ON THE FORMATION OF MICROSTRUCTURE FOR SINGULARLY PERTURBED PROBLEMS WITH $2, 3$ OR 4 PREFERRED GRADIENTS

JANUSZ GINSTER

ABSTRACT. In this manuscript, singularly perturbed energies with 2, 3 or 4 preferred gradients subject to incompatible Dirichlet boundary conditions are studied. This extends results on models for martensitic microstructures in shape-memory alloys ($N = 2$), a continuum approximation for the $J_1 - J_3$-model for discrete spin systems ($N = 4$), and models for crystalline surfaces with N different facets (general N). On a unit square, scaling laws are proven with respect to two parameters, one measuring the transition cost between different preferred gradients, the other measuring the incompatibility of the set of preferred gradients and the boundary conditions. By a change of coordinates, the latter can also be understood as as an incompatibility of a variable domain with a fixed set of preferred gradients. Moreover, it is shown how simple building blocks and covering arguments lead to upper bounds on the energy and solutions to the differential inclusion problem on general Lipschitz-domains.

1. Introduction

We consider a singularly perturbed energy for $2, 3$ or 4 preferred gradients and study the formation of microstructure for incompatible boundary conditions in terms of scaling laws for the minimal energy. More precisely, we consider the energy

$$E_{\sigma,\gamma,N}(u) = \int_{(0,1)^2} \operatorname{dist}(\nabla u, K_{\gamma,N})^2 \, dx + \sigma |D^2 u|((0,1)^2), \tag{1.1}$$

where $\sigma > 0$, $N \in \{2, 3, 4\}$ and the set of preferred gradients is defined by $K_{\gamma,N} = \left\{ e^{i\gamma + 2i\pi \frac{k}{N}} : k = 0, \dots, N-1 \right\}$ for $\gamma \in (-\pi, \pi]$. Our main result concerns a scaling law for the minimal energy subject to incompatible boundary conditions with respect to σ and γ.

Proving scaling laws for the minimal energy has been proven useful to explain the formation of patterns in a variety of problems in which the energy is non-quasiconvex and identifying the minimizers by analytical or numerical methods is not possible, c.f. [34]. Often the formation of patterns is related to the competition of a part of the energy that favors rather uniform structures and a non-quasiconvex part of the energy that favors oscillations on fine scales. Constructions of good competitors for the upper bound often involve branching construction that refine in a self-similar manner. A non-exhaustive list of references where this technique has been successfully applied includes [2, 6, 7, 10, 16, 17, 20, 25, 32, 37, 44] for martensitic microstructure, [36, 38] for compliance minimization, [12, 13, 26, 27, 28, 33, 40, 41, 46] for micromagnetism, [11, 14, 24] for type-I-superconductors, [3, 4, 5, 19] for compressed thin elastic films, [23, 25] for dislocation patterns and [30, 31] for helimagnets.

Energies of the form (1.1) appear in different contexts. For $N = 2$ and $\gamma = \pi/2$ the energy (1.1) reads as

$$\int_{(0,1)^2} |\partial_1 u|^2 + \min\{|\partial_2 + 1|, |\partial_2 u - 1|\}^2 \, dx + \sigma |D^2 u|((0,1)^2),$$

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which is a variant of the energy studied by Kohn and Müller in [35] for pattern formation in shape-memory alloys, see also [17, 49]. In particular, in [49] it is shown that

$$\min_{u(0,\cdot)=0} E_{\sigma,\pi/2,2}(u) \sim \min\{1, \sigma^{2/3}\}. \tag{1.2}$$

Here, the term of order 1 corresponds to very uniform configurations such as $u = 0$, whereas the bound of order $\sigma^{2/3}$ can be shown by a self-similarly refining branching construction.

For $N = 4$ and $\gamma = \pi/4$ the energy (1.1) is a variant of the energy that appears in certain parameter regimes (after appropriate rescaling) as the continuum approximation of the discrete $J_1 - J_3$-model

$$F_{\varepsilon,\alpha}(\varphi) = -\alpha \sum_{i,j\in\varepsilon\mathbb{Z}^2\cap(0,1)^2:|i-j|=\varepsilon} \varphi(i)\cdot\varphi(j) + \sum_{i,j\in\varepsilon\mathbb{Z}^2\cap(0,1)^2:|i-j|=2\varepsilon} \varphi(i)\cdot\varphi(j)$$

associated to a spin field $\varphi : \varepsilon\mathbb{Z}^2 \cap (0,1)^2 \to S^1$, see [15, 30] for the heuristic argument and [29] for a rigorous derivation via Γ-convergence. A corresponding scaling law is studied in [30] which reads as

$$\min_{u(0,\cdot)=0} E_{\sigma,\pi/4,4} \sim \min\{1, \sigma(|\log\sigma| + 1)\}. \tag{1.3}$$

Again, the term of order 1 is associated to rather uniform structures such as $u = 0$, whereas the bound $\sigma(|\log\sigma| + 1)$ corresponds again to a self-similarly refining branching construction. However, the scales of the construction are very different from the one for $N = 2$. In particular, in contrast to $N = 2$, for $N = 4$ it is possible to construct branching configurations such that (up to an interpolation region of order σ) $\nabla u \in K_{\pi/4,N}$.

For general N the energy functional (1.1) is very closely related to the (small slope approximation of the) free energy of a faceted crystalline surface in $\mathbb{R}^3$ with N different facets parameterized by $z = u(x, y)$

$$F_\sigma[u] = \int W(\nabla u) + \frac{\sigma^2}{2}|\Delta u|^2\, dxdy,$$

where W has N distinct minima in S^1, see, for example, [48] and the references therein. Typically, replacing the term $\sigma|D^2u|((0,1)^2)$ by $\sigma^2 \int_{(0,1)^2} |D^2u|^2\, dxdy$ does no qualitatively change corresponding scaling law results, see [49] and [45]. Hence, up to replacing the Laplacian by a full Hessian it is to be expected that the scaling law results presented in this paper are also valid for F_σ above (c.f. also the discussion in [30]). A study of the dynamics of F_σ, coarsening rates and simulations can, for example, be found in [47, 48].

For general $\gamma \in (-\pi, \pi]$, the energy (1.1) can by a simple change of variables alternatively be written as

$$\int_{Q_{\gamma_0-\gamma}} \operatorname{dist}(\nabla u, K_{\gamma_0,N})^2 + \sigma|D^2u|(Q_\gamma),$$

where $Q_{\gamma_0-\gamma}$ evolves from $(0,1)^2$ by a counterclockwise rotation with angle $\gamma_0 - \gamma$. Hence, our studies can be understood as a generalization of the scaling laws (1.2) and (1.3) (cf. [35, 49] and [30]) either to different domains or to a different set of preferred gradients.

Our main result is the following scaling law

$$\min_{u(0,\cdot)=0} E_{\sigma,\gamma,N}(u) \sim \begin{cases} \min\{|\sin(\gamma)|^2, |\sin(\gamma)|^3 + \sigma, \sigma^{2/3}|\sin(\gamma)|\} & \text{if } N = 2, \\ \min\left\{|\sin(\gamma)|^2, \sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)I|}\right)\right\} & \text{if } N = 3, 4, \end{cases}$$

where the range of γ is by symmetry considerations restricted to the interval around 0 such that $|\sin(\gamma)| \le |\sin(\gamma + \frac{2\pi k}{N})|$, $0 \le k \le N - 1$. The first term $|\sin(\gamma)|^2$ corresponds for all N to a uniform structure, i.e., $u(x, y) = x$. The second term for $N = 2$ still corresponds to a rather uniform structure, where the function u behaves as $u(x, y) = x$ close to $x = 0$ and has one transition to $u(x, y) = -\cos(\gamma)x - \sin(\gamma)y$ shortly after, see Figure 1. For the last term, branching structures play a role. The corresponding upper bounds can be proven using variants of the self-similar constructions for $\gamma = \pi/2$ ($N = 2$) and $\gamma = \pi/4$ ($N = 4$), for a sketch see Figure 3 for $N = 2$ and Figure 6, 8 and 7 for $N = 3, 4$. Similarly to before, there are significant differences between the constructions for $N = 2$ and $N = 3, 4$, mainly stemming from the fact that for $N > 2$ it is possible to construct self-similarly refining competitors such that (up to a small interpolation region) $\nabla u \in K_{\gamma,N}$. For

$N = 2$, this is not possible as $\nabla u \in K_{\gamma,N}$ implies that u is constant in direction $e^{i\gamma+i\pi/2}$. This is also reflected in the different scaling laws for $N = 2$ and $N = 3, 4$. It is to be expected that a similar behavior a for $N = 3, 4$ is also true for $N > 4$. It would then be interesting to understand the scaling behavior with respect to the number of preferred gradients N.

Additionally, for $N = 3, 4$ we discuss how upper bounds can be constructed using simple building blocks and covering arguments. More precisely, the structures of $K_{\gamma,N}$ allow us to construct functions $u : T \to \mathbb{R}$ (where T is a (rotated) square if $N = 4$ or a particular triangle for $N = 3$) such that $u = 0$ on ∂T, $\nabla u \in K_{\gamma,N}$ a.e. and $|D^2 u|(T) \le C$ (see Figure 9), c.f. also [8, 9, 18, 22, 39, 42, 43] and the references therein for constructions in the significantly more complicated vectorial setting. Then simple covering arguments allow us to construct upper bounds on more general domains and give rise to solutions of the differential inclusion $u = 0$ on $\partial\Omega$ and $\nabla u \in K_{\gamma,N}$. Similarly to [43] (c.f. also [30]), by interpolation inequalities regularity of these solutions can be established in certain fractional Sobolev spaces.

In the following, we will fix notation and state the precise model under consideration. In Section 3 we will present our main results and discuss the organization of the proofs.

2. Notation and setting of the problem

We will write C or c for generic constants that may change from line to line but do not depend on the problem parameters. We write log to denote the natural logarithm. We write $A \sim B$ for $A, B \in \mathbb{R}$ if there exists a universal constant $C > 0$ such that $\frac{1}{C}A \le B \le CA$. For the ease of notation, we always identify vectors with their transposes. Moreover, we will identify $\mathbb{C}$ with $\mathbb{R}^2$ and denote by e_1 and e_2 the two canonical basis vectors for $\mathbb{R}^2$.

For a measurable set $B \subseteq \mathbb{R}^n$ with $n = 1, 2$, we use the notation $\mid B \mid$ or $\mathcal{L}^n(B)$ to denote its n-dimensional Lebesgue measure. In addition, for $B \subseteq \mathbb{R}^2$ we write *conv*$(B) \subseteq \mathbb{R}^2$ for its convex hull and *int*(B) for its interior. For $\gamma \in [-\pi, \pi]$ and $N \in \{2, 3, 4\}$ we set

$$K_{\gamma,N} = \left\{ e^{i\gamma + 2i\pi\frac{l}{N}} : 0 \le l \le N-1 \right\} \subseteq \mathbb{R}^2.$$

The set of admissible functions is defined as

$$\mathcal{A} := \left\{ u \in W^{1,2}((0,1)^2) : \ \nabla u \in BV((0,1)^2),\ u(0,\cdot) = 0 \right\}$$

For $\sigma > 0$, $N \in \{2, 3, 4\}$ and $\gamma \in [-\pi, \pi]$ we consider the functional $E_{\sigma,\gamma,N} : \mathcal{A} \to [0, \infty)$ by

$$E_{\sigma,\gamma,N}(u) = \int_{(0,1)^2} \operatorname{dist}^2(\nabla u, K_{\gamma,N})\, dx + \sigma \mid D^2 u \mid (\Omega).$$

The expression $|D^2 u|(\Omega)$ in the second term of the functional $E_{\sigma,\gamma,N}$ denotes the total variation of the vector measure $D^2 u$. By symmetry considerations it will be enough to consider the angles $\gamma \in [-\pi/2, \pi/2]$ such that $|e^{i\gamma} \cdot e_2| = |\sin(\gamma)|$ is minimal within the set $K_{\gamma,N}$ i.e., $\gamma \in \Gamma_N$, where $\Gamma_N = [-\pi/2, \pi/2]$ for $N = 2$, $\Gamma_N = [-\pi/6, \pi/6]$ for $N = 3$, and $\Gamma_N = [-\pi/4, \pi/4]$ for $N = 4$.

Note that $u \in \mathcal{A}$ in particular implies that $u \in W^{1,1}((0,1)^2)$ and $\nabla u \in BV$. Hence, u has a continuous representative on the closed square $[0,1]^2$, see e.g. [23, Lemma 9]. We will always identify such functions with their continuous representatives.

For an open set $B \subseteq \mathbb{R}^2$ and $u \in W^{1,2}(B)$ with $\nabla u \in BV(B)$, we use the notation $E_{\sigma,\gamma,N}(u; B)$ for the energy on B, i.e.,

$$E_{\sigma,\gamma,N}(u; B) = \int_B \operatorname{dist}^2(\nabla u, K_{\gamma,N})\, dx + \sigma |D^2 u|(B).$$

In addition for $x \in (0, 1)$, $I \subseteq (0, 1)$ and $u \in \mathcal{A}$

$$E_{\sigma,\gamma,N}(u; \{x\} \times I) = \int_I \operatorname{dist}^2(\nabla u(x, y), K_{\gamma,N})\, dy + \sigma |\partial_2 \nabla u(x, \cdot)|(I).$$

Note that since $\nabla u \in BV((0,1)^2)$ this formula makes sense for almost every $x \in (0,1)$ in the sense of slicing of BV-functions, see [1]. Similarly, we write for $y \in (0,1)$ and $u \in \mathcal{A}$

$$E_{\sigma,\gamma,N}(u; I \times \{y\}) = \int_I \mathrm{dist}^2(\nabla u(x,y), K_{\gamma,N})\, dx + \sigma |\partial_1 \nabla u(\cdot,y)|(I).$$

Eventually, we define for $B \subseteq (0,1)^2$ open with Lipschitz boundary the set

$$\mathcal{A}_0(B) := \left\{ u \in W^{1,2}(B) :\ \nabla u \in BV(B),\ u = 0 \text{ on } \partial B \right\}.$$

3. Main result

Our main result is the following scaling law for the minimal energy.

Theorem 3.1. There exists constants $C, c > 0$ such that for all $\sigma > 0$ and $\gamma \in \Gamma_N$ it holds:

(1) If $N = 2$ then

$$c \min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\} \leq \min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \leq C \min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\}.$$

(2) If $N = 3$ or $N = 4$ then

$$c \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||} + 1\right), |\sin(\gamma)|^2\right\} \leq \min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \leq C \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||} + 1\right), |\sin(\gamma)|^2\right\}.$$

The proof of the theorem is split into different sections. Bounds that are valid for all $N \in \{2,3,4\}$ are collected in Secion 4. Specific upper and lower bounds for $N = 2$ are proven in Section 5. The lower bound for $N = 3,4$ can be found in Proposition 6.3 in Section 6. The corresponding upper bound is shown in Proposition 6.4 in Section 6.

Below, we identify the different regimes that appear in the bounds in Theorem 3.1.

Remark 1. (1) We note that it holds for $\gamma \in \Gamma_2$

- if $\sigma \geq |\sin(\gamma)|^2$ then
$$\min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\} = |\sin(\gamma)|^2;$$
- if $|\sin(\gamma)|^3 \leq \sigma \leq |\sin(\gamma)|^2$ then
$$\frac{1}{2}(\sigma + |\sin(\gamma)|^3) \leq \sigma \leq \min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\} \leq \sigma + |\sin(\gamma)|^3;$$
- if $\sigma \leq |\sin(\gamma)|^3$ then
$$\frac{1}{2}(\sigma^{2/3}|\sin(\gamma)| + \sigma) \leq \sigma^{2/3}|\sin(\gamma)| \leq \min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\} \leq \sigma^{2/3}|\sin(\gamma)| + \sigma.$$

Hence, we find that

$$\min\left\{\sigma^{2/3}|\sin(\gamma)| + \sigma, \sigma + |\sin(\gamma)|^3, |\sin(\gamma)|^2\right\} \sim \begin{cases} |\sin(\gamma)|^2 & \text{if } \sigma \geq |\sin(\gamma)|^2, \\ \sigma & \text{if } |\sin(\gamma)|^2 \geq \sigma \geq |\sin(\gamma)|^3, \\ |\sin(\gamma)|\sigma^{2/3} & \text{if } |\sin(\gamma)|^3 \geq \sigma. \end{cases}$$

(2) For $N = 3,4$ and $\gamma \in \Gamma_N$ it holds

- if $\sigma \geq |\sin(\gamma)|^2$ then

$$|\sin(\gamma)|^2 = \min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right), |\sin(\gamma)|^2\right\}.$$

- if $\sigma \leq |\sin(\gamma)|^2$ then

$$\begin{aligned}\frac{|\log\sigma|}{|\log|\sin(\gamma)||} - 3 \leq \frac{|\log\sigma/|\sin(\gamma)|^3|}{|\log|\sin(\gamma)||} &\leq \frac{|\sin(\gamma)|^3}{\sigma\,|\log|\sin(\gamma)||}\\ &\leq \frac{|\sin(\pi/4)|\,|\sin(\gamma)|^2}{\sigma\,|\log|\sin(\pi/4)||} = \sqrt{2}\frac{|\sin(\gamma)|^2}{\sigma\log 2},\end{aligned}$$

since the mapping $t \to \frac{t}{|\log(t)|}$ is increasing for $t \in (0,1)$. Hence, it follows

$$\frac{1}{10}\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right) \leq \frac{1}{5}\sigma\left(\frac{|\sin(\gamma)|^2}{\sigma}+4\right) \leq \frac{|\sin(\gamma)|^2}{5} + \frac{4}{5}\sigma \leq |\sin(\gamma)|^2$$

and consequently,

$$\min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right), |\sin(\gamma)|^2\right\} \geq \frac{1}{10}\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right).$$

In particular, we have

$$\min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right), |\sin(\gamma)|^2\right\} \sim \begin{cases} |\sin(\gamma)|^2 & \text{if } \sigma \geq |\sin(\gamma)|^2,\\ \sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right) & \text{if } \sigma \leq |\sin(\gamma)|^2.\end{cases}$$

Additionally, we also consider for $N = 3,4$ the minimization problem subject to $u = 0$ on $\partial(0,1)^2$. In this situation the scaling laws for $N = 3$ or $N = 4$ differ due to different incompatibilities of $K_{\gamma,N}$ with respect to the full boundary of $(0,1)^2$.

Theorem 3.2. There exists constants $C, c > 0$ such that it holds for all $\gamma \in \Gamma_4$

$$c\min\left\{1, \sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right)\right\} \leq \min_{u\in\mathcal{A}_0((0,1)^2)} E_{\sigma,\gamma,4}(u) \leq C\min\left\{1, \sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right)\right\}.$$

Moreover, it holds for all $\gamma \in \Gamma_3$

$$c\min\{1, \sigma(|\log\sigma|+1)\} \leq \min_{u\in\mathcal{A}_0} E_{\sigma,\gamma,3}(u) \leq C\min\{1, \sigma(|\log\sigma|+1)\}.$$

Upper bounds can be shown by means of optimal coverings with suitable building blocks. The same arguments lead to upper bounds and solutions to the differential inclusion problem on general Lipschitz domains.

Proposition 3.3. Let $N = 3,4$ and $\Omega \subseteq \mathbb{R}^2$ a Lipschitz domain. Then the following hold:

(1) There exists a constant $C_\Omega > 0$ such that

$$\min_{u\in\mathcal{A}_0(\Omega)} E_{\sigma,\gamma,N}(u;\Omega) \leq C_\Omega\sigma(|\log\sigma|+1).$$

(2) There exists $u \in W_0^{1,\infty}(\Omega)$ such that $\nabla u \in BV_{loc}(\Omega; K_{\gamma,N})$ and $\nabla u \in W^{s,q}$ for all $0 < s < 1$, $q \in (0,\infty)$ satisfying $\frac{1}{q} > s$.

The proofs are discussed in Section 7.

4. Preliminaries

We first show the energy bounds that are true for all $N \in \{2,3,4\}$. The argument in the lower bound are closely related to the ones in [30, Lemma 5] (c.f. also [31]).

Proposition 4.1. There exists $c_A > 0$ such that for all $\gamma \in \Gamma_N$, $N \in \{2,3,4\}$, it holds

$$c_A \min\{|\sin(\gamma)|^2, \sigma\} \leq \min_{u \in \mathcal{A}} E_{\sigma,\gamma,N}(u) \leq |\sin(\gamma)|^2.$$

Proof. Step 1: Upper bound. Consider $u : (0,1)^2 \to \mathbb{R}$ defined by $u(x,y) = \cos(\gamma)x$. Then

$$E_{\sigma,\gamma,N}(u_1) \leq |\cos(\gamma)e_1 - e^{i\gamma}|^2 = |\sin(\gamma)|^2.$$

Step 2: Lower bounds. Let $u \in \mathcal{A}$. Without loss of generality, we may assume that it holds $E_{\sigma,\gamma,N}(u) \leq \frac{1}{48}\min\{|\sin(\gamma)|^2, \sigma\}$ (otherwise there is nothing to show). By a standard slicing and Fubini-type argument, we find $\bar{x}, y_1, y_2 \in (0,1)$ such that $y_2 - y_1 \geq 1/2$,

$$\begin{aligned} &\int_0^1 \operatorname{dist}(\nabla u(s,y_i), K_{\gamma,N})^2\, ds + \sigma|\partial_1 \nabla u|((0,1)\times\{y_i\}), \leq 4E_{\sigma,\gamma,N}(u) \\ \text{and } &\int_0^1 \operatorname{dist}(\nabla u(\bar{x},t), K_{\gamma,N})^2\, dt + \sigma|\partial_2 \nabla u|(\{\bar{x}\}\times(0,1)) \leq E_{\sigma,\gamma,N}(u). \end{aligned}$$

In particular, there exists $t \in (0,1)$ and $\xi \in K_{\gamma,N}$ such that $|\nabla u(t,y_1) - \xi| \leq \frac{2}{\sqrt{48}}|\sin(\gamma)|$. Additionally, it holds $|\partial_1 \nabla u|((0,1)\times\{y_1\}), |\partial_1 \nabla u|((0,1)\times\{y_2\}), |\partial_2 \nabla u|(\{\bar{x}\}\times(0,1)) \leq \frac{4}{\sigma}E_{\sigma,\gamma,N}(u) \leq \frac{1}{12}$. Consequently, it holds for almost all $s \in (0,1)$

$$|\nabla u(s,y_1) - \xi|, |\nabla u(s,y_2) - \xi|, |\nabla u(\bar{x},s) - \xi| \leq \frac{1}{\sqrt{12}}|\sin(\gamma)| + \frac{1}{4} \leq \frac{7}{12} \leq \frac{1}{\sqrt{2}}.$$

Since the distance between different points in $K_{\gamma,N}$ is at least $\sqrt{2}$, it follows for almost all $s \in (0,1)$ that

$$\begin{aligned} |\nabla u(s,y_1) - \xi| &= \operatorname{dist}(\nabla u(s,y_1), K_{\gamma,N}), \\ |\nabla u(s,y_2) - \xi| &= \operatorname{dist}(\nabla u(s,y_2), K_{\gamma,N}), \\ \text{and } |\nabla u(\bar{x},s) - \xi| &= \operatorname{dist}(\nabla u(\bar{x},s), K_{\gamma,N}). \end{aligned}$$

Then we estimate using $u(0,\cdot) = 0$

$$\begin{aligned} |u(\bar{x},y_2) - u(\bar{x},y_1)| &\leq \int_0^{\bar{x}} |\partial_1 u(s,y_2) - \xi_1| + |\partial_1 u(s,y_1) - \xi_1|\, ds \\ &\leq \int_0^1 \operatorname{dist}(\nabla u(s,y_2), K_{\gamma,N}) + \operatorname{dist}(\nabla u(s,y_1), K_{\gamma,N})\, ds \\ &\leq \left(\int_0^1 \operatorname{dist}(\nabla u(s,y_2), K_{\gamma,N})^2\, ds\right)^{1/2} + \left(\int_0^1 \operatorname{dist}(\nabla u(s,y_1), K_{\gamma,N})^2\, ds\right)^{1/2} \\ &\leq 4\sqrt{E_{\sigma,\gamma,N}(u)}. \end{aligned}$$

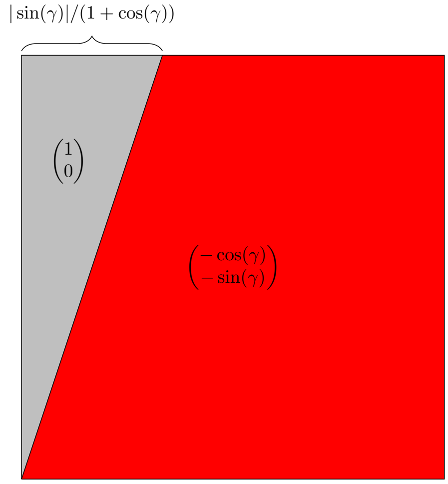

FIGURE 1. Sketch of the construction with an energy of order $|\sin(\gamma)|^3 + \sigma$ for $\gamma \leq 0$.

On the other hand,

$$\begin{aligned}|u(\bar{x}, y_2) - u(\bar{x}, y_1)| &= \left|\int_{y_1}^{y_2} \partial_2 u(\bar{x}, s) - \xi_2 + \xi_2 \, ds\right| \\ &\geq \frac{1}{2}|\xi_2| - \int_0^1 \operatorname{dist}(\nabla u(\bar{x}, s), K_{\gamma,N})\, ds \\ &\geq \frac{1}{2}|\sin(\gamma)| - \left(\int_0^1 \operatorname{dist}(\nabla u(\bar{x}, s), K_{\gamma,N})^2 \, ds\right)^{1/2} \\ &\geq \frac{1}{2}|\sin(\gamma)| - \frac{1}{\sqrt{48}}|\sin(\gamma)| \geq \frac{1}{4}|\sin(\gamma)|,\end{aligned}$$

where we used the definition of Γ_N for the second inequality. Consequently, we find

$$E_{\sigma,\gamma,N}(u) \geq \frac{1}{16^2}|\sin(\gamma)|^2 \geq \frac{1}{16^2}\min\{\sigma, |\sin(\gamma)|^2\}.$$

□

5. The 2-Gradient Problem

In this section we prove the scaling law for $N = 2$ claimed in Theorem 3.1. The additional upper in the regime $\sigma \leq |\sin(\gamma)|^3$ is proven using a variant of the celebrated Kohn-Müller branching construction developed in [35], c.f. also [21, 49]. The lower bound uses a variant of the argument from [17], c.f. also [21, 49].

Proof of Theorem 3.1 for $N = 2$. Step 1: Upper bounds.

First assume $\gamma \in [-\pi/2, 0]$ ($\gamma \in (0, \pi/2]$ can be treated analogously) and consider the function $u : \Omega \to \mathbb{R}$ defined by (see also Fig. 1)

$$u(x, y) = \begin{cases} x & \text{if } (x, y) \cdot v \leq 0, \\ e^{i\gamma + i\pi} \cdot (x, y) & \text{if } (x, y) \cdot v \geq 0, \end{cases}$$

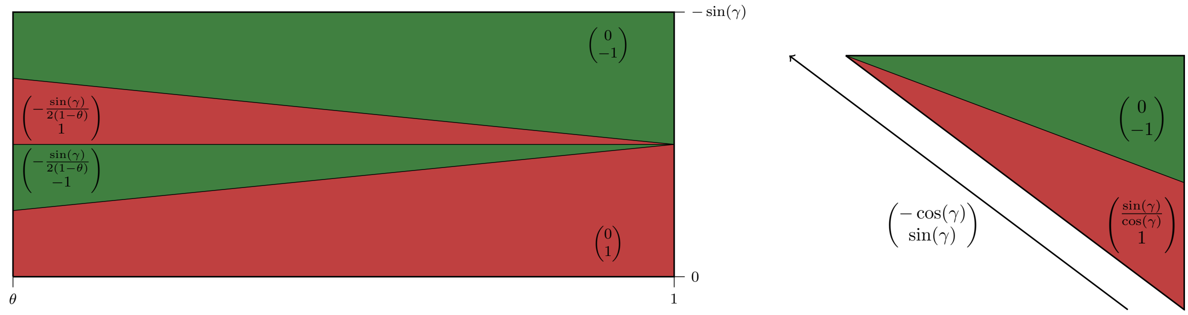

FIGURE 2. Left: Sketch of the building block for the branching construction on S_γ for $N = 2$. Right: Sketch of the gradient of the function u_∂ used to moderate between the sawtooth functions (in vertical direction) achieved by the branching construction and the boundary condition $u = 0$ on the line $conv\{(-\cos(\gamma), \sin(\gamma)), (0, 0)\}$. This construction is used if $0 \leq \gamma \leq \pi/4$ is relatively small. The red and green regions indicate that the y-derivative is ± 1, respectively.

where $v = (-\sin(\gamma), 1 + \cos(\gamma))$. Then

$$\begin{aligned} E_{\sigma,\gamma,2}(u) &\leq \mathcal{L}^2\left(\left\{(x,y) \in (0,1)^2 : (x,y)\cdot v \leq 0\right\}\right)|e_1 - e^{i\gamma}|^2 + 2\sigma\mathcal{H}^1\left(\left\{(x,y) \in (0,1)^2 : (x,y)\cdot v = 0\right\}\right) \\ &\leq C(|\sin(\gamma)|^3 + \sigma). \end{aligned}$$

For the second inequality, note that $|e_1 - e^{i\gamma}|^2 = |1-\cos(\gamma)|^2 + |\sin(\gamma)|^2 \leq C|\gamma|^4 + |\sin(\gamma)|^2 \leq C|\sin(\gamma)|^2$. Moreover, note that $\mathcal{L}^2\left(\left\{(x,y) \in (0,1)^2 : (x,y)\cdot v \leq 0\right\}\right) \leq \frac{|\sin(\gamma)|}{1+\cos(\gamma)} \leq |\sin(\gamma)|$.
By Remark 1 and the upper bound in Proposition 4.1, it remains to show that there exists $u : (0,1)^2 \to \mathbb{R}$ such that $E_{\sigma,\gamma,2}(u) \leq C|\sin(\gamma)|\sigma^{2/3}$ if $\sigma \leq |\sin(\gamma)|^3$. In the following, we will assume $0 \leq \gamma \leq \frac{\pi}{2}$ ($\gamma < 0$ can be treated similarly). We first present a modified version of the self-similar Kohn-Müller branching construction, see [21, 35, 49], on the domain (see Figure 3 for a sketch)

$$S_\gamma = conv\left\{\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \begin{pmatrix}1\\ \sin(\gamma)\end{pmatrix}, \begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\right\},$$

i.e., we will construct a function $u : S_\gamma \to \mathbb{R}$ such that $u = 0$ on the line $conv\left\{\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \begin{pmatrix}0\\ 0\end{pmatrix}\right\}$ and $E_{\sigma,\pi/2,2}(u; S_\gamma) \leq C|\sin(\gamma)|\sigma^{2/3}$. We will need two slightly different constructions for $0 \leq \gamma \leq \frac{\pi}{4}$ and $\frac{\pi}{4} \leq \gamma \leq \frac{\pi}{2}$, respectively, see Figure 3. As in the original construction, we fix for the proof some $\theta \in (\frac{1}{4}, \frac{1}{2})$.

Step 1a: Branching on S_γ for $\gamma \in (0, \pi/4]$:
Consider the building block for the branching construction $u_{bb} : (\theta, 1) \times \mathbb{R} \to \mathbb{R}$ defined for $(x, y) \in (\theta, 1) \times (0, \sin(\gamma))$ as (see Figure 3)

$$u_{bb}(x,y) = \begin{cases} y & \text{if } 0 \leq y \leq \frac{\sin(\gamma)}{4} + \sin(\gamma)\frac{x-\theta}{4(1-\theta)}, \\ -y + \sin(\gamma)\frac{x-\theta}{2(1-\theta)} + \frac{\sin(\gamma)}{2} & \text{if } \frac{\sin(\gamma)}{4} + \sin(\gamma)\frac{x-\theta}{4(1-\theta)} \leq y \leq \frac{\sin(\gamma)}{2}, \\ y + \sin(\gamma)\frac{x-\theta}{2(1-\theta)} - \frac{\sin(\gamma)}{2} & \text{if } \frac{\sin(\gamma)}{2} \leq y \leq \frac{3\sin(\gamma)}{4} - \sin(\gamma)\frac{x-\theta}{4(1-\theta)}, \\ -y + \sin(\gamma) & \text{if } \frac{3\sin(\gamma)}{4} - \sin(\gamma)\frac{x-\theta}{4(1-\theta)} \leq y \leq \sin(\gamma). \end{cases}$$

and extended $\sin(\gamma)$-periodically in y. Note that $u(\theta, y) = \frac{1}{2}u(1, 2y)$ and $u(\cdot, 0) = u(\cdot, \sin(\gamma)) = 0$.

Additionally, define the function $u_\partial : conv\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \begin{pmatrix}0\\ \sin(\gamma)\end{pmatrix}\right\} =: \Delta_\gamma \to \mathbb{R}$ as (see Figure 2)

$$u_\partial(x,y) = \begin{cases} \frac{\sin(\gamma)}{\cos(\gamma)}x + y & \text{if } y \leq \sin(\gamma) - \frac{\sin(\gamma)}{2\cos(\gamma)}(x+\cos(\gamma)), \\ -y + \sin(\gamma) & \text{if } y \geq \sin(\gamma) - \frac{\sin(\gamma)}{2\cos(\gamma)}(x+\cos(\gamma)). \end{cases}$$

Now, let $K, M \in \mathbb{N}$ be fixed. Then we define the function $u_K : S_\gamma \to \mathbb{R}$ as follows (see Figure 2):

(1) Let $0 \leq m \leq M-1$. Then define for $(x,y) \in (1 - \frac{m}{M}\cos(\gamma), 1) \times (\frac{m\sin(\gamma)}{M}, \frac{(m+1)\sin(\gamma)}{M})$

$$u_K(x,y) = \frac{1}{M} u_{bb}(1, My).$$

(2) Let $0 \leq k \leq K$, $0 \leq m \leq M-1$, $a \in \{0,1\}^k$ and

$$(x,y) \in \left(\theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}, \theta^k - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}\right)$$
$$\times \left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M} + 2^{-k}\frac{\sin(\gamma)}{M}\right).$$

Then define

$$u_K(x,y) = \frac{1}{M} 2^{-k} u_{bb}\left(\theta^{-k}\left(x + \frac{m}{M}\cos(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}\right), 2^k M y\right).$$

(3) Let $0 \leq k \leq K$, $0 \leq m \leq M-1$, $a \in \{0,1\}^k$ and

$$(x,y) \in \left(\theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M} - 2^{-k-1}\frac{\cos(\gamma)}{M}, \theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}\right)$$
$$\times \left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M} + 2^{-k-1}\frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M} + 2^{-k}\frac{\sin(\gamma)}{M}\right).$$

Then define u_K constant in x-direction as

$$u_K(x,y) = \frac{1}{M} 2^{-k} u_{bb}(\theta, 2^k M y).$$

(4) Let $a \in \{0,1\}^{K+1}$, $0 \leq m \leq M-1$ and

$$(x,y) \in \left(-\frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{K+1} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}, \theta^{K+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{K+1} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}\right)$$
$$\times \left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{K+1} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{K+1} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M} + 2^{-K-1}\frac{\sin(\gamma)}{M}\right).$$

Then define u_K constant in x-direction as

$$u_K(x,y) = \frac{1}{M} 2^{-K} u_{bb}(\theta, 2^K M y).$$

(5) Let $\ell \in \{0, \dots, 2^{K+1}M - 1\}$ and

$$(x,y) \in conv\left\{\frac{\ell}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \frac{\ell+1}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \frac{\ell}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix} + \frac{1}{2^{K+1}M}\begin{pmatrix}0\\ \sin(\gamma)\end{pmatrix}\right\}.$$

Then we define

$$u_K(x,y) = \frac{1}{M} 2^{-K-1} u_\partial\left(2^{K+1}M\left(x + \frac{\ell}{2^{K+1}M}\cos(\gamma)\right), 2^{K+1}M\left(y - \frac{\ell}{2^{K+1}M}\sin(\gamma)\right)\right).$$

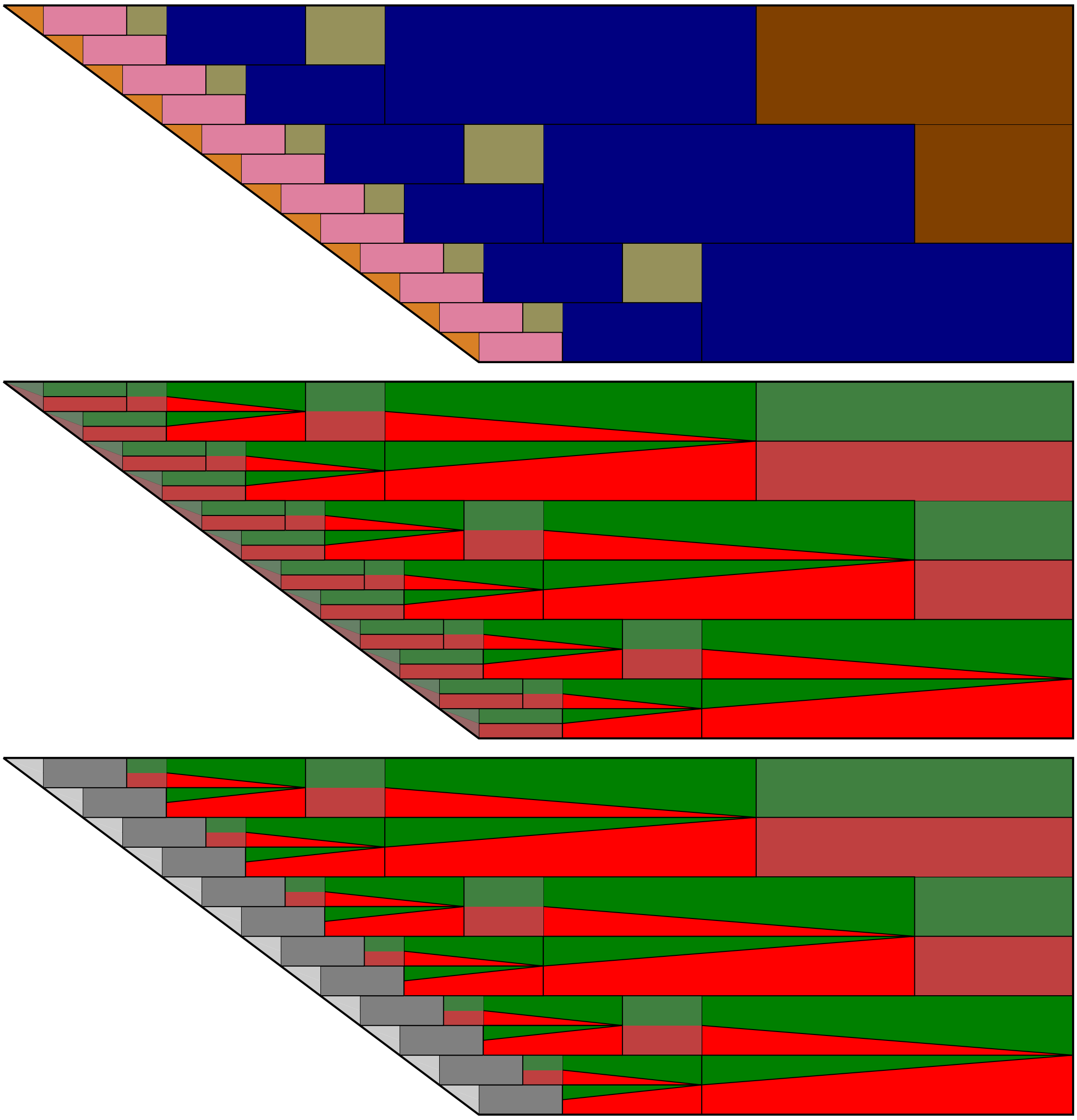

FIGURE 3. Sketch of the Kohn-Müller type branching constructions. Top: Sketch of the regions appearing in the definition of u_K in (1) (brown), (2) (blue), (3) (beige), (4) (pink), and (5) (orange) on the domain S_γ. In the regions (1) and (3) the function is constant in x-direction. Middle: Sketch for small γ. The interpolation between the boundary values and the branching construction uses an extension to region (4) which is constant in x and the function u_∂ as a building block in region (5). The regions with $\partial_2 u_K = 1$ is colored in red, the region with $\partial_2 u_K = -1$ is colored in green. Bottom: Sketch for large γ. In region (4) a linear interpolation in x to $u_K = 0$ is used. Then u_K is the extended by 0 into region (5). The regions with $\partial_2 u_K = 1$ is colored in red, the region with $\partial_2 u_K = -1$ is colored in green.

We note that by definition we have $u_K(\cdot,0) = u_K(\cdot,\sin(\gamma)) = 0$ and $u_K(-t\cos(\gamma), t\sin(\gamma)) = 0$ for all $t \in (0,1)$. Moreover, we estimate using that $\cos(\gamma) \geq \cos(\pi/4) > 0$, $\partial_1 u = 0$ a.e. in regions (1), (3) and (4), and $\partial_2 u_K \in \{\pm 1\}$ a.e.

$$
\begin{aligned}
&\int_{S_\gamma} (\partial_1 u_K)^2 + \min\{|\partial_2 u_K - 1|, |\partial_2 u_K + 1|\}^2 \, dxdy \\
\leq &\sum_{k=0}^{K} \frac{1}{M^2}(4\theta)^{-k} \int_\theta^1 \int_0^{\sin(\gamma)} (\partial_1 u_{bb}(x,y))^2 \, dxdy + \frac{1}{M} 2^{-K-1} \int_{\Delta_\gamma} (\partial_1 u_\partial(x,y))^2 \, dxdy \\
\leq &C \left(\frac{1}{M^2} \sin(\gamma)^3 + \frac{1}{M} 2^{-K-1} \sin(\gamma)^3 \right)
\end{aligned} \tag{5.1}
$$

and

$$
\begin{aligned}
|D^2 u_K|(S_\gamma) &\leq |\partial_2\partial_2 u_K|(S_\gamma) + 2|\partial_2\partial_1 u_K|(S_\gamma)| + |\partial_1\partial_1 u_K|(S_\gamma) \\
&\leq C \left(\sum_{k=0}^{K+1} M 2^k \theta^k + K + 1 + \sin(\gamma) K + \sum_{k=0}^{K+1} 2^{-k}\theta^{-k} \sin(\gamma) \right) \\
&\leq C \left(M + K + 2^{-K}\theta^{-K} \sin(\gamma) \right) \\
&\leq C \left(M + K + 2^K \sin(\gamma) \right).
\end{aligned} \tag{5.2}
$$

Eventually, choose $M = \lceil \sigma^{-1/3} \sin(\gamma) \rceil \leq 2\sigma^{-1/3}\sin(\gamma)$ and K such that $K = \lceil \frac{\log(\sigma^{-1/3}\sin(\gamma))}{\log(2)} \rceil \leq 2 \frac{\sigma^{-1/3}\sin(\gamma)}{\log(2)}$ (recall that $\sigma \leq \sin(\gamma)^3$). It then follows from (5.1) and (5.2)

$$
\begin{aligned}
E_{\sigma,\pi/2,2}(u_K; S_\gamma) = &\int_{S_\gamma} (\partial_1 u_K)^2 + \min\{|\partial_2 u_K - 1|, |\partial_2 u_K + 1|\}^2 \, dxdy + \sigma |D^2 u_K|(S_\gamma) \\
\leq &C \left(\sigma^{2/3} \sin(\gamma) + \sigma^{1/3} 2^{-K} \sin(\gamma)^2 + \sigma K + \sigma 2^K \sin(\gamma) \right) \\
\leq &C \sigma^{2/3} \sin(\gamma).
\end{aligned}
$$

Step 1b: Branching on S_γ for $\gamma \in (\pi/4, \pi/2]$:
Next, consider $\frac{\pi}{4} < \gamma \leq \frac{\pi}{2}$. Again, fix $K, M \in \mathbb{N}$. In the regions (1), (2), (3) we define the function $u_K : S_\gamma \to \mathbb{R}$ as before. Precisely (see Figure 2),

(1) Let $0 \leq m \leq M-1$. Then define for $(x,y) \in (1 - \frac{m}{M}\cos(\gamma), 1) \times (\frac{m\sin(\gamma)}{M}, \frac{(m+1)\sin(\gamma)}{M})$

$$
u_K(x,y) = \frac{1}{M} u_{bb}(1, My).
$$

(2) Let $0 \leq k \leq K$, $0 \leq m \leq M-1$, $a \in \{0,1\}^k$ and

$$
\begin{aligned}
(x,y) \in &\left(\theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}, \theta^k - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M} \right) \\
\times &\left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\sin(\gamma)}{M} + 2^{-k}\frac{\sin(\gamma)}{M} \right).
\end{aligned}
$$

Then define

$$
u_K(x,y) = \frac{1}{M} 2^{-k} u_{bb} \left(\theta^{-k} \left(x + \frac{m}{M}\cos(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M} \right), 2^k M y \right).
$$

(3) Let $0 \leq k \leq K$, $0 \leq m \leq M-1$, $a \in \{0,1\}^k$ and

$$(x,y) \in \left(\theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M} - 2^{-k-1}\frac{\cos(\gamma)}{M}, \theta^{k+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{k} a_{k'} 2^{-k'} \frac{\cos(\gamma)}{M}\right)$$
$$\times \left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'}\frac{\sin(\gamma)}{M} + 2^{-k-1}\frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{k} a_{k'} 2^{-k'}\frac{\sin(\gamma)}{M} + 2^{-k}\frac{\sin(\gamma)}{M}\right).$$

Then define

$$u_K(x,y) = \frac{1}{M} 2^{-k} u_{bb}(\theta, 2^k M y).$$

However, in this case we simply use linear interpolation in x to a vertical interface in region (4) and then extend u_K by 0 in region (5). Precisely:

(4) Let $a \in \{0,1\}^{K+1}$, $0 \leq m \leq M-1$ and

$$(x,y) \in \left(-\frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{K+1} a_{k'} 2^{-k'}\frac{\cos(\gamma)}{M}, \theta^{K+1} - \frac{m}{M}\cos(\gamma) - \sum_{k'=1}^{K+1} a_{k'} 2^{-k'}\frac{\cos(\gamma)}{M}\right)$$
$$\times \left(\frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{K+1} a_{k'} 2^{-k'}\frac{\sin(\gamma)}{M}, \frac{m}{M}\sin(\gamma) + \sum_{k'=1}^{K+1} a_{k'} 2^{-k'}\frac{\sin(\gamma)}{M} + 2^{-K-1}\frac{\sin(\gamma)}{M}\right).$$

Then define

$$u_K(x,y) = \theta^{-K-1}\left(x + \frac{m}{M}\cos(\gamma) + \sum_{k'=1}^{K+1} a_{k'} 2^{-k'}\frac{\cos(\gamma)}{M}\right)\frac{1}{M} 2^{-K} u_{bb}(\theta, 2^K M y).$$

(5) Let $\ell \in \{0,\dots,2^{K+1}M-1\}$ and

$$(x,y) \in conv\left\{\frac{\ell}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \frac{\ell+1}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix}, \frac{\ell}{2^{K+1}M}\begin{pmatrix}-\cos(\gamma)\\ \sin(\gamma)\end{pmatrix} + \frac{1}{2^{K+1}M}\begin{pmatrix}0\\ \sin(\gamma)\end{pmatrix}\right\}.$$

Then we define

$$u_K(x,y) = 0.$$

We only compute the energy contributions from the regions (4) and (5). The contribution in (4) for the term $\sigma|D^2u_K|$ can be estimated by a term of the form $C\sigma(M2^K\theta^K + 1)$, whereas the other term can be estimated by $C((4\theta)^{-K}M^{-2}\sin(\gamma)^3 + \theta^K\sin(\gamma))$ since the building block u_{bb} is bounded by $\sin(\gamma)$. Additionally, in region (5) we obtain an energy less than $2^{-K}\frac{1}{M}$. Hence, we find

$$\begin{aligned}
&\int_{S_\gamma} (\partial_1 u_K)^2 + \min\{|\partial_2 u_K - 1|, |\partial_2 u_K + 1|\}^2\, dxdy + \sigma|D^2 u_K|(S_\gamma)\\
\leq & C\left(\frac{1}{M^2} + 2^{-K}\frac{1}{M} + \theta^K\right) + C\sigma(M + K + 2^K)\\
\leq & C\left(\frac{1}{M^2} + 2^{-K}\frac{1}{M} + \sigma(M + 2^K)\right).
\end{aligned}$$

Then choose $M = \lceil \sigma^{-1/3} \rceil$ and $K = \frac{\log \sigma^{-1/3}}{\log(2)} \leq 2\frac{\sigma^{-1/3}}{\log(2)}$. It follows that

$$\begin{aligned}
E_{\sigma,\pi/2,2}(u_K; S_\gamma) = & \int_{S_\gamma} (\partial_1 u_N)^2 + \min\{|\partial_2 u_N - 1|, |\partial_2 u_N + 1|\}^2\, dxdy + \sigma|D^2 u_N|(S_\gamma)\\
&\leq C\sigma^{2/3} \leq C\sigma^{2/3}\sin(\gamma),
\end{aligned}$$

since $\pi/4 \leq \gamma \leq \pi/2$.

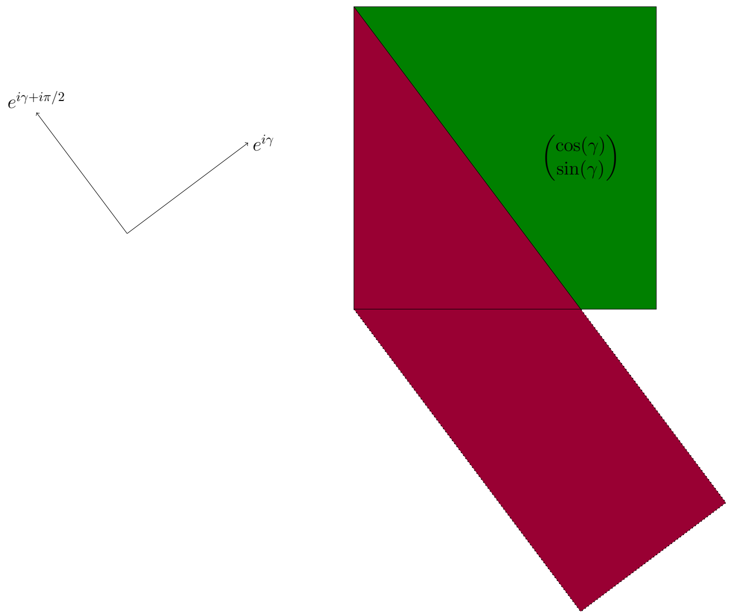

FIGURE 4. Sketch of the construction of the test function u in Step 1c. The purple area extends T_γ and is a rotated version of S_γ. In this region, we use a rotated version of the branching constructions from Step 1a and 1b in the proof of the upper bound of Theorem 3.1 for $N = 2$, Step 1 in the proof of Proposition 6.4 ($N = 4$) and Step 2 in the proof of Proposition 6.4 ($N = 3$).

Step 1c: Branching on $(0,1)^2$*:*
Define the set

$$T_\gamma = \left\{(x,y) \in (0,1)^2 : (x, y-1)\cdot e^{i\gamma} \le 0\right\}.$$

Let $w : S_\gamma \to \mathbb{R}$ be the function from step 1a or 1b according to the case $\gamma \in (\pi/4, \pi/2]$ or $\gamma \in (0, \pi/4]$, respectively. Note that $w(x, \sin(\gamma)) = 0$ for all $x \in (-\cos(\gamma), 1)$. Then we define (see Figure 4)

$$u(x,y) = \begin{cases} e^{i\gamma}\cdot(x, y-1) & \text{if } (x,y) \notin T_\gamma, \\ w(R_\gamma(x, y-1)) & \text{if } (x,y) \in T_\gamma, \end{cases}$$

where R_γ is the rotation by the angle $\pi/2 - \gamma$. Using step 1, we obtain that

$$E_{\sigma,\gamma,2}(u) \le E_{\sigma,\pi/2,2}(w; S_\gamma) + 4\sigma \le C|\sin(\gamma)|\sigma^{2/3} + 4\sigma \le C|\sin(\gamma)|\sigma^{2/3},$$

where we used that $\sigma^{1/3} \le |\sin(\gamma)|$.

Step 2: Lower bound:
Let $u \in \mathcal{A}$ and assume for simplicity $-\pi/2 \le \gamma \le 0$. Again, note that for $\sigma \ge |\sin(\gamma)|^3$ the lower bound was already shown in Proposition 4.1, c.f. Remark 1. Hence, we assume that $\sigma \le |\sin(\gamma)|^3$. Now, let $t := \frac{1}{4}\sigma^{1/3}|\sin(\gamma)|^{-1} \le 1/4$, $\xi := e^{i\gamma}$ and $\xi^\perp := e^{i\gamma+i\pi/2}$. Then find $a \in (t, \frac{1}{2})$ such that (see Figure 5)

$$E_{\sigma,\gamma,2}\left(u; (0,1)^2 \cap \{(0,y) + r\xi^\perp : y \in (a-t, a), r \in (0,1)\}\right) \le 4t\, E_{\sigma,\gamma,2}(u).$$

Let us write $S := (0,1)^2 \cap \{(0,y) + r\xi^\perp : y \in (a-t,a), r \in (0,1)\}$. Now, we write for $r \in (0, 1/2)$

$$L_r = \left\{a + r\xi^\perp + s\xi : s \in (0, |\sin(\gamma)|t)\right\} \subseteq S.$$

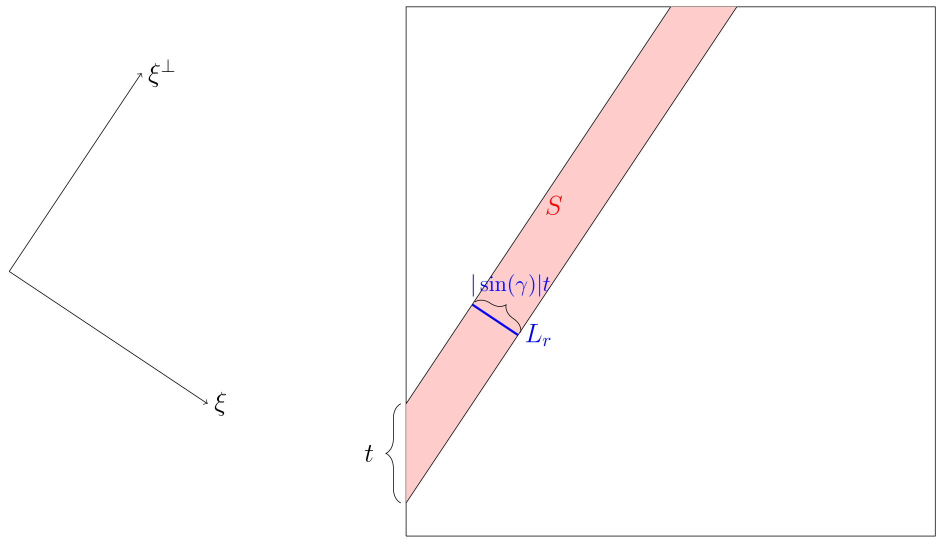

FIGURE 5. Sketch of the choices made in the proof of the lower bound in Theorem 3.1 for $N = 2$.

Then find by a slicing and Fubini-type argument $r \in (0, 1/2)$ such that

$$\int_{L_r} \operatorname{dist}(\nabla u, K_{\gamma,N})^2 \, d\mathcal{H}^1 + \sigma |\partial_\xi \nabla u|(L_r) \leq 2E_{\sigma,\gamma,2}(u; S) \leq (8t)E_{\sigma,\gamma,2}(u). \tag{5.3}$$

Now one of the following options holds true:

(1) $\operatorname{dist}(\nabla u, K_{\gamma,N}) = |\nabla u - \xi|$ almost everywhere on L_r,
(2) $\operatorname{dist}(\nabla u, K_{\gamma,N}) = |\nabla u + \xi|$ almost everywhere on L_r,
(3) $|\partial_\xi \nabla u|(L_r) \geq 1/4$,
(4) $\operatorname{dist}(\nabla u, K_{\gamma,N}) \geq 1/4$ almost everywhere on L_r.

If (3) or (4) hold true, we obtain from (5.3)

$$\min\{\sigma/4, |\sin(\gamma)|t/16\} \leq 8tE_{\sigma,\gamma,2}(u). \tag{5.4}$$

Let us now assume that (1) holds true (case (2) can be treated similarly). By the triangle inequality we obtain

$$\frac{1}{4}(|\sin(\gamma|t)^2 = \frac{1}{4}\mathcal{H}^1(L_r)^2 \leq \min_{c\in\mathbb{R}} \|u - \xi \cdot (x,y) - c\|_{L^1(L_r)} + \|u\|_{L^1(L_r)}.$$

On the other hand, we estimate using Poincaré's inequality

$$\begin{aligned}\min_{c\in\mathbb{R}} \|u(x,y) - \xi \cdot (x,y) - c\|_{L^1(L_r)} &\leq (|\sin(\gamma)|t)\|\partial_\xi u(x,y) - 1\|_{L^1(L_r)} \\ &\leq (|\sin(\gamma)|t)^{3/2}\|\nabla u(x,y) - \xi\|_{L^2(L_r)} \leq \sqrt{8}(|\sin(\gamma)|t)^{3/2} t^{1/2} E_{\sigma,\gamma,2}(u)^{1/2}\end{aligned}$$

and

$$\|u\|_{L^1(L_r)} \leq \|\partial_{\xi^\perp} u\|_{L^1(S)} \leq (|\sin(\gamma)|t)^{1/2}\|\operatorname{dist}(\nabla u, K_{\gamma,N})\|_{L^2(S)} \leq 2(|\sin(\gamma)|t)^{1/2} t^{1/2} E_{\sigma,\gamma,2}(u)^{1/2}.$$

Consequently, we obtain

$$E_{\sigma,\gamma,2}(u) \geq \frac{1}{128}\min\{|\sin(\gamma)|, |\sin(\gamma)|^3 t^2\}.$$

Together with (5.4) this yields

$$E_{\sigma,\gamma,N}(u) \geq \frac{1}{128} \min\{\sigma t^{-1}, |\sin(\gamma)|, |\sin(\gamma)|^3 t^2\}\} \geq \frac{1}{2048}\sigma^{2/3}|\sin(\gamma)|,$$

since $|\sin(\gamma)| \geq |\sin(\gamma)|^3 \geq |\sin(\gamma)|\sigma^{2/3}$. □

6. The 3- and 4-Gradient Problem

6.1. **The lower bound.** We split the proof of the lower bound into different steps depending on the size of $|\sin(\gamma)|$. The general strategy is similar to the one used in [30], c.f. also [31].

We start with proving the lower bound for all relatively large (in absolute value) angles $\gamma \in \Gamma_N$. Here, the term $|\log|\sin(\gamma)||$ in the denominator of the lower bound will not play a role as it is uniformly bounded.

Proposition 6.1. Let $N = 3, 4$ and $\gamma_0 \in \Gamma_N$ with $\gamma_0 > 0$. Then there exists a constant $c_B > 0$ such that for all $\gamma \in \Gamma_0$ with $|\gamma| \geq \gamma_0$ it holds

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \geq c_B \min\{\sigma(|\log\sigma| + 1), 1\}.$$

Proof. Let $u \in \mathcal{A}$. We fix $\alpha_0 = \min\{\sin(\gamma_0), \cos(\pi/4), 1/16\}$ and $c_1 = \frac{\alpha_0^2}{256} \leq 1$. We assume that $E_{\sigma,\gamma,N}(u) \leq \frac{c_1^{1/2}}{16} \min\{\sigma(|\log\sigma| + 1), 1\}$. Otherwise there is nothing to show.

Let $0 < s \leq c_1$. Then find an interval $I \subseteq (0, 1)$ of length $\frac{s}{c_1}$ such that

$$E_{\sigma,\gamma,N}(u; (0, 1) \times I) \leq \frac{8s}{c_1} E_{\sigma,\gamma,N}(u)$$

$$\text{and } \int_I \operatorname{dist}(\nabla u(s, t), K_{\gamma,N})^2\, dt + \sigma|\partial_2\nabla u(s, \cdot)|(I) \leq \frac{8s}{c_1}\left(\int_0^1 \operatorname{dist}(\nabla u(s, t), K_{\gamma,N})^2\, dt + \sigma|\partial_2\nabla u(s, \cdot)|(0, 1)\right).$$

Then one of the following three conditions has to hold on I:

(1) $|\partial_2\nabla u(s, \cdot)|(I) \geq \frac{1}{4}$,
(2) $\exists\xi \in K_{\gamma,N}$ such that $|\nabla u(s, t) - \xi| \leq \frac{1}{2}$ for almost all $t \in I$,
(3) $\int_I \operatorname{dist}(\nabla u(s, t), K_{\gamma,N})^2\, dt \geq \frac{1}{16}|I|$.

If (1) or (3) is true, we obtain

$$\int_I \operatorname{dist}(\nabla u(s, t), K_{\gamma,N})^2\, dt + \sigma|\partial_2\nabla u(s, \cdot)|(I) \geq \min\{\sigma/4, |I|/16\}.$$

Lastly, we assume that (2) is true. By the triangle inequality, we observe (recall that $|\xi_2| \geq \sin(\gamma_0)$ by definition of Γ_N)

$$\frac{\alpha_0}{4}|I|^2 \leq \min_{c\in\mathbb{R}} \|\xi_2 y - c\|_{L^1(I)} \leq \|u(s, \cdot)\|_{L^1(I)} + \min_{c\in\mathbb{R}} \|u(s, y) - \xi_2 y - c\|_{L^1(I)}.$$

Let us first assume that $\frac{\alpha_0}{8}|I|^2 \leq \|u(s, \cdot)\|_{L^1(I)}$. We estimate

$$\begin{aligned}\frac{\alpha_0}{8}|I|^2 \leq \int_I |u(s, t)|\, dt &\leq \int_0^s \int_I \operatorname{dist}(\nabla u(s, t), K_{\gamma,N})\, dtds + s|I| \\ &\leq E_{\sigma,\gamma,N}(u; (0, 1) \times I)^{1/2} s^{1/2}|I|^{1/2} + s|I|.\end{aligned}$$

By the definition of c_1 it follows $s \leq \frac{\alpha_0}{16}|I|$. Consequently, we obtain

$$\frac{\alpha_0}{16}|I|^2 \leq E_{\sigma,\gamma,N}(u)^{1/2}(8|I|)^{1/2}\left(\frac{\alpha_0}{16}\right)^{1/2}|I|,$$

which in turn yields

$$\frac{1}{8c_1^{1/2}} s \leq \frac{\alpha_0}{128}|I| \leq E_{\sigma,\gamma,N}(u) \leq \frac{c_1^{1/2}}{16} \min\{1, \sigma(|\log\sigma| + 1)\},$$

i.e., $s \leq \frac{c_1}{2} \min\{1, \sigma(|\log\sigma| + 1)\}$.

On the other hand, if $\frac{\alpha_0}{8}|I|^2 \leq \min_{c\in\mathbb{R}} \|u(s,y)-\xi_2 y - c\|_{L^1(I)}$ we estimate using Poincaré's and Hölder's inequality

$$\begin{aligned}\frac{\alpha_0}{8}|I|^2 &\leq \min_{c\in\mathbb{R}} \|u(s,y)-\xi_2 y - c\|_{L^1(I)} \\ &\leq |I|\, \|\partial_2 u(s,y) - \xi_2\|_{L^1(I)} \\ &\leq |I|^{3/2} \left(\int_I \operatorname{dist}(\nabla u(s,y), K_{\gamma,N})^2\, dy \right)^{1/2}.\end{aligned}$$

Here, we used that (2) implies that $|\nabla u(s,t) - \xi| = \operatorname{dist}(\nabla u(s,t), K_{\gamma,N})$ for a.e. $t \in I$. It follows that

$$\frac{\alpha_0^2}{64}|I| \leq \int_I \operatorname{dist}(\nabla u(s,y), K_{\gamma,N})^2\, dy.$$

Combining the cases (1), (2) and (3) we obtain for all $\frac{c_1}{2}\min\{1, \sigma(|\log\sigma|+1)\} \leq s \leq c_1$

$$\begin{aligned}8|I| \left(\int_0^1 \operatorname{dist}(\nabla u(s,y), K_{\gamma,N})^2\, dy + \sigma|\partial_2 \nabla u(s,\cdot)|(0,1) \right) &\geq \int_I \operatorname{dist}(\nabla u(s,y), K_{\gamma,N})^2\, dy + \sigma|\partial_2\nabla u(s,\cdot)|(I) \\ &\geq \min\left\{ \frac{\sigma}{4}, \frac{\alpha_0^2}{64}|I| \right\}.\end{aligned}$$

Consequently, we find

$$\begin{aligned}E_{\sigma,\gamma,N}(u) &\geq \int_{\frac{c_1}{2}\min\{1,\sigma(|\log\sigma|+1)\}}^{c_1} \left(\int_0^1 \operatorname{dist}(\nabla u(s,y), K_{\gamma,N})^2\, dy + \sigma|\partial_2\nabla u(s,\cdot)|(0,1) \right) ds \\ &\geq \int_{\frac{c_1}{2}\min\{1,\sigma(|\log\sigma|+1)\}}^{c_1} \min\left\{ \frac{c_1\sigma}{32 s}, \frac{\alpha_0^2}{512} \right\} ds. \end{aligned} \tag{6.1}$$

Let us first assume $\sigma \leq \sigma_0$ for some fixed $0 < \sigma_0 \leq 1$. In addition, note that $\frac{\alpha_0^2}{512} \geq \frac{\alpha_0^2}{32}\frac{c_1\sigma}{32s}$ for all $s \geq \frac{c_1}{2}\sigma$. Consequently, (6.1) implies that

$$\begin{aligned}E_{\sigma,\gamma,N}(u) \geq& \frac{\alpha_0^2}{32} \int_{\frac{c_1}{2}\sigma(|\log\sigma|+1)}^{c_1} \frac{c_1\sigma}{32s}\, ds \\ =& \frac{c_1\alpha_0^2}{1024}\sigma\left(|\log\sigma| + \log(2) - \log(|\log\sigma|+1) \right) \\ \geq& \frac{c_1\alpha_0^2\log(2)}{2048}\,\sigma\left(|\log\sigma|+1\right)\end{aligned}$$

if σ_0 is small enough.

On the other hand, assume $\sigma > \sigma_0$. Then we it follows from (6.1)

$$E_{\sigma,\gamma,N}(u) \geq \int_{c_1/2}^{c_1} \min\{\frac{\sigma_0}{32}, \frac{\alpha_0^2}{512}\}\, dt = \frac{c_1}{2}\min\{\frac{\sigma_0}{32}, \frac{\alpha_0^2}{512}\}.$$

Hence, setting $c_B = \min\{\frac{\frac{c_1\sigma_0}{64}, c_1\alpha_0^2\log(2)}{2048}\}$ the assertion follows.

□

Next, we prove the lower bound for all small (in absolute value) angles and small σ. The proof is similar to the proof of [30, Lemma 6], see also [31].

Proposition 6.2. Let $N = 3, 4$. There exists $\pi/4 > \gamma_0 > 0$, $K \in \mathbb{N}$ and a constant $c_D > 0$ such that for all $\gamma \in \Gamma_N$ with $|\gamma| \leq \gamma_0$ and $\sigma \leq |\sin(\gamma)|^K$ it holds

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \geq c_D\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||} + 1 \right).$$

Proof. Step 1: Preliminaries. Fix $K=32$. Moreover, let $\gamma_0>0$ be such that $k\sin(\gamma_0)^{k/4}\le\frac{1}{3\cdot256\cdot9\cdot48\cdot120}$ for all $k\ge K$. Let $u\in\mathcal{A}$. Let $k>K$ and assume that $\sigma\in(|\sin(\gamma)|^k,|\sin(\gamma)|^{k-1})$. Then $\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\le k+1\le 2k$. Hence, it suffices to prove that $E_{\sigma,\gamma,N}(u)\ge ck\sigma$ for a universal constant $c>0$. In particular, we may assume that $E_{\sigma,\gamma,N}(u)\le k\sigma$.

Next, note that by the choice of γ_0 it follows for all $|\gamma|\le\gamma_0$ that $|\sin(\gamma)|\le\frac14$ which implies $\frac12|\sin(\gamma)|^i-\frac32|\sin(\gamma)|^{i+1}\ge\frac18|\sin(\gamma)|^i$. Then find for $i\le\lfloor k/4\rfloor$ a point $x_i\in\left(\frac12|\sin(\gamma)|^i,\frac32|\sin(\gamma)|^i\right)$ such that $\nabla u(x_i,\cdot)\in BV((0,1);\mathbb{R}^2)$ and

$$E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))\le|\sin(\gamma)|^{-i}E_{\sigma,\gamma,N}\left(u;\left(\frac12|\sin(\gamma)|^i,\frac32|\sin(\gamma)|^i\right)\times(0,1)\right).$$

Claim: For all $i\le\lfloor k/4\rfloor$ there exists a constant $c>0$ such that it holds

$$|\sin(\gamma)|^iE_{\sigma,\gamma,N}(u,\{x_i\}\times(0,1))+E_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1))\ge c\sigma.$$

Before we prove the claim, we briefly show that it implies the desired lower bound. Indeed, we have

$$\begin{aligned}E_{\sigma,\gamma,N}(u)&\ge\frac12\sum_{i=1}^{\lfloor k/4\rfloor}E_{\sigma,\gamma,N}\left(u,(\frac12|\sin(\gamma)|^i,\frac32|\sin(\gamma)|^i)\times(0,1)\right)+E_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1))\\&\ge\frac12\sum_{i=1}^{\lfloor k/4\rfloor}|\sin(\gamma)|^iE_{\sigma,\gamma,N}(u,\{x_i\}\times(0,1))+E_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1))\ge\frac{c}{16}k\sigma.\end{aligned}$$

Hence, it only remains to show the claim.

Step 2: Estimates on vertical slices. Let $t=120|\sin(\gamma)|^i$. Then find $y\in(0,1)$ with $(y,y+t)\subseteq(0,1)$ such that

$$\begin{aligned}&E_{\sigma,\gamma,N}(u;(0,1)\times(y,y+t))\le48tE_{\sigma,\gamma,N}(u),\\&E_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(y,y+t))\le48tE_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1))\\&E_{\sigma,\gamma,N}(u;\{x_i\}\times(y,y+t))\le48tE_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))\\\text{and }&E_{\sigma,\gamma,N}(u;\{x_{i+1}\}\times(y,y+t))\le48tE_{\sigma,\gamma,N}(u;\{x_{i+1}\}\times(0,1)).\end{aligned}$$

Then one of the following holds in $(y,y+t)$ (recall that the minimal distance between two points in $K_{\gamma,N}$ is $\sqrt2$)

(1) $|\partial_2\nabla u(x_i,\cdot)|(y,y+t)\ge\frac14$,
(2) $\exists\xi\in K_{\gamma,N}$ such that $\operatorname{dist}(\nabla u(x_i,\cdot),K_{\gamma,N})\le3|\nabla u(x_i,\cdot)-\xi|$.

Suppose first that (1) holds. Then

$$\frac{\sigma}{4}\le E_{\sigma,\gamma,N}(u;\{x_i\}\times(y,y+t))\le48t\,E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))=48\cdot120\,|\sin(\gamma)|^iE_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))$$

and thus the claim follows in this case.

For the rest of the proof we will now assume that (2) holds. First, recall that by the definition of Γ_N we have $|\xi_2|\ge|\sin(\gamma)|$. By the triangle inequality we obtain

$$\frac14|\xi_2|t^2\le\|u(x_i,s+t/2)-u(x_i,s)-\xi_2t/2\|_{L^1(y,y+t/2)}+\|u(x_i,s+t/2)-u(x_i,s)\|_{L^1(y,y+t/2)}.$$

Hence, it holds one of the following:

(a) $\frac18|\xi_2|t^2\le\|u(x_i,s+t/2)-u(x_i,s)-\xi_2t/2\|_{L^1(y,y+t/2)}$,
(b) $\frac18|\xi_2|t^2\le\|u(x_i,s+t/2)-u(x_i,s)\|_{L^1(y,y+t/2)}$.

Let us first assume that (a) holds. We estimate

$$
\begin{aligned}
(6.2)\qquad \left|\int_y^{y+t/2} u(x_i,s+t/2)-u(x_i,s)-\xi_2 t/2\,ds\right| &\leq \int_y^{y+t/2}\int_s^{s+t/2} |\partial_2 u(x_i,r)-\xi_2|\,dr\,ds \\
&\leq t^{1/2}\int_y^{y+t/2}\left(\int_y^{y+t}|\partial_2 u(x_i,r)-\xi_2|^2\,dr\right)^{1/2} ds \\
&\leq \sqrt{9\cdot 48}t^2\, E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))^{1/2} \\
&\leq \frac{1}{16}t^2|\sin(\gamma)| \leq \frac{1}{16}|\xi_2|t^2.
\end{aligned}
$$

For the third estimate we used that by the choice of x_i and γ_0 it holds $E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1)) \leq |\sin(\gamma)|^{-i}k\sigma \leq k|\sin(\gamma)|^{k-i-2}|\sin(\gamma)|^2 \leq \frac{1}{9\cdot 48\cdot 256}|\sin(\gamma)|^2$. Next, observe that it holds by (2) and Poincaré's inequality for $a=\frac{2}{t}\int_y^{y+t/2} u(x_i,s+t/2)-u(x_i,s)-\xi_2 t/2\,ds$

$$
\begin{aligned}
(6.3)\qquad \|u(x_i,s+t/2)-u(x_i,s)-\xi_2 t/2-a\|_{L^1(y,y+t/2)} &\leq t\|\partial_2 u(x_i,s+t/2)-\partial_2 u(x_i,s)\|_{L^1(y,y+t/2)} \\
&\leq t\|\partial_y u(x_i,s)-\xi_2\|_{L^1(y,y+t)} \\
&\leq 3t^{3/2}E_{\sigma,\gamma,N}(u;\{x_i\}\times(y,y+t))^{1/2} \\
&\leq \sqrt{9\cdot 48}t^2 E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))^{1/2}.
\end{aligned}
$$

Consequently, since $\|u(x_i+t/2,s)-u(x_i,s)-\xi_2 t/2\|_{L^1(y,y+t/2)} \geq \frac{1}{8}|\xi_2|t^2$ it follows from (6.2) and (6.3) that

$$
432\cdot t^4 E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1)) \geq \left(\frac{1}{8}|\xi_2|t^2-|a|t/2\right)^2 \geq \frac{t^4}{256}|\xi_2|^2 \geq \frac{t^4}{256}|\sin(\gamma)|^2 \geq \frac{t^4}{256}|\sin(\gamma)|^{-i}\sigma,
$$

which yields the claim if (2) and (a) hold.

Hence, from now on we will assume that (b) holds i.e., $\|u(x_i,s)-u(x_i,s+t/2)\|_{L^1(y,y+t/2)} \geq \frac{1}{8}|\xi_2|t^2$.

Step 3: An estimate for horizontal difference quotients. First, observe that it holds for all $s\in(y,y+t/2)$

$$
\begin{aligned}
|u(x_i,s+t/2)-u(x_i,s)| &\leq |\xi_2|t/2+3t^{1/2}E_{\sigma,\gamma,N}(u;\{x_i\}\times(0,1))^{1/2} \\
&\leq |\xi_2|t/2+3t^{1/2}\left(|\sin(\gamma)|^{-i}k\sigma\right)^{1/2} \leq |\xi_2|t,
\end{aligned}
$$

where we used that $|\sin(\gamma)|^{-i}k\sigma \leq |\sin(\gamma)|^{k-i-1}k \leq \frac{1}{9\cdot 4}|\sin(\gamma)|^2 \leq \frac{1}{9\cdot 4}|\xi_2|^2$. Next, define

$$
A_i = \left\{ s\in(y,y+t) : E_{\sigma,\gamma,N}(u;(0,1)\times\{s\}) \leq \frac{80}{t}E_{\sigma,\gamma,N}(u;(0,1)\times(y,y+t)) \right\}
$$

Then $\mathcal{L}^1(A_i) \geq \frac{79}{80}t$. For $s\in A_i$ we estimate

$$
\begin{aligned}
|u(x_{i+1},s)| &\leq x_{i+1}+x_{i+1}^{1/2}E_{\sigma,\gamma,N}(u;(0,1)\times\{s\})^{1/2} \\
&\leq x_{i+1}+x_{i+1}^{1/2}\left(\frac{80}{t}E_{\sigma,\gamma,N}(u;(0,1)\times(y,y+t))\right)^{1/2} \\
&\leq x_{i+1}+\sqrt{80\cdot 48}\cdot x_{i+1}^{1/2}E_{\sigma,\gamma,N}(u)^{1/2} \\
&\leq 2x_{i+1} \\
&\leq \frac{1}{40}|\sin(\gamma)|\,t \leq \frac{1}{40}|\xi_2|\,t.
\end{aligned}
$$

For the fourth inequality we used that we have $k\sigma \leq k|\sin(\gamma)|^{k-1-(i+1)}|\sin(\gamma)|^{i+1} \leq \frac{1}{2\cdot 48\cdot 80}|\sin(\gamma)|^{i+1} \leq \frac{1}{48\cdot 80}x_{i+1}$. Next, find for $s \in (y, y+t)$ a value $\bar{s} \in A_i$ such that $|s-\bar{s}| \leq \frac{1}{80}t$. Then we obtain

$$\begin{aligned}|u(x_{i+1},s)| \leq&|u(x_{i+1},s)-u(x_{i+1},\bar{s})|+|u(x_{i+1},\bar{s})|\\ \leq&|\xi_2|\,|s-\bar{s}|+3|s-\bar{s}|^{1/2}E_{\sigma,\gamma,N}(u;\{x_{i+1}\}\times(y,y+t))^{1/2}+\frac{1}{40}|\xi_2|\,t\\ \leq&\frac{1}{20}|\xi_2|t, \end{aligned}\tag{6.4}$$

where we used similarly to above that it holds $E_{\sigma,\gamma,N}(u;\{x_{i+1}\}\times(y,y+t)) \leq \frac{1}{9\cdot 80}|\sin(\gamma)|^2 t$. In particular, we obtain for almost all $s \in (y, y+t/2)$ that

$$\begin{aligned}&|u(x_i,s)-u(x_i,s+t/2)-u(x_{i+1},s)+u(x_{i+1},s+t/2)|\\ \leq&|u(x_i,s)-u(x_i,s+t/2)|+|u(x_{i+1},s)|+|u(x_{i+1},s+t/2)|\\ \leq&2|\xi_2|t.\end{aligned}\tag{6.5}$$

On the other hand, it holds by (b) and (6.4) that

$$\begin{aligned}&\|u(x_i,s)-u(x_i,s+t/2)-u(x_{i+1},s)+u(x_{i+1},s+t/2)\|_{L^1(y,y+t/2)}\\ \geq&\|u(x_i,s)-u(x_i,s+t/2)\|_{L^1(y,y+t/2)}-\|u(x_{i+1},s)\|_{L^1(y,y+t/2)}-\|u(x_{i+1},s+t/2)\|_{L^1(y,y+t/2)}\\ \geq&\frac{1}{8}|\xi_2|t^2-\frac{1}{20}|\xi_2|t^2\\ \geq&\frac{1}{16}|\xi_2|t^2.\end{aligned}\tag{6.6}$$

We define

$$S := \left\{ s \in (y, y+t/2) : \frac{1}{16}|\xi_2|t \leq |u(x_i,s)-u(x_i,s+t/2)-u(x_{i+1},s)+u(x_{i+1},s+t/2)| \leq 2|\xi_2|t \right\}.$$

Combining (6.5) and (6.6) yields

$$\frac{1}{16}|\xi_2|t^2 \leq \left(\frac{t}{2}-\mathcal{L}^1(S)\right)\frac{1}{16}|\xi_2|t+\mathcal{L}^1(S)2|\xi_2|t \leq \frac{1}{32}|\xi_2|t^2+2\mathcal{L}^1(S)|\xi_2|t,$$

which implies $\mathcal{L}^1(S) \geq \frac{t}{64}$. Next, note that $x_i - x_{i+1} \leq \frac{3}{2}|\sin(\gamma)|^i$. Hence, for a subset $(y, y+t/2)$ of size at least $\frac{t}{64}$ it holds

$$5|\xi_2| \leq \left|\frac{u(x_i,s)-u(x_{i+1},s)}{x_i-x_{i+1}}-\frac{u(x_i,s+t/2)-u(x_{i+1},s+t/2)}{x_i-x_{i+1}}\right|.$$

It follows for a subset of $(y, y+t)$ whose measure is at least $\frac{t}{64}$ that

$$\left|\frac{u(x_i,s)-u(x_{i+1},s)}{x_i-x_{i+1}}-\xi_1\right| \geq \frac{5}{2}|\sin(\gamma)|.\tag{6.7}$$

Step 4: Lower bound for $E_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1)$. Let us now fix such an $s \in (y, y+t)$ for which (6.7) holds. Moreover, let us assume that $|\partial_1\nabla u(\cdot,s)|(x_{i+1},x_i) \leq 1/10$. By (2) it follows that $|\nabla u(x,s)-\xi| \leq 6\,\mathrm{dist}(\nabla u(x,s),K_{\gamma,N})$ for a.e. $x \in (x_{i+1},x_i)$. Recalling that $x_i - x_{i+1} \geq \frac{1}{2}|\sin(\gamma)|^i - \frac{3}{2}|\sin(\gamma)|^{i+1} \geq \frac{1}{8}|\sin(\gamma)|^i$

it follows that

$$\begin{aligned}
&\int_{x_{i+1}}^{x_i} \operatorname{dist}(\nabla u(x,s), K_{\gamma,N})^2\,dx \\
\geq &\frac{1}{36}\int_{x_{i+1}}^{x_i} |\partial_1 u(x,s)-\xi_1|^2\,ds \\
\geq &\frac{1}{36(x_i-x_{i+1})}\left(\int_{x_{i+1}}^{x_i} \partial_1 u(x,s)-\xi_1\,ds\right)^2 \\
= &\frac{1}{36}(x_i-x_{i+1})\left(\frac{u(x_i,s)-u(x_{i+1},s)}{x_i-x_{i+1}}-\xi_1\right)^2 \\
\geq &\frac{25}{32\cdot 36}|\sin(\gamma)|^{i+2} \geq \frac{25}{32\cdot 36}\sigma.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
48tE_{\sigma,\gamma,N}(u;(x_{i+1},x_i)\times(0,1)) &\geq \int_y^{y+t}\int_{x_{i+1}}^{x_i} \operatorname{dist}(\nabla u(x,s),K_{\gamma,N})^2\,dx + \sigma|\partial_1\nabla u(\cdot,s)|(x_{i+1},x_i) \\
&\geq \frac{t}{64}\,\sigma\min\left\{\frac{25}{32\cdot 36},\frac{1}{10}\right\},
\end{aligned}$$

which shows the claim if (2) and (b) hold. This finishes the proof of the claim. □

Eventually, we can now prove the lower bound stated in Theorem 3.1 for $N=3,4$.

Proposition 6.3. Let $N=3,4$. Then there exists a constant $c>0$ such that for all $\gamma\in\Gamma_N$ it holds

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \geq c\min\left\{|\sin(\gamma)|^2, \sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right)\right\}.$$

Proof. Let us first assume that $\sigma\geq|\sin(\gamma)|^2$. By Proposition 4.1 it follows

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u)\geq c_A\min\{\sigma,|\sin(\gamma)|^2\} = c_A|\sin(\gamma)|^2.$$

Next, assume that $\sigma\leq|\sin(\gamma)|^2$ and let $\gamma_0>0$ and $K\geq 2$ be as in Proposition 6.2. Then by Proposition 6.1 there exists $c_B>0$ such that for all $\gamma\in\Gamma_N$ with $|\gamma|\geq\gamma_0$,

$$E_{\sigma,\gamma,N}(u)\geq c_B\min\{\sigma(|\log\sigma|+1),1\}\geq c_B\min\{|\log|\sin(\gamma_0)||,1\}\,\min\left\{\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)||}+1\right),|\sin(\gamma)|^2\right\}.$$

For $\gamma\in\Gamma_N$ with $|\gamma|\leq\gamma_0$ and $\sigma\leq|\sin(\gamma)|^K$ it holds by Proposition 6.2

$$E_{\sigma,\gamma,N}(u)\geq c_D\sigma\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)\,|}+1\right).$$

Eventually, we notice that for $|\sin(\gamma)|^2\geq\sigma\geq|\sin(\gamma)|^K$ it holds by Proposition 4.1

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u)\geq c_A\min\{\sigma,|\sin(\gamma)|^2\} = \frac{c_A}{K+1}\left(\frac{|\log\sigma|}{|\log|\sin(\gamma)|\,|}+1\right).$$

This concludes by Remark 1 the proof of the lower bound for $c=\min\left\{\frac{c_A}{K+1}, c_B\min\{1,|\log|\sin(\gamma_0)||\}, c_D\right\}$. □

6.2. **The upper bound.** In this section we prove the upper bound stated in Theorem 3.1 for $N = 3, 4$. By Remark 1 it remains to consider the case $\sigma \leq |\sin(\gamma)|^3$.

Proposition 6.4. Let $N = 3, 4$. Then there exists $C > 0$ such that for all $\sigma > 0$ and $\gamma \in \Gamma_N$ it holds

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \leq C \min\left\{ |\sin(\gamma)|^2, \sigma\left(\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1\right)\right\}.$$

Proof. By Proposition 4.1 we find for $\sigma \geq |\sin(\gamma)|^2$

$$\min_{u\in\mathcal{A}} E_{\sigma,\gamma,N}(u) \leq C_A|\sin(\gamma)|^2 \leq C_A \min\left\{ |\sin(\gamma)|^2, \sigma\left(\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1\right)\right\}.$$

Hence, it remains to consider the case $|\sin(\gamma)|^2 \geq \sigma$. Additionally, we will only treat the case $\gamma \geq 0$. As for $N = 2$, we define the set

$$S_\gamma = conv\left\{ \begin{pmatrix} -\cos(\gamma) \\ -\sin(\gamma) \end{pmatrix}, \begin{pmatrix} 1 \\ -\sin(\gamma) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Again, we will first use a self-similar branching construction to define a function on S_γ. However, in this setting the constructed competitor will have gradients in $K_{\pi/2,N}$ in a large part of the domain (c.f. the branching constructions in [30]). Then - up to a rotation - we will use this function on the set $\{(x, y) \in (0, 1)^2 : (x, y - 1) \cdot e^{i\gamma} \leq 0\}$ and glue it to a function with constant gradient $e^{i\gamma}$ on the rest of $(0, 1)^2$.

Step 1: Branching Construction on S_γ for $N = 4$ and $\gamma \in (0, \pi/4) \subseteq \Gamma_4$.

Step 1a : Building block for S_γ. Let $m = \lceil|\sin(\gamma)|^{-1}\rceil$ and define the set

$$B_\gamma^{(4)} = \bigcup_{k=1}^{m} \left(\frac{\sin(\gamma)}{m} - \frac{k-1}{m}\cos(\gamma), \sin(\gamma)\right) \times \left(\frac{k-1}{m}\sin(\gamma), \frac{k}{m}\sin(\gamma)\right).$$

Then there exists a function $V^{(4)} : B_\gamma^{(4)} \to \mathbb{R}^2$ such that (see Figure 6)

(a) $V^{(4)}(x, y) \in K_{\pi/2,4}$ for a.e. $(x, y) \in B_\gamma^{(4)}$,
(b) $|D^2V^{(4)}| \leq C(m \sin(\gamma) + 1) \leq C$,
(c) For the second component of $V^{(4)}$ it holds $V_2^{(4)}(\frac{\sin(\gamma)}{m} - \frac{k-1}{m}\cos(\gamma), y) = V_2^{(4)}(1, my - (k - 1)\sin(\gamma))$ for all $y \in \left(\frac{k-1}{m}\sin(\gamma), \frac{k}{m}\sin(\gamma)\right)$,
(d) $V^{(4)}$ is curl-free,
(e) $V_1^{(4)}(x, y) = 0$ for all $(x, y) \in \partial B_\gamma$ such that e_1 is tangent to ∂B_γ at (x, y),
(f) It holds $V_2^{(4)}(\sin(\gamma), \cdot) = -\mathbb{1}_{(0,\sin(\gamma)/2)} + \mathbb{1}_{(\sin(\gamma)/2,\sin(\gamma))}$.

Step 1b : Interpolation to boundary conditions. We define $V_\partial : conv\left\{ \begin{pmatrix} -\cos(\gamma) \\ \sin(\gamma) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(\gamma) \end{pmatrix} \right\} \to \mathbb{R}^2$ as (c.f. Figure 2 on the right)

$$V_\partial(x, y) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } y \geq \sin(\gamma) - \frac{\sin(\gamma)}{2\cos(\gamma)}(x + \cos(\gamma)) \\ \begin{pmatrix} -\frac{\sin(\gamma)}{\cos(\gamma)} \\ -1 \end{pmatrix} & \text{if } y \leq \sin(\gamma) - \frac{\sin(\gamma)}{2\cos(\gamma)}(x + \cos(\gamma)). \end{cases}$$

Step 1c : Branching construction. For $K \in \mathbb{N}$ we define the following function $V_K : S_\gamma \to \mathbb{R}^2$ as follows (see Figure 7)

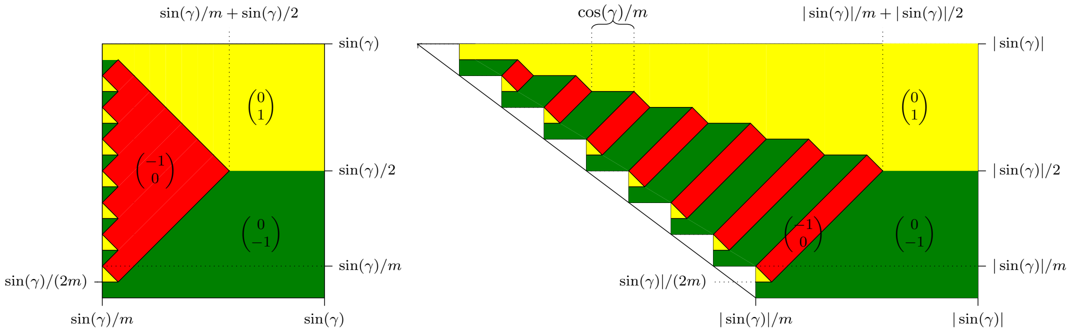

FIGURE 6. Sketch for the construction of the building block for $N = 4$ and $m = 8$. The different appearing gradients are color-coded. Left: A version of the construction of the building block for a rectangle of height $\sin(\gamma)$. It is immediate that $|D^2V^{(3)}| \leq C(\sin(\gamma)m + 1)$. Right: Construction of the building block for S_γ, $N = 4$ and $m = 8$. The essential difference to the construction on the left is that after each branching of the construction an extra horizontal gap of length $\cos(\gamma)/m$ has to be bridged by horizontal interfaces creating an extra interface of length $\cos(\gamma)/m$ in horizontal direction and at most $\sqrt{2}\sin(\gamma)$ in diagonal direction. Hence, the total surfaces created in the construction can be estimated up to a constant by $|\sin(\gamma)|m + 1$.

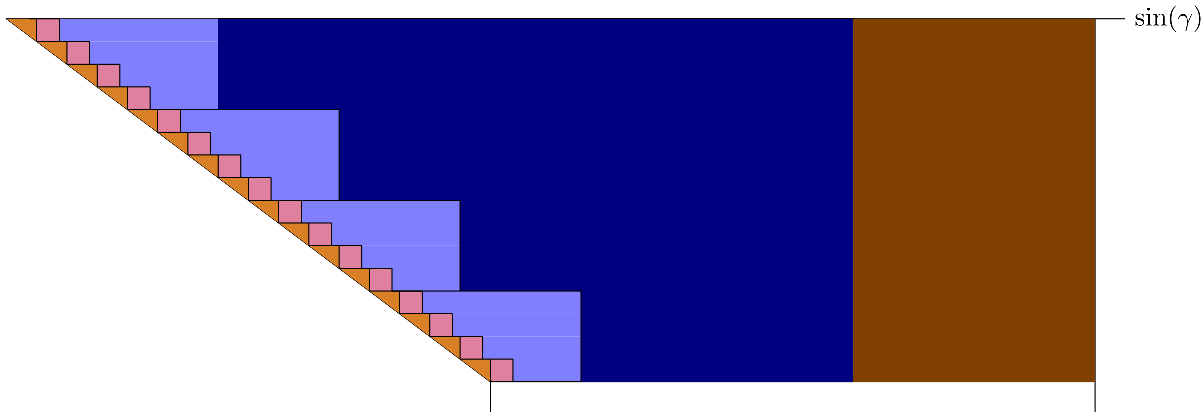

FIGURE 7. Sketch of the branching construction for $N \in \{3, 4\}$, $K = 1$ and $m = 4$. The different regions of in the definition of $V_k^{(N)}$ are colored in brown (1), blue (2) (darkblue for $k = 0$, lightblue for $k = 1$), pink (3) and orange (4).

(1) For $x \in (\sin(\gamma), 1)$, we define

$$V_K^{(4)}(x, y) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } y \geq 1/2, \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } y \leq 1/2. \end{cases}$$

(2) Let $0 \le k \le K$ and $l \in \{0, \dots, m^k - 1\}$. Then we define for

$$(x,y) \in \begin{pmatrix} -\frac{l}{m^k}\cos(\gamma) \\ \frac{l}{m^k}\sin(\gamma) \end{pmatrix} + \frac{1}{m^k} B_\gamma^{(4)}$$

the function $V_K^{(4)}$ as

$$V_K^{(4)}(x,y) = V\left(m^k(x + \frac{l}{m^k}\cos(\gamma)), m^k(y - \frac{l}{m^k}\sin(\gamma))\right).$$

(3) Let $l \in \{0, \dots, m^{K+1} - 1\}$. We define for

$$(x,y) \in \left(-\frac{l}{m^{K+1}}\cos(\gamma), -\frac{l}{m^{K+1}}\cos(\gamma) + \frac{\sin(\gamma)}{m^{K+1}}\right) \times \left(\frac{l}{m^{K+1}}\sin(\gamma), \frac{l+1}{m^{K+1}}\sin(\gamma)\right)$$

the constant extension

$$V_K^{(4)}(x,y) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } y \ge \frac{l+1/2}{m^{K+1}}\sin(\gamma), \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } y \le \frac{l+1/2}{m^{K+1}}\sin(\gamma). \end{cases}.$$

(4) Let $l \in \{0, \dots, m^{K+1} - 1\}$. We define for

$$(x,y) \in conv\left\{ \begin{pmatrix} -\frac{l}{m^{K+1}}\cos(\gamma) \\ \frac{l}{m^{K+1}}\sin(\gamma) \end{pmatrix}, \begin{pmatrix} -\frac{l}{m^{K+1}}\cos(\gamma) \\ \frac{l+1}{m^{K+1}}\sin(\gamma) \end{pmatrix}, \begin{pmatrix} -\frac{l+1}{m^{K+1}}\cos(\gamma) \\ \frac{l+1}{m^{K+1}}\sin(\gamma) \end{pmatrix} \right\}$$

the function $V_K^{(4)}$ as

$$V_K^{(4)}(x,y) = V_\partial\left(m^{K+1}(x + \frac{l}{m^{K+1}}\cos(\gamma)), m^{K+1}(y - \frac{l}{m^{K+1}}\sin(\gamma))\right).$$

We observe that $V_K^{(4)}$ is curl-free as $V_K^{(4)}$ is curl-free and $\nu \parallel ((V_K^{(4)})^- - (V_K^{(4)})^+)$ on its jump set $J_{V_K^{(4)}}$, where ν is the measure-theoretic normal to $J_{V_K^{(4)}}$, due to (c), (e) and (f). Let $u_K : S_\gamma \to \mathbb{R}$ be a corresponding primitive with $u_K(0,0) = 0$. It holds by construction that $u(-t\cos(\gamma), t\sin(\gamma)) = 0$ for all $t \in (0,1)$ and $u(t, \sin(\gamma)) = 0$ for all $t \in (-\cos(\gamma), 1)$ (c.f. (e)). Next, we estimate using (a), (b) and $\cos(\gamma) \ge \cos(\pi/4) > 0$

$$\begin{aligned} \int_{S_\gamma} \operatorname{dist}\left(\nabla u_K, K_{\pi/2,4}\right)^2 dx + \sigma|D^2 u_K|(S_\gamma) &\le C\frac{|\sin(\gamma)|^3}{\cos(\gamma) m^{K+1}} + C\sigma\left(K + |\sin(\gamma)| + 1\right) \\ &\le C\left(|\sin(\gamma)|^K + \sigma K\right). \end{aligned}$$

Choosing $K = \left\lceil \frac{|\log\sigma|}{|\log\sin(\gamma)|} \right\rceil$ yields the estimate

$$E_{\sigma,\pi/2,4}(u; S_\gamma) = \int_{S_\gamma} \operatorname{dist}\left(\nabla u_K, K_{\pi/2,N}\right)^2 dx + \sigma|D^2 u_K|(S_\gamma) \le C\sigma\left(\frac{|\log\sigma|}{|\log\sin(\gamma)|}| + 1\right).$$

Step 2: Branching construction on S_γ for $N = 3$ and $\gamma \in (0, \pi/6) \subseteq \Gamma_3$.
The construction for $N = 3$ is very similar to the one for $N = 4$ but needs a different building block.
Step 2a : Building block for S_γ. Let $m = \lceil |\sin(\gamma)|^{-1} \rceil$ and define the set

$$B_\gamma^{(3)} = \bigcup_{k=1}^{m} \left(\frac{2\sin(\gamma)}{\sqrt{3}m} - \frac{k-1}{m}\cos(\gamma), \frac{2\sin(\gamma)}{\sqrt{3}}\right) \times \left(\frac{k-1}{m}\sin(\gamma), \frac{k}{m}\sin(\gamma)\right).$$

Then there exists a function $V^{(3)} : B_\gamma^{(3)} \to \mathbb{R}^2$ such that (see Figure 8

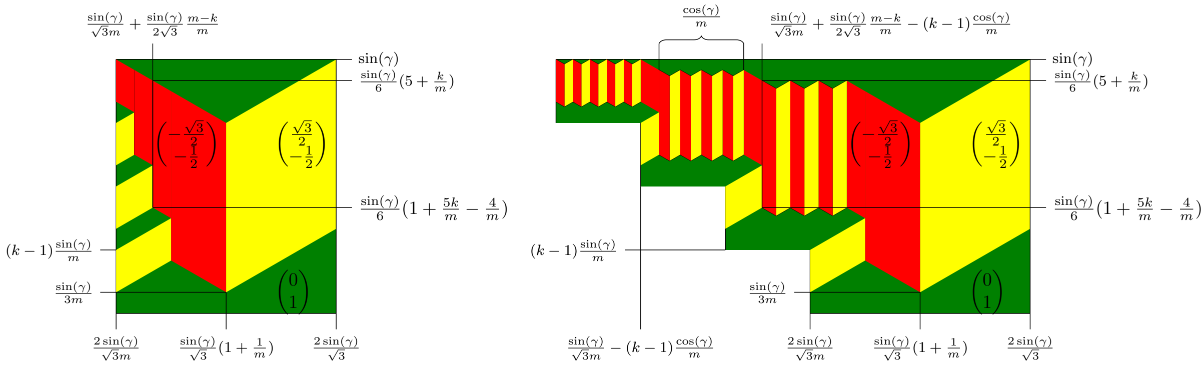

FIGURE 8. Sketch for the construction of the building block for $N = 3$ and $m = 4$. The different appearing gradients are color-coded, the appearing diagonals have slope $\pm\frac{1}{\sqrt{3}}$. Left: A version of the construction of the building block for a rectangle of height $\sin(\gamma)$. It is immediate that $|D^2V^{(3)}| \leq C\sin(\gamma)(m+1)$. Right: Construction of the building block for S_γ, $N = 3$ and $m = 4$. The essential difference to the construction on the left is that after each branching of the construction an extra horizontal gap of length $\cos(\gamma)/m$ has to be bridged by sawtooth patterns. In each of those bridging steps the number of sawteeth can be chosen proportional to the quotient $\frac{\cos(\gamma)}{m} / \left(\frac{\sin(\gamma)}{6}(1 + \frac{5k}{m} - \frac{4}{m}) - (k-1)\frac{\sin(\gamma)}{m}\right) \leq C\frac{m}{(m-k+2)\sin(\gamma)}$ (in this way $V^{(3)}$ stays $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ on the horizontal parts of the boundary of $B_\gamma^{(3)}$). The corresponding height of each of those vertical interfaces is $\sin(\gamma)\frac{2(m-k+1)}{3m}$. Hence, the total surfaces created in the construction can be estimated by $C(\sin(\gamma)m + 1)$.

(a') $V^{(3)}(x,y) \in K_{\pi/2,3}$ for a.e. $(x,y) \in B_\gamma^{(3)}$,
(b') $|D^2V^{(3)}| \leq C(m|\sin(\gamma)| + 1) \leq C$,
(c') For the second component of $V^{(3)}$ it holds $V_2^{(3)}(\frac{2\sin(\gamma)}{\sqrt{3}m} - \frac{k-1}{m}\cos(\gamma), y) = V_2^{(3)}(\frac{2\sin(\gamma)}{\sqrt{3}}, my - (k-1)\sin(\gamma))$ for all $y \in \left(\frac{k-1}{m}\sin(\gamma), \frac{k}{m}\sin(\gamma)\right)$,
(d') $V^{(3)}$ is curl-free,
(e') For the first component of $V^{(3)}$ it holds $V_1^{(3)}(x,y) = 0$ for all $(x,y) \in \partial B_\gamma$ such that e_1 is tangent to ∂B_γ at (x,y),
(f') It holds $V_2^{(3)}(\frac{2\sin(\gamma)}{\sqrt{3}}, \cdot) = \mathbb{1}_{(0,\sin(\gamma)/3)} - \frac{1}{2}\mathbb{1}_{(\sin(\gamma)/3,\sin(\gamma))}$.

Step 2b : Interpolation to boundary conditions. We define $V_\partial^{(3)} : conv\left\{\begin{pmatrix} -\cos(\gamma) \\ \sin(\gamma) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(\gamma) \end{pmatrix}\right\} \to \mathbb{R}^2$ as

$$V_\partial(x,y) = \begin{cases} \begin{pmatrix} \frac{\sin(\gamma)}{\cos(\gamma)} \\ 1 \end{pmatrix} & \text{if } y \leq \sin(\gamma) - \frac{2\sin(\gamma)}{3\cos(\gamma)}(x + \cos(\gamma)) \\ \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix} & \text{if } y \geq \sin(\gamma) - \frac{2\sin(\gamma)}{3\cos(\gamma)}(x + \cos(\gamma)). \end{cases}$$

Step 2c : Branching construction. For $K \in \mathbb{N}$ we define the function $V_K^{(3)} : S_\gamma \to \mathbb{R}^2$ as follows (see Figure 7)

(1) Let $x \in (\frac{2\sin(\gamma)}{\sqrt{3}}, 1)$. Then we define

$$V_K^{(3)}(x,y) = \begin{cases} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} & \text{if } \frac{\sin(\gamma)}{3} + \frac{1}{\sqrt{3}}(x - \frac{2\sin(\gamma)}{\sqrt{3}}) \le y, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{else.} \end{cases}$$

(2) Let $0 \le k \le K$ and $l \in \{0, \ldots, m^k - 1\}$. Then we define for

$$(x,y) \in \left(-\frac{l}{m^k}\cos(\gamma), \frac{l}{m^k}\sin(\gamma)\right) + \frac{1}{m^k} B_\gamma^{(3)}$$

the function $V_K^{(3)}$ as

$$V_K^{(3)}(x,y) = V^{(3)}\left(m^k(x + \frac{l}{m^k}\cos(\gamma)), m^k(y - \frac{l}{m^k}\sin(\gamma))\right).$$

(3) Let $l \in \{0, \ldots, m^{K+1} - 1\}$. We define for

$$(x,y) \in \left(-\frac{l}{m^{K+1}}\cos(\gamma), -\frac{l}{m^{K+1}}\cos(\gamma) + \frac{2\sin(\gamma)}{\sqrt{3}m^{K+1}}\right) \times \left(\frac{l}{m^{K+1}}\sin(\gamma), \frac{l+1}{m^{K+1}}\sin(\gamma)\right)$$

the function $V_K^{(3)}$ as

$$V_K^{(3)}(x,y) = \begin{cases} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} & \text{if } y \ge \frac{l}{m^{K+1}}\sin(\gamma) + \frac{1}{\sqrt{3}}(x + \frac{l}{m^{K+1}}\cos(\gamma) - \frac{\sin(\gamma)}{\sqrt{3}m^{K+1}}) \\ & \text{and } y \le \frac{l+2/3}{m^{K+1}}\sin(\gamma) + \frac{1}{\sqrt{3}}(x + \frac{l}{m^{K+1}}\cos(\gamma) - \frac{\sin(\gamma)}{\sqrt{3}m^{K+1}}) \\ & \text{and } x \ge -\frac{l}{m^{K+1}}\cos(\gamma) + \frac{\sin(\gamma)}{\sqrt{3}m^{K+1}}, \\ \begin{pmatrix} \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} & \text{if } y \ge \frac{l+1/3}{m^{K+1}}\sin(\gamma) - \frac{1}{\sqrt{3}}(x + \frac{l}{m^{K+1}}\cos(\gamma)) \\ & \text{and } y \le \frac{l+1}{m^{K+1}}\sin(\gamma) - \frac{1}{\sqrt{3}}(x + \frac{l}{m^{K}}\cos(\gamma)) \\ & \text{and } x \le -\frac{l}{m^{K+1}}\cos(\gamma) + \frac{\sin(\gamma)}{\sqrt{3}m^{K+1}}, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{else.} \end{cases}.$$

(4) Let $l \in \{0, \ldots, m^{K+1} - 1\}$. We define for

$$(x,y) \in conv\left\{\begin{pmatrix} -\frac{l}{m^{K+1}}\cos(\gamma) \\ \frac{l}{m^{K+1}}\sin(\gamma) \end{pmatrix}, \begin{pmatrix} -\frac{l}{m^{K+1}}\cos(\gamma) \\ \frac{l+1}{m^{K+1}}\sin(\gamma) \end{pmatrix}, \begin{pmatrix} -\frac{l+1}{m^{K+1}}\cos(\gamma) \\ \frac{l+1}{m^{K+1}}\sin(\gamma) \end{pmatrix}\right\}$$

the function $V_K^{(3)}$ as

$$V_K^{(3)}(x,y) = V_\partial^{(3)}\left(m^{K+1}(x + \frac{l}{m^{K+1}}\cos(\gamma)), m^{K+1}(y - \frac{l}{m^{K+1}}\sin(\gamma))\right).$$

As in the case $N = 4$ we observe that $V_K^{(3)}$ is curl-free. Let $u_K^{(3)} : S_\gamma \to \mathbb{R}$ be a corresponding primitive with $u_K^{(3)}(0,0) = 0$. As for $N = 4$, it holds by construction that $u_K(-t\cos(\gamma), t\sin(\gamma)) = 0$ for all $t \in (0,1)$ and $u_K(t, \sin(\gamma)) = 0$ for all $t \in (-\cos(\gamma), 1)$. Next, we estimate similarly to before

$$\begin{aligned} \int_{S_\gamma} \operatorname{dist}\left(\nabla u_K^{(3)}, K_{\pi/2,3}\right)^2 dx + \sigma|D^2 u_K^{(3)}|(S_\gamma) &\le C\frac{1}{m^K} + C\sigma\left(K + \sin(\gamma) + 1\right) \\ &\le C\left(\sin(\gamma)^K + \sigma K\right). \end{aligned}$$

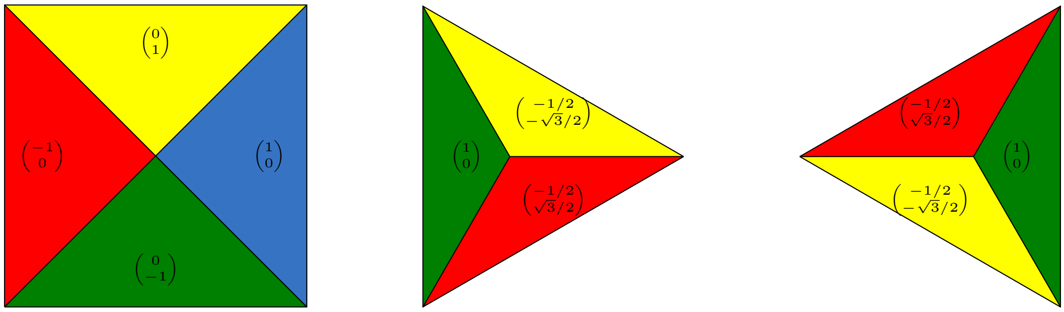

FIGURE 9. Sketch of the functions constructed in Lemma 7.0.1 for Q_0 (left), $T_0^{(1)}$ (middle) and $T_0^{(2)}$ (right).

Again, choosing $K = \left\lceil \frac{|\log\sigma|}{|\log\sin(\gamma)|} \right\rceil$ yields the estimate

$$E_{\sigma,\pi/2,3}(u_K; S_\gamma) = \int_{S_\gamma} \operatorname{dist}\left(\nabla u_K^{(3)}, K_{\pi/2,3}\right)^2 dx + \sigma|D^2 u_K^{(3)}|(S_\gamma) \leq C\sigma\left(\frac{|\log\sigma|}{|\log\sin(\gamma)|} + 1\right).$$

Step 3: Branching construction on $(0,1)^2$ *for* $N = 3, 4$. We define $u : (0,1)^2 \to \mathbb{R}$ as (c.f. also Figure 4)

$$u(x,y) = \begin{cases} e^{i\gamma} \cdot (x, y-1) & \text{if } e^{i\gamma} \cdot (x, y-1) \geq 0, \\ u^{(N)}(e^{-i(\gamma-\pi/2)}(x,y)) & \text{if } e^{i\gamma} \cdot (x, y-1) \leq 0, \end{cases}$$

where $u^{(N)}$ is the function constructed in Step 1 and Step 2 for $N = 3$ or $N = 4$, respectively. Then we conclude by Step 1c and 2c, respectively, that

$$E_{\sigma,\gamma,N}(u) \leq E_{\sigma,\pi/2,N}(u^{(N)}) + C\sigma \leq C\sigma\left(\frac{|\log\sigma|}{|\log\sin(\gamma)\,|} + 1\right) + C\sigma \leq C\sigma\left(\frac{|\log\sigma|}{|\log\sin(\gamma)\,|} + 1\right).$$

□

7. AN UPPER BOUND BY COVERING AND SOLUTIONS TO THE DIFFERENTIAL INCLUSION

In this section, we discuss how upper bounds for the energy $E_{\sigma,\gamma,N}(\cdot,\Omega)$ for $N = 3,4$ and a bounded Lipschitz-domain $\Omega \subseteq \mathbb{R}^2$ can be shown using a good covering of Ω with specific building blocks that allow for simple functions which only use the preferred gradients and satisfy zero boundary conditions. Iterating this construction leads to solutions of the differential inclusion $\nabla u \in K_{\gamma,N}$ and $u = 0$ on $\partial\Omega$ whose regularity can be controlled through interpolation. We start by constructing the building blocks.

Lemma 7.0.1. (1) Let $Q_\gamma = e^{i\gamma}(-1/2, 1/2)^2$ be the unit cube centered at 0 and with two sides parallel to $e^{i\gamma}$. Then there exists a function $u \in W_0^{1,\infty}(Q_\gamma)$ such that $\nabla u \in BV(Q_\gamma; K_{\gamma,4})$ and $|D^2 u|(Q_\gamma) \leq 4$.

(2) Let $T_\gamma^{(1)} = conv\left\{e^{i\gamma}, e^{i\gamma + i\frac{2\pi}{3}}, e^{i\gamma+i\frac{4\pi}{3}}\right\}$ and $T_\gamma^{(2)} = conv\left\{-e^{i\gamma}, -e^{i\gamma+i\frac{2\pi}{3}}, -e^{i\gamma+i\frac{4\pi}{3}}\right\}$. Then there exist functions $u^{(i)} \in W_0^{1,\infty}(T_\gamma^{(i)})$ such that $\nabla u^{(i)} \in BV(T_\gamma^{(i)}, K_{\gamma,N})$ and $|D^2 u^{(i)}| = 3\sqrt{3}$.

Proof. For a visualization of the constructions for $\gamma = 0$, see Figure 9.

For (1), note that by a change of coordinates it is enough to construct a function for $\gamma = 0$ and $Q_0 = (-1/2, 1/2)^2$. Then define $u : (-1/2, 1/2)^2 \to \mathbb{R}$ as

$$u(x,y) = \begin{cases} x - \frac{1}{2} & \text{, if } x \geq 0, -x \leq y \leq x, \\ -x - \frac{1}{2} & \text{, if } x \leq 0, x \leq y \leq -x, \\ y - \frac{1}{2} & \text{, if } y \geq 0, -y \leq x \leq y, \\ -y - \frac{1}{2} & \text{, if } y \leq 0, y \leq x \leq -y. \end{cases}$$

The claimed properties follow directly.

For (2) it suffices again to consider the case $\gamma = 0$. Then we define

$$u^{(1)}(x,y) = \begin{cases} x + \frac{1}{2} & \text{if } (x,y) \in conv \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}, \\ -\frac{1}{2}x - \frac{\sqrt{3}}{2}y + \frac{1}{2} & \text{if } (x,y) \in conv \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \right\}, \\ -\frac{1}{2}x + \frac{\sqrt{3}}{2}y + \frac{1}{2} & \text{if } (x,y) \in conv \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\} \end{cases}$$

and

$$u^{(2)}(x,y) = -u^{(1)}(-x,y).$$

All claimed properties can be immediately verified. □

Next, we discuss a way of covering $\Omega \subseteq \mathbb{R}^2$ that will allow for simple construction of test functions using the functions from the previous lemma.

Definition 7.1. Let $\Omega \subseteq \mathbb{R}^2$, $N = 3, 4$, $\gamma \in (-\pi, \pi)$ and $\theta \in (0,1)$. We say that a sequence of families $\mathcal{F}_K \subseteq \mathcal{P}(\Omega)$ is a (N, γ, θ)-covering of Ω if there exists $C_{N,\gamma,\theta} > 0$ such that the following properties are satisfied:

1. the family $\mathcal{F}_K$ consists of pairwise disjoint sets;
2. $\mathcal{F}_K \subseteq \mathcal{F}_{K+1}$ for every $K \in \mathbb{N}$;
3. (a) If $N = 3$: if $S \in \mathcal{F}_K$ then $S = a + \lambda T_\gamma^{(i)} \subseteq \Omega$ for some $i = 1, 2$, $a \in \mathbb{R}^2$ and $\lambda > 0$;
   (b) If $N = 4$: if $S \in \mathcal{F}_K$ then $S = a + \lambda Q_\gamma \subseteq \Omega$ for some $a \in \mathbb{R}^2$ and $\lambda > 0$;
4. for every $K \in \mathbb{N}$ it holds $\mathcal{L}^2(\Omega \setminus \bigcup_{S \in \mathcal{F}_K} S) \leq C_{N,\gamma,\theta}\theta^K$;
5. for every $K \in \mathbb{N}$ it holds $\sum_{a + \lambda S \in \mathcal{F}_K} \lambda \leq C_{N,\gamma,\theta} K$.

If a domain Ω allows for a covering of the above type this leads to good competitors for the energy $E_{\sigma,\gamma,N}(\cdot; \Omega)$ and solutions to the differential inclusion.

Proposition 7.1. Let $\Omega \subseteq \mathbb{R}^2$ be open, $N = 3, 4$. Let $\gamma \in \Gamma_N$ and $\theta \in (0,1)$ such that Ω possesses a (N, γ, θ)-cover.

1. Then it holds
$$\min_{u \in \mathcal{A}_0(\Omega)} E_{\sigma,\gamma,N}(u; \Omega) \leq C C_{N,\gamma,\theta}\, \sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1 \right),$$
where $C > 0$ is a universal constant independent from θ and γ.
2. Then there exists a function $u \in W_0^{1,\infty}(\Omega)$ such that $\nabla u \in K_{\gamma,N}$ a.e. and $\nabla u \in W^{s,q}(\Omega)$ for all $0 < s < 1$, $q \in (0, \infty)$ such that $\frac{1}{q} > s$.

Proof. Let $(\mathcal{F}_K)_K$ be a (N, γ, θ)-cover for Ω. Let $K \in \mathbb{N}$. If $N = 3$ then define the function $u_K^{(3)} : (0,1)^2 \to \mathbb{R}$ as

$$u_K^{(3)}(x,y) = \begin{cases} \lambda u^{(i)}\left(\frac{(x,y) - a}{\lambda}\right) & \text{if } (x,y) \in a + \lambda T_\gamma^{(i)} \text{ for } i \in \{1,2\},\ a + \lambda T_\gamma^{(i)} \in \mathcal{F}_K, \\ 0 & \text{if } (x,y) \in (0,1)^2 \setminus \bigcup_{S \in \mathcal{F}_K} S, \end{cases}$$

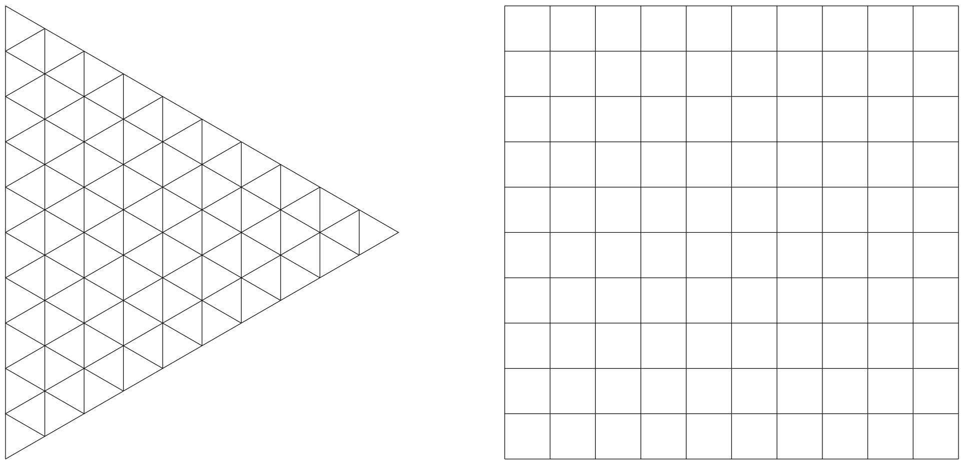

FIGURE 10. Sketch of the lattices $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$ for $\gamma = 0$.

where $u^{(i)} : T_\gamma^{(i)} \to \mathbb{R}$ are the functions from Lemma 7.0.1 (2). If $N = 4$ then define the function $u_K^{(4)} : (0,1)^2 \to \mathbb{R}$ as

$$u_K^{(4)}(x,y) = \begin{cases} \lambda u\left(\frac{(x,y)-a}{\lambda}\right) & \text{if } (x,y) \in a + \lambda Q_\gamma \text{ for } a + \lambda Q_\gamma \in \mathcal{F}_K, \\ 0 & \text{if } (x,y) \in (0,1)^2 \setminus \bigcup_{S \in \mathcal{F}_K} S, \end{cases}$$

where $u : Q_\gamma \to \mathbb{R}$ is the function from lemma 7.0.1 (1).

Then $u_K^{(N)} \in \mathcal{A}_0$. Moreover, it holds by the properties of $\mathcal{F}_K$ and u (cf. Lemma 7.0.1) for $N = 4$ that

$$E_{\sigma,\gamma,N}(u_K^{(4)}) \leq \mathcal{L}^2\left(\Omega \setminus \bigcup_{S \in \mathcal{F}_K} S\right) + \sigma \sum_{a + \lambda \cdot Q_\gamma \in \mathcal{F}_K} \left(\lambda |D^2 u|(Q_\gamma) + 2\lambda \mathcal{H}^1(\partial Q_\gamma)\right) \leq C C_{4,\gamma,\theta}\left(\theta^K + \sigma K\right).$$

Then choose $K = \lceil \frac{|\log \sigma|}{|\log \theta|} \rceil \in [\frac{|\log \sigma|}{|\log \theta|}, \frac{|\log \sigma|}{|\log \theta|} + 1]$ which yields the desired upper bound. A similar calculation shows the upper bound for $N = 3$. This shows (1). For (2) we note that as in [30, Proposition 12], we can use an interpolation argument (c.f. also [43, Corollary 2.1]) to show that the sequence of functions $u_K^{(N)}$ as constructed above has a limit $u^N \in W_0^{1,\infty}(\Omega)$ with the claimed properties. $\square$

Next, we show that every Lipschitz domain allows for a $(N, \gamma, \frac{1}{2})$-cover for $N = 3, 4$ and $\gamma \in (-\pi, \pi)$.

Lemma 7.1.1. Let $N = 3, 4$, $\gamma \in (-\pi, \pi)$ and $\Omega \subseteq \mathbb{R}^2$ an open, bounded set with Lipschitz boundary. Then there exists a $(N, \gamma, \frac{1}{2})$-cover of Ω. Moreover, the constants $C_{N,\gamma,1/2}$ are uniformly bounded.

Proof. Consider the lattices $\mathcal{L}^{(3)} = \left\{l_1 e^{i\gamma + i\pi/2} + l_2 e^{i\gamma + i\pi/6} : l_1, l_2 \in \mathbb{Z}\right\}$ and $\mathcal{L}^{(4)} = \{l_1 e^{i\gamma} + l_2 e^{i\gamma + i\pi/2} : l_1, l_2 \in \mathbb{Z}\}$, see Figure 10. We define the families $\mathcal{F}_K^{(N)}$ inductively. Let

$$\mathcal{F}_1^{(3)} = \left\{\text{int}(conv\{i,j,k\}) : i,j,k \in \mathcal{L}^{(3)}, |i-j| = |i-k| = |j-k| = 1 \text{ and } \text{int}(conv\{i,j,k\}) \subseteq \Omega\right\},$$

$$\mathcal{F}_1^{(4)} = \left\{\text{int}(conv\{h,i,j,k\}) : h,i,j,k \in \mathcal{L}^{(4)}, |i-j| = |j-k| = |k-h| = |h-i| = 1 \text{ and } conv\{h,i,j,k\} \subseteq \Omega\right\}.$$

Assume that $\mathcal{F}_K^{(3)}$ and $\mathcal{F}_K^{(4)}$ are already defined then define

$$\mathcal{F}_{K+1}^{(3)} = \mathcal{F}_K^{(3)} \cup \{ \operatorname{int}(conv\{i,j,k\}) : i,j,k \in 2^{-K}\mathcal{L}^{(3)}, |i-j| = |i-k| = |j-k| = 2^{-K}, \\ \operatorname{int}(conv\{i,j,k\}) \subseteq \Omega \text{ and } \operatorname{int}(conv\{i,j,k\}) \cap T = \emptyset \text{ for all } T \in \mathcal{F}_K^{(3)} \}$$

and

$$\mathcal{F}_{K+1}^{(4)} = \mathcal{F}_K^{(4)} \cup \{ \operatorname{int}(conv\{h,i,j,k\}) : h,i,j,k \in 2^{-K}, |i-j| = |j-k| = |k-h| = |h-i| = 2^{-K}, \\ \operatorname{int}(conv\{h,i,j,k\}) \subseteq \Omega \text{ and } \operatorname{int}(conv\{i,j,k\}) \cap = \emptyset \text{ for all } T \in \mathcal{F}_K^{(3)} \}$$

Then, by definition, these families only consist of scaled and translated versions of the triangles $T_\gamma^{(i)}$ for $N=3$ and the rotated square Q_γ for $N=4$, respectively. It follows for $(x,y) \notin \bigcup_{S \in \mathcal{F}_K^{(N)}} S$ that $(x,y) \in B_{C2^{-K}}(\partial\Omega)$ and therefore by the regularity of $\partial\Omega$ that

$$\mathcal{L}^2(\Omega \setminus \bigcup_{S \in \mathcal{F}_K^{(N)}} S) \leq C2^{-K}. \tag{7.1}$$

This implies also that $\#\mathcal{F}_{K+1}^{(N)} \setminus \mathcal{F}_K^{(N)} \leq C2^K$. In particular,

$$\sum_{a+\lambda S \in \mathcal{F}_K^{(N)}} \lambda \leq C\#\mathcal{F}_1^{(N)} + C\sum_{k=1}^{K-1} \#\mathcal{F}_{k+1}^{(N)} \setminus \mathcal{F}_k^{(N)} \cdot 2^{-k} \leq CK. \tag{7.2}$$

Note that the constant $C>0$ in (7.1) and (7.2) can be chosen independently from γ. □

As a direct consequence of the Lemma 7.1.1 and Proposition 7.1 we find the following corollary.

Corollary 7.1.1. Let $N = 3,4$ and $\Omega \subseteq \mathbb{R}^2$ be an open bounded set with Lipschitz boundary. Then the following hold:

(1) There exists a constant $C>0$ (depending on Ω) such that

$$\min_{u \in \mathcal{A}_0(\Omega)} E_{\sigma,\gamma,N}(u;\Omega) \leq C\sigma\left(|\log\sigma| + 1\right).$$

(2) There exists $u \in W_0^{1,\infty}(\Omega)$ such that $\nabla u \in BV_{loc}(\Omega; K_{\gamma,N})$ and $\nabla u \in W^{s,q}$ for all $0<s<1$, $q \in (0,\infty)$ satisfying $\frac{1}{q} > s$.

Proof. The only property that does not follow directly from Lemma 7.1.1 and Proposition 7.1 is the fact that it holds for the solution $u \in W_0^{1,\infty}(\Omega)$ of the differential inclusion constructed in Proposition 7.1 based on the covering from Lemma 7.1.1 that $\nabla u \in BV_{loc}(\Omega)$. However, in the proof of Lemma 7.1.1 it can be seen that for all $U \subset\subset \Omega$ there exists $K \in \mathbb{N}$ such that $U \subseteq \bigcup_{S \in \mathcal{F}_K} S$. In particular, it follows from the associated construction in the proof of Proposition 7.1 that $|\nabla u|(U) \leq CC_{N,\gamma,1/2}K$ i.e., $\nabla u \in BV_{loc}(\Omega)$. □

However, clearly the result above is not optimal in terms of the minimal scaling with respect to the angle γ. Next, we show that for $N=4$ the unit square allows for a much better covering. Precisely, $(0,1)^2$ has a $(4,\gamma,|\sin(\gamma)|)$-cover with uniformly bounded constants.

Lemma 7.1.2. For every $\gamma \in (-\pi/4, \pi/4)$ the square $(0,1)^2$ possesses a $(4,\gamma,|\sin(\gamma)|)$-cover. Moreover, the constant $C_{4,\gamma,|\sin\gamma|}$ can be uniformly bounded.

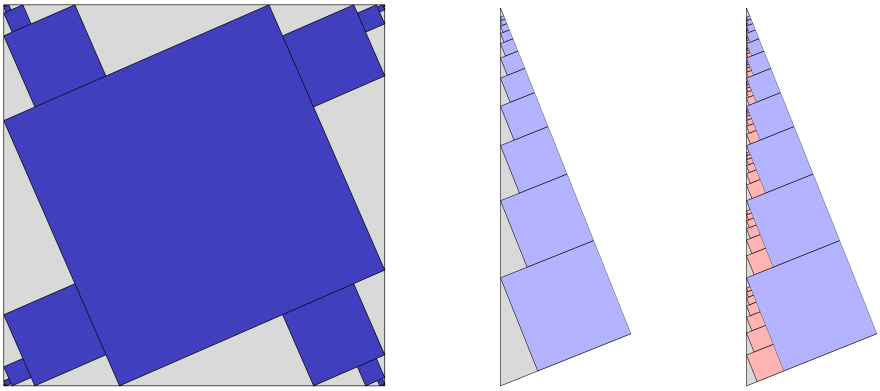

FIGURE 11. Sketch of the covering of $(0,1)^2$ described in Lemma 7.1.2. Left: the set $\mathcal{F}_1$ (blue). Middle: the set $\tilde{\mathcal{F}}_1$ (lightblue). Right: the set $\tilde{\mathcal{F}}_2$ (red).

Proof. Let us assume that $\gamma > 0$. For a visualization of the construction, see Figure 11. For $\ell \in \mathbb{N}$, let $r_\ell = \frac{\sin(\gamma)}{\cos(\gamma)+\sin(\gamma)} \frac{\cos(\gamma)}{1+\sin(\gamma)\cos(\gamma)} \left(\frac{\cos(\gamma)\sin(\gamma)}{1+\cos(\gamma)\sin(\gamma)}\right)^\ell$. Then define the families

$$\mathcal{F}_1^{(1)} = \left\{ \begin{pmatrix} \frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \\ 1-\frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \end{pmatrix} + r_\ell Q_\gamma : \ell \in \mathbb{N} \right\},$$
$$\mathcal{F}_1^{(2)} = \left\{ \begin{pmatrix} 1-\frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \\ 1-\frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \end{pmatrix} + r_\ell Q_\gamma : \ell \in \mathbb{N} \right\},$$
$$\mathcal{F}_1^{(3)} = \left\{ \begin{pmatrix} 1-\frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \\ \frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \end{pmatrix} + r_\ell Q_\gamma : \ell \in \mathbb{N} \right\},$$
$$\mathcal{F}_1^{(4)} = \left\{ \begin{pmatrix} \frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \\ \frac{r_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \end{pmatrix} + r_\ell Q_\gamma : \ell \in \mathbb{N} \right\}.$$

Eventually, we set

$$\mathcal{F}_1 = \bigcup_{i=1}^4 \mathcal{F}_1^{(i)} \cup \left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \frac{1}{\cos(\gamma)+\sin(\gamma)} Q_\gamma \right).$$

Note that $\mathcal{F}_1$ consists of pairwise disjoint sets. Then one computes

$$\mathcal{L}^2((0,1)^2 \setminus \bigcup_{Q \in \mathcal{F}^1} Q) = 1 - \frac{1}{(\cos(\gamma)+\sin(\gamma))^2} - 4\sum_{\ell=0}^\infty r_\ell^2 = \frac{2\sin(\gamma)\cos(\gamma)}{1+2\sin(\gamma)\cos(\gamma)} - 4\frac{\sin(\gamma)^2\cos(\gamma)^2}{(1+2\sin(\gamma)\cos(\gamma))^2}$$
$$= \frac{2\sin(\gamma)\cos(\gamma)}{(1+2\sin(\gamma)\cos(\gamma))^2} \leq 2\sin(\gamma)$$

and

$$\sum_{a+\lambda Q \in \mathcal{F}^1} \lambda = \frac{1}{\cos(\gamma)+\sin(\gamma)} + 4\sum_{\ell=0}^\infty r_\ell = \frac{1+4\sin(\gamma)\cos(\gamma)}{\cos(\gamma)+\sin(\gamma)} \leq C.$$

This shows all the needed properties of $\mathcal{F}_1$. To construct $\mathcal{F}_K$, we notice that $(0,1)^2 \setminus \bigcup_{Q\in\mathcal{F}_1} Q$ can be written as the disjoint union $\bigcup_{k\in\mathbb{N}} T_k$ where T_k is a rotated (by multiples of $\frac{\pi}{2}$), dilated and translated version of the triangle $T = conv\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \sin(\gamma)\begin{pmatrix} \cos(\gamma) \\ \sin(\gamma) \end{pmatrix} \right\}$ such that $\sum_{k\in\mathbb{N}} \mathcal{H}^1(\partial T_k) \leq C$, see Figure 11. In order to construct the families $\mathcal{F}_K$ for $K \geq 2$ inductively, it is then enough to show that there exists a disjoint family $\mathcal{F}$

of dilated and translated versions of Q_γ such that $T \setminus \bigcup_{a+\lambda Q_\gamma \in \mathcal{F}} a + \lambda Q_\gamma$ can be written as the disjoint union of translated and dilated versions of T such that (see Figure 11)

$$\sum_{a+\lambda Q_\gamma \in \mathcal{F}} \lambda \leq 1 \text{ and } \mathcal{L}^2(T \setminus \bigcup_{a+\lambda Q_\gamma \in \mathcal{F}} a + \lambda Q_\gamma) \leq \sin(\gamma)\mathcal{L}^2(T).$$

Then one can inductively define families $\tilde{F}_K$ for T such that (cf. Figure 11)

$$\tilde{\mathcal{F}}_K \subseteq \tilde{\mathcal{F}}_{K-1}, \qquad \sum_{a+\lambda Q_\gamma \in \tilde{\mathcal{F}}_K} \lambda \leq K \text{ and } \mathcal{L}^2(T \setminus \bigcup_{a+\lambda Q_\gamma \in \tilde{\mathcal{F}}_K} a + \lambda Q_\gamma) \leq \sin(\gamma)^K \mathcal{L}^2(T).$$

Eventually, the families $\mathcal{F}_K$ are simply given by $\mathcal{F}_1$ together with the corresponding families $\tilde{\mathcal{F}}_{K-1}$ for the triangles T_k.

Hence, it remains to show the existence of the family $\mathcal{F}$. For $\ell \in \mathbb{N}$, we define $\tilde{r}_\ell = \frac{\sin(\gamma)\cos(\gamma)}{\sin(\gamma)+\cos(\gamma)} \left(\frac{\cos(\gamma)}{\sin(\gamma)+\cos(\gamma)}\right)^\ell$ and $\delta_\ell = \left(\frac{\cos(\gamma)}{\sin(\gamma)+\cos(\gamma)}\right)^\ell$. Then set (cf. Figure 11)

$$\mathcal{F} = \left\{ \begin{pmatrix} \frac{\tilde{r}_\ell}{2}(\cos(\gamma)+\sin(\gamma)) \\ \frac{\tilde{r}_\ell}{2}(\cos(\gamma)+\sin(\gamma)) + \tilde{r}_\ell \frac{\sin(\gamma)^2}{\cos(\gamma)} + \sum_{k=0}^{\ell-1} \frac{\sin(\gamma)}{\sin(\gamma)+\cos(\gamma)} \delta_k \end{pmatrix} + \tilde{r}_\ell Q_\gamma : \ell \in \mathbb{N} \right\}.$$

It follows that $\mathcal{F}$ is a pairwise disjoint family. Moreover, we compute (recall that $\cos(\gamma) \geq \cos(\pi/4) \geq 1/2$)

$$\begin{aligned} \mathcal{L}^2(T \setminus \bigcup_{Q \in \mathcal{F}} Q) = \frac{\sin(\gamma)\cos(\gamma)}{2} - \sum_{k=0}^{\infty} \tilde{r}_\ell^2 &= \frac{\sin(\gamma)\cos(\gamma)}{2}\left(1 - 2\frac{\sin(\gamma)\cos(\gamma)}{\sin(\gamma)^2 + 2\sin(\gamma)\cos(\gamma)}\right) \\ &\leq \mathcal{L}^2(T)\frac{\sin(\gamma)}{2\cos(\gamma)} \leq \sin(\gamma)\mathcal{L}^2(T) \end{aligned}$$

and

$$\sum_{a+\lambda Q_\gamma \in \mathcal{F}} \lambda = \sum_{\ell=0}^{\infty} \tilde{r}_\ell = \cos(\gamma) \leq 1.$$

□

Remark 2. Similarly, it can be shown that there are families $\mathcal{F}_K$ consisting of dilated and translated versions of the building blocks $T_\gamma^{(1)}$ and $T_\gamma^{(2)}$ satisfying (1) - (3) when $(0,1)^2$ is replaced by, for example, $T_0^{(1)}$ or $T_0^{(2)}$.

Combining the previous result and Lemma 7.1.2 allows us to prove a scaling law for $\min_{u \in \mathcal{A}_0((0,1)^2)} E_{\sigma,\gamma,4}(u)$, Theorem 3.2.

Proof of Thm. 3.2. Step 1: Upper bounds. Clearly, $u = 0$ satisfies $u \in \mathcal{A}_0$ and $E_{\sigma,\gamma,N}(u) = 1$. The other upper bounds follow directly from Corollary 7.1.1 for $N = 3$ and from Proposition 7.1 and Lemma 7.1.2 for $N = 4$.

Step 2: Lower bounds. Let us first consider $N = 3$. First, note that for $\gamma \in \Gamma_3$ with $|\gamma| \geq \pi/12$ the lower bound follows from Proposition 6.1. On the other hand, for $|\gamma| \leq \pi/12$ the lower bound follows from a similar argument and the fact that $u(\cdot, 0) = 0$ and for $\gamma \in \Gamma_3$ it holds for $\xi \in K_{\gamma,3}$ that $|\xi_1| \geq \min\{\cos(\gamma), |\cos(\gamma + 2\pi/3)|, |\cos(\gamma + 4\pi/3)|\} \geq |\cos(7\pi/12))| > 0$.

Next, let $N = 4$. First we claim that

$$\min_{u \in \mathcal{A}_0} E_{\sigma,\gamma,4}(u) \geq \frac{1}{6}\min\{1, \sigma\}. \tag{7.3}$$

Let $u \in \mathcal{A}_0$. We may assume that $E_{\sigma,\gamma,4}(u) \leq \frac{1}{6}\min\{1,\sigma\}$ (otherwise there is nothing to show). Now, find $\bar{x} \in (0,1)$ and $\bar{y} \in (0,1)$ such that

$$\int_0^1 \operatorname{dist}(\nabla u(x,\bar{y}), K_{\gamma,4})^2\, dx + \sigma|\partial_1 \nabla u(\cdot,\bar{y})|(0,1) \leq E_{\sigma,\gamma,4}(u) \tag{7.4}$$
$$\text{and } \int_0^1 \operatorname{dist}(\nabla u(\bar{x},y), K_{\gamma,4})^2\, dy + \sigma|\partial_2 \nabla u(\bar{x},\cdot)|(0,1) \leq E_{\sigma,\gamma,4}(u).$$

As $\nabla u \in BV((0,1)^2;\mathbb{R}^2)$ and $u = 0$ on $\partial(0,1)^2$, we have in the sense of traces $\partial_2 u(0,y) = 0$ and $\partial_1 u(x,0) = 0$. Using (7.4) we obtain the estimate $|\nabla u(x,\bar{y})| \leq 3/6 = 1/2$ for a.e. $x \in (0,1)$. Hence, it follows that

$$\frac{1}{4} \leq \int_0^1 \operatorname{dist}(\nabla u(x,\bar{y}), K_{\gamma,N})^2\, dx \leq E_{\sigma,\gamma,N}(u).$$

This shows claim (7.3). Then the lower bound for $\sigma \geq 1$ follows immediately. Next, fix $K \in \mathbb{N}$ from Proposition 6.2 and notice that for $1 \geq \sigma \geq |\sin(\gamma)|^K$ it holds

$$\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1 \leq K + 1.$$

Consequently it follows from estimate (7.3)

$$\min_{u \in \mathcal{A}_0} E_{\sigma,\gamma,4}(u) \geq \frac{1}{6}\min\{1,\sigma\} \geq \frac{1}{6(K+1)} \min\left\{1, \sigma\left(\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1\right)\right\}.$$

Eventually, we note that the lower bound for $\sigma \leq |\sin(\gamma)|^K$ follows from Proposition 6.2. This finishes the proof for $N = 4$.

□

Remark 3. One can argue similarly to show that there exist $C, c > 0$ such that for $i = 1, 2$ and $\gamma \in \Gamma_3$, c.f. Remark 2,

$$c\min\left\{1, \sigma\left(\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1\right)\right\} \leq \min_{u \in \mathcal{A}_0} E_{\sigma,\gamma,3}(u; T_0^{(i)}) \leq C\min\left\{1, \sigma\left(\frac{|\log \sigma|}{|\log|\sin(\gamma)||} + 1\right)\right\}.$$

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Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10117 Berlin, Germany