A BOURGAIN-BREZIS-MIRONESCU TYPE RESULT FOR THE FRACTIONAL RELATIVISTIC SEMINORM

VINCENZO AMBROSIO AND FARES ESSEBEI

Abstract. We establish a version of the Bourgain-Brezis-Mironescu type formula on the limit as $s \to 1^-$ of the fractional relativistic seminorm.

1. Introduction

Let $N \geq 2$ and $s \in (0,1)$. The fractional relativistic operator (see [2, 3])

$$(-\Delta+1)^s \tag{1.1}$$

is defined in Fourier space by setting

$$\mathcal{F}(-\Delta+1)^s u(\xi) = (|\xi|^2+1)^s \mathcal{F}u(\xi) \quad \text{for all } u \in \mathcal{S}(\mathbb{R}^N), \tag{1.2}$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions. Equivalently, (1.1) may be defined for all $u \in \mathcal{S}(\mathbb{R}^N)$ via singular integral as

$$(-\Delta+1)^s u(x) = C_{N,s}\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\mathrm{dy} + u(x) \tag{1.3}$$

for every $x \in \mathbb{R}^N$, where P.V. stands for the Cauchy principal value,

$$C_{N,s} := 2^{-\frac{N+2s}{2}+1}\pi^{-\frac{N}{2}}2^{2s}\frac{s(1-s)}{\Gamma(2-s)}, \tag{1.4}$$

and $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}\,\mathrm{d}x$, for $t>0$, is the usual Gamma function. Here $K_\nu(r)$, with $r>0$, denotes the modified Bessel function of the third kind of order $\nu \in \mathbb{R}$ (see [5, 9, 18]). It is well-known that $K_\nu(r)$ is an analytic function of r, $K_\nu(r)$ is an entire function of ν and it is symmetric with respect to ν, that is, $K_{-\nu}(r) = K_\nu(r)$. Moreover, fixed $\nu>0$, the function $r \mapsto r^{-\nu}K_\nu(r)$ is decreasing. We also have the following integral representation for K_ν:

$$K_\nu(r) = 2^{-\nu-1}r^\nu \int_0^{+\infty} e^{-t}e^{-\frac{r^2}{4t}}t^{-\nu-1}\mathrm{dt}.$$

Finally, K_ν satisfies the following asymptotic formulas:

$$K_\nu(r) \sim \frac{\Gamma(\nu)}{2}\left(\frac{r}{2}\right)^{-\nu} \quad \text{as } r \to 0^+, \text{ for } \nu > 0, \tag{1.5}$$

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$$K_\nu(r) \sim \sqrt{\frac{\pi}{2}} r^{-\frac{1}{2}} e^{-r} \quad \text{as } r \to +\infty, \text{ for } \nu \in \mathbb{R}. \tag{1.6}$$

We recall that when $s = \frac{1}{2}$, the operator (1.1) is strictly related to $\mathcal{H} = \sqrt{-\Delta + 1} - 1$ which has a relevant role in relativistic quantum mechanics because it corresponds to the kinetic energy of a relativistic particle with unit mass (see [7]). The study of $\mathcal{H}$ has been strongly influenced by several works on the stability of relativistic matter (see [11]). The operator (1.1) is very important in the theory of the so-called interpolation spaces of Bessel potentials and finds application in harmonic analysis and partial differential equations (see [10, 16]). There exists also a deep connection between (1.1) and the theory of Lévy processes (see [7, 14]). Note that when $-\Delta + 1$ is replaced by $-\Delta$, (1.1) boils down to the fractional Laplacian operator $(-\Delta)^s$ (see [8]) defined for all $u \in \mathcal{S}(\mathbb{R}^N)$ via Fourier transform by

$$\mathcal{F}(-\Delta)^s u(\xi) = |\xi|^{2s} \mathcal{F}u(\xi),$$

or via singular integral by

$$(-\Delta)^s u(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{dy},$$

where $c_{N,s}$ is given by (see [17, Remark 5.2])

$$c_{N,s} \coloneqq \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{-\Gamma(-s)} = \pi^{-\frac{N}{2}} 2^{2s} \Gamma\left(\frac{N+2s}{2}\right) \frac{s(1-s)}{\Gamma(2-s)} = 2^{\frac{N+2s}{2}-1} \Gamma\left(\frac{N+2s}{2}\right) C_{N,s}. \tag{1.7}$$

It is well-known that for all $u \in C_0^\infty(\mathbb{R}^N)$ it holds $(-\Delta)^s u \to -\Delta u$ as $s \to 1^-$ (see [8, Proposition 4.4-(ii)]). On the other hand, it is natural to expect that (1.1) converges pointwise to $-\Delta + 1$ as $s \to 1^-$ along smooth functions. Indeed, this fact can be easily deduced from (1.2), alternatively by using (1.3) in order to show the consistency in the definition of the constant $C_{N,s}$ (see [4, Theorem 1.1]).

Motivated by the above discussion, we study the limiting behavior as $s \to 1^-$ of the following fractional relativistic seminorm

$$\frac{C_{N,s}}{2} \omega_N \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|) \, \mathrm{dx} \, \mathrm{dy},$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and ω_N is the volume of the unit ball in $\mathbb{R}^N$. We recall that in [6] the authors showed that for all $u \in H^1(\Omega)$ it holds

$$\lim_{s \to 1^-} (1-s) \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, \mathrm{dx} \, \mathrm{dy} = K_N \int_\Omega |\nabla u|^2 \, \mathrm{dx}, \tag{1.8}$$

where

$$K_N \coloneqq \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\omega \cdot \mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega), \tag{1.9}$$

here $\mathbb{S}^{N-1}$ denotes the $(N-1)$-dimensional unit sphere in $\mathbb{R}^N$, $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure, $\mathbf{e}$ is any unit vector in $\mathbb{R}^N$, and $\cdot$ stands for the scalar product in $\mathbb{R}^N$. The formula (1.8) says that $H^1(\Omega)$ can be seen as a continuous limit of the spaces $H^s(\Omega)$ as $s \to 1^-$ provided that one considers on $H^s(\Omega)$ the seminorm

$$\left((1-s)\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

Subsequently, this result has been extended in the fractional magnetic framework in [15] (see also [13]). The purpose of this paper is to obtain an analogue of (1.8) in the fractional relativistic setting. More precisely, we prove the following result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then, for every $u \in H^1(\Omega)$, the following formula is true:*

$$\lim_{s\to 1^-} \frac{C_{N,s}}{2}\omega_N \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = 2K_N \int_\Omega |\nabla u|^2\,\mathrm{d}x\,. \tag{1.10}$$

We emphasize that the factor in front of the double integral in (1.10) is different from the one in (1.8), because of the presence of the modified Bessel function K_ν. We also provide the following variant of Theorem 1.1 for functions belonging to $H^1_0(\Omega)$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then, for every $u \in H^1_0(\Omega)$, the following formula is valid:*

$$\lim_{s\to 1^-} \frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = 2K_N \int_\Omega |\nabla u|^2\,\mathrm{d}x\,. \tag{1.11}$$

The proofs of Theorems 1.1 and 1.2 are inspired by [6, 15]. Nevertheless, due to the appearance of the modified Bessel function K_ν, several technical difficulties arise in our study. To overcome these problems, we will exploit some crucial properties of K_ν that will be fundamental to accomplish our results. Finally, we would like to mention [1] for some limiting formulas in the setting of fractional Sobolev spaces on the torus.

2. Proofs of the main results

Let us first prove the following useful lemma.

Lemma 2.1. *For every $\delta > 0$, we have*

$$\lim_{s\to 1^-} \frac{C_{N,s}}{2}\omega_N \int_0^\delta \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1}\,\mathrm{d}r = 1, \tag{2.1}$$

and

$$\lim_{s\to 1^-} \frac{C_{N,s}}{2}\omega_N \int_\delta^{+\infty} \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1}\,\mathrm{d}r = 0. \tag{2.2}$$

Proof. We start by recalling the following formula (see [18, formula (8) at pag.388])

$$\int_0^{+\infty} K_\nu(r) r^{\mu-1}\,\mathrm{dr} = 2^{\mu-2}\,\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right) \quad \text{for all } \mu>\nu. \tag{2.3}$$

Then, choosing $\mu=\frac{N+4-2s}{2}$ and $\nu=\frac{N+2s}{2}$ in (2.3), we see that

$$\int_0^{+\infty} \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1}\,\mathrm{dr} = 2^{\frac{N-2s}{2}}\Gamma(1-s)\Gamma\left(\frac{N+2}{2}\right).$$

Exploiting (1.4), $\omega_N=\frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ and that $\Gamma(t+1)=t\Gamma(t)$ for all $t>0$, it follows that

$$\begin{aligned}
\frac{C_{N,s}}{2}\omega_N \int_0^{+\infty} \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1}\,\mathrm{dr} &= \frac{C_{N,s}}{2}\omega_N 2^{\frac{N-2s}{2}}\Gamma(1-s)\Gamma\left(\frac{N}{2}+1\right) \\
&= \pi^{-\frac{N}{2}}\frac{s(1-s)}{\Gamma(2-s)}\omega_N\Gamma(1-s)\Gamma\left(\frac{N}{2}+1\right) \\
&= \frac{s(1-s)}{\Gamma(2-s)}\Gamma(1-s) \\
&= s.
\end{aligned}$$

As a result,

$$\lim_{s\to1^-}\frac{C_{N,s}}{2}\omega_N \int_0^{+\infty} \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1}\,\mathrm{dr} = 1. \tag{2.4}$$

Now, using (1.6), we can find $C_0>0$ and $R_0>0$, independent of s, such that

$$K_{\frac{N+2s}{2}}(r) \le C_0 r^{-\frac{1}{2}}e^{-r} \quad \text{for all } r>R_0.$$

Note that, for all $r>R_0$, it holds

$$\frac{K_{\frac{N+2s}{2}}(r) r^{N+1}}{r^{\frac{N+2s}{2}}} \le C_0 r^{\frac{N+1}{2}-s}e^{-r} \le C_0 C'_{N,s} r^{-2},$$

where

$$C'_{N,s} \coloneqq \left(\frac{N+1}{2}-s+2\right)^{\frac{N+1}{2}-s+2} e^{-\left(\frac{N+1}{2}-s+2\right)}.$$

Since

$$\lim_{s\to1^-} C'_{N,s} = \left(\frac{N+1}{2}+1\right)^{\frac{N+1}{2}+1} e^{-\left(\frac{N+1}{2}+1\right)},$$

there exists a constant $L_1>0$, independent of s, such that

$$C'_{N,s} \le L_1 \quad \text{for all } s \text{ near } 1.$$

Hence,

$$\frac{K_{\frac{N+2s}{2}}(r) r^{N+1}}{r^{\frac{N+2s}{2}}} \le L_1 r^{-2},$$

for all $r > R_0$ and s near 1. Let $R > \max\{R_0, \delta\}$. Because $r \mapsto \frac{K_\nu(r)}{r^\nu}$ is decreasing in $(0, +\infty)$, we see that, for all $r \in [\delta, R]$,

$$\frac{K_{\frac{N+2s}{2}}(r) r^{N+1}}{r^{\frac{N+2s}{2}}} \le \frac{K_{\frac{N+2s}{2}}(\delta)}{\delta^{\frac{N+2s}{2}}} R^{N+1} \le \frac{K_{\frac{N+2s}{2}}(\delta)}{\delta^{\frac{N+2s}{2}}} R^{N+3} r^{-2}.$$

Recalling that $K_\nu(r)$ is an entire function of ν, we have

$$\lim_{s \to 1^-} \frac{K_{\frac{N+2s}{2}}(\delta)}{\delta^{\frac{N+2s}{2}}} = \frac{K_{\frac{N+2}{2}}(\delta)}{\delta^{\frac{N+2}{2}}},$$

and so we can find $L_2 > 0$ independent of s such that

$$\frac{K_{\frac{N+2s}{2}}(\delta)}{\delta^{\frac{N+2s}{2}}} \le L_2 \quad \text{for all } s \text{ near } 1.$$

Consequently, for all $r \in [\delta, R]$ and s near 1,

$$\frac{K_{\frac{N+2s}{2}}(r) r^{N+1}}{r^{\frac{N+2s}{2}}} \le L_2 R^{N+3} r^{-2}.$$

Therefore, there exists $L_3 > 0$, independent of s, such that, for all $r \ge \delta$ and s near 1,

$$\frac{K_{\frac{N+2s}{2}}(r) r^{N+1}}{r^{\frac{N+2s}{2}}} \le L_3 r^{-2}. \tag{2.5}$$

On the other hand, exploiting (1.7) and $\lim_{s \to 1^-} c_{N,s} = 0$ (see [8, Corollary 4.2]), we obtain

$$\lim_{s \to 1^-} C_{N,s} = \lim_{s \to 1^-} \frac{2^{-\frac{N+2s}{2}+1}}{\Gamma(\frac{N+2s}{2})} c_{N,s} = 0. \tag{2.6}$$

Then (2.5) and (2.6) imply

$$\lim_{s \to 1^-} \frac{C_{N,s}}{2} \omega_N \int_\delta^{+\infty} \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1} \, \mathrm{dr} = 0.$$

Using the above limit and (2.4), we can infer that

$$\lim_{s \to 1^-} \frac{C_{N,s}}{2} \omega_N \int_0^\delta \frac{K_{\frac{N+2s}{2}}(r)}{r^{\frac{N+2s}{2}}} r^{N+1} \, \mathrm{dr} = 1.$$

The proof of the lemma is now complete. □

Proof of Theorem 1.1. Let $r_\Omega := \mathrm{diam}(\Omega)$. Let us consider a radial cut-off $\psi_0 \in C_0^\infty(\mathbb{R}^N)$, with $\psi_0(t) = 1$ for $t < r_\Omega$ and $\psi_0(t) = 0$ for $t > 2r_\Omega$. By construction, $\psi_0(|x-y|) = 1$ for every $x, y \in \Omega$. Let $(s_n)_{n \in \mathbb{N}} \subset (0,1)$ be a sequence such that $s_n \to 1^-$ as $n \to +\infty$ and select the following family of radial functions

$$\rho_n(x) := \frac{C_{N,s_n}}{2} \omega_N \frac{\psi_0(|x|)}{|x|^{\frac{N+2s_n-4}{2}}} K_{\frac{N+2s_n}{2}}(|x|) \quad \text{for } x \in \mathbb{R}^N \text{ and } n \in \mathbb{N}. \tag{2.7}$$

Our aim is to apply [15, Theorem 2.5] (see also [6, Theorem 2]). More precisely, if we verify that $(\rho_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n\to+\infty}\int_0^{+\infty}\rho_n(r)r^{N-1}\mathrm{dr}=1, \tag{2.8}$$

and, for every $\delta>0$,

$$\lim_{n\to+\infty}\int_\delta^{+\infty}\rho_n(r)r^{N-1}\mathrm{dr}=0, \tag{2.9}$$

then we can conclude that

$$\lim_{n\to+\infty}\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^2}\rho_n(x-y)\,\mathrm{dx}\,\mathrm{dy}=2K_N\int_\Omega|\nabla u|^2\,\mathrm{dx}, \tag{2.10}$$

where K_N is the constant given in (1.9). First we check (2.8). Note that

$$\begin{aligned}\int_0^{+\infty}\rho_n(r)r^{N-1}\,\mathrm{dr}&=\int_0^{r_\Omega}\rho_n(r)r^{N-1}\,\mathrm{dr}+\int_{r_\Omega}^{+\infty}\rho_n(r)r^{N-1}\,\mathrm{dr}\\&=\frac{C_{N,s_n}}{2}\omega_N\int_0^{r_\Omega}\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}\\&\quad+\frac{C_{N,s_n}}{2}\omega_N\int_{r_\Omega}^{+\infty}\psi_0(r)\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}\,.\end{aligned}$$

Thanks to Lemma 2.1, we have that (2.1) implies

$$\lim_{n\to+\infty}\frac{C_{N,s_n}}{2}\omega_N\int_0^{r_\Omega}\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}=1,$$

and (2.2) yields

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{C_{N,s_n}}{2}\omega_N\int_{r_\Omega}^{+\infty}\psi_0(r)\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}\\&\le\limsup_{n\to+\infty}\frac{C_{N,s_n}}{2}\omega_N\int_{r_\Omega}^{+\infty}\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}=0,\end{aligned}$$

whence (2.8) is true. In a similar fashion, for every $\delta>0$, we see that

$$\begin{aligned}\limsup_{n\to+\infty}\int_\delta^{+\infty}\rho_n(r)r^{N-1}\,\mathrm{dr}&=\limsup_{n\to+\infty}\frac{C_{N,s_n}}{2}\omega_N\int_\delta^{+\infty}\psi_0(r)\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}\\&\le\limsup_{n\to+\infty}\frac{C_{N,s_n}}{2}\omega_N\int_\delta^{+\infty}\frac{K_{\frac{N+2s_n}{2}}(r)}{r^{\frac{N+2s_n}{2}}}r^{N+1}\,\mathrm{dr}=0,\end{aligned}$$

where we have again exploited (2.2) in Lemma 2.1. Therefore, the assertion (2.9) is proved and so (1.10) is valid.

□

In order to demonstrate Theorem 1.2, we recall the following result contained in [15].

Lemma 2.2. [15, Lemma 2.1] *For any compact $V \subset \mathbb{R}^N$ with $\Omega \Subset V$, there exists $C = C(V) > 0$ such that*

$$\int_{\mathbb{R}^N} |u(y+h) - u(y)|^2 \, \mathrm{dy} \le C|h|^2 \left\|u\right\|^2_{H^1(\mathbb{R}^N)},$$

for all $u \in H^1(\mathbb{R}^N)$ such that $u = 0$ on V^c and for all $h \in \mathbb{R}^N$ with $|h| \le 1$.

Now we establish the following result.

Lemma 2.3. *Let $u \in H^1_0(\Omega)$. Then, for all s near 1, we have*

$$C_{N,s} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|) \, \mathrm{dx} \, \mathrm{dy} \le C \left\|u\right\|^2_{H^1(\Omega)},$$

where C depends only on Ω.

Proof. Pick $u \in C_0^\infty(\Omega)$. By Lemma 2.2, we have that, for all $h \in \mathbb{R}^N$ with $|h| \le 1$,

$$\int_{\mathbb{R}^N} |u(y+h) - u(y)|^2 \, \mathrm{dy} \le C|h|^2 \left\|u\right\|^2_{H^1(\Omega)},$$

where $C > 0$ depends only on Ω. Hence,

$$\begin{aligned}
&C_{N,s} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)) \, \mathrm{dx} \, \mathrm{dy} \\
&= C_{N,s} \iint_{\mathbb{R}^{2N}} \frac{|u(y+h) - u(y)|^2}{|h|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|h|) \mathrm{dh} \, \mathrm{dy} \\
&= C_{N,s} \int_{\mathbb{R}^N} \frac{K_{\frac{N+2s}{2}}(|h|)}{|h|^{\frac{N+2s}{2}}} \left(\int_{\mathbb{R}^N} |u(y+h) - u(y)|^2 \mathrm{dy} \right) \mathrm{dh} \\
&\le C C_{N,s} \left\|u\right\|^2_{H^1(\Omega)} \int_{\{|h| \le 1\}} \frac{K_{\frac{N+2s}{2}}(|h|)}{|h|^{\frac{N+2s}{2}}} |h|^2 \mathrm{dh} + 4 C_{N,s} \left\|u\right\|^2_{L^2(\Omega)} \int_{\{|h|>1\}} \frac{K_{\frac{N+2s}{2}}(|h|)}{|h|^{\frac{N+2s}{2}}} \mathrm{dh}.
\end{aligned}$$

Arguing as in the proof of (2.2) in Lemma 2.1, we can see that

$$4 C_{N,s} \left\|u\right\|^2_{L^2(\Omega)} \int_{\{|h|>1\}} \frac{K_{\frac{N+2s}{2}}(|h|)}{|h|^{\frac{N+2s}{2}}} \mathrm{dh} \le C_2 \left\|u\right\|^2_{L^2(\Omega)},$$

for all s near 1, where $C_2 > 0$ is independent of s. On the other hand, using (2.1) in Lemma 2.1, we deduce that

$$C C_{N,s} \left\|u\right\|^2_{H^1(\Omega)} \int_{\{|h| \le 1\}} \frac{K_{\frac{N+2s}{2}}(|h|)}{|h|^{\frac{N+2s}{2}}} |h|^2 \mathrm{dh} \le C_3 \left\|u\right\|^2_{H^1(\Omega)},$$

for all s near 1, where $C_3 > 0$ is independent of s. Consequently, for all s near 1,

$$C_{N,s} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|) \, \mathrm{dx} \, \mathrm{dy} \le C \left\|u\right\|^2_{H^1(\Omega)}.$$

Since $C_0^\infty(\Omega)$ is dense in $H^1_0(\Omega)$, we obtain the assertion. $\square$

At this point we have all tools needed to provide the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $u \in C_0^\infty(\Omega)$. Because $u = 0$ outside Ω, we get

$$\begin{aligned}
&\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{C_{N,s}}{2}\omega_N \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + C_{N,s}\omega_N \int_\Omega \int_{\mathbb{R}^N\setminus\Omega} \frac{|u(x)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Invoking Theorem 1.1, we infer that

$$\lim_{s\to 1^-} \frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = 2K_N \int_\Omega |\nabla u|^2\,\mathrm{d}x + \lim_{s\to 1^-} R_s,$$

where

$$R_s \coloneqq C_{N,s}\,\omega_N \int_\Omega \int_{\mathbb{R}^N\setminus\Omega} \frac{|u(x)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Since $u \in C_0^\infty(\Omega)$, we have that $\mathrm{dist}(\partial K, \partial\Omega) > 0$, where $K \coloneqq \mathrm{supp}(u)$. Recalling that the function $r \mapsto \frac{K_\nu(r)}{r^\nu}$ is decreasing in $(0,+\infty)$, it follows that

$$\int_\Omega \left(\int_{\mathbb{R}^N\setminus\Omega} \frac{|u(x)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}y \right) \mathrm{d}x \le \|u\|_{L^2(\Omega)}^2 \int_{\mathbb{R}^N\setminus\Omega} \frac{K_{\frac{N+2s}{2}}(\delta_K(y))}{\delta_K(y)^{\frac{N+2s}{2}}}\,\mathrm{d}y,$$

where $\delta_K(y) \coloneqq \mathrm{dist}(y, \partial K)$. Without loss of generality, we can assume that $0 \in \Omega$. Put $R \coloneqq \mathrm{dist}(\mathbb{R}^N \setminus \Omega, \partial K) > 0$. Then there exists a constant $\bar{C} = \bar{C}(\Omega, R) > 0$ such that $\delta_K(y) \ge \bar{C}|y|$ for all $y \in \mathbb{R}^N \setminus \Omega$, and so

$$\int_{\mathbb{R}^N\setminus\Omega} \frac{K_{\frac{N+2s}{2}}(\delta_K(y))}{\delta_K(y)^{\frac{N+2s}{2}}}\,\mathrm{d}y \le \int_{\mathbb{R}^N\setminus\Omega} \frac{K_{\frac{N+2s}{2}}(\bar{C}|y|)}{(\bar{C}|y|)^{\frac{N+2s}{2}}}\,\mathrm{d}y\,.$$

Because $0 \in \Omega$, we can argue as in the proof of (2.2) in Lemma 2.1 to see that

$$\lim_{s\to 1^-} C_{N,s}\,\omega_N \int_{\mathbb{R}^N\setminus\Omega} \frac{K_{\frac{N+2s}{2}}(\bar{C}|y|)}{(\bar{C}|y|)^{\frac{N+2s}{2}}}\,\mathrm{d}y = 0.$$

Thus, $R_s \to 0$ as $s \to 1^-$ and this implies that (1.10) holds whenever $u \in C_0^\infty(\Omega)$. Assume now $u \in H_0^1(\Omega)$. Hence we can find a sequence $(\phi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\|\nabla\phi_n - \nabla u\|_{L^2(\Omega)} \to 0$ as $n \to +\infty$. Consequently, $\|\nabla\phi_n\|_{L^2(\Omega)} \to \|\nabla u\|_{L^2(\Omega)}$ as $n \to +\infty$. Fix $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$\big|\|\nabla u\|_{L^2(\Omega)} - \|\nabla\phi_n\|_{L^2(\Omega)}\big| \le \varepsilon. \tag{2.11}$$

In light of Lemma 2.3, we know that, for all s near 1 and $n \geq n_0$,

(2.12)

$$\left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|(\phi_n - u)(x) - (\phi_n - u)(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \leq C \left\|\nabla \phi_n - \nabla u\right\|_{L^2(\Omega)},$$

where C is independent of n and s. Therefore, using (2.11) and (2.12), we have, for all s near 1 and $n \geq n_0$,

$$\begin{aligned}
&\Bigg| \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|\phi_n(x) - \phi_n(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \\
&\qquad - \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \Bigg| \\
&\leq \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|(\phi_n - u)(x) - (\phi_n - u)(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \leq C\varepsilon,
\end{aligned}$$

from which

$$\begin{aligned}
&\left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|\phi_n(x) - \phi_n(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} - C\varepsilon \\
&\leq \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|\phi_n(x) - \phi_n(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} + C\varepsilon.
\end{aligned}$$

Exploiting the fact that (1.11) is valid for ϕ_n, we can pass to the limit as $s \to 1^-$ in the above inequality to see that, for all $n \geq n_0$,

$$\begin{aligned}
\left(2K_N \int_\Omega |\nabla \phi_n|^2 \mathrm{dx}\right)^{\frac{1}{2}} - C\varepsilon &\leq \lim_{s \to 1^-} \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \\
&\leq \left(2K_N \int_\Omega |\nabla \phi_n|^2 \mathrm{dx}\right)^{\frac{1}{2}} + C\varepsilon.
\end{aligned}$$

Letting $n \to +\infty$, it follows from (2.11) that

$$\begin{aligned}
\left(2K_N \int_\Omega |\nabla u|^2 \mathrm{dx}\right)^{\frac{1}{2}} - C\varepsilon &\leq \lim_{s \to 1^-} \left(\frac{C_{N,s}}{2}\omega_N \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{dx}\,\mathrm{dy}\right)^{\frac{1}{2}} \\
&\leq \left(2K_N \int_\Omega |\nabla u|^2 \mathrm{dx}\right)^{\frac{1}{2}} + C\varepsilon.
\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, we can conclude that (1.11) is true for all $u \in H^1_0(\Omega)$. $\square$

Remark 2.4. *We briefly discuss the case $\Omega = \mathbb{R}^N$. According to* [3, Theorem 2.1], *we know that for all $u \in H^s(\mathbb{R}^N)$ it holds*

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \frac{2}{C_{N,s}} \int_{\mathbb{R}^N} \left[|(-\Delta+1)^{\frac{s}{2}} u|^2 - u^2\right] \mathrm{d}x$$
$$= \frac{2}{C_{N,s}} \int_{\mathbb{R}^N} \left[(|\xi|^2+1)^s - 1\right] |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi.$$

Then we deduce that for all $u \in H^1(\mathbb{R}^N)$

$$\begin{aligned}
&\lim_{s\to 1^-} (1-s) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{s\to 1^-} \frac{2(1-s)}{C_{N,s}} \int_{\mathbb{R}^N} \left[(|\xi|^2+1)^s - 1\right] |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi \\
&= \lim_{s\to 1^-} \frac{\Gamma(2-s)(2\pi)^{\frac{N}{2}}}{s2^s} \int_{\mathbb{R}^N} \left[(|\xi|^2+1)^s - 1\right] |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi \\
&= \frac{(2\pi)^{\frac{N}{2}}}{2} \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi =: \kappa_N \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x,
\end{aligned}$$

which is comparable with (see [6, Remark 2] *or* [8, Remark 4.3]*)*

$$\lim_{s\to 1^-} (1-s) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = K_N \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x\,.$$

On the other hand, we have the following asymptotic behavior as $s \to 0^+$:

$$\begin{aligned}
&\lim_{s\to 0^+} s \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(|x-y|)\,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{s\to 0^+} \frac{2s}{C_{N,s}} \int_{\mathbb{R}^N} \left[(|\xi|^2+1)^s - 1\right] |\mathcal{F}u(\xi)|^2\,\mathrm{d}\xi \\
&= \lim_{s\to 0^+} \frac{\Gamma(2-s)2^{\frac{N}{2}-s}\pi^{\frac{N}{2}}}{1-s} \int_{\mathbb{R}^N} \left[(|\xi|^2+1)^s - 1\right] |\mathcal{F}u(\xi)|^2\,\mathrm{d}\xi \\
&= (2\pi)^{\frac{N}{2}} \cdot 0 = 0,
\end{aligned}$$

which is completely different from (see [12, Theorem 3] *or* [8, Remark 4.3]*)*

$$\lim_{s\to 0^+} s \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \omega_{N-1} \int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x\,.$$

Roughly speaking, this phenomenon is due to the fact that $(-\Delta+1)^s u - u \to 0$ as $s \to 0^+$ while $(-\Delta)^s u \to u$ as $s \to 0^+$.

References

[1] V. Ambrosio, *On some convergence results for fractional periodic Sobolev spaces*, Opuscula Math. **40** (2020), no. 1, 5–20.

[2] V. Ambrosio, *Nonlinear fractional Schrödinger equations in $\mathbb{R}^N$*, Birkhäuser, 2021.

[3] V. Ambrosio, *On the fractional relativistic Schrödinger operator*, J. Differential Equations **308** (2022), 327–368.

[4] V.Ambrosio, H.Bueno, A.H.Medeiros, G.A.Pereira, *On the convergence of the fractional relativistic Schrödinger operator*, submitted.

[5] N.Aronszajn, K.T.Smith, *Theory of Bessel potentials. I*, Ann. Inst. Fourier (Grenoble) **11** (1961), 385–475.

[6] J.Bourgain, H.Brezis, P.Mironescu, *Another look at Sobolev spaces*, Optimal control and partial differential equations, 439–455, IOS, Amsterdam, 2001.

[7] R.Carmona, W.C.Masters, B.Simon, *Relativistic Schrödinger operators: Asymptotic behavior of the eigenfunctions*, J. Func. Anal **91** (1990), 117–142.

[8] E.Di Nezza, G.Palatucci, E.Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[9] A.Erdélyi, W.Magnus, F.Oberhettinger, F.G.Tricomi, *Higher transcendental functions. Vol. II*, Based on notes left by Harry Bateman. Reprint of the 1953 original. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. xviii+396 pp.

[10] L.Hörmander, *The analysis of linear partial differential operators. III. Pseudo-differential operators*, Reprint of the 1994 edition. Classics in Mathematics. Springer, Berlin, 2007. viii+525 pp.

[11] E.H.Lieb, R.Seiringer, *The stability of matter in quantum mechanics*, Cambridge University Press, Cambridge, 2010. xvi+293 pp.

[12] V.Maz'ya, T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), no. 2, 230–238.

[13] A.Pinamonti, M.Squassina, E.Vecchi, *Magnetic BV-functions and the Bourgain-Brezis-Mironescu formula*, Adv. Calc. Var. **12** (2019), no. 3, 225–252.

[14] M.Ryznar, *Estimate of Green function for relativistic α-stable processes*, Potential Analysis, **17**, (2002), 1–23.

[15] M.Squassina, B.Volzone, *Bourgain-Brezis-Mironescu formula for magnetic operators*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 8, 825–831.

[16] E.Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp.

[17] P.R.Stinga, J.L.Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122.

[18] G.N.Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944. vi+804 pp.

Vincenzo Ambrosio
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona (Italy)
Email address: v.ambrosio@staff.univpm.it

Fares Essebei
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona (Italy)
Email address: f.essebei@staff.univpm.it