned}|x_\tau(t) - \boldsymbol{S}_t x_0| &\le |\boldsymbol{S}_t^\tau x_0 - \boldsymbol{S}_t^\tau y_0| + |\boldsymbol{S}_t^\tau y_0 - \boldsymbol{S}_t y_0| + |\boldsymbol{S}_t y_0 - \boldsymbol{S}_t x_0| \\ &\le e^{\frac{\lambda}{1-\lambda\tau}t}|x_0-y_0| + C(\lambda,t)|\boldsymbol{B}^\circ y_0|\sqrt{\tau} + e^{\lambda t}|x_0-y_0| \\ &\le \left(e^{\frac{\lambda^+}{1-\lambda\tau}T} + e^{\lambda^+ T}\right)|x_0-y_0| + \sup_{t\in[0,T]} C(\lambda,t)|\boldsymbol{B}^\circ y_0|\sqrt{\tau},\end{aligned}$$

where we have used (A.8) for $\boldsymbol{B}$ and $\boldsymbol{B}_\tau$ and (A.6). Passing first to $\sup_{t\in[0,T]}$, then to the limit as $\tau \downarrow 0$ and finally to the infimum w.r.t. $y_0 \in \mathrm{D}(\boldsymbol{B})$, gives the sought uniform convergence of x_τ to $t \mapsto \boldsymbol{S}_t x_0$ in $[0,T]$. The argument for $\boldsymbol{J}_\tau x_\tau$ is similar: for every $t \in [0,T]$ and every $y_0 \in \mathrm{D}(\boldsymbol{B})$

we estimate

$$\begin{aligned}|\boldsymbol{J}_\tau x_\tau(t)-\boldsymbol{S}_t x_0| &\le |\boldsymbol{J}_\tau x_\tau(t)-\boldsymbol{J}_\tau\boldsymbol{S}_t x_0|+|\boldsymbol{J}_\tau\boldsymbol{S}_t x_0-\boldsymbol{J}_\tau\boldsymbol{S}_t y_0|+|\boldsymbol{J}_\tau\boldsymbol{S}_t y_0-\boldsymbol{S}_t y_0|+|\boldsymbol{S}_t y_0-\boldsymbol{S}_t x_0|\\ &\le \frac{1}{1-\lambda\tau}|x_\tau(t)-\boldsymbol{S}_t x_0|+\left(\frac{e^{\lambda t}}{1-\lambda\tau}+e^{\lambda t}\right)|x_0-y_0|+\tau|\boldsymbol{B}_\tau\boldsymbol{S}_t y_0|\\ &\le \frac{1}{1-\lambda\tau}|x_\tau(t)-\boldsymbol{S}_t x_0|+\left(\frac{e^{\lambda t}}{1-\lambda\tau}+e^{\lambda t}\right)|x_0-y_0|+\frac{\tau e^{\lambda t}}{1-\lambda\tau}|\boldsymbol{B}^\circ y_0|\\ &\le \frac{1}{1-\lambda\tau}\sup_{t\in[0,T]}|x_\tau(t)-\boldsymbol{S}_t x_0|+\left(\frac{e^{\lambda^+ T}}{1-\lambda\tau}+e^{\lambda^+ T}\right)|x_0-y_0|+\frac{\tau e^{\lambda^+ T}}{1-\lambda\tau}|\boldsymbol{B}^\circ y_0|\end{aligned}$$

where we have used the $(1-\lambda\tau)^{-1}$-Lipschitzianeity of $\boldsymbol{J}_\tau$ coming from Theorem A.2(1), (A.8) for $\boldsymbol{B}$, the definition of $\boldsymbol{B}_\tau$, Theorem A.3(5) and Theorem A.5(4) applied to $\boldsymbol{B}$ (notice that this is possible since $y_0\in \mathrm{D}(\boldsymbol{B})$). Passing first to $\sup_{t\in[0,T]}$, then to the limit as $\tau\downarrow 0$ and finally to the infimum w.r.t. $y_0\in\mathrm{D}(\boldsymbol{B})$, concludes the proof of the claim.

Claim 3. *For every $T>0$ there exists a constant $M>0$ (not depending on τ) such that $|\boldsymbol{B}_\tau x_\tau(T)|\le M$ for every $0<\tau<1/\lambda^+$.*
We fix some $y\in\operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ and we apply Claim 1 to $(x,v):=(\boldsymbol{J}_\tau x_\tau(t),\boldsymbol{B}_\tau x_\tau(t))\in\boldsymbol{B}$, with $t\in[0,T]$ and $0<\tau<1/\lambda^+$ so that

$$\varrho\,|\boldsymbol{B}_\tau x_\tau(t)|\le-\frac12\frac{\mathrm{d}}{\mathrm{d}t}|x_\tau(t)-y|^2+M\varrho+M|\boldsymbol{J}_\tau x_\tau(t)-y|+\lambda^+\left(|\boldsymbol{J}_\tau x_\tau(t)-y|+\varrho\right)^2.$$

Integrating in $[0,T]$ and using Theorem A.5(4) applied to $\boldsymbol{B}_\tau$, we get

$$\varrho\,|\boldsymbol{B}_\tau x_\tau(T)|\,I_{\frac{\lambda}{1-\lambda\tau}}(T)\le\frac12|x_0-y|^2+M\varrho T+\int_0^T\left[M|\boldsymbol{J}_\tau x_\tau(t)-y|+\lambda^+\left(|\boldsymbol{J}_\tau x_\tau(t)-y|+\varrho\right)^2\right]\mathrm{d}t.$$

By Claim 2, the right hand side of the previous inequality is uniformly bounded (w.r.t. $\tau\in(0,1/\lambda^+)$) so that we conclude the proof of the claim.

Claim 4. *Proof of points (1), (2) and (3).*
By Claim 3, we have that for every $t>0$, up to an unrelabeled subsequence, $\boldsymbol{B}_\tau x_\tau(t)\rightharpoonup v$ for some $v\in\mathcal{H}$. By Claim 2, we have that $\boldsymbol{J}_\tau x_\tau(t)\to\boldsymbol{S}_t x_0$ so that we deduce by Theorem A.3(1) that $\boldsymbol{S}_t x_0\in\mathrm{D}(\boldsymbol{B})$; this proves (2). We can then fix some $y\in\operatorname{int}(\mathrm{D}(\boldsymbol{B}))$ and apply Claim 1 to $(x,v):=(x(t),\dot x_+(t))$, $t>0$, where $\dot x_+(t)$ is the right derivative of $t\mapsto x(t)$ at t. Indeed, since $\boldsymbol{S}_t x_0=\boldsymbol{S}_{t-\delta}(\boldsymbol{S}_\delta x_0)$ and $\boldsymbol{S}_\delta x_0\in\mathrm{D}(\boldsymbol{B})$ for every $0<\delta<t$ by (2), we can apply Theorem A.5(3) to get that $(x(t),\dot x_+(t))\in\boldsymbol{B}$. We then obtain

$$\varrho|\dot x_+(t)|\le-\frac12\frac{\mathrm{d}}{\mathrm{d}t}|x(t)-y|^2+M\varrho+M|x(t)-y|+\lambda^+(|x(t)-y|+\varrho)^2.$$

Integrating the above inequality in $[s,1]$ for any $0<s<1$, we get

$$\varrho\int_s^1|\dot x_+(t)|\,\mathrm{d}t\le\frac12|x_0-y|^2+M\varrho+\int_s^1\left[M|x(t)-y|+\lambda^+(|x(t)-y|+\varrho)^2\right]\mathrm{d}t.$$

Thanks to (A.8) and Theorem A.5(4) we have that for every $t\in[s,1]$ it holds

$$|x(t)-y|\le e^{\lambda t}|x_0-y|+|\boldsymbol{S}_t y-y|\le e^{\lambda^+}(|x_0-y|+|\boldsymbol{B}^\circ y|).$$

This proves that there exists some constant $C>0$ (depending solely on x_0,λ,y,ϱ and M) such that

$$\int_s^1|\dot x_+(t)|\,\mathrm{d}t\le C\qquad\text{for every } s\in(0,1).$$

Being the constant independent on s, we conclude that x is absolutely continuous in $(0,1)$; using also Theorem A.5, this proves (1). To prove (3), it is enough to use the above estimate with Theorem A.5(3),(4). □

Proposition A.8. *Let $\boldsymbol{B}$ be a maximal λ-dissipative operator, let $\mathcal{Y} \subset \mathcal{H}$ be a closed subspace and suppose that $\mathcal{Y}$ is invariant for the resolvent of $\boldsymbol{B}$, i.e. $\boldsymbol{J}_\tau x \in \mathcal{Y}$ for every $x \in \mathcal{Y}$. Then the operator $\boldsymbol{B}_\mathcal{Y} := \boldsymbol{B} \cap (\mathcal{Y} \times \mathcal{Y})$ has the following properties:*

(i) $\boldsymbol{B}_\mathcal{Y}$ *is maximal λ-dissipative in $\mathcal{Y}$;*
(ii) the resolvent (resp. the semigroup) of $\boldsymbol{B}$ coincides with the resolvent (resp. the semigroup) of $\boldsymbol{B}_\mathcal{Y}$ when restricted to $\mathcal{Y}$.
(iii) $\mathrm{D}(\boldsymbol{B}_\mathcal{Y}) = \mathrm{D}(\boldsymbol{B}) \cap \mathcal{Y}$;
(iv) $\overline{\mathrm{D}(\boldsymbol{B}_\mathcal{Y})} = \overline{\mathrm{D}(\boldsymbol{B})} \cap \mathcal{Y}$;
(v) $(\boldsymbol{B}_\mathcal{Y})^\circ x = \boldsymbol{B}^\circ x$ *for every* $x \in \mathrm{D}(\boldsymbol{B}_\mathcal{Y})$.

Proof. It is clear that the restriction of $\boldsymbol{J}_\tau$, the resolvent of $\boldsymbol{B}$, to $\mathcal{Y}$ provides the resolvent operator for $\boldsymbol{B}_\mathcal{Y}$ and it is a $(1-\lambda\tau)^{-1}$-Lipschitz map defined on the whole $\mathcal{Y}$: by Theorem A.2(1), $\boldsymbol{B}_\mathcal{Y}$ is maximal λ-dissipative in $\mathcal{Y}$. This proves (i) and (ii), also using the exponential formula (cf. Theorem A.6). To prove (iii), it is enough to show the inclusion "$\supset$": if $x \in \mathrm{D}(\boldsymbol{B}) \cap \mathcal{Y}$, then $(\boldsymbol{J}_\tau x - x)/\tau \in \mathcal{Y}$ is bounded by Theorem A.3(5) and, by the same result together with (ii), it must be that $x \in \mathrm{D}(\boldsymbol{B}_\mathcal{Y})$. The inclusion "$\subset$" in (iv) follows by (iii), while the inclusion "$\supset$" follows simply noticing that, if $x \in \overline{\mathrm{D}(\boldsymbol{B})} \cap \mathcal{Y}$, then $\boldsymbol{J}_\tau x \to x$ by Theorem A.3(4) and $\boldsymbol{J}_\tau x \in \mathrm{D}(\boldsymbol{B}) \cap \mathcal{Y} = \mathrm{D}(\boldsymbol{B}_\mathcal{Y})$. Assertion (v) follows again by Theorem A.3(5). □

Corollary A.9. *Let $\boldsymbol{B}$ be a maximal λ-dissipative operator and suppose that $\mathcal{H}$ has finite dimension. Then the conclusions of Theorem A.7 hold.*

Proof. Up to a translation, we can assume that $0 \in \mathrm{D}(\boldsymbol{B})$. Let $\mathcal{Y}$ be the subspace generated by $\mathrm{D}(\boldsymbol{B})$. Since $\mathcal{H}$ is finite dimensional, then $\mathcal{Y}$ is closed. We can thus apply Proposition A.8 and obtain that $\boldsymbol{B}_\mathcal{Y} := \boldsymbol{B} \cap (\mathcal{Y} \times \mathcal{Y})$ is maximal λ-dissipative in $\mathcal{Y}$, has the same domain of $\boldsymbol{B}$ and its semigroup coincides with the semigroup generated by $\boldsymbol{B}$. Since $\mathcal{H}$ is finite dimensional, the relative interior of $\mathrm{co}(\mathrm{D}(\boldsymbol{B}_\mathcal{Y}))$ in $\mathcal{Y}$ is nonempty and thus we conclude by Theorem A.3(3) that the relative interior of $\mathrm{D}(\boldsymbol{B}_\mathcal{Y})$ in $\mathcal{Y}$ is nonempty, so that we can apply Theorem A.7 to $\boldsymbol{B}_\mathcal{Y}$ and obtain the conclusion of such theorem for the semigroup generated by $\boldsymbol{B}$. □

Corollary A.10. *Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be maximal λ-dissipative operators with $\overline{\mathrm{D}(\boldsymbol{B}_1)} = \overline{\mathrm{D}(\boldsymbol{B}_2)}$ and let $\boldsymbol{S}^1_t$ and $\boldsymbol{S}^2_t$ be the semigroups of Lipschitz transformations associated to $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ respectively given by Theorem A.6. If for every $x \in \overline{\mathrm{D}(\boldsymbol{B}_1)} = \overline{\mathrm{D}(\boldsymbol{B}_2)}$ there exists $\delta > 0$ such that $\boldsymbol{S}^1_t x = \boldsymbol{S}^2_t x$ for every $0 \le t < \delta$, then $\boldsymbol{B}_1 = \boldsymbol{B}_2$.*

Proof. This can be proven as in [Bré73, Theorem 4.1]: let $x \in \mathrm{D}(\boldsymbol{B}_1)$ and let $y \in \mathrm{D}(\boldsymbol{B}_2)$; by hypotesis, we can find some $\delta > 0$ such that $\boldsymbol{S}^1_t x = \boldsymbol{S}^2_t x$ and $\boldsymbol{S}^1_t y = \boldsymbol{S}^2_t y$ for every $0 \le t < \delta$. Thus, for every $0 \le t < \delta$, we have

$$\begin{aligned}\langle \frac{\boldsymbol{S}_t x - x}{t} - \frac{\boldsymbol{S}_t y - y}{t}, x - y\rangle &\le \frac{1}{t}|\boldsymbol{S}_t x - \boldsymbol{S}_t y||x-y| - \frac{1}{t}|x-y|^2 \\ &\le \frac{e^{\lambda t} - 1}{t}|x-y|^2,\end{aligned}$$

where we have used that $\boldsymbol{S}_t := \boldsymbol{S}^1_t = \boldsymbol{S}^2_t$ is $e^{\lambda t}$-Lipschitz by (A.8). Passing to the limit as $t \downarrow 0$ and using Theorem A.5(3), we get that

$$\langle \boldsymbol{B}^\circ_1 x - \boldsymbol{B}^\circ_2 y, x - y\rangle \le \lambda |x-y|^2.$$

By (A.2) we get that $\mathrm{D}(\boldsymbol{B}_1) = \mathrm{D}(\boldsymbol{B}_2)$ and thus that $\boldsymbol{B}^\circ_1 = \boldsymbol{B}^\circ_2$. By (A.3) we thus get that $\boldsymbol{B}_1 = \boldsymbol{B}_2$. □

The following proposition is a slight generalization of [Att79, Lemma 2.3] but we report its proof for the reader's convenience.

Proposition A.11. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be maximal λ-dissipative and let $\boldsymbol{G} \subset \boldsymbol{B}$ be s.t. $\mathrm{D}(\boldsymbol{G})$ is dense in $\mathrm{D}(\boldsymbol{B})$. Then for every $x \in \mathrm{int}\,(\mathrm{D}(\boldsymbol{B}))$ it holds*

$$\boldsymbol{B}x = \overline{\mathrm{co}}\left(\{v \in \mathcal{H} \mid \exists (x_n, v_n) \in \boldsymbol{G} \text{ s.t. } x_n \to x,\, v_n \rightharpoonup v\}\right). \tag{A.12}$$

Proof. Let $x \in \mathrm{int}\,(\mathrm{D}(\boldsymbol{B}))$ and let us define

$$\boldsymbol{M}x := \overline{\mathrm{co}}\left(\{v \in \mathcal{H} \mid \exists (x_n, v_n) \in \boldsymbol{G} \text{ s.t. } x_n \to x,\, v_n \rightharpoonup v\}\right).$$

If $(x_n, v_n) \in \boldsymbol{G} \subset \boldsymbol{B}$ with $x_n \to x$ and $v_n \rightharpoonup v$, by λ-dissipativity of $\boldsymbol{B}$, we have that

$$\langle v_n - w, x_n - y\rangle \le \lambda |x_n - y|^2 \quad \forall\,(y, w) \in \boldsymbol{B}.$$

Passing to the limit we get

$$\langle v - w, x - y\rangle \le \lambda |x - y|^2 \quad \forall\,(y, w) \in \boldsymbol{B},$$

so that $v \in \boldsymbol{B}x$ by (A.2). This, together with the closure and convexity of $\boldsymbol{B}x$ given by Theorem A.3(2), proves that $\boldsymbol{M}x \subset \boldsymbol{B}x$. Let us prove the other inclusion by contradiction: suppose that there is some $v \in \boldsymbol{B}x$ s.t. $v \notin \boldsymbol{M}x$. The sets $\{v\}$ and $\boldsymbol{M}x$ are disjoint, closed, convex and $\{v\}$ is also compact. By Hahn-Banach theorem we can find some $z \in \mathcal{H}$ with $|z| = 1$ s.t.

$$\langle v, z\rangle > \langle u, z\rangle \quad \forall\, u \in \boldsymbol{M}x. \tag{A.13}$$

Since $x \in \mathrm{int}\,(\mathrm{D}(\boldsymbol{B}))$, if we define $z_n := x + z/n$, we have that $z_n \in \mathrm{int}\,(\mathrm{D}(\boldsymbol{B}))$ for n sufficiently large. We can thus find $x_n \in \mathrm{D}(\boldsymbol{G})$ s.t. $|x_n - z_n| < n^{-2}$. Clearly $x_n \to x$ and it is easy to check that $(x_n - x)/|x_n - x| \to z$. Since $x_n \in \mathrm{D}(\boldsymbol{G})$, we can find $v_n \in \boldsymbol{G}x_n$. Since $\boldsymbol{B}$ is maximal, it is locally bounded (cf. Theorem A.3(3)) at x. Being $\boldsymbol{G} \subset \boldsymbol{B}$ and being $x_n \to x$, the sequence (v_n) is bounded so that, up to an unrelabeled subsequence, it converges weakly to some point $u \in \mathcal{H}$. By λ-dissipativity of $\boldsymbol{B}$ we have

$$\langle v - v_n, x - x_n\rangle \le \lambda |x - x_n|^2 \quad \forall\, n \in \mathbb{N},$$

so that, dividing by $|x_n - x|$ and passing to the limit, we obtain

$$\langle v - u, z\rangle \le 0,$$

a contradiction with (A.13) since, obviously, $u \in \boldsymbol{M}x$. □

The following proposition is an immediate consequence of [Qi83, Theorem 1] and Remark A.1.

Proposition A.12. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be λ-dissipative with open non empty convex domain. Then there exists a unique maximal λ-dissipative $\hat{\boldsymbol{B}} \supset \boldsymbol{B}$ with $\mathrm{D}(\hat{\boldsymbol{B}}) \subset \overline{\mathrm{D}(\boldsymbol{B})}$ and it is characterized by*

$$\hat{\boldsymbol{B}} = \left\{(x, v) \in \overline{\mathrm{D}(\boldsymbol{B})} \times \mathcal{H} \mid \langle v - w, x - y\rangle \le \lambda |x - y|^2 \quad \forall\,(y, w) \in \boldsymbol{B}\right\}.$$

As a consequence of Propositions A.11 and A.12 we can prove the following.

Theorem A.13. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be λ-dissipative with*

$$C := \overline{\mathrm{D}(\boldsymbol{B})} \text{ is convex}, \quad \mathrm{int}\,(\mathrm{D}(\boldsymbol{B})) \neq \emptyset.$$

Then there exists a unique maximal λ-dissipative $\hat{\boldsymbol{B}} \supset \boldsymbol{B}$ with $\mathrm{D}(\hat{\boldsymbol{B}}) \subset C$ and it is characterized by

$$\hat{\boldsymbol{B}} = \left\{(x, v) \in C \times \mathcal{H} \mid \langle v - w, x - y\rangle \le \lambda |x - y|^2 \quad \forall\,(y, w) \in \boldsymbol{B}\right\}. \tag{A.14}$$

Moreover, for every $x \in \mathrm{int}\left(\mathrm{D}(\hat{\boldsymbol{B}})\right)$ it holds

$$\hat{\boldsymbol{B}}x = \overline{\mathrm{co}}\left(\{v \in \mathcal{H} \mid \exists (x_n, v_n) \in \boldsymbol{B} \text{ s.t. } x_n \to x,\, v_n \rightharpoonup v\}\right). \tag{A.15}$$

Finally

$$\mathrm{int}\,(C) = \mathrm{int}\left(\mathrm{D}(\hat{\boldsymbol{B}})\right) \subset \mathrm{D}(\hat{\boldsymbol{B}}) \subset \overline{\mathrm{D}(\hat{\boldsymbol{B}})} = C. \tag{A.16}$$

Proof. Let $\boldsymbol{B}'$ be a λ-dissipative maximal extension of $\boldsymbol{B}$ with $\mathrm{D}(\boldsymbol{B}') \subset C$, whose existence is granted by Theorem A.2(2); by λ-dissipativity of $\boldsymbol{B}'$ and since $\boldsymbol{B} \subset \boldsymbol{B}'$, then $\boldsymbol{B}' \subset \hat{\boldsymbol{B}}$, where $\hat{\boldsymbol{B}}$ is defined as in (A.14). We need to prove the other inclusion.
Since $\mathrm{D}(\boldsymbol{B}) \subset \mathrm{D}(\boldsymbol{B}') \subset C$, we have that $\overline{\mathrm{D}(\boldsymbol{B}')} = C$. Moreover, being $\boldsymbol{B}'$ maximal λ-dissipative and being the interior of its domain nonempty, we have by Theorem A.3(3) that

$$\mathrm{int}\left(\mathrm{D}(\boldsymbol{B}')\right) \text{ is convex }, \quad \mathrm{int}\left(\mathrm{D}(\boldsymbol{B}')\right) = \mathrm{int}\left(\overline{\mathrm{D}(\boldsymbol{B}')}\right) = \mathrm{int}\,(C)\,.$$

It is then clear that $\boldsymbol{B}_0 := \boldsymbol{B}' \cap (\mathrm{int}\,(\mathrm{D}(\boldsymbol{B}')) \times \mathcal{H})$ is λ-dissipative with open and nonempty convex domain so that, by Proposition A.12, there exists a unique maximal λ-dissipative $\boldsymbol{B}'' \supset \boldsymbol{B}_0$ with $\mathrm{D}(\boldsymbol{B}'') \subset \overline{\mathrm{D}(\boldsymbol{B}_0)} = \overline{\mathrm{int}\,(\mathrm{D}(\boldsymbol{B}'))} = \overline{\mathrm{int}\,(C)} = C$ (C is convex) and it is characterized by

$$\boldsymbol{B}'' = \left\{(x,v) \in C \times \mathcal{H} \mid \langle v - w, x - y\rangle \le \lambda |x-y|^2 \quad \forall\,(y,w) \in \boldsymbol{B}_0\right\}. \tag{A.17}$$

Since $\boldsymbol{B}' \supset \boldsymbol{B}_0$, $\boldsymbol{B}'$ is maximal λ-dissipative and $\mathrm{D}(\boldsymbol{B}') \subset C$, it must be that $\boldsymbol{B}' = \boldsymbol{B}''$.
By (A.17), we need to prove that

$$\hat{\boldsymbol{B}} \subset \left\{(x,v) \in C \times \mathcal{H} \mid \langle v - w, x - y\rangle \le \lambda |x-y|^2 \quad \forall\,(y,w) \in \boldsymbol{B}_0\right\}. \tag{A.18}$$

To this aim we apply Proposition A.11 to the maximal λ-dissipative $\boldsymbol{B}'$ and its subset $\boldsymbol{B}$ noticing that $\mathrm{D}(\boldsymbol{B})$ is dense in $\mathrm{D}(\boldsymbol{B}')$. In this way, we obtain that

$$\boldsymbol{B}_0 y = \overline{\mathrm{co}}\left(\overline{\boldsymbol{B}} y\right), \quad y \in \mathrm{D}(\boldsymbol{B}_0), \tag{A.19}$$

where

$$\overline{\boldsymbol{B}} y = \{u \in \mathcal{H} \mid \exists (y_n, u_n) \in \boldsymbol{B} \text{ s.t. } y_n \to y,\, u_n \rightharpoonup u\}\,.$$

If $(x,v) \in \hat{\boldsymbol{B}}$ and $(y,w) \in \mathrm{D}(\boldsymbol{B}_0) \times \mathcal{H}$ is such that $w \in \overline{\boldsymbol{B}} y$, we can find a sequence $(y_n, u_n) \in \boldsymbol{B}$ s.t. $y_n \to y$ and $u_n \rightharpoonup w$; then, by the very definition of $\hat{\boldsymbol{B}}$, we have

$$\langle v - u_n, x - y_n\rangle \le \lambda |x - y_n|^2 \quad \forall\, n \in \mathbb{N},$$

so that, passing to the limit, we get

$$\langle v - w, x - y\rangle \le \lambda |x-y|^2.$$

This proves that, if $(x,v) \in \hat{\boldsymbol{B}}$, then

$$\langle v - w, x - y\rangle \le \lambda |x-y|^2 \quad \forall\, w \in \overline{\boldsymbol{B}} y, \quad \forall\, y \in \mathrm{D}(\boldsymbol{B}_0). \tag{A.20}$$

Finally, if $(x,v) \in \hat{\boldsymbol{B}}$ and $(y,w) \in \boldsymbol{B}_0$, we can find a sequence $(N_n)_n \subset \mathbb{N}$, numbers $(\alpha_i^n)_{i=1}^{N_n} \subset [0,1]$ and points $(w_i^n)_{i=1}^{N_n} \subset \overline{\boldsymbol{B}} y$ s.t.

$$\sum_{i=1}^{N_n} \alpha_i^n = 1 \ \forall n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \sum_{i=1}^{N_n} \alpha_i^n w_i^n = w.$$

By (A.20)

$$\langle v - w_i^n, x - y\rangle \le \lambda |x-y|^2 \quad \forall i = 1, \dots, N_n, \quad \forall n \in \mathbb{N},$$

so that, multiplying by α_i^n and summing up w.r.t. i we obtain

$$\langle v - \sum_{i=1}^{N_n} \alpha_i^n w_i^n, x - y\rangle \le \lambda |x-y|^2 \quad \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\langle v - w, x - y\rangle \le \lambda |x-y|^2,$$

so that (A.18) holds. Finally notice that (A.15) is already stated in (A.19) since we just proved that $\boldsymbol{B}' = \boldsymbol{B}'' = \hat{\boldsymbol{B}}$. □

As a consequence, we have the following corollary.

Corollary A.14. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be as in Theorem A.13 and let $\boldsymbol{G} : \mathrm{int}\,(C) \to \mathcal{H}$ be a single-valued selection of the maximal λ-dissipative extension $\hat{\boldsymbol{B}}$ of $\boldsymbol{B}$. Then the unique maximal λ-dissipative extension of $\boldsymbol{G}$ with domain included in C, $\hat{\boldsymbol{G}}$, coincides with $\hat{\boldsymbol{B}}$ and in particular*

$$(x, v) \in \hat{\boldsymbol{B}} \Leftrightarrow x \in C, \langle v - \boldsymbol{G}y, x - y\rangle \le \lambda|x - y|^2 \quad \forall y \in \mathrm{int}\,(C)\,. \tag{A.21}$$

Let us consider a different situation when we do not assume that $\mathrm{D}(\boldsymbol{B})$ contains interior points but there exists a subset D dense in $\mathrm{D}(\boldsymbol{B})$ which is invariant with respect to the resolvent map $\boldsymbol{J}_\tau$, i.e.

$$\overline{D} \supset \mathrm{D}(\boldsymbol{B}) \text{ and } \quad \forall\, x \in D,\ 0 < \tau < 1/\lambda^+ \quad \exists\, x_\tau \in D:\ x_\tau - \tau \boldsymbol{B} x_\tau \ni x. \tag{A.22}$$

Since $\boldsymbol{B}$ is λ-dissipative, the point x_τ solving the inclusion in (A.22) is unique and defines a map $\boldsymbol{J}_\tau : D \to D \cap \mathrm{D}(\boldsymbol{B})$.

Lemma A.15. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be λ-dissipative with $C := \overline{\mathrm{D}(\boldsymbol{B})}$ convex, let us assume that $D \subset \mathcal{H}$ satistifies (A.22), and let us set $\boldsymbol{B}_0 := \boldsymbol{B} \cap (D \times \mathcal{H})$. The following hold:*

(1) $\boldsymbol{B}$ admits a unique maximal λ-dissipative extension $\hat{\boldsymbol{B}}$ with $\mathrm{D}(\hat{\boldsymbol{B}}) \subset C$ characterized by

$$\hat{\boldsymbol{B}} = \Big\{(x, v) \in C \times \mathcal{H} \mid \langle v - v_0, \rangle \le \lambda|x - x_0|^2 \textit{ for every } (x_0, v_0) \in \boldsymbol{B}_0\Big\}. \tag{A.23}$$

(2) If moreover the interior of $\overline{D}$ contains C, we have

$$\hat{\boldsymbol{B}} = \Big\{(x, v) \in \mathcal{H} \times \mathcal{H} : \exists\, (x_n, v_n) \in \boldsymbol{B}_0 : x_n \to x, v_n \to v \quad \textit{as } n \to \infty\Big\}. \tag{A.24}$$

Proof. We first prove Claim (1). Let $\boldsymbol{B}'$ be any maximal λ-dissipative extension of $\boldsymbol{B}$ with domain included in C (whose existence is granted by Theorem A.2(2)) and let $\boldsymbol{J}'_\tau$ be the resolvent associated with $\boldsymbol{B}'$. By dissipativity of $\boldsymbol{B}'$ and since $\boldsymbol{B}_0 \subset \boldsymbol{B} \subset \boldsymbol{B}'$, we have that $\boldsymbol{B}' \subset \hat{\boldsymbol{B}}$ defined as in (A.23). We need to prove the other inclusion.
Clearly, the restriction of $\boldsymbol{J}'_\tau$ to D coincides with $\boldsymbol{J}_\tau$; since $\boldsymbol{J}'_\tau$ is Lipschitz and D is dense in C, it is the unique Lipschitz extension of $\boldsymbol{J}_\tau$ to $\overline{D} \supset C$.
If $(x, v) \in \hat{\boldsymbol{B}}$, (A.23) and the fact that for every $y \in D$, $\frac{1}{\tau}(\boldsymbol{J}_\tau y - y) \in \boldsymbol{B}\boldsymbol{J}_\tau y$ yield by density that

$$\langle v - \tau^{-1}(\boldsymbol{J}'_\tau y - y), x - \boldsymbol{J}'_\tau y\rangle \le \lambda|x - \boldsymbol{J}'_\tau y|^2 \quad \forall\, y \in \mathrm{D}(\boldsymbol{B}'), \quad \forall\, 0 < \tau < 1/\lambda^+, \tag{A.25}$$

and passing to the limit as $\tau \downarrow 0$ we obtain that

$$\langle v - \boldsymbol{B}'^\circ y, x - y\rangle \le \lambda|x - y|^2 \quad \forall\, y \in \mathrm{D}(\boldsymbol{B}'), \tag{A.26}$$

where we also used Theorem A.3(4), (5). We can then apply (A.3) and conclude that $(x, v) \in \boldsymbol{B}'$.

We prove Claim (2). Since $\overline{\boldsymbol{B}_0} \subset \hat{\boldsymbol{B}}$, it is sufficient to prove the opposite inclusion $\hat{\boldsymbol{B}} \subset \overline{\boldsymbol{B}_0}$. Let $(x, v) \in \hat{\boldsymbol{B}}$, let $0 < \tau < 1/\lambda^+$ and set $y := x - \tau v$. Clearly $\boldsymbol{J}'_\tau y = x$; since $\overline{D}$ contains a neighborhood of every element of $\mathrm{D}(\hat{\boldsymbol{B}}) \subset C$, for sufficiently small $\tau > 0$ there exists a sequence $(y_n)_n \subset D$ converging to y as $n \to \infty$. Setting $x_n := \boldsymbol{J}'_\tau y_n$ and $v_n := (x_n - y_n)/\tau \in \boldsymbol{B}x_n$, we clearly have $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} v_n = v$. □

Corollary A.16. *Let $\boldsymbol{B} \subset \mathcal{H} \times \mathcal{H}$ be maximal λ-dissipative, let us assume that $D \subset \mathcal{H}$ satistifies (A.22) and the interior of $\overline{D}$ contains $C := \overline{\mathrm{D}(\boldsymbol{B})}$. The following hold:*

(1) For every $x \in \mathrm{D}(\boldsymbol{B})$ there exists a sequence $x_n \in D \cap \mathrm{D}(\boldsymbol{B})$ converging to x such that $\boldsymbol{B}^\circ x_n \to \boldsymbol{B}^\circ x$ as $n \to \infty$.

(2) $\boldsymbol{B}$ can be determined by the restriction of the minimal section $\boldsymbol{B}^\circ$ to D i.e.

$$\boldsymbol{B} = \Big\{(x, v) \in \overline{\mathrm{D}(\boldsymbol{B})} \times \mathcal{H} \mid \langle v - \boldsymbol{B}^\circ x_0, \rangle \le \lambda|x - x_0|^2 \textit{ for every } x_0 \in D \cap \mathrm{D}(\boldsymbol{B})\Big\}. \tag{A.27}$$

Proof. We first prove Claim (1). Since $\boldsymbol{B}$ is maximal λ-dissipative, the closure of its domain C is convex (see Theorem A.3(4)). We can thus apply the second claim of the previous Lemma A.15 (in this case $\hat{\boldsymbol{B}} = \boldsymbol{B}$) to find a sequence $(x_n, v_n) \in \boldsymbol{B} \cap (D \times \mathcal{H})$ such that $x_n \to x$ and $v_n \to \boldsymbol{B}^\circ x$. Let us first prove that $\boldsymbol{B}^\circ x_n \rightharpoonup \boldsymbol{B}^\circ x$ weakly in $\mathcal{H}$ as $n \to \infty$: extracting an unrelabeled subsequence, since $|\boldsymbol{B}^\circ x_n| \le |v_n|$ is bounded, we can suppose that there exists an increasing subsequence $k \mapsto n(k)$ and an element $v \in \mathcal{H}$ such that $\boldsymbol{B}^\circ x_{n(k)} \rightharpoonup v$ as $k \to \infty$. Since the graph of $\boldsymbol{B}$ is strongly-weakly closed (cf. Theorem A.3(1)), we deduce that $(x, v) \in \boldsymbol{B}$ so that $|v| \ge |\boldsymbol{B}^\circ x|$. On the other hand, the lower semicontinuity of the norm yields

$$|\boldsymbol{B}^\circ x| \le |v| \le \liminf_{k\to\infty} |\boldsymbol{B}^\circ x_{n(k)}| \le \limsup_{k\to\infty} |\boldsymbol{B}^\circ x_{n(k)}| \le \limsup_{k\to\infty} |v_{n(k)}| = |\boldsymbol{B}^\circ x|.$$

We deduce that $\boldsymbol{B}^\circ x_{n(k)} \rightharpoonup \boldsymbol{B}^\circ x$ and $\lim_{k\to\infty} |\boldsymbol{B}^\circ x_{n(k)}| = |\boldsymbol{B}^\circ x|$ so that the convergence is also strong. Since the starting (unrelabeled) subsequence was arbitrary, we deduce the strong convergence of the whole sequence.
Claim (2) now follows easily by approximation using the previous claim and Theorem A.3(6). □

Appendix B. Borel partitions and almost optimal couplings

In this appendix we summarize some of the results of [CSS23b] related to standard Borel spaces, Borel partitions and optimal couplings between probability measures that have been used throughout the whole paper. We refer to [CSS23b, Section 3] for the proofs.

Definition B.1. *A* standard Borel space $(\Omega, \mathcal{B})$ *is a measurable space that is isomorphic (as a measurable space) to a Polish space. Equivalently, there exists a Polish topology τ on Ω such that the Borel sigma algebra generated by τ coincides with $\mathcal{B}$. We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{B})$ is nonatomic if $\mathbb{P}(\{\omega\}) = 0$ for every $\omega \in \Omega$ (notice that $\{\omega\} \in \mathcal{B}$ since it is compact in any Polish topology on Ω).*

If $(\Omega, \mathcal{B})$ is a standard Borel space endowed with a nonatomic probability measure $\mathbb{P}$, we denote by $\mathrm{S}(\Omega, \mathcal{B}, \mathbb{P})$ the class of $\mathcal{B}$-$\mathcal{B}$-measurable maps $g : \Omega \to \Omega$ which are essentially injective and measure-preserving, meaning that there exists a full $\mathbb{P}$-measure set $\Omega_0 \in \mathcal{B}$ such that g is injective on Ω_0 and $g_\sharp \mathbb{P} = \mathbb{P}$. If $\mathcal{A} \subset \mathcal{B}$ is a sigma algebra on Ω we denote by $\mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{A})$ the subset of $\mathrm{S}(\Omega, \mathcal{B}, \mathbb{P})$ of $\mathcal{A} - \mathcal{A}$ measurable maps.
We will often use the notation

$$I_N := \{0, \dots, N-1\}, \quad N \in \mathbb{N},\ N \ge 1$$

while $\mathrm{Sym}(I_N)$ denotes the set of permutations of I_N i.e. bijective maps $\sigma : I_N \to I_N$. We will consider the partial order on $\mathbb{N}$ given by

$$m \prec n \quad \Leftrightarrow \quad m \mid n$$

where $m \mid n$ means that $n/m \in \mathbb{N}$. We write $m \precneqq n$ if $m \prec n$ and $m \ne n$.
This first result shows a correspondence between permutations and measure-preserving isomorphisms.

Lemma B.2. *Let $(\Omega, \mathcal{B})$ be a standard Borel space endowed with a nonatomic probability measure $\mathbb{P}$, and let $\mathfrak{P}_N = \{\Omega_{N,k}\}_{k\in I_N} \subset \mathcal{B}$ be a N*-partition *of $(\Omega, \mathcal{B})$ for some $N \in \mathbb{N}$, i.e.*

$$\bigcup_{k\in I_N} \Omega_{N,k} = \Omega, \quad \Omega_{N,k} \cap \Omega_{N,h} = \emptyset \text{ if } h, k \in I_N,\ h \ne k;$$

assume moreover that $\mathbb{P}(\Omega_{N,k}) = \mathbb{P}(\Omega)/N$ for every $k \in I_N$. If $\sigma \in \mathrm{Sym}(I_N)$, there exists a measure-preserving isomorphism $g \in \mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \sigma(\mathfrak{P}_N))$ such that

$$(g_k)_\sharp \mathbb{P}|_{\Omega_{N,k}} = \mathbb{P}|_{\Omega_{N,\sigma(k)}} \quad \forall k \in I_N,$$

where g_k is the restriction of g to $\Omega_{N,k}$.

We introduce now the notion of *refined* standard Borel measure space which turns out to be useful when dealing with approximation of general measures with discrete ones.

Definition B.3. *Let $(\Omega, \mathcal{B})$ be a standard Borel space endowed with a nonatomic probability measure $\mathbb{P}$, and let $\mathfrak{N} \subset \mathbb{N}$ be an unbounded directed set w.r.t. $\prec$. We say that a collection of partitions $(\mathfrak{P}_N)_{N\in\mathfrak{N}}$ of Ω, with corresponding sigma algebras $\mathcal{B}_N := \sigma(\mathfrak{P}_N)$, is a* $\mathfrak{N}$-segmentation of $(\Omega, \mathcal{B}, \mathbb{P})$ *if*

(1) $\mathfrak{P}_N = \{\Omega_{N,k}\}_{k\in I_N}$ is a N-partition of $(\Omega, \mathcal{B})$ for every $N \in \mathfrak{N}$,
(2) $\mathbb{P}(\Omega_{N,k}) = \mathbb{P}(\Omega)/N$ for every $k \in I_N$ and every $N \in \mathfrak{N}$,
(3) if $M \mid N = KM$ then $\bigcup_{k=0}^{K-1} \Omega_{N,mK+k} = \Omega_{M,m}$, $m \in I_M$,
(4) $\sigma(\{\mathcal{B}_N \mid N \in \mathfrak{N}\}) = \mathcal{B}$.

In this case we call $(\Omega, \mathcal{B}, \mathbb{P}, (\mathfrak{P}_N)_{N\in\mathfrak{N}})$ a $\mathfrak{N}$-refined standard Borel probability space.

Proposition B.4. *For any standard Borel space $(\Omega, \mathcal{B})$ endowed with a nonatomic probability measure $\mathbb{P}$ and any unbounded directed set $\mathfrak{N} \subset \mathbb{N}$ w.r.t. $\prec$, there exists a $\mathfrak{N}$-segmentation of $(\Omega, \mathcal{B}, \mathbb{P})$. If $\mathfrak{N} \subset \mathbb{N}$ is an unbounded directed subset w.r.t. $\prec$, then there exists a totally ordered cofinal sequence $(b_n)_n \subset \mathfrak{N}$ satisfying*

- *$b_n \precneqq b_{n+1}$ for every $n \in \mathbb{N}$,*
- *for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $N \mid b_n$.*

In particular, for every $\mathfrak{N}$-refined standard Borel measure space $(\Omega, \mathcal{B}, \mathfrak{m}, (\mathfrak{P}_N)_{N\in\mathfrak{N}})$ it holds that $(\mathcal{B}_{b_n})_{n\in\mathbb{N}}$ is a filtration on $(\Omega, \mathcal{B})$,

$$\text{for every } N \in \mathfrak{N} \text{ there exists } n \in \mathbb{N} \text{ such that } \mathcal{B}_N \subset \mathcal{B}_{b_n}, \tag{B.1}$$

and $\sigma(\{\mathcal{B}_{b_n} \mid n \in \mathbb{N}\}) = \mathcal{B}$.
For every every separable Hilbert space X, we thus have that

$$\bigcup_{N\in\mathfrak{N}} L^2(\Omega, \mathcal{B}_N, \mathfrak{m}; \mathsf{X}) \text{ is dense in } L^2(\Omega, \mathcal{B}, \mathfrak{m}; \mathsf{X}). \tag{B.2}$$

The next theorem contains approximation results for couplings by means of maps in different situations.

Theorem B.5. *Let $(\Omega, \mathcal{B}, \mathbb{P}, (\mathfrak{P}_N)_{N\in\mathfrak{N}})$ be a $\mathfrak{N}$-refined standard Borel probability space. Then:*

(1) For every $\boldsymbol{\gamma} \in \Gamma(\mathbb{P}, \mathbb{P})$ there exist a totally ordered strictly increasing sequence $(N_n)_n \subset \mathfrak{N}$ and maps $g_n \in \mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{B}_{N_n})$ such that, for every separable Hilbert space X and every $X, Y \in L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ it holds

$$(X, Y)_\sharp(\boldsymbol{i}_\Omega, g_n)_\sharp\mathbb{P} \to (X \otimes Y)_\sharp\boldsymbol{\gamma} \text{ in } \mathcal{P}_2(\mathsf{X}^2). \tag{B.3}$$

(2) If X is a separable Hilbert space and $X, X' \in L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$, then for every $\boldsymbol{\mu} \in \Gamma(X_\sharp\mathbb{P}, X'_\sharp\mathbb{P})$ there exist a totally ordered strictly increasing sequence $(N_n)_n \subset \mathfrak{N}$ and maps $g_n \in \mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{B}_{N_n})$ such that

$$(X, X' \circ g_n)_\sharp\mathbb{P} \to \boldsymbol{\mu} \text{ in } \mathcal{P}_2(\mathsf{X}^2). \tag{B.4}$$

In particular, if $X_\sharp\mathbb{P} = X'_\sharp\mathbb{P}$, there exist a totally ordered strictly increasing sequence $(N_n)_n \subset \mathfrak{N}$ and maps $g_n \in \mathrm{S}(\Omega, \mathcal{B}, \mathbb{P}; \mathcal{B}_{N_n})$ such that $X' \circ g_n \to X$ in $L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ as $n \to \infty$.

Finally, if $(\Omega, \mathcal{B})$ is a standard Borel space endowed with a nonatomic probability measure $\mathbb{P}$, X is a separable Hilbert space, $\mu, \nu \in \mathcal{P}_2(\mathsf{X})$ and $X \in L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ is s.t. $X_\sharp\mathbb{P} = \mu$, then, for every $\varepsilon > 0$, there exists $Y \in L^2(\Omega, \mathcal{B}, \mathbb{P}; \mathsf{X})$ s.t. $Y_\sharp\mathbb{P} = \nu$ and

$$|X - Y|_{L^2(\Omega,\mathcal{B},\mathbb{P};\mathsf{X})} \le W_2(\mu, \nu) + \varepsilon.$$

References

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows In Metric Spaces and in the Space of Probability Measures.* Lectures in Mathematics. ETH Zürich. Birkhäuser Basel, 2008.

[AK22] Y. Averboukh and D. Khlopin. "Pontryagin maximum principle for the deterministic mean field type optimal control problem via the Lagrangian approach". In: *arXiv:2207.01892* (2022).

[Amb+21] L. Ambrosio, M. Fornasier, M. Morandotti, and G. Savaré. "Spatially Inhomogeneous Evolutionary Games". In: *Communications on Pure and Applied Mathematics* 74.7 (2021), pp. 1353–1402.

[AMQ21] Y. Averboukh, A. Marigonda, and M. Quincampoix. "Extremal shift rule and viability property for mean field-type control systems". In: *J. Optim. Theory Appl.* 189.1 (2021), pp. 244–270.

[Att79] H. Attouch. "Familles d'opérateurs maximaux monotones et mesurabilite". In: *Ann. Mat. Pura Appl. (4)* 120 (1979), pp. 35–111.

[Ave22] Y. V. Averboukh. "A mean field type differential inclusion with upper semicontinuous right-hand side". In: *Vestn. Udmurt. Univ. Mat. Mekh. Komp'yut. Nauki* 32.4 (2022), pp. 489–501.

[Bén72] P. Bénilan. "Solutions intégrales d'équations d'évolution dans un espace de Banach". In: *C. R. Acad. Sci. Paris Sér. A-B* 274 (1972), A47–A50.

[BF21a] B. Bonnet and H. Frankowska. "Necessary optimality conditions for optimal control problems in Wasserstein spaces". In: *Appl. Math. Optim.* 84.suppl. 2 (2021), S1281–S1330.

[BF21b] B. Bonnet and H. Frankowska. "Differential inclusions in Wasserstein spaces: the Cauchy-Lipschitz framework". In: *J. Differential Equations* 271 (2021), pp. 594–637.

[BF22a] Z. Badreddine and H. Frankowska. "Solutions to Hamilton-Jacobi equation on a Wasserstein space". In: *Calc. Var. Partial Differential Equations* 61.1 (2022), Paper No. 9, 41.

[BF22b] Z. Badreddine and H. Frankowska. "Viability and invariance of systems on metric spaces". In: *Nonlinear Analysis* 225 (2022), pp. 113–133.

[BF22c] B. Bonnet and H. Frankowska. "Viability and Exponentially Stable Trajectories for Differential Inclusions in Wasserstein Spaces". In: *2022 IEEE 61st Conference on Decision and Control (CDC).* 2022, pp. 5086–5091.

[BF23] B. Bonnet and H. Frankowska. "Carathéodory theory and a priory estimates for continuity inclusions in the space of probability measures". In: *arXiv:2302.00963* (2023).

[Bre10] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Universitext. Springer New York, 2010.

[Bré73] H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973, pp. vi+183.

[BW09] H. H. Bauschke and X. Wang. "The kernel average for two convex functions and its application to the extension and representation of monotone operators". In: *Trans. Amer. Math. Soc.* 361.11 (2009), pp. 5947–5965.

[BW10] H. H. Bauschke and X. Wang. "Firmly nonexpansive and Kirszbraun-Valentine extensions: a constructive approach via monotone operator theory". In: *Nonlinear analysis and optimization I. Nonlinear analysis.* Vol. 513. Contemp. Math. Amer. Math. Soc., Providence, RI, 2010, pp. 55–64.

[Cam+21] F. Camilli, G. Cavagnari, R. De Maio, and B. Piccoli. "Superposition principle and schemes for measure differential equations". In: *Kinet. Relat. Models* 14.1 (2021), pp. 89–113.

[Car13] P. Cardaliaguet. "Notes on Mean Field Games". From P.-L. Lions' lectures at College de France, link to the notes. 2013.

[Cav+22] G. Cavagnari, S. Lisini, C. Orrieri, and G. Savaré. "Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems: equivalence and gamma-convergence". In: *J. Differential Equations* 322 (2022), pp. 268–364.

[CB18] L. Chizat and F. Bach. "On the Global Convergence of Gradient Descent for Over-parameterized Models using Optimal Transport". In: *Advances in neural information processing systems* (2018), pp. 3036–3056.

[CCR11] J. A. Cañizo, J. A. Carrillo, and J. Rosado. "A well-posedness theory in measures for some kinetic models of collective motion". In: *Math. Models Methods Appl. Sci.* 21.3 (2011), pp. 515–539.

[CD18] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I.* Vol. 83. Probability Theory and Stochastic Modelling. Mean field FBSDEs, control, and games. Springer, Cham, 2018, pp. xxv+713.

[CM22] R. Capuani and A. Marigonda. "Constrained Mean Field Games Equilibria as Fixed Point of Random Lifting of Set-Valued Maps". In: *IFAC-PapersOnLine* 55.30 (2022). 25th International Symposium on Mathematical Theory of Networks and Systems MTNS 2022, pp. 180–185.

[CMP20] G. Cavagnari, A. Marigonda, and B. Piccoli. "Generalized dynamic programming principle and sparse mean-field control problems". In: *J. Math. Anal. Appl.* 481.1 (2020), pp. 123–137, 45.

[CMQ21] G. Cavagnari, A. Marigonda, and M. Quincampoix. "Compatibility of state constraints and dynamics for multiagent control systems". In: *J. Evol. Equ.* 21.4 (2021), pp. 4491–4537.

[CSS23a] G. Cavagnari, G. Savaré, and G. E. Sodini. "Dissipative probability vector fields and generation of evolution semigroups in Wasserstein spaces". In: *Probab. Theory Related Fields* 185.3-4 (2023), pp. 1087–1182.

[CSS23b] G. Cavagnari, G. Savaré, and G. E. Sodini. "Extension of monotone operators and Lipschitz maps invariant for a group of isometries". In: *arXiv:2305.04678* (2023).

[DOr+06] M. R. D'Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. Chayes. "Self-propelled particles with soft-core interactions: Patterns, stability, and collapse". In: *Phys. Rev. Lett.* 96 (2006), pp. 104302–1/4.

[For+19] M. Fornasier, S. Lisini, C. Orrieri, and G. Savaré. "Mean-field optimal control as gamma-limit of finite agent controls". In: *European J. Appl. Math.* 30.6 (2019), pp. 1153–1186.

[FSS22] M. Fornasier, G. E. Sodini, and G. Savaré. "Density of subalgebras of Lipschitz functions in metric Sobolev spaces and applications to Sobolev-Wasserstein spaces". In: *arXiv:2209.00974* (2022).

[GT19] W. Gangbo and A. Tudorascu. "On differentiability in the Wasserstein space and well-posedness for Hamilton-Jacobi equations". In: *J. Math. Pures Appl. (9)* 125 (2019), pp. 119–174.

[JKO98a] R. Jordan, D. Kinderlehrer, and F. Otto. "The variational formulation of the Fokker-Planck equation". In: *SIAM J. Math. Anal.* 29.1 (1998), pp. 1–17.

[JKO98b] R. Jordan, D. Kinderlehrer, and F. Otto. "The variational formulation of the Fokker-Planck equation". In: *SIAM J. Math. Anal.* 29.1 (1998), pp. 1–17.

[JMQ20] C. Jimenez, A. Marigonda, and M. Quincampoix. “Optimal control of multiagent systems in the Wasserstein space”. In: *Calc. Var. Partial Differential Equations* 59.2 (2020), Paper No. 58, 45.

[Lio07] P.-L. Lions. “Théorie des jeux à champs moyen et applications”. In: Lectures at the Collège de France (2007).

[McC97] R. J. McCann. “A convexity principle for interacting gases”. In: *Adv. Math.* 128.1 (1997), pp. 153–179.

[NS06] R. H. Nochetto and G. Savaré. “Nonlinear evolution governed by accretive operators in Banach spaces: error control and applications”. In: *Math. Models Methods Appl. Sci.* 16.3 (2006), pp. 439–477.

[NS09] L. Natile and G. Savaré. “A Wasserstein approach to the one-dimensional sticky particle system”. In: *SIAM J. Math. Anal.* 41.4 (2009), pp. 1340–1365.

[NS21] E. Naldi and G. Savaré. “Weak topology and Opial property in Wasserstein spaces, with applications to gradient flows and proximal point algorithms of geodesically convex functionals”. In: *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 32.4 (2021), pp. 725–750.

[NSV00] R. H. Nochetto, G. Savaré, and C. Verdi. “A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations”. In: *Comm. Pure Appl. Math.* 53.5 (2000), pp. 525–589.

[Ott01] F. Otto. “The geometry of dissipative evolution equations: the porous medium equation”. In: *Comm. Partial Differential Equations* 26.1-2 (2001), pp. 101–174.

[Pic18] B. Piccoli. “Measure differential inclusions”. In: *2018 IEEE Conference on Decision and Control (CDC)* (2018), pp. 1323–1328.

[Pic19] B. Piccoli. “Measure differential equations”. In: *Arch. Ration. Mech. Anal.* 233.3 (2019), pp. 1289–1317.

[Pic23] B. Piccoli. “Control of multi-agent systems: results, open problems, and applications”. In: *arXiv:2302.12308* (2023).

[Pog16] N. Pogodaev. “Optimal control of continuity equations”. In: *NoDEA Nonlinear Differential Equations Appl.* 23.2 (2016), Art. 21, 24.

[Qi83] L. Qi. “Uniqueness of the maximal extension of a monotone operator”. English. In: *Nonlinear Anal., Theory Methods Appl.* 7 (1983), pp. 325–332.

[Szn91] A.-S. Sznitman. “Topics in propagation of chaos”. In: *École d’Été de Probabilités de Saint-Flour XIX—1989.* Vol. 1464. Lecture Notes in Math. Springer, Berlin, 1991, pp. 165–251.

Giulia Cavagnari: Politecnico di Milano, Dipartimento di Matematica, Piazza Leonardo Da Vinci 32, 20133 Milano (Italy)
Email address: giulia.cavagnari@polimi.it

Giuseppe Savaré: Bocconi University, Department of Decision Sciences and BIDSA, Via Roentgen 1, 20136 Milano (Italy)
Email address: giuseppe.savare@unibocconi.it

Giacomo Enrico Sodini: Institut für Mathematik - Fakultät für Mathematik - Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien (Austria)
Email address: giacomo.sodini@univie.ac.at