Γ-CONVERGENCE AND INTEGRAL REPRESENTATION FOR A CLASS OF FREE DISCONTINUITY FUNCTIONALS

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Dedicated to Giuseppe Buttazzo on the occasion his 70th birthday.

Abstract. We study the Γ-limits of sequences of free discontinuity functionals with linear growth, assuming that the surface energy density is bounded. We determine the relevant properties of the Γ-limit, which lead to an integral representation result by means of integrands obtained by solving some auxiliary minimum problems on small cubes.

Keywords: free discontinuity problems, Γ-convergence, integral representation, blow-up method.

2020 Mathematics Subject Classification: 49J45, 49Q20, 74Q05, 74E30.

1. Introduction

The purpose of this paper is to study the Γ-limits of sequences of integral functionals of the form

$$E^{f,g}(u,A):=\int_A f(x,\nabla u)dx+\int_A f^\infty\big(x,\frac{dD^cu}{d|D^cu|}\big)d|D^cu|+\int_{A\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}, \quad (1.1)$$

defined for all bounded open subsets A of $\mathbb{R}^d$, $d\geq 1$, and for all functions u in a suitable function space contained in the space $GBV(A)$ of generalized functions with bounded variation on A, for which we refer to [1, 2]. More precisely, we shall assume that u belongs to the space $GBV_\star(A)$ introduced in [8]. Hereafter ∇u is the approximate gradient of u, D^cu is the Cantor part of the distributional gradient Du of u when $u\in BV(A)$, and a suitable measure that extends this notion when $u\in GBV_\star(A)$, $|D^cu|$ is the variation of the measure D^cu, $\frac{dD^cu}{d|D^cu|}$ is the Radon-Nykodim derivative of D^cu with respect to $|D^cu|$, J_u is the jump set of u, $[u]$ denotes the amplitude of the jump, ν_u is the approximate unit normal to J_u, and $\mathcal{H}^{d-1}$ is the Hausdorff measure of dimension $d-1$.

We assume that $f(x,\xi)$ has linear growth with respect to ξ (see Definition 3.6) and that f^∞ is its recession function with respect to ξ (see Definition 3.8). As for the function g, we assume that it is bounded and satisfies the inequalities $c|\zeta|\leq g(x,\zeta,\nu)\leq C|\zeta|$ when $|\zeta|$ is small, for suitable constants $0<c<C$ (see Definition 3.7). These hypotheses on g are natural in cohesive models in fracture mechanics, as for instance the Dugdale model [10], (see also [5]) where $g(\zeta)=\min\{c|\zeta|,\kappa\}$.

The boundedness of g is the main difference with respect to the paper [6], where the inequalities $c|\zeta|\leq g(x,\zeta,\nu)\leq C|\zeta|$ are assumed to hold for every $\zeta\in\mathbb{R}$. Thanks to this hypothesis, in [6] the problem is studied in the space $BV(A)$. On the contrary, when g is bounded the functional $E^{f,g}(u,A)$ does not control the norm of u in $BV(A)$, because there is no control on the amplitude of the jump. As a consequence, the Γ-limit of a sequence of functionals of the form (1.1) can be finite also out of $BV(A)$. This forces us to study the problem in $GBV_\star(A)$ and to consider the topology of convergence in measure as the

Preprint SISSA 05/2023/MATE.

underlying topology for Γ-convergence, since a bound on $E^{f,g}(u,A)$ provides compactness only in this topology.

We introduce a class $\mathcal{F}$ of volume integrands f (see Definition 3.6) and a class $\mathcal{G}$ of bounded surface integrands g (see Definition 3.7), and study the properties of Γ-limits of sequences E^{f_k,g_k} with $f_k\in\mathcal{F}$ and $g_k\in\mathcal{G}$. In particular, we prove that if $E^{f_k,g_k}(\cdot,A)$ Γ-converges to $E(\cdot,A)$ for every bounded open set A, then the natural extension of E to Borel sets satisfies the following property (see Theorem 3.16 and Remark 4.2): for every bounded open set A and every $u\in GBV_\star(A)$ the set function $B\mapsto E(u,B)$ is a measure on the Borel subsets of A, that can be decomposed as sum of three measures:

$$E(u,B)=E^a(u,B)+E^c(u,B)+E^j(u,B)\,,$$

where $E^a(u,\cdot)$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, $E^c(u,\cdot)$ is absolutely continuous with respect to $|D^cu|$, and $E^j(u,\cdot)$ is absolutely continuous with respect to the restriction to J_u of the Hausdorff measure $\mathcal{H}^{d-1}$. Moreover, (see Theorems 3.16 and 6.3) we prove that there exist a function $f\in\mathcal{F}$ and a function $g\in\mathcal{G}$ such that for every bounded open set A we have

$$E^a(u,B)=\int_B f(x,\nabla u)dx\quad\text{and}\quad E^j(u,B)=\int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1} \tag{1.2}$$

for every $u\in GBV_\star(A)$ and every Borel subset B of A. As for the Cantor part, we have the integral representation

$$E^c(u,B)=\int_B f^\infty\big(x,\frac{dD^cu}{d|D^cu|}\big)d|D^cu|$$

under an additional continuity assumption of E with respect to translations (see Theorem 6.7).

Since the Γ-convergence considered in this paper refers to the topology of convergence in measure, it is convenient to extend the functionals introduced in (1.1) to functionals, still denoted by $E^{f,g}$ (see Definition 3.10), defined for every measurable function $u\colon\mathbb{R}^d\to\mathbb{R}$ and for every Borel set $B\subset\mathbb{R}^d$ in such a way that for every u the set function $E^{f,g}(u,\cdot)$ is a measure on the Borel σ-algebra of $\mathbb{R}^d$, and $E^{f,g}(u,A)=+\infty$ if A is a bounded open set in $\mathbb{R}^d$ and $u|_A\notin GBV_\star(A)$.

To prove the results of our paper we introduce (see Definition 3.1) a class $\mathfrak{E}$ of functionals, defined for every measurable function u on $\mathbb{R}^d$ and every Borel set $B\subset\mathbb{R}^d$, which contains all functionals $E^{f,g}$ with $f\in\mathcal{F}$ and $g\in\mathcal{G}$. All functionals $E\in\mathfrak{E}$ are local, i.e., if A is a bounded open set and $u=v$ $\mathcal{L}^d$-a.e. on A, then $E(u,A)=E(v,A)$. Moreover, for every u the set function $E(u,\cdot)$ is a Borel measure on the Borel σ-algebra of $\mathbb{R}^d$.

We prove the following compactness result (see Theorem 3.16): if E_k is a sequence in $\mathfrak{E}$, then there exists a subsequence, not relabelled, and a functional $E\in\mathfrak{E}$ such that $E_k(\cdot,A)$ Γ-converges to $E(\cdot,A)$ for every bounded open set $A\subset\mathbb{R}^d$.

Since, by definition, $u\in GBV_\star(A)$ if and only if the truncations of u belong to $BV(A)$ and satisfy suitable estimates, in order to reduce our problem to $BV(A)$ it is crucial that the definition of $\mathfrak{E}$ implies that for every $E\in\mathfrak{E}$ we have careful estimates on the difference between the values of E on u and on its truncations (see (g) in Definition 3.1 and Remark 3.4).

The main difficulty in the proof of the compactness result is to show that for the limit functional E the set function $E(u,\cdot)$ is a Borel measure for every u. Thanks to the estimates mentioned above this is done first for bounded BV functions and then extended to arbitrary functions. Finally, the decomposition $E(u,B)=E^a(u,B)+E^c(u,B)+E^j(u,B)$ for $E\in\mathfrak{E}$ follows from the upper bounds in the definition of $\mathfrak{E}$.

A second result of our paper concerns the integral representation of functionals in $\mathfrak{E}$ (see Theorem 6.3). We prove that if $E\in\mathfrak{E}$ and for every bounded open set $A\subset\mathbb{R}^d$ the functional $E(\cdot,A)$ is lower semicontinuous with respect to convergence in measure, then there exist two functions, $f \in \mathcal{F}$ and $g \in \mathcal{G}$, such that (1.2) holds. Moreover, we show that for every $x \in \mathbb{R}^d$ we can determine $f(x,\xi)$ and $g(x,\zeta,\nu)$ by considering the minimum values of E on small cubes centered at x, with suitable boundary conditions depending on ξ, ζ, and ν, and taking the limits, after suitable rescalings, as the size of the cubes tends to 0 (see Definition 4.12 and Theorem 6.3).

These compactness and integral representation results, together with the characterisation of f and g, will be applied in a future work to the study of stochastic homogenisation problems for this class of free discontinuity functionals.

The integral representation result (1.2) is well-known in $BV(A)$, under the additional assumption that there exist two positive constants c and C such that

$$c|Du|(A) \leq E(u,A) \leq C(\mathcal{L}^d(A) + |Du|(A))$$

for every bounded open set $A \subset \mathbb{R}^d$ and every $u \in BV(A)$, see [4]. Since the functions $g \in \mathcal{G}$ are bounded, the functional $E^{f,g}$ cannot satisfy this estimate, and hence our definition of the class $\mathfrak{E}$ cannot imply such a condition. To obtain the integral representation result for $E \in \mathfrak{E}$ we first introduce for every $\varepsilon > 0$ the functional E_ε defined by

$$E_\varepsilon(u,A) = E(u,A) + \varepsilon|Du|(A)$$

for every bounded open set A and every $u \in BV(A)$. From the results in [4] we deduce that for every bounded open set A we have

$$E^a_\varepsilon(u,B) = \int_B f_\varepsilon(x,\nabla u)dx \quad \text{and} \quad E^j_\varepsilon(u,B) = \int_{B\cap J_u} g_\varepsilon(x,[u],\nu_u)d\mathcal{H}^{d-1}$$

for every $u \in BV(A)$ and every Borel subset B of A (see Theorem 6.1). Taking the limit as $\varepsilon \to 0+$ we get (1.2) for every $u \in BV(A)$. The extension to $GBV_\star(A)$ can be obtained using the estimates on the difference between the values of E on u and on its truncations.

Thanks to the characterisation of f_ε and g_ε given in [4], the integrands f and g can be obtained as limits of rescaled minimum values of E_ε on small cubes, as the size of the cubes and the parameter ε tend to zero. To prove that f and g can be obtained directly as limits of rescaled minimum values of E on small cubes we use a technical result (see Lemma 4.16), which allows us to estimate of the L^∞-norm of suitable quasi-minimisers of the minimum problems for E.

The characterisation of the pointwise values of f and g by means of minimum problems on small cubes is also used to prove that f and g satisfy the inequalities that define $\mathcal{F}$ and $\mathcal{G}$, respectively (see Theorem 5.1).

The result for the Cantor part, under the additional assumption of continuity with respect to translations, is obtained using the same line of proof (see Theorem 6.7).

In the last part the paper we fix a bounded open set $\Omega \subset \mathbb{R}^d$ and we study the convergence of minimum values and of quasi-minimisers of some minimum problems in Ω for functionals in $\mathfrak{E}$, under the assumption of Γ-convergence. The first one concerns

$$\min_{u\in GBV_\star(\Omega)} \Big(E^{f,g}(u,\Omega) + \int_\Omega \psi(x,u)dx\Big),$$

where $f \in \mathcal{F}$, $g \in \mathcal{G}$, and $\psi\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$a_1|s|^p - a_2 \leq \psi(x,s) \leq a_3|s|^p + a_4 \quad \text{for } \mathcal{L}^d\text{-a.e. in } \Omega \text{ and every } s \in \mathbb{R} \tag{1.3}$$

for some $p \geq 1$, $a_1 > 0$, $a_2 \geq 0$, $a_3 > 0$, and $a_4 \geq 0$. We prove (see Theorem 7.1) that the Γ-convergence of $E^{f_k,g_k}(\cdot,\Omega)$ to $E^{f,g}(\cdot,\Omega)$ implies the convergence of the corresponding minimum values and, up to a subsequence, the convergence in $L^p(\Omega)$ of the quasi-minimisers to a minimiser of the limit problem.

When Ω has a Lipschitz boundary and $f_k \in \mathcal{F}$ and $g_k \in \mathcal{G}$ are two given sequences, we also consider the following weak formulation of the minimum problems with Dirichlet boundary condition :

$$\min_{u\in GBV_\star(\Omega)} \Big(E^{f_k,g_k}(u,\Omega)+\tilde c\int_{\partial\Omega}|\mathrm{tr}_\Omega u-\varphi|\wedge 1 d\mathcal{H}^{d-1}\Big)\,, \tag{1.4}$$

where $\tilde c$ is a positive constant and $\varphi\in L^1(\partial\Omega)$. To determine the limit problem we introduce the functionals $\tilde E_k$ defined by

$$\tilde E_k(u,A):=E^{f,g}(u,A\cap\Omega)+\tilde c\Big(\int_{A\setminus\Omega}|\nabla u|dx+|D^cu|(A\setminus\Omega)+\int_{(A\cap J_u)\setminus\Omega}|[u]|\wedge 1 d\mathcal{H}^{d-1}\Big)$$

and we assume that there exists $\hat E\in\mathfrak{E}$ such that $\tilde E_k(\cdot,A)$ Γ-converge to $\hat E(\cdot,A)$. By the integral representation results previously mentioned, there exists $\hat g\in\mathcal{G}$ such that for every bounded open set $A\subset\mathbb{R}^d$

$$\hat E^j(u,B)=\int_{B\cap\partial\Omega\cap J_u}\hat g(x,[u],\nu_u)d\mathcal{H}^{d-1}$$

for every $u\in GBV_\star(A)$ and every Borel subset B of A. Under an additional assumption on g_k (see (7.43)), which is always satisfied when g_k is even with respect to ζ, we prove that the limit problem of (1.4) is

$$\min_{u\in GBV_\star(\Omega)} \Big(\hat E(u,\Omega)+\int_{\partial\Omega}\hat g(x,\varphi-\mathrm{tr}_\Omega u,\nu_\Omega)d\mathcal{H}^{d-1}\Big)\,, \tag{1.5}$$

where ν_Ω is the outer unit normal to Ω. More precisely (see Corollary 7.15), we prove that the minimum values of (1.4) converge to the minimum value of (1.5) and that, up to a subsequence, we can construct a sequence of quasi-minimisers of (1.4) which converges in measure to a minimiser of (1.5).

2. Notation and preliminaries

We begin by introducing some notation.

(a) Throughout this paper $d\geq 1$ is fixed integer. The Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. We set $\mathbb{S}^{d-1}:=\{\nu\in\mathbb{R}^d:|\nu|=1\}$ and $\mathbb{S}^{d-1}_\pm:=\{\nu\in\mathbb{S}^{d-1}:\pm\nu_{i(\nu)}>0\}$, where $i(\nu)$ is the largest $i\in\{1,\dots,d\}$ such that $\nu_i\neq 0$. Note that $\mathbb{S}^{d-1}=\mathbb{S}^{d-1}_+\cup\mathbb{S}^{d-1}_-$.

(b) Given an open set $A\subset\mathbb{R}^d$, let $\mathcal{A}(A)$ be the collection of all open subsets of A and let $\mathcal{A}_c(A):=\{A'\in\mathcal{A}(A):A'\subset\subset A\}$, where $A'\subset\subset A$ means that A' is relatively compact in A. Given a Borel set $B\subset\mathbb{R}^d$, $\mathcal{B}(B)$ denotes the σ-algebra of all Borel measurable subsets of B.

(c) For every $x\in\mathbb{R}^d$ and $\rho>0$ let $Q(x,\rho):=\{y\in\mathbb{R}^d:|(y-x)\cdot e_i|<\rho/2,$ for every $i=1,\dots,d\}$, where $(e_i)_{i=1,\dots,d}$ is the canonical basis in $\mathbb{R}^d$, and $\cdot$ denotes the scalar product.

(d) For every $\nu\in\mathbb{S}^{d-1}$ we fix a rotation $R_\nu\colon\mathbb{R}^d\to\mathbb{R}^d$ such that $R_\nu(e_d)=\nu$. We assume that R_{e_d} is the identity, that the restrictions of the function $\nu\mapsto R_\nu$ to the sets $\mathbb{S}^{d-1}_\pm$ are continuous, and that $R_\nu(Q(0,\rho))=R_{-\nu}(Q(0,\rho))$ for every $\nu\in\mathbb{S}^{d-1}$ and every $\rho>0$.

(e) For every $\lambda>0$, $\nu\in\mathbb{S}^{d-1}$, $x\in\mathbb{R}^d$, and $\rho>0$ let $Q^\lambda_\nu(x,\rho)$ be the rectangle defined by

$$Q^\lambda_\nu(x,\rho):=x+R_\nu((-\tfrac{\lambda\rho}{2},\tfrac{\lambda\rho}{2})^{d-1}\times(-\tfrac{\rho}{2},\tfrac{\rho}{2}))\,. \tag{2.1}$$

(f) For every $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, $\nu\in\mathbb{S}^{d-1}$ we define the functions $\ell_\xi\colon\mathbb{R}^d\to\mathbb{R}$ and $u_{x,\zeta,\nu}\colon\mathbb{R}^d\to\mathbb{R}$ by

$$\ell_\xi(y):=\xi\cdot y\,,$$

$$u_{x,\zeta,\nu}(y):=\begin{cases}\zeta & \text{if } (y-x)\cdot\nu\geq 0\,,\\ 0 & \text{if } (y-x)\cdot\nu<0\,;\end{cases}$$

moreover, we set $\Pi_x^\nu = \{y \in \mathbb{R}^d : (y-x)\cdot\nu = 0\}$.

(g) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and an $\mathcal{L}^d$-measurable function $u\colon A \to \overline{\mathbb{R}}$, we say that $a \in \overline{\mathbb{R}}$ is the approximate limit of u as $y \to x \in A$ if for every neighbourhood U of a we have

$$\lim_{\rho\to 0+} \frac{\mathcal{L}^d(\{y \in A \cap Q(x,\rho) : u(y) \notin U\})}{\rho^d} = 0\,;$$

the same definition is meaningful also if $x \in \partial A$ provided $\lim_{\rho\to 0+} \frac{\mathcal{L}^d(A\cap Q(x,\rho))}{\rho^d} > 0$; moreover, the set of points $x \in A$ where the approximate limit $\tilde{u}(x)$ exists and is finite is a Borel subset of A, and the function $x \mapsto \tilde{u}(x)$ is a Borel function defined on it; we say that $\xi \in \mathbb{R}^d$ is the approximate gradient of u at x if the approximate limit of $\frac{u(y)-u(x)-\xi\cdot(y-x)}{|y-x|}$ as $y \to x$ is equal to 0.

(h) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and an $\mathcal{L}^d$-measurable function $u\colon A \to \overline{\mathbb{R}}$, the jump set J_u is the set of all points $x \in A$ for which there exist $u^+(x), u^-(x) \in \overline{\mathbb{R}}$, with $u^+(x) \neq u^-(x)$, and $\nu_u(x) \in \mathbb{S}^{d-1}$ such that $u^\pm(x)$ is the approximate limit as $y \to x$ of the restriction of u to the set $\{y \in A : \pm(y-x)\cdot\nu_u(x) > 0\}$. It is easy to see that the triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely defined up to a swap of the first two terms and a change of sign in the third one. For every $x \in J_u$ we set $[u](x) := u^+(x) - u^-(x)$. It can be proved that J_u is a Borel set and that, if we choose ν_u so that $\nu_u(x) \in \mathbb{S}^{d-1}_+$ for every $x \in J_u$, then the functions $u^+, u^-, [u]\colon J_u \to \overline{\mathbb{R}}$ and $\nu_u\colon J_u \to \mathbb{S}^{d-1}$ are Borel functions.

(i) For every $A \in \mathcal{A}(\mathbb{R}^d)$ and $u \in BV(A)$ let Du be the distributional gradient of u, which can be decomposed as the sum of three $\mathbb{R}^d$-valued measures:

$$Du = D^a u + D^c u + D^j u\,,$$

where $D^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$, $D^c u$ is singular with respect to the Lebesgue measure and vanishes on all $B \in \mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B) < +\infty$, and $D^j u$ is concentrated on the jump set J_u of u. The approximate gradient of u at x exists for $\mathcal{L}^d$-a.e. $x \in A$ and is denoted by $\nabla u(x)$; it is known that the function ∇u coincides $\mathcal{L}^d$-a.e. in A with the density of $D^a u$ with respect to $\mathcal{L}^d$. Moreover, it is known that $D^j u = [u]\nu_u \mathcal{H}^{d-1} \llcorner J_u$, where for every measure μ the measure $\mu \llcorner E$ is defined by $\mu \llcorner E(B) := \mu(E \cap B)$. For these and related fine properties of BV functions we refer to [2].

Given $B \in \mathcal{B}(\mathbb{R}^d)$, $u\colon B \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, and $m > 0$ the truncation $u^{(m)}$ of u is defined as

$$u^{(m)}(x) := (u(x) \wedge m) \vee (-m)\,,$$

where $a \wedge b$ and $a \vee b$ denote the minimum and the maximum between a and b, respectively.

Let us now recall the definition of $GBV_\star(A)$ introduced in [8, Definition 3.1].

Definition 2.1. Given $A \in \mathcal{A}_c(\mathbb{R}^d)$, the space $GBV_\star(A)$ is defined as the space of functions $u\colon A \to \mathbb{R}$ such that $u^{(m)} \in BV(A)$ for every $m > 0$ and

$$\sup_{m>0} \Big(\int_A |\nabla u^{(m)}| dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \Big) < +\infty\,.$$

It follows immediately from the definition that $GBV_\star(A) \cap L^\infty(A) \subset BV(A) \subset GBV_\star(A) \subset GBV(A)$, where the last space is defined in [2, Definition 4.26]. For the reader's convenience in the following theorem we summarize the main properties of $GBV_\star$ functions.

Theorem 2.2. *Let $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in GBV_\star(A)$. Then the following properties hold:*

(a) *the approximate limit $\tilde{u}(x)$ of u as $y \to x$ is finite for $\mathcal{H}^{d-1}$-a.e. $x \in A \setminus J_u$; moreover, $u^+(x)$ and $u^-(x)$ are finite for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$;*

(b) *there exists a function* $\nabla u \in L^1(A;\mathbb{R}^d)$ *such that for* $\mathcal{L}^d$*-a.e.* $x \in A$ *the vector* $\nabla u(x)$ *is the approximate gradient of* u *at* x*; moreover* $\nabla u(x) = \nabla u^{(m)}(x)$ *for every* $m > 0$ *and* $\mathcal{L}^d$*-a.e.* $x \in \{|u| \leq m\}$*;*

(c) *there exists a unique* $\mathbb{R}^d$*-valued Radon measure on* A*, denoted by* $D^c u$*, such that for every* $m > 0$ $D^c u(B) = D^c u^{(m)}(B)$ *for every Borel set* $B \subset \{x \in A \setminus J_u : \tilde{u}(x)$ *exists and* $|\tilde{u}(x)| \leq m\}$ *and* $D^c u(B) = 0$ *for every Borel set* B *such that* $\mathcal{H}^{d-1}(B \setminus J_u) = 0$*; moreover, for every* $B \in \mathcal{B}(A)$ *we have*

$$D^c u^{(m)}(B) \to D^c u(B) \quad \text{as } m \to +\infty\,, \tag{2.2}$$

$$\lim_{m\to+\infty} |D^c u^{(m)}|(B) = \sup_{m>0} |D^c u^{(m)}|(B) = |D^c u|(B)\,; \tag{2.3}$$

finally, $D^c u$ *is singular with respect to the Lebesgue measure* $\mathcal{L}^d$ *and* $D^c u(B) = 0$ *for every* $B \in \mathcal{B}(A)$ *with* $\mathcal{H}^{d-1}(B) < +\infty$*;*

(d) *for every* $m > 0$ *we have* $J_{u^{(m)}} \subset J_u$ *up to a set of* $\mathcal{H}^{d-1}$*-measure zero and* $|[u^{(m)}]| \leq |[u]|$ $\mathcal{H}^{d-1}$*-a.e. on* $J_{u^{(m)}} \cap J_u$*; moreover, for* $\mathcal{H}^{d-1}$*-a.e.* $x \in J_u$*, there exists* $m_x \in \mathbb{N}$ *such that* $x \in J_{u^{(m)}}$ *for every* $m \in \mathbb{N}$ *with* $m \geq m_x$ *and* $[u^{(m)}](x) \to [u](x)$ *as* $m \to +\infty$ *with* $m \in \mathbb{N}$*; finally,* $\mathcal{H}^{d-1} \llcorner J_u$ *is* σ*-finite;*

(e) *if, in addition,* A *has Lipschitz boundary, then for* $\mathcal{H}^{d-1}$*-a.e.* $x \in \partial A$ *the approximate limit of* u *at* x *exists and is finite; its value is denoted by* $(\mathrm{tr}_A u)(x)$ *and the function* $\mathrm{tr}_A u$*,* $\mathcal{H}^{d-1}$*-a.e. defined on* ∂A*, is called the trace of* u *on* ∂A*.*

Proof. Property (a) is proved in [8, Theorem 3.8]. The properties in (b) are proved in [8, Proposition 2.6 (b) and Proposition 3.3]. The properties in (c) can be found in [8, Theorem 2.7 and Proposition 2.9], except for the last one, which follows from (2.2), (2.3), and the analogous property for BV functions. The properties in (d) are proved in [8, Proposition 2.6 (c)] except for the last one, which is a consequence of the previous properties and the corresponding property for BV functions.

To prove (e) we fix $A' \in \mathcal{A}_c(\mathbb{R}^d)$ with $A \subset\subset A'$ and consider the function $v\colon A' \to \mathbb{R}$ defined by $v(x) = u(x)$, if $x \in A$, and $v(x) = 0$, if $x \in A' \setminus A$. Let $\partial^r A$ denote the set of such points $x \in \partial A$ where the outer unit normal vector $\nu_A(x)$ is well-defined. This is the unique unit vector satisfying

$$\frac{\mathcal{L}^d(A \cap \{y \in B_\rho(x) : (y-x)\cdot\nu_A(x) > 0\})}{\rho^d} \to 0 \tag{2.4}$$

$$\frac{\mathcal{L}^d(\{y \in B_\rho(x) : (y-x)\cdot\nu_A(x) < 0\} \setminus A)}{\rho^d} \to 0 \tag{2.5}$$

as $\rho \to 0+$. Since A has Lipschitz boundary, we have $\mathcal{H}^{d-1}(\partial A \setminus \partial^r A) = 0$. To conclude the proof it is enough to prove that for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^r A \cap J_v$ and for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^r A \setminus J_v$ the approximate limit of u at x exists.

By (a) for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^r A \cap J_v$ there exist $v^+(x)$, $v^-(x)$ in $\mathbb{R}$, with $v^+(x) \neq v^-(x)$, and $\nu_v(x) \in \mathbb{S}^{d-1}$ such that $v^\pm(x)$ are the approximate limits of v as $y \to x$ in $\{y \in A' : \pm(y-x)\cdot\nu_v(x) > 0\}$. This implies that $\nu_v(x) = \pm\nu_A(x)$. Indeed, otherwise there would exist two open cones C^+ and C^- with vertex x such that $A \cap (C^\pm \cap B_\rho(x)) = \emptyset$ for $\rho > 0$ small enough, and $C^\pm \subset \{y \in \mathbb{R}^d : \pm(y-x)\cdot\nu_v(x) > 0\}$. Since $v = 0$ on $C^\pm \cap B_\rho(x) \subset A'$ for $\rho > 0$ small enough, we deduce that $v^\pm(x) = 0$, which contradicts the fact that $v^+(x) \neq v^-(x)$. Hence we may assume that $\nu_v(x) = -\nu_A(x)$. Since, by (2.4) and (2.5), the symmetric difference between A and $\{y \in A' : (y-x)\cdot\nu_v(x) > 0\}$ has density zero at x, we have that $v^+(x)$ is the approximate limit at x of the restriction of v to A, which shows that $v^+(x)$ is the approximate limit of u at x.

By (a) for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^r A \setminus J_v$ there exist the approximate limit of v at x. Since u is a restriction of v this implies the existence of the approximate limit of u at x. $\square$

We now present some properties of the function space $GBV_\star(A)$, which will be used in the sequel.

Theorem 2.3. *Let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then the following properties hold:*

(a) *$GBV_\star(A)$ is a vector space;*
(b) *$GBV_\star(A)$ is a lattice, i.e., $u, v \in GBV_\star(A)$ implies $u \vee v, u \wedge v \in GBV_\star(A)$;*
(c) *if $u \in GBV_\star(A)$, $w \in BV(A)$, and $m > 0$, then $(u \vee (w-m)) \wedge (w+m)$ belongs to $BV(A)$.*

Proof. Property (a) is proved in [8, Theorem 3.9].

To prove (b) we fix u and $v \in GBV_\star(A)$. For every $m > 0$ we have $(u \vee v)^{(m)} = u^{(m)} \vee v^{(m)}$. We claim that this implies

$$|D((u \vee v)^{(m)})|(B) \leq |Du^{(m)}|(B) + |Dv^{(m)}|(B) \tag{2.6}$$

for every $B \in \mathcal{B}(A)$. It is enough to prove (2.6) when B is open. In this case the inequality is trivial if $u^{(m)}$ and $v^{(m)}$ belong to $W^{1,1}(A)$ and follows by approximation in the general case. We conclude that (2.6) holds for every $B \in \mathcal{B}(A)$. This implies that $|D^a((u \vee v)^{(m)})|(B) \leq |D^a u^{(m)}|(B) + |D^a v^{(m)}|(B)$ and $|D^c((u \vee v)^{(m)})|(B) \leq |D^c u^{(m)}|(B) + |D^c v^{(m)}|(B)$ for every $B \in \mathcal{B}(A)$. Hence $|\nabla (u \vee v)^{(m)}| \leq |\nabla u^{(m)}| + |\nabla v^{(m)}|$ $\mathcal{L}^d$-a.e. in A. Moreover, since $|a \vee b - c \vee d| \leq |a-c| + |b-d|$ for every $a, b, c, d \in \mathbb{R}$, we have $|[(u \vee v)^{(m)}]| \leq |[u^{(m)}]| + |[v^{(m)}]|$. These inequalities imply that

$$\begin{aligned}
\int_A |\nabla (u \vee v)^{(m)}| dx + |D^c (u \vee v)^{(m)}|(A) + \int_{J_{(u \vee v)^{(m)}}} |[(u \vee v)^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \\
\leq \int_A |\nabla u^{(m)}| dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 d\mathcal{H}^{d-1} \\
+ \int_A |\nabla v^{(m)}| dx + |D^c v^{(m)}|(A) + \int_{J_{v^{(m)}}} |[v^{(m)}]| \wedge 1 d\mathcal{H}^{d-1}
\end{aligned}$$

for every $m > 0$. The conclusion $u \vee v \in GBV_\star(A)$ follows now from Definition 2.1. The same arguments hold for $u \wedge v$.

To prove (c) it is enough to observe that $(u \vee (w-m)) \wedge (w+m) = w + (u-w)^{(m)}$. The conclusion now follows from (a) and from the definition of $GBV_\star(A)$. $\square$

If $u, v \in GBV_\star(A)$ coincide on an open set $U \subset A$, then their approximate gradients and the measures $D^c u$ and $D^c v$ coincide on U. The following lemma can be considered as an extension of this property to arbitrary Borel subsets of A.

Lemma 2.4. *Let $A \in \mathcal{A}_c(\mathbb{R}^d)$, let $u, v \in GBV_\star(A)$, and let $E \in \mathcal{B}(A)$ with $E \cap (J_u \cup J_v) = \emptyset$. Suppose that $\tilde{u} = \tilde{v}$ $\mathcal{H}^{d-1}$-a.e. in E. Then $\nabla u = \nabla v$ $\mathcal{L}^d$-a.e. in E and $D^c u(B) = D^c v(B)$ for every Borel set $B \subset E$.*

Proof. Let $w := u - v$. By [8, Theorem 3.9] we have that $w \in GBV_\star(A)$, $E \cap J_w = \emptyset$, and $\tilde{w} = 0$ $\mathcal{H}^{d-1}$-a.e. in E. For every $m > 0$ we have that $w^{(m)} \in BV(A)$ and since $\widetilde{w^{(m)}} = \tilde{w}^{(m)}$ (see, e.g., [8, (2.2)]), we conclude that $\widetilde{w^{(m)}} = 0$ $\mathcal{H}^{d-1}$-a.e. in E. By [8, Lemma 2.3] we obtain $\nabla w^{(m)} = 0$ $\mathcal{L}^d$-a.e. in E and $D^c w^{(m)}(B) = 0$ for every Borel set $B \subset E$. Passing to the limit as $m \to +\infty$ and using (b) and (c) in Theorem 2.2 we obtain that $\nabla w = 0$ $\mathcal{L}^d$-a.e. in E and $D^c w(B) = 0$ for every Borel set $B \subset E$. Using [8, Proposition 3.10] we conclude that $\nabla u = \nabla v$ $\mathcal{L}^d$-a.e. in E and $D^c u(B) = D^c v(B)$ for every Borel set $B \subset E$. $\square$

In the following definition we introduce a functional that will play an important role in this paper.

Definition 2.5. The functional $V\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ is defined in the following way. For every $A\in\mathcal{A}_c(\mathbb{R}^d)$ we set

$$V(u,A):=\int_A|\nabla u|dx+|D^cu|(A)+\int_{A\cap J_u}|[u]|\wedge 1d\mathcal{H}^{d-1}\quad\text{if } u|_A\in GBV_\star(A)\,, \tag{2.7}$$

and $V(u,A):=+\infty$ otherwise; the definition is then extended to $\mathcal{A}(\mathbb{R}^d)$ by setting

$$V(u,A):=\sup\{V(u,A'):A'\in\mathcal{A}_c(\mathbb{R}^d)\cap\mathcal{A}(A)\}\quad\text{for } A\in\mathcal{A}(\mathbb{R}^d)\,, \tag{2.8}$$

and to $\mathcal{B}(\mathbb{R}^d)$ by setting

$$V(u,B):=\inf\{V(u,A):A\in\mathcal{A}(\mathbb{R}^d),\ B\subset A\}\quad\text{for } B\in\mathcal{B}(\mathbb{R}^d)\,. \tag{2.9}$$

Remark 2.6. It follows immediately from the definition that if $A\in\mathcal{A}(\mathbb{R}^d)$ and $u|_A\in BV(A)$ then $V(u,B)\le|Du|(B)$ for every $B\in\mathcal{B}(A)$.

In the following remark we reformulate the definition of $GBV_\star(A)$ in terms of the behaviour of V on the truncations $u^{(m)}$.

Remark 2.7. Let $u\in L^0(\mathbb{R}^d)$ and $A\in\mathcal{A}_c(\mathbb{R}^d)$. Then $u|_A\in GBV_\star(A)$ if and only if $\sup_{m>0}V(u^{(m)},A)<+\infty$. In this case Theorem 2.2 gives $V(u,A)=\sup_{m>0}V(u^{(m)},A)$.

The following proposition shows that the set function $V(u,\cdot)$ is inner regular in $\mathcal{A}(\mathbb{R}^d)$.

Proposition 2.8. *Let $u\in L^0(\mathbb{R}^d)$ and $A\in\mathcal{A}(\mathbb{R}^d)$. Then*

$$V(u,A)=\sup_{A'\in\mathcal{A}_c(A)}V(u,A')\,. \tag{2.10}$$

Proof. By (2.8) it suffices to prove (2.10) when $A\in\mathcal{A}_c(\mathbb{R}^d)$. By monotonicity it is enough to prove that

$$V(u,A)\le\sup_{A'\in\mathcal{A}_c(A)}V(u,A') \tag{2.11}$$

when the right-hand side S of (2.11) is finite. For every $m>0$ and $A'\in\mathcal{A}_c(A)$, by Remark 2.7 we have

$$\int_{A'}|\nabla u^{(m)}|dx+|D^cu^{(m)}|(A')+\int_{A'\cap J_{u^{(m)}}}|[u^{(m)}]|\wedge 1d\mathcal{H}^{d-1}=V(u^{(m)},A')\le S\,, \tag{2.12}$$

hence $u^{(m)}\in BV(A')$. This implies that $u^{(m)}\in BV_{loc}(A)$. To prove that $u^{(m)}\in BV(A)$ we have to estimate the jump part. Let $J^1_{u^{(m)}}:=\{x\in J_{u^{(m)}}:|[u^{(m)}](x)|\ge 1\}$. Then for every $A'\in\mathcal{A}_c(A)$

$$\begin{aligned}\int_{A'\cap J_{u^{(m)}}}|[u^{(m)}]|d\mathcal{H}^{d-1}&=\int_{A'\cap J^1_{u^{(m)}}}|[u^{(m)}]|d\mathcal{H}^{d-1}+\int_{A'\cap(J_{u^{(m)}}\setminus J^1_{u^{(m)}})}|[u^{(m)}]|d\mathcal{H}^{d-1}\\&\le 2m\mathcal{H}^{d-1}(A'\cap J^1_{u^{(m)}})+\int_{A'\cap(J_{u^{(m)}}\setminus J^1_{u^{(m)}})}|[u^{(m)}]|\wedge 1d\mathcal{H}^{d-1}\\&\le(2m+1)\int_{A'\cap J_{u^{(m)}}}|[u^{(m)}]|\wedge 1d\mathcal{H}^{d-1}\le(2m+1)S\,,\end{aligned}$$

where in the last inequality we used (2.12). Taking into account the other terms in (2.12) we obtain $|Du^{(m)}|(A')\le(2m+2)S$.

Passing to the supremum for $A'\in\mathcal{A}_c(A)$ we deduce that $|Du^{(m)}|(A)\le(2m+2)S$, hence $u^{(m)}\in BV(A)$ and

$$V(u^{(m)},A)=\int_A|\nabla u^{(m)}|dx+|D^cu^{(m)}|(A)+\int_{A\cap J_{u^{(m)}}}|[u^{(m)}]|\wedge 1d\mathcal{H}^{d-1}\le S\,.$$

Since $m>0$ is arbitrary, by Remark 2.7 we obtain that $u\in GBV_\star(A)$ and that (2.11) holds. □

Remark 2.9. For every $A \in \mathcal{A}(\mathbb{R}^d)$ the functional $u \mapsto V(u,A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$. This is an immediate conseguence of [3, Theorem 2.1], see also [8, Theorem 6.1].

3. A Γ-compact class of local functionals related to $GBV_\star$

Throughout the paper we fix five constants $c_1, \dots, c_5 \geq 0$ and a bounded continuous function $\sigma\colon [0,+\infty) \to [0,+\infty)$, such that

$$0 < c_1 \leq 1 \leq c_3 \leq c_5\,, \tag{3.1}$$

$$\sigma(0) = 0 \quad \text{and} \quad \sigma(t) \geq c_3(t \wedge 1) \quad \text{for every } t \geq 0\,. \tag{3.2}$$

In the following definition we introduce the class of functionals we are interested in.

Definition 3.1. Let $\mathfrak{E}$ denote the class of functionals $E\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ that satisfy the following properties:

(a) E is local on $\mathcal{A}(\mathbb{R}^d)$, i.e., $E(u,A) = E(v,A)$ if $A \in \mathcal{A}(\mathbb{R}^d)$, $u,v \in L^0(\mathbb{R}^d)$, and $u = v$ $\mathcal{L}^d$-a.e. in A;

(b) for every $u \in L^0(\mathbb{R}^d)$ the function $E(u,\cdot)\colon \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ is a nonnegative Borel measure and

$$E(u,B) = \inf\{E(u,A) : A \in \mathcal{A}(\mathbb{R}^d),\ B \subset A\} \tag{3.3}$$

for every $B \in \mathcal{B}(\mathbb{R}^d)$;

(c1) for every $u \in L^0(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$c_1 V(u,B) - c_2\mathcal{L}^d(B) \leq E(u,B)\,; \tag{3.4}$$

(c2) for every $u \in L^0(\mathbb{R}^d)$ and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$E(u,B) \leq c_3 V(u,B) + c_4\mathcal{L}^d(B)\,; \tag{3.5}$$

(d) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $a \in \mathbb{R}$ we have

$$E(u+a,B) = E(u,B)\,; \tag{3.6}$$

(e) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $\xi \in \mathbb{R}^d$ we have

$$E(u+\ell_\xi,B) \leq E(u,B) + c_5|\xi|\mathcal{L}^d(B)\,; \tag{3.7}$$

(f) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$E(u+u_{x,\zeta,\nu},B) \leq E(u,B) + \sigma(|\zeta|)\mathcal{H}^{d-1}(B \cap \Pi^\nu_x)\,; \tag{3.8}$$

(g) for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$, with $w_1 \leq w_2$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$, we have

$$E((u \vee w_1) \wedge w_2, B) \leq E(u,B) + c_3\int_{B^u_{12}} |\nabla w_1| \vee |\nabla w_2|\,dx + c_4\mathcal{L}^d(B^u_{12})\,, \tag{3.9}$$

where $B^u_{12} = \{x \in B : u(x) \notin [w_1(x), w_2(x)]\}$.

The following remarks highlight some properties of functionals in $\mathfrak{E}$ that will be used in the sequel.

Remark 3.2. Let $E \in \mathfrak{E}$, $u,v \in L^0(\mathbb{R}^d)$, $A \in \mathcal{A}(\mathbb{R}^d)$, and $B \in \mathcal{B}(A)$. If $u = v$ $\mathcal{L}^d$-a.e. in A, then $E(u,B) = E(v,B)$. Indeed,

$$E(u,B) = \inf_{\substack{A' \in \mathcal{A}(\mathbb{R}^d)\\ B \subset A' \subset A}} E(u,A') = \inf_{\substack{A' \in \mathcal{A}(\mathbb{R}^d)\\ B \subset A' \subset A}} E(v,A') = E(v,B)$$

where the first and the last equalities follow from (3.3), while the second one follows from the locality property (a).

Remark 3.3. Let $E \in \mathfrak{E}$, $A \in \mathcal{A}(\mathbb{R}^d)$, and $u \in L^0(A)$. For every $B \in \mathcal{B}(A)$ we can define $E(u,B)$ by extending u to a function $v \in L^0(\mathbb{R}^d)$ and setting $E(u,B) := E(v,B)$. The value $E(u,B)$ does not depend on the extension thanks to Remark 3.2.

Remark 3.4. If $m > 0$ is a constant, by choosing $w_1 := -m$ and $w_2 := m$, it follows from (g) that for every $u \in L^0(\mathbb{R}^d)$, $B \in \mathcal{B}(\mathbb{R}^d)$, and every $m > 0$ we have

$$E(u^{(m)}, B) \leq E(u, B) + c_4 \mathcal{L}^d(B \cap \{|u| > m\}) \,. \tag{3.10}$$

Remark 3.5. By (c1), (c2), and the definition of V it follows that for every $u \in L^0(\mathbb{R}^d)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have

$$E(u, A) < +\infty \qquad \Longleftrightarrow \qquad u|_A \in GBV_\star(A) \,.$$

Let us now provide a typical example of integral functionals that belong to $\mathfrak{E}$. To this end we introduce two classes of functions.

Definition 3.6. Let $\mathcal{F}$ be the set of functions

$$f \colon \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$$

that satisfy the following conditions:

(f1) f is Borel measurable;
(f2) $c_1|\xi| - c_2 \leq f(x, \xi)$ for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$;
(f3) $f(x, \xi) \leq c_3|\xi| + c_4$ for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$;
(f4) $|f(x, \xi_1) - f(x, \xi_2)| \leq c_5|\xi_1 - \xi_2|$ for every $x \in \mathbb{R}^d$ and every $\xi_1, \xi_2 \in \mathbb{R}^d$.

Definition 3.7. Let $\mathcal{G}$ be the set of functions

$$g \colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0, +\infty)$$

that satisfy the following conditions:

(g1) g is Borel measurable;
(g2) $c_1(|\zeta| \wedge 1) \leq g(x, \zeta, \nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g3) $g(x, \zeta, \nu) \leq c_3(|\zeta| \wedge 1)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g4) $|g(x, \zeta_1, \nu) - g(x, \zeta_2, \nu)| \leq \sigma(|\zeta_1 - \zeta_2|)$ for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g5) $g(x, -\zeta, -\nu) = g(x, \zeta, \nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, $\nu \in \mathbb{S}^{d-1}$;
(g6) for every $x \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ the function $\zeta \mapsto g(x, \zeta, \nu)$ is non-decreasing on $[0, +\infty)$ and non-increasing on $(-\infty, 0]$.

We recall the definition of the recession function.

Definition 3.8. For every $f \colon \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ the recession function $f^\infty \colon \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$ (with respect to ξ) is defined by

$$f^\infty(x, \xi) := \limsup_{t \to +\infty} \frac{f(x, t\xi)}{t} \tag{3.11}$$

for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^d$.

Remark 3.9. For every $x \in \mathbb{R}^d$ the function $\xi \mapsto f^\infty(x, \xi)$ is positively homogeneous of degree 1. Moreover, if $\xi \mapsto f(x, \xi)$ is convex on $\mathbb{R}^d$, then the upper limit in (3.11) is a limit (see, e.g., [18, Theorem 8.5]). If f satisfies (f2) and (f3), then

$$c_1|\xi| \leq f^\infty(x, \xi) \leq c_3|\xi| \quad \text{for every } \xi \in \mathbb{R}^d \,, \tag{3.12}$$

while if f satisfies (f4), then

$$|f^\infty(x, \xi_1) - f^\infty(x, \xi_2)| \leq c_5|\xi_1 - \xi_2| \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^d \,. \tag{3.13}$$

We are now in a position to introduce the integral functionals associated with the integrands $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

Definition 3.10. Given $f \in \mathcal{F}$ and $g \in \mathcal{G}$ we define the functional $E^{f,g} \colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ in the following way: if $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u|_A \in GBV_\star(A)$ we set

$$E^{f,g}(u, A) := \int_A f(x, \nabla u) dx + \int_A f^\infty\big(x, \frac{dD^c u}{d|D^c u|}\big) d|D^c u| + \int_{A \cap J_u} g(x, [u], \nu_u) d\mathcal{H}^{d-1}, \tag{3.14}$$

while we set $E^{f,g}(u,A) := +\infty$ if $u|_A \notin GBV_\star(A)$. The definition is then extended to $\mathcal{A}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u,A) := \sup\{E^{f,g}(u,A') : A' \in \mathcal{A}_c(A)\} \quad \text{for } A \in \mathcal{A}(\mathbb{R}^d)\,, \tag{3.15}$$

and to $\mathcal{B}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u,B) := \inf\{E^{f,g}(u,A) : A \in \mathcal{A}(\mathbb{R}^d),\ B \subset A\} \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d)\,. \tag{3.16}$$

In the following proposition we show that the functionals $E^{f,g}$ belong to $\mathfrak{E}$.

Proposition 3.11. *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then the functional $E^{f,g}\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ belongs to the class $\mathfrak{E}$. Moreover, if $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in GBV_\star(A)$, then*

$$E^{f,g}(u,B) := \int_B f(x,\nabla u)dx + \int_B f^\infty\big(x, \frac{dD^c u}{d|D^c u|}\big)d|D^c u| + \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1} \tag{3.17}$$

for every $B \in \mathcal{B}(A)$, where $E^{f,g}(u,B)$ is defined according to Remark 3.3.

Proof. By construction $E^{f,g}$ satisfies condition (a). To prove that it satisfies condition (b) we fix $u \in L^0(\mathbb{R}^d)$ and observe that the set function $E^{f,g}(u,\cdot)$ is increasing, subadditive, superadditive, and inner regular on $\mathcal{A}(\mathbb{R}^d)$. Therefore, by the De Giorgi-Letta criterion [9] (see also [7, Theorem 14.23]), the extension of $E^{f,g}(u,\cdot)$ to $\mathcal{B}(\mathbb{R}^d)$ given by (3.16) is a Borel measure.

To prove (3.17) let us fix $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in GBV_\star(A)$, and consider an arbitrary extension $v \in L^0(\mathbb{R}^d)$ of u. Then by (f3), (g3), and (3.12) we have $E^{f,g}(v,A) < +\infty$. For every $B \in \mathcal{B}(A)$ let $\mu(B)$ be given by the right-hand side of (3.17). Then μ and $E^{f,g}(v,\cdot)$ are bounded nonnegative measures defined on $\mathcal{B}(A)$, which coincide on $\mathcal{A}(A)$ by the definition of $E^{f,g}$. We conclude that they coincide on $\mathcal{B}(A)$, which shows that (3.17) holds for every $B \in \mathcal{B}(A)$.

By (f2), (f3), (g2), (g3), and (3.12) for every $u \in L^0(\mathbb{R}^d)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have that

$$c_1 V(u,A) - c_2\mathcal{L}^d(A) \le E^{f,g}(u,A)\,, \tag{3.18}$$

$$E^{f,g}(u,A) \le c_3 V(u,A) + c_4\mathcal{L}^d(A)\,. \tag{3.19}$$

These inequalities are extended to Borel sets by (2.8), (2.9), (3.15), and (3.16), hence $E^{f,g}$ satisfies (c1) and (c2). Condition (d) can be easily checked. Moreover, (e) and (f) follow from (f4) and (g4), respectively.

To prove condition (g) we fix $u \in L^0(\mathbb{R}^d)$ and $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$ with $w_1 \le w_2$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$. By property (b) it is enough to prove (3.9) when B is a bounded open set. Let us fix $A \in \mathcal{A}_c(\mathbb{R}^d)$ such that $E^{f,g}(u,A) < +\infty$. By definition we have that $u|_A \in GBV_\star(A)$.

We set $v := (u \vee w_1) \wedge w_2$. By Proposition 2.3 we have that $v|_A \in GBV_\star(A)$. We observe that $\nabla v = \nabla u$ $\mathcal{L}^d$-a.e. in $\{x \in A : w_1(x) \le u(x) \le w_2(x)\}$, $\nabla v = \nabla w_1$ $\mathcal{L}^d$-a.e. in $\{x \in A : u(x) < w_1(x)\}$, and $\nabla v = \nabla w_2$ $\mathcal{L}^d$-a.e. in $\{x \in A : u(x) > w_2(x)\}$. Therefore, by (f3) we have that

$$\int_A f(x,\nabla v)dx \le \int_A f(x,\nabla u)dx + c_3\int_{A^u_{12}} |\nabla w_1| \vee |\nabla w_2| dx + c_4\mathcal{L}^d(A^u_{12})\,, \tag{3.20}$$

where $A^u_{12} := \{x \in A : u(x) \notin [w_1(x), w_2(x)]\}$.

Since $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$, the approximate limits

$$\tilde{w}_1(x)\,,\ \tilde{w}_2(x) \quad \text{exist at } \mathcal{H}^{d-1}\text{-a.e. point } x \in \mathbb{R}^d\,, \tag{3.21}$$

see, e.g., [11, Theorem 1 in Chapter 4.8 and Theorem 3 in Section 5.6.3]. Let $\tilde{B} := \{x \in A : \tilde{u}(x),\ \tilde{w}_1(x),\ \tilde{w}_2(x)$ exist and are finite$\}$ and $N = (A \setminus J_u) \setminus \tilde{B}$. By (g) and (h) at the beginning of Section 2, $\tilde{B}$ and N are Borel sets and by Theorem 2.2 (a) we have $\mathcal{H}^{d-1}(N) = 0$. Let $E := \{x \in \tilde{B} : \tilde{u}(x) \in [\tilde{w}_1(x), \tilde{w}_2(x)]\}$ and let χ_E be its indicator function defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \in \mathbb{R}^d \setminus E$.

Let us prove that

$$D^c v = \chi_E D^c u \quad \text{as measures on } \mathcal{B}(A)\,. \tag{3.22}$$

Since $\tilde v = (\tilde u \vee \tilde w_1) \wedge \tilde w_2$ in $\tilde B$, we have $\tilde v = \tilde u$ in E. Therefore, Lemma 2.4 gives that $D^c v(B) = D^c u(B)$ for every $B \in \mathcal{B}(E)$.

Let $E_1 := \{x \in \tilde B : \tilde u(x) < \tilde w_1(x)\}$ and $E_2 := \{x \in \tilde B : \tilde u(x) > \tilde w_2(x)\}$. Since $\tilde v = \tilde w_1$ on E_1, by Lemma 2.4 we have $D^c v(B) = D^c w_1(B) = 0$ for every $B \in \mathcal{B}(E_1)$. Similarly we have $D^c v(B) = 0$ for every $B \in \mathcal{B}(E_2)$. By Theorem 2.2 (c) $D^c v(B) = 0$ for every $B \subset J_u \cup N$, and thus the proof of (3.22) is concluded.

Since $f^\infty(x, 0) = 0$ we deduce from (3.22) that

$$\int_A f^\infty\Big(x, \frac{dD^c v}{d|D^c v|}\Big) d|D^c v| \le \int_A f^\infty\Big(x, \frac{dD^c u}{d|D^c u|}\Big) d|D^c u|\,. \tag{3.23}$$

We consider now the surface integral in $E^{f,g}$. If $x \in \tilde B$ from the equality $\tilde v = (\tilde u \vee \tilde w_1) \wedge \tilde w_2$ we deduce that $\tilde v(x)$ exists, hence $x \notin J_v$. Therefore $J_v \cap A \subset A \setminus \tilde B$, and by (3.21) and Theorem 2.2 (a) this implies that for $\mathcal{H}^{d-1}$-a.e. $x \in J_v$ the approximate limits $\tilde w_1(x)$ and $\tilde w_2(x)$ exist, $x \in J_u$, and we can choose $\nu_v(x) = \nu_u(x)$; this leads to $v^+ = (u^+ \vee \tilde w_1) \wedge \tilde w_2$ and $v^- = (u^- \vee \tilde w_1) \wedge \tilde w_2$ $\mathcal{H}^{d-1}$-a.e. in J_v. Therefore $[v]$ has the same sign as $[u]$ and $|[v]| \le |[u]|$ $\mathcal{H}^{d-1}$-a.e. in J_v. By the monotonicity property (g6), we obtain $g(x, [v], \nu_v) \le g(x, [u], \nu_u)$ $\mathcal{H}^{d-1}$-a.e. in J_v hence

$$\int_{A \cap J_v} g(x, [v], \nu_v) d\mathcal{H}^{d-1} \le \int_{A \cap J_u} g(x, [u], \nu_u) d\mathcal{H}^{d-1}\,. \tag{3.24}$$

By (3.20), (3.23), and (3.24) we obtain

$$E^{f,g}(v, A) \le E^{f,g}(u, A) + c_3 \int_{A^u_{12}} |\nabla w_1| \vee |\nabla w_2| dx + c_4 \mathcal{L}^d(A^u_{12})\,.$$

This concludes the proof. □

Since Γ-limits are lower semicontinuous, it is natural to introduce the class of functionals in $\mathfrak{E}$ that satisfy this property, which also plays a crucial role in the integral representation.

Definition 3.12. Let $\mathfrak{E}_{sc}$ denote the class of functionals in $\mathfrak{E}$ which satisfy the following property: for every $A \in \mathcal{A}(\mathbb{R}^d)$ the functional $E(\cdot, A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$.

Remark 3.13. If $E \in \mathfrak{E}$ and $E(\cdot, A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$ for every $A \in \mathcal{A}_c(\mathbb{R}^d)$, then $E \in \mathfrak{E}_{sc}$. Indeed, for every $A \in \mathcal{A}(\mathbb{R}^d)$ we have $E(u, A) = \sup_{A' \in \mathcal{A}_c(A)} E(u, A')$ by property (b) in Definition 3.1.

Remark 3.14. Let $E \in \mathfrak{E}_{sc}$ and $A \in \mathcal{A}(\mathbb{R}^d)$. Thanks to Remark 3.3 we can define $E(u, A)$ for every $u \in BV(A)$ and the functional $E(\cdot, A)\colon BV(A) \to [0, +\infty)$ is $L^1(A)$-lower semicontinuous.

Remark 3.15. Taking $f(x, \xi) := |\xi|$ and $g(x, \zeta, \nu) := |\zeta| \wedge 1$ and recalling (3.1) and (3.2), we obtain from Proposition 3.11 that the functional $V : L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ belongs to the class $\mathfrak{E}$. By Remark 2.9 it follows that $V \in \mathfrak{E}_{sc}$.

We now state the main result of this section.

Theorem 3.16. *Let E_k be a sequence in $\mathfrak{E}$. Then there exist a subsequence, not relabelled, and a functional $E \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$.*

Proof. For every $A \in \mathcal{A}(\mathbb{R}^d)$ let

$$E'(\cdot, A) := \Gamma\text{-}\liminf_{k\to\infty} E_k(\cdot, A) \quad \text{and} \quad E''(\cdot, A) := \Gamma\text{-}\limsup_{k\to\infty} E_k(\cdot, A)\,, \tag{3.25}$$

$$E'_-(\cdot, A) := \sup_{A' \in \mathcal{A}_c(A)} E'(\cdot, A') \quad \text{and} \quad E''_-(\cdot, A) := \sup_{A' \in \mathcal{A}_c(A)} E''(\cdot, A')\,, \tag{3.26}$$

where Γ-liminf and Γ-limsup are considered with respect to the topology of $L^0(\mathbb{R}^d)$. It is clear from the definition that for every $u \in L^0(\mathbb{R}^d)$ the set functions $E'(u,\cdot)$, $E''(u,\cdot)$, $E'_-(u,\cdot)$, and $E''_-(u,\cdot)$ are increasing with respect to set inclusion. By [7, Theorem 8.5] there exists a subsequence, not relabelled, such that $E'_-(u,A) = E''_-(u,A)$ for every $u \in L^0(\mathbb{R}^d)$ and every $A \in \mathcal{A}(\mathbb{R}^d)$. We define $E\colon L^0(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ by

$$E(u,A) = E'_-(u,A) = E''_-(u,A) \quad \text{if } A \in \mathcal{A}(\mathbb{R}^d)\,, \tag{3.27}$$

$$E(u,B) := \inf\{E(u,A) : A \in \mathcal{A}(\mathbb{R}^d), B \subset A\} \quad \text{if } B \in \mathcal{B}(\mathbb{R}^d)\,. \tag{3.28}$$

We want to prove that $E \in \mathfrak{E}_{sc}$ and that $E(u,A) = E'(u,A) = E''(u,A)$ for every $u \in L^0(\mathbb{R}^d)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$.

By [7, Proposition 16.15] the functional E is local (property (a)). The most difficult point in the proof is to obtain that E satisfies the measure property (b). To this end we will use a characterization of measures introduced by De Giorgi and Letta [9] (see also [7, Theorem 14.23]), which requires subadditivity, superadditivity, and inner regularity. In our case superadditivity follows from [7, Proposition 16.12] and property (b) for E_k, while inner regularity is obvious from the definition. The subadditivity will be obtained through a sequence of technical lemmas. □

We begin with a weak form of subadditivity for E'' for the truncated function $u^{(m)}$.

Lemma 3.17. *Let E_k be a sequence in $\mathfrak{E}$, let E'' be defined by* (3.25)*, let $u \in L^0(\mathbb{R}^d)$, and let $m > 0$. Let $A', A, B \in \mathcal{A}_c(\mathbb{R}^d)$ with $A' \subset\subset A$. Then*

$$E''(u^{(m)}, A' \cup B) \le E''(u^{(m)}, A) + E''(u^{(m)}, B)\,. \tag{3.29}$$

To prove the lemma we need the following results.

Lemma 3.18. *Let E_k be a sequence in $\mathfrak{E}$, let $u \in L^\infty(\mathbb{R}^d)$, let $m > \|u\|_{L^\infty(\mathbb{R}^d)}$, let u_k be a sequence in $L^0(\mathbb{R}^d)$ converging to u in $L^0(\mathbb{R}^d)$, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then there exists a sequence $\varepsilon_k \to 0$ such that*

$$E_k(u_k^{(m)}, A) \le E_k(u_k, A) + \varepsilon_k\,.$$

Proof. By (g) and Remark 3.4 we have

$$E_k(u_k^{(m)}, A) \le E_k(u_k, A) + \varepsilon_k \tag{3.30}$$

with $\varepsilon_k := c_4 \mathcal{L}^d(A \cap \{|u_k| > m\})$. Since $u_k \to u$ in $L^0(\mathbb{R}^d)$ and $\|u\|_{L^\infty(\mathbb{R}^d)} < m$, we conclude that $\varepsilon_k \to 0$. □

We also need the following version of the fundamental estimate commonly used to obtain subadditivity.

Lemma 3.19. *Let E_k be a sequence in $\mathfrak{E}$, let $A', A'', A, B \in \mathcal{A}_c(\mathbb{R}^d)$ with $A' \subset\subset A'' \subset\subset A$, let $u \in L^1_{loc}(\mathbb{R}^d)$, and let $v_k, w_k \in L^1_{loc}(\mathbb{R}^d)$, converging to u in $L^1_{loc}(\mathbb{R}^d)$ as $k \to \infty$ and such that $v_k|_A \in BV(A)$ and $w_k|_B \in BV(B)$ for every k. Then for every $\eta > 0$ there exists a sequence $\varphi_k \in C_c^\infty(\mathbb{R}^d)$, with $0 \le \varphi_k \le 1$ in $\mathbb{R}^d$, $\operatorname{supp}\varphi_k \subset A''$, and $\varphi_k = 1$ in a neighbourhood of $\overline{A'}$, such that, setting*

$$u_k := \varphi_k v_k + (1 - \varphi_k) w_k\,,$$

we have that $u_k \to u$ in $L^1_{loc}(\mathbb{R}^d)$, $u_k|_{A' \cup B} \in BV(A' \cup B)$,

$$u_k = v_k \ \mathcal{L}^d\text{-a.e. in } A' \quad \text{and} \quad u_k = w_k \ \mathcal{L}^d\text{-a.e. in } B \setminus A''\,, \tag{3.31}$$

$$\limsup_{k\to\infty} E_k(u_k, A' \cup B) \le (1+\eta) \limsup_{k\to\infty} \Big(E_k(v_k, A) + E_k(w_k, B)\Big) + \eta\,. \tag{3.32}$$

Proof. Given $n \in \mathbb{N}$ we fix a finite family of open sets $A' \subset\subset A^0 \subset\subset \cdots \subset\subset A^n \subset\subset A''$ and for $i = 1, \dots, n$ we fix a cut-off function φ^i between A^{i-1} and A^i, i.e., $\varphi^i \in C_c^\infty(\mathbb{R}^d)$ with $0 \le \varphi \le 1$, $\operatorname{supp} \varphi^i \subset A^i$, and $\varphi^i = 1$ in a neighbourhood of $\overline{A^{i-1}}$. For $i = 1, \dots, n$ and $k \in \mathbb{N}$ we set

$$u_k^i := \varphi^i v_k + (1 - \varphi^i) w_k \,. \tag{3.33}$$

Note that $\varphi^i v_k \in BV(A)$ and $(1 - \varphi^i) w_k \in BV(B)$. Actually, $\varphi^i v_k \in BV(\mathbb{R}^d)$, since $\operatorname{supp} \varphi^i \subset A''$, and $(1 - \varphi^i) w_k \in BV(A' \cup B)$, since $\varphi^i = 1$ in a neighbourhood of $\overline{A}'$. We conclude that $u_k^i \in BV(A' \cup B)$.

Using properties (a) and (b) for E_k we obtain

$$E_k(u_k^i, A' \cup B) \le E_k(v_k, A) + E_k(w_k, B \setminus \overline{A}') + E_k(u_k^i, S_i) \,, \tag{3.34}$$

where $S_i := (A^i \setminus \overline{A^{i-1}}) \cap B$. By (c2) we have

$$\begin{aligned} E_k(u_k^i, S_i) &\le c_3 V(u_k^i, S_i) + c_4 \mathcal{L}^d(S_i) \\ &= c_3 \int_{S_i} |\nabla u_k^i| dx + c_3 |D^c u_k^i|(S_i) + c_3 \int_{S_i \cap J_{u_k^i}} |[u_k^i]| \wedge 1 d\mathcal{H}^{d-1} + c_4 \mathcal{L}^d(S_i) \\ &\le c_3 \int_{S_i} (|\nabla v_k| + |\nabla w_k| + |\nabla \varphi^i|\, |w_k - v_k|) dx + c_3 |D^c v_k|(S_i) + c_3 |D^c w_k|(S_i) \\ &\quad + c_3 \int_{S_i \cap J_{v_k}} |[v_k]| \wedge 1 d\mathcal{H}^{d-1} + c_3 \int_{S_i \cap J_{w_k}} |[w_k]| \wedge 1 d\mathcal{H}^{d-1} + c_4 \mathcal{L}^d(S_i) \\ &= c_3 V(v_k, S_i) + c_3 V(w_k, S_i) + c_3 \int_{S_i} |\nabla \varphi^i|\, |w_k - v_k| dx + c_4 \mathcal{L}^d(S_i) \,, \end{aligned} \tag{3.35}$$

where in the last inequality we used the fact that $|\varphi^i [v_k] + (1 - \varphi^i)[w_k]| \le |[v_k]| + |[w_k]|$.

From (c1) and the previous inequality it follows that

$$E_k(u_k^i, S_i) \le \frac{c_3}{c_1} E_k(v_k, S_i) + \frac{c_3}{c_1} E_k(w_k, S_i) + M_n \int_{S_i} |w_k - v_k| dx + C \mathcal{L}^d(S_i) \,, \tag{3.36}$$

where

$$M_n := c_3 \max_{1 \le i \le n} \|\nabla \varphi^i\|_{L^\infty(\mathbb{R}^d)} \quad \text{and} \quad C := c_4 + 2 c_2 c_3 / c_1 \,.$$

Let $S := (A'' \setminus A') \cap B$. Since the sets S_i are pairwise disjoint, from (3.36) we get

$$\sum_{i=1}^n \big(\frac{c_3}{c_1} E_k(v_k, S_i) + \frac{c_3}{c_1} E_k(w_k, S_i) + C \mathcal{L}^d(S_i) \big) \le \frac{c_3}{c_1} E_k(v_k, S) + \frac{c_3}{c_1} E_k(w_k, S) + C \mathcal{L}^d(S) \,,$$

hence there exists $i_k \in \{1, \dots, n\}$ such that

$$\frac{c_3}{c_1} E_k(v_k, S_{i_k}) + \frac{c_3}{c_1} E_k(w_k, S_{i_k}) + C \mathcal{L}^d(S_{i_k}) \le \frac{c_3}{n c_1} E_k(v_k, S) + \frac{c_3}{n c_1} E_k(w_k, S) + \frac{C}{n} \mathcal{L}^d(S) \,.$$

Given $\eta > 0$ we choose n such that $\frac{c_3}{n c_1} < \eta$ and $\frac{C}{n} \mathcal{L}^d(S) < \eta$. For every k let $\varphi_k = \varphi^{i_k}$ and $u_k := u_k^{i_k}$. Then (3.31) holds. By (3.36) and the previous inequalities we have

$$E_k(u_k, S_{i_k}) \le \eta E_k(v_k, S) + \eta E_k(w_k, S) + M_n \int_S |w_k - v_k| dx + \eta \,, \tag{3.37}$$

which, together with (3.34), gives

$$E_k(u_k, A' \cup B) \le (1 + \eta) \big(E_k(v_k, A) + E_k(w_k, B) \big) + M_n \int_S |w_k - v_k| dx + \eta \,.$$

Since $v_k, w_k \to u$ in $L^1_{loc}(\mathbb{R}^d)$, the integral term in the previous inequality tends to 0 and taking the $\limsup$ as $k \to \infty$ we obtain (3.32). $\square$

Proof of Lemma 3.17. It is not restrictive to assume that $E''(u^{(m)},A)$ and $E''(u^{(m)},B)$ are finite. By the definition of Γ- $\limsup$ there exist two sequences $v_k, w_k \in L^0(\mathbb{R}^d)$ converging to $u^{(m)}$ in $L^0(\mathbb{R}^d)$ such that

$$E''(u^{(m)},A) = \limsup_{k\to\infty} E_k(v_k,A) \quad \text{and} \quad E''(u^{(m)},B) = \limsup_{k\to\infty} E_k(w_k,B)\,.$$

By Lemma 3.18

$$\limsup_{k\to\infty} E_k(v_k^{(2m)},A) \le E''(u^{(m)},A) \quad \text{and} \quad \limsup_{k\to\infty} E_k(w_k^{(2m)},B) \le E''(u^{(m)},B)\,. \tag{3.38}$$

By Remark 3.5 we have $v_k^{(2m)}|_A \in BV(A)$ and $w_k^{(2m)}|_B \in BV(B)$. Since $v_k^{(2m)}$ and $w_k^{(2m)}$ converge to $u^{(m)}$ in $L^1_{loc}(\mathbb{R}^d)$, by Lemma 3.19 for every $\eta > 0$ there exists $u_k \in L^1_{loc}(\mathbb{R}^d)$ with $u_k|_{A'\cup B} \in BV(A'\cup B)$, such that $u_k \to u^{(m)}$ in $L^1_{loc}(\mathbb{R}^d)$ and

$$\limsup_{k\to\infty} E_k(u_k, A'\cup B) \le (1+\eta)\limsup_{k\to\infty}(E_k(v_k^{(2m)},A) + E_k(w_k^{(2m)},B)) + \eta\,. \tag{3.39}$$

By the definition of Γ- $\limsup$, (3.38), and (3.39) we obtain

$$E''(u^{(m)},A'\cup B) \le (1+\eta)\big(E''(u^{(m)},A) + E''(u^{(m)},B)\big) + \eta\,,$$

Taking the limit as $\eta \to 0+$ we obtain (3.29). □

To obtain the same result without truncations we use the following lemma.

Lemma 3.20. *Let E_k be a sequence in $\mathfrak{E}$, let E'' be defined by* (3.25)*, let $u \in L^0(\mathbb{R}^d)$, let $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$ with $w_1 \le w_2$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then*

$$E''((u\wedge w_1)\vee w_2, A) \le E''(u,A) + c_3\int_{A_{12}^u} |\nabla w_1|\vee|\nabla w_2|dx + c_4\mathcal{L}^d(A_{12}^u)\,, \tag{3.40}$$

where $A_{12}^u := \{x\in A : u(x) \notin [w_1(x), w_2(x)]\}$.

Proof. It is not restrictive to assume that $E''(u,A) < +\infty$, otherwise inequality (3.40) is trivial. By the definition of Γ-limsup there exists a sequence u_k in $L^0(\mathbb{R}^d)$ with $u_k \to u$ in $L^0(\mathbb{R}^d)$ such that

$$\limsup_{k\to\infty} E_k(u_k,A) \le E''(u,A)\,. \tag{3.41}$$

Passing to a subsequence, not relabelled, we may assume that $u_k \to u$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$. Let $v_k \in L^0(\mathbb{R}^d)$ be defined by $v_k = (u_k\wedge w_1)\vee w_2$. By property (g) for E_k we have

$$E_k(v_k,A) \le E_k(u_k,A) + c_3\int_{A_{12}^{u_k}} |\nabla w_1|\vee|\nabla w_2|dx + c_4\mathcal{L}^d(A_{12}^{u_k})\,, \tag{3.42}$$

where $A_{12}^{u_k} := \{x\in A : u_k(x)\notin[w_1(x),w_2(x)]\}$. Since $u_k \to u$ $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$ we have that

$$\limsup_{k\to\infty} \chi_{A_{12}^{u_k}} \le \chi_{A_{12}^u} \quad \mathcal{L}^d\text{-a.e. in } \mathbb{R}^d\,.$$

By the Fatou Lemma this implies

$$\limsup_{k\to\infty}\Big(c_3\int_{A_{12}^{u_k}}|\nabla w_1|\vee|\nabla w_2|dx + c_4\mathcal{L}^d(A_{12}^{u_k})\Big) \le c_3\int_{A_{12}^u}|\nabla w_1|\vee|\nabla w_2|dx + c_4\mathcal{L}^d(A_{12}^u)\,. \tag{3.43}$$

On the other hand, since $v_k \to (u\wedge w_1)\vee w_2$ in $L^0(\mathbb{R}^d)$ we obtain

$$E''((u\wedge w_1)\vee w_2, A) \le \limsup_{k\to\infty} E_k(v_k,A)\,. \tag{3.44}$$

By (3.41)-(3.44) we obtain (3.40). □

Remark 3.21. If $m > 0$ is a constant, by choosing $w_1 := -m$ and $w_2 := m$, it follows from Lemma 3.20 that for every $u \in L^0(\mathbb{R}^d)$, and $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have

$$E''(u^{(m)},A) \le E''(u,A) + c_4\mathcal{L}^d(A\cap\{|u|>m\})\,. \tag{3.45}$$

The following lemma allows us to pass to the limit when the truncation parameter m tends to $+\infty$.

Lemma 3.22. *Let E_k be a sequence in $\mathfrak{E}$, let E'' be defined by* (3.25)*, let $u \in L^0(\mathbb{R}^d)$, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then for every $m > 0$ we have*

$$\lim_{m\to+\infty} E''(u^{(m)}, A) = E''(u, A)\,. \tag{3.46}$$

Proof. By Remark 3.21 we have

$$E''(u^{(m)}, A) \le E''(u, A) + c_4\mathcal{L}^d(A \cap \{|u| > m\})$$

for every $m > 0$. Hence

$$\limsup_{m\to+\infty} E''(u^{(m)}, A) \le E''(u, A)\,.$$

The inequality

$$\liminf_{m\to+\infty} E''(u^{(m)}, A) \ge E''(u, A)$$

follows from the lower semicontinuity of $E''(\cdot, A)$, since $u^{(m)} \to u$ in $L^0(\mathbb{R}^d)$ as $m \to +\infty$. ☐

We are now able to prove a weak form of subadditivity for the functional E'' on an arbitrary function $u \in L^0(\mathbb{R}^d)$.

Lemma 3.23. *Let E_k be a sequence in $\mathfrak{E}$, let E'' be defined by* (3.25)*, let $u \in L^0(\mathbb{R}^d)$, and let $A', A, B \in \mathcal{A}_c(\mathbb{R}^d)$ with $A' \subset\subset A$. Then*

$$E''(u, A' \cup B) \le E''(u, A) + E''(u, B)\,. \tag{3.47}$$

Proof. The conclusion follows from Lemmas 3.17 and 3.22. ☐

We are now in a position to obtain the subadditivity of E.

Lemma 3.24. *Let E_k be a sequence in $\mathfrak{E}$, let E be defined by* (3.27) *and* (3.28)*, let $u \in L^0(\mathbb{R}^d)$, and let $A, B \in \mathcal{A}(\mathbb{R}^d)$. Then*

$$E(u, A \cup B) \le E(u, A) + E(u, B)\,. \tag{3.48}$$

Proof. Thanks to the previous lemma the result can be obtained arguing as in the proof of [7, Lemma 18.4]. ☐

The following lemma proves property (c1) for E.

Lemma 3.25. *Let E_k be a sequence in $\mathfrak{E}$, let E be defined by* (3.27) *and* (3.28)*, let $u \in L^0(\mathbb{R}^d)$, and let $B \in \mathcal{B}(\mathbb{R}^d)$. Then*

$$c_1 V(u, B) - c_2\mathcal{L}^d(B) \le E(u, B)\,. \tag{3.49}$$

Proof. By (2.9), (2.10), (3.27), and (3.28) it is enough to prove (3.49) for relatively compact open sets. Let us fix $A \in \mathcal{A}_c(\mathbb{R}^d)$ with $E(u, A) < +\infty$. By (3.27) for every $A' \in \mathcal{A}_c(A)$ we have $E(u, A') \le E(u, A) < +\infty$. By (c1) for E_k and the lower semicontinuity of V (see Lemma 2.9) we have that $c_1 V(u, A') - c_2\mathcal{L}^d(A') \le E'(u, A')$. Taking the limit as $A' \nearrow A$, by (2.10) and (3.27) we obtain $c_1 V(u, A) - c_2\mathcal{L}^d(A) \le E(u, A)$, which concludes the proof. ☐

The following lemma shows that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_k(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$.

Lemma 3.26. *Let E_k be a sequence in $\mathfrak{E}$, let E', E'', and E be defined by* (3.25), (3.27), *and* (3.28)*, let $u \in L^0(\mathbb{R}^d)$, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then $E(u, A) = E'(u, A) = E''(u, A)$.*

Proof. Since $E(u,A) \le E'(u,A) \le E''(u,A)$, it remains to prove that $E''(u,A) \le E(u,A)$. This inequality is trivial if $E(u,A) = +\infty$, therefore we assume $E(u,A) < +\infty$, which implies $u|_A \in GBV_\star(A)$ by Lemma 3.25 and the definition of V. Given $\varepsilon > 0$ we fix a compact set $K \subset A$ such that

$$c_3 V(u, A \setminus K) + c_4 \mathcal{L}^d(A \setminus K) < \varepsilon . \tag{3.50}$$

Since the Γ-$\limsup$ is smaller than the pointwise $\limsup$ (see [7, Proposition 5.1]), from (c2) for E_k we obtain

$$E''(u, A \setminus K) \le c_3 V(u, A \setminus K) + c_4 \mathcal{L}^d(A \setminus K) < \varepsilon . \tag{3.51}$$

We now fix $A', A'' \in \mathcal{A}_c(\mathbb{R}^d)$ such that $K \subset A' \subset\subset A'' \subset\subset A$ and apply Lemma 3.23 with $B = A \setminus K$, so that $A' \cup B = A$. We obtain

$$E''(u,A) \le E''(u,A'') + E''(u, A \setminus K) \le E(u,A) + \varepsilon ,$$

where in the second inequality we used (3.27) and (3.51). As $\varepsilon \to 0+$ we obtain the desired inequality. □

Proof of Theorem 3.16 (continuation). By Lemma 3.26 we have that

$$E(\cdot, A) = E'(\cdot, A) = E''(\cdot, A) = \Gamma\text{-}\lim_{k\to\infty} E_k(\cdot, A) \tag{3.52}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$.

We already proved that E is local (property (a)). To prove (b) we fix $u \in L^0(\mathbb{R}^d)$ and apply the De Giorgi-Letta criterion, see [7, Theorem 14.23]. By (3.28), to prove that $E(u, \cdot)$ is a measure on $\mathcal{B}(\mathbb{R}^d)$ it is enough to show that $E(u, \cdot)$ is subadditive, superadditive and inner regular on $\mathcal{A}(\mathbb{R}^d)$. Subadditivity is proved in Lemma 3.24, while superadditivity follows from [7, Proposition 16.12] and property (b) for E_k. Since we already observed that $E(u, \cdot)$ is inner regular, the proof of (b) is complete.

Property (c1) is proved in Lemma 3.25. Since the Γ-$\limsup$ is smaller than the pointwise $\limsup$ (see [7, Proposition 5.1]), from (c2) for E_k we obtain

$$E(u,A) = E''(u,A) \le c_3 V(u,A) + c_4 \mathcal{L}^d(A) , \tag{3.53}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$. By inner regularity the inequality holds for every $A \in \mathcal{A}(\mathbb{R}^d)$, and by (2.9) and (3.28) the inequality continues to hold for every $B \in \mathcal{B}(\mathbb{R}^d)$, thus concluding the proof of (c2).

The invariance property (d) and the estimates (e) and (f) for E follow from the same properties for E_k, using the definition of Γ-limit. Finally, property (g) on $\mathcal{A}_c(\mathbb{R}^d)$ follows from Lemma 3.20 and (3.52) and can be extended to $\mathcal{B}(\mathbb{R}^d)$ as in the proof of (c2). This concludes the proof of the fact that $E \in \mathfrak{E}$.

From general properties of Γ-limits it follows that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the functional $E(\cdot, A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$ (see, e.g., [7, Proposition 6.8]). By Remark 3.13 this proves that $E \in \mathfrak{E}_{sc}$. □

The case of integral functionals is considered in the following corollary.

Corollary 3.27. *For every $k \in \mathbb{N}$ let $f_k \in \mathcal{F}$, $g_k \in \mathcal{G}$, and E^{f_k, g_k} as in* (3.14)-(3.16)*. Then there exists a subsequence, not relabelled, and a functional $E \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E^{f_k,g_k}(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$.*

Proof. It is enough to apply Proposition 3.11 and Theorem 3.16. □

4. Towards the integral representation

In this section we investigate some properties of the functionals in $\mathfrak{E}$ that are instrumental in the proof of the integral representation results that will be obtained in Section 6.

Definition 4.1. Let $E\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ be a functional satisfying properties (b) and (c2) in Definition 3.1. Let $u\in L^0(\mathbb{R}^d)$ and $A\in\mathcal{A}_c(\mathbb{R}^d)$ with $u|_A\in GBV_\star(A)$. We define the measures $E^a(u,\cdot)$, $E^s(u,\cdot)$, $E^c(u,\cdot)$, and $E^j(u,\cdot)$ on $\mathcal{B}(A)$ in the following way:

$$E^a(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d\,, \tag{4.1}$$

$$E^s(u,\cdot) \text{ is the singular part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d\,, \tag{4.2}$$

$$E^c(u,B):=E^s(u,B\setminus J_u) \text{ for every } B\in\mathcal{B}(A)\,, \tag{4.3}$$

$$E^j(u,B):=E^s(u,B\cap J_u)=E(u,B\cap J_u) \text{ for every } B\in\mathcal{B}(A)\,. \tag{4.4}$$

Remark 4.2. The following properties hold:

$$E(u,\cdot)=E^a(u,\cdot)+E^c(u,\cdot)+E^j(u,\cdot) \quad \text{in } \mathcal{B}(A)\,, \tag{4.5}$$

$$E^c(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } |D^cu|\,, \tag{4.6}$$

$$E^j(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } \mathcal{H}^{d-1}\llcorner J_u. \tag{4.7}$$

Property (4.5) follows immediately from the definition.

Since the measures $E^s(u,\cdot)$ and $|D^cu|$ are singular with respect to $\mathcal{L}^d$, there exists $N\in\mathcal{B}(A)$ with $\mathcal{L}^d(N)=0$ such that $E^s(u,A\setminus N)=|D^cu|(A\setminus N)=0$. Therefore, if $B\in\mathcal{B}(A)$ and $|D^cu|(B)=0$ we have that

$$E^c(u,B)=E^s(u,B\setminus J_u)=E(u,B\cap N\setminus J_u)\le c_3V(u,B\cap N\setminus J_u)=0\,,$$

hence $E^c(u,\cdot)$ is absolutely continuous with respect to $|D^cu|$.

On the other hand, by Theorem 2.2(c) we have $|D^cu|(J_u)=0$, hence $|D^cu|$ is concentrated on $N\setminus J_u$. Since $E^a(u,N\setminus J_u)=0$ and $E^j(u,N\setminus J_u)=0$, the measure $E^a(u,\cdot)+E^j(u,\cdot)$ is singular with respect to $|D^cu|$. We conclude that $E^c(u,\cdot)$ is the absolutely continuous part of $E(u,\cdot)$ with respect to $|D^cu|$.

Finally, by (c2) in Definition 3.1 for every $B\in\mathcal{B}(A)$ we have

$$E^j(u,B)=E(u,B\cap J_u)\le c_3V(u,B\cap J_u)=c_3\int_{B\cap J_u}|[u]|\wedge 1d\mathcal{H}^{d-1}\,,$$

hence $E^j(u,\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{d-1}\llcorner J_u$. Observing that the measure $E^a(u,\cdot)+E^c(u,\cdot)$ is singular with respect to $\mathcal{H}^{d-1}\llcorner J_u$, we conclude that $E^j(u,\cdot)$ is the absolutely continuous part of $E(u,\cdot)$ with respect to $\mathcal{H}^{d-1}\llcorner J_u$.

Our proofs of the integral representation theorems considered in the next sections rely on the results of [4] about functions $u\in BV(A)$. Since we want to extend them to $GBV_\star(A)$, it is important to approximate the values of a functional on a function $u\in GBV_\star(A)$ with the corresponding values on its truncations $u^{(m)}$. This is done in the following proposition.

Proposition 4.3. *Assume that $E\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ satisfies (3.10) and properties (b) and (c2) in Definition 3.1. Suppose also that $E(\cdot,A)$ is lower semicontinuous in $L^0(\mathbb{R}^d)$ for every $A\in\mathcal{A}_c(\mathbb{R}^d)$. Let $u\in L^0(\mathbb{R}^d)$ and $A\in\mathcal{A}_c(\mathbb{R}^d)$ with $u|_A\in GBV_\star(A)$. Then*

$$\lim_{m\to+\infty}E(u^{(m)},B)=E(u,B)\,, \tag{4.8}$$

$$\lim_{m\to+\infty}E^a(u^{(m)},B)=E^a(u,B)\,, \tag{4.9}$$

$$\lim_{m\to+\infty}E^s(u^{(m)},B)=E^s(u,B)\,, \tag{4.10}$$

$$\lim_{m\to+\infty}E^c(u^{(m)},B)=E^c(u,B)\,, \tag{4.11}$$

$$\lim_{m\to+\infty}E^j(u^{(m)},B)=E^j(u,B)\,, \tag{4.12}$$

for every $B \in \mathcal{B}(A)$.

Proof. By (3.10) we have that

$$\limsup_{m\to+\infty} E(u^{(m)}, B) \leq E(u, B)$$

for every $B \in \mathcal{B}(A)$. By lower semicontinuity we have also

$$\liminf_{m\to+\infty} E(u^{(m)}, U) \geq E(u, U)$$

for every $U \in \mathcal{A}(A)$. Hence

$$\lim_{m\to+\infty} E(u^{(m)}, U) = E(u, U)$$

for every $U \in \mathcal{A}(A)$.

Equality (4.8) follows from Lemma 4.4 below, while (4.9) and (4.10) follow from Lemma 4.5 below. As for (4.12) we observe that

$$E^j(u^{(m)}, B) = E^s(u^{(m)}, B \cap J_{u^{(m)}}) \quad \text{and} \quad E^j(u, B) = E^s(u, B \cap J_u)\,.$$

By Theorem 2.2(d) there exists a sequence $N_m \in \mathcal{B}(A)$ such that $\mathcal{H}^{d-1}(N_m \Delta J_{u^{(m)}}) = 0$ and $N_m \nearrow J_u$. By (c2) we have $E^j(u^{(m)}, B) = E^s(u^{(m)}, B \cap N_m)$. Let us fix $k \in \mathbb{N}$. For every $m \geq k$ we have

$$E^s(u^{(m)}, B \cap N_k) \leq E^s(u^{(m)}, B \cap N_m) \leq E^s(u^{(m)}, B \cap J_u)\,.$$

By (4.10) we have

$$E^s(u, B \cap N_k) \leq \liminf_{m\to+\infty} E^s(u^{(m)}, B \cap N_m) \leq \limsup_{m\to+\infty} E^s(u^{(m)}, B \cap N_m) \leq E^s(u, B \cap J_u)\,.$$

Passing to the limit as $k \to \infty$ we obtain (4.12). Equality (4.11) follows by difference. ☐

To conclude the proof of Proposition 4.3 it remains to prove the following lemmas.

Lemma 4.4. *Let* $A \in \mathcal{A}(\mathbb{R}^d)$ *and for every* $k \in \mathbb{N}$ *let* $\mu_k, \mu \colon \mathcal{B}(A) \to [0, +\infty)$ *be finite-valued Borel measures. Assume that*

$$\lim_{k\to\infty} \mu_k(U) = \mu(U) \quad \textit{for every } U \in \mathcal{A}(A)\,. \tag{4.13}$$

Then

$$\lim_{k\to\infty} \mu_k(B) = \mu(B) \quad \textit{for every } B \in \mathcal{B}(A)\,. \tag{4.14}$$

Proof. Let us fix $B \in \mathcal{B}(A)$. For every $\varepsilon > 0$ there exists $U \in \mathcal{A}(A)$ with $U \supset B$ such that $\mu(U) \leq \mu(B) + \varepsilon$. By (4.13) we have $\limsup_k \mu_k(B) \leq \lim_k \mu_k(U) \leq \mu(B) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary we conclude that $\limsup_k \mu_k(B) \leq \mu(B)$. The same property for $A \setminus B$ gives $\limsup_k \mu_k(A \setminus B) \leq \mu(A \setminus B)$. Since $\lim_k(\mu_k(B) + \mu_k(A \setminus B)) = \mu(B) + \mu(A \setminus B)$, we conclude that (4.14) holds. ☐

Lemma 4.5. *Let* $A \in \mathcal{A}(\mathbb{R}^d)$, *for every* $k \in \mathbb{N}$ *let* $\mu_k, \mu \colon \mathcal{B}(A) \to [0, +\infty)$ *be finite-valued Borel measures, let* μ_k^a, μ^a *be their absolutely continuous parts with respect to* $\mathcal{L}^d$, *and let* μ_k^s, μ^s *be their singular parts with respect to* $\mathcal{L}^d$. *Assume that*

$$\lim_{k\to\infty} \mu_k(B) = \mu(B) \quad \textit{for every } B \in \mathcal{B}(A)\,.$$

Then

$$\lim_{k\to\infty} \mu_k^a(B) = \mu^a(B) \quad \textit{and} \quad \lim_{k\to\infty} \mu_k^s(B) = \mu^s(B)$$

for every $B \in \mathcal{B}(A)$.

Proof. Let $N \in \mathcal{B}(A)$ with $\mathcal{L}^d(N) = 0$ be such that μ^s and μ^s_k are concentrated on N for every k. Then for every $B \in \mathcal{B}(A)$ we have

$$\lim_{k\to\infty} \mu^a_k(B) = \lim_{k\to\infty} \mu_k(B \setminus N) = \mu(B \setminus N) = \mu^a(B)\,.$$

The property for μ^s_k and μ^s is obtained by difference. □

One of the difficulties in the proof of the integral representation is that the functional $E(\cdot, B)$, in general, does not decrease under truncations. However, the following proposition shows, in particular, that the singular part $E^s(\cdot, B)$ is always decreasing under truncations.

Proposition 4.6. *Assume that* $E\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ *satisfies properties (b), (c2), and (g) in Definition 3.1. Let* $u \in L^0(\mathbb{R}^d)$*,* $A \in \mathcal{A}_c(\mathbb{R}^d)$*, with* $u|_A \in GBV_\star(A)$*, and let* $w_1, w_2 \in W^{1,1}_{loc}(\mathbb{R}^d)$*, with* $w_1 \le w_2$ $\mathcal{L}^d$*-a.e. in* $\mathbb{R}^d$*. Then*

$$E^s((u \vee w_1) \wedge w_2, B) \le E^s(u, B) \tag{4.15}$$

for every $B \in \mathcal{B}(A)$*.*

Proof. Let $N \in \mathcal{B}(A)$ be a set with $\mathcal{L}^d(N) = 0$ such that $E^s((u \vee w_1) \wedge w_2, \cdot)$ and $E^s(u, \cdot)$ are concentrated on N. Then

$$\begin{aligned} E^s((u \vee w_1) \wedge w_2, B) &= E((u \vee w_1) \wedge w_2, B \cap N) \\ \le E(u, B \cap N) + c_3 \int_{B\cap N} |\nabla w_1| \vee |\nabla w_2| dx &+ c_4\mathcal{L}^d(B \cap N) = E^s(u, B)\,, \end{aligned}$$

thus concluding the proof. □

The results in [4] cannot be applied directly to the restriction of the functional E to $BV(A)$ since they require a lower bound of the form $E(u, A) \ge c|Du|(A)$ for some constant $c > 0$, which does not hold under our hypotheses. For this reason we consider the functionals introduced in the following definition.

Definition 4.7. Let $E \in \mathfrak{E}$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$. For every $\varepsilon > 0$ we define the functional $E_\varepsilon\colon BV(A) \times \mathcal{B}(A) \to [0, +\infty)$ by setting

$$E_\varepsilon(u, B) := E(u, B) + \varepsilon|Du|(B) \tag{4.16}$$

for every $u \in BV(A)$ and every $B \in \mathcal{B}(A)$, where $E(u, B)$ is defined thanks to Remark 3.3.

Remark 4.8. By (c1) we have

$$\varepsilon|Du|(A) - c_2\mathcal{L}^d(A) \le E_\varepsilon(u, A)\,, \tag{4.17}$$

while by (c2) and Remark 2.6 we have

$$E_\varepsilon(u, A) \le (c_3 + \varepsilon)|Du|(A) + c_4\mathcal{L}^d(A)\,. \tag{4.18}$$

Finally, note that E_ε satisfies the analogue of properties (a), (b), and (d) in A.

Definition 4.9. Let $E \in \mathfrak{E}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $\varepsilon > 0$. Given $u \in BV(A)$ we define $E^a_\varepsilon(u, \cdot), E^s_\varepsilon(u, \cdot), E^c_\varepsilon(u, \cdot), E^j_\varepsilon(u, \cdot)\colon \mathcal{B}(A) \to [0, +\infty)$ as in Definition 4.1 starting from $E_\varepsilon(u, \cdot)$.

The integrands appearing in the integral representation results in [4] are constructed using the minimum problems considered in the following definition.

Definition 4.10. Let $A \in \mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundary and $w \in BV(A)$. Given an arbitrary functional $E(\cdot, A)\colon BV(A) \to [0, +\infty]$, we define (see [4])

$$m^E(w, A) := \inf\{E(u, A) : u \in BV(A)\,,\ \mathrm{tr}_A u = \mathrm{tr}_A w\ \ \mathcal{H}^{d-1}\text{-a.e. on } \partial A\}\,, \tag{4.19}$$

where $\mathrm{tr}_A v$ denotes the trace on ∂A of a function $v \in BV(A)$.

The following lemma compares the values of m^E on different sets.

Lemma 4.11. *Let $A', A \in \mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundaries and $A' \subset\subset A$, let $w \in BV(A)$, and let $E\colon BV(A) \times \mathcal{B}(A) \to [0,+\infty]$ be a functional such that*

(a) *E is local in $\mathcal{A}(A)$, i.e., $E(u,U) = E(v,U)$ for every $U \in \mathcal{A}(A)$ and $u, v \in BV(A)$ with $u = v$ $\mathcal{L}^d$-a.e. in U;*
(b) *for every $u \in BV(A)$ the set function $E(u,\cdot)$ is additive on $\mathcal{B}(A)$;*
(c) *there exist $c_3' > 0$ and $c_4' \geq 0$ such that $E(u,B) \leq c_3'|Du|(B) + c_4'\mathcal{L}^d(B)$ for every $u \in BV(A)$ and $B \in \mathcal{B}(A)$.*

Then

$$m^E(w,A) \leq m^E(w,A') + c_3'|Dw|(A \setminus A') + c_4'\mathcal{L}^d(A \setminus A'). \tag{4.20}$$

Proof. Let us fix $\eta > 0$. By the definition of $m^E(w,A')$ there exists $u \in BV(A')$ such that $\mathrm{tr}_{A'}u = \mathrm{tr}_{A'}w$ $\mathcal{H}^{d-1}$-a.e. on $\partial A'$ and

$$E(u,A') \leq m^E(w,A') + \eta. \tag{4.21}$$

Let $v\colon A \to \mathbb{R}$ be defined by $v = u$ in A' and $v = w$ in $A \setminus A'$. Since $\mathrm{tr}_A v = \mathrm{tr}_A w$ $\mathcal{H}^{d-1}$-a.e. on ∂A, we have $m^E(w,A) \leq E(v,A)$. By (b) $E(v,A) = E(v,A') + E(v,A \setminus A')$. Moreover, by (a) $E(v,A') = E(u,A')$ and by (c) $E(v,A \setminus A') \leq c_3'|Dv|(A \setminus \overline{A}') + c_3'|Dv|(\partial A') + c_4'\mathcal{L}^d(A \setminus A') = c_3'|Dw|(A \setminus \overline{A}') + c_3' \int_{\partial A'} |[v]| d\mathcal{H}^{d-1} + c_4'\mathcal{L}^d(A \setminus A')$. Since $\mathrm{tr}_{A'}v = \mathrm{tr}_{A'}u = \mathrm{tr}_{A'}w$ $\mathcal{H}^{d-1}$-a.e. on $\partial A'$, while $v = w$ in $A \setminus A'$, we obtain that $\int_{\partial A'} |[v]| d\mathcal{H}^{d-1} = \int_{\partial A'} |[w]| d\mathcal{H}^{d-1} = |Dw|(\partial A')$.

Combining these inequalities and (4.21) we obtain

$$m^E(w,A) \leq m^E(w,A') + \eta + c_3'|Dw|(A \setminus A') + c_4'\mathcal{L}^d(A \setminus A'),$$

which gives (4.20) by the arbitrariness of $\eta > 0$. □

We now define the integrands that will be used in Section 6 in the integral representation results for functionals in $\mathfrak{E}_{sc}$.

Definition 4.12. Given $E \in \mathfrak{E}$ we define the integrands $f\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$, and $g\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$ by setting

$$f(x,\xi) := \limsup_{\rho \to 0+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d}, \tag{4.22}$$

$$g(x,\zeta,\nu) := \limsup_{\rho \to 0+} \frac{m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}. \tag{4.23}$$

To obtain the integral representation result for E^a and E^j we shall first prove, for every $\varepsilon > 0$, an integral representation for the functionals E^a_ε and E^j_ε introduced in Definition 4.9 using the integrands given in the following definition.

Definition 4.13. Let $E \in \mathfrak{E}$ and let $\varepsilon > 0$. We define the integrands $f_\varepsilon\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$, and $g_\varepsilon\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{S}^{d-1} \to [0,+\infty)$ by setting

$$f_\varepsilon(x,\xi) := \limsup_{\rho \to 0+} \frac{m^{E_\varepsilon}(\ell_\xi, Q(x,\rho))}{\rho^d}, \tag{4.24}$$

$$g_\varepsilon(x,\zeta,\nu) := \limsup_{\rho \to 0+} \frac{m^{E_\varepsilon}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}. \tag{4.25}$$

Remark 4.14. It follows immediately from the definitions that $\varepsilon \mapsto f_\varepsilon(x,\xi)$ and $\varepsilon \mapsto g_\varepsilon(x,\zeta,\nu)$ are non-decreasing and that

$$f(x,\xi) \leq f_\varepsilon(x,\xi) \quad \text{and} \quad g(x,\zeta,\nu) \leq g_\varepsilon(x,\zeta,\nu) \tag{4.26}$$

for every $\varepsilon > 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$.

The following proposition shows that, under an additional assumption, f and f^∞ are the limit of f_ε and f_ε^∞ as $\varepsilon \to 0+$. In Theorem 6.2 we shall see that this assumption is satisfied by all functionals in the class $\mathfrak{E}_{sc}$.

Proposition 4.15. *Let $E \in \mathfrak{E}$ and let f and f_ε be defined by* (4.22) *and* (4.24)*. Suppose that there exists a function $\check{f} \in \mathcal{F}$ such that*

$$E^a(u, A) = \int_A \check{f}(x, \nabla u)\, dx \tag{4.27}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and every $u \in BV(A)$. Then for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$

$$f(x,\xi) = \lim_{\varepsilon\to 0+} f_\varepsilon(x,\xi) \quad \text{for every } \xi \in \mathbb{R}^d\,, \tag{4.28}$$

$$f^\infty(x,\xi) = \lim_{\varepsilon\to 0+} f_\varepsilon^\infty(x,\xi) \quad \text{for every } \xi \in \mathbb{R}^d\,. \tag{4.29}$$

If, in addition, $\check{f}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$, then (4.28) *and* (4.29) *hold for every $x \in \mathbb{R}^d$.*

To prove the proposition we need the following lemma, which contains a more general result for the rectangles $Q_\nu^\lambda(x,\rho)$ defined by (2.1). For every $\xi \in \mathbb{R}^d$ we set

$$c_\xi := \frac{c_2 + c_4 + 1}{c_1} d^{1/2} + |\xi| d^{1/2}\,. \tag{4.30}$$

Lemma 4.16. *Let $E \in \mathfrak{E}$. Suppose that there exists a function $\check{f} \in \mathcal{F}$ such that* (4.27) *holds for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and every $u \in BV(A)$. Then there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$, $\lambda \geq 1$, $\nu \in \mathbb{S}^{d-1}$, and $\eta > 0$ there exists $\rho_{\nu,\eta}^\lambda(x) > 0$ with the following property: for every $0 < \rho < \rho_{\nu,\eta}^\lambda(x)$ and every $\xi \in \mathbb{R}^d$ there exists $u \in BV(Q_\nu^\lambda(x,\rho)) \cap L^\infty(Q_\nu^\lambda(x,\rho))$ satisfying $\|u - \ell_\xi\|_{L^\infty(Q_\nu^\lambda(x,\rho))} \leq c_\xi \lambda\rho$, $\mathrm{tr}_{Q_\nu^\lambda(x,\rho)} u = \mathrm{tr}_{Q_\nu^\lambda(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu^\lambda(x,\rho)$, and*

$$E(u, Q_\nu^\lambda(x,\rho)) \leq m^E(\ell_\xi, Q_\nu^\lambda(x,\rho)) + \eta\lambda^{d-1}\rho^d\,. \tag{4.31}$$

If, in addition, $\check{f}$ is continuous in $\mathbb{R}^d \times \mathbb{R}^d$, then $N = \emptyset$. Finally, if $\check{f}$ does not depend on x, then $\rho_{\nu,\eta}^\lambda(x) = +\infty$.

Proof. Let us fix $0 < \eta < 1$. We claim that there exists a Borel function $\psi_\eta \colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\check{f}(x, \psi_\eta(x)) \leq \check{f}(x,\xi) + \eta/3 \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d\,. \tag{4.32}$$

To prove this, let $\mu \colon \mathbb{R}^d \to [0, +\infty)$ be the Borel function defined by

$$\mu(x) := \inf_{\xi\in\mathbb{Q}^d} \check{f}(x,\xi) = \inf_{\xi\in\mathbb{R}^d} \check{f}(x,\xi) \quad \text{for every } x \in \mathbb{R}^d\,, \tag{4.33}$$

let (ξ_i) be an enumeration of $\mathbb{Q}^d$, and let (B_i) be the sequence of Borel sets defined inductively by

$$B_1 := \{x \in \mathbb{R}^d : \check{f}(x,\xi_1) \leq \mu(x) + \eta/3\}\,,$$
$$B_i := \{x \in \mathbb{R}^d \setminus (B_1 \cup \cdots \cup B_{i-1}) : \check{f}(x,\xi_i) \leq \mu(x) + \eta/3\} \quad \text{for } i > 1\,.$$

Observing that (B_i) is a partition of $\mathbb{R}^d$, we obtain that the function $\psi_\eta \colon \mathbb{R}^d \to \mathbb{R}^d$ defined by $\psi_\eta(x) = \xi_i$ for $x \in B_i$ is Borel measurable and satisfies (4.32).

By (f2) and (f3) in Definition 3.6 we have

$$|\psi_\eta(x)| \leq \frac{c_2 + c_4 + 1}{c_1} \quad \text{for every } x \in \mathbb{R}^d\,. \tag{4.34}$$

By the Lebesgue Differentiation Theorem for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ we have

$$\lim_{\rho\to 0+} \frac{1}{\lambda^{d-1}\rho^d} \int_{Q_\nu^\lambda(x,\rho)} |\psi_\eta(y) - \psi_\eta(x)| dy = 0$$

for every $\lambda \geq 1$ and every $\nu \in \mathbb{S}^{d-1}$. By (f4)

$$\lim_{\rho \to 0+} \frac{1}{\lambda^{d-1}\rho^d} \int_{Q_\nu^\lambda(x,\rho)} |\check{f}(y, \psi_\eta(y)) - \check{f}(y, \psi_\eta(x))| dy = 0 \,,$$

therefore, for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ there exists $\rho_{\nu,\eta}^\lambda(x) > 0$ such that

$$\int_{Q_\nu^\lambda(x,\rho)} |\check{f}(y, \psi_\eta(y)) - \check{f}(y, \psi_\eta(x))| dy \leq \eta \lambda^{d-1} \rho^d / 3 \tag{4.35}$$

for every $0 < \rho < \rho_{\nu,\eta}^\lambda(x)$. This implies that there exists $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that (4.35) holds for every $x \in \mathbb{R}^d \backslash N$, $\lambda \geq 1$, $\nu \in \mathbb{S}^{d-1}$, $\eta \in \mathbb{Q} \cap (0,1)$, and $0 < \rho < \rho_{\nu,\eta}^\lambda(x)$.

Let us fix $x, \lambda, \nu, \eta, \rho$ as above and let $\xi \in \mathbb{R}^d$. By the definition of m^E there exists $v \in BV(Q_\nu^\lambda(x,\rho))$ such that $\mathrm{tr}_{Q_\nu^\lambda(x,\rho)} v = \mathrm{tr}_{Q_\nu^\lambda(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu^\lambda(x,\rho)$, and

$$E(v, Q_\nu^\lambda(x,\rho)) \leq m^E(\ell_\xi, Q_\nu^\lambda(x,\rho)) + \eta \lambda^{d-1} \rho^d / 3 \,. \tag{4.36}$$

Let $m_1 := \xi \cdot x - c_\xi \lambda \rho / 2$ and $m_2 := \xi \cdot x + c_\xi \lambda \rho / 2$, where c_ξ is defined in (4.30). For every $y \in \mathbb{R}^d$ we set $w_1(y) := \psi_\eta(x) \cdot (y - x) + m_1$ and $w_2(y) := \psi_\eta(x) \cdot (y - x) + m_2$. Note that $w_2 - w_1 = m_2 - m_1 = c_\xi \lambda \rho$.

Moreover, for every $y \in Q_\nu^\lambda(x,\rho)$ we set

$$u(y) := (v(y) \vee w_1(y)) \wedge w_2(y) \,. \tag{4.37}$$

Then $u \in BV(Q_\nu^\lambda(x,\rho)) \cap L^\infty(Q_\nu^\lambda(x,\rho))$. For every $y \in Q_\nu^\lambda(x,\rho)$ we have $w_1(y) \leq u(y) \leq w_2(y)$ and $w_1(y) \leq \ell_\xi(y) \leq w_2(y)$, hence $\|u - \ell_\xi\|_{L^\infty(Q_\nu^\lambda(x,\rho))} \leq c_\xi \lambda \rho$ and $\mathrm{tr}_{Q_\nu^\lambda(x,\rho)} u = \mathrm{tr}_{Q_\nu^\lambda(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu^\lambda(x,\rho)$.

Let $B := \{y \in Q_\nu^\lambda(x,\rho) : w_1 \leq v \leq w_2\}$. Since, by a well-known property of approximate gradients we have $\nabla v(y) = \nabla u(y)$ $\mathcal{L}^d$-a.e. in B and $\nabla u(y) = \nabla w_1(y) = \nabla w_2(y) = \psi_\eta(x)$ $\mathcal{L}^d$-a.e. in $Q_\nu^\lambda(x,\rho) \setminus B$, by (4.27) we have

$$\begin{aligned} E^a(u, Q_\nu^\lambda(x,\rho)) &= \int_{Q_\nu^\lambda(x,\rho)} \check{f}(y, \nabla u(y)) dy = \int_{Q_\nu^\lambda(x,\rho) \cap B} \check{f}(y, \nabla v(y)) dy + \int_{Q_\nu^\lambda(x,\rho) \setminus B} \check{f}(y, \psi_\eta(x)) dy \\ &\leq \int_{Q_\nu^\lambda(x,\rho) \cap B} \check{f}(y, \nabla v(y)) dy + \int_{Q_\nu^\lambda(x,\rho) \setminus B} \check{f}(y, \psi_\eta(y)) dy + \int_{Q_\nu^\lambda(x,\rho) \setminus B} (\check{f}(y, \psi_\eta(x)) - \check{f}(y, \psi_\eta(y))) dy \\ &\leq \int_{Q_\nu^\lambda(x,\rho)} \check{f}(y, \nabla v(y)) dy + 2\eta \lambda^{d-1} \rho^d / 3 = E^a(v, Q_\nu^\lambda(x,\rho)) + 2\eta \lambda^{d-1} \rho^d / 3 \,, \end{aligned} \tag{4.38}$$

where in the last inequality we used (4.32) and (4.35).

By Proposition 4.6 (using also Remark 3.3) we have

$$E^s(u, Q_\nu^\lambda(x,\rho)) \leq E^s(v, Q_\nu^\lambda(x,\rho)) \,, \tag{4.39}$$

which together with (4.36) and (4.38) gives $E(u, Q_\nu^\lambda(x,\rho)) \leq E(v, Q_\nu^\lambda(x,\rho)) + 2\eta \lambda^{d-1} \rho^d / 3 \leq m^E(w, Q_\nu^\lambda(x,\rho)) + \eta \lambda^{d-1} \rho^d$, thus concluding the proof of (4.31).

If, in addition, $\check{f}$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$, we fix an arbitrary $x \in \mathbb{R}^d$. By (f2) and (f4) the function $\xi \mapsto \check{f}(x, \xi)$ has a minimum point $\psi(x) \in \mathbb{R}^d$. By (f3) we have

$$\check{f}(x, \psi(x)) \leq \check{f}(x, 0) \leq c_4 \,, \tag{4.40}$$

hence (f2) gives

$$|\psi(x)| \leq \frac{c_2 + c_4}{c_1} \,. \tag{4.41}$$

Let us fix $\eta > 0$. Since $\check{f}$ is continuous, for every $\lambda \geq 1$ there exists $\rho_\eta^\lambda(x) > 0$ such that

$$|\check{f}(y, \xi) - \check{f}(x, \xi)| \leq \eta / 3 \tag{4.42}$$

for every $y \in Q_\nu^\lambda(x, \rho_\eta^\lambda(x))$ for some $\nu \in \mathbb{S}^{d-1}$, and every $\xi \in \mathbb{R}^d$, with $|\xi| \leq (c_2 + c_4)/c_1$. Given $\nu \in \mathbb{S}^{d-1}$, we claim that

$$\check{f}(y, \psi(x)) \leq \check{f}(y, \xi) + 2\eta/3 \tag{4.43}$$

for every $y \in Q_\nu^\lambda(x, \rho_\eta^\lambda(x))$ and every $\xi \in \mathbb{R}^d$. To prove this, let us fix $y \in Q_\nu^\lambda(x, \rho_\eta^\lambda(x))$. If $|\xi| \leq (c_2 + c_4)/c_1$, the minimality of $\psi(x)$, together with (4.41) and (4.42), gives

$$\check{f}(y, \psi(x)) \leq \check{f}(x, \psi(x)) + \eta/3 \leq \check{f}(x, \xi) + \eta/3 \leq \check{f}(y, \xi) + 2\eta/3 .$$

If, instead, $|\xi| \geq (c_2 + c_4)/c_1$, by (f2) we have $\check{f}(y, \xi) \geq c_4$, which, together with (4.40), (4.41), and (4.42), yields

$$\check{f}(y, \psi(x)) \leq \check{f}(x, \psi(x)) + \eta/3 \leq c_4 + \eta/3 \leq \check{f}(y, \xi) + \eta/3 .$$

In both cases we have (4.43).

Given $\lambda \geq 1$, $\nu \in \mathbb{S}^{d-1}$, and $0 < \rho < \rho_\eta^\lambda(x)$, we consider the functions v and u satisfying (4.36) and (4.37), with $\psi_\eta(x)$ replaced by $\psi(x)$ in the definition of w_1 and w_2. Arguing as in the proof of (4.38), by (4.27) we obtain

$$\begin{aligned} E^a(u, Q_\nu^\lambda(x, \rho)) &= \int_{Q_\nu^\lambda(x,\rho)} \check{f}(y, \nabla u(y)) dy = \int_{Q_\nu^\lambda(x,\rho) \cap B} \check{f}(y, \nabla v(y)) dy + \int_{Q_\nu^\lambda(x,\rho) \setminus B} \check{f}(y, \psi(x)) dy \\ &\leq \int_{Q_\nu^\lambda(x,\rho)} \check{f}(y, \nabla v(y)) dy + 2\eta\lambda^{d-1}\rho^d/3 = E^a(v, Q_\nu^\lambda(x, \rho)) + 2\eta\lambda^{d-1}\rho^d/3 , \end{aligned} \tag{4.44}$$

where in the last inequality we used (4.43). Inequality (4.31) follows now from the previous inequality, using (4.36) and (4.39).

Finally, if $\check{f}$ does not depend on x we repeat the previous arguments taking $\psi(x)$ independent of x. Then (4.42) and (4.43) clearly hold with $\eta = 0$ and $\rho_\eta^\lambda(x) = +\infty$. Consequently $E^a(u, Q_\nu^\lambda(x, \rho)) \leq E^a(v, Q_\nu^\lambda(x, \rho))$ for every $\rho > 0$ and the conclusion follows. $\square$

Proof of Proposition 4.15. Let $N \in \mathcal{B}(\mathbb{R}^d)$ be the set with $\mathcal{L}^d(N) = 0$ introduced in Lemma 4.16. By (4.26), to prove (4.28) and (4.29) we have only to show that for every $x \in \mathbb{R}^d \setminus N$ (or $x \in \mathbb{R}^d$ if $\check{f}$ is continuous) and every $\xi \in \mathbb{R}^d$ we have

$$\inf_{\varepsilon > 0} f_\varepsilon(x, \xi) \leq f(x, \xi) , \tag{4.45}$$

$$\inf_{\varepsilon > 0} f_\varepsilon^\infty(x, \xi) \leq f^\infty(x, \xi) . \tag{4.46}$$

Let us fix $\eta > 0$, $x \in \mathbb{R}^d \setminus N$ (or an arbitrary $x \in \mathbb{R}^d$ if $\check{f}$ is continuous), $\xi \in \mathbb{R}^d$, and $t \geq 1$. Let $c_{t\xi}$ be the constant introduced in (4.30) corresponding to $t\xi$. By the definition of f (see (4.22)) there exists $r_\eta(x, t\xi) > 0$, with $c_{t\xi} r_\eta(x, t\xi) < 1/2$, such that for every $0 < \rho < r_\eta(x, t\xi)$ we have

$$\frac{m^E(\ell_{t\xi}, Q(x, \rho))}{t\rho^d} \leq \frac{f(x, t\xi)}{t} + \eta . \tag{4.47}$$

We apply Lemma 4.16 with $t\xi$ instead of ξ and we find a constant $\rho_\eta(x, t\xi) \in (0, r_\eta(x, t\xi))$ such that for every $\rho \in (0, \rho_\eta(x, t\xi))$ there exists $u \in BV(Q(x, \rho)) \cap L^\infty(Q(x, \rho))$, with $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} \ell_{t\xi}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x, \rho)$ and $\|u - \ell_{t\xi}\|_{L^\infty(Q(x,\rho))} \leq c_{t\xi}\rho < 1/2$, such that

$$E(u, Q(x, \rho)) \leq m^E(\ell_{t\xi}, Q(x, \rho)) + \eta\rho^d .$$

Together with (4.47) this inequality gives

$$\frac{E(u, Q(x, \rho))}{t\rho^d} \leq \frac{f(x, t\xi)}{t} + 2\eta . \tag{4.48}$$

Since $\|u-\ell_{t\xi}\|_{L^\infty(Q(x,\rho))}<1/2$ we have $|[u]|<1$ $\mathcal{H}^{d-1}$-a.e. on J_{u_ρ}. Therefore, recalling the definition of E_ε (see (4.16)), for every $\varepsilon>0$ we have

$$\begin{aligned}E_\varepsilon(u,Q(x,\rho))&\le E(u,Q(x,\rho))+\varepsilon\int_{Q(x,\rho)}|\nabla u|dx+\varepsilon|D^cu|(Q(x,\rho))+\varepsilon\int_{J_u}|[u]|\wedge 1d\mathcal{H}^{d-1}\\&\le\Big(1+\frac{\varepsilon}{c_1}\Big)E(u,Q(x,\rho))+\frac{\varepsilon c_2}{c_1}\rho^d\,,\end{aligned}\tag{4.49}$$

where in the last inequality we used (c1) in Definition 3.1. Let us fix $\varepsilon_\eta>0$ such that $\varepsilon_\eta/c_1<\eta$ and $\varepsilon_\eta c_2/c_1<\eta$. Therefore, the previous chain of inequalities together with (4.48) yields

$$\frac{E_\varepsilon(u,Q(x,\rho))}{t\rho^d}<(1+\eta)\Big(\frac{f(x,t\xi)}{t}+2\eta\Big)+\eta$$

for every $0<\rho<\rho_\eta(x,t\xi)$ and every $0<\varepsilon<\varepsilon_\eta$.

Since $u\in BV(Q(x,\rho))$ and $\mathrm{tr}_{Q(x,\rho)}u=\mathrm{tr}_{Q(x,\rho)}\ell_{t\xi}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, recalling the definition of m^{E_ε}, from the previous inequality we deduce that

$$\frac{m^{E_\varepsilon}(\ell_{t\xi},Q(x,\rho))}{t\rho^d}<(1+\eta)\Big(\frac{f(x,t\xi)}{t}+2\eta\Big)+\eta$$

for every $0<\rho<\rho_\eta(x,t\xi)$ and every $0<\varepsilon<\varepsilon_\eta$. Taking the limsup as $\rho\to0+$ we obtain

$$\frac{f_\varepsilon(x,t\xi)}{t}\le(1+\eta)\Big(\frac{f(x,t\xi)}{t}+2\eta\Big)+\eta\quad\text{for }\mathcal{L}^d\text{-a.e. }x\in\mathbb{R}^d\tag{4.50}$$

and for every $\xi\in\mathbb{R}^d$, $t\ge1$, and $0<\varepsilon<\varepsilon_\eta$. In particular, for $t=1$ we have

$$f_\varepsilon(x,\xi)\le(1+\eta)(f(x,\xi)+2\eta)+\eta\quad\text{for }\mathcal{L}^d\text{-a.e. }x\in\mathbb{R}^d\tag{4.51}$$

and for every $\xi\in\mathbb{R}^d$ and $0<\varepsilon<\varepsilon_\eta$. Passing to the lim sup in (4.50) as $t\to+\infty$ we obtain

$$f_\varepsilon^\infty(x,\xi)\le(1+\eta)(f^\infty(x,\xi)+2\eta)+\eta\quad\text{for }\mathcal{L}^d\text{-a.e. }x\in\mathbb{R}^d\tag{4.52}$$

and for every $\xi\in\mathbb{R}^d$ and $0<\varepsilon<\varepsilon_\eta$. To obtain (4.45) and (4.46) it is enough to take the infimum in (4.51) and (4.52) for $\varepsilon\in(0,\varepsilon_\eta)$, and then the limit as $\eta\to0+$. □

We now prove the same result for the function g defined in (4.23).

Proposition 4.17. *Let $E\in\mathfrak{E}$, and let g and g_ε be defined by (4.23) and (4.25). Then*

$$g(x,\zeta,\nu)=\lim_{\varepsilon\to0+}g_\varepsilon(x,\zeta,\nu)\tag{4.53}$$

for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$.

Proof. Let us fix x, ζ, and ν as in the statement and let $\hat g(x,\zeta,\nu)$ be the right-hand side of (4.53). By (4.26) we have only to prove that

$$\hat g(x,\zeta,\nu)\le g(x,\zeta,\nu)\,.\tag{4.54}$$

Let us fix $\eta>0$. By the definition of g there exists $r_\eta(x)>0$ such that

$$\frac{m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}\le g(x,\zeta,\nu)+\eta\tag{4.55}$$

for every $0<\rho<r_\eta(x)$. By the definition of m^E, for every $0<\rho<r_\eta(x)$ there exists $u\in BV(Q_\nu(x,\rho))$, with $\mathrm{tr}_{Q_\nu(x,\rho)}u=\mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, such that

$$\frac{E(u,Q_\nu(x,\rho))}{\rho^{d-1}}\le g(x,\zeta,\nu)+2\eta\,.\tag{4.56}$$

Let us fix $m>|\zeta|\vee\frac12$. By (3.10) we have

$$E(u^{(m)},Q_\nu(x,\rho))\le E(u,Q_\nu(x,\rho))+c_4\rho^d\,.\tag{4.57}$$

Together with (4.56) this inequality gives

$$\frac{E(u^{(m)}, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta,\nu) + 2\eta + c_4\rho\,. \tag{4.58}$$

Let $J_u^1 := \{x \in J_u : |[u](x)| \ge 1\}$. By (4.16) and (4.57) for every $\varepsilon > 0$ we have

$$\begin{aligned}
E_\varepsilon(u^{(m)}, Q_\nu(x,\rho)) &\le E(u^{(m)}, Q_\nu(x,\rho)) + \varepsilon\int_{Q_\nu(x,\rho)} |\nabla u^{(m)}|\,dx \\
&\quad + \varepsilon|D^c u^{(m)}|(Q_\nu(x,\rho)) + \varepsilon\int_{J_u\cap Q_\nu(x,\rho)} |[u]|\wedge(2m)d\mathcal{H}^{d-1} \\
&\le E(u, Q_\nu(x,\rho)) + c_4\rho^d + \varepsilon\int_{Q_\nu(x,\rho)} |\nabla u|\,dx + \varepsilon|D^c u|(Q_\nu(x,\rho)) \\
&\quad + \varepsilon 2m\mathcal{H}^{d-1}(J_u^1\cap Q_\nu(x,\rho)) + \varepsilon\int_{(J_u\setminus J_u^1)\cap Q_\nu(x,\rho)} |[u]|d\mathcal{H}^{d-1} \\
&\le E(u, Q_\nu(x,\rho)) + c_4\rho^d + \varepsilon\int_{Q_\nu(x,\rho)} |\nabla u|\,dx + \varepsilon|D^c u|(Q_\nu(x,\rho)) \\
+\varepsilon 2m\int_{J_u\cap Q_\nu(x,\rho)} |[u]|\wedge 1 d\mathcal{H}^{d-1} &\le \big(1+\varepsilon\frac{2m}{c_1}\big)E(u,Q_\nu(x,\rho)) + (c_4 + \varepsilon\frac{2mc_2}{c_1})\rho^d\,,
\end{aligned}$$

where in the last inequality we used (c1) in Definition 3.1. We can find $\varepsilon_\eta > 0$ such that $\varepsilon_\eta\frac{2m}{c_1} < \eta$ and $\varepsilon_\eta\frac{2mc_2}{c_1} < \eta$. Therefore, the previous chain of inequalities together with (4.56) yields

$$\frac{E_\varepsilon(u^{(m)}, Q_\nu(x,\rho))}{\rho^{d-1}} \le (1+\eta)(g(x,\zeta,\nu) + 2\eta) + (c_4+\eta)\rho$$

for every $0 < \rho < r_\eta(x)$ and every $0 < \varepsilon < \varepsilon_\eta$.

Since $u^{(m)} \in BV(Q_\nu(x,\rho))$ and $\mathrm{tr}_{Q_\nu(x,\rho)} u^{(m)} = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, recalling the definition of m^{E_ε}, from the previous inequality we deduce that

$$\frac{m^{E_\varepsilon}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \le (1+\eta)(g(x,\zeta,\nu) + 2\eta) + (c_4+\eta)\rho$$

for every $0 < \rho < r_\eta(x)$ and every $0 < \varepsilon < \varepsilon_\eta$. Taking the $\limsup$ as $\rho \to 0+$ and using the definition of g_ε (see (4.25)) we obtain

$$g_\varepsilon(x,\zeta,\nu) \le (1+\eta)(g(x,\zeta,\nu) + 2\eta)$$

for every $\eta > 0$ and every $0 < \varepsilon < \varepsilon_\eta$. Taking the limit, first as $\varepsilon \to 0+$ and then as $\eta \to 0+$, we obtain (4.54). □

5. Properties of the integrands f and g

In this section we shall prove the following result.

Theorem 5.1. *Let $E \in \mathfrak{E}$, let f and g be defined by (4.22) and (4.23), and let $\hat f\colon \mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$ be defined by*

$$\hat f(x,\xi) = \inf_{\varepsilon>0} f_\varepsilon(x,\xi) = \lim_{\varepsilon\to 0+} f_\varepsilon(x,\xi) \quad \textit{for every } x\in\mathbb{R}^d \textit{ and } \xi\in\mathbb{R}^d, \tag{5.1}$$

where f_ε is introduced in (4.24). Then $f \in \mathcal{F}$, $\hat f \in \mathcal{F}$, and $g \in \mathcal{G}$.

The proof of the theorem relies on several technical lemmas, in which we tacitly assume that $E \in \mathfrak{E}$, $\varepsilon > 0$, and f, g, f_ε, and g_ε are defined by (4.22), (4.23), (4.24), and (4.25). To obtain the Borel measurability of these functions we use the following lemma, which provides general conditions under which we can restrict a $\limsup$ to a countable set.

Lemma 5.2. *Let* $\kappa_1 \geq 0$, $\kappa_2 \geq 0$, *let* $\psi\colon (0,1) \to (0,+\infty)$ *be a function such that*

$$\psi(\rho) \leq \psi(r) + \kappa_1(1-(\tfrac{r}{\rho})^{d-1}) + \kappa_2(1-(\tfrac{r}{\rho})^{d}) \quad \textit{for every } 0<r<\rho<1\,, \tag{5.2}$$

and let D *be a dense subset of* $(0,1)$. *Then*

$$\limsup_{\substack{\rho\to 0+\\ \rho\in D}} \psi(\rho) = \limsup_{\rho\to 0+} \psi(\rho)\,. \tag{5.3}$$

Proof. It is enough to prove that

$$\limsup_{\rho\to 0+} \psi(\rho) \leq \limsup_{\substack{\rho\to 0+\\ \rho\in D}} \psi(\rho)\,. \tag{5.4}$$

Let λ be the right-hand side of (5.4) and let $\varepsilon>0$. Then there exists $0<\delta<1$ such that

$$\psi(r) \leq \lambda+\varepsilon \qquad \text{for every } r\in D\cap(0,\delta)\,. \tag{5.5}$$

We claim that $\psi(\rho)\leq\lambda+\varepsilon$ for every $\rho\in(0,\delta)$. Indeed, given $\rho\in(0,\delta)$, by (5.2) and (5.5) we have

$$\psi(\rho) \leq \lambda+\varepsilon+\kappa_1(1-(\tfrac{r}{\rho})^{d-1}) + \kappa_2(1-(\tfrac{r}{\rho})^{d})$$

for every $0<r<\rho$ with $r\in D$. Passing to the limit as $r\to\rho-$ we obtain $\psi(\rho)\leq\lambda+\varepsilon$ for every $\rho\in(0,\delta)$, which implies that

$$\limsup_{\rho\to 0+} \psi(\rho) \leq \lambda+\varepsilon\,.$$

By the arbitrariness of $\varepsilon>0$ this implies (5.4). □

We are now ready to begin the proof of property (f1).

Lemma 5.3. *Let* $\xi\in\mathbb{R}^d$. *Then* $x\mapsto f(x,\xi)$ *and* $x\mapsto f_\varepsilon(x,\xi)$ *are Borel functions on* $\mathbb{R}^d$.

Proof. For every $x\in\mathbb{R}^d$ and for every $\rho>0$ let

$$\varphi(x,\rho) := m^E(\ell_\xi, Q(x,\rho))\,. \tag{5.6}$$

By Lemma 4.11 we have

$$\varphi(x,\rho_2) - (c_3|\xi|+c_4)\rho_2^d \leq \varphi(x,\rho_1) - (c_3|\xi|+c_4)\rho_1^d \tag{5.7}$$

for every $0<\rho_1<\rho_2$. This shows that the function

$$\rho \mapsto \frac{\varphi(x,\rho)}{\rho^d}$$

satisfies (5.2) with $\kappa_1=0$ and $\kappa_2=c_3|\xi|+c_4$. By Lemma 5.2 this implies that

$$f(x,\xi) = \limsup_{\substack{\rho\to 0+\\ \rho\in\mathbb{Q}}} \frac{\varphi(x,\rho)}{\rho^d}\,. \tag{5.8}$$

Moreover, by (5.7) the function $\rho\mapsto\varphi(x,\rho)$ has locally bounded variation in $(0,+\infty)$ for every $x\in\mathbb{R}^d$. This implies that for every $x\in\mathbb{R}^d$ and $r>0$ there exists

$$\varphi(x,r+) := \lim_{\rho\to r+} \varphi(x,\rho)\,.$$

Using (5.7) we see that

$$\varphi(x,\rho) \leq \varphi(x,r+) + (c_3|\xi|+c_4)(\rho^d-r^d) \tag{5.9}$$

for every $0<r<\rho$. Moreover, it is obvious that

$$\sup_{\substack{0<r<\delta\\ r\in\mathbb{Q}}} \frac{\varphi(x,r+)}{r^d} = \sup_{\substack{0<r<\delta\\ r\in\mathbb{Q}}} \frac{\varphi(x,r)}{r^d}\,.$$

Together with (5.8) this implies that

$$f(x,\xi) = \limsup_{\substack{r\to 0+\\ r\in\mathbb{Q}}} \frac{\varphi(x,r+)}{r^d}\,. \tag{5.10}$$

Let us fix $r>0$. We claim that the function $x\mapsto\varphi(x,r+)$ is lower semicontinuous. Let $x_k\to x$. Let us fix $\rho_2>\rho_1>r$. For k large enough we have $Q(x_k,\rho_1)\subset\subset Q(x,\rho_2)$. By Lemma 4.11 we have

$$\varphi(x,\rho_2)\le\varphi(x_k,\rho_1)+(c_3|\xi|+c_4)(\rho_2^d-\rho_1^d)\,.$$

By (5.9) we obtain

$$\varphi(x,\rho_2)\le\varphi(x_k,r+)+(c_3|\xi|+c_4)(\rho_2^d-r^d)\,,$$

Hence

$$\varphi(x,\rho_2)\le\liminf_{k\to\infty}\varphi(x_k,r+)+(c_3|\xi|+c_4)(\rho_2^d-r^d)\,,$$

and taking the limit as $\rho_2\to r+$ we obtain

$$\varphi(x,r+)\le\liminf_{k\to\infty}\varphi(x_k,r+)\,,$$

which proves the lower semicontinuity of $x\mapsto\varphi(x,r+)$. By (5.10) we conclude that $x\mapsto f(x,\xi)$ is a Borel function. The same proof holds for f_ε. ☐

In the next result we show that f satisfies property (f4) and f_ε satisfies the same property with a different constant.

Lemma 5.4. *Let $x\in\mathbb{R}^d$ and $\xi_1,\xi_2\in\mathbb{R}^d$. Then*

$$|f(x,\xi_1)-f(x,\xi_2)|\le c_5|\xi_1-\xi_2| \quad \textit{and} \quad |f_\varepsilon(x,\xi_1)-f_\varepsilon(x,\xi_2)|\le(c_5+\varepsilon)|\xi_1-\xi_2|\,. \tag{5.11}$$

Proof. Let us fix $\rho>0$. By the definition of m^E there exists $u_1\in BV(Q(x,\rho))$ with $\mathrm{tr}_{Q(x,\rho)}u_1=\mathrm{tr}_{Q(x,\rho)}\ell_{\xi_1}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$ such that

$$E(u_1,Q(x,\rho))\le m^E(\ell_{\xi_1},Q(x,\rho))+\rho^{d+1}\,.$$

Let $u_2:=u_1-\ell_{\xi_1}+\ell_{\xi_2}$. Since $\mathrm{tr}_{Q(x,\rho)}u_2=\mathrm{tr}_{Q(x,\rho)}\ell_{\xi_2}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, by (3.7) we have

$$\begin{aligned} m^E(\ell_{\xi_2},Q(x,\rho))&\le E(u_2,Q(x,\rho))\le E(u_1,Q(x,\rho))+c_5|\xi_1-\xi_2|\rho^d\\ &\le m^E(\ell_{\xi_1},Q(x,\rho))+\rho^{d+1}+c_5|\xi_1-\xi_2|\rho^d\,.\end{aligned}$$

Dividing by ρ^d and taking the $\limsup$ as $\rho\to 0+$ we obtain

$$f(x,\xi_2)-f(x,\xi_1)\le c_5|\xi_1-\xi_2|\,.$$

Exchanging the roles of ξ_1 and ξ_2 we obtain the first inequality in (5.11). The proof for f_ε is similar. ☐

Corollary 5.5. *The functions f and f_ε are Borel measurable on $\mathbb{R}^d\times\mathbb{R}^d$.*

Proof. The result follows from Lemmas 5.3 and 5.4 ☐

The following lemma provides the lower estimate (f2) for f and f_ε.

Lemma 5.6. *Let $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$. Then*

$$f_\varepsilon(x,\xi)\ge f(x,\xi)\ge c_1|\xi|-c_2\,. \tag{5.12}$$

To prove the lemma we use the following result about one-dimensional problems.

Lemma 5.7. *Let $I = (a,b)$ be a bounded open interval in $\mathbb{R}$, let $s,t \in \mathbb{R}$, and let $\Psi\colon BV(I) \to [0,+\infty)$ be defined by*

$$\Psi(u) := \int_I |\nabla u| dx + |D^c u|(I) + \sum_{x\in J_u} |[u](x)| \wedge 1 = |Du|(I \setminus J_u) + \sum_{x\in J_u} |[u](x)| \wedge 1 \,. \tag{5.13}$$

Then

$$\inf_{\substack{u\in BV(I)\\ u(a)=s,\, u(b)=t}} \Psi(u) \geq |t-s| \wedge 1 \,. \tag{5.14}$$

Proof. For every $u \in BV(I)$ we have

$$|Du(I)| \wedge 1 \leq |Du|(I) \wedge 1 \leq |Du|(I \setminus J_u) + \sum_{x\in J_u} (|[u](x)| \wedge 1) \,.$$

If, in addition, $u(a) = s$, and $u(b) = t$, then $Du(I) = t-s$, and the previous chain of inequalities gives (5.14). □

To prove Lemma 5.6 we use a slicing argument. Given $A \in \mathcal{A}_c(\mathbb{R}^d)$, $u \in BV(A)$, $\nu \in \mathbb{S}^{d-1}$, and $y \in \Pi_0^\nu$, we define

$$A_y^\nu := \{t \in \mathbb{R} : y + t\nu \in A\} \tag{5.15}$$

and $u_y^\nu\colon A_y^\nu \to \mathbb{R}$ by

$$u_y^\nu(t) := u(y+t\nu) \quad \text{for every } t \in A_y^\nu \,. \tag{5.16}$$

Proof of Lemma 5.6. Since the first inequality in (5.12) is given by (4.26), we only have to prove the second one. Let $\rho > 0$. By the definition of $m^E(\ell_\xi, Q(x,\rho))$ there exists $u \in BV(Q(x,\rho))$, with $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, such that $E(u, Q(x,\rho)) \leq m^E(\ell_\xi, Q(x,\rho)) + \rho^{d+1}$.

Let $\nu := \xi/|\xi|$. By the results on slicing for BV functions (see [2, Theorem 3.108]) we have that

$$V(u, Q(x,\rho)) \geq \int_{C_\nu(x,\rho)} \Psi_y^\nu(u_y^\nu) d\mathcal{H}^{d-1}(y) \,,$$

where $C_\nu(x,\rho)$ is the orthogonal projection of the cube $Q(x,\rho)$ onto Π_0^ν, and Ψ_y^ν is the functional defined by (5.13) with $I := Q(x,\rho)_y^\nu$.

Since $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} \ell_\xi$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q(x,\rho)$, we have $\mathrm{tr}_{Q(x,\rho)_y^\nu} u_y^\nu = \mathrm{tr}_{Q(x,\rho)_y^\nu} (\ell_\xi)_y^\nu$ in $\partial(Q(x,\rho)_y^\nu)$, therefore, by Lemma 5.7 we obtain that $\Psi_y^\nu(u_y^\nu) \geq |\xi| \mathcal{L}^1(Q(x,\rho)_y^\nu)$ for $\rho > 0$ small enough. Hence integrating over $C_\nu(x,\rho)$, by Fubini's Theorem we obtain

$$V(u, Q(x,\rho)) \geq |\xi| \rho^d \,.$$

By (c1) in Definition 3.1 and by the choice of u, this shows that $m^E(\ell_\xi, Q(x,\rho)) + \rho^{d+1} \geq c_1 |\xi| \rho^d - c_2 \rho^d$. Dividing by ρ^d and taking the $\limsup$ as $\rho \to 0+$ we obtain (5.12). □

We now prove the upper estimate (f3) for f and a corresponding estimate for f_ε.

Lemma 5.8. *Let $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$. Then*

$$f(x,\xi) \leq c_3 |\xi| + c_4 \quad \text{and} \quad f_\varepsilon(x,\xi) \leq (c_3 + \varepsilon) |\xi| + c_4 \,. \tag{5.17}$$

Proof. For every $\rho > 0$, by (c2) in Definition 3.1 we have

$$m^E(\ell_\xi, Q(x,\rho)) \leq E(\ell_\xi, Q(x,\rho)) \leq c_3 |\xi| \rho^d + c_4 \rho^d \,.$$

Hence, dividing by ρ^d and taking the $\limsup$ as $\rho \to 0+$ we obtain the first inequality in (5.17). The inequality for f_ε is proved in the same way. □

In the next lemmas we shall establish the required properties of the function g defined by (4.23). To prove the Borel measurability we use the following lemma, which gives an estimate of the dependence of $m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho))$ on x, ν, and ρ.

Lemma 5.9. *There exists a continuous function* $\omega\colon [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *with* $\omega(0,0)=0$ *such that for every* $x_1,x_2\in\mathbb{R}^d$*,* $\zeta\in\mathbb{R}$*,* $\nu_1,\nu_2\in\mathbb{S}^{d-1}$*, and* $0<\rho_1<\rho_2$ *the inclusion* $Q_{\nu_1}(x_1,\rho_1)\subset\subset Q_{\nu_2}(x_2,\rho_2)$ *implies*

$$\begin{aligned} m^E(u_{x_2,\zeta,\nu_2},Q_{\nu_2}(x_2,\rho_2)) &\le m^E(u_{x_1,\zeta,\nu_1},Q_{\nu_1}(x_1,\rho_1))+c_3|\zeta|(\rho_2^{d-1}-\rho_1^{d-1})\\ &+c_4(\rho_2^d-\rho_1^d)+c_3|\zeta|\,\omega(\tfrac{|x_1-x_2|}{\rho_1},|\nu_1-\nu_2|)\,\rho_1^{d-1}\,. \end{aligned}\tag{5.18}$$

The same inequality holds for m^{E_ε}*, with* c_3 *replaced by* $c_3+\varepsilon$*.*

Proof. Let $x_1,x_2\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, $\nu_1,\nu_2\in\mathbb{S}^{d-1}$, and $0<\rho_1<\rho_2$ be as in the statement. Given $\eta>0$ by the definition of m^E there exists $u_1\in BV(Q_{\nu_1}(x_1,\rho_1))$, with $\mathrm{tr}_{Q_{\nu_1}(x_1,\rho_1)}u_1=\mathrm{tr}_{Q_{\nu_1}(x_1,\rho_1)}u_{x_1,\zeta,\nu_1}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_{\nu_1}(x_1,\rho_1)$, such that

$$E(u_1,Q_{\nu_1}(x_1,\rho_1))\le m^E(u_{x_1,\zeta,\nu_1},Q_{\nu_1}(x_1,\rho_1))+\eta\,. \tag{5.19}$$

Let $u_2\in BV(Q_{\nu_2}(x_2,\rho_2))$ be the function defined by

$$u_2(y):=\begin{cases}u_1(y) & \text{if } y\in Q_{\nu_1}(x_1,\rho_1)\\ u_{x_2,\zeta,\nu_2}(y) & \text{if } y\in Q_{\nu_2}(x_2,\rho_2)\setminus Q_{\nu_1}(x_1,\rho_1)\,.\end{cases}\tag{5.20}$$

Since $\mathrm{tr}_{Q_{\nu_2}(x_2,\rho_2)}u_2=\mathrm{tr}_{Q_{\nu_2}(x_2,\rho_2)}u_{x_2,\zeta,\nu_2}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_{\nu_2}(x_2,\rho_2)$, we have

$$\begin{aligned} m^E(u_{x_2,\zeta,\nu_2},Q_{\nu_2}(x_2,\rho_2)) &\le E(u_2,Q_{\nu_2}(x_2,\rho_2))\\ =E(u_1,Q_{\nu_1}(x_1,\rho_1)) &+E(u_2,Q_{\nu_2}(x_2,\rho_2)\setminus Q_{\nu_1}(x_1,\rho_1))\,, \end{aligned}\tag{5.21}$$

where for the equality we used the locality and the measure property of E (see (a) and (b) in Definition 3.1). By (c2) in the same definition we have

$$E(u_2,Q_{\nu_2}(x_2,\rho_2)\setminus Q_{\nu_1}(x_1,\rho_1))\le c_3|Du_2|(Q_{\nu_2}(x_2,\rho_2)\setminus Q_{\nu_1}(x_1,\rho_1))+c_4(\rho_2^d-\rho_1^d)\,. \tag{5.22}$$

Let

$$\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1):=\{y\in\partial Q_{\nu_1}(x_1,\rho_1):\mathrm{sign}((y-x_1)\cdot\nu_1)\neq\mathrm{sign}((y-x_2)\cdot\nu_2)\}\,. \tag{5.23}$$

Since $\mathrm{tr}_{Q_{\nu_1}(x_1,\rho_1)}u_1=\mathrm{tr}_{Q_{\nu_1}(x_1,\rho_1)}u_{x_1,\zeta,\nu_1}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_{\nu_1}(x_1,\rho_1)$, by (5.20) we have $J_{u_2}\cap\partial Q_{\nu_1}(x_1,\rho_1)=\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)$, hence

$$\begin{aligned} |Du_2|(Q_{\nu_2}(x_2,\rho_2)\setminus Q_{\nu_1}(x_1,\rho_1)) &=|Du_2|(Q_{\nu_2}(x_2,\rho_2)\setminus\overline{Q}_{\nu_1}(x_1,\rho_1))+|Du_2|(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1))\\ &=\int_{(Q_{\nu_2}(x_2,\rho_2)\setminus\overline{Q}_{\nu_1}(x_1,\rho_1))\cap J_{u_2}}|[u_2]|d\mathcal{H}^{d-1}+\int_{\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)}|[u_2]|d\mathcal{H}^{d-1}\\ &\le|\zeta|(\rho_2^{d-1}-\mathcal{H}^{d-1}(\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_2}_{x_2})+\mathcal{H}^{d-1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)))\,. \end{aligned}\tag{5.24}$$

For $a\ge0$ and $b\ge0$ we set

$$\omega(a,b):=\max_{\substack{|x_1-x_2|\le a\\ |\nu_1-\nu_2|\le b}}\mathcal{H}^{d-1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(1))\,. \tag{5.25}$$

By continuity the maximum exists, ω is a continuous function on $[0,+\infty)\times[0,+\infty)$, and $\omega(0,0)=0$. By rescaling we obtain that

$$\mathcal{H}^{d-1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1))\le\omega(\tfrac{|x_1-x_2|}{\rho_1},|\nu_1-\nu_2|)\,\rho_1^{d-1}\,. \tag{5.26}$$

Let $\pi^{\nu_1}_{x_1}$ be the orthogonal projection of $\mathbb{R}^d$ onto $\Pi^{\nu_1}_{x_1}$. We claim that

$$\pi^{\nu_1}_{x_1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)\cup(\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_2}_{x_2}))=\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_1}_{x_1}\,.$$

To prove the claim let us fix $y\in\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_1}_{x_1}$. If there exists $t\in[-\rho_1/2,\rho_1/2]$ such that $y+t\nu_1\in\Pi^{\nu_2}_{x_2}$, then $y\in\pi^{\nu_1}_{x_1}(\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_2}_{x_2})$. If, instead, for every $t\in[-\rho_1/2,\rho_1/2]$ we have $y+t\nu_1\notin\Pi^{\nu_2}_{x_2}$, then $\mathrm{sign}((y+t\nu_1-x_2)\cdot\nu_2)$ is constant on $[-\rho_1/2,\rho_1/2]$. On the other hand $\mathrm{sign}((y\pm\rho_1\nu_1/2-x_1)\cdot\nu_1)=\pm1$. By the definition of $\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)$ this implies that either $y+\rho_1\nu_1/2$ or $y-\rho_1\nu_1/2$ belong to $\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1)$ and hence $y\in\pi^{\nu_1}_{x_1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1))$. This concludes the proof of the claim, which implies that

$$\mathcal{H}^{d-1}(\overline{Q}_{\nu_1}(x_1,\rho_1)\cap\Pi^{\nu_2}_{x_2})\geq\rho_1^{d-1}-\mathcal{H}^{d-1}(\Sigma^{\nu_1,\nu_2}_{x_1,x_2}(\rho_1))\,. \tag{5.27}$$

Therefore (5.19), (5.21), (5.22), and (5.24), (5.26), and (5.27) give

$$\begin{aligned}m^E(u_{x_2,\zeta,\nu_2},Q_{\nu_2}(x_2,\rho_2))&\leq m^E(u_{x_1,\zeta,\nu_1},Q_{\nu_1}(x_1,\rho_1))+c_3|\zeta|(\rho_2^{d-1}-\rho_1^{d-1})\\&+c_4(\rho_2^d-\rho_1^d)+2c_3|\zeta|\,\omega(\tfrac{|x_1-x_2|}{\rho_1},|\nu_1-\nu_2|)\,\rho_1^{d-1}+\eta\,.\end{aligned}$$

Taking the limit as $\eta\to 0$ we obtain (5.18) with ω therein replaced by 2ω. ☐

The following lemma provides the Borel measurability of g and g_ε for a fixed ζ.

Lemma 5.10. *Let $\zeta\in\mathbb{R}$. Then the functions $(x,\nu)\mapsto g(x,\zeta,\nu)$ and $(x,\nu)\mapsto g_\varepsilon(x,\zeta,\nu)$ are Borel measurable on $\mathbb{R}^d\times\mathbb{S}^{d-1}$.*

Proof. For every $\rho>0$ we set

$$\varphi(x,\nu,\rho):=m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho))\,. \tag{5.28}$$

By Lemma 4.11 we have

$$\varphi(x,\nu,\rho_2)-c_3\rho_2^{d-1}-c_4\rho_2^d\leq\varphi(x,\nu,\rho_1)-c_3\rho_1^{d-1}-c_4\rho_1^d\,. \tag{5.29}$$

Hence the function

$$\rho\mapsto\frac{\varphi(x,\nu,\rho)}{\rho^{d-1}}$$

satisfies (5.2) with $\kappa_1=c_3$ and $\kappa_2=c_4$. By Lemma 5.2 this implies that

$$g(x,\zeta,\nu)=\limsup_{\substack{\rho\to0+\\ \rho\in\mathbb{Q}}}\frac{\varphi(x,\nu,\rho)}{\rho^{d-1}}\,. \tag{5.30}$$

Moreover, by (5.29), for every $x\in\mathbb{R}^d$ and $\nu\in\mathbb{S}^{d-1}$ the function $\rho\mapsto\varphi(x,\nu,\rho)$ has bounded variation in $(0,+\infty)$. This implies that for every $x\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, and $r>0$ there exists

$$\varphi(x,\nu,r+):=\lim_{\rho\to r+}\varphi(x,\nu,\rho)\,.$$

Using (5.29) we see that

$$\varphi(x,\nu,\rho)\leq\varphi(x,\nu,r+)+c_3(\rho^{d-1}-r^{d-1})+c_4(\rho^d-r^d) \tag{5.31}$$

for every $0<r<\rho$. Moreover, it is obvious that

$$\sup_{\substack{0<r<\delta\\ r\in\mathbb{Q}}}\frac{\varphi(x,\nu,r+)}{r^{d-1}}=\sup_{\substack{0<r<\delta\\ r\in\mathbb{Q}}}\frac{\varphi(x,\nu,r)}{r^{d-1}}$$

for every $\delta>0$. By (5.30) this implies that

$$g(x,\zeta,\nu)=\limsup_{\substack{r\to0+\\ r\in\mathbb{Q}}}\frac{\varphi(x,\nu,r+)}{r^{d-1}}\,. \tag{5.32}$$

Let us fix $x_0\in\mathbb{R}^d$, $\nu_0\in\mathbb{S}^{d-1}_+$, and $r>0$. We claim that the function $(x,\nu)\mapsto\varphi(x,\nu,r+)$ is lower semicontinuous in $\mathbb{R}^d\times\mathbb{S}^{d-1}_+$ at (x_0,ν_0). Let $x_k\to x_0$ in $\mathbb{R}^d$ and $\nu_k\to\nu_0$ in $\mathbb{S}^{d-1}_+$. Let us fix $r<\rho_1<\rho_2$. For k large enough we have $Q_{\nu_k}(x_k,\rho_1)\subset\subset Q_{\nu_0}(x_0,\rho_2)$ (see (d) and (e) at the beginning of Section 2).

By (5.18)

$$\begin{aligned}\varphi(x_0,\nu_0,\rho_2)&\leq\varphi(x_k,\nu_k,\rho_1)+c_3(\rho_2^{d-1}-\rho_1^{d-1})\\&+c_4(\rho_2^d-\rho_1^d)+c_3\omega(\tfrac{|x_0-x_k|}{\rho_1},|\nu_0-\nu_k|)\,\rho_1^{d-1}\,.\end{aligned} \tag{5.33}$$

By (5.31) applied to $x = x_k$, $\nu = \nu_k$, and $\rho = \rho_1$, we obtain

$$\begin{aligned}\varphi(x_0,\nu_0,\rho_2) &\leq \varphi(x_k,\nu_k,r+) + c_3(\rho_1^{d-1} - r^{d-1}) + c_4(\rho_1^d - r^d)\\ &+c_3(\rho_2^{d-1} - r^{d-1}) + c_4(\rho_2^d - r^d) + c_3\,\omega\big(\tfrac{|x-x_k|}{\rho_1}, |\nu-\nu_k|\big)\,\rho_1^{d-1}\,.\end{aligned}$$

Hence

$$\varphi(x_0,\nu_0,\rho_2) \leq \liminf_{k\to\infty}\varphi(x_k,\nu_k,r+)+c_3(\rho_1^{d-1}-r^{d-1})+c_4(\rho_1^d-r^d)+c_3(\rho_2^{d-1}-r^{d-1})+c_4(\rho_2^d-r^d)$$

and taking the limit as $\rho_1,\rho_2 \to r+$ we obtain

$$\varphi(x_0,\nu_0,r+) \leq \liminf_{k\to\infty}\varphi(x_k,\nu_k,r+)\,,$$

which proves the lower semicontinuity of $(x,\nu)\mapsto\varphi(x,\nu,r+)$ in $\mathbb{R}^d\times\mathbb{S}^{d-1}_+$. By (5.32) we deduce that $(x,\nu)\mapsto g(x,\zeta,\nu)$ is a Borel function in $\mathbb{R}^d\times\mathbb{S}^{d-1}_+$. The same argument holds for $\mathbb{R}^d\times\mathbb{S}^{d-1}_-$ and this leads to the result for g. The proof for g_ε is similar. □

The following lemma provides the uniform continuity of g and g_ε with respect to ζ.

Lemma 5.11. *Let $x\in\mathbb{R}^d$, $\zeta_1,\zeta_2\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$. Then*

$$|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \leq \sigma(|\zeta_1-\zeta_2|) \tag{5.34}$$

$$|g_\varepsilon(x,\zeta_1,\nu) - g_\varepsilon(x,\zeta_2,\nu)| \leq \sigma(|\zeta_1-\zeta_2|) + \varepsilon|\zeta_1-\zeta_2|\,. \tag{5.35}$$

Proof. Let us fix $\rho>0$. By the definition of m^E there exists $u_1\in BV(Q_\nu(x,\rho))$, with $\mathrm{tr}_{Q_\nu(x,\rho)}u_1 = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta_1,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, such that

$$E(u_1,Q_\nu(x,\rho)) \leq m^E(u_{x,\zeta_1,\nu},Q_\nu(x,\rho)) + \rho^d\,.$$

Let $u_2 := u_1 - u_{x,\zeta_1,\nu} + u_{x,\zeta_2,\nu} = u_1 + u_{x,\zeta_2-\zeta_1,\nu}$. Since $\mathrm{tr}_{Q_\nu(x,\rho)}u_2 = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta_2,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, by (3.8) we have

$$\begin{aligned}m^E(u_{x,\zeta_2,\nu},Q_\nu(x,\rho)) &\leq E(u_2,Q_\nu(x,\rho)) \leq E(u_1,Q_\nu(x,\rho)) + \sigma(|\zeta_1-\zeta_2|)\rho^{d-1}\\ &\leq m^E(u_{x,\zeta_1,\nu},Q_\nu(x,\rho)) + \rho^d + \sigma(|\zeta_1-\zeta_2|)\rho^{d-1}\,.\end{aligned}$$

Dividing by ρ^{d-1} and taking the $\limsup$ as $\rho\to 0+$ we obtain (5.34). The proof of (5.35) is similar. □

We are now in a position to obtain the measurability of g and g_ε.

Corollary 5.12. *The functions g and g_ε are Borel measurable on $\mathbb{R}^d\times\mathbb{R}\times\mathbb{S}^{d-1}$.*

Proof. The result follows from Lemmas 5.10 and 5.11. □

The following lemma provides the lower estimate for g.

Lemma 5.13. *Let $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}$, and $\nu\in\mathbb{S}^{d-1}$. Then*

$$g(x,\zeta,\nu) \geq c_1(|\zeta|\wedge 1)\,. \tag{5.36}$$

Proof. Let $\rho>0$ and let $\partial_\nu Q_\nu(x,\rho)$ be the union of the two faces of $Q_\nu(x,\rho)$ that are orthogonal to ν. Then, by (c1) of Definition 3.1,

$$m^E(u_{x,\zeta,\nu},Q_\nu(x,\rho)) \geq \inf_{\substack{u\in BV(Q_\nu(x,\rho))\\ \mathrm{tr}_{Q_\nu(x,\rho)}u=\mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta,\nu}\\ \mathcal{H}^{d-1}\text{-a.e. on } \partial_\nu Q_\nu(x,\rho)}} c_1 V(u,Q_\nu(x,\rho)) - c_2\rho^d\,. \tag{5.37}$$

Using the notation introduced in (5.15) and (5.16), by [2, Theorem 3.108] for every $u\in BV(Q_\nu(x,\rho))$ we have

$$V(u,Q_\nu(x,\rho)) \geq \int_{C_\nu(x,\rho)}\Psi(u^\nu_y)d\mathcal{H}^{d-1}(y)\,,$$

where $C_\nu(x,\rho)$ is the orthogonal projection onto Π^ν_0 of the cube $Q_\nu(x,\rho)$ and Ψ is the functional defined by (5.13) with $a := x\cdot\nu - \rho/2$ and $b := x\cdot\nu - \rho/2$. If, in addition,

$\mathrm{tr}_{Q_\nu(x,\rho)}u = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial_\nu Q_\nu(x,\rho)$, we have also $u^\nu_y(a)=0$ and $u^\nu_y(b)=\zeta$ (in the sense of traces in dimension 1) for $\mathcal{H}^{d-1}$-a.e. $y \in C_\nu(x,\rho)$. Therefore, by Lemma 5.7 we obtain that $\Psi(u^\nu_y) \geq |\zeta| \wedge 1$. Together with (5.37) this shows that $m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \geq c_1(|\zeta|\wedge 1)\rho^{d-1} - c_2\rho^d$. Dividing by ρ^{d-1} and taking the lim sup as $\rho \to 0+$ we obtain (5.36). □

We now prove the upper estimate for g and g_ε.

Lemma 5.14. *Let $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. Then*

$$g(x,\zeta,\nu) \leq c_3(|\zeta| \wedge 1) \quad \textit{and} \quad g_\varepsilon(x,\zeta,\nu) \leq c_3(|\zeta| \wedge 1) + \varepsilon|\zeta|\,. \tag{5.38}$$

Proof. For every $\rho > 0$, by (c2) of Definition 3.1 we have

$$m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \leq E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \leq c_3(|\zeta| \wedge 1)\rho^{d-1} + c_4\rho^d\,.$$

Dividing by ρ^{d-1} and taking the lim sup as $\rho \to 0+$ we obtain the first inequality in (5.38). The inequality for g_ε is obtained in a similar way. □

The following lemma proves the symmetry property of g.

Lemma 5.15. *Let $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. Then*

$$g(x,-\zeta,-\nu) = g(x,\zeta,\nu)\,. \tag{5.39}$$

Proof. Since $Q_\nu(x,\rho) = Q_{-\nu}(x,\rho)$ and $\zeta + u_{x,-\zeta,-\nu} = u_{x,\zeta,\nu}$, the conclusion follows from (3.6) and from the definition of g. □

The following result shows the monotonicity of g with respect to ζ.

Lemma 5.16. *Let $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}$, and $\nu \in \mathbb{S}^{d-1}$. Assume that $0 \leq \zeta_1 \leq \zeta_2$ or that $\zeta_2 \leq \zeta_1 \leq 0$. Then*

$$g(x,\zeta_1,\nu) \leq g(x,\zeta_2,\nu)\,. \tag{5.40}$$

Proof. We prove the result when $0 \leq \zeta_1 \leq \zeta_2$, the other case being analogous. Let us fix $\rho > 0$. By the definition of m^E there exists $u_2 \in BV(Q_\nu(x,\rho))$ such that $\mathrm{tr}_{Q_\nu(x,\rho)}u_2 = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta_2,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$ and $E(u_2, Q_\nu(x,\rho)) \leq m^E(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \rho^d$. Let $u_1 := u_2^{(\zeta_1)}$. Since $\mathrm{tr}_{Q_\nu(x,\rho)}u_1 = \mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta_1,\nu}$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q_\nu(x,\rho)$, by (3.10) we have

$$\begin{aligned} m^E(u_{x,\zeta_1,\nu}, Q_\nu(x,\rho)) &\leq E(u_1, Q_\nu(x,\rho)) \leq E(u_2, Q_\nu(x,\rho)) + c_4\rho^d \\ &\leq m^E(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + (c_4+1)\rho^d\,. \end{aligned}$$

Dividing by ρ^{d-1} and taking the limsup as $\rho \to 0+$ we obtain (5.40). □

We conclude this section by showing that the previous lemmas prove all properties mentioned in Theorem 5.1.

Proof of Theorem 5.1. Property (f1) for f and f_ε is proved in Corollary 5.5. Properties (f2), (f3), and (f4), for f and the analogous properties for f_ε are proved in Lemmas 5.6, 5.8, and 5.4, respectively. Hence $f \in \mathcal{F}$ and taking the limit as $\varepsilon \to 0+$ we obtain also that $\hat{f} \in \mathcal{F}$.

Property (g1) for g and g_ε is proved in Corollary 5.12. Properties (g2), (g3), (g4), (g5), and (g6) for g are proved in Lemmas 5.13, 5.14, 5.11, 5.15, and 5.16, respectively. □

6. Integral representation

In this section we shall prove first an integral representation on $GBV_\star$ of the functionals E^a and E^j defined in (4.1) and (4.4). The full integral representation for E requires an additional hypothesis (see (6.16) below) and will be obtained at the end of the section. We begin with an integral representation on BV for the functionals E^a_ε and E^j_ε introduced in Definition 4.9.

Theorem 6.1. *Let $E \in \mathfrak{E}_{sc}$, let $A \in \mathcal{A}_c(\mathbb{R}^d)$, and let $\varepsilon > 0$. Let $E^a_\varepsilon, E^j_\varepsilon \colon BV(A) \times \mathcal{B}(A) \to [0, +\infty)$ be the functionals introduced in Definition 4.9 and let f_ε and g_ε be the integrands introduced in Definition 4.13. Then*

$$E^a_\varepsilon(u, B) = \int_B f_\varepsilon(x, \nabla u) dx \,, \tag{6.1}$$

$$E^j_\varepsilon(u, B) = \int_{B \cap J_u} g_\varepsilon(x, [u], \nu_u) d\mathcal{H}^{d-1} \,, \tag{6.2}$$

for every $u \in BV(A)$ and every $B \in \mathcal{B}(A)$.

Proof. By (a), (b), and (c2) in Definition 3.1 and by Remarks 3.14 and 4.8 the functional E_ε satisfies all hypotheses of [4, Theorem 3.7]. In the proof of that theorem, recalling also [4, Remark 3.8(1)], it is shown that for every $u \in BV(A)$

$$\frac{dE_\varepsilon(u, \cdot)}{d\mathcal{L}^d}(x) = f_\varepsilon(x, \nabla u(x)) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \,, \tag{6.3}$$

where

$$\frac{dE_\varepsilon(u, \cdot)}{d\mathcal{L}^d}(x) := \lim_{\rho \to 0+} \frac{E_\varepsilon(u, Q(x, \rho))}{\rho^d} \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d$$

and f_ε is obtained using (4.24). By the differentiation theory for Radon measures $\frac{dE_\varepsilon(u,\cdot)}{d\mathcal{L}^d}$ is the density of $E^a_\varepsilon(u, \cdot)$ with respect to $\mathcal{L}^d$. Therefore, by integration (6.3) gives (6.1).

Moreover, in the proof of [4, Theorem 3.7] it is shown also that for every $u \in BV(A)$ with $\mathcal{H}^{d-1}(J_u) < +\infty$

$$\frac{dE_\varepsilon(u, \cdot)}{d\mathcal{H}^{d-1} \llcorner J_u}(x) = g_\varepsilon(x, [u](x), \nu_u(x)) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J_u \,, \tag{6.4}$$

where

$$\frac{dE_\varepsilon(u, \cdot)}{d\mathcal{H}^{d-1} \llcorner J_u}(x) := \lim_{\rho \to 0+} \frac{E_\varepsilon(u, Q_{\nu_u(x)}(x, \rho))}{\rho^{d-1}} \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J_u \tag{6.5}$$

and g_ε is obtained using (4.25). To prove (6.2) let us fix $\eta > 0$ and consider $J^\eta_u := \{x \in J_u : |[u](x)| \geq \eta\}$. Then, from [2, (3.90)] it follows easily that $\mathcal{H}^{d-1}(J^\eta_u) < +\infty$. Since J^η_u is also $(\mathcal{H}^{d-1}, d-1)$-countably rectifiable (see [2, Theorem 3.78]), we infer that

$$\lim_{\rho \to 0+} \frac{\mathcal{H}^{d-1}(J^\eta_u \cap Q_{\nu(x)}(x, \rho))}{\rho^{d-1}} = 1 \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J^\eta_u \tag{6.6}$$

(see [12, Theorem 3.2.19]). From (6.4)-(6.6) we get

$$\lim_{\rho \to 0+} \frac{E_\varepsilon(u, Q_{\nu_u(x)}(x, \rho))}{\mathcal{H}^{d-1} \llcorner J^\eta_u(Q_{\nu_u(x)})} = g_\varepsilon(x, [u](x), \nu_u(x)) \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J^\eta_u \,.$$

We observe that the absolutely continuous part of $E_\varepsilon(u, \cdot)$ with respect to $\mathcal{H}^{d-1} \llcorner J^\eta_u$ coincides on J^η_u with the measure $E^j_\varepsilon(u, \cdot)$ introduced in Definition 4.9. Therefore, by a general version of the differentiation theory for Radon measures based on Morse's covering theorem (see [17] and [13, Sections 1.2.1-1.2.2]) we obtain that $g_\varepsilon(x, [u], \nu_u)$ is the density of $E^j_\varepsilon(u, \cdot)$ on J^η_u. Integrating, we obtain (6.2) for every Borel set $B \subset J^\eta_u$. The case of a general B can be obtained passing to the limit as $\eta \to 0+$. $\square$

Let $\hat{f}\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,+\infty)$ be defined by (5.1). By Theorem 5.1 $\hat{f} \in \mathcal{F}$. We now prove an integral representation result for E^a and E^j on $BV(A)$.

Theorem 6.2. *Let $E \in \mathfrak{E}_{sc}$, let $A \in \mathcal{A}_c(\mathbb{R}^d)$, and let $\hat{f}$ and g be defined by* (5.1) *and* (4.23)*, respectively. Then*

$$E^a(u,B) = \int_B \hat{f}(x,\nabla u)dx\,, \tag{6.7}$$

$$E^j(u,B) = \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,, \tag{6.8}$$

for every $u \in BV(A)$ and every $B \in \mathcal{B}(A)$.

Proof. To prove (6.7) we observe that $E^a_\varepsilon(u,B) = E^a(u,B) + \varepsilon\int_B |\nabla u|dx$ for every $u \in BV(A)$ and every $B \in \mathcal{B}(A)$, hence

$$E^a(u,B) = \lim_{\varepsilon\to 0+} E^a_\varepsilon(u,B) = \inf_{\varepsilon>0} E^a_\varepsilon(u,B)\,.$$

Therefore (6.7) follows from (5.1) and (6.1), recalling the upper bound (5.17) for f_ε.

Notice also that $E^j_\varepsilon(u,B) = E^j(u,B) + \varepsilon\int_{B\cap J_u} |[u]|d\mathcal{H}^{d-1}$, hence

$$E^j(u,B) = \lim_{\varepsilon\to 0+} E^j_\varepsilon(u,B) = \inf_{\varepsilon>0} E^j_\varepsilon(u,B)\,.$$

Therefore (6.8) follows from (4.53) and (6.2), recalling the upper bound (5.38) for g_ε. $\square$

We are now in a position to provide an integral representation result for E^a and E^j on $GBV_\star(A)$.

Theorem 6.3. *Let $E \in \mathfrak{E}_{sc}$, let f and g be defined by* (4.22) *and* (4.23)*, respectively, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. Then*

$$E^a(u,B) = \int_B f(x,\nabla u)dx\,, \tag{6.9}$$

$$E^j(u,B) = \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,, \tag{6.10}$$

for every $u \in GBV_\star(A)$ and every $B \in \mathcal{B}(A)$.

Proof. Let us fix $u \in GBV_\star(A)$ and $B \in \mathcal{B}(A)$. For every $m > 0$ we have $u^{(m)} \in BV(A)$. By Theorem 6.2 we can apply Proposition 4.15 and we obtain that the function $\hat{f}$ defined by (5.1) coincides with the function f defined by (4.22). Therefore, Theorem 6.2 gives

$$E^a(u^{(m)},B) = \int_B f(x,\nabla u^{(m)})dx \tag{6.11}$$

$$E^j(u^{(m)},B) = \int_{B\cap J_{u^{(m)}}} g(x,[u^{[m]}],\nu_{u^{(m)}})d\mathcal{H}^{d-1} \tag{6.12}$$

We pass to the limit in the left-hand side of (6.11) as $m \to +\infty$ thanks to (4.9). As for the right-hand side, by Theorem 2.2(b) we have

$$\int_B f(x,\nabla u^{(m)})dx = \int_{B\cap\{|u|\le m\}} f(x,\nabla u)dx + \int_{B\cap\{|u|>m\}} f(x,0)dx \to \int_B f(x,\nabla u)dx\,,$$

where we used the fact that u has finite values and f satisfies (f3).

In (6.12) we pass to the limit in the left-hand side by (4.12). To deal with the right-hand side we note that by Theorem 2.2(d) and (g4) we have

$$g(x,[u^{(m)}],\nu_{u^{(m)}})1_{J_{u^{(m)}}} \to g(x,[u],\nu_u) \quad \mathcal{H}^{d-1}\text{-a.e. in } J_u$$

and from (g3) we obtain $g(x,[u^{(m)}],\nu_{u^{(m)}}) \leq c_3(|[u^{(m)}]| \wedge 1) \leq c_3(|[u]| \wedge 1)$ $\mathcal{H}^{d-1}$-a.e. in J_u. Since $u \in GBV_\star(A)$, we have $c_3 \int_{J_u} |[u]| \wedge 1 d\mathcal{H}^{d-1} < +\infty$. Hence we can apply the Lebesgue Dominated Convergence Theorem and we obtain

$$\int_{B\cap J_{u^{(m)}}} g(x,[u^{(m)}],\nu_{u^{(m)}})d\mathcal{H}^{d-1} \to \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,.$$

This shows that the right-hand side of (6.12) converges to the the right-hand side of (6.10) and concludes the proof. □

As a consequence of the integral representation of E^a we obtain the convexity of f with respect to ξ.

Corollary 6.4. *Under the assumptions of Theorem 6.3 for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ the function $\xi \mapsto f(x,\xi)$ is convex on $\mathbb{R}^d$.*

Proof. Since $E \in \mathfrak{E}_{sc}$, for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the functional $u \mapsto E(u,A)$ is lower semicontinuous in $GBV_\star(A)$ with respect to convergence in $L^0(\mathbb{R}^d)$, hence it is lower semicontinuous in $W^{1,1}(A)$ with respect to the weak convergence of $W^{1,1}(A)$. The integral representation (6.9) implies that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the functional $u \mapsto \int_A f(x,\nabla u)dx$ is lower semicontinuous in $W^{1,1}(A)$ with respect to the weak convergence of $W^{1,1}(A)$. The convexity of $\xi \mapsto f(x,\xi)$ for $\mathcal{L}^d$-a.e. $x \in A$ follows from a well known property of the integrands of lower semicontinuous functionals on $W^{1,1}(A)$ (see, e.g., [16]). The arbitrariness of $A \in \mathcal{A}_c(\mathbb{R}^d)$ allows us to conclude the proof. □

We consider now the problem of a full integral representation for E, which includes its Cantor part E^c. To this end we assume that the functional $E \in \mathfrak{E}_{sc}$ satisfies an additional property which is clearly satisfied whenever E is invariant under translations.

Definition 6.5 (Translation operators)**.** For every $z \in \mathbb{R}^d$ we set

$$\tau_z x := x + z \quad \text{for every } x \in \mathbb{R}^d\,, \tag{6.13}$$

$$\tau_z B := B + z = \{x + z : x \in B\} \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d)\,. \tag{6.14}$$

Given $v \in L^0(\mathbb{R}^d)$ we define $\tau_z v \in L^0(\mathbb{R}^d)$ by

$$\tau_z v(x) := v(x - z) \qquad \text{for every } x \in \mathbb{R}^d\,. \tag{6.15}$$

Note that if $u \in GBV_\star(A)$ for some $A \in \mathcal{A}_c(\mathbb{R}^d)$, then $\tau_z u \in GBV_\star(\tau_z A)$.

The following proposition shows that the functions f and g defined by (4.22) and (4.23) are continuous with respect to x when $E \in \mathfrak{E}_{sc}$ satisfies a continuity estimate with respect to translations.

Proposition 6.6. *Let $E \in \mathfrak{E}_{sc}$. Assume that there exists a modulus of continuity ω such that*

$$|E(\tau_z u,\tau_z A) - E(u,A)| \leq \omega(|z|)(E(\tau_z u,\tau_z A) + E(u,A) + \mathcal{L}^d(A)) \tag{6.16}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$, $u \in GBV_\star(A)$, and $z \in \mathbb{R}^d$. Let f and g be defined by (4.22) *and* (4.23)*, respectively. Then f is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and $\xi \mapsto f(x,\xi)$ is convex on $\mathbb{R}^d$ for every $x \in \mathbb{R}^d$, while $(x,\zeta) \mapsto g(x,\zeta,\nu)$ is continuous on $\mathbb{R}^d \times \mathbb{R}$ for every $\nu \in \mathbb{S}^{d-1}$. Moreover the recession function f^∞ defined by* (3.11) *is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and*

$$|f(x,\xi) - f(y,\xi)| \leq \omega(|x-y|)(f(x,\xi) + f(y,\xi) + 1) \tag{6.17}$$

$$|g(x,\zeta,\nu) - g(y,\zeta,\nu)| \leq \omega(|x-y|)(g(x,\zeta,\nu) + g(y,\zeta,\nu)) \tag{6.18}$$

$$|f^\infty(x,\xi) - f^\infty(y,\xi)| \leq \omega(|x-y|)(f^\infty(x,\xi) + f^\infty(y,\xi))\,, \tag{6.19}$$

for every $x,y \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\zeta \in \mathbb{R}$ and $\nu \in \mathbb{S}^{d-1}$. Finally, for every $\varepsilon > 0$ all these properties are satisfied by the functions f_ε and g_ε defined by (4.24) *and* (4.25)*.*

Proof. Exchanging the roles of x and y we see that (6.17) and (6.18) are equivalent to

$$f(x,\xi) \le f(y,\xi) + \omega(|x-y|)(f(x,\xi) + f(y,\xi) + 1)\,,$$
$$g(x,\zeta,\nu) \le g(y,\zeta,\nu) + \omega(|x-y|)(g(x,\zeta,\nu) + g(y,\zeta,\nu))\,,$$

which follow immediately from (6.16) and the definitions of f and g.

By Corollary 6.4 for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ the function $\xi \mapsto f(x,\xi)$ is convex. From (6.17) we deduce that this property holds for every $x \in \mathbb{R}^d$. Since f satisfies (f4) and g satisfies (g4) by Theorem 5.1, the continuity of f follows from (6.17) and the continuity of $(x,\zeta) \mapsto g(x,\zeta,\nu)$ follows from (6.18). Inequality (6.19) is an elementary consequence of (3.11) and (6.17).

The properties of f_ε and g_ε are proved in the same way. □

We are now in a position to state the main result of this section: the integral representation of $E(u,A)$ for $u \in GBV_\star(A)$.

Theorem 6.7. *Let $E \in \mathfrak{E}_{sc}$ and let f, f^∞, and g be defined by* (4.22), (3.11), *and* (4.23), *respectively. Assume that E satisfies property* (6.16). *Then $E = E^{f,g}$. In particular, for every $A \in A_c(\mathbb{R}^d)$ we have*

$$E(u,B) = \int_B f(x,\nabla u)dx + \int_B f^\infty\Big(x, \frac{dD^cu}{d|D^cu|}\Big)d|D^cu| + \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1} \quad (6.20)$$

for every $u \in GBV_\star(A)$ and every $B \in \mathcal{B}(A)$.

Proof. By Definitions 3.1 and 3.10 it is enough to prove (6.20). Let us fix $A \in A_c(\mathbb{R}^d)$. Since, by Remark 4.2 and Theorem 6.3, we have

$$E(u,B) = \int_B f(x,\nabla u)dx + E^c(u,B) + \int_{B\cap J_u} g(x,[u],\nu_u)d\mathcal{H}^{d-1}\,, \quad (6.21)$$

in order to complete the proof it remains to show that

$$E^c(u,B) = \int_B f^\infty\Big(x, \frac{dD^cu}{d|D^cu|}\Big)d|D^cu| \quad (6.22)$$

for every $u \in GBV_\star(A)$ and every $B \in \mathcal{B}(A)$.

Let us first consider $\varepsilon > 0$ and E_ε and f_ε defined by (4.16) and (4.24). We now prove that the Cantor part E^c_ε of E_ε satisfies

$$E^c_\varepsilon(u,B) = \int_B f^\infty_\varepsilon\Big(x, \frac{dD^cu}{d|D^cu|}\Big)d|D^cu| \quad (6.23)$$

for every $u \in BV(A)$ and $B \in \mathcal{B}(A)$.

To this end, let $z \in \mathbb{R}^d$. By Definitions 4.7 and 6.5, for every $\varepsilon > 0$ we have $E_\varepsilon(\tau_z u, \tau_z A) - E_\varepsilon(u,A) = E(\tau_z u,\tau_z A) - E(u,A)$, hence E_ε satisfies (6.16). By (a), (b), and (c2)in Definition 3.1 and by Remarks 3.14 and 4.8 the functional E_ε defined by (4.16) satisfies the hypotheses of [4, Theorem 3.12]. Therefore the integral representation formula (6.23) holds.

By (4.3) and Definition 4.9 for every $u \in BV(A)$ and every $B \in \mathcal{B}(A)$ we have

$$E^c(u,B) = \lim_{\varepsilon\to 0+} E^c_\varepsilon(u,B)\,. \quad (6.24)$$

By (4.29) for $\mathcal{L}^d$-a.e. $x \in A$ we have

$$f^\infty(x,\xi) = \lim_{\varepsilon\to 0+} f^\infty_\varepsilon(x,\xi) \quad \text{for every } \xi \in \mathbb{R}^d\,. \quad (6.25)$$

By Proposition 6.6 this property holds for every $x \in A$. By (5.17) we have

$$0 \le f^\infty_\varepsilon(x,\xi) \le (c_3+\varepsilon)|\xi| \quad \text{for every } x \in \mathbb{R}^d\,. \quad (6.26)$$

Equality (6.22) for $u \in BV(A)$ and $B \in \mathcal{B}(A)$ is obtained by passing to the limit as $\varepsilon \to 0+$ in (6.23), using (6.24) for the left-hand side and using (6.25) for the right-hand side, observing that we can apply the Dominated Convergence Theorem by (6.26).

Let us fix $u \in GBV_\star(A)$ and $B \in \mathcal{B}(A)$. For every $m > 0$ by the previous step we have

$$E^c(u^{(m)}, B) = \int_B f^\infty\Big(x, \frac{dD^c u^{(m)}}{d|D^c u^{(m)}|}\Big) d|D^c u^{(m)}| \,. \tag{6.27}$$

By Theorem 2.2(c) the measures $D^c u^{(m)}$ and $D^c u$ coincide on all Borel subsets of $\{|\tilde u| \le m\}$, while by Lemma 2.4 the measure $|D^c u^{(m)}|$ vanishes on all Borel subsets of A that do not intersect $\{|\tilde u| \le m\}$. This implies that the integral on the right-hand side of (6.27) coincides with

$$\int_{B \cap \{|\tilde u| \le m\}} f^\infty\Big(x, \frac{dD^c u}{d|D^c u|}\Big) d|D^c u|$$

which converges to

$$\int_B f^\infty\Big(x, \frac{dD^c u}{d|D^c u|}\Big) d|D^c u|$$

since $\tilde u$ is finite $|D^c u|$-a.e. in A by Theorem 2.2(a).

Thanks to (4.11) we can pass to the limit also in the left-hand side of (6.27) and this gives (6.22) for $u \in GBV_\star(A)$ and $B \in \mathcal{B}(A)$. □

7. Convergence of minima

We conclude the paper with two results concerning the convergence of minimum values of some minimum problems related to the functional $E^{f,g}$.

7.1. **Convergence of absolute minimisers.**

In this subsection we fix $\Omega \in \mathcal{A}_c(\mathbb{R}^d)$ and a Carathéodory function $\psi \colon \Omega \times \mathbb{R} \to \mathbb{R}$. We assume that there exist $p \ge 1$, $a_1 > 0$, $a_2 \ge 0$, $a_3 > 0$, and $a_4 \ge 0$ such that

$$a_1 |s|^p - a_2 \le \psi(x,s) \le a_3 |s|^p + a_4 \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}\,, \tag{7.1}$$

and we define $\Psi \colon L^p(\Omega) \to \mathbb{R}$ by $\Psi(u) := \int_\Omega \psi(x,u)dx$ for every $u \in L^p(\Omega)$.

The following theorem shows the convergence of minima of $E_k(\cdot,\Omega) + \Psi$ for a Γ-convergent sequence E_k in $\mathfrak{E}$.

Theorem 7.1. *Let E_k be a sequence in $\mathfrak{E}$ and let $E \in \mathfrak{E}$. Assume that $E_k(\cdot,\Omega)$ Γ-converges to $E(\cdot,\Omega)$ with respect to the topology of $L^0(\Omega)$. Then*

(a) *the minimum problem*

$$\min_{v \in GBV_\star(\Omega) \cap L^p(\Omega)} \big(E(v,\Omega) + \Psi(v)\big) \tag{7.2}$$

has a solution;

(b) *we have*

$$\min_{v \in GBV_\star(\Omega) \cap L^p(\Omega)} \big(E(v,\Omega) + \Psi(v)\big) = \lim_{k\to\infty} \inf_{v \in GBV_\star(\Omega) \cap L^p(\Omega)} \big(E_k(v,\Omega) + \Psi(v)\big)\,; \tag{7.3}$$

(c) *if u_k is a sequence in $GBV_\star(\Omega) \cap L^p(\Omega)$ such that*

$$E_k(u_k,\Omega) + \Psi(u_k) \le \inf_{v \in GBV_\star(\Omega) \cap L^p(\Omega)} \big(E_k(v,\Omega) + \Psi(v)\big) + \varepsilon_k, \tag{7.4}$$

for some sequence $\varepsilon_k \to 0$, then there exist a subsequence of u_k, not relabelled, that converges in $L^p(\Omega)$ to a minimum point u of (7.2).

To prove the theorem we use the following result.

Lemma 7.2. *Under the assumptions of Theorem 7.1, for every $z \in GBV_\star(\Omega) \cap L^p(\Omega)$ there exists a sequence $z_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that*

$$z_k \to z \quad \textit{in } L^p(\Omega)\,, \tag{7.5}$$

$$\limsup_{k\to\infty} E_k(z_k,\Omega) \le E(z,\Omega)\,. \tag{7.6}$$

Proof. Let us fix $z \in GBV_\star(\Omega) \cap L^p(\Omega)$. By Γ-convergence and by Lemma 3.18 for every $m \in \mathbb{N}$ there exists a sequence $z_k^m \in BV(\Omega) \cap L^\infty(\Omega)$ with $\|z_k^m\|_{L^\infty(\Omega)} \leq m+1$, converging to $z^{(m)}$ in $L^0(\Omega)$ such that

$$\limsup_{k\to\infty} E_k(z_k^m, \Omega) \leq E(z^{(m)}, \Omega)\,.$$

Therefore, for every $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that for every $k \geq k_m$ we have

$$E_k(z_k^m, \Omega) \leq E(z^{(m)}, \Omega) + \tfrac{1}{m} \leq E(z, \Omega) + c_4 \mathcal{L}^d(\{|z| \geq m\}) + \tfrac{1}{m},$$
$$\|z_k^m - z^{(m)}\|_{L^p(\Omega)} \leq \tfrac{1}{m}\,,$$

where we used also Remark 3.4. It is not restrictive to assume that k_m is strictly increasing with respect to m. Therefore, setting $z_k := z_k^m$ for $k_m \leq k < k_{m+1}$ we have

$$E_k(z_k, \Omega) \leq E(z, \Omega) + c_4 \mathcal{L}^d(\{|z| \geq m\}) + \tfrac{1}{m}\,,$$
$$\|z_k - z\|_{L^p(\Omega)} \leq \|z^{(m)} - z\|_{L^p(\Omega)} + \tfrac{1}{m}\,,$$

for $k \geq k_m$. Since $z^{(m)} \to z$ in $L^p(\Omega)$, we conclude that (7.5) and (7.6) hold. ☐

Proof of Theorem 7.1. Let v_k be a minimizing sequence of (7.2). By (c2) in Definition 3.1 and (7.1), the sequence v_k is bounded in $L^p(\Omega)$ and $V(v_k, \Omega)$ is bounded. By the compactness theorem in $GBV_\star$, proved in [8, Theorem 3.11], there exist a subsequence, not relabelled, and a function $v_0 \in GBV_\star(\Omega)$ such that $v_k \to v_0$ in $L^0(\Omega)$. The boundedness of v_k in $L^p(\Omega)$ implies that $v_0 \in L^p(\Omega)$. Since $E(\cdot, \Omega)$ is a Γ-limit, it is lower semicontinuous with respect to the topology of $L^0(\Omega)$ (see [7, Proposition 6.8]), hence $E(v_0, \Omega) \leq \liminf_{k\to\infty} E(v_k, \Omega)$. By the Fatou Lemma we have also $\Psi(v_0) \leq \liminf_{k\to\infty} \Psi(v_k)$. These inequalities lead to

$$E(v_0, \Omega) + \Psi(v_0) \leq \lim_{k\to\infty} \big(E(v_k, \Omega) + \Psi(v_k)\big) = \inf_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E(v, \Omega) + \Psi(v)\big)\,.$$

This proves that v_0 is a minimiser of $E(\cdot, \Omega) + \Psi(\cdot)$ and concludes the proof of (a).

Let us prove that

$$\min_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E(v, \Omega) + \Psi(v)\big) \;\geq\; \limsup_{k\to\infty} \inf_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E_k(v, \Omega) + \Psi(v)\big). \tag{7.7}$$

Let $z \in GBV_\star(\Omega) \cap L^p(\Omega)$ be a minimiser of (7.2). By Lemma 7.2 there exists a sequence $z_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that $z_k \to z$ in $L^p(\Omega)$ and

$$E(z, \Omega) \geq \limsup_{k\to\infty} E_k(z_k, \Omega)\,.$$

The continuity of Ψ on $L^p(\Omega)$ gives

$$E(z, \Omega) + \Psi(z) \geq \limsup_{k\to\infty} \big(E_k(z_k, \Omega) + \Psi(z_k)\big)\,.$$

Since the left-hand side of the previous equality coincides with the left-hand side of (7.7), while the right-hand side of the previous equality is greater than or equal to the right-hand side of (7.7), we conclude that (7.7) holds.

To complete the proof of (7.3) it remains to show that

$$\min_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E(v, \Omega) + \Psi(v)\big) \;\leq\; \liminf_{k\to\infty} \inf_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E_k(v, \Omega) + \Psi(v)\big). \tag{7.8}$$

Passing to a subsequence, not relabelled, we may assume that the lim inf in the right-hand side is a limit, which is finite by (7.7).

Let u_k be a sequence in $GBV_\star(\Omega) \cap L^p(\Omega)$ satisfying (7.4). Then

$$\lim_{k\to\infty} \big(E_k(u_k, A) + \Psi(u_k)\big) = \lim_{k\to\infty} \inf_{v \in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E_k(v, \Omega) + \Psi(v)\big). \tag{7.9}$$

By (c2) in Definition 3.1 and (7.1), the sequence u_k is bounded in $L^p(\Omega)$ and $V(u_k, \Omega)$ is bounded. By the compactness theorem in $GBV_\star$, proved in [8, Theorem 3.11], there exist a subsequence, not relabelled, and a function $u \in GBV_\star(\Omega)$ such that $u_k \to u$ in $L^0(\Omega)$. Since

u_k is bounded in $L^p(\Omega)$ we deduce that $u \in L^p(\Omega)$. By Γ-convergence we have $E(u,\Omega) \leq \liminf_{k\to\infty} E_k(u_k,\Omega)$. By the Fatou Lemma we have also $\Psi(u) \leq \liminf_{k\to\infty} \Psi(u_k)$, hence

$$E(u,\Omega) + \Psi(u) \leq \lim_{k\to\infty} \big(E_k(u_k,\Omega) + \Psi(u_k)\big) = \min_{v\in GBV_\star(\Omega)\cap L^p(\Omega)} \big(E(v,\Omega) + \Psi(v)\big)\,.$$

This inequality together with (7.9) proves (7.8), and concludes the proof of (b). Moreover, it shows that u is a minimiser of $E(\cdot,\Omega) + \Psi(\cdot)$.

To complete the proof of (c) it remains to show that $u_k \to u$ in $L^p(\Omega)$. We observe that the minimum property of u, together with (7.3) and (7.4), implies that

$$E(u,\Omega) + \Psi(u) = \lim_{k\to\infty} \big(E_k(u_k,\Omega) + \Psi(u_k)\big)\,.$$

Since $E(u,\Omega) \leq \liminf_{k\to\infty} E_k(u_k,\Omega)$ and $\Psi(u) \leq \liminf_{k\to\infty} \Psi(u_k)$ we deduce that

$$\Psi(u) = \lim_{k\to\infty} \Psi(u_k)\,. \tag{7.10}$$

Since $|u_k - u|^p \leq 2^{p-1}|u_k|^p + 2^{p-1}|u|^p \leq (2^{p-1}/a_1)(\psi(x,u_k) + a_2) + 2^{p-1}|u|^p$ and $\psi(x,u_k) \to \psi(x,u)$ in measure, by (7.10) we can apply the generalized version of the Dominated Convergence Theorem and we obtain $u_k - u \to 0$ in $L^p(\Omega)$, which concludes the proof of (c).

□

7.2. **Dirichlet boundary condition.** In this subsection we fix $\Omega \in \mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundary and a function $\varphi \in L^1(\partial\Omega)$. Given a functional $E \in \mathfrak{E}$, the naive formulation of the minimum problem with Dirichlet boundary condition is

$$\min_{\substack{v\in GBV_\star(\Omega)\\ \mathrm{tr}_\Omega v = \varphi\ \mathcal{H}^{d-1}\text{-a.e. on } \partial\Omega}} E(v,\Omega)\,,$$

where $\mathrm{tr}_\Omega v$ is the trace on $\partial\Omega$ defined in Theorem 2.2(e). It is known that, since the functional $E(\cdot,\Omega)$ has linear growth, this problem has in general no solution, even if $E \in \mathfrak{E}_{sc}$. Simple examples of nonexistence are known for the functional V, even when φ is smooth. The usual way to overcome this difficulty is to replace the condition $\mathrm{tr}_\Omega v = \varphi$ $\mathcal{H}^{d-1}$-a.e. on $\partial\Omega$ by a penalization term. The most common one leads to the following minimum problem

$$\min_{v\in GBV_\star(\Omega)} \Big(E(v,\Omega) + \tilde c \int_{\partial\Omega} |\mathrm{tr}_\Omega v - \varphi| \wedge 1 d\mathcal{H}^{d-1}\Big)\,, \tag{7.11}$$

where $\tilde c$ is a positive constant.

To study this problem we fix a set $\tilde\Omega \in A_c(\mathbb{R}^d)$ with $\Omega \subset\subset \tilde\Omega$, and a function $w \in W^{1,1}(\tilde\Omega)$ such that φ is the trace of w on $\partial\Omega$, whose existence is granted by Gagliardo's Theorem [15]. Problem (7.11) is equivalent to

$$\min_{\substack{v\in GBV_\star(\tilde\Omega)\\ v = w\ \mathcal{L}^d\text{-a.e. in } \tilde\Omega\setminus\Omega}} \tilde E(v,\tilde\Omega)\,, \tag{7.12}$$

where $\tilde E$ is given by the following definition.

Definition 7.3. Given a constant $\tilde c > 0$ and a functional $E \in \mathfrak{E}$, let $\tilde E\colon L^0(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ be the functional defined by

$$\tilde E(u,B) = E(u, B\cap\Omega) + \tilde c\, V(u, B\setminus\Omega)\,.$$

In the rest of the paper we fix a constant $\tilde c$ with $c_1 \leq \tilde c \leq c_3$, so that $\tilde E \in \mathfrak{E}$ as we shall see in Proposition 7.5 below.

Remark 7.4. Problems (7.11) and (7.12) are equivalent in the following sense: the quasi-minimisers are the same, i.e., for every $\varepsilon > 0$ a function $u \in GBV_\star(\tilde\Omega)$ with $u = w$ $\mathcal{L}^d$-a.e. in $\tilde\Omega\setminus\Omega$ satisfies

$$\tilde E(u,\tilde\Omega) \leq \inf_{\substack{v\in GBV_\star(\tilde\Omega)\\ v = w\ \mathcal{L}^d\text{-a.e. in } \tilde\Omega\setminus\Omega}} \tilde E(v,\tilde\Omega) + \varepsilon$$

if and only if the restriction of u to Ω belongs to $GBV_\star(\Omega)$ and

$$E(u,\Omega)+\tilde{c}\int_{\partial\Omega}|\mathrm{tr}_\Omega u-\varphi|\wedge 1d\mathcal{H}^{d-1}\leq \inf_{v\in GBV_\star(\Omega)}\Big(E(v,\Omega)+\tilde{c}\int_{\partial\Omega}|\mathrm{tr}_\Omega v-\varphi|\wedge 1d\mathcal{H}^{d-1}\Big)+\varepsilon\,.$$

Moreover, the infima of the two problems differ by the constant $\tilde{c}V(w,\tilde{\Omega}\setminus\overline{\Omega})$.

In this section we consider a sequence of functionals in $\mathfrak{E}$ of the form E^{f_k,g_k} introduced in Definition 3.10, with $f_k\in\mathcal{F}$ and $g_k\in\mathcal{G}$, and we study the asymptotic behaviour of the minimum problems

$$\min_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in }\tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_k,g_k}(v,\tilde{\Omega})\,. \tag{7.13}$$

Applying the Compactness Theorem 3.16 to the sequence $\tilde{E}^{f_k,g_k}$, we can find a subsequence, not relabelled, and a functional $\hat{E}\in\mathfrak{E}_{sc}$ such that for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ the sequence $\tilde{E}^{f_k,g_k}(\cdot,A)$ Γ-converges to $\hat{E}(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Under an additional assumption on g_k, which is always satisfied when g_k is even with respect to ζ, we shall prove in Theorem 7.14 that the minimum problem

$$\min_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in }\tilde{\Omega}\setminus\Omega}} \hat{E}(v,\tilde{\Omega})\,. \tag{7.14}$$

has a solution, that the sequence of infima in (7.13) converge to the minimum value of (7.14), and that there exists a suitable subsequence of quasi-minimisers of (7.13) that converges in $L^0(\tilde{\Omega})$ to a minimiser of (7.14).

We now prepare the technical tools that are used to obtain these results.

Proposition 7.5. *If $E\in\mathfrak{E}$, then $\tilde{E}\in\mathfrak{E}$.*

Proof. Since $V\in\mathfrak{E}$ the locality property (a) in Definition 3.1 follows from Remark 3.2. Recalling (3.1), (3.2), and the inequalities $c_1\leq\tilde{c}\leq c_3$, the other properties in Definition 3.1 are trivial, except (3.3). It is enough to prove it when $\tilde{E}(u,B)<+\infty$. In this case, by (3.3) for E and V, for every $\varepsilon>0$ there exist $A_1,A_2\in\mathcal{A}(\mathbb{R}^d)$ with $B\cap\Omega\subset A_1$ and $B\setminus\Omega\subset A_2$, such that

$$E(u,A_1)\leq E(u,B\cap\Omega)+\varepsilon<+\infty\quad\text{and}\quad V(u,A_2)\leq V(u,B\setminus\Omega)+\varepsilon<+\infty\,.$$

It is not restrictive to assume that $A_1\subset\Omega$. Since $V(u,A_2\cap\Omega)<+\infty$, there exists a compact set $K\subset A_2\cap\Omega$ such that

$$V(u,(A_2\cap\Omega)\setminus K)<\varepsilon\,.$$

Let $A:=A_1\cup(A_2\setminus K)$. Then $A\in\mathcal{A}(\mathbb{R}^d)$, $B\subset A$, and

$$\begin{aligned}\tilde{E}(u,A)&=E(u,A_1\cup((A_2\setminus K)\cap\Omega))+\tilde{c}V(u,A_2\setminus\Omega)\\&\leq E(u,A_1)+c_3V(u,(A_2\cap\Omega)\setminus K)+\tilde{c}V(u,A_2)\\&\leq E(u,B\cap\Omega)+\varepsilon+c_3\varepsilon+\tilde{c}V(u,B\setminus\Omega)+\tilde{c}\varepsilon\\&=\tilde{E}(u,B)+(1+c_3+\tilde{c})\varepsilon\,.\end{aligned}\tag{7.15}$$

By the arbitrariness of ε we obtain (3.3) for $\tilde{E}$. □

Remark 7.6. If $f\in\mathcal{F}$, $g\in\mathcal{G}$, and $E=E^{f,g}$ (see Definition 3.10), then $\tilde{E}^{f,g}=E^{\tilde{f},\tilde{g}}$, where

$$\tilde{f}(x,\xi)=\begin{cases}f(x,\xi) & \text{if } x\in\Omega,\ \xi\in\mathbb{R}^d\,,\\ \tilde{c}|\xi| & \text{if } x\in\mathbb{R}^d\setminus\Omega,\ \xi\in\mathbb{R}^d\,,\end{cases}\tag{7.16}$$

$$\tilde{g}(x,\zeta,\nu)=\begin{cases}g(x,\zeta,\nu) & \text{if } x\in\Omega,\ \zeta\in\mathbb{R},\ \nu\in\mathbb{S}^{d-1}\,,\\ \tilde{c}(|\zeta|\wedge 1) & \text{if } x\in\mathbb{R}^d\setminus\Omega,\ \zeta\in\mathbb{R},\ \nu\in\mathbb{S}^{d-1}\,.\end{cases}\tag{7.17}$$

Remark 7.7. Let E_k be a sequence in $\mathfrak{E}$. Then, by Proposition 7.5 and Theorem 3.16 there exist a subsequence, not relabelled, and a functional $\hat{E} \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $\tilde{E}_k(\cdot, A)$ Γ-converges to $\hat{E}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. By the integral representation result in Theorem 6.3 there exists $\hat{g} \in \mathcal{G}$ such that, recalling that φ is the trace of w on $\partial\Omega$, we have

$$\hat{E}(u, A \cap \partial\Omega) = \hat{E}^j(u, A \cap \partial\Omega) = \int_{A \cap \partial\Omega} \hat{g}(x, \varphi - \mathrm{tr}_\Omega u, \nu_\Omega) d\mathcal{H}^{d-1} \tag{7.18}$$

for every $u \in GBV_\star(A)$ with $u = w$ $\mathcal{L}^d$-a.e. in $A \setminus \Omega$, where ν_Ω is the outer normal to Ω. Therefore, since $\hat{E}(u, \tilde{\Omega}) = \hat{E}(u, \Omega) + \hat{E}(u, \partial\Omega) + \hat{E}(w, \tilde{\Omega} \setminus \overline{\Omega})$ and $\hat{E}(w, \tilde{\Omega} \setminus \overline{\Omega}) = \tilde{c}V(w, \tilde{\Omega} \setminus \overline{\Omega})$, we obtain

$$\min_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v = w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega} \setminus \Omega}} \hat{E}(v, \tilde{\Omega})$$

$$= \min_{v \in GBV_\star(\Omega)} \Big(\hat{E}(v, \Omega) + \int_{\partial\Omega} \hat{g}(x, \varphi - \mathrm{tr}_\Omega v, \nu_\Omega) d\mathcal{H}^{d-1} \Big) + \tilde{c}V(w, \tilde{\Omega} \setminus \overline{\Omega}) \,. \tag{7.19}$$

Theorem 7.8. *Let E_k be a sequence in $\mathfrak{E}$. Assume that there exists $\hat{E} \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $\tilde{E}_k(\cdot, A)$ Γ-converges to $\hat{E}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Let $u \in BV(\tilde{\Omega})$ with $u = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$. Then there exists a sequence v_k in $BV(\tilde{\Omega})$, with $v_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$, such that $v_k \to u$ in $L^1(\tilde{\Omega})$ and*

$$\hat{E}(u, \tilde{\Omega}) = \lim_{k \to \infty} \tilde{E}_k(v_k, \tilde{\Omega}) \,. \tag{7.20}$$

The following example shows that in general $\hat{E}$ can not be written as $\tilde{E}$ for some $E \in \mathfrak{E}$.

Example 7.9. Assume that $d = 1$, $\Omega = (-1, 1)$, $\tilde{\Omega} = (-2, 2)$, $c_1 < \tilde{c} \leq c_3$. Let us fix $\hat{c}$ with $c_1 < \hat{c} < \tilde{c}$, and let $f_k(x, \xi) := |\xi|$, and $g_k(x, \zeta, \nu) = a_k(x)(|\zeta| \wedge 1)$, with

$$a_k(x) := \begin{cases} c_1 & \text{if } x \in (1, 1 + \frac{1}{k}), \\ \hat{c} & \text{if } x \in (1 - \frac{1}{k}, 1], \\ c_3 & \text{otherwise.} \end{cases}$$

Then for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E^{f_k, g_k}(\cdot, A)$ Γ-converges to $E^{f,g}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$, where $f(x, \xi) := |\xi|$ and $g(x, \zeta, \nu) = a(x)(|\zeta| \wedge 1)$, with

$$a(x) := \begin{cases} c_1 & \text{if } x = 1, \\ c_3 & \text{otherwise,} \end{cases}$$

while $E^{\tilde{f}_k, \tilde{g}_k}(\cdot, A)$ Γ-converges to $E^{f, \hat{g}}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$, where $\hat{g}(x, \zeta, \nu) = \hat{a}(x)(|\zeta| \wedge 1)$, with

$$\hat{a}(x) := \begin{cases} \hat{c} & \text{if } x = 1, \\ c_3 & \text{otherwise.} \end{cases}$$

Since $\tilde{E}(u, \{1\}) = \tilde{c}(|[u](1)| \wedge 1)$ we deduce that $\hat{E}$ can not be of the form $\tilde{E}$ for some $E \in \mathfrak{E}$.

In the proof of Theorem 7.8 we shall use the following one-dimensional result.

Lemma 7.10. *Let $I = (a, b)$ and $\tilde{I} = (\tilde{a}, \tilde{b})$ be bounded open intervals in $\mathbb{R}$ with $I \subset\subset \tilde{I}$, let $s, t \in \mathbb{R}$, and let $\Psi \colon BV(\tilde{I}) \to [0, +\infty)$ be defined by*

$$\Psi(u) := \int_I |\nabla u| dx + |D^c u|(I) + \sum_{x \in J_u \cap [a,b]} |[u](x)| \wedge 1 = |Du|(I \setminus J_u) + \sum_{x \in J_u \cap [a,b]} |[u](x)| \wedge 1 \,, \tag{7.21}$$

for every $u \in BV(\tilde{I})$*. Then*

$$\inf_{\substack{v \in BV(\tilde{I}) \\ v(a-)=s,\, v(b+)=t}} \Psi(v) \geq |t-s| \wedge 1\,, \tag{7.22}$$

where $v(a-) := \lim_{\rho \to 0+} \frac{1}{\rho}\int_{a-\rho}^{a} v(x)dx$ *and* $v(b+) := \lim_{\rho \to 0+} \frac{1}{\rho}\int_{b}^{b+\rho} v(x)dx$*.*

Proof. It is enough to adapt the proof of Lemma 5.7. □

Proof of Theorem 7.8. We claim that there exists a sequence u_k in $BV(\tilde{\Omega})$ converging to u in $L^1(\tilde{\Omega})$ such that

$$\limsup_{k\to\infty} \tilde{E}_k(u_k, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega})\,. \tag{7.23}$$

Indeed, by the definition of Γ-limit there exists a sequence z_k in $L^0(\mathbb{R}^d)$ converging to u in $L^0(\mathbb{R}^d)$ and such that

$$\hat{E}(u, \tilde{\Omega}) = \lim_{k\to\infty} \tilde{E}_k(z_k, \tilde{\Omega})\,. \tag{7.24}$$

By Remark 3.5 it is not restrictive to assume that $z_k \in GBV_\star(\tilde{\Omega})$ for every $k \in \mathbb{N}$. For every $m > 0$ let $u^m := w + (u-w)^{(m)} = (u \vee (w-m)) \wedge (w+m)$ and $z_k^m := w + (z_k - w)^{(m)} = (z_k \vee (w-m)) \wedge (w+m)$. Since $GBV_\star(\tilde{\Omega})$ is a vector space, we have $z_k - w \in GBV_\star(\tilde{\Omega})$, hence $(z_k - w)^{(m)} \in BV(\tilde{\Omega})$, which gives $z_k^m \in BV(\tilde{\Omega})$. Moreover $z_k^m \to u^m$ in $L^1(\tilde{\Omega})$ as $k \to \infty$. By (g) in Definition 3.1 we have that

$$\tilde{E}_k(z_k^m, \tilde{\Omega}) \leq \tilde{E}(z_k, \tilde{\Omega}) + c_3 \int_{\{|z_k - w| \geq m\}} |\nabla w| dx + c_4 \mathcal{L}^d(\{|z_k - w| \geq m\})\,.$$

Using (7.24) and Fatou Lemma to estimate the last two terms, for every $m > 0$ we obtain

$$\limsup_{k\to\infty} \tilde{E}_k(z_k^m, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega}) + \varepsilon_m\,, \tag{7.25}$$

where

$$\varepsilon_m := c_3 \int_{\{|u-w|>m\}} |\nabla w| dx + c_4 \mathcal{L}^d(\{|u-w| > m\}) \to 0 \quad \text{as } m \to +\infty. \tag{7.26}$$

Inequality (7.25) implies that for every $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that for every $k \geq k_m$ we have $\|z_k^m - u^m\|_{L^1(\tilde{\Omega})} \leq \frac{1}{m}$ and $\tilde{E}_k(z_k^m, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega}) + \varepsilon_m + \frac{1}{m}$. It is not restrictive to assume that $k_m < k_{m+1}$ for every m. For every $k \geq k_1$ we set $u_k := z_k^m$ for $k_m \leq k < k_{m+1}$. Then $\|u_k - u\|_{L^1(\tilde{\Omega})} \leq \frac{1}{m} + \|u^m - u\|_{L^1(\tilde{\Omega})}$ and $\tilde{E}_k(u_k, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega}) + \varepsilon_m + \frac{1}{m}$ for $k_m \leq k < k_{m+1}$. Since $u^m \to u$ in $L^1(\tilde{\Omega})$ as $m \to +\infty$, it follows that $u_k \to u$ in $L^1(\tilde{\Omega})$ as $k \to \infty$, and since $\varepsilon_m \to 0$ we also have $\limsup_{k\to\infty} \tilde{E}_k(u_k, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega})$, which concludes the proof of (7.23).

We now define $v_k \in BV(\tilde{\Omega})$ by

$$v_k = \begin{cases} u_k & \text{in } \Omega\,, \\ w & \text{in } \tilde{\Omega} \setminus \Omega\,, \end{cases} \tag{7.27}$$

and observe that $v_k \to u$ in $L^1(\tilde{\Omega})$. By the definition of Γ-limit

$$\hat{E}(u, \tilde{\Omega}) \leq \liminf_{k\to\infty} \tilde{E}_k(v_k, \tilde{\Omega})\,,$$

hence in order to prove (7.20) we have only to show that

$$\limsup_{k\to\infty} \tilde{E}_k(v_k, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega})\,. \tag{7.28}$$

Recalling (7.23), this will be done by estimating $\tilde{E}_k(v_k, \tilde{\Omega})$ in terms of $\tilde{E}_k(u_k, \tilde{\Omega})$.

Given $0 < \eta < 1$ we fix an open set Ω_1 with C^1 boundary such that $\Omega \subset\subset \Omega_1 \subset\subset \tilde\Omega$ and

$$V(w, \overline{\Omega}_1 \setminus \overline{\Omega}) < \eta\,. \tag{7.29}$$

By the definition of $\tilde E_k$ we have

$$\tilde E_k(v_k, \tilde\Omega) = E_k(u_k, \Omega) + \tilde c\, V(v_k, \partial\Omega) + \tilde c\, V(w, \overline{\Omega}_1 \setminus \overline{\Omega}) + \tilde c\, V(w, \widetilde{\Omega} \setminus \overline{\Omega}_1)\,. \tag{7.30}$$

By the lower semicontinuity of $V(\cdot, \tilde\Omega \setminus \overline{\Omega}_1)$

$$V(w, \tilde\Omega \setminus \overline{\Omega}_1) \le V(u_k, \tilde\Omega \setminus \overline{\Omega}_1) + \delta_k\,, \tag{7.31}$$

for a suitable sequence $\delta_k \to 0+$. We observe that (7.29)-(7.31) give

$$\tilde E_k(v_k, \tilde\Omega) \le \tilde E_k(u_k, \Omega \cup (\widetilde{\Omega} \setminus \overline{\Omega}_1)) + \tilde c\, V(v_k, \partial\Omega) + \tilde c \delta_k + \tilde c \eta\,. \tag{7.32}$$

Therefore it remains to estimate $\tilde c\, V(v_k, \partial\Omega)$ in terms of $\tilde E_k(u_k, \overline{\Omega}_1 \setminus \Omega)$. We proceed first with the case $d = 1$, and then we shall use a slicing argument to obtain the general case.

Case $\mathbf{d = 1}$. Since Ω has Lipschitz boundary it is enough to prove the result when $\Omega = (a, b)$ and $\tilde\Omega = (\tilde a, \tilde b)$. Given $\eta > 0$ we choose $a_1 \in (\tilde a, a)$ and $b_1 \in (b, \tilde b)$ such that

$$\int_{a_1}^{a} |\nabla w| dx < \eta\,, \qquad \int_{b}^{b_1} |\nabla w| dx < \eta\,, \qquad |u_k(a_1) - w(a_1)| < \varepsilon_k\,, \qquad |u_k(b_1) - w(b_1)| < \varepsilon_k\,,$$

with $\varepsilon_k \to 0+$. It is not restrictive to assume that u_k is continuous at a_1 and b_1. We observe that

$$V(v_k, \{a\}) = |u_k(a+) - w(a)| \wedge 1$$

and by Lemma 7.10 we have also

$$|u_k(a+) - u_k(a_1)| \wedge 1 \le V(u_k, [a_1, a])\,.$$

Combining these inequalities we obtain

$$\begin{aligned} V(v_k, \{a\}) &\le V(u_k, [a_1, a]) + |u_k(a_1) - w(a_1)| + |w(a_1) - w(a)| \\ &\le V(u_k, [a_1, a]) + \varepsilon_k + \eta\,. \end{aligned} \tag{7.33}$$

Similarly we prove that

$$V(v_k, \{b\}) \le V(u_k, [b, b_1]) + \varepsilon_k + \eta\,. \tag{7.34}$$

Therefore, by (7.32)-(7.34)

$$\tilde E_k(v_k, (\tilde a, \tilde b)) \le \tilde E_k(u_k, (\tilde a, \tilde b)) + \tilde c(2\varepsilon_k + 3\eta + \delta_k)\,.$$

Passing to the limsup as $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \tilde E_k(v_k, (\tilde a, \tilde b)) \le \limsup_{k\to\infty} \tilde E_k(u_k, (\tilde a, \tilde b)) + 3\tilde c \eta\,, \tag{7.35}$$

and (7.28) follows from (7.23) and the arbitrariness of η, thus concluding the proof in the case $d = 1$. ☐

To deal with the case $d > 1$ we need the following lemma, which provides some useful properties of sets with Lipschitz boundary. We observe that these properties are obvious when the boundary is C^1. For every $B \subset \partial\Omega$, $\nu \in \mathbb{S}^{d-1}$, and $\varepsilon > 0$ we set

$$C^\nu_\varepsilon(B) := \{x + t\nu : x \in B,\ 0 \le t \le \varepsilon\}.$$

Let π^ν denote the orthogonal projection from $\mathbb{R}^d$ into the hyperplane Π^ν_0 introduced in (f) at the beginning of Section 2.

Lemma 7.11. *Let Ω_1 be an open subset of $\mathbb{R}^d$ with $\Omega \subset\subset \Omega_1$ and let $\eta > 0$. Then there exist $\varepsilon > 0$, a finite family K_i, $i = 1, \dots, n$, of compact subsets of $\partial\Omega$, and a finite family ν_i, $i = 1, \dots, n$, in $\mathbb{S}^{d-1}$ such that*

(a) $\mathcal{H}^{d-1}(\partial\Omega \setminus \bigcup_{i=1}^n K_i) < \eta$,
(b) $\pi^{\nu_i} : \mathbb{R}^d \to \Pi^{\nu_i}_0$ *is injective on* K_i,

(c) $|\nu_\Omega - \nu_i| < \eta$ $\mathcal{H}^{d-1}$*-a.e. in* K_i*, where* ν_Ω *denotes the unit outer normal to* Ω*,*
(d) *the sets* $C_\varepsilon^{\nu_i}(K_i)$ *are pairwise disjoint and contained in* Ω_1*,*
(e) $C_\varepsilon^{\nu_i}(K_i) \cap \Omega = \emptyset$.

When the boundary is not of class C^1, the proof of this result involves a lot of technical arguments and is given in the Appendix.

Proof of Theorem 7.8 (continuation). **Case d > 1.** We recall that $0 < \eta < 1$ and Ω_1 have been introduced in (7.29). By property (a) of Lemma 7.11 the function v_k introduced in (7.27) satisfies

$$V(v_k, \partial\Omega) \leq \sum_{i=1}^n \int_{K_i} |\mathrm{tr}_\Omega(u_k) - w| \wedge 1 d\mathcal{H}^{d-1} + \eta\,. \tag{7.36}$$

On each K_i we proceed by slicing in the direction ν_i. By property (b) in Lemma 7.11 for every $y \in H_i := \pi^{\nu_i}(K_i)$ there exists a unique $a_i(y) \in \mathbb{R}$ such that $y + a_i(y)\nu_i \in K_i$. We observe that, using the notation introduced in (5.15) we have $(C_\varepsilon^{\nu_i}(K_i))_y^{\nu_i} = [a_i(y), a_i(y)+\varepsilon]$.

By [2, Theorem 3.108] we have $\mathrm{tr}_\Omega(u_k)(y + a_i(y)\nu_i) = (u_k)_y^{\nu_i}(a_i(y)-)$, while $w(y + a_i(y)\nu_i) = (w)_y^{\nu_i}(a_i(y))$ for $\mathcal{H}^{d-1}$-a.e. $y \in H_i$. Then by the area formula (see, e.g., [11, Section 3.3])

$$\begin{aligned}&\int_{K_i} (|\mathrm{tr}_\Omega(u_k)(x) - w(x)| \wedge 1)\nu_i \cdot \nu_\Omega(x) d\mathcal{H}^{d-1}(x)\\ &= \int_{H_i} |(u_k)_y^{\nu_i}(a_i(y)-) - (w)_y^{\nu_i}(a_i(y))| \wedge 1 d\mathcal{H}^{d-1}(y)\,.\end{aligned}$$

Since by (c) in Lemma 7.11 we have $1 - \eta \leq \nu_i \cdot \nu_\Omega(x)$, we obtain

$$\int_{K_i} |\mathrm{tr}_\Omega(u_k) - w| \wedge 1 d\mathcal{H}^{d-1} \leq \frac{1}{1-\eta} \int_{H_i} |(u_k)_y^{\nu_i}(a_i(y)-) - (w)_y^{\nu_i}(a_i(y))| \wedge 1 d\mathcal{H}^{d-1}(y) \tag{7.37}$$

On the other hand, for $\mathcal{H}^{d-1}$-a.e. $y \in H_i$ and for $\mathcal{L}^1$-a.e. $\sigma \in (0, \varepsilon)$ by the triangle inequality we can write

$$\begin{aligned}|(u_k)_y^{\nu_i}(a_i(y)-) - w_y^{\nu_i}(a_i(y))| \wedge 1 &\leq |(u_k)_y^{\nu_i}(a_i(y)-) - (u_k)_y^{\nu_i}(a_i(y) + \sigma)| \wedge 1\\ &+|(u_k)_y^{\nu_i}(a_i(y) + \sigma) - w_y^{\nu_i}(a_i(y) + \sigma)| + |w_y^{\nu_i}(a_i(y) + \sigma) - w_y^{\nu_i}(a_i(y))|\,.\end{aligned} \tag{7.38}$$

By Lemma 7.10, for $\mathcal{H}^{d-1}$-a.e. $y \in H_i$ and for $\mathcal{L}^1$-a.e. $\sigma \in (0, \varepsilon)$ we have

$$|(u_k)_y^{\nu_i}(a_i(y)-) - (u_k)_y^{\nu_i}(a_i(y) + \sigma)| \wedge 1 \leq \Psi_y^\sigma((u_k)_y^{\nu_i}) \leq \Psi_y^\varepsilon((u_k)_y^{\nu_i})\,,$$

where Ψ_y^σ is the function introduced in (7.21) corresponding to $(a, b) = (a_i(y), a_i(y) + \sigma)$. Hence for $\mathcal{L}^1$-a.e. $\sigma \in (0, \varepsilon)$ integrating on H_i we obtain

$$\begin{aligned}\int_{H_i} |(u_k)_y^{\nu_i}(a_i(y)-) - (u_k)_y^{\nu_i}(a_i(y) + \sigma)| \wedge 1 d\mathcal{H}^{d-1} &\leq \int_{H_i} \Psi_y^\varepsilon((u_k)_y^{\nu_i}) d\mathcal{H}^{d-1}\\ &\leq V(u_k, C_\varepsilon^{\nu_i}(K_i))\,,\end{aligned} \tag{7.39}$$

where in the last inequality we used a general result on slicing (see [2, Theorem 3.108]).

Moreover, for $\mathcal{H}^{d-1}$-a.e. $y \in H_i$ and for $\mathcal{L}^1$-a.e. $\sigma \in (0, \varepsilon)$, since $w \in W^{1,1}(\tilde{\Omega})$,

$$|w_y^{\nu_i}(a_i(y) + \sigma) - w_y^{\nu_i}(a_i(y))| \leq \int_0^\varepsilon |(\nabla w)_y^{\nu_i}(a_i(y) + t)| dt\,.$$

Integrating (7.38) on H_i and using Fubini Theorem we obtain that for $\mathcal{L}^1$-a.e. $\sigma \in (0, \varepsilon)$

$$\begin{aligned}&\int_{H_i} [(u_k)_y^{\nu_i}(a_i(y)-) - (w)_y^{\nu_i}(a_i(y))] \wedge 1 d\mathcal{H}^{d-1}(y) \leq V(u_k, C_\varepsilon^{\nu_i}(K_i))\\ &+ \int_{H_i} |(u_k)_y^{\nu_i}(a_i(y) + \sigma) - w_y^{\nu_i}(a_i(y) + \sigma)| d\mathcal{H}^{d-1} + \int_{C_\varepsilon^{\nu_i}(K_i)} |\nabla w| dx\,.\end{aligned}$$

Integrating with respect to σ on $(0,\varepsilon)$ and dividing by ε we obtain

$$\int_{H_i} [(u_k)^{\nu_i}_y(a_i(y)-) - (w)^{\nu_i}_y(a_i(y))] \wedge 1 d\mathcal{H}^{d-1}(y) \leq V(u_k, C^{\nu_i}_\varepsilon(K_i))$$
$$+\frac{1}{\varepsilon}\int_{C^{\nu_i}_\varepsilon(K_i)} |u_k - w| dx + \int_{C^{\nu_i}_\varepsilon(K_i)} |\nabla w| dx\,,$$

where in the second term we used Fubini Theorem.

Since the sets $C^{\nu_i}_\varepsilon(K_i)$ are pairwise disjoint and contained in $\Omega_1 \setminus \Omega$ by (d) and (e) in Lemma 7.11, summing for $i = 1, \dots, n$ and using (7.29), (7.36), (7.37), and the last inequality, we obtain

$$V(v_k, \partial\Omega) \leq \frac{1}{1-\eta}\Big(V(u_k, \Omega_1 \setminus \Omega) + \frac{1}{\varepsilon}\int_{\Omega_1\setminus\Omega} |u_k - w| dx + \eta\Big) + \eta\,.$$

Recalling (7.32) this implies

$$\tilde{E}_k(v_k, \tilde{\Omega}) \leq \frac{1}{1-\eta}\tilde{E}_k(u_k, \tilde{\Omega}) + \frac{\tilde{c}}{1-\eta}\Big(\frac{1}{\varepsilon}\int_{\Omega_1\setminus\Omega} |u_k - w| dx + \eta\Big) + \tilde{c}(2\eta + \delta_k)\,. \tag{7.40}$$

Passing to the limsup as $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \tilde{E}_k(v_k, \tilde{\Omega}) \leq \frac{1}{1-\eta}\limsup_{k\to\infty} \tilde{E}_k(u_k, \tilde{\Omega}) + \frac{\tilde{c}\eta}{1-\eta} + 2\tilde{c}\eta\,.$$

Recalling (7.23), by the arbitrariness of η we obtain (7.28), which concludes the proof of the theorem. □

We now prove an inequality concerning the minimum values of (7.13) and (7.14).

Proposition 7.12. *Let E_k be a sequence in $\mathfrak{E}$. Assume that there exists $\hat{E} \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $\tilde{E}_k(\cdot, A)$ Γ-converges to $\hat{E}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then*

$$\inf_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega}) \;\geq\; \limsup_{k\to\infty} \inf_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}_k(v, \tilde{\Omega})\,. \tag{7.41}$$

Proof. Given $\eta > 0$ there exists $u \in GBV_\star(\tilde{\Omega})$, with $u = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$, such that

$$\hat{E}(u, \tilde{\Omega}) \leq \inf_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega}) + \eta\,.$$

We proceed as at the beginning of the proof of Theorem 7.8. Let $m \in \mathbb{N}$ be such that $\varepsilon_m < \eta$, where ε_m is introduced in (7.26), and let $u^m := w + (u-w)^{(m)}$. Then $u^m \in BV(\tilde{\Omega})$ and $u^m = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$. By (g) in Definition 3.1 we have that $\hat{E}(u_m, \tilde{\Omega}) \leq \hat{E}(u, \tilde{\Omega}) + \eta$, hence

$$\hat{E}(u_m, \tilde{\Omega}) \leq \inf_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega}) + 2\eta\,. \tag{7.42}$$

By Theorem 7.8 applied to u^m there exists a sequence $v_k \in BV(\tilde{\Omega})$, with $v_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$, such that $v_k \to u^m$ in $L^1(\tilde{\Omega})$ as $k \to \infty$ and

$$\hat{E}(u^m, \tilde{\Omega}) = \lim_{k\to\infty} \tilde{E}_k(v_k, \tilde{\Omega}) \geq \limsup_{k\to\infty} \inf_{\substack{v\in GBV_\star(\tilde{\Omega})\\ v=w\ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}_k(v, \tilde{\Omega})\,.$$

This inequality together with (7.42) gives (7.41) by the arbitrariness of η. □

To prove an inequality for the $\liminf$ we need the following compactness result for bounded sequences u_k in $GBV_\star$. Note that in this setting we have to modify the sequence u_k, because the bound (7.44) below does not provide a control on $\int_{J_{u_k}} |[u_k]| d\mathcal{H}^{d-1}$ and consequently we cannot obtain a control of the L^1-norms of u_k. We show that this modification does not increase the values of the energies $E^{f,g}$ for $f \in \mathcal{F}$ and $g \in \mathcal{G}$, provided that g satisfies the following condition: there exists $\kappa \geq 1$ such that

$$\kappa|\zeta_1| \leq |\zeta_2| \quad \Longrightarrow \quad g(x,\zeta_1,\nu) \leq g(x,\zeta_2,\nu) \text{ for every } x \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{7.43}$$

Note that by (g6) this condition is satisfied with $\kappa = 1$ if g is even with respect to ζ.

Theorem 7.13. *Let $(u_k) \subset GBV_\star(\tilde{\Omega})$ be a sequence such that $u_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$ and*

$$V(u_k, \tilde{\Omega}) \leq M \tag{7.44}$$

for some constant $M > 0$ independent of k, and let $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for every k. Then there exist a subsequence of u_k, not relabelled, and a sequence y_k in $GBV_\star(\tilde{\Omega})$ such that

(1) *$y_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$,*
(2) *for every $f \in \mathcal{F}$ and every $g \in \mathcal{G}$ satisfying (7.43) we have*
$$E^{f,g}(y_k, \tilde{\Omega}) \leq E^{f,g}(u_k, \tilde{\Omega}) + \varepsilon_k\,,$$
where $E^{f,g}$ is defined by (3.14),
(3) *the sequence y_k converges in $L^0(\tilde{\Omega})$ to some function $y \in GBV_\star(\tilde{\Omega})$ with $y = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$.*

Proof. The proof is obtained by adapting the arguments of [8, Section 5], which is based on the results of [14] for a different function space. More precisely, in [8, Theorems 5.3, 5.5, and Corollary 5.4] we replace the functional $\mathcal{G}^g_{\Gamma_0}$ by the functional $E^{f,g}$ and then we apply the compactness result for $GBV_\star(\tilde{\Omega})$ [8, Theorem 3.11]. The only change in the proofs regards the inequalities (5.26) and (5.27) in the proof of [8, Theorem 5.3], which are replaced by

$$\int_{\tilde{\Omega}} f^\infty(x, \frac{dD^c v}{d|D^c v|}) d|D^c v| \leq \int_{\tilde{\Omega}} f^\infty(x, \frac{dD^c u}{d|D^c u|}) d|D^c u|\,, \tag{7.45}$$

$$\int_{J_v} g(x,[v],\nu_v) d\mathcal{H}^{d-1} \leq \int_{J_u} g(x,[u],\nu_u) d\mathcal{H}^{d-1} + \theta C_{M,\tilde{\Omega}}\,, \tag{7.46}$$

where in this step of the proof u is a suitable function in $GBV_\star(\tilde{\Omega})$ and $v \in BV(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega})$ satisfies

$$v := \sum_{j=1}^{J} (u - t_j)\chi_{P_j}$$

for a suitable choice of the constants t_j and of the pairwise disjoint sets P_j of finite perimeter.

To prove (7.45) we set $m := \|v\|_{L^\infty(\tilde{\Omega})} + \sum_j |t_j|$, so that $|u| \leq m$ $\mathcal{L}^d$-a.e. in P_j. Then $v = \sum_{j=1}^J (u^{(m)} - t_j)\chi_{P_j}$, hence, by [8, Lemma 2.4] we have $D^c v = \sum_{j=1}^J D^c u^{(m)} \chi_{P_j^{(1)}}$, where $P_j^{(1)}$ is the set of points of Lebesgue density one for P_j. Since $|\tilde{u}| \leq m$ $\mathcal{H}^{d-1}$-a.e. in $P_j^{(1)} \setminus J_u$, we have $D^c u = D^c u^{(m)}$ as measures in $P_j^{(1)}$ by [8, Definition 2.8]. Therefore $D^c v = \sum_{j=1}^J D^c u \chi_{P_j^{(1)}}$, which implies that

$$\int_{\tilde{\Omega}} f^\infty(x, \frac{dD^c v}{d|D^c v|}) d|D^c v| = \sum_{j=1}^{J} \int_{P_j^{(1)}} f^\infty(x, \frac{dD^c u}{d|D^c u|}) d|D^c u| \leq \int_{\tilde{\Omega}} f^\infty(x, \frac{dD^c u}{d|D^c u|}) d|D^c u|\,,$$

concluding the proof of (7.45).

Inequality (7.46) can be obtained arguing as in the proof of (20) in [14, Theorem 3.2]. Note that these arguments require also property (7.43). $\square$

We are now in a position to prove the main result concerning the convergence of the minimum values of (7.13) to the minimum value of (7.14) and the convergence of a subsequence of the corresponding quasi-minimisers.

Theorem 7.14. *Let f_k be a sequence in $\mathcal{F}$ and let g_k be a sequence in $\mathcal{G}$ such that* (7.43) *holds for g_k with a constant κ independent of k. Assume that there exists $\hat{E} \in \mathfrak{E}_{sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $\tilde{E}^{f_k,g_k}(\cdot, A)$ Γ-converges to $\hat{E}(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then*

(a) *the minimum problem*

$$\min_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega}) \tag{7.47}$$

has a solution;

(b) *we have*

$$\min_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega}) = \lim_{k\to\infty} \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_k,g_k}(v, \tilde{\Omega}) . \tag{7.48}$$

(c) *given a sequence $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for every k, there exist a subsequence of $\tilde{E}^{f_k,g_k}$, not relabelled, and a sequence u_k in $GBV_\star(\tilde{\Omega})$ such that*
   (1) *$u_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$,*
   (2) *we have*
$$\tilde{E}^{f_k,g_k}(u_k, \tilde{\Omega}) \leq \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_k,g_k}(v, \tilde{\Omega}) + \varepsilon_k ,$$
   (3) *u_k converge in $L^0(\tilde{\Omega})$ to a minimum point u of* (7.47).

Proof. It is obvious that there exists $v_k \in GBV_\star(\tilde{\Omega})$, with $v_k = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$, such that

$$\tilde{E}^{f_k,g_k}(v_k, \tilde{\Omega}) \leq \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_k,g_k}(v, \tilde{\Omega}) + \varepsilon_k . \tag{7.49}$$

We fix a subsequence v_{k_j} of v_k such that

$$\liminf_{k\to\infty} \tilde{E}^{f_k,g_k}(v_k, \tilde{\Omega}) = \lim_{j\to\infty} \tilde{E}^{f_{k_j},g_{k_j}}(v_{k_j}, \tilde{\Omega}) . \tag{7.50}$$

By (7.41) and (7.49) it follows that the sequence $\tilde{E}^{f_{k_j},g_{k_j}}(v_{k_j}, \tilde{\Omega})$ is bounded, hence by (c1) in Definition 3.1, inequality (7.44) holds for the subsequence v_{k_j}. By Theorem 7.13 there exist a further subsequence, not relabelled, and a sequence u_j in $GBV_\star(\tilde{\Omega})$ such that $u_j = w$ $\mathcal{L}^d$-a.e. in $\tilde{\Omega} \setminus \Omega$,

$$\tilde{E}^{f_{k_j},g_{k_j}}(u_j, \tilde{\Omega}) \leq \tilde{E}^{f_{k_j},g_{k_j}}(v_{k_j}, \tilde{\Omega}) + \varepsilon_{k_j} \leq \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_{k_j},g_{k_j}}(v, \tilde{\Omega}) + 2\varepsilon_{k_j} , \tag{7.51}$$

and u_j converge in $L^0(\tilde{\Omega})$ to a function $u \in GBV_\star(\tilde{\Omega})$. By Γ-convergence, using (7.50) and (7.51) we obtain

$$\hat{E}(u, \tilde{\Omega}) \leq \liminf_{j\to\infty} \tilde{E}^{f_{k_j},g_{k_j}}(u_j, \tilde{\Omega}) \leq \liminf_{k\to\infty} \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \tilde{E}^{f_k,g_k}(v, \tilde{\Omega}) .$$

Combining this inequality and (7.41) with the obvious inequality

$$\hat{E}(u, \tilde{\Omega}) \geq \inf_{\substack{v \in GBV_\star(\tilde{\Omega}) \\ v=w \ \mathcal{L}^d\text{-a.e. in } \tilde{\Omega}\setminus\Omega}} \hat{E}(v, \tilde{\Omega})$$

we obtain (a)-(c) in the statement. $\square$

The following corollary reformulates Theorem 7.14 in terms of minimum problems of the form (7.11) and (7.19).

Corollary 7.15. *Under the hypotheses of Theorem 7.14 let $\hat g \in \mathcal{G}$ be the function such that* (7.18) *holds. Then*

(a) *the minimum problem*

$$\min_{v\in GBV_\star(\Omega)} \Big(\hat E(v,\Omega) + \int_{\partial\Omega} \hat g(x,\varphi - \mathrm{tr}_\Omega v, \nu_\Omega) d\mathcal{H}^{d-1}\Big) \tag{7.52}$$

has a solution;

(b) *we have*

$$\min_{v\in GBV_\star(\Omega)} \Big(\hat E(v,\Omega) + \int_{\partial\Omega} \hat g(x,\varphi - \mathrm{tr}_\Omega v, \nu_\Omega) d\mathcal{H}^{d-1}\Big) \tag{7.53}$$

$$= \lim_{k\to\infty} \inf_{v\in GBV_\star(\Omega)} \Big(E^{f_k,g_k}(v,\Omega) + \tilde c \int_{\partial\Omega} |\varphi - \mathrm{tr}_\Omega v| \wedge 1 d\mathcal{H}^{d-1}\Big)\,. \tag{7.54}$$

(c) *given a sequence $\varepsilon_k \to 0$ with $\varepsilon_k > 0$ for every k, there exist a subsequence of $\tilde E^{f_k,g_k}$, not relabelled, and a sequence u_k in $GBV_\star(\Omega)$ such that*

(1) $$E^{f_k,g_k}(u_k,\Omega) + \tilde c \int_{\partial\Omega} |\varphi - \mathrm{tr}_\Omega u_k| \wedge 1 d\mathcal{H}^{d-1}$$

$$\le \inf_{v\in GBV_\star(\Omega)} \Big(E^{f_k,g_k}(v,\Omega) + \tilde c \int_{\partial\Omega} |\varphi - \mathrm{tr}_\Omega v| \wedge 1 d\mathcal{H}^{d-1}\Big) + \varepsilon_k\,, \tag{7.55}$$

(2) *u_k converge in $L^0(\Omega)$ to a minimum point u of* (7.52).

Proof. The result follows from the previous theorem taking into account Remarks 7.4 and 7.7. □

8. Appendix

We now provide the detailed proof of Lemma 7.11 in the general case of a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary.

Proof of Lemma 7.11. Given $\xi \in \mathbb{S}^{d-1}$, $B \subset \Pi_0^\xi$, and $I \subset \mathbb{R}$, we set

$$B \times_\xi I := \{y + t\xi : y \in B\,,\ t \in I\} = \{x \in \mathbb{R}^d : \pi^\xi(x) \in B,\ x\cdot\xi \in I\}.$$

Since Ω has Lipschitz boundary, for every $x_0 \in \partial\Omega$ there exist $\xi \in \mathbb{S}^{d-1}$, a relatively open set $U \subset \Pi_0^\xi$ containing $\pi^\xi(x_0)$, an open interval $I \subset \mathbb{R}$ containing $x_0 \cdot \xi$, and a Lipschitz function $\varphi\colon U \to I$ such that

$$(U \times_\xi I) \cap \partial\Omega = \{y + \varphi(y)\xi : y \in U\}\,, \tag{8.1}$$

$$(U \times_\xi I) \cap \Omega = \{y + t\xi : y \in U,\ t \in I,\ t < \varphi(y)\}\,. \tag{8.2}$$

Given $B \subset U$ and a function $\psi\colon U \to \mathbb{R}$, the graph and the subgraph of ψ over B are denoted by

$$\Gamma_\psi^\xi(B) := \{y + \psi(y)\xi : y \in B\}\,, \quad \text{and} \quad S_\psi^\xi(B,I) := \{y + t\xi : y \in B,\ t \in I,\ t < \psi(y)\}\,.$$

If ψ is differentiable at $y \in U$, its gradient $\nabla\psi(y)$ is an element of $\Pi_0^\xi \subset \mathbb{R}^d$.

Note that $x \in S_\varphi^\xi(U,I)$ if and only if $x\cdot\xi \in I$ and $x\cdot\xi < \varphi(\pi^\xi(x))$. Therefore, by (8.2) for $x \in U \times_\xi I$ we have

$$x \in \Omega \iff x\cdot\xi < \varphi(\pi^\xi(x))\,. \tag{8.3}$$

If $x \in \partial\Omega \cap (U \times_\xi I)$ and φ is differentiable at $\pi^\xi(x)$ then the outer unit normal to $\partial\Omega$ at x is given by

$$\nu_\Omega(x) = \frac{\xi - \nabla\varphi(\pi^\xi(x))}{\sqrt{1 + |\nabla\varphi(\pi^\xi(x))|^2}}\,. \tag{8.4}$$

Moreover, if $\nu \in \mathbb{S}^{d-1}$ and $\nu \cdot \nu_\Omega(x) > 0$, then for $|t|$ small we have

$$x + t\nu \in \Omega \iff t < 0\,. \tag{8.5}$$

Indeed, for $|t|$ small we have $x + t\nu \in U \times_\xi I$, and by (8.3) we have

$$x + t\nu \in \Omega \iff x \cdot \xi + t\nu \cdot \xi < \varphi(\pi^\xi(x) + t\pi^\xi(\nu))\,. \tag{8.6}$$

Since φ is differentiable at $\pi^\xi(x)$ and $\varphi(\pi^\xi(x)) = x\cdot\xi$ by (8.1), from the previous equivalence we obtain that

$$x + t\nu \in \Omega \iff t\nu \cdot \xi < t\nabla\varphi(\pi^\xi(x))\pi^\xi(\nu) + o(t)\,;$$

from (8.4) it follows that

$$x + t\nu \in \Omega \iff t\nu \cdot \nu_\Omega(x) < o(t)\,,$$

and since $\nu \cdot \nu_\Omega(x) > 0$ for $|t|$ small this can happen if and only if $t < 0$.

By a direct consequence of Whitney's Extension Theorem (see, e.g., [11, Section 6.6.1]) given $\sigma > 0$ there exist a compact set $H \subset U \subset \Pi^\xi_0$ and a C^1 function $\psi\colon \overline{U} \to I$ such that

$$\mathcal{H}^{d-1}(U \setminus H) < \sigma\,,\ \ \psi(y) = \varphi(y)\,,\ \text{ and } \nabla\psi(y) = \nabla\varphi(y) \text{ for every } y \in H\,, \tag{8.7}$$

meaning, in particular, that φ is differentiable at every $y \in H$. Hence $\Gamma^\xi_\varphi(H) = \Gamma^\xi_\psi(H)$, and $S^\xi_\varphi(H, I) = S^\xi_\psi(H, I)$. Moreover, for every $x \in \Gamma^\xi_\varphi(H)$ we have

$$\nu_\Omega(x) = \frac{\xi - \nabla\varphi(y)}{\sqrt{1 + |\nabla\varphi(y)|^2}} = \frac{\xi - \nabla\psi(y)}{\sqrt{1 + |\nabla\psi(y)|^2}}\,, \tag{8.8}$$

where $y = \pi^\xi(x) \in H$ and $\nabla\varphi(y) = \nabla\psi(y) \in \Pi^\xi_0$.

Let $x \in \Gamma^\xi_\varphi(H) = \Gamma^\xi_\psi(H)$. Then $x = y + \psi(y)\xi$, with $y = \pi^\xi(x)$. Let $\nu \in \mathbb{S}^{d-1}$ with $|\nu - \nu_\Omega(x)| < \eta$. We claim that $x + t\nu \notin \Omega$ for $t > 0$ sufficiently small.

We observe that

$$\mathcal{H}^{d-1}(\Gamma^\xi_\varphi(U) \setminus \Gamma^\xi_\psi(H)) = \mathcal{H}^{d-1}(\Gamma^\xi_\varphi(U \setminus H)) \le (1 + L^2)^{1/2}\sigma\,, \tag{8.9}$$

where L is the Lipschitz constant of φ.

Since $\Gamma^\xi_\psi(U)$ is a C^1 manifold of dimension $d-1$, for every $x \in \Gamma^\xi_\psi(U)$ we can represent $\Gamma^\xi_\psi(U)$ in a neighbourhood of x as the graph of a C^1 function defined on the tangent space at x. More precisely, let $y = \pi^\xi(x)$ and let $\nu := \frac{\xi - \nabla\psi(y)}{\sqrt{1+|\nabla\psi(y)|^2}}$ be the unit normal to $\Gamma^\xi_\psi(U)$ at x pointing towards the exterior of $S^\xi_\psi(U, I)$. There exist a relatively open set $V \subset \Pi^\nu_0$ containing y, an open interval $J \subset \mathbb{R}$ containing $x \cdot \nu$, and a C^1 function $\omega\colon \overline{V} \to J$ with $\nabla\omega(y) = 0$ such that $V \times_\nu J \subset U \times_\xi I$,

$$\Gamma^\xi_\psi(U) \cap (V \times_\nu J) = \Gamma^\nu_\omega(V) \quad\text{and}\quad S^\xi_\psi(U, I) \cap (V \times_\nu J) = S^\nu_\omega(V, J)\,. \tag{8.10}$$

For every $x' \in \Gamma^\nu_\omega(V)$ and $s > 0$ small enough we have $x' + s\nu \in (V \times_\nu J) \setminus S^\nu_\omega(V, J) = (V \times_\nu J) \setminus S^\xi_\psi(U, I)$. Recalling (8.2), this shows that, if $x' \in \Gamma^\nu_\omega(V)$ and $\pi^\xi(x' + s\nu) \in H$, then $x' + s\nu \notin \Omega$ for $s > 0$ sufficiently small.

Let us note that, by taking V and J small, we can guarantee the smallness of $\nabla\omega$ and of the oscillation of $\nabla\psi$ on the projection of $\Gamma^\xi_\psi(U) \cap (V \times_\nu J)$ onto Π^ξ_0. Therefore, by (8.8) we can choose V and J so that

$$|\nabla\omega(y')| < 1 \quad \text{for every } y' \in V\,, \tag{8.11}$$

$$|\nu_\Omega(x) - \nu_\Omega(x')| < \eta \quad \text{for every } x, x' \in \Gamma^\xi_\varphi(H) \cap (V \times_\nu J)\,. \tag{8.12}$$

By compactness there exists a finite family $(x_i)_{i=1,\dots,m}$ in $\partial\Omega$ such that the corresponding U_i, I_i, ξ_i, and φ_i satisfy (8.1), (8.2), and

$$\partial\Omega = \bigcup_{i=1}^m \Gamma^{\xi_i}_{\varphi_i}(U_i)\,. \tag{8.13}$$

Let $B_1 := \Gamma^{\xi_1}_{\varphi_1}(U_1)$, $B_2 := \Gamma^{\xi_2}_{\varphi_2}(U_2) \setminus B_1$, ..., $B_m := \Gamma^{\xi_m}_{\varphi_m}(U_m) \setminus \bigcup_{i=1}^{m-1} B_i$. Then each B_i is a Borel set, the sets B_i are pairwise disjoint, and $\partial\Omega = \bigcup_{i=1}^m B_i$. Therefore there exist compact sets $F_i \subset B_i$ such that

$$\mathcal{H}^{d-1}(\partial\Omega \setminus \bigcup_{i=1}^m F_i) < \eta\,.$$

There exists a family of pairwise disjoint relatively open subsets V_i' of $\partial\Omega$ such that $F_i \subset V_i' \subset \Gamma^{\xi_i}_{\varphi_i}(U_i)$. Let $U_i' := \pi^{\xi_i} V_i'$. Then U_i' are relatively open subsets of $\Pi^{\xi_i}_0$, $U_i' \subset U_i$, $\Gamma^{\xi_i}_{\varphi_i}(U_i') = V_i'$ are pairwise disjoint, and

$$\mathcal{H}^{d-1}(\partial\Omega \setminus \bigcup_{i=1}^m \Gamma^{\xi_i}_{\varphi_i}(U_i')) < \eta\,. \tag{8.14}$$

We then apply the argument involving Whitney's Extension Theorem to U_i' and we find compact sets H_i and C^1 functions $\psi_i \colon \overline{U}_i' \to I_i$ that satisfy (8.7) with $U = U_i'$, and $\varphi = \varphi_i$, and $\sigma = \eta(1+L^2)^{-1/2}/m$, where L is the largest Lipschitz constant of the functions φ_i. In particular,

$$\varphi_i \text{ is differentiable at every point of } H_i\,, \tag{8.15}$$

$$\mathcal{H}^{d-1}(U_i' \setminus H_i) < \eta(1+L^2)^{-1/2}/m\,. \tag{8.16}$$

Let $K_i := \Gamma^{\xi_i}_{\psi_i}(H_i)$. Using also (8.9) and (8.14) we obtain

$$\mathcal{H}^{d-1}(\partial\Omega \setminus \bigcup_{i=1}^m K_i) < 2\eta\,. \tag{8.17}$$

Let us fix $i \in \{1, \dots, m\}$. By compactness there exists a finite family of points $(x^i_j)_{j=1,\dots,n_i}$ in $K_i = \Gamma^{\xi_i}_{\varphi_i}(H_i)$ such that, setting $\nu^i_j := \nu_\Omega(x^i_j)$, there exist relatively open set $V^i_j \subset \Pi^{\nu^i_j}_0$, open intervals J^i_j, and C^1 functions $\omega^i_j \colon \overline{V}^i_j \to J^i_j$ such that (8.10) hold with $\xi = \xi_i$, $\psi = \psi_i$, $U = U_i'$, $V = V^i_j$, $\nu = \nu^i_j$, $J = J^i_j$, and $\omega = \omega^i_j$, and

$$K_i \subset \bigcup_{j=1}^{n_j} \Gamma^{\nu^i_j}_{\omega^i_j}(V^i_j).$$

Arguing as before we can construct pairwise disjoint compact sets F^i_j such that

$$F^i_j \subset \Gamma^{\nu^i_j}_{\omega^i_j}(V^i_j) \cap K_i \subset \partial\Omega\,, \tag{8.18}$$

$$\mathcal{H}^{d-1}\big(K_i \setminus \bigcup_{j=1}^{n_i} F^i_j\big) < \frac{\eta}{m}\,. \tag{8.19}$$

Let us fix $\varepsilon > 0$ such that for every $i, h = 1, \dots, m$, $j = 1, \dots, n_i$, and $k = 1, \dots, n_h$,

$$2\varepsilon \text{ is smaller than the distance between } F^i_j \text{ and } F^h_k, \tag{8.20}$$

$$C^{\nu^i_j}_\varepsilon(F^i_j) \subset V^i_j \times_{\nu^i_j} J^i_j \subset U_i' \times_{\xi_i} I_i\,. \tag{8.21}$$

This implies, in particular, that

$$x \in F^i_j \text{ and } t \in (0, \varepsilon] \quad \Longrightarrow \quad x + t\nu^i_j \notin F^i_j\,. \tag{8.22}$$

Moreover, the sets $C^{\nu^i_j}_\varepsilon(F^i_j)$ are pairwise disjoint, and

$$(\partial\Omega \setminus F^i_j) \cap C^{\nu^i_j}_\varepsilon(F^i_j) \subset \Gamma^{\xi_i}_{\varphi_i}(U_i') \setminus \bigcup_{k=1}^{n_i} F^i_k = \big(\Gamma^{\xi_i}_{\varphi_i}(U_i' \setminus H_i)\big) \cup \big(K_i \setminus \bigcup_{k=1}^{n_i} F^i_k\big)\,. \tag{8.23}$$

To obtain property (e) we have to reduce the sets F^i_j. Since the compact sets $C^{\nu^i_j}_\varepsilon(F^i_j)$ are pairwise disjoint, they can be separated by pairwise disjoint open sets. Since $\mathcal{H}^{d-1}(\partial\Omega) < +\infty$, we can choose a family A^i_j, $j = 1,\dots,n_i$ of pairwise disjoint open sets in $\mathbb{R}^d$ such that $\partial\Omega \cap A^i_j \supset (\partial\Omega \setminus F^i_j) \cap C^{\nu^i_j}_\varepsilon(F^i_j)$, and

$$\mathcal{H}^{d-1}\Big((\partial\Omega \cap A^i_j) \setminus \big((\partial\Omega \setminus F^i_j) \cap C^{\nu^i_j}_\varepsilon(F^i_j)\big)\Big) < \frac{\eta}{mn_i}\,. \tag{8.24}$$

We set

$$B^i_j := \pi^{\nu^i_j}(\partial\Omega \cap A^i_j) \supset \pi^{\nu^i_j}\big((\partial\Omega \setminus F^i_j) \cap C^{\nu^i_j}_\varepsilon(F^i_j)\big)$$

and observe that B^i_j is a Borel set, since $\partial\Omega\cap A^i_j$ can be written as the union of an increasing sequence of compact sets. Since the sets $(\partial\Omega\setminus F^i_j)\cap C^{\nu^i_j}_\varepsilon(F^i_j)$ are pairwise disjoint, by (8.23) and (8.24) we have $\sum_{j=1}^{n_i}\mathcal{H}^{d-1}(B^i_j) \le \mathcal{H}^{d-1}(\Gamma^{\xi_i}_{\varphi_i}(U'_i\setminus H_i)) + \mathcal{H}^{d-1}\big(K_i\setminus\bigcup_{j=1}^{n_i}F^i_j\big) + \frac{\eta}{m}$. Using (8.9) with $\sigma = \eta(1+L^2)^{-1/2}/m$ and (8.16) for the first term in the right-hand side and (8.19) for the second one we obtain

$$\sum_{j=1}^{n_i}\mathcal{H}^{d-1}(B^i_j) < \frac{3\eta}{m}\,. \tag{8.25}$$

For every $j \in \{1,\dots,n_i\}$ let H^i_j be a compact set contained in $\pi^{\nu^i_j}(F^i_j) \setminus B^i_j$ with $\mathcal{H}^{d-1}((\pi^{\nu^i_j}(F^i_j)\setminus B^i_j)\setminus H^i_j) < \frac{\eta}{mn_i}$, and let

$$K^i_j := \Gamma^{\nu^i_j}_{\omega^i_j}(H^i_j) \subset F^i_j\,.$$

We now check that the family K^i_j, ν^i_j satisfies properties (a)-(e) in the statement. It follows immediately from the definition that $\pi^{\nu^i_j}$ is injective on K^i_j, which proves property (b). Since by (8.11) the Lipschitz constants of ω^i_j are smaller than 1, we have

$$\mathcal{H}^{d-1}(F^i_j \setminus K^i_j) \le \mathcal{H}^{d-1}((\pi^{\nu^i_j}(F^i_j)\setminus H^i_j) \le \frac{\eta}{mn_i} + \mathcal{H}^{d-1}(B^i_j)\,,$$

hence, by (8.25)

$$\sum_{j=1}^{n_i}\mathcal{H}^{d-1}(F^i_j\setminus K^i_j) \le \frac{4\eta}{m}\,. \tag{8.26}$$

Since the sets $C^{\nu^i_j}_\varepsilon(F^i_j)$ are pairwise disjoint, so are the sets $C^{\nu^i_j}_\varepsilon(K^i_j)$, and this proves the first part of property (d). The second part is obvious when $\varepsilon > 0$ is smaller than the distance between Ω and $\mathbb{R}^d\setminus\Omega_1$.

We now want to prove that $C^{\nu^i_j}_\varepsilon(K^i_j)\cap\Omega = \emptyset$. We argue by contradiction. Assume there exists $x \in C^{\nu^i_j}_\varepsilon(K^i_j)\cap\Omega$. Then $x = x' + t\nu^i_j$ with $x' \in K^i_j$ and $0 < t \le \varepsilon$. We recall that by (8.15) and (8.18) the function φ_i is differentiable at $y' := \pi^{\xi_i}x'$ and by (8.12) we have $\nu_\Omega(x')\cdot\nu^i_j > 0$. Hence, by (8.5) it follows that $x' + s\nu^i_j \notin \Omega$ if $s > 0$ is small. Therefore there exists $t_0 \in (0,t)$ such that $x' + t_0\nu^i_j \in (\partial\Omega\setminus F^i_j)\cap C^{\nu^i_j}_\varepsilon(F^i_j)$.

Since $\pi^{\nu^i_j}(x'+t_0\nu^i_j) \in B^i_j$, and $\pi^{\nu^i_j}(x) \in H^i_j$, the equality $\pi^{\nu^i_j}(x'+t_0\nu^i_j) = \pi^{\nu^i_j}(x)$ contradicts the fact that $B^i_j \cap H^i_j = \emptyset$ by construction. This proves (e).

To prove (a) we write

$$\mathcal{H}^{d-1}(\partial\Omega\setminus\bigcup_{i=1}^m\bigcup_{j=1}^{n_i}K^i_j) \le \mathcal{H}^{d-1}(\partial\Omega\setminus\bigcup_{i=1}^m K_i) + \sum_{i=1}^m\mathcal{H}^{d-1}(K_i\setminus\bigcup_{j=1}^{n_i}K^i_j)\,.$$

We have

$$\mathcal{H}^{d-1}(K_i \setminus \bigcup_{j=1}^{n_i} K_j^i) \leq \mathcal{H}^{d-1}\big(K_i \setminus \bigcup_{j=1}^{n_i} F_j^i\big) + \sum_{j=1}^{n_i} \mathcal{H}^{d-1}\big(F_j^i \setminus K_j^i\big)$$

By (8.17), (8.19), and (8.26) we conclude that

$$\mathcal{H}^{d-1}(\partial\Omega \setminus \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} K_j^i) < 7\eta\,,$$

which proves (a). By (8.12) property (c) also holds. □

Acknowledgements. This paper is based on work supported by the National Research Project (PRIN 2017) "Variational Methods for Stationary and Evolution Problems with Singularities and Interfaces", funded by the Italian Ministry of Education, University, and Research. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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