THE LEAST GRADIENT PROBLEM WITH DIRICHLET AND NEUMANN BOUNDARY CONDITIONS

SAMER DWEIK

Abstract. In this paper, we consider the BV least gradient problem with Dirichlet condition on a part $\Gamma \subset \partial\Omega$ and Neumann boundary condition on its complementary part $\partial\Omega\backslash\Gamma$. We will show that in the plane this problem is equivalent to an optimal transport problem with import/export taxes on $\partial\Omega\backslash\Gamma$. Thanks to this equivalence, we will be able to show existence and uniqueness of a solution to this mixed least gradient problem and, we will also prove some Sobolev regularity on this solution. We note that these results generalize those in [7], where we studied the pure Dirichlet version of this problem.

1. Introduction

The least gradient problem with Dirichlet condition consists in minimizing the total variation of the vector measure Du among all BV functions u on an open domain $\Omega \subset \mathbb{R}^d$ such that the trace of u on the boundary is given by a function $g \in L^1(\partial\Omega)$ (see, for instance, [2, 16, 18, 26]):

$$\inf\left\{\int_\Omega |Du| \,:\, u \in BV(\Omega),\, u_{|\partial\Omega} = g\right\}. \tag{1.1}$$

The author of [15] proves existence of a solution to Problem (1.1) in the case where g is in $BV(\partial\Omega)$ and Ω is strictly convex. While the authors of [27] showed by a counter-example that Problem (1.1) may have no solutions as soon as $g \notin BV(\partial\Omega)$. In addition, a solution may not exist if Ω is not strictly convex. In [26], the authors prove existence and uniqueness of a solution u to Problem (1.1) provided that $g \in C(\partial\Omega)$. On the other hand, the authors of [23, 24, 6] have studied Problem (1.1) but in the case where Ω is just convex. More precisely, they proved under some strong assumptions on the boundary datum g, that Problem (1.1) reaches a minimum.

Now, we assume that $g \in BV(\partial\Omega)$ and $d = 2$. Then, in [9, 16], the authors prove that Problem (1.1) is equivalent to the following minimal flow formulation:

$$\inf\left\{\int_{\overline{\Omega}} |v| \,:\, v \in \mathcal{M}(\bar{\Omega}, \mathbb{R}^2),\ \nabla\cdot v = 0 \ \text{ and } \ v\cdot n = f := \partial_\tau g \ \text{ on } \partial\Omega\right\}, \tag{1.2}$$

where $\partial_\tau g$ denotes the tangential derivative of g and the divergence condition $\nabla \cdot v = 0$ and $v \cdot n = f$ on $\partial\Omega$ (where $n := R_{\frac{\pi}{2}}\tau$ is the outward normal vector to $\partial\Omega$ and $R_{\frac{\pi}{2}}$ denotes the rotation with angle $\frac{\pi}{2}$ around the origin) should be understood in the weak form $\int_{\overline{\Omega}} \nabla\phi\cdot \mathrm{d}v = \int_{\partial\Omega} \phi\, \mathrm{d}f$, for all $\phi \in C^1(\overline{\Omega})$. More precisely, one can show that $\inf\,(1.1) = \inf\,(1.2)$. Moreover, if u is a solution for Problem (1.1) then $v := R_{\frac{\pi}{2}}Du$ solves Problem (1.2). On the other hand, if v is an optimal flow for Problem (1.2) such that $|v|$ gives zero mass to the boundary, then the function u such that $v = R_{\frac{\pi}{2}}Du$ turns out to be a solution for Problem (1.1). It is also well known (see, for instance, [25]) that Problem (1.2) is equivalent to the following Monge-Kantorovich problem:

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma \,:\, \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\, (\Pi_x)_{\#}\gamma = f^+ \ \text{ and } \ (\Pi_y)_{\#}\gamma = f^-\right\}, \tag{1.3}$$

where f^+ and f^- are the positive and negative parts of f. In addition, we note that Problem (1.3) has a dual formulation, which is the following:

$$\sup\left\{\int_{\overline{\Omega}} w\,\mathrm{d}(f^+ - f^-)\,:\,w \in \mathrm{Lip}_1(\overline{\Omega})\right\}. \tag{1.4}$$

If γ is an optimal transport plan for Problem (1.3) then the vector measure v_γ defined as follows

$$< v_\gamma, \xi >= \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \xi((1-t)x+ty)\cdot(x-y)\,\mathrm{d}t\,\mathrm{d}\gamma(x,y), \quad \text{for all} \;\; \xi \in C(\overline{\Omega},\mathbb{R}^2), \tag{1.5}$$

is a minimizer for Problem (1.2). We note also that $v_\gamma = |v_\gamma|\,\nabla w$, where w is a Kantorovich potential (i.e. a maximizer of the dual problem (1.4)), since one can show that for any pair $(x,y) \in \mathrm{spt}(\gamma)$, w is differentiable in the interior of the transport ray $[x,y]$ and its gradient ∇w is given by the opposite unit direction of $[x,y]$. In particular, this means that transport rays cannot intersect at an interior point. In addition, any minimizer v of Problem (1.2) is exactly of this form $v = v_\gamma$, for some optimal transport plan γ (we refer the reader to [25] for detailed proofs of these results). The measure $\sigma_\gamma := |v_\gamma|$ is called a *transport density* and it plays a special role in the optimal transport theory, since it represents the amount of transport taking place in each region of Ω. In other words, we have

$$< \sigma_\gamma, \varphi >= \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \varphi((1-t)x+ty)|x-y|\,\mathrm{d}t\,\mathrm{d}\gamma(x,y), \;\; \text{for all} \;\; \varphi \in C(\overline{\Omega}). \tag{1.6}$$

The properties of this transport density σ_γ have been studied in several works. In [14, 25], the authors proved that σ_γ is unique (which means that it does not depend on the choice of the optimal transport plan γ) and it is in $L^1(\Omega)$ as soon as f^+ or f^- is absolutely continuous with respect to the Lebesgue measure. On the other hand, the authors of [3, 4, 5, 25] proved that the transport density σ belongs to $L^p(\Omega)$ as soon as f^+ and f^- are both in $L^p(\Omega)$, for all $p \in [1,\infty]$.

On the other hand, the least gradient problem with Neumann boundary condition has been considered in [22, 20]. In other words, the authors studied the following minimization problem:

$$\inf\left\{\int_\Omega |Du| - \int_{\partial\Omega} \psi\,u\,\mathrm{d}\mathcal{H}^1\,:\,u \in BV(\Omega)\right\}, \tag{1.7}$$

where $\psi \in L^\infty(\partial\Omega)$ with $\int_{\partial\Omega}\psi = 0$. More precisely, Problem (1.7) reaches a minimum (which has to be clearly equal zero) as soon as the datum ψ is small enough, that is $||\psi||_\star \le 1$ where the norm $||\cdot||_\star$ is equivalent to $||\cdot||_{L^\infty(\partial\Omega)}$ and it is defined as follows:

$$||\psi||_\star := \sup\left\{\frac{\int_{\partial\Omega}\psi\,u}{\int_\Omega |Du|}\,:\,u \in BV(\Omega)\right\}.$$

To be more precise, if $||\psi||_\star < 1$ then $u = 0$ is the unique solution for Problem (1.7) while if $||\psi||_\star = 1$, then there are infinitely many minimizers. If $||\psi||_\star > 1$, the minimal value will be $-\infty$ and so, a solution u does not exist. However, given a bounded function ψ on $\partial\Omega$ then it is not clear how to check whether the assumption $||\psi||_\star \le 1$ is well satisfied or not! We see that

$$||\psi||_\star \le \Lambda||\psi||_\infty$$

where Λ is the best constant of the Sobolev trace embedding $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$ for functions with vanishing mean value over Ω. But again, this constant Λ is unknown.

In this paper, we are mainly concerned in studying the least gradient problem with Dirichlet condition imposed on an open connected arc $\Gamma \subset \partial\Omega$ and Neumann boundary condition on its complementary part $\partial\Omega\backslash\Gamma$:

$$\inf\left\{ \int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u \,\mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega),\ u_{|\Gamma} = g\right\}, \tag{1.8}$$

where ψ is a bounded function on $\partial\Omega\backslash\Gamma$, $g \in BV(\Gamma)$ and $u_{|\Gamma} = g$ is in the sense that there is an L^1 extension $\tilde{g}$ of g to $\partial\Omega$ such that $u_{|\partial\Omega} = \tilde{g}$. Notice that if u is a solution for Problem (1.8), then u solves the following 1−Laplacian PDE with mixed Dirichlet and Neumann boundary conditions (see [19]):

$$\begin{cases} \nabla\cdot[\frac{Du}{|Du|}] = 0 & \text{in } \ \Omega, \\ u = g & \text{on } \ \Gamma, \\ \frac{Du}{|Du|}\cdot n = \psi & \text{on } \ \partial\Omega\backslash\Gamma. \end{cases}$$

On the other hand, we note that the relaxed version of Problem (1.8) is given by the following (see [18]):

$$\inf\left\{ \int_\Omega |Du| + \int_\Gamma |u-g| - \int_{\partial\Omega\backslash\Gamma} \psi\, u \,\mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega)\right\}. \tag{1.9}$$

However, it is not easy to show existence of a solution to (1.9) since for an arbitrary bounded function ψ on $\partial\Omega\backslash\Gamma$, the functional may not be lower semicontinuous and so, a solution may not exist. Yet, Problem (1.8) has been already studied in [7] but in the particular case when $\psi = 0$. But, we note that it is not immediate to extend the results of [7] to the case of a general bounded function ψ. Inspired by [7, 16], we will show that Problem (1.8) is equivalent to the following minimal flow formulation:

$$\inf\left\{ \int_{\overline{\Omega}} |v| + \int_{\partial\Omega\backslash\Gamma} \phi \,\mathrm{d}\chi : \ v \in \mathcal{M}(\overline{\Omega},\mathbb{R}^2),\ \chi\in\mathcal{M}(\partial\Omega\backslash\Gamma),\ \nabla\cdot v = 0,\ v\cdot n = f+\chi \ \text{ on } \ \partial\Omega\right\}, \tag{1.10}$$

where $f = \partial_\tau g$ and ϕ is a Lipschitz function on $\partial\Omega\backslash\Gamma$ such that $\psi = \partial_\tau\phi$. On the other hand, we will also show that Problem (1.10) is equivalent to the following import/export optimal transport problem:

$$\inf\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\mathrm{d}\gamma + \int_{\partial\Omega\backslash\Gamma} \phi\,\mathrm{d}[(\Pi_x)_\#\gamma] - \int_{\partial\Omega\backslash\Gamma} \phi\,\mathrm{d}[(\Pi_y)_\#\gamma] \;:\; \gamma\in\Pi(f^+,f^-)\right\}, \tag{1.11}$$

where

$$\Pi(f^+,f^-) := \left\{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}) : (\Pi_x)_\#\gamma = f^+ + \chi^+,\ (\Pi_y)_\#\gamma = f^- + \chi^-,\ \chi^\pm\in\mathcal{M}^+(\partial\Omega\backslash\Gamma)\right\}.$$

In [10], the authors have studied the transport problem from a diffuse measure $f^+ \in \mathcal{M}^+(\overline{\Omega})$ to the boundary $\partial\Omega$. More generally, the import/export transport problem from/to $\partial\Omega$ has been already considered in [11, 17]. Here, we study the mass transportation problem between two masses f^+ and f^- on $\Gamma \subset \partial\Omega$ (which do not have a priori the same total mass) with the possibility of transporting some mass from/to the arc $\partial\Omega\backslash\Gamma$, paying the transport cost $|x-y|$ for each unit of mass that moves from a point x to another one y plus an import tax $\phi(x)$ for each unit of mass that enters at the point $x \in \partial\Omega\backslash\Gamma$ and $-\phi(y)$ for each unit of mass that comes out from a point $y \in \partial\Omega\backslash\Gamma$. This means that we can use $\partial\Omega\backslash\Gamma$ as an infinite reserve/repository, we can take as much mass as we wish from $\partial\Omega\backslash\Gamma$ or send back as much mass as we want provided we pay the import/export taxes.

Thanks to the equivalence between Problems (1.8), (1.10) & (1.11), we will show existence and uniqueness of a solution u to Problem (1.8) and, we will also study its $W^{1,p}$ regularity. In the particular case $\psi = 0$, we have already proved in [7] existence of a solution u for this problem (1.8) provided that Γ is strictly convex and $g \in BV(\Gamma)$. Moreover, the solution u is unique as soon as $g \in C(\Gamma)$. In addition, there are several Sobolev estimates on this solution u, under some geometric assumptions on $\partial\Omega$. In this paper, we extend these results to some class of bounded functions ψ. To the best of our knowledge, all these results of existence, uniqueness, and $W^{1,p}$ regularity (with $\psi \neq 0$) are completely new, in the sense that in the literature there are no results concerning at least the existence of a solution to the mixed least gradient problem (1.8). As a last interesting point, we mention that most of the proofs in the general case $\psi \neq 0$ are not a mere translation of those given in [7] where $\psi = 0$.

This paper is organized as follows. In Section 2, we will prove existence and uniqueness of a solution u to another (equivalent) version of Problem (1.8) (see Problem (2.1) below) by showing equivalence with the import/export transport problem from/to $\partial\Omega\backslash\Gamma$. In Section 3, we will study the Sobolev regularity of this solution by studying the summablity of the transport density in the import/export transport problem. Finally, Section 4 summarizes the applications of these results to the least gradient problem with Dirichlet and Neumann boundary conditions (1.8).

2. On the existence and uniqueness of a solution to the mixed least gradient problem

Throughout the paper, $\Omega \subset \mathbb{R}^2$ is assumed to be an open bounded contractible set with Lipschitz boundary and Γ is an open connected subset of $\partial\Omega$. Let g be a BV function on Γ and $\phi^{\pm}$ be two continuous functions on $\partial\Omega\backslash\Gamma$. Then, we consider the following problem:

(2.1)
$$\inf\left\{\int_\Omega |Du| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}[\partial_\tau u]^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}[\partial_\tau u]^- \;:\; u \in BV(\Omega),\ u_{|\partial\Omega} \in BV(\partial\Omega),\ u_{|\Gamma} = g\right\},$$

where $\partial_\tau u$ denotes the tangential derivative of the trace of u (so, $\partial_\tau u$ is a measure on $\partial\Omega$ since we assume that $u_{|\partial\Omega} \in BV(\partial\Omega)$, which is of course not satisfied by any function $u \in BV(\Omega)$ but here it is an additional constraint on u), $[\partial_\tau u]^+$ and $[\partial_\tau u]^-$ are the positive and negative parts of $\partial_\tau u$. The aim of this section is to prove existence and uniqueness of a solution u to this problem (2.1). The idea is similar to the one used in [7]. We prove some equivalence between Problem (2.1) and an optimal transport problem. More precisely, we will show that Problem (2.1) is equivalent to the following minimal flow formulation:

(2.2)
$$\inf_{v \in \mathcal{M}(\overline{\Omega},\mathbb{R}^2),\, \chi \in \mathcal{M}(\partial\Omega\backslash\Gamma)}\left\{\int_{\overline{\Omega}} |v| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}\chi^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}\chi^- : \nabla\cdot v = 0,\ v\cdot n = f + \chi \text{ on } \partial\Omega\right\},$$

where $f = \partial_\tau g$, $\mathcal{M}(\overline{\Omega},\mathbb{R}^2)$ is the set of vector measures over $\overline{\Omega}$ and, $\mathcal{M}(\partial\Omega\backslash\Gamma)$ is the set of measures on $\partial\Omega\backslash\Gamma$. On the other hand, we show that Problem (2.2) is also equivalent to the following optimal transport problem with import/export taxes on $\partial\Omega\backslash\Gamma$ (we note that in [7], $\partial\Omega\backslash\Gamma$ was assumed to be a "free" Dirichlet region which is equivalent to say that $\phi^{\pm} = 0$, while here we have to pay some taxes $\phi^{\pm}$ in order to import/export masses from/to $\partial\Omega\backslash\Gamma$):

$$(2.3)\quad \inf\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \,\mathrm{d}\gamma + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}[(\Pi_x)_\#\gamma] - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}[(\Pi_y)_\#\gamma] \;:\; \gamma \in \Pi(f^+, f^-)\right\}.$$

We recall that in [17, 11, 12] the authors have already studied this import/export transport problem but in the case where the import/export region is the whole boundary $\partial\Omega$ and $f^\pm$ are two densities in the interior of Ω. In the sequel, we will analyse Problem (2.3) in details. More precisely, we will decompose Problem (2.3) into three classical transport problems: a transport problem from Γ to Γ, an export transport problem with tax ϕ^- from Γ to $\partial\Omega\backslash\Gamma$ and, an import transport problem with tax ϕ^+ from $\partial\Omega\backslash\Gamma$ to Γ.

First of all, we need to assume that the pair (ϕ^+,ϕ^-) satisfies the following condition:

$$\phi^-(y)-\phi^+(x)\le |x-y|,\quad \text{for all } x,\, y\in\partial\Omega\backslash\Gamma. \tag{2.4}$$

In fact, this is a natural assumption on (ϕ^+,ϕ^-) since it means that we do not need to transport mass from $\partial\Omega\backslash\Gamma$ onto $\partial\Omega\backslash\Gamma$. Thanks to this condition, one can show existence of a solution to Problem (2.3).

Proposition 2.1. *Under the condition (2.4), Problem (2.3) has an optimal transport plan γ. In addition, we either have $\gamma(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)=0$ or $\phi^-(y)-\phi^+(x)=|x-y|$, for $\gamma-$a.e. $(x,y)\in\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma$. In particular, there is always an optimal transport plan γ such that $\gamma(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)=0$.*

Proof. First, we note that the proof of this proposition is quite similar to the one in [11, Proposition 2.1] but we introduce it here just for the sake of completeness. Let $(\gamma_k)_k$ be a minimizing sequence in Problem (2.3). We define $\widetilde{\gamma_k}:=\gamma_k\cdot 1_{(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)^c}$. It is easy to check that $\widetilde{\gamma_k}\in\Pi(f^+,f^-)$. Moreover, we have

$$\begin{aligned}
&\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\mathrm{d}\gamma_k+\int_{\partial\Omega\backslash\Gamma}\phi^+\,\mathrm{d}[(\Pi_x)_\#\gamma_k]-\int_{\partial\Omega\backslash\Gamma}\phi^-\,\mathrm{d}[(\Pi_y)_\#\gamma_k]\\
&=\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\mathrm{d}\widetilde{\gamma_k}+\int_{\partial\Omega\backslash\Gamma}\phi^+\mathrm{d}[(\Pi_x)_\#\widetilde{\gamma_k}]-\int_{\partial\Omega\backslash\Gamma}\phi^-\mathrm{d}[(\Pi_x)_\#\widetilde{\gamma_k}]\\
&\quad+\int_{\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma}[|x-y|+\phi^+(x)-\phi^-(y)]\mathrm{d}\gamma_k.
\end{aligned}\tag{2.5}$$

Thanks to (2.4), we infer that $(\widetilde{\gamma_k})_k$ is also a minimizing sequence in Problem (2.3). Since $\widetilde{\gamma_k}\in\Pi(f^+,f^-)$ and $\widetilde{\gamma_k}(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)=0$, then

$$\widetilde{\gamma_k}(\overline{\Omega}\times\overline{\Omega})\le f^+(\Gamma)+f^-(\Gamma).$$

Hence, up to a subsequence, $\widetilde{\gamma_k}\rightharpoonup\gamma$, for some $\gamma\in\Pi(f^+,f^-)$. In fact, $(\Pi_x)_\#\widetilde{\gamma_k}=f^++\chi_k^+$ and $(\Pi_y)_\#\widetilde{\gamma_k}=f^-+\chi_k^-$, where $\chi_k^\pm\in\mathcal{M}^+(\partial\Omega\backslash\Gamma)$. And, we see that $\chi_k^\pm\rightharpoonup\chi^\pm$ where $\chi^\pm\in\mathcal{M}^+(\partial\Omega\backslash\Gamma)$. Then, $(\Pi_x)_\#\gamma=f^++\chi^+$ and $(\Pi_y)_\#\gamma=f^-+\chi^-$. This yields that γ minimizes Problem (2.3) since

$$\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\mathrm{d}\widetilde{\gamma_k}+\int_{\partial\Omega\backslash\Gamma}\phi^+\mathrm{d}\chi_k^+-\int_{\partial\Omega\backslash\Gamma}\phi^-\mathrm{d}\chi_k^-\to\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\mathrm{d}\gamma+\int_{\partial\Omega\backslash\Gamma}\phi^+\mathrm{d}\chi^+-\int_{\partial\Omega\backslash\Gamma}\phi^-\mathrm{d}\chi^-.$$

Finally, the second statement follows directly from (2.5), the fact that $\widetilde{\gamma}:=\gamma\cdot 1_{(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)^c}$ is always admissible in (2.3) and, the optimality of γ. □

Let γ be an optimal transport plan in Problem (2.3) with $\gamma(\partial\Omega\backslash\Gamma\times\partial\Omega\backslash\Gamma)=0$. Let χ^+ and χ^- be the two nonnegative measures on $\partial\Omega\backslash\Gamma$ such that $(\Pi_x)_\#\gamma=f^++\chi^+$ and $(\Pi_y)_\#\gamma=f^-+\chi^-$. It is clear that γ also minimizes

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma\;:\;(\Pi_x)_\#\gamma=f^++\chi^+\quad\text{and}\quad(\Pi_y)_\#\gamma=f^-+\chi^-\right\}.$$

Set

$$\gamma(\Gamma,\Gamma)=\gamma_{|\Gamma\times\Gamma},\quad \gamma(\Gamma,\partial\Omega\backslash\Gamma)=\gamma_{|\Gamma\times\partial\Omega\backslash\Gamma},\quad \gamma(\partial\Omega\backslash\Gamma,\Gamma)=\gamma_{|\partial\Omega\backslash\Gamma\times\Gamma},$$

and

$$\nu^+=(\Pi_x)_\#[\gamma(\Gamma,\partial\Omega\backslash\Gamma)],\quad \nu^-=(\Pi_y)_\#[\gamma(\partial\Omega\backslash\Gamma,\Gamma)].$$

Then, we consider the following problems:

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma\,:\,(\Pi_x)_\#\gamma=f^+-\nu^+\ \text{ and }\ (\Pi_y)_\#\gamma=f^--\nu^-\right\}, \tag{2.6}$$

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma-\int_{\partial\Omega\backslash\Gamma}\phi^-\,\mathrm{d}(\Pi_y)_\#\gamma\,:\,(\Pi_x)_\#\gamma=\nu^+\ \text{ and }\ \mathrm{spt}[(\Pi_y)_\#\gamma]\subset\partial\Omega\backslash\Gamma\right\}, \tag{2.7}$$

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma+\int_{\partial\Omega\backslash\Gamma}\phi^+\,\mathrm{d}(\Pi_x)_\#\gamma\,:\,\mathrm{spt}[(\Pi_x)_\#\gamma]\subset\partial\Omega\backslash\Gamma\ \text{ and }\ (\Pi_y)_\#\gamma=\nu^-\right\}. \tag{2.8}$$

Similarly to [7, Proposition 3.3], it is not difficult to prove that the transport plans $\gamma(\Gamma,\Gamma)$, $\gamma(\Gamma,\partial\Omega\backslash\Gamma)$ & $\gamma(\partial\Omega\backslash\Gamma,\Gamma)$ minimize Problems (2.6), (2.7) & (2.8) respectively (this follows directly from the linearity of the functional and the fact that $\gamma=\gamma(\Gamma,\Gamma)+\gamma(\Gamma,\partial\Omega\backslash\Gamma)+\gamma(\partial\Omega\backslash\Gamma,\Gamma)$). In order to characterize these two optimal transport plans $\gamma(\Gamma,\partial\Omega\backslash\Gamma)$ and $\gamma(\partial\Omega\backslash\Gamma,\Gamma)$, we define the following multivalued map $\tilde{T}^{\pm}$ (notice that $\tilde{T}^{\pm}$ is the classical projection map onto $\partial\Omega\backslash\Gamma$ as soon as $\phi^{\pm}=0$):

$$\tilde{T}^{\pm}(x)=\mathrm{argmin}\{|x-y|\pm\phi^{\pm}(y)\,:\,y\in\partial\Omega\backslash\Gamma\},\quad \text{for every }\ x\in\mathbb{R}^2.$$

Now, we introduce the following:

Definition 2.1. *Assume that $\Gamma\subset\partial\Omega$. Then, we say that $\partial\Omega\backslash\Gamma$ is visible from the arc Γ if for every $x\in\Gamma$ and $y\in\partial\Omega\backslash\Gamma$ such that $]x,y[\cap\partial\Omega\backslash\Gamma=\emptyset$, we have $]x,y[\subset\Omega$.*

In the sequel, we will say that the assumption (H) holds if and only if we have the following statement:

$$\text{(H)}\qquad \Gamma\ \text{ is strictly convex}$$

and

$$\Omega\ \text{ is convex }\ \text{or}\ \text{ the convex hull of }\Gamma\text{ is contained in }\overline{\Omega},\ \partial\Omega\backslash\Gamma\text{ is visible from }\Gamma,$$

$$\text{and }\ \phi^{\pm}\ \text{are }\lambda-\text{Lip}\ \text{ with }\lambda<1.$$

Lemma 2.2. *Assume that (H) holds. Then, we have $]x,y[\subset\Omega$ for all $y\in\tilde{T}^{\pm}(x)$. Moreover, there is a countable set $D^{\pm}\subset\Gamma$ such that $\tilde{T}^{\pm}$ is single valued on $\Gamma\backslash D^{\pm}$.*

Proof. Let $D\subset\Gamma$ be the set of points x such that $\tilde{T}^-(x)$ is not a singleton. For every $x\in D$, let us denote by $T_1(x)$ and $T_2(x)$ two different elements of $\tilde{T}^-(x)$. Let Δ_x be the interior of the region delimited by $[x,T_1(x)]$, $[x,T_2(x)]$ and $\partial\Omega\backslash\Gamma$. First, we claim that $\Delta_x\subset\Omega$ with $\mathcal{L}^2(\Delta_x)>0$, where $\mathcal{L}^2$ denotes the Lebesgue measure on $\mathbb{R}^2$. If Ω is convex then we clearly have $\Delta_x\subset\Omega$ and $\mathcal{L}^2(\Delta_x)>0$ since Γ is an open strictly convex arc of $\partial\Omega$. Now, assume that $\partial\Omega\backslash\Gamma$ is visible from Γ and ϕ^- is $\lambda-\text{Lip}$ with $\lambda<1$. Assume there is a point $z\in[x,T_1(x)[\cap\partial\Omega\backslash\Gamma$. Then, we have

$$|x-T_1(x)|-\phi^-(T_1(x))\leq|x-z|-\phi^-(z).$$

Hence,

$$|z - T_1(x)| \leq \phi^-(T_1(x)) - \phi^-(z),$$

which is a contradiction since ϕ^- is $\lambda-$Lip with $\lambda < 1$. This implies that $[x, T_1(x)[\cap \partial\Omega\backslash\Gamma = \emptyset$. Since $\partial\Omega\backslash\Gamma$ is visible from Γ, then one has $]x, T_1(x)[\subset \Omega$ (and, $]x, T_2(x)[\subset \Omega$). This yields again that $\Delta_x \subset \Omega$ and $\mathcal{L}^2(\Delta_x) > 0$.

On the other hand, we claim that these sets $\{\Delta_x\}_{x\in D}$ are disjoint. For this aim, we just need to show that $\tilde{T}(z) = \{T_1(x)\}$, for every $z \in]x, T_1(x)[$ and $x \in \Gamma$. Assume that Ω is convex. For all $y \in \partial\Omega\backslash\Gamma$, one has

$$|z-T_1(x)|-\phi^-(T_1(x)) = |x-T_1(x)|-|x-z|-\phi^-(T_1(x)) \leq |x-y|-|x-z|-\phi^-(y) < |z-y|-\phi^-(y),$$

where the last inequality comes from the fact that x, z and y are not aligned. Now, assume that ϕ^- is $\lambda-$Lip with $\lambda < 1$. If x, z and y are aligned, then we clearly have

$$|x - T_1(x)| - \phi^-(T_1(x)) < |x - y| - \phi^-(y).$$

But, this implies again that

$$|z - T_1(x)| - \phi^-(T_1(x)) < |z - y| - \phi^-(y).$$

Assume that x, $x' \in \Gamma$ and $\Delta_x \cap \Delta_{x'} \neq \emptyset$. Then, there is a point $z \in]x, T_1(x)[\cap]x', T_1(x')[$. But, $\tilde{T}(z) = \{T_1(x)\} = \{T_1(x')\}$, which is a contradiction since $\Delta_x \cap \Delta_{x'} \neq \emptyset$. Hence, the second claim is also proved. Consequently, the set D is at most countable. $\square$

On the other hand, it is clear that the graph of $\tilde{T}^\pm$ is closed (thanks to the continuity of $\phi^\pm$) and so, $\tilde{T}^\pm$ admits a Borel selector function which will be denoted by $T^\pm$. Now, we are ready to give a characterization of $\gamma(\Gamma, \partial\Omega\backslash\Gamma)$ and $\gamma(\partial\Omega\backslash\Gamma, \Gamma)$. More precisely, we have the following:

Proposition 2.3. *The transport plans $(Id, T^-)_\#\nu^+$ and $(T^+, Id)_\#\nu^-$ minimize Problems (2.7) & (2.8), respectively. Moreover, for $\gamma(\Gamma, \partial\Omega\backslash\Gamma)-$ a.e. (x, y), $y \in \tilde{T}^-(x)$ and, for $\gamma(\partial\Omega\backslash\Gamma, \Gamma)-$ a.e. (x, y), $x \in \tilde{T}^+(y)$. In addition, if (H) holds and $f^\pm$ are atomless (i.e. $f^\pm(\{x\}) = 0$, for all $x \in \Gamma$), then $\gamma(\Gamma, \partial\Omega\backslash\Gamma) = (Id, T^-)_\#\nu^+$ and $\gamma(\partial\Omega\backslash\Gamma, \Gamma) = (T^+, Id)_\#\nu^-$.*

Proof. Let us prove that for $\gamma(\Gamma, \partial\Omega\backslash\Gamma)-$ a.e. (x, y), $y \in \tilde{T}^-(x)$ (in the same way, we prove that for $\gamma(\partial\Omega\backslash\Gamma, \Gamma)-$ a.e. (x, y), $x \in \tilde{T}^+(y)$). For this aim, assume that this is not the case. Then, we get

$$\int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \, \mathrm{d}[\gamma(\Gamma, \partial\Omega\backslash\Gamma)] - \int_{\partial\Omega\backslash\Gamma} \phi^- \, \mathrm{d}[(\Pi_y)_\#\gamma(\Gamma, \partial\Omega\backslash\Gamma)] = \int_{\overline{\Omega}\times\overline{\Omega}} [|x-y| - \phi^-(y)] \, \mathrm{d}[\gamma(\Gamma, \partial\Omega\backslash\Gamma)]$$

$$> \int_{\overline{\Omega}\times\overline{\Omega}} [|x - T^-(x)| - \phi^-(T^-(x))] \, \mathrm{d}[\gamma(\Gamma, \partial\Omega\backslash\Gamma)]$$

$$= \int_{\overline{\Omega}\times\overline{\Omega}} |x - y| \, \mathrm{d}[(Id, T^-)_\#\nu^+] - \int_{\partial\Omega\backslash\Gamma} \phi^- \, \mathrm{d}[(\Pi_y)_\#[(Id, T^-)_\#\nu^+]].$$

But, this is a contradiction since $\gamma(\Gamma, \partial\Omega\backslash\Gamma)$ minimizes Problem (2.9) and $(Id, T^-)_\#\nu^+$ is admissible in (2.9). This shows at the same time that $(Id, T^-)_\#\nu^+$ is a minimizer in (2.9). Now, assume that f^+ is atomless. Then by Lemma 2.2, for $\gamma(\Gamma, \partial\Omega\backslash\Gamma)-$ a.e. (x, y), we have $y \in \tilde{T}^-(x) = \{T^-(x)\}$ and so, $\gamma(\Gamma, \partial\Omega\backslash\Gamma) = (Id, T^-)_\#\nu^+$. $\square$

In particular, under the assumption that $f^\pm$ are atomless, we see that $\gamma(\Gamma, \partial\Omega\backslash\Gamma)$ and $\gamma(\partial\Omega\backslash\Gamma, \Gamma)$ minimize the following Kantorovich problems, respectively:

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x - y| \, \mathrm{d}\gamma \, : \, (\Pi_x)_\#\gamma = \nu^+ \quad \text{and} \quad (\Pi_y)_\#\gamma = T^-_\#\nu^+ \right\} \tag{2.9}$$

and

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma\,:\,(\Pi_x)_{\#}\gamma=T^+_{\#}\nu^-\ \text{ and }\ (\Pi_y)_{\#}\gamma=\nu^-\right\}. \tag{2.10}$$

On the other hand, the key point in the proof of existence of a solution to Problem (2.3) is to show that there are no transport rays gliding on the boundary. More precisely, we have the following:

Proposition 2.4. *Assume that (H) holds. Then, for $\gamma-$a.e. (x,y), we have $]x,y[\subset\Omega$. In particular, the transport density σ_γ associated with γ (see (1.6)) is well defined and, it gives zero mass to $\partial\Omega$ (i.e. $\sigma_\gamma[\partial\Omega]=0$).*

Proof. Thanks to (H), it is clear that $]x,y[\subset\Omega$ for $\gamma(\Gamma,\Gamma)-$a.e. (x,y). From Lemma 2.2, we also have $]x,y[\subset\Omega$ for $\gamma(\Gamma,\partial\Omega\backslash\Gamma)$ (resp. $\gamma(\partial\Omega\backslash\Gamma,\Gamma))-$a.e. (x,y). Hence, $]x,y[\subset\Omega$ for $\gamma-$a.e. (x,y). Recalling (1.6), this implies that σ_γ is well defined and, we have

$$\sigma_\gamma[\partial\Omega]=\int_{\partial\Omega\times\partial\Omega}\mathcal{H}^1(\partial\Omega\cap[x,y])\,\mathrm{d}\gamma(x,y)=0.\quad\square$$

It is also possible to show that there is at least one special optimal transport plan γ such that the corresponding transport density σ_γ is well defined and has zero mass on $\partial\Omega$, without assuming that the convex hull of Γ is contained in $\overline{\Omega}$ but we need instead to reinforce the assumptions on $\phi^\pm$. In the sequel, we will say that the assumption (H$'$) holds if we have the following statement:

(H$'$) Γ is strictly convex, $\partial\Omega\backslash\Gamma$ is visible from Γ and, $\phi^+=\phi^-$ is $1-\mathrm{Lip}$ on $\partial\Omega\backslash\Gamma$.

Proposition 2.5. *Assume that (H$'$) holds. Then, there is an optimal transport plan γ for Problem (2.3) such that for $\gamma-$a.e. (x,y), we have $]x,y[\subset\Omega$. In particular, $\sigma_\gamma[\partial\Omega]=0$.*

Proof. Set $E:=\{(x,y)\in\Gamma\times\Gamma\ :]x,y[\subset\Omega\}$. Let γ be an optimal transport plan in (2.3) with $(\Pi_x)_{\#}\gamma=f^++\chi^+$ and $(\Pi_y)_{\#}\gamma=f^-+\chi^-$. Then, we define $\gamma^\star:=\gamma(\Gamma,\Gamma)_{|E}+P^+_{\#}[\gamma(\Gamma,\Gamma)_{|E^c}]+P^-_{\#}[\gamma(\Gamma,\Gamma)_{|E^c}]+\gamma(\Gamma,\partial\Omega\backslash\Gamma)+\gamma(\partial\Omega\backslash\Gamma,\Gamma)$, where the maps P^+ and P^- are defined on E^c as follows:

$$P^+(x,y)=(x,y')\ \text{ such that }\ y'\in]x,y[\cap\partial\Omega\backslash\Gamma \text{ and }]x,y'[\subset\Omega$$

and

$$P^-(x,y)=(x',y)\ \text{ such that }\ x'\in]x,y[\cap\partial\Omega\backslash\Gamma \text{ and }]x',y[\subset\Omega.$$

First, it is not difficult to check that $\gamma^\star\in\Pi(f^+,f^-)$. Moreover, thanks to the fact that $\phi^+=\phi^-$ is $1-\mathrm{Lip}$, we have the following:

$$\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma^\star+\int_{\partial\Omega\backslash\Gamma}\phi^+\,\mathrm{d}[(\Pi_x)_{\#}\gamma^\star]-\int_{\partial\Omega\backslash\Gamma}\phi^-\,\mathrm{d}[(\Pi_y)_{\#}\gamma^\star]$$

$$=\int_E|x-y|\,\mathrm{d}[\gamma(\Gamma,\Gamma)]+\int_{E^c}[|x-y'|+|x'-y|+\phi^+(x')-\phi^-(y')]\,\mathrm{d}[\gamma(\Gamma,\Gamma)]+\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma(\Gamma,\partial\Omega\backslash\Gamma)$$

$$+\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma(\partial\Omega\backslash\Gamma,\Gamma)+\int_{\partial\Omega\backslash\Gamma}\phi^+\,\mathrm{d}\chi^+-\int_{\partial\Omega\backslash\Gamma}\phi^-\,\mathrm{d}\chi^-$$

$$\leq\int_E|x-y|\,\mathrm{d}[\gamma(\Gamma,\Gamma)]+\int_{E^c}[|x-y'|+|x'-y|+|x'-y'|]\,\mathrm{d}[\gamma(\Gamma,\Gamma)]+\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma(\Gamma,\partial\Omega\backslash\Gamma)$$

$$+\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,\mathrm{d}\gamma(\partial\Omega\backslash\Gamma,\Gamma)+\int_{\partial\Omega\backslash\Gamma}\phi^+\,\mathrm{d}\chi^+-\int_{\partial\Omega\backslash\Gamma}\phi^-\,\mathrm{d}\chi^-$$

$$\leq \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| \mathrm{d}\gamma + \int_{\partial\Omega\setminus\Gamma} \phi^+ \,\mathrm{d}[(\Pi_x)_\# \gamma] - \int_{\partial\Omega\setminus\Gamma} \phi^- \,\mathrm{d}[(\Pi_y)_\# \gamma].$$

Yet, we recall that γ is an optimal transport plan in Problem (2.3) and so, $\gamma^\star$ is also a minimizer. By definition, we have that $]x,y[\subset \Omega$, for $\gamma^\star-$a.e. (x,y). But, this yields that $\sigma_{\gamma^\star}[\partial\Omega]=0$. □

Thanks to Propositions 2.4 & 2.5, one can always find a "good" optimal transport plan γ such that $]x,y[\subset \Omega$, for $\gamma-$a.e. (x,y) (so, σ_γ is well defined and $\sigma_\gamma[\partial\Omega]=0$), provided that one of the assumptions (H) or (H$'$) is well satisfied. Now, we are ready to prove some equivalence between Problems (2.2) & (2.3).

Proposition 2.6. *Assume that (H) or (H$'$) holds. Let γ be a "good" optimal transport plan in* (2.3) *with $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. Then, we have the following:*

(1) The minimal values of (2.2) & (2.3) *coincide, i.e.* $\min(2.2) = \min(2.3)$.

(2) Let v_γ be the vector measure in (1.5)*. Then, (v_γ,χ) solves Problem* (2.2) *and* $|v_\gamma|[\partial\Omega]=0$.

(3) If (v,χ) is a minimizer for Problem (2.2)*, then there is an optimal transport plan γ in* (2.3) *such that v_γ is well defined and $v=v_\gamma$ with $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$.*

Proof. We will show statements (1) & (2) simultaneously. Since γ is a "good" optimal transport plan, then the vector measure v_γ (see (1.5)) is well defined. Moreover, (v_γ,χ) is admissible in (2.2) since, for all $\varphi\in C^1(\overline{\Omega})$, we have

$$< v_\gamma, \nabla\varphi >= \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \nabla\varphi((1-t)x+ty)\cdot(x-y)\,\mathrm{d}t\,\mathrm{d}\gamma(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}} [\varphi(x)-\varphi(y)]\,\mathrm{d}\gamma(x,y)$$

$$= \int_{\partial\Omega} \varphi\,\mathrm{d}[(f^+ + \chi^+) - (f^- + \chi^-)].$$

Moreover, we have

$$\int_{\overline{\Omega}} |v_\gamma| + \int_{\partial\Omega\setminus\Gamma} \phi^+\,\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\,\mathrm{d}\chi^- = \sigma_\gamma(\overline{\Omega}) + \int_{\partial\Omega\setminus\Gamma} \phi^+\,\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\,\mathrm{d}\chi^-$$

$$= \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}\gamma + \int_{\partial\Omega\setminus\Gamma} \phi^+\,\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\,\mathrm{d}\chi^- = \min(2.3). \tag{2.11}$$

But, we claim that

$$\min(2.2) \geq \min(2.3).$$

Similarly to [11, Proposition 2.2], one can show that Problem (2.3) has a dual formulation which is the following:

$$\sup\left\{ \int_{\overline{\Omega}} \varphi\,\mathrm{d}(f^- - f^+) \,:\, \varphi\in \mathrm{Lip}_1(\overline{\Omega}),\ \phi^- \leq \varphi \leq \phi^+ \ \text{ on }\ \partial\Omega\setminus\Gamma \right\}. \tag{2.12}$$

If φ is a smooth admissible function in Problem (2.12) and (v,χ) is admissible in Problem (2.2), then we have

$$\int_{\overline{\Omega}} |v| \geq -\int_{\overline{\Omega}} \nabla\varphi\cdot \mathrm{d}v = -\int_{\partial\Omega} \varphi\,\mathrm{d}[f+\chi] = -\int_{\Gamma} \varphi\,\mathrm{d}f - \int_{\partial\Omega\setminus\Gamma} \varphi\,\mathrm{d}\chi^+ + \int_{\partial\Omega\setminus\Gamma} \varphi\,\mathrm{d}\chi^-$$

$$\geq -\int_\Gamma \varphi\,\mathrm{d}f - \int_{\partial\Omega\setminus\Gamma} \phi^+\,\mathrm{d}\chi^+ + \int_{\partial\Omega\setminus\Gamma} \phi^-\,\mathrm{d}\chi^-.$$

Hence,

$$\min(2.2) \geq \sup(2.12) = \min(2.3).$$

Recalling (2.11), we infer that $\min(2.2) = \min(2.3)$ and (v_γ, χ) solves Problem (2.2). Now, let us prove statement (3). Let (v, χ) be a minimizer in (2.2). In particular, we see that v solves

$$\min\left\{ \int_{\overline{\Omega}} |v| \,:\, v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2),\ \nabla\cdot v = 0 \ \text{ and } \ v\cdot n = f + \chi \ \text{ on } \ \partial\Omega \right\}. \tag{2.13}$$

In order to show that there is an optimal transport plan γ such that $v = v_\gamma$ with $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$, the idea will be to adapt the proofs of [25, Theorem 4.13] or [10, Proposition 2.4]. First of all, we need to introduce some objects that generalize both σ_γ and v_γ. Let $\mathcal{C}$ be the set of absolutely continuous curves $w : [0,1] \mapsto \overline{\Omega}$. We call traffic plan any positive measure Q on $\mathcal{C}$ such that $(e_0)_\# Q = f^+ + \chi^+$ and $(e_1)_\# Q = f^- + \chi^-$, where $e_0(w) := w(0)$ and $e_1(w) = w(1)$. Following [10, 25], we define the traffic intensity $i_Q \in \mathcal{M}^+(\overline{\Omega})$ and the traffic flow $v_Q \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ as follows:

$$< i_Q, \varphi >= \int_{\mathcal{C}} \int_0^1 \varphi(w(t))|w'(t)|\,\mathrm{d}t\,\mathrm{d}Q(w), \quad \text{for all } \ \varphi \in C(\overline{\Omega}),$$

and

$$< v_Q, \xi >= -\int_{\mathcal{C}} \int_0^1 \xi(w(t))\cdot w'(t)\,\mathrm{d}t\,\mathrm{d}Q(w), \quad \text{for all } \ \xi \in C(\overline{\Omega}, \mathbb{R}^2).$$

It is easy to see that $|v_Q| \leq i_Q$, $\nabla\cdot v_Q = 0$ and $v_Q\cdot n = f + \chi$. Moreover, by [10, Lemma 2.2], if $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ is such that $\nabla\cdot v = 0$ and $v\cdot n = f + \chi$, then there is a traffic plan Q such that $|v - v_Q|(\overline{\Omega}) + i_Q(\overline{\Omega}) = |v|(\overline{\Omega})$. Since v minimizes (2.13), then we have

$$\int_{\overline{\Omega}} |v| \leq \int_{\overline{\Omega}} |v_Q| \leq \int_{\overline{\Omega}} i_Q.$$

Hence, $v = v_Q$ and $|v| = i_Q$. Thanks to the fact that the pair (v, χ) minimizes (2.2) and $(\Pi_x)_\#[(e_0, e_1)_\# Q] = f^+ + \chi^+$ and $(\Pi_y)_\#[(e_0, e_1)_\# Q] = f^- + \chi^-$, one has

$$\min(2.2)$$

$$= \int_{\overline{\Omega}} |v| + \int_{\partial\Omega\setminus\Gamma} \phi^+\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\mathrm{d}\chi^- = \int_{\overline{\Omega}} i_Q + \int_{\partial\Omega\setminus\Gamma} \phi^+\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\mathrm{d}\chi^-$$

$$= \int_{\mathcal{C}} \int_0^1 |w'(t)|\,\mathrm{d}t\,\mathrm{d}Q(w) + \int_{\partial\Omega\setminus\Gamma} \phi^+\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\mathrm{d}\chi^-$$

$$\geq \int_{\mathcal{C}} |w(0) - w(1)|\,\mathrm{d}Q(w) + \int_{\partial\Omega\setminus\Gamma} \phi^+\mathrm{d}\chi^+ - \int_{\partial\Omega\setminus\Gamma} \phi^-\mathrm{d}\chi^-$$

$$= \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,\mathrm{d}[(e_0,e_1)_\# Q] + \int_{\partial\Omega\setminus\Gamma} \phi^+\,\mathrm{d}[(\Pi_x)_\#[(e_0,e_1)_\# Q]] - \int_{\partial\Omega\setminus\Gamma} \phi^-\,\mathrm{d}[(\Pi_y)_\#[(e_0,e_1)_\# Q]]$$

$$\geq \min(2.3).$$

Yet, statement (1) implies that the above inequalities are actually equalities. In particular, Q must be concentrated on line segments and the transport plan $\gamma := (e_0, e_1)_\# Q$ minimizes (2.3). Consequently, we have $v = v_Q = v_\gamma$. □

On the other hand, one can also show some equivalence between Problems (2.1) & (2.2). More precisely, we have the following:

Proposition 2.7. *Assume that $g \in BV(\Gamma)$ and let f be the tangential derivative of g (i.e. $f = \partial_\tau g$). Then, we have the following statements:*

(1) The minimal values of (2.1) & (2.2) coincide, i.e. $\min(2.1) = \min(2.2)$.

(2) Let u be a solution for Problem (2.1) with $u_{|\partial\Omega} = \tilde{g}$. Set $v := R_{\frac{\pi}{2}} Du$ and $\chi := [\partial_\tau \tilde{g}]_{|\partial\Omega\backslash\Gamma}$. Then, (v,χ) solves Problem (2.2).

(3) Moreover, if (v,χ) is a minimizer in Problem (2.2) with $|v|[\partial\Omega] = 0$ then there exists a BV function u such that $v = R_{\frac{\pi}{2}} Du$ and, u turns out to be a solution for Problem (2.1).

Proof. First, we prove statement (1). For every $h \in BV(\partial\Omega\backslash\Gamma)$, we denote by $\tilde{g}_h$ a BV extension of g to $\partial\Omega$ such that $\tilde{g}_h = h$ on $\partial\Omega\backslash\Gamma$. Then, we have

$$\inf\left\{ \int_\Omega |Du| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}[\partial_\tau u]^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}[\partial_\tau u]^- \,:\, u \in BV(\Omega),\, u_{|\partial\Omega} \in BV(\partial\Omega),\, u_{|\Gamma} = g \right\}$$

$$= \inf_{h \in BV(\partial\Omega\backslash\Gamma)} \left\{ \inf\left\{ \int_\Omega |Du| : u \in BV(\Omega),\, u_{|\partial\Omega} = \tilde{g}_h \right\} + \int_{\partial\Omega\backslash\Gamma} \phi^+ \mathrm{d}[\partial_\tau \tilde{g}_h]^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \mathrm{d}[\partial_\tau \tilde{g}_h]^- \right\}.$$

But, by [13, Theorem 3.4] and the fact that Ω is assumed to be contractible and $\tilde{g}_h \in BV(\partial\Omega)$, we have the following equality:

$$\inf\left\{ \int_\Omega |Du| \,:\, u \in BV(\Omega),\, u_{|\partial\Omega} = \tilde{g}_h \right\}$$

$$= \inf\left\{ \int_{\overline{\Omega}} |v| \,:\, v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2),\; \nabla \cdot v = 0 \;\text{ and }\; v \cdot n = \tilde{f}_h \;\text{ on }\; \partial\Omega \right\},$$

where $\tilde{f}_h := \partial_\tau \tilde{g}_h$. Yet, it is clear that $\tilde{f}_h = f + \chi$, for some $\chi \in \mathcal{M}(\partial\Omega\backslash\Gamma)$. Then, we get the following:

$$\inf(2.1) = \inf_{\chi \in \mathcal{M}(\partial\Omega\backslash\Gamma)} \left\{ \inf\left\{ \int_{\overline{\Omega}} |v| \,:\, v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2),\; \nabla \cdot v = 0 \text{ and } v \cdot n = f + \chi \text{ on } \partial\Omega \right\} \right.$$
$$\left. + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}\chi^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}\chi^- \right\} = \inf(2.2).$$

Now, we prove statement (2). Let u be a minimizer in (2.1) with $u_{|\partial\Omega} = \tilde{g}$. First, let us check that the pair (v,χ), where $v = R_{\frac{\pi}{2}} Du$ and $\chi = [\partial_\tau \tilde{g}]_{|\partial\Omega\backslash\Gamma}$, is admissible in (2.2). For all $\varphi \in C^1(\overline{\Omega})$, we have

$$\int_\Omega R_{-\frac{\pi}{2}} \nabla\varphi \cdot Du = \int_{\partial\Omega} [R_{-\frac{\pi}{2}} \nabla\varphi \cdot n] u \,\mathrm{d}\mathcal{H}^1 = -\int_{\partial\Omega} u \,\partial_\tau \varphi \,\mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega} \varphi \,\mathrm{d}[\partial_\tau u].$$

Yet, $\partial_\tau u = f + \chi$. Then, we get

$$\int_\Omega \nabla\varphi \cdot \mathrm{d}v = \int_{\partial\Omega} \varphi \,\mathrm{d}[f + \chi], \;\text{ for all }\; \varphi \in C^1(\overline{\Omega}).$$

Moreover, we have

$$\int_{\overline{\Omega}} |v| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}\chi^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}\chi^- = \int_\Omega |Du| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}[\partial_\tau u]^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}[\partial_\tau u]^-$$
$$= \min(2.1) = \min(2.2).$$

Then, (v,χ) solves Problem (2.2). It remains to prove statement (3). Let (v,χ) be a solution to Problem (2.2) with $|v|[\partial\Omega] = 0$. Let us extend v by 0 outside Ω. Set $v_\varepsilon = v * \rho_\varepsilon$, where

ρ_ε is a sequence of mollifiers. First, it is clear that $\nabla \cdot v_\varepsilon = 0$. Let u_ε be a smooth function such that $\nabla u_\varepsilon = R_{-\frac{\pi}{2}} v_\varepsilon$. Up to adding a constant, one can assume that $\int_\Omega u_\varepsilon = 0$ and then, we have

$$\int_\Omega |u_\varepsilon| \,\mathrm{d}x \leq C \int_\Omega |\nabla u_\varepsilon| \,\mathrm{d}x \leq C \int_\Omega |v| \,\mathrm{d}x.$$

Then, we get

$$\|u_\varepsilon\|_{W^{1,1}(\Omega)} \leq (C+1) \int_\Omega |v|.$$

Hence, up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges weakly* in $BV(\Omega)$ to some function u. And, we have $Du = R_{-\frac{\pi}{2}} v$. Moreover, $u_\varepsilon \to u$ strictly in BV since $|v_\varepsilon| \rightharpoonup |v|$. Thanks to the continuity of the trace map with respect to the strict convergence in BV, we get that

$$\int_\Omega \nabla\varphi \cdot \mathrm{d}[R_{\frac{\pi}{2}} Du] = \lim_{\varepsilon\to 0} \int_\Omega R_{\frac{\pi}{2}} \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\partial\Omega} [R_{\frac{\pi}{2}} \nabla u_\varepsilon \cdot n]\varphi \,\mathrm{d}\mathcal{H}^1 = \lim_{\varepsilon\to 0} \int_{\partial\Omega} \partial_\tau u_\varepsilon \,\varphi \,\mathrm{d}\mathcal{H}^1$$

$$= -\lim_{\varepsilon\to 0} \int_{\partial\Omega} u_\varepsilon \,\partial_\tau\varphi \,\mathrm{d}\mathcal{H}^1 = -\int_{\partial\Omega} u \,\partial_\tau\varphi \,\mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega} \varphi \,\mathrm{d}[\partial_\tau u], \text{ for all } \varphi \in C^1(\overline{\Omega}).$$

Yet, $v = R_{\frac{\pi}{2}} Du$, $\nabla \cdot v = 0$ and $v \cdot n = f + \chi$. This implies that $\partial_\tau u = f + \chi$. Hence, up to adding a constant, one can assume that $u_{|\Gamma} = g$. In addition, u solves Problem (2.1) since

$$\int_\Omega |Du| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}[\partial_\tau u]^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}[\partial_\tau u]^- = \int_{\overline{\Omega}} |v| + \int_{\partial\Omega\backslash\Gamma} \phi^+ \,\mathrm{d}\chi^+ - \int_{\partial\Omega\backslash\Gamma} \phi^- \,\mathrm{d}\chi^-$$

$$= \min\,(2.2) = \min\,(2.1).$$

□

Consequently, we get equivalence between Problems (2.1), (2.2) & (2.3). Finally, we are in a position to prove existence of a solution for Problem (2.1). To be more precise, we have the following existence result (always under the assumption that (2.4) is well satisfied):

Theorem 2.8. *Assume that (H) or (H′) holds. Then, there exists a function $u \in BV(\Omega)$ which attains the infimum in Problem (2.1).*

Proof. Let γ be a "good" optimal transport plan in (2.3) with $(\Pi_x)_\#\gamma = f^+ + \chi^+$ and $(\Pi_y)_\#\gamma = f^- + \chi^-$. Thanks to Proposition 2.6, we know that (v_γ, χ) solves Problem (2.2) and $|v_\gamma|[\partial\Omega] = \sigma_\gamma[\partial\Omega] = 0$ (recall Propositions 2.4 & 2.5). Hence, by Proposition 2.7, we infer that there is a BV function u such that $v_\gamma = R_{\frac{\pi}{2}} Du$ and, this u is in fact a solution to Problem (2.1). □

Now, we study the uniqueness of the solution u in (2.1). For this aim, we need to restrict our assumption (2.4). Let us assume that there exists $\lambda < 1$ such that

$$\phi^-(y) - \phi^+(x) \leq \lambda|x-y|, \quad \text{for all } x, y \in \partial\Omega\backslash\Gamma. \tag{2.14}$$

Under the assumption (2.14), one can prove uniqueness of the optimal transport plan in (2.3). Then, we have the following:

Proposition 2.9. *Assume that (H) or (H′) holds and, f^+ and f^- are atomless. Then, there is a unique "good" optimal transport plan γ in Problem (2.3). In addition, Problem (2.2) has a unique minimizer.*

Proof. Let γ be an optimal transport plan in (2.3). Thanks to Proposition 2.1, it is easy to see that the condition (2.14) yields that $\gamma(\partial\Omega\backslash\Gamma \times \partial\Omega\backslash\Gamma) = 0$. Let us decompose again γ into $\gamma(\Gamma,\Gamma) := \gamma_{|\Gamma\times\Gamma}$, $\gamma(\Gamma,\partial\Omega\backslash\Gamma) := \gamma_{|\Gamma\times\partial\Omega\backslash\Gamma}$ and $\gamma(\partial\Omega\backslash\Gamma,\Gamma) := \gamma_{|\partial\Omega\backslash\Gamma\times\Gamma}$. Moreover, we set $\nu^+ = (\Pi_x)_\#[\gamma(\Gamma,\partial\Omega\backslash\Gamma)]$ and $\nu^- = (\Pi_y)_\#[\gamma(\partial\Omega\backslash\Gamma,\Gamma)]$. First of all, one can show that there are two sets $A^\pm \subset \Gamma$ such that $A^\pm$ are two countable union of connected arcs and $\nu^\pm = f^\pm \cdot \chi_{A^\pm}$. This follows from the fact that $0 \le \nu^\pm \le f^\pm$ while the set of points which are transported at the same time to Γ and $\partial\Omega\backslash\Gamma$ is at most countable; we refer the reader to [7, Lemma 3.8] for more details. In order to show uniqueness of the optimal transport plan γ, we proceed as in [7, Proposition 3.9] and so, it is sufficient to show that these three parts of γ are all induced by maps. Indeed, the functional in (2.3) is linear in γ and the constraint $\Pi(f^+,f^-)$ is convex. This means that if γ_1 and γ_2 minimize (2.3) then $\frac{\gamma_1+\gamma_2}{2}$ is aslo a minimizer in (2.3); but this yields to a contradiction as soon as we prove that the three corresponding parts of any optimal transport plan γ are induced by maps. From Proposition 2.3 and the fact that f^+ and f^- are atomless, we know that $\gamma(\Gamma,\partial\Omega\backslash\Gamma) = (Id,T^-)_\#\nu^+$ and $\gamma(\partial\Omega\backslash\Gamma,\Gamma) = (T^+,Id)_\#\nu^-$. It remains to show that $\gamma(\Gamma,\Gamma)$ is also induced by a map. Let $D \subset \Gamma$ be the set of points that belong to two different transport rays. For every $x \in D$, let us denote by $R_x^\pm$ two different transport rays from x to Γ. Let $\Delta_x \subset \Omega$ be the region delimited by R_x^+, R_x^- and Γ. Since transport rays cannot intersect at an interior point, then we see that these sets $\{\Delta_x\}_x$ must be disjoint with $\mathcal{L}^2(\Delta_x) > 0$. This implies that the set D is at most countable. Hence, thanks again to the fact that f^+ is atomless, we get that $f^+(D) = 0$. In other words, for f^+-a.e. $x \in \Gamma$, there is a unique transport ray R_x starting at x and intersecting Γ at exactly one point (recall that Γ is strictly convex). But, this means that $\gamma(\Gamma,\Gamma)$ is also induced by a map. The second statement follows immediately from Proposition 2.6. □

Finally, we are ready to state our result on the uniqueness of the solution u in Problem (2.1). Hence, we conclude this section by the following (we always assume that (2.14) is well satisfied):

Theorem 2.10. *Assume that (H) or (H′) holds. Then, the solution u of Problem (2.1) is unique provided that $g \in C(\Gamma)$.*

Proof. Let u be a minimizer in (2.1). Thanks to Proposition 2.7, we know that the pair (v,χ), where $v = R_{\frac{\pi}{2}}Du$ and $\chi = [\partial_\tau u]_{|\partial\Omega\backslash\Gamma}$, is a minimizer in Problem (2.2). On the other hand, since $g \in C(\Gamma)$ then $f = \partial_\tau g$ is atomless. But so, by Proposition 2.9, (v,χ) is the unique minimizer in (2.2). This implies that the solution u of Problem (2.1) is also unique. □

3. Sobolev regularity on the solution of the mixed least gradient problem

In this section, we study the $W^{1,p}$ regularity of the solution u in Problem (2.1). Thanks to Proposition 2.7, this is equivalent to study the L^p summability of the optimal flow v in (2.2) or equivalently, the L^p summability of the transport density σ in Problem (2.3) (i.e. between $f^+ + \chi^+$ and $f^- + \chi^-$, where $\chi^\pm$ represent the import/export masses on $\partial\Omega\backslash\Gamma$). We recall that studying the L^p summability of σ between two singular measures (i.e. if $f^\pm \notin L^p(\Omega)$) is a delicate question! However, the authors of [9] proved that the transport density σ, between two measures $f^\pm$ on $\partial\Omega$, is in $L^p(\Omega)$ as soon as $f^\pm \in L^p(\partial\Omega)$ with $p \le 2$ and Ω is uniformly convex. Moreover, they introduced a counter-example to the L^p summability of σ for $p > 2$. Yet, they also showed some L^p estimates on σ for $p > 2$ provided that $f^\pm$ are smooth enough. Anyway, in Problem (2.3), the measures χ^+ and χ^- are unknown and so, it is not clear whether $\chi^\pm \in L^p(\partial\Omega\backslash\Gamma)$ or not. Before proving our L^p estimates on σ, we need to introduce the following:

Definition 3.1. *We say that $\Gamma \subset \partial\Omega$ is uniformly convex if there exists $R < \infty$ such that, for every $x \in \Gamma$ and every unit vector n in the exterior normal cone to Ω at x, we have $\Gamma \subset B(z, R)$ with $z = x - Rn$.*

In the sequel, we will always assume that (H) or (H′) holds, $f^\pm$ are at least in $L^1(\Gamma)$ (so, $f^\pm$ are atomless) and (2.14) is well satisfied. Hence, by Proposition 2.9, we know that the optimal transport plan γ in (2.3) is unique. Let us decompose again γ into three parts: $\gamma(\Gamma,\Gamma)$, $\gamma(\Gamma,\partial\Omega\backslash\Gamma)$ and $\gamma(\partial\Omega\backslash\Gamma,\Gamma)$. In addition, let $\sigma(\Gamma,\Gamma)$, $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ and $\sigma(\partial\Omega\backslash\Gamma,\Gamma)$ be the transport densities associated with $\gamma(\Gamma,\Gamma)$, $\gamma(\Gamma,\partial\Omega\backslash\Gamma)$ and $\gamma(\partial\Omega\backslash\Gamma,\Gamma)$, respectively. If σ is the transport density associated with γ, then it is clear that $\sigma = \sigma(\Gamma,\Gamma) + \sigma(\Gamma,\partial\Omega\backslash\Gamma) + \sigma(\partial\Omega\backslash\Gamma,\Gamma)$. Thanks to [9], we have the following:

Proposition 3.1. *Assume that Γ is uniformly convex. Then, the transport density $\sigma(\Gamma,\Gamma)$ belongs to $L^p(\Omega)$ provided that $f^\pm \in L^p(\Gamma)$ with $p \le 2$ or $f^\pm \in C^{0,\alpha}(\Gamma)$ with $0 < \alpha \le 1$ and $p = \frac{2}{1-\alpha}$ (with $p = \infty$ for $\alpha = 1$). Moreover, $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ (resp. $\sigma(\partial\Omega\backslash\Gamma,\Gamma)$) is in $L^p(\Omega)$ as soon as $f^+ \in L^p(\Gamma)$ (resp. $f^- \in L^p(\Gamma)$) and $p < 2$. In particular, $\sigma \in L^p(\Omega)$ as soon as $f^\pm \in L^p(\Gamma)$ and $p < 2$.*

Proof. First, we recall that $\sigma(\Gamma,\Gamma)$ is the transport density between $f^+ - \nu^+ = f^+ \cdot \chi_{A^+}$ and $f^- - \nu^- = f^- \cdot \chi_{A^-}$ (where $A^\pm \subset \Gamma$ are two countable union of connected arcs). Hence, thanks to [9, Proposition 3.3], $\sigma(\Gamma,\Gamma)$ belongs to $L^p(\Omega)$ provided that $f^\pm \in L^p(\Gamma)$ and $p \le 2$. Moreover, by [9, Proposition 3.5 & Remark 5.10], one can show that $\sigma(\Gamma,\Gamma)$ is in $L^p(\Omega)$ for $p > 2$ as soon as $f^\pm \in C^{0,\alpha}(\Gamma)$ with $\alpha = 1 - \frac{2}{p}$. On the other hand, $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ is the transport density between ν^+ and $T^-_\#\nu^+$. So, again by [9, Proposition 3.2 & Remark 5.10], $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ belongs to $L^p(\Omega)$ as soon as $f^+ \in L^p(\Gamma)$ and $p < 2$. Similarly, we have $\sigma(\partial\Omega\backslash\Gamma,\Gamma) \in L^p(\Omega)$ provided that $f^- \in L^p(\Gamma)$ and $p < 2$. □

Now, we will try to extend our L^p estimates on the transport density σ to the case $p \ge 2$. Recalling Proposition 3.1, we just need to study the L^p summability of $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ (it will be the same for $\sigma(\partial\Omega\backslash\Gamma,\Gamma)$). We recall that $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ is the transport density between ν^+ and $T^-_\#\nu^+$. In the sequel, we will denote by $\Gamma^\pm \subset \Gamma$ the set of points x such that $T^\mp(x)$ is not an endpoint of $\partial\Omega\backslash\Gamma$. Then, we have the following:

Proposition 3.2. *Assume that $\mathrm{spt}(\nu^+) \subset \Gamma^+$, $\partial\Omega\backslash\Gamma$ is $C^{1,1}$, ϕ^+ is $\lambda-$Lip with $\lambda < 1$ and, $\phi^+ \in C^{1,1}(\partial\Omega\backslash\Gamma)$. Hence, the transport density $\sigma(\Gamma,\partial\Omega\backslash\Gamma)$ is in $L^p(\Omega)$ provided that $\nu^+ \in L^p(\Gamma)$, for all $p \in [1,\infty]$.*

Proof. First of all, we mention that the proof of this proposition is similar to the one in [7, Proposition 4.7]. Recalling the definition of the transport density (1.6), we have by Proposition 2.3 that

$$< \sigma(\Gamma,\partial\Omega\backslash\Gamma), \varphi >= \int_\Gamma \int_0^1 \varphi((1-t)x + tT^-(x))|x - T^-(x)| \,\mathrm{d}t \,\mathrm{d}\nu^+(x), \quad \text{for all } \ \varphi \in C(\overline{\Omega}).$$

Fix $x_0 \in \mathrm{spt}(\nu^+)$. Let $\Gamma_0 \subset \Gamma^+$ be an arc around x_0. Let $\tilde\alpha(s) := (s, \alpha(s))$, $s \in [-\varepsilon,\varepsilon]$, be a parametrization of the image of Γ_0 by T^- and, $\beta(s) := (\beta_1(s), \beta_2(s))$ be a parametrization of Γ_0 such that $\alpha(0) = \alpha'(0) = 0$ and $T^-(\beta(s)) = \tilde\alpha(s)$, for every $s \in [-\varepsilon,\varepsilon]$. Now, let Δ be the set of all the transport rays $[\beta(s), \tilde\alpha(s)]$, $s \in [-\varepsilon,\varepsilon]$. For all $y \in \Delta$, we see that there exists a unique pair $(s,t) \in [-\varepsilon,\varepsilon] \times [0,1]$ such that

$$y = ((1-t)\beta_1(s) + ts, (1-t)\beta_2(s) + t\alpha(s)).$$

For all $\varphi \in C(\Delta)$, we have

$$< \sigma(\Gamma, \partial\Omega\backslash\Gamma), \varphi >:= \int_{-\varepsilon}^{\varepsilon}\int_0^1 \varphi((1-t)\beta_1(s)+ts, (1-t)\beta_2(s)+t\alpha(s))\, l(s)\,|\beta'(s)|\,\nu^+(\beta(s))\,\mathrm{d}t\,\mathrm{d}s,$$

where

$$l(s) := |\beta(s) - \tilde{\alpha}(s)|, \ \ \forall\ s \in [-\varepsilon, \varepsilon].$$

Hence,

$$< \sigma(\Gamma, \partial\Omega\backslash\Gamma), \varphi >= \int_\Omega \varphi(y)\, \frac{l(s)\,|\beta'(s)|\,\nu^+(\beta(s))}{J(s,t)}\,\mathrm{d}y, \quad \text{for all} \ \ \varphi \in C(\Delta),$$

where

$$J(s,t) := |\det[D_{(s,t)}(y_1,y_2)]| = (\beta_1(s)-s, \beta_2(s)-\alpha(s))\cdot[(1-t)(-\beta_2'(s), \beta_1'(s)) + t(-\alpha'(s),1)].$$

Then,

$$\sigma(\Gamma, \partial\Omega\backslash\Gamma)[y] = \frac{l(s)\,|\beta'(s)|\,\nu^+(\beta(s))}{J(s,t)}, \ \ \text{for a.e. } y \in \Delta.$$

Now, we claim that there is a uniform constant C (which does not depend on ε) such that $\frac{|\beta'(s)|}{J(s,t)} \leq C$. Thanks to [8, Lemma 2.1], we have

$$\frac{\beta(s)-\tilde{\alpha}(s)}{|\beta(s)-\tilde{\alpha}(s)|} = \partial_\tau\phi^+(\tilde{\alpha}(s))\,\tau(\tilde{\alpha}(s)) - \sqrt{1-\partial_\tau\phi^+(\tilde{\alpha}(s))^2}\,n(\tilde{\alpha}(s)), \tag{3.1}$$

where $n(\tilde{\alpha}(s))$ is the unit exterior normal vector to $\partial\Omega\backslash\Gamma$ at $\tilde{\alpha}(s)$ and $\tau(\tilde{\alpha}(s)) := R_{-\frac{\pi}{2}}[n(\tilde{\alpha}(s))]$ is the unit tangent vector to $\partial\Omega\backslash\Gamma$ at $\tilde{\alpha}(s)$. Hence, it is easy to see that we have the following inequality:

$$(\beta_1(s)-s, \beta_2(s)-\alpha(s))\cdot(-\alpha'(s),1) \geq \sqrt{1-\lambda^2}\ \mathrm{dist}(\mathrm{spt}(\nu^+), \partial\Omega\backslash\Gamma). \tag{3.2}$$

Let $\tilde{\beta}(r) := (\tilde{\beta}_1(r), \tilde{\beta}_2(r))$, $r \in [-\delta,\delta]$, be a smooth parametrization of Γ_0 such that $|\tilde{\beta}'| = 1$ and $\tilde{\beta}_1' > 0$. For every $s \in [-\varepsilon,\varepsilon]$, let $r(s) \in [-\delta,\delta]$ be such that $T^-(\tilde{\beta}(r(s))) = \tilde{\alpha}(s)$. Thanks to the fact that (H) or (H$'$) holds, it is not difficult to see that there is a uniform geometric constant $c > 0$ such that

$$(\beta_1(s)-s, \beta_2(s)-\alpha(s))\cdot(-\tilde{\beta}_2'(r(s)), \tilde{\beta}_1'(r(s))) \geq c. \tag{3.3}$$

Assume that Γ_0 as well as its image by T^- and ϕ^+ are smooth. Then, we claim that the map $s \mapsto r(s)$ is Lipschitz. Hence, combining (3.2) & (3.3), we infer that

$$J(s,t) \geq c[(1-t)r'(s)+t].$$

Consequently,

$$\frac{|\beta'(s)|}{J(s,t)} \leq c^{-1}\frac{r'(s)}{(1-t)r'(s)+t} \leq 2c^{-1}\max\{r'(s),1\}. \tag{3.4}$$

On the other hand, thanks to [11, Proposition 2.2], it is well known that the dual problem of (2.9) is the following:

$$\sup\left\{\int_\Gamma w\,\mathrm{d}\nu^+ \ :\ w \in \mathrm{Lip}_1(\overline{\Omega}),\ w = \phi^+ \ \text{ on } \ \partial\Omega\backslash\Gamma\right\}. \tag{3.5}$$

We recall that $\gamma = (Id, T^-)_{\#}\nu^+$ is the unique optimal transport plan in (2.9). Moreover, the Kantorovich potential w in (3.5) is clearly given by the following:

$$w(x) = \min\{|x-y| + \phi^+(y) \,:\, y \in \partial\Omega\backslash\Gamma\}, \ \text{ for every } \ x \in \Gamma.$$

Now, we see that

$$(\tilde{\beta}(r(s)) - \tilde{\alpha}(s)) \cdot R_{\frac{\pi}{2}}[Dw(\tilde{\alpha}(s))] = 0, \ \text{ for all } s \in [-\varepsilon, \varepsilon]. \tag{3.6}$$

Thanks to [8, Proposition 2.2, Lemma 2.1 & Lemma 2.3], w is C^2 on $\partial\Omega\backslash\Gamma$ and, we have the following:

$$Dw(\tilde{\alpha}(s)) = Dw(\tilde{\alpha}(0)) + D^2w(\tilde{\alpha}(0))(\tilde{\alpha}(s) - \tilde{\alpha}(0)) + o(|\tilde{\alpha}(s) - \tilde{\alpha}(0)|)$$

and

$$Dw(\tilde{\alpha}(0)) = \left(\partial_\tau\phi^+(\tilde{\alpha}(0)), \sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}\right).$$

Let us denote by κ the curvature on $\partial\Omega\backslash\Gamma$. Then, by [8, Proposition 2.2], we also have the following:

$$D^2w(\tilde{\alpha}(0))$$
$$= -\frac{K(\tilde{\alpha}(0))}{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} \begin{bmatrix} 1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2 & -\partial_\tau\phi^+(\tilde{\alpha}(0))\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} \\ -\partial_\tau\phi^+(\tilde{\alpha}(0))\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} & \partial_\tau\phi^+(\tilde{\alpha}(0))^2 \end{bmatrix}$$

where

$$K(\tilde{\alpha}(0)) = \sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}\,\kappa(\tilde{\alpha}(0)) - \partial^2_{\tau\tau}\phi^+(\tilde{\alpha}(0)) + \partial_n\phi^+(\tilde{\alpha}(0))\kappa(\tilde{\alpha}(0)).$$

Hence, one has

$$Dw(\tilde{\alpha}(s)) = \begin{bmatrix} \partial_\tau\phi^+(\tilde{\alpha}(0)) \\ \sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} \end{bmatrix} - \frac{K(\tilde{\alpha}(0))}{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} \begin{bmatrix} [1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2]s \\ -\partial_\tau\phi^+(\tilde{\alpha}(0))\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}\, s \end{bmatrix} + o(s)$$

$$= \begin{bmatrix} \partial_\tau\phi^+(\tilde{\alpha}(0)) - K(\tilde{\alpha}(0))s + o(s) \\ \sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} + \frac{K(\tilde{\alpha}(0))}{\sqrt{1-\partial_\tau\phi^+(\tilde{\alpha}(0))^2}}\partial_\tau\phi^+(\tilde{\alpha}(0))s + o(s) \end{bmatrix}.$$

By (3.6), we get

$$[s - \tilde{\beta}_1(r(s))]\left[\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} + \frac{K(\tilde{\alpha}(0))}{\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}}\partial_\tau\phi^+(\tilde{\alpha}(0))s\right]$$
$$+ [\tilde{\beta}_2(r(s)) - \alpha(s)]\left[\partial_\tau\phi^+(\tilde{\alpha}(0)) - K(\tilde{\alpha}(0))s\right] + o(s) = 0.$$

But,

$$\tilde{\beta}(r) = \tilde{\beta}(0) + \tilde{\beta}'(0)r + o(r) = \left(\partial_\tau\phi^+(\tilde{\alpha}(0)), \sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}\right)l(0) + \tilde{\beta}'(0)r + o(r).$$

Therefore,

$$[s - \partial_\tau\phi^+(\tilde{\alpha}(0))\, l(0) - \tilde{\beta}_1'(0)\, r(s)]\left[\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2} + \frac{K(\tilde{\alpha}(0))}{\sqrt{1 - \partial_\tau\phi^+(\tilde{\alpha}(0))^2}}\partial_\tau\phi^+(\tilde{\alpha}(0))s\right]$$

$$+\left[\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}\,l(0)+\tilde\beta_2'(0)r(s)-\alpha(s)\right]\left[\partial_\tau\phi^+(\tilde\alpha(0))-K(\tilde\alpha(0))s\right]+o(s)=0.$$

Then, we get

$$\left[\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}(1-K(\tilde\alpha(0))l(0))-\frac{K(\tilde\alpha(0))}{\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}}\partial_\tau\phi^+(\tilde\alpha(0))^2 l(0)\right]s$$

$$-\left[\tilde\beta_1'(0)\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}-\tilde\beta_2'(0)\partial_\tau\phi^+(\tilde\alpha(0))+\frac{\tilde\beta_1'(0)\partial_\tau\phi^+(\tilde\alpha(0))K(\tilde\alpha(0))}{\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}}s+K(\tilde\alpha(0))\tilde\beta_2'(0)s\right]r(s)$$
$$+o(s)=0.$$

Hence, we have

$$r(s)=\frac{1-\partial_\tau\phi^+(\tilde\alpha(0))^2-K(\tilde\alpha(0))\,l(0)}{\tilde\beta_1'(0)\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}-\tilde\beta_2'(0)\partial_\tau\phi^+(\tilde\alpha(0))}s+o(s).$$

By (3.3), we have

$$\tilde\beta_1'(0)\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}-\tilde\beta_2'(0)\partial_\tau\phi^+(\tilde\alpha(0))=\left(\partial_\tau\phi^+(\tilde\alpha(0)),\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}\right)\cdot(-\tilde\beta_2'(0),\tilde\beta_1'(0))$$
$$=Dw(\tilde\alpha(0))\cdot R_{\frac{\pi}{2}}[\tilde\beta'(0)]\geq c.$$

Recalling (3.4), we get that

$$\frac{|\beta'(s)|}{J(s,t)}\leq 2c^{-1}\left(\frac{1-\partial_\tau\phi^+(\tilde\alpha(0))^2-K(\tilde\alpha(0))\,l(0)}{\tilde\beta_1'(0)\sqrt{1-\partial_\tau\phi^+(\tilde\alpha(0))^2}-\tilde\beta_2'(0)\,\partial_\tau\phi^+(\tilde\alpha(0))}+1\right)\leq C.$$

Therefore, we get

$$||\sigma(\Gamma,\partial\Omega\backslash\Gamma)||^p_{L^p(\Delta)}=\int_{-\varepsilon}^{\varepsilon}\int_0^1\frac{l(s)^p|\beta'(s)|^p\nu^+(\beta(s))^p}{J(s,t)^{p-1}}\mathrm{d}t\mathrm{d}s$$

$$=\int_{-\varepsilon}^{\varepsilon}\int_0^1 l(s)^p\left(\frac{|\beta'(s)|}{J(s,t)}\right)^{p-1}\nu^+(\beta(s))^p|\beta'(s)|\,\mathrm{d}t\,\mathrm{d}s\leq C^p\int_{-\varepsilon}^{\varepsilon}\nu^+(\beta(s))^p|\beta'(s)|\mathrm{d}s=C^p||\nu^+||^p_{L^p(\Gamma_0)}.$$

Hence,

$$||\sigma(\Gamma,\partial\Omega\backslash\Gamma)||_{L^p(\Omega)}\leq C||f^+||_{L^p(\Gamma)}.$$

□

Consequently, under the assumption that (H) or (H′) holds and (2.14) is well satisfied, we have the following:

Proposition 3.3. *Assume that Γ is uniformly convex, $\mathrm{spt}(f^\pm)\subset\Gamma^\pm$, $\partial\Omega\backslash\Gamma$ is $C^{1,1}$, $\phi^\pm$ are $\lambda-$Lip with $\lambda<1$ and, $\phi^\pm\in C^{1,1}(\partial\Omega\backslash\Gamma)$. Then, the transport density σ is in $L^2(\Omega)$ as soon as $f^\pm\in L^2(\Gamma)$ and, $\sigma\in L^p(\Omega)$ for $p>2$ provided that $f^\pm\in C^{0,\alpha}(\Gamma)$ with $p=\frac{2}{1-\alpha}$. In particular, σ belongs to $L^\infty(\Omega)$ if $f^\pm$ are Lipschitz on Γ.*

Proof. This follows immediately from Propositions 3.1 & 3.2. □

Finally, we conclude this section by the following Sobolev regularity on the solution u of the mixed least gradient problem (2.1).

Proposition 3.4. *Assume that Γ is uniformly convex. Then, the solution u of Problem (2.1) belongs to $W^{1,p}(\Omega)$ as soon as $g \in W^{1,p}(\Gamma)$ with $p < 2$. In addition, assume that $\mathrm{spt}([\partial_\tau g]^{\pm}) \subset \Gamma^{\pm}$, $\partial\Omega\backslash\Gamma$ is $C^{1,1}$, $\phi^{\pm}$ are $\lambda-$Lip with $\lambda < 1$ and, $\phi^{\pm} \in C^{1,1}(\partial\Omega\backslash\Gamma)$. Then, $u \in W^{1,2}(\Omega)$ provided that $g \in W^{1,2}(\Gamma)$. For $p > 2$, $u \in W^{1,p}(\Omega)$ as soon as $g \in C^{1,\alpha}(\Gamma)$ with $p = \frac{2}{1-\alpha}$. And, u is Lipschitz as soon as $g \in C^{1,1}(\Gamma)$.*

Proof. Thanks to Proposition 2.7, the pair $(v := R_{\frac{\pi}{2}}Du\,,\, \chi := [\partial_\tau u]_{|\partial\Omega\backslash\Gamma})$ is a solution to Problem (2.2). Yet, by Proposition 2.6, $|v|$ is nothing else than the transport density σ in Problem (2.3). Hence, Propositions 3.1 & 3.3 conclude the proof. $\square$

4. Applications to the least gradient problem with Dirichlet and Neumann boundary conditions

In this section, we apply all the results of the previous sections to prove existence and uniqueness of a solution u to Problem (1.8) and, to give $W^{1,p}$ estimates on this solution u. First, we consider the following problem:

$$\inf\left\{\int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u \,\mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega),\; u_{|\partial\Omega} \in BV(\partial\Omega),\; u_{|\Gamma} = g\right\}. \tag{4.1}$$

This is exactly Problem (1.8) but with the additional constraint $u_{|\partial\Omega} \in BV(\partial\Omega)$. Let $\gamma : [0,L] \mapsto \partial\Omega\backslash\Gamma$ be a unit parametrization of $\partial\Omega\backslash\Gamma$ and $\psi \in L^\infty(\partial\Omega\backslash\Gamma)$. Then, we introduce the following constant:

$$\Lambda_\psi = \sup\left\{\frac{|\int_{l_1}^{l_2}\psi(\gamma(s))\,\mathrm{d}s|}{|\gamma(l_2)-\gamma(l_1)|} \;:\; 0 \le l_1 < l_2 \le L\right\}.$$

Hence, we have the following results:

Theorem 4.1. *Assume that Γ is strictly convex, $\partial\Omega\backslash\Gamma$ is visible from Γ, $g \in BV(\Gamma)$ and, ψ is a bounded function on $\partial\Omega\backslash\Gamma$ with $\Lambda_\psi \le 1$. Then, Problem (4.1) reaches a minimum.*

Proof. First of all, we define the Lipschitz function ϕ on $\partial\Omega\backslash\Gamma$ such that $\psi = \partial_\tau\phi$ as follows:

$$\phi(\gamma(l)) := \int_0^l \psi(\gamma(s))\,\mathrm{d}s, \quad \text{for all } \; l \in [0,L].$$

Then, we see that $\Lambda_\psi \le 1$ implies that ϕ is a $1-$Lip function on $\partial\Omega\backslash\Gamma$. Indeed, we have the following:

$$|\phi(\gamma(l_1)) - \phi(\gamma(l_2))| = \left|\int_{l_1}^{l_2}\psi(\gamma(s))\,\mathrm{d}s\right| \le \Lambda_\psi|\gamma(l_1)-\gamma(l_2)|.$$

By integration by parts, we have

$$\begin{aligned}&\inf\left\{\int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma}\psi\, u\,\mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega),\; u_{|\partial\Omega} \in BV(\partial\Omega),\; u_{|\Gamma} = g\right\}\\ &= \inf\left\{\int_\Omega |Du| + \int_{\partial\Omega\backslash\Gamma}\phi\,\partial_\tau u\,\mathrm{d}\mathcal{H}^1 - \phi(\gamma(L))\,g(\gamma(L)^-) : u \in BV(\Omega),\, u_{|\partial\Omega} \in BV(\partial\Omega),\, u_{|\Gamma} = g\right\}.\end{aligned} \tag{4.2}$$

Thanks to Theorem 2.8, we see that Problem (4.2) has a solution u, which turns out to be clearly a solution to Problem (4.1). $\square$

In the following example, we will show that if the assumption that $\Lambda_\psi \le 1$ is not well satisfied, then a solution to Problem (1.8) does not exist. More precisely, the minimal value is not finite!

Example 4.1.1. *Let Ω be the bounded domain with $\Gamma := \{(x_1, x_2) : x_1^2 + x_2^2 = 1, x_2 \geq 0\}$, $\partial\Omega\backslash\Gamma := [-1,1] \times \{0\}$ and $g = 0$ on Γ. Fix $\varepsilon > 0$, then we set $\psi := (1+\varepsilon)\chi_{|[0,1]\times\{0\}}$ and, let ϕ be such that $\psi = \partial_\tau \phi$. It is clear that $\Lambda_\psi = 1 + \varepsilon$. For every $n \in \mathbb{N}$, we define $v_n := n < -1, 0 > \cdot \mathcal{H}^1_{|[0,1]\times\{0\}}$. Then, we see that v_n is admissible in Problem (2.2) since we have*

$$\int_{\overline{\Omega}} v_n \cdot \nabla\varphi = n \int_{[0,1]\times\{0\}} \nabla\varphi(x) \cdot < -1, 0 > \, \mathrm{d}\mathcal{H}^1(x) = n[\varphi(0,0) - \varphi(1,0)] = \int_{\partial\Omega} \varphi \, \mathrm{d}[\chi_n^+ - \chi_n^-],$$

for all $\varphi \in C^1(\overline{\Omega})$, where $\chi_n^+ := n\,\delta_{(0,0)}$ and $\chi_n^- := n\,\delta_{(1,0)}$. Moreover, we have the following:

$$\int_{\overline{\Omega}} |v_n| + \int_{\partial\Omega\backslash\Gamma} \phi \, \mathrm{d}[\chi_n^+ - \chi_n^-] = n[1 + \phi(0,0) - \phi(1,0)] = -\varepsilon\, n \to -\infty.$$

Then, $\inf$ (2.2) $= -\infty$. *Recalling Proposition 2.7, this also implies that* $\inf$ (2.1) $= -\infty$. *In particular,* $\inf$ (1.8) $= -\infty$ *and so, a solution u for Problem (1.8) does not exist!*

On the other hand, we have the following uniqueness result:

Theorem 4.2. *Assume that Γ is strictly convex, $\partial\Omega\backslash\Gamma$ is visible from Γ and, $\psi \in L^\infty(\partial\Omega\backslash\Gamma)$ with $\Lambda_\psi < 1$. Then, Problem (4.1) has a unique solution provided that $g \in BV(\Gamma) \cap C(\Gamma)$.*

Proof. This follows immediately from Proposition 2.10. □

In addition, we get the following Sobolev regularity on the solution u:

Proposition 4.3. *Assume that Γ is uniformly convex, $\partial\Omega\backslash\Gamma$ is visible from Γ and, $\psi \in L^\infty(\partial\Omega\backslash\Gamma)$ with $\Lambda_\psi < 1$. Then, the solution u of Problem (4.1) is in $W^{1,p}(\Omega)$ as soon as $g \in W^{1,p}(\Gamma)$ with $p < 2$. In addition, assume that $\mathrm{spt}([\partial_\tau g]^\pm) \subset \Gamma^\pm$, $\partial\Omega\backslash\Gamma$ is $C^{1,1}$ and, ψ is Lipschitz. Then, $u \in W^{1,2}(\Omega)$ provided that $g \in W^{1,2}(\Gamma)$. For $p > 2$, $u \in W^{1,p}(\Omega)$ as soon as $g \in C^{1,\alpha}(\Gamma)$ with $\alpha = 1 - \frac{2}{p}$. In particular, u is Lipschitz as soon as $g \in C^{1,1}(\Gamma)$.*

Proof. This follows immediately from Proposition 3.4, using the fact that $\psi = \partial_\tau \phi$. □

Finally, it remains to show that Problems (1.8) & (4.1) are completely equivalent. We recall that Problem (1.8) is given by

$$\inf\left\{ \int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u \, \mathrm{d}\mathcal{H}^1 \; : \; u \in BV(\Omega), \; u_{|\Gamma} = g \right\}.$$

Let $u \in BV(\Omega)$ such that $u_{|\partial\Omega} = \tilde{g}$ where $\tilde{g} = g$ on Γ. Let $(\tilde{g}_n)_n \subset BV(\partial\Omega)$ be such that $\tilde{g}_n = g$ on Γ and $\tilde{g}_n \to \tilde{g}$ in $L^1(\partial\Omega)$. Thanks to [1], we may find a sequence $(w_n)_n$ in $BV(\Omega)$ satisfying

$$w_{n|\partial\Omega} = \tilde{g}_n - \tilde{g} \quad \text{and} \quad \int_\Omega |Dw_n| \leq \int_{\partial\Omega} |\tilde{g}_n - \tilde{g}| + \frac{1}{n}.$$

Now, set $u_n := u + w_n$. It is clear that $u_n \in BV(\Omega)$ with $u_n = \tilde{g}_n$ on $\partial\Omega$. So, u_n is admissible in (4.1). Moreover, we have the following

$$\left| \int_\Omega |Du_n| - \int_\Omega |Du| \right| \leq \int_\Omega |Dw_n| \leq \int_{\partial\Omega} |\tilde{g}_n - \tilde{g}| + \frac{1}{n}.$$

Then, we have

$$\int_\Omega |Du_n| - \int_{\partial\Omega\backslash\Gamma} \psi\, u_n \, \mathrm{d}\mathcal{H}^1 \to \int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u \, \mathrm{d}\mathcal{H}^1.$$

Hence, we get that

$$\inf\left\{\int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u\, \mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega),\ u_{|\Gamma} = g\right\}$$
$$\leq \inf\left\{\int_\Omega |Du| - \int_{\partial\Omega\backslash\Gamma} \psi\, u\, \mathrm{d}\mathcal{H}^1 \;:\; u \in BV(\Omega),\ u_{|\partial\Omega} \in BV(\partial\Omega),\ u_{|\Gamma} = g\right\}.$$

Yet, it is obvious that the other inequality also holds. Then, we infer that Problems (1.8) & (4.1) have the same minimal values, i.e. inf (1.8) = inf (4.1). In particular, we get immediately the following existence result:

Theorem 4.4. *Under the assumptions of Theorem 4.1, Problem (1.8) admits a solution u. Moreover, the trace of u is in $BV(\partial\Omega)$.*

In addition, we claim that if the assumptions of Theorem 4.2 hold and u solves Problem (1.8), then we must have $u_{|\partial\Omega} \in BV(\partial\Omega)$. Thanks to Theorem 4.2, this will imply that the solution u of Problem (1.8) is unique as soon as $g \in C(\Gamma)$. The rest of the paper is dedicated to proving this claim. Fix u a solution in (1.8). Let Ω' be an open bounded domain containing Ω such that $\partial\Omega \cap \partial\Omega' = \partial\Omega\backslash\Gamma$, $\tilde{g} \in BV(\Omega'\backslash\overline{\Omega})$ be a function with trace g on Γ and, $\tilde{u}$ be the BV extension of u to Ω' with $\tilde{u} = \tilde{g}$ on $\Omega'\backslash\overline{\Omega}$. First, it is easy to see that

$$\int_{\Omega'} |D\tilde{u}| \leq \int_{\Omega'} |D(\tilde{u} + v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v\, \mathrm{d}\mathcal{H}^1,$$

for any function $v \in BV(\Omega')$ such that $\mathrm{spt}(v) \subset \overline{\Omega}\backslash\Gamma$. For every $s \in \mathbb{R}$, we define the super-level set $E_s := \{x \in \Omega' : \tilde{u}(x) \geq s\}$. Then, we claim that

$$\int_{\Omega'} |D\chi_{E_s}| \leq \int_{\Omega'} |D(\chi_{E_s} + v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v\, \mathrm{d}\mathcal{H}^1, \tag{4.3}$$

for any function $v \in BV(\Omega')$ such that $\mathrm{spt}(v) \subset \overline{\Omega}\backslash\Gamma$. Yet, if (4.3) holds, then we clearly have

$$Per(E_s) - \int_{E_s \cap \partial\Omega\backslash\Gamma} \psi\, \mathrm{d}\mathcal{H}^1 \leq Per(E) - \int_{E \cap \partial\Omega\backslash\Gamma} \psi\, \mathrm{d}\mathcal{H}^1, \tag{4.4}$$

for all $E \subset \Omega'$ such that $\overline{E\Delta E_s} \subset \overline{\Omega}\backslash\Gamma$. In particular, we have

$$Per(E_s) \leq Per(E), \tag{4.5}$$

for all $E \subset \Omega'$ such that $\overline{E\Delta E_s} \subset \Omega$. Hence, if α is a connected component of the level set ∂E_s in Ω, then by (4.5) α must be a segment $[x_s, y_s]$. Now, assume that $y_s := \gamma(l^\star)$ is an interior point of $\partial\Omega\backslash\Gamma$ (i.e. $0 < l^\star < L$). Let $\varepsilon > 0$ be small enough such that $\overline{B(y_s,\varepsilon)} \cap \Gamma = \emptyset$. Set $z_s := [x_s, y_s] \cap \partial B(y_s, \varepsilon)$. Recalling (4.4), it is not difficult to see that depending on the monotonicity of g at the point x_s, we either have

$$|z_s - y_s| + \phi(y_s) \leq |z_s - \gamma(l)| + \phi(\gamma(l)), \text{ for all } 0 \leq l \leq L,$$

or

$$|z_s - y_s| - \phi(y_s) \leq |z_s - \gamma(l)| - \phi(\gamma(l)), \text{ for all } 0 \leq l \leq L.$$

Hence, we either have $y_s = T^+(x_s)$ or $y_s = T^-(x_s)$. Since $u(y_s) = g(x_s)$, this implies that $u_{|\partial\Omega\backslash\Gamma} \in BV(\partial\Omega\backslash\Gamma)$ with $|\partial_\tau u|(\partial\Omega\backslash\Gamma) \leq |\partial_\tau g|(\Gamma)$. Finally, it remains to prove (4.3). For this aim, the idea will be to follow the proof of [2, Theorem 1]. Fix $r \in \mathbb{R}$, then we define $u_1 := \max\{\tilde{u} - r, 0\}$ and $u_2 := \min\{\tilde{u}, r\}$. We see that $u_1 + u_2 = \tilde{u}$ and, we have $\int_{\Omega'} |Du_1| + \int_{\Omega'} |Du_2| = \int_{\Omega'} |D\tilde{u}|$. Hence, we get

$$\begin{aligned}\int_{\Omega'} |Du_1| + \int_{\Omega'} |Du_2| &= \int_{\Omega'} |D\tilde{u}| \leq \int_{\Omega'} |D(\tilde{u}+v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1 \\ &\leq \int_{\Omega'} |D(u_1+v)| + \int_{\Omega'} |Du_2| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1\end{aligned}$$

and so,

$$\int_{\Omega'} |Du_1| \leq \int_{\Omega'} |D(u_1+v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1,$$

for any function $v \in BV(\Omega')$ such that $\mathrm{spt}(v) \subset \overline{\Omega}\backslash\Gamma$. In the same way, we see that we also have

$$\int_{\Omega'} |Du_2| \leq \int_{\Omega'} |D(u_2+v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1.$$

Hence, for $s \in \mathbb{R}$ and $\varepsilon > 0$, we infer that the function $u_{s,\varepsilon} := \frac{1}{\varepsilon}\min\{\max\{\tilde{u} - s, 0\}, \varepsilon\}$ also satisfies

$$\int_{\Omega'} |Du_{s,\varepsilon}| \leq \int_{\Omega'} |D(u_{s,\varepsilon}+v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1,$$

for any function $v \in BV(\Omega')$ such that $\mathrm{spt}(v) \subset \overline{\Omega}\backslash\Gamma$. If $\mathcal{L}^2(\{x \in \Omega' : \tilde{u}(x) = s\}) = 0$, then it is clear that $u_{s,\varepsilon} \to \chi_{E_s}$ in $L^1(\Omega')$, when $\varepsilon \to 0$. Moreover, we see that $u_{s,\varepsilon | \partial\Omega\backslash\Gamma} \to \chi_{E_s | \partial\Omega\backslash\Gamma}$ in $L^1(\partial\Omega\backslash\Gamma)$. If $\mathcal{L}^2(\{x \in \Omega' : \tilde{u}(x) = s\}) > 0$, then there will be a sequence $s_n \to s$ with $s_n < s$ and $\mathcal{L}^2(\{x \in \Omega' : \tilde{u}(x) = s_n\}) = 0$, for all n. Yet, it is easy to see that $\chi_{E_{s_n}} \to \chi_{E_s}$ in $L^1(\Omega')$ and $\chi_{E_{s_n} | \partial\Omega\backslash\Gamma} \to \chi_{E_s | \partial\Omega\backslash\Gamma}$ in $L^1(\partial\Omega\backslash\Gamma)$, when $n \to \infty$. But, we also know that $u_{s_n,\varepsilon} \to \chi_{E_{s_n}}$ in $L^1(\Omega')$ and $u_{s_n,\varepsilon | \partial\Omega\backslash\Gamma} \to \chi_{E_{s_n} | \partial\Omega\backslash\Gamma}$ in $L^1(\partial\Omega\backslash\Gamma)$, when $\varepsilon \to 0$. Hence, by a diagonal argument, we infer that there is a sequence of functions $\{u_\varepsilon\}$ with $u_\varepsilon \to \chi_{E_s}$ in $L^1(\Omega)$, $u_{\varepsilon | \partial\Omega\backslash\Gamma} \to \chi_{E_s | \partial\Omega\backslash\Gamma}$ in $L^1(\partial\Omega\backslash\Gamma)$ and such that, for every ε, we have

$$\int_{\Omega'} |Du_\varepsilon| \leq \int_{\Omega'} |D(u_\varepsilon+v)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1, \tag{4.6}$$

for any function $v \in BV(\Omega')$ such that $\mathrm{spt}(v) \subset \overline{\Omega}\backslash\Gamma$. Now, in order to pass to the limit in (4.6) and get (4.3), we will adapt the proof of [21, Theorem 3]. Let A be a set such that $\partial\Omega\backslash\Gamma \subset A$, $\overline{A} \subset \overline{\Omega}\backslash\Gamma$, $A \cap \Omega$ is open, $\mathrm{spt}(v) \subset A$ and, such that the following holds:

$$\lim_{\varepsilon\to 0} \int_{\partial A\cap\Omega'} |u_\varepsilon - \chi_{E_s}| = 0.$$

For each ε, set $v_\varepsilon := [\chi_{E_s} + v - u_\varepsilon]\chi_A$. It is clear that $v_\varepsilon \in BV(\Omega')$ and $\mathrm{spt}(v_\varepsilon) \subset \overline{\Omega}\backslash\Gamma$. Hence, we have

$$\int_{\Omega'} |Du_\varepsilon| \leq \int_{\Omega'} |D(u_\varepsilon+v_\varepsilon)| - \int_{\partial\Omega\backslash\Gamma} \psi\, v_\varepsilon \,\mathrm{d}\mathcal{H}^1. \tag{4.7}$$

Yet,

$$\begin{aligned}\int_{\Omega'} |D(u_\varepsilon+v_\varepsilon)| &= \int_{\Omega'} |D(u_\varepsilon + [\chi_{E_s} + v - u_\varepsilon]\chi_A)| \\ &= \int_{\Omega'\cap A} |D(\chi_{E_s}+v)| + \int_{\Omega'\backslash\overline{A}} |Du_\varepsilon| + \int_{\partial A\cap\Omega'} |\chi_{E_s} - u_\varepsilon|\end{aligned}$$

and

$$\int_{\partial\Omega\backslash\Gamma} \psi\, v_\varepsilon \,\mathrm{d}\mathcal{H}^1 = \int_{\partial\Omega\backslash\Gamma} \psi\, \chi_{E_s} \,\mathrm{d}\mathcal{H}^1 + \int_{\partial\Omega\backslash\Gamma} \psi\, v \,\mathrm{d}\mathcal{H}^1 - \int_{\partial\Omega\backslash\Gamma} \psi\, u_\varepsilon \,\mathrm{d}\mathcal{H}^1.$$

Recalling (4.7), we get

$$\int_{\Omega'} |Du_\varepsilon| \le \int_{\Omega'\cap A} |D(\chi_{E_s}+v)| + \int_{\Omega'\setminus\overline{A}} |Du_\varepsilon| + \int_{\partial A\cap\Omega'} |\chi_{E_s}-u_\varepsilon| - \int_{\partial\Omega\setminus\Gamma} \psi\,\chi_{E_s}\,\mathrm{d}\mathcal{H}^1 - \int_{\partial\Omega\setminus\Gamma} \psi\, v\,\mathrm{d}\mathcal{H}^1 + \int_{\partial\Omega\setminus\Gamma} \psi\, u_\varepsilon\,\mathrm{d}\mathcal{H}^1. \tag{4.8}$$

Consequently, we have

$$\int_{\Omega'\cap A} |Du_\varepsilon| \le \int_{\Omega'\cap A} |D(\chi_{E_s}+v)| - \int_{\partial\Omega\setminus\Gamma} \psi\, v\,\mathrm{d}\mathcal{H}^1 + \int_{\partial A\cap\Omega'} |\chi_{E_s}-u_\varepsilon| + \int_{\partial\Omega\setminus\Gamma} \psi\,(u_\varepsilon-\chi_{E_s})\mathrm{d}\mathcal{H}^1.$$

Hence,

$$\liminf_{\varepsilon\to 0} \int_{\Omega'\cap A} |Du_\varepsilon| \le \int_{A\cap\Omega'} |D(\chi_{E_s}+v)| - \int_{\partial\Omega\setminus\Gamma} \psi\, v\,\mathrm{d}\mathcal{H}^1.$$

But, $u_\varepsilon \to \chi_{E_s}$ in $L^1(\Omega')$. Then, by the lower semicontinuity of the total variation, we have

$$\int_{\Omega'\cap A} |D\chi_{E_s}| \le \liminf_{\varepsilon\to 0} \int_{\Omega'\cap A} |Du_\varepsilon|.$$

So, we get that

$$\int_{\Omega'\cap A} |D\chi_{E_s}| \le \int_{\Omega'\cap A} |D(\chi_{E_s}+v)| - \int_{\partial\Omega\setminus\Gamma} \psi\, v\,\mathrm{d}\mathcal{H}^1.$$

This concludes the proof of our claim (4.3). We finish this paper by recalling then the following results:

Theorem 4.5. *Under the assumptions of Theorem 4.2, the solution of Problem (1.8) is unique provided that $g \in BV(\Gamma)\cap C(\Gamma)$.*

Theorem 4.6. *Assume that Γ is uniformly convex. Let u be the unique solution of Problem (1.8). Then, we have*

$$g \in W^{1,p}(\Gamma) \Rightarrow u \in W^{1,p}(\Omega), \ \ \textit{for all } p<2.$$

In addition, assume that $\partial\Omega\setminus\Gamma$ is $C^{1,1}$, ψ is Lipschitz and, $\mathrm{spt}([\partial_\tau g]^{\pm}) \subset \Gamma^{\pm}$, where $\Gamma^{\pm}\subset\Gamma$ is the set of points x such that $T^{\mp}(x)$ is an interior point of $\partial\Omega\setminus\Gamma$. Then, we have the following statements:

$$\begin{cases} g \in W^{1,2}(\Gamma) \Rightarrow u \in W^{1,2}(\Omega), \\ g \in C^{1,\alpha}(\Gamma) \Rightarrow u \in W^{1,\frac{2}{1-\alpha}}(\Omega), \quad \forall\ \alpha \in]0,1[, \\ g \in C^{1,1}(\Gamma) \Rightarrow u \in \mathrm{Lip}(\Omega). \end{cases}$$

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Faculty of Mathematics, Informatics and Mechanics, The University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland.

Email address: `s.dweik@uw.edu.pl`