

Energy-minimizing torus-valued maps with prescribed singularities, Plateau's problem, and BV-lifting

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Abstract

In this paper, we investigate the relation between energy-minimizing torus-valued maps with prescribed singularities, the lifting problem for torus-valued maps in the space BV, and Plateau's problem for vectorial currents, in codimension one. First, we show that the infimum of the $W^{1,1}$ -seminorm among all maps with values in the k -dimensional flat torus and prescribed topological singularities S is equal to the minimum of the mass among all *normal* \mathbb{R}^k -currents, of codimension one, bounded by S . Then, we show that the minimum of the BV-energy among all liftings of a given torus-valued $W^{1,1}$ -map \mathbf{u} can be expressed in terms of the minimum mass among all *integral* \mathbb{Z}^k -currents, of codimension one, bounded by the singularities of \mathbf{u} . As a byproduct of our analysis, we provide a bound for the solution of the integral Plateau problem, in codimension one, in terms of Plateau's problem for normal currents.

Keywords: Torus-valued maps, lifting problems, Plateau's problem, G -currents.

Mathematics Subject Classification: 49Q10, 49Q15, 49Q20, 58E20.

1 Introduction

In the seminal paper of Brezis, Coron, and Lieb [12], the authors showed the equivalence between the energy of harmonic maps with prescribed singularities and *minimal connections* between those points. More precisely, in the ambient space \mathbb{R}^d , $d \geq 2$, given distinct points P_1, \dots, P_k and N_1, \dots, N_k (which stand for the topological singularities), we consider the set of sphere-valued maps

$$V := \left\{ \mathbf{u} \in C^1(\mathbb{R}^d \setminus \{P_1, \dots, P_k, N_1, \dots, N_k\}, \mathbb{S}^{d-1}), \deg(\mathbf{u}, P_i) = 1, \deg(\mathbf{u}, N_i) = -1 \right\},$$

where $\deg(\mathbf{u}, P)$ denotes the degree of \mathbf{u} on a small sphere around the point P . Then, there holds

$$\inf \left\{ \int_{\mathbb{R}^d} |\nabla \mathbf{u}|^{d-1} dx \mid \mathbf{u} \in V \right\} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} L, \quad (1)$$

where α_{d-1} is the surface area of the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d and L is the *minimal connection* between P_1, \dots, P_k , and N_1, \dots, N_k i.e. the solution of a Monge-Kantorovich problem in which

P_1, \dots, P_k , and N_1, \dots, N_k play the role of marginals. More precisely, let S_k be the set of all the permutations of $\{1, \dots, k\}$. Then, L is defined as

$$L := \min_{\sigma \in S_k} \sum_{i=1}^k |P_i - N_{\sigma(i)}|. \quad (2)$$

The equality (1) carries over, with few modifications, to the case the points $P_1, \dots, P_k, N_1, \dots, N_k$ are not distinct; see [12] for the details. This result was later recast and generalised by Almgren, Browder, and Lieb [3], who interpreted the minimal connection L as a solution of Plateau's problem — that is, a minimizer of the mass among all 1-dimensional integral currents T whose boundary is defined by the point singularities at P_i, N_i :

$$L = \inf \left\{ \mathbb{M}(T) \mid \partial T = \sum_{i=1}^k ([P_i] - [N_i]) \right\}. \quad (3)$$

Here, $\mathbb{M}(T)$ denotes the mass of the current T and ∂T its boundary.

Following [12], Baldo et al. recently provided [5] a connection between energy-minimizing maps with values in a Cartesian product of spheres ($\mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \times \dots \times \mathbb{S}^{d-1}$, k -times), having prescribed topological singularities at a finite number of points, and a branched transportation problem i.e. a *Gilbert-Steiner* problem with the singular points as sources. The analysis in [5] is based on a reformulation of the branched transportation problem as a Plateau's problem for 1-dimensional integral currents with coefficients in the normed group $(\mathbb{Z}^k, \|\cdot\|_p)$, where $1 \leq p \leq \infty$ and $\|\cdot\|_p$ is the ℓ_p -norm on \mathbb{Z}^k , as introduced in [26, 25]. More precisely, given points P_1, \dots, P_k, P_{k+1} in \mathbb{R}^d with $k \geq 1$, Baldo et al. considered

$$M_p := \inf \left\{ \mathbb{M}_p(T) : \partial T = e_1 \delta_{P_1} + e_2 \delta_{P_2} + \dots + e_k \delta_{P_k} - (e_1 + e_2 + \dots + e_k) \delta_{P_{k+1}} \right\}, \quad (4)$$

where T is a 1-dimensional current with coefficients in the normed group $(\mathbb{Z}^k, \|\cdot\|_p)$ and $\mathbb{M}_p(T)$ is the p -mass of T (we refer the reader to [26, 25] and Section 2 below for more details). Now, let O_i be the set of maps $u \in W_{\text{loc}}^{1,d-1}(\mathbb{R}^d; \mathbb{S}^{d-1})$ that are constant in a neighbourhood of infinity and whose Jacobian determinant is given by $Ju = \frac{\alpha_{d-1}}{d} (\delta_{P_i} - \delta_{P_{k+1}})$ in the sense of distributions (see for instance [24, 2]). In [5], the authors considered a class of functionals E_p , defined on $O_1 \times \dots \times O_k$, which depend on p and are comparable to the L^{d-1} -seminorm of the gradient (more precisely, the energy functional E_p must satisfy the requirements of [5, Definition 13]). Assuming a minimizer of (4) admits a calibration (see Definition 2.11), they showed that

$$\inf_{O_1 \times \dots \times O_k} E_p = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} M_p. \quad (5)$$

Motivated by [12, 5], in this work we investigate the connections between energy-minimizing maps with values in the k -dimensional *flat torus* ($\mathbb{T}^k := \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, k -times) and Plateau's problems for currents in codimension 1, with coefficients in $(\mathbb{Z}^k, \|\cdot\|_p)$ or $(\mathbb{R}^k, \|\cdot\|_p)$. It is worth mentioning that the study of maps with values in the flat torus arises naturally in several physical contexts. For instance, \mathbb{T}^2 -valued maps are considered in two-component Ginzburg-Landau models, in which the order parameter is a complex vector-valued map accounting for ferromagnetic or antiferromagnetic effects in high-temperature superconductors (see e.g. [1] and the references therein).

Another motivation for our work is the study of Plateau's problem for vectorial currents, in codimension one. More precisely, we would like to compare the minimal mass between Plateau's

problem for *normal currents* and *integral currents*, with coefficients in the normed Abelian groups $(\mathbb{R}^k, \|\cdot\|_p)$, $(\mathbb{Z}^k, \|\cdot\|_p)$ respectively. It is well-known [17, 5.10] that in case $k = 1$, i.e. for classical currents, given a $(d - 2)$ -dimensional integral flat boundary S in \mathbb{R}^d with coefficients in \mathbb{Z} , we have the following equality:

$$\mathbb{P}_{\mathbb{Z}}(S) = \mathbb{P}_{\mathbb{R}}(S), \quad (6)$$

where $\mathbb{P}_{\mathbb{Z}}(S)$ is the minimal mass among all the integral currents whose boundary is S and $\mathbb{P}_{\mathbb{R}}(S)$ is the minimal mass among all the normal currents whose boundary is S .

Then, it is natural to ask whether the equality (6) extends to vectorial currents, with $k > 1$. More precisely, given a $(d - 2)$ -dimensional integral boundary S in \mathbb{R}^d with coefficients in \mathbb{Z}^k , let

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) := \inf \left\{ \mathbb{M}_p(T) : T \text{ is an } \textit{integral} (d - 1)\text{-current} \right. \\ \left. \text{with coefficients in } (\mathbb{Z}^k, \|\cdot\|_p), \partial T = S \right\} \quad (\text{P}_{\mathbb{Z}^k})$$

and let

$$\mathbb{P}_{\mathbb{R}^k,p}(S) := \inf \left\{ \mathbb{M}_p(T) : T \text{ is a } \textit{normal} (d - 1)\text{-current} \right. \\ \left. \text{with coefficients in } (\mathbb{R}^k, \|\cdot\|_p), \partial T = S \right\}. \quad (\text{P}_{\mathbb{R}^k})$$

Here, \mathbb{M}_p denotes the mass of a current, taken with respect to the ℓ^p norm $\|\cdot\|_p$. The definition immediately implies $\mathbb{P}_{\mathbb{Z}^k,p}(S) \geq \mathbb{P}_{\mathbb{R}^k,p}(S)$. However, the equality does *not* hold true, in general: in [10, Example 4.2], the authors gave an example (with $d = 2$, $k = 4$, $p = \infty$) where

$$\mathbb{P}_{\mathbb{Z}^k,\infty}(S) > \mathbb{P}_{\mathbb{R}^k,\infty}(S). \quad (7)$$

(see also Remark 3.9 below for more details).

Instead, one could try to derive a bound on $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ in terms of $\mathbb{P}_{\mathbb{R}^k,p}(S)$. We observe that we always have the following inequality:

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq k \mathbb{P}_{\mathbb{R}^k,p}(S). \quad (8)$$

To see this, let (S_1, \dots, S_k) be the components of S . For any i , let T_i be an integral current of minimal mass among all integral currents bounded by S_i . Then, on the one hand, (T_1, \dots, T_k) is an admissible competitor for problem $\mathbb{P}_{\mathbb{Z}^k,p}(S)$, so $\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq \sum_{i=1}^k \mathbb{M}(T_i)$. On the other hand, Federer's result [17, 5.10] implies that T_i also minimizes the mass among the normal currents bounded by S_i . Therefore, we have $\mathbb{M}(T_i) = \mathbb{P}_{\mathbb{R}}(S_i) \leq \mathbb{P}_{\mathbb{R}^k,p}(S)$ for any i and any p , and (8) follows. In Theorem C below, we provide a different bound, in terms of a constant that grows *sublinearly* as a function of k (when p is finite). More precisely, we prove that

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq (2k^{1-\frac{1}{p}} - 1) \mathbb{P}_{\mathbb{R}^k,p}(S). \quad (9)$$

The constant factor in front of the right-hand side is independent on the dimension of the ambient space \mathbb{R}^d : it depends only on the number of components k and on p . We observe that the inequality (9) is stronger than (8) when $p < \infty$ and k is large enough — for instance, when $1 \leq p \leq 2$ and $k > 1$ (because then $2k^{1-\frac{1}{p}} - 1 \leq 2k^{1/2} - 1 < k$). Moreover, the inequality (9) is sharp when $k = 1$ or $p = 1$ (for in this case, it reduces to $\mathbb{P}_{\mathbb{Z}^k,p}(S) = \mathbb{P}_{\mathbb{R}^k,p}(S)$), but it is not sharp when $k > 1$ and $p = \infty$ (for in this case, the estimate (8) is stronger than (9)). We do not know whether (8) is sharp for $k > 1$, $1 < p < \infty$. In fact, to the best of our knowledge, we are not aware of any inequality that provides an estimate for $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ in terms of $\mathbb{P}_{\mathbb{R}^k,p}(S)$, other than (8) and (9).

Main results. Throughout the paper, we will consider currents in \mathbb{R}^d with coefficients in the normed space $(\mathbb{R}^k, \|\cdot\|_p)$, where $p \in [1, \infty]$, as defined by Marchese and Massaccesi [26]. (For convenience of the reader, we recall the definition in Section 2 below). We consider a $(d-2)$ -dimensional current S , with coefficients in \mathbb{Z}^k . We will always assume that S is an \mathbb{Z}^k -integral flat boundary — that is, there exists a $(d-1)$ -dimensional, rectifiable current T , with multiplicity in \mathbb{Z}^k , such that $\partial T = S$ — and that S has compact support. Moreover, we will often make a (rather mild) assumption on the support of S . More precisely, if (S_1, \dots, S_k) are the components of S , we require that

$$\text{for each } i = 1, \dots, k, \quad \mathbb{R}^d \setminus \text{spt } S_i \text{ is connected and } \mathcal{H}^{d-1}(\text{spt } S_i) = 0, \quad (10)$$

where $\text{spt } S_i$ denotes the support of S_i and \mathcal{H}^{d-1} the Hausdorff measure of dimension $d-1$. However, we do not need to assume the condition (10) in Theorem C. Let (S_1, \dots, S_k) be the components of S . For each index $i = 1, \dots, k$, we consider the space G_i , defined as the subset of $W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{S}^1)$ consisting of the functions u_i that are constant outside an open ball of radius $r_i = r_i(u_i)$ and are such that $\star J(u_i) = \pi S_i$. (Here, $J(u_i)$ denotes the distributional Jacobian of u_i and \star is the Hodge star operator; see Section 2 for details.) We define $\mathcal{Q}_S = G_1 \times G_2 \times \dots \times G_k$. As it turns out, if S has finite mass and the assumption (10) is satisfied, then the set \mathcal{Q}_S is nonempty — see Remark 3.6.

Let $p \in [1, \infty]$ be fixed. In this paper, we shall consider two functionals. The first one is the $W^{1,1}$ -harmonic energy, defined by

$$H_p(\mathbf{u}) := \int_{\mathbb{R}^d} \|(|\nabla u_1|, \dots, |\nabla u_k|)\|_p dx \quad (11)$$

for any $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S$. Correspondingly, we set

$$\mathbb{H}_p(S) := \inf \{H_p(\mathbf{u}) : \mathbf{u} \in \mathcal{Q}_S\}. \quad (\text{H})$$

The second functional we consider is given by

$$E_p(\mathbf{u}) := \int_{\mathbb{R}^d} |\nabla \mathbf{u}(x)|_{\text{nucl},p} dx \quad (12)$$

for any $\mathbf{u} \in \mathcal{Q}_S$, where $|\cdot|_{\text{nucl},p}$ is the so-called p -nuclear norm. This is a suitable norm on the space of linear operators $\mathbb{R}^d \rightarrow \mathbb{C}^k$, defined in (3.4). We set

$$\mathbb{E}_p(S) := \inf \{E_p(\mathbf{u}) : \mathbf{u} \in \mathcal{Q}_S\}. \quad (\text{E})$$

The functional H_p has the advantage of being rather simple to compute, because its integrand is defined component-wise. This will make it a handy tool in our analysis of the lifting problem, in the second part of the paper (see Section 4 below). On the other hand, the functional E_p turns to be quite a natural object to consider, because the nuclear norm of the gradient $|\nabla \mathbf{u}|_{\text{nucl},p}$ equals the mass of a suitable vector field, depending on \mathbf{u} (see Lemma 3.7 below). In particular, E_p “is suitably related to \mathbb{M} and H_p ”, in the sense of [5, Definition 13]. In the first part of the paper, we investigate the relationship between Problems (H), (E) and $\mathbb{P}_{\mathbb{R}^k,p}(S)$.

Theorem A. *Let S be a \mathbb{Z}^k -integral flat boundary, of dimension $(d-2)$. Assume that S has finite mass and compact support and the condition (10) is satisfied. Then, for any $p \in [1, \infty]$, one has*

$$\mathbb{H}_p(S) \leq \mathbb{E}_p(S) = 2\pi \mathbb{P}_{\mathbb{R}^k,p}(S) \leq \mathbb{H}_1(S) \leq k^{1-\frac{1}{p}} \mathbb{H}_p(S). \quad (13)$$

Remark 1. In the planar case, i.e. when $d = 2$, the first part of present paper is in the same spirit as [5]. In particular, our functional \mathbb{E} satisfies the requirements of [5, Definition 13]. Furthermore, we are able to prove the equivalence between the problems (E) and $\mathbb{P}_{\mathbb{R}^k,p}(S)$ thanks to the special structure of the target manifold. It turns out that the main result of [5] is not always true in case without assuming the existence of calibration: in Remark 3.9, by combining Theorem A with Example 4.2 from [10], we provide a counterexample for the equality (5) in case $p = \infty$, $k = 4$, and $d = 2$. Instead, in Theorem 4.6 below, we give a necessary and sufficient condition for the equality $\mathbb{H}_p(S) = 2\pi \mathbb{P}_{\mathbb{Z}^k,p}(S)$ to hold, in terms of the lifting problem for torus-valued maps.

In the second part of the paper, we investigate the relationship between Plateau's problem and the *lifting* problems for torus-valued maps of bounded variation. Given $\mathbf{u} \in \mathcal{Q}_S$, we will say that $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \text{BV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ is a BV-lifting of \mathbf{u} if $u_j = e^{i\theta_j}$ a.e. in \mathbb{R}^d , for any index $j = 1, \dots, k$. Any map $\mathbf{u} \in \mathcal{Q}_S$ admits a BV-lifting [19, 15, 21]. Now, given $u \in \mathcal{Q}_S$, we consider the following problem:

$$\mathbb{L}_p(S, \mathbf{u}) := \inf \left\{ \int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p \, d\mathcal{H}^{d-1}(x) : \boldsymbol{\theta} \text{ is a BV-lifting of } \mathbf{u} \right\}, \quad (\text{L})$$

where $S(\boldsymbol{\theta})$ is the $(d-1)$ -rectifiable jump set of $\boldsymbol{\theta}$, and $\boldsymbol{\theta}^+(x)$, $\boldsymbol{\theta}^-(x)$ stand for the right and left approximate limits of $\boldsymbol{\theta}$ at almost every point $x \in S(\boldsymbol{\theta})$ (we refer the reader to Section 4 for more details). As we will prove later on (in Lemma 4.4), the quantity $\mathbb{L}_p(S, \mathbf{u})$ actually depends on S , but not on the choice of $\mathbf{u} \in \mathcal{Q}_S$, so we can write $\mathbb{L}_p(S)$ instead of $\mathbb{L}_p(S, \mathbf{u})$. The relation between the problems (L) and ($\mathbb{P}_{\mathbb{Z}^k}$) is encoded in the following theorem:

Theorem B. *Let S be a \mathbb{Z}^k -integral flat boundary, of dimension $(d-2)$. Assume that S has finite mass and compact support and the condition (10) is satisfied. Then, for any $p \in [1, \infty]$ one has*

$$\mathbb{L}_p(S) = 2\pi \mathbb{P}_{\mathbb{Z}^k,p}(S). \quad (14)$$

Finally, combining Theorem A and Theorem B with suitable bounds on the norm of BV-liftings (see [15]), we obtain a bound for $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ in terms of $\mathbb{P}_{\mathbb{R}^k,p}(S)$.

Theorem C. *Let S be a \mathbb{Z}^k -integral flat boundary, of dimension $(d-2)$, of compact support. Then, for all $p \in [1, \infty]$, we have*

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq (2k^{1-\frac{1}{p}} - 1) \mathbb{P}_{\mathbb{R}^k,p}(S). \quad (15)$$

Note that Theorem C does *not* rely on the assumption that S has finite mass, nor that it satisfies (10). We have removed these assumptions by an approximation procedure.

Remark 2. In case $k = 1$, when we are working with \mathbb{S}^1 -valued maps, the quantities defined in ($\mathbb{P}_{\mathbb{Z}^k}$), ($\mathbb{P}_{\mathbb{R}^k}$), (L), (H) and (E) do not depend on p . Moreover, from Theorem A, Theorem B and Theorem C we deduce (dropping the dependence of p from the notation) that

$$\mathbb{H}(S) = \mathbb{E}(S) = \mathbb{L}(S) = 2\pi \mathbb{P}_{\mathbb{R}}(S) = 2\pi \mathbb{P}_{\mathbb{Z}}(S). \quad (16)$$

The equality $\mathbb{H}(S) = 2\pi \mathbb{P}_{\mathbb{Z}}(S)$ was first conjectured by Brezis, Coron and Lieb [12, Equation (8.23)]. The same authors gave a proof of this equality in a particular case, namely, when $d = 3$ and the boundary S is a planar curve. Later on, Almgren, Browder and Lieb [3] proved it in greater generality, using the coarea formula (see also the recent book by Brezis and Mironescu [13], in particular Chapters 2 and 3). In our paper, we provide an alternative proof, which is based on the existence of BV-liftings instead of the coarea formula.

Remark 3. We stress that our arguments do *not* depend on Federer’s result [17] that $\mathbb{P}_{\mathbb{Z}}(S) = \mathbb{P}_{\mathbb{R}}(S)$ in case $k = 1$. In fact, Theorem C provides an alternative proof for this fact.

The paper is organized as follows. In Section 2, we recall basic notions in Geometric Measure Theory which will be used in the paper. Section 3 is devoted to the proof Theorem A. In Section 4 we address the lifting problem for torus-valued maps in the space BV, and we prove Theorems B and C. In addition, still in Section 4, we provide a necessary and sufficient condition for the problems (H) and $(P_{\mathbb{Z}^k})$ to be equivalent. An appendix, containing the proof of some technical results, completes the paper.

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2 Notations and preliminaries

2.1 Preliminary notions

2.1.1 Currents with coefficients in normed groups and Plateau’s problems

Throughout this entire paper, we work in the ambient space \mathbb{R}^d , with $d \geq 2$. We denote by $\mathbb{B}_r^d(x)$ the open ball in \mathbb{R}^d with center at $x \in \mathbb{R}^d$ and radius $r > 0$, and by $\bar{\mathbb{B}}_r^d(x)$ its closure. For each integer $k > 0$, $\mathbb{S}^{k-1} = \partial\mathbb{B}_1^k(0)$ is the unit sphere in \mathbb{R}^k and $\alpha_k = \mathcal{H}^{k-1}(\mathbb{S}^{k-1})$ is its surface area, where \mathcal{H}^k is the k -dimensional Hausdorff measure. We denote by \mathcal{L}^k the Lebesgue measure in \mathbb{R}^k , and we use the notation $|E|$ to denote the Lebesgue measure of the measurable set E for convenience.

We follow [27] for basic notions and terminologies regarding currents with coefficients in a normed group. Let $k \geq 1$ be an integer and let $1 \leq p \leq \infty$. The normed groups that we consider in this paper are $(\mathbb{R}^k, \|\cdot\|_p)$ (corresponding to normal currents) and $(\mathbb{Z}^k, \|\cdot\|_p)$ (corresponding to integral currents), where $\|\cdot\|_p$ is the ℓ_p -norm in \mathbb{R}^k , defined for $z = (z_1, \dots, z_k) \in \mathbb{R}^k$ by

$$\|z\|_p := \begin{cases} \left(\sum_{j=1}^k |z_j|^p\right)^{\frac{1}{p}} & \text{in case } p \in [1, \infty), \\ \max\{|z_1|, \dots, |z_k|\} & \text{in case } p = \infty. \end{cases} \quad (2.17)$$

For the convenience of the reader, we recall here the main definitions.

Given an integer m with $1 \leq m \leq d$, we denote by $\Lambda_m(\mathbb{R}^d)$ the space of m -vectors and by $\Lambda^m(\mathbb{R}^d)$ the space of m -covectors in \mathbb{R}^d . Let $\{e_1, e_2, \dots, e_d\}$ be an orthonormal basis of \mathbb{R}^d , and let $\{dx_1, dx_2, \dots, dx_d\}$ be its dual basis. Given a multi-index $I = \{i_1, \dots, i_m\}$ such that $i_1 < i_2 < \dots < i_m$, we use the notation $e_I := e_{i_1} \wedge \dots \wedge e_{i_m}$ and $dx_I := dx_{i_1} \wedge \dots \wedge dx_{i_m}$, where \wedge is the wedge product. Then, we introduce the following operators:

$$\#: \Lambda^m(\mathbb{R}^d) \rightarrow \Lambda_m(\mathbb{R}^d), \quad \flat: \Lambda_m(\mathbb{R}^d) \rightarrow \Lambda^m(\mathbb{R}^d). \quad (2.18)$$

The operator $\#$ is the identification between m -covectors and m -vectors induced by Euclidean metric i.e. for any m -covector $w = \sum_I a_I dx_I$, the corresponding vector $w^\#$ is defined by

$$w^\# := \sum_I a_I e_I, \quad (2.19)$$

where the sum is taken over all multi-indices $I = \{i_1, \dots, i_m\}$ such that $i_1 < i_2 < \dots < i_m$. The operator \flat is the inverse of $\#$: for any m -vector $w = \sum_I a_I e_I$, the corresponding covector w^\flat is defined by

$$w^\flat := \sum_I a_I dx_I, \quad (2.20)$$

where again the sum is taken over all multi-indices $I = \{i_1, \dots, i_m\}$ such that $i_1 < i_2 < \dots < i_m$.

Definition 2.4. An $(\mathbb{R}^k)^*$ -valued m -covector on \mathbb{R}^d is a bilinear map

$$w: \Lambda_m(\mathbb{R}^d) \times \mathbb{R}^k \longrightarrow \mathbb{R}.$$

An \mathbb{R}^k -valued m -vector on \mathbb{R}^d is a bilinear map

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We denote by $\Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ and $\Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ the spaces of $(\mathbb{R}^k)^*$ -valued m -covectors and (\mathbb{R}^k) -valued m -vectors on \mathbb{R}^d , respectively. These spaces are dual each other. We observe that $(\mathbb{R}^k)^*$ -valued covectors and \mathbb{R}^k -vectors can be identified with k -tuples of classical covectors and vectors. More precisely, let $\{e_1, e_2, \dots, e_k\}$ be an orthonormal basis of \mathbb{R}^k , and let $\{e_1^*, e_2^*, \dots, e_k^*\}$ be its dual. Then, each $(\mathbb{R}^k)^*$ -valued m -covector on \mathbb{R}^d can be expressed as $w = w_1 e_1^* + \dots + w_k e_k^*$, where for any index $i = 1, \dots, k$, w_i is a classical m -covector in \mathbb{R}^d , defined by $w_i := \langle w; \cdot, e_i \rangle$. Similarly, each \mathbb{R}^k -valued m -vector on \mathbb{R}^d can be expressed as $v = v_1 e_1 + \dots + v_k e_k$ where v_i is a classical m -vector in \mathbb{R}^d defined by $v_i := \langle v; \cdot, e_i^* \rangle$. The operators $\#, \flat$ can be extended to operators $\Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d) \rightarrow \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ and $\Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d) \rightarrow \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ respectively, by applying $\#, \flat$ component-wise.

Given $p \in [1, \infty]$, the space $\Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ is equipped with the p -comass norm, defined by

$$|w|_{\text{com}, p} := \sup \left\{ \|\langle w; \tau, \cdot \rangle\|_p^* : |\tau| \leq 1, \tau \text{ is simple } m\text{-vector} \right\} \quad (2.21)$$

for any $w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$. Here, $\|\cdot\|$ is the Euclidean norm on $\Lambda^m(\mathbb{R}^d)$, $\Lambda_m(\mathbb{R}^d)$. We identify the dual space $((\mathbb{R}^k)^*, \|\cdot\|_p^*)$ with $(\mathbb{R}^k, \|\cdot\|_{p^*})$ where $p^* \in [1, \infty]$ is the conjugate exponent of p , defined in such a way that $\frac{1}{p^*} + \frac{1}{p} = 1$ (on the understanding that $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$). The space $\Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ is equipped with the pre-dual norm or p -mass norm, defined as

$$|v|_{\text{mass}, p} := \sup \left\{ \langle w, v \rangle : w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d), |w|_{\text{com}, p} \leq 1 \right\} \quad (2.22)$$

for any $v \in \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$. In case $k = 1$, the mass and comass norms are independent of p (i.e. $|v|_{\text{mass}, p} = |v|_{\text{mass}, 1}$, $|w|_{\text{com}, p} = |w|_{\text{com}, 1}$ for any $p \in [1, \infty]$, $v \in \Lambda_m(\mathbb{R}^d)$, $w \in \Lambda^m(\mathbb{R}^d)$) and reduce to the classical mass and comass. In this case, we write $|v|_{\text{mass}} := |v|_{\text{mass}, 1}$, $|w|_{\text{com}} := |w|_{\text{com}, 1}$ for any $v \in \Lambda_m(\mathbb{R}^d)$, $w \in \Lambda^m(\mathbb{R}^d)$.

Remark 2.5. (i) The mass norm of a vector $v \in \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ can be equivalently characterized as

$$|v|_{\text{mass}, p} = \inf \left\{ \sum_{i=1}^l \|z_i\|_p |v_i| : v = \sum_{i=1}^l z_i \otimes v_i, z_i \in \mathbb{R}^k, v_i \text{ is a simple } m\text{-vector in } \mathbb{R}^d \right\} \quad (2.23)$$

(for a proof, see [16, § 1.8.1] for the scalar case $k = 1$ and Lemma A.2 in the appendix for the case $k > 1$).

(ii) Let $v \in \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ be a vector with components $v_1, \dots, v_k \in \Lambda_m(\mathbb{R}^d)$. If $1 \leq p < \infty$, then there holds

$$\left(\sum_{i=1}^k |v_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^k |v_i|_{\text{mass}}^p \right)^{\frac{1}{p}} \leq |v|_{\text{mass}, p} \leq \sum_{i=1}^k |v_i|_{\text{mass}}. \quad (2.24)$$

See Lemma A.6 in the appendix for the proof of (2.24), as well as more general results on the comparison between different norms of a given \mathbb{R}^k -valued m -vector or $(\mathbb{R}^k)^*$ -valued m -covector.

Definition 2.6. An $(\mathbb{R}^k)^*$ -valued m -dimensional differential form, or $(\mathbb{R}^k)^*$ -valued m -form, defined on \mathbb{R}^d is a map

$$\omega: \mathbb{R}^d \longrightarrow \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d).$$

As before, ω can be represented as $\omega = \omega_1 e_1^* + \dots + \omega_k e_k^*$, and the regularity of ω is inherited from the regularity of its components ω_i , for $i = 1, \dots, k$. Let $\omega = \omega_1 e_1^* + \dots + \omega_k e_k^* \in C^1(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{m-1}(\mathbb{R}^d))$. We denote

$$d\omega := d\omega_1 e_1^* + \dots + d\omega_k e_k^*,$$

where $d\omega_i$ is the differential of ω_i . By definition, we have $d\omega \in C(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$. We define the support of ω as $\text{supp } \omega := \text{cl}\{x \in \mathbb{R}^d : \omega(x) \neq 0\}$, where cl indicates the closure of a set. We can now define the notion of currents with coefficients in \mathbb{R}^k .

Definition 2.7. An m -dimensional current, or m -current, T with coefficients in $(\mathbb{R}^k, \|\cdot\|_p)$ is a linear and continuous map

$$T: C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)) \longrightarrow \mathbb{R}.$$

The continuity of T is understood with respect to the (locally convex) topology on the space of test forms $C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$, which is analogous to the standard topology on $C_c^\infty(\mathbb{R}^d; \mathbb{R})$.

Observe that a vector-valued current can be written in components, and that we will often use the notation $T = (T_1, \dots, T_k)$, where T_i is the classical current defined by

$$T_i(\omega) := T(\omega e_i^*) \quad \text{for any } \omega \in C_c^\infty(\mathbb{R}^d; \Lambda^m(\mathbb{R}^d)). \quad (2.25)$$

The mass of T is defined as

$$\mathbb{M}_p(T) := \sup \left\{ T(\omega) : \omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)), \|\omega\|_{\text{com}, p} \leq 1 \right\},$$

where $\|\omega\|_{\text{com}, p} = \sup_{x \in \mathbb{R}^d} |\omega(x)|_{\text{com}, p}$. When $k = 1$, \mathbb{M}_p is equal to the mass of classical currents for any $p \in [1, \infty]$, therefore we shall denote it by \mathbb{M} .

Moreover, we define the boundary ∂T of T as an $(m-1)$ -dimensional current, such that

$$\partial T(\omega) := T(d\omega) \quad (2.26)$$

for any $\omega \in C_c^\infty(\mathbb{R}^d, \Lambda_{\mathbb{R}^k}^{m-1}(\mathbb{R}^d))$. The boundary operator acts component-wise. The space of \mathbb{R}^k -currents of dimension m is denoted by $\mathcal{D}'(\mathbb{R}^d, \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)) = (C_c^\infty(\mathbb{R}^d, \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)))'$. In case $k = 1$, we write $\mathcal{D}'(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d))$ for the space of classical currents. The support of the current T , denoted by $\text{spt}(T)$, is defined by:

$$\text{spt}(T) := \mathbb{R}^d \setminus \cup \left\{ W : W \text{ is open, } T(\varphi) = 0 \text{ whenever } \varphi \in C_c^\infty(W; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)) \right\}. \quad (2.27)$$

Weak* convergence of a sequence of currents, $T^\ell \rightharpoonup^* T$, is defined by duality with the space of test forms $C_c^\infty(\mathbb{R}^d; \Lambda^m(\mathbb{R}^d))$. It turns out that a sequence of \mathbb{R}^k -valued currents converge weakly* if and only if the j -th components of the sequence converge weakly* to the j -th component of the limit, for any index j .

Definition 2.8. A current T is said to be an \mathbb{R}^k -normal current if $\mathbb{M}_p(T) + \mathbb{M}_p(\partial T) < \infty$.

Definition 2.9. A m -dimensional rectifiable current with coefficients in $(\mathbb{Z}^k, \|\cdot\|_p)$ (or a \mathbb{Z}^k -rectifiable current) is a m -dimensional current such that there exists a m -dimensional oriented rectifiable set $\Sigma \subset \mathbb{R}^d$, an approximate tangent vectorfield $\tau: \Sigma \rightarrow \Lambda_m(\mathbb{R}^d)$, and a density function $\theta: \Sigma \rightarrow \mathbb{Z}^k$ such that

$$T(\omega) = \int_{\Sigma} \langle \omega(x); \tau(x), \theta(x) \rangle d\mathcal{H}^m(x) \quad (2.28)$$

for every $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$. We say that T is an integral current if both T and ∂T are \mathbb{Z}^k -rectifiable currents. A m -dimensional polyhedral current is a finite union of m -dimensional oriented simplexes Σ_i , each equipped with constant multiplicity $\sigma_i \in \mathbb{Z}^k$, such that for any $i \neq j$, the intersection between Σ_i and Σ_j is either empty or a common face. In addition, an m -dimensional current T is said to be a cycle if $\partial T = 0$; T is called a boundary if there exists an $(m+1)$ -dimensional current R such that $T = \partial R$.

We write $T := \llbracket \Sigma, \tau, \theta \rrbracket$ for the rectifiable current defined in (2.28).

Remark 2.10. Given an integrable vector field $\tau = (\tau_1, \dots, \tau_k) \in L^1(\mathbb{R}^d; \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d))$, integration against τ defines a current T — that is, each component of $T = (T_1, \dots, T_k)$ is represented by integration against $\tau_i \in L^1(\mathbb{R}^d; \Lambda_m(\mathbb{R}^d))$, $T_i = \tau_i \wedge \mathcal{L}^d$. Then, classical arguments in measure theory show that the current T has finite mass and

$$\mathbb{M}_p(T) = \int_{\mathbb{R}^d} |\tau(x)|_{\text{mass}, p} d\mathcal{L}^d(x).$$

In a similar spirit, given an m -dimensional rectifiable current T with coefficients in $(\mathbb{Z}^k, \|\cdot\|_p)$, the mass of T can be written as

$$\mathbb{M}_p(T) = \int_{\Sigma} \|\theta(x)\|_p d\mathcal{H}^m(x).$$

Moreover, T can be expressed as (T_1, \dots, T_k) , where each component T_i is a m -dimensional classical rectifiable current with coefficients in \mathbb{Z} (see [27]).

We now recall Plateau's problem in the setting of currents with coefficients in the groups $(\mathbb{R}^k, \|\cdot\|_p)$, $(\mathbb{Z}^k, \|\cdot\|_p)$. Given a $(d-2)$ -dimensional \mathbb{Z}^k -integral flat boundary S with compact support, we can define Plateau's problem for S in integral currents as

$$\mathbb{P}_{\mathbb{Z}^k, p}(S) := \inf \left\{ \mathbb{M}_p(T) : T \text{ is a rectifiable } \mathbb{Z}^k\text{-current of dimension } (d-1), \partial T = S \right\}. \quad (2.29)$$

The Plateau problem for S in normal currents is

$$\mathbb{P}_{\mathbb{R}^k, p}(S) := \inf \left\{ \mathbb{M}_p(T) : T \text{ is a normal } \mathbb{R}^k\text{-current of dimension } (d-1), \partial T = S \right\}. \quad (2.30)$$

One of the advantages of the approach based on the theory of currents is the possibility to define the (dual) notion of calibration, which provides us with a tool to check whether a given configuration is a minimizer. We briefly recall the definition of calibration, which will be useful in the next sections.

Definition 2.11. Consider a \mathbb{Z}^k -rectifiable current of dimension m , $T = \llbracket \Sigma, \tau, \theta \rrbracket$ in the ambient space \mathbb{R}^d . A smooth $(\mathbb{R}^k)^*$ -valued m -dimensional differential form ω in \mathbb{R}^d is a calibration for T if the following conditions hold:

- (i) $\langle \omega(x); \tau(x), \theta(x) \rangle = \|\theta(x)\|_p$ for \mathcal{H}^m -a.e $x \in \Sigma$;
- (ii) ω is closed, i.e, $d\omega = 0$;
- (iii) for every $x \in \mathbb{R}^d$, for every simple unit vector $t \in \Lambda_m(\mathbb{R}^d)$ and for every $h \in \mathbb{Z}^k$, we have that

$$\langle \omega(x); t, h \rangle \leq \|h\|_p.$$

If a current T admits a calibration, then it is a minimizer of the mass among all normal \mathbb{R}^k currents its homology class. (See for instance [27] for a proof of this claim). Therefore, the existence of a calibration for an integral \mathbb{Z}^k -current T is a sufficient condition to prove that $\mathbb{P}_{\mathbb{Z}^k, p}(\partial T) = \mathbb{P}_{\mathbb{R}^k, p}(\partial T)$. The question of whether or not a minimizing current admits a calibration is a delicate issue (see for instance [27, Section 1.4, Section 3.1.2] and references therein, as well as [5, Section 2.1 and Example 16]).

2.1.2 Push-forward of currents

We recall the notion of push-forward of currents. The push-forward is first introduced for classical currents (see e.g. [18, Section 7.4.2]), then extended to currents with coefficients in the group $(\mathbb{R}^k, \|\cdot\|_p)$. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^l$ be a smooth function, and let ω be a smooth, compactly supported m -differential form in \mathbb{R}^l . We define the pull-back of ω , $f^\#(\omega)$, as a smooth differential form in \mathbb{R}^d , given by

$$f^\#(\omega)(x)(v_1 \wedge \dots \wedge v_m) := \omega(f(x))(df(v_1) \wedge \dots \wedge df(v_m)) \quad (2.31)$$

for any $x \in \mathbb{R}^d$ and any simple m -vector $v = v_1 \wedge \dots \wedge v_m$. Next, let T be a classical normal current in \mathbb{R}^d . We assume again that $f: \mathbb{R}^d \rightarrow \mathbb{R}^l$ is smooth and that $f|_{\text{spt}(T)}$ is proper (i.e., for any compact set $A \subset \mathbb{R}^l$, $f^{-1}(A) \cap \text{spt}(T)$ is compact). Then, we define the push-forward of the current T , $f_\#(T)$, to be the m -dimensional current in \mathbb{R}^l given by

$$f_\#(T)(\omega) := T(\psi f^\#(\omega)) \quad (2.32)$$

for any $\omega \in C_c^\infty(\mathbb{R}^l; \Lambda^m(\mathbb{R}^l))$, where ψ is any compactly supported, smooth function in \mathbb{R}^d that is equal to 1 in a neighborhood of $\text{spt}(T) \cap \text{supp } f^\#(\omega)$. It can be checked that the right-hand side of (2.32) is independent of the choice of ψ , and that

$$\partial f_\#(T) = f_\#(\partial T). \quad (2.33)$$

This notion can be extended to the case f is a Lipschitz map (see [18, Section 7.4.2, Lemma 7.4.3] for the details). Moreover, given a Lipschitz map f (with Lipschitz constant Λ) such that the restriction of f to the support of T is proper, there holds

$$\mathbb{M}(f_\#(T)) \leq \Lambda \mathbb{M}(T) \quad (2.34)$$

(see again [18, Section 7.4.2, Lemma 7.4.3]). The push-forward can be defined in case $T = (T_1, \dots, T_k)$ is a current with coefficients in the group $(\mathbb{R}^k, \|\cdot\|_p)$, $(\mathbb{Z}^k, \|\cdot\|_p)$ by working component-wise, i.e. $f_\#(T) := (f_\#(T_1), \dots, f_\#(T_k))$.

2.1.3 The Hodge-star operator and the codifferential

The Hodge-star operator (on a Riemannian manifold of dimension d) is usually defined as a map between (classical) m -covectors and $(d - m)$ -covectors. Here, instead, we find it convenient to regard it as an operator between vectors and covectors. These approaches are essentially equivalent; the operator we consider here reduces to the classical one, up to composition with the isomorphisms $\#$, \flat . However, for the convenience of the reader, we recall the definitions.

We define the Hodge-star operator as a map

$$\star: \Lambda^m(\mathbb{R}^d) \rightarrow \Lambda_{d-m}(\mathbb{R}^d),$$

in the following way: for any $\beta \in \Lambda^m(\mathbb{R}^d)$, $\star\beta$ is the unique element of $\Lambda_{d-m}(\mathbb{R}^d)$ such that

$$\langle \alpha, \star\beta \rangle := \langle \beta \wedge \alpha, e_1 \wedge \dots \wedge e_d \rangle \quad \text{for every } \alpha \in \Lambda^{d-m}(\mathbb{R}^d). \quad (2.35)$$

Remark 2.12. There is a variant definition of Hodge star operator which is defined by

$$\langle \alpha, \star\beta \rangle := \langle \alpha \wedge \beta, e_1 \wedge \dots \wedge e_d \rangle \quad \text{for every } \alpha \in \Lambda^{d-m}(\mathbb{R}^d) \quad (2.36)$$

for any $\beta \in \Lambda^m(\mathbb{R}^d)$. The two definitions agree up to a sign. In this paper, we choose the definition in (2.35) for the compatibility in the sign involving the Hodge star operator, pre-jacobian, and distributional Jacobian later (see for instance (2.15)).

An operator $\Lambda_m(\mathbb{R}^d) \rightarrow \Lambda^{d-m}(\mathbb{R}^d)$, still denoted \star for simplicity, is defined in a way completely analogous to (2.35). The Hodge-star operator can be described explicitly as follows: for any multi-index I , there holds

$$\star dx_I = \sigma(I, I') e_{I'} \quad \text{and} \quad \star e_I = \sigma(I, I') dx_{I'} \quad (2.37)$$

where I' is the set of indices that are not contained in I and $\sigma(I', I)$ is the sign of the permutation (I', I) . From (2.35) and (2.37), it follows that \star is an isometry and that

$$\star \star \beta = (-1)^{m(d-m)} \beta \quad (2.38)$$

whenever β is an m -vector or an m -covector. As a consequence, we have

$$\langle \alpha, \star\beta \rangle = (-1)^{m(d-m)} \langle \star\alpha, \beta \rangle \quad (2.39)$$

for any $\alpha \in \Lambda^{d-m}(\mathbb{R}^d)$, $\beta \in \Lambda^m(\mathbb{R}^d)$ as well as any $\alpha \in \Lambda_{d-m}(\mathbb{R}^d)$, $\beta \in \Lambda_m(\mathbb{R}^d)$.

The Hodge-star operator extends to an operator between form-valued distributions and (classical) currents. More precisely, let $\mathcal{D}'(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d)) = (C_c^\infty(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d)))'$ be the space of m -dimensional form-valued distributions and let $\mathcal{D}'(\mathbb{R}^d, \Lambda_{d-m}(\mathbb{R}^d)) = (C_c^\infty(\mathbb{R}^d, \Lambda^{d-m}(\mathbb{R}^d)))'$ be the space of $(d - m)$ -dimensional classical currents. (Both $C_c^\infty(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d))$ and $C_c^\infty(\mathbb{R}^d, \Lambda^{d-m}(\mathbb{R}^d))$ are equipped with the locally convex topology of test functions.) In view of (2.39), it is natural to define

$$\star: \mathcal{D}'(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d)) \rightarrow \mathcal{D}'(\mathbb{R}^d, \Lambda_{d-m}(\mathbb{R}^d)), \quad \star: \mathcal{D}'(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d)) \rightarrow \mathcal{D}'(\mathbb{R}^d, \Lambda^{d-m}(\mathbb{R}^d))$$

by duality, in the following way: for any $\omega \in \mathcal{D}'(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d))$, the associated current $\star\omega$ is given by

$$\star\omega(\tau) := (-1)^{m(d-m)} \omega(\star\tau) \quad (2.40)$$

for any $\tau \in C_c^\infty(\mathbb{R}^d, \Lambda^{d-m}(\mathbb{R}^d))$. Similarly, given a current $S \in \mathcal{D}'(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d))$, we define the form-valued distribution $\star S$ as

$$\star S(v) := (-1)^{m(d-m)} S(\star v) \quad (2.41)$$

for any $v \in C_c^\infty(\mathbb{R}^d, \Lambda_{d-m}(\mathbb{R}^d))$. The Hodge-star operator may be used to define the codifferential operator d^* :

$$d^* := (-1)^{d(m-1)+1} \star d \star: C_c^\infty(\mathbb{R}^d; \Lambda_m(\mathbb{R}^d)) \rightarrow C_c^\infty(\mathbb{R}^d; \Lambda_{m-1}(\mathbb{R}^d)). \quad (2.42)$$

While usually d^* is defined as an operator between forms, here we regard d^* as an operator between vector fields. This is consistent with our definition of the Hodge operator \star , which maps vectors to covectors and vice-versa. In spite of this unusual choice, the codifferential is still the formal adjoint of the exterior differential d — that is, for any $\alpha \in C_c^\infty(\mathbb{R}^d; \Lambda^{m-1}(\mathbb{R}^d))$ and $\beta \in C_c^\infty(\mathbb{R}^d; \Lambda_m(\mathbb{R}^d))$, we have

$$\int_{\mathbb{R}^d} \langle d\alpha(x), \beta(x) \rangle dx = \int_{\mathbb{R}^d} \langle \alpha(x), d^*\beta(x) \rangle dx. \quad (2.43)$$

We define the differential in the sense of distributions as follows: given $\omega \in \mathcal{D}'(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d))$, we define $d\omega \in \mathcal{D}'(\mathbb{R}^d, \Lambda^{m+1}(\mathbb{R}^d))$ by

$$d\omega(v) := \omega(d^*v) \quad (2.44)$$

for any test vector-field $v \in C_c^\infty(\mathbb{R}^d, \Lambda_{m+1}(\mathbb{R}^d))$. By comparing (2.44) with the definition of the boundary operator (2.26), we deduce that the boundary and the codifferential agree on smooth vector fields. More precisely, if we identify a vector field $v \in C_c^\infty(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d))$ with a current (defined by integration against v), then ∂v is the current carried by d^*v . (For instance, on 1-dimensional currents, $\partial = -\text{div}$ in the sense of distributions.) Moreover, the Hodge-star operator exchanges differentials and boundaries — that is, for any $\omega \in \mathcal{D}'(\mathbb{R}^d, \Lambda^m(\mathbb{R}^d))$, and any $T \in \mathcal{D}'(\mathbb{R}^d, \Lambda_m(\mathbb{R}^d))$, there holds

$$\star(d\omega) = (-1)^{m-1} \partial(\star\omega), \quad (2.45)$$

$$d(\star T) = - \star(\partial T). \quad (2.46)$$

Equations (2.45) and (2.46) are a rather direct consequence of the definitions and properties we have recalled in this section, and we skip their proof.

For vector-valued currents, everything we said in this section applies component-wise. For instance, if $w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ and w can be written component-wise as $w = (w_1, \dots, w_k)$, we define $\star w := (\star w_1, \dots, \star w_k)$. In a similar way if $v = (v_1, \dots, v_k) \in \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$, we set $\star v := (\star v_1, \dots, \star v_k)$. The following property holds: if $\beta \in \Lambda^m(\mathbb{R}^d)$ (respectively, $\beta \in \Lambda_m(\mathbb{R}^d)$), $z \in (\mathbb{R}^k)^*$ (respectively, $z \in \mathbb{R}^k$) and $I: \mathbb{R}^k \rightarrow (\mathbb{R}^k)^*$ is the isomorphism induced by the canonical basis in \mathbb{R}^k , then $\star(z \otimes \beta) = I^{-1}(z) \otimes \star\beta$ (respectively, $\star(z \otimes \beta) = I(z) \otimes \star\beta$).

2.2 Convolution of currents

2.2.1 Convolution of currents with mollifiers

We recall the definition and basic properties of convolution for currents. We will be interested only in very special cases, i.e. convolution with a sequence of mollifiers and with a vector field; for a more general treatment, we refer e.g. to [2].

In some of our technical results, we need a radially symmetric, scale-invariant sequence of mollifiers. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a *radial* function (i.e. $\rho(x) = \bar{\rho}(|x|)$ for any $x \in \mathbb{R}^d$, for some function $\bar{\rho}: \mathbb{R} \rightarrow \mathbb{R}$) such that $0 \leq \rho \leq 1$, $\text{spt}(\rho) \subseteq \bar{\mathbb{B}}_1^d(0)$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. For $\varepsilon > 0$, we define

$$\rho_\varepsilon(x) := \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^d. \quad (2.47)$$

Then, ρ_ε is a symmetric mollifier in \mathbb{R}^d , with $\text{spt}(\rho_\varepsilon) \subset \bar{\mathbb{B}}_\varepsilon^d(0)$ and $\int_{\mathbb{R}^d} \rho_\varepsilon(x) dx = 1$. Let $\omega = \omega_1 e_1^* + \dots + \omega_k e_k^* \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$ be a test form. The convolution of ω and ρ_ε is defined component-wise as

$$(\omega * \rho_\varepsilon)(x) := \int_{\mathbb{R}^d} \omega(x-y) \rho_\varepsilon(y) dy = \sum_{i=1}^k \left(\int_{\mathbb{R}^d} \omega_i(x-y) \rho_\varepsilon(y) dy \right) e_i^*, \quad (2.48)$$

for any $x \in \mathbb{R}^d$. Let $T = (T_1, \dots, T_k)$ be an \mathbb{R}^k -normal current of dimension m in \mathbb{R}^d . We define the convolution between T and ρ_ε by

$$(T * \rho_\varepsilon)(\omega) := T(\omega * \rho_\varepsilon) \quad (2.49)$$

for any $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$. As the convolution for test forms is defined component-wise, we can think of $T * \rho_\varepsilon$ as being defined component-wise, too (for the convolution of classical currents with a mollifier, see for instance [18, Section 7.3, Definition 7.3.2]). Then, $T * \rho_\varepsilon = (T_{1,\varepsilon}, \dots, T_{k,\varepsilon})$, where each component $T_{i,\varepsilon} = T_i * \rho_\varepsilon$ is represented by a smooth m -vector-valued function in \mathbb{R}^d .

Lemma 2.13. *For each \mathbb{R}^k -normal current T , the following properties hold:*

- (i) $T * \rho_\varepsilon \xrightarrow{*} T$, that is, for each $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$, we have $T * \rho_\varepsilon(\omega) \rightarrow T(\omega)$;
- (ii) $\partial(T * \rho_\varepsilon) = \partial T * \rho_\varepsilon$,
- (iii) for any $p \in [1, \infty]$ fixed, we have $\mathbb{M}_p(T * \rho_\varepsilon) \rightarrow \mathbb{M}_p(T)$ as $\varepsilon \rightarrow 0$.

Proof. Property (i) is deduced from properties of classical currents (see e.g. [18, Section 7.3, Lemma 7.3.3]) by reasoning component-wise. Property (ii) comes from the fact that

$$d(\omega * \rho_\varepsilon) = d\omega * \rho_\varepsilon. \quad (2.50)$$

To prove Property (iii), we apply Jensen's inequality. Let $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$. We have

$$\begin{aligned} |\omega * \rho_\varepsilon(x)|_{\text{com},p} &= \left| \int_{\mathbb{R}^d} \omega(x-y) \rho_\varepsilon(y) dy \right|_{\text{com},p} \\ &\leq \int_{\mathbb{R}^d} |\omega(x-y)|_{\text{com},p} \rho_\varepsilon(y) dy \\ &\leq \|\omega\|_{\text{com},p} \int_{\mathbb{R}^d} \rho_\varepsilon(y) dy \\ &= \|\omega\|_{\text{com},p}. \end{aligned} \quad (2.51)$$

Therefore,

$$\|\omega * \rho_\varepsilon\|_{\text{com},p} \leq \|\omega\|_{\text{com},p} \quad (2.52)$$

for any $p \in [1, \infty]$. In turn, (2.52) implies that

$$\mathbb{M}_p(T) \geq \mathbb{M}_p(T * \rho_\varepsilon). \quad (2.53)$$

On the other hand, by the Property (i) we know that $T * \rho_\varepsilon \xrightarrow{*} T$, and by lower semicontinuity of \mathbb{M}_p , we deduce

$$\liminf_{\varepsilon \rightarrow 0} \mathbb{M}_p(T * \rho_\varepsilon) \geq \mathbb{M}_p(T). \quad (2.54)$$

From the inequalities (2.53) and (2.54), we obtain (iii). \square

In a similar way, we can define the convolution between T and a bounded measure μ on \mathbb{R}^d , as

$$(T * \mu)(\omega) := T(\omega * \mu), \quad (2.55)$$

where $(\omega * \mu)(x) := \int_{\mathbb{R}^d} \omega(x - y) d\mu(y)$ for any test form $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d))$ and any $x \in \mathbb{R}^d$.

2.2.2 Convolution of currents with vector-valued maps and differential forms

Let $R = (R_1, \dots, R_d)$ be a function in $L^1(\mathbb{R}^d, \mathbb{R}^d)$ with compact support such that $\operatorname{div} R$ is a bounded measure on \mathbb{R}^d , and let $T = (T_1, \dots, T_k)$ be an m -dimensional normal current, with coefficients in the normed group $(\mathbb{R}^k, \|\cdot\|_p)$. We aim to define the object $T * R$, which will be an \mathbb{R}^k -normal current of dimension $m + 1$. First, we briefly recall the contraction of covectors with vectors. Let β be an h -covector and v be an k -vector with $h > k$. We define the contraction $\beta \lrcorner v$ to be an $(h - k)$ -covector, given by

$$\langle \beta \lrcorner v, w \rangle := \langle \beta, (v \wedge w) \rangle \quad (2.56)$$

for any $w \in \Lambda_{h-k}(\mathbb{R}^d)$. Now, we define the convolution $T * R$ by its action on a test form $\omega = (\omega_1, \dots, \omega_k) \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{m+1}(\mathbb{R}^d))$, as

$$(T * R)(\omega) := \sum_{i=1}^k T_i(\bar{R} * \omega_i), \quad (2.57)$$

where \bar{R} is the vector field given by $\bar{R}(x) := R(-x)$, and the convolution $\bar{R} * \omega_i$ is defined by

$$(\bar{R} * \omega_i)(x) := \int_{\mathbb{R}^d} \omega_i(x - y) \lrcorner \bar{R}(y) dy \quad (2.58)$$

for any $x \in \mathbb{R}^d$. We observe that, for each i , $\bar{R} * \omega_i$ is a smooth m -dimensional form with compact support, since ω_i is a smooth differential form with compact support and R also has compact support. Therefore, the right-hand side of (2.57) is well-defined. From (2.57), one has

$$\begin{aligned} |T * R(\omega)| &\leq \sum_{i=1}^k \mathbb{M}(T_i) \|R * \omega_i\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \sum_{i=1}^k \mathbb{M}_p(T) \|R * \omega_i\|_{L^\infty(\mathbb{R}^d)} \\ &\leq k \sqrt{k} \mathbb{M}_p(T) \|R\|_{L^1(\mathbb{R}^d)} \end{aligned} \quad (2.59)$$

for any $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{m+1}(\mathbb{R}^d))$ such that $|\omega|_{\text{com},p} \leq 1$. Therefore,

$$\mathbb{M}_p(T * R) \leq k\sqrt{k} \mathbb{M}_p(T) \|R\|_{L^1(\mathbb{R}^d)}. \quad (2.60)$$

Moreover, there holds

$$\partial(T * R) = -\text{div } R * T + (-1)^m R * \partial T \quad (2.61)$$

in the sense of distributions. The equality (2.61) follows by the analogous identity for classical currents (see [2, Section 2.8, Equation (2.5)]), applied component-wise.

Lemma 2.14. *Let T be a normal \mathbb{R}^k -current of dimension m . Let $R \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ be a vector field of compact support. Then, the current $T * R$ can be identified with an element of $L^1(\mathbb{R}^d, \Lambda_{m+1, \mathbb{R}^k}(\mathbb{R}^d))$. Moreover, if R is smooth in $\mathbb{R}^d \setminus \{0\}$, then (the vector field corresponding to) $T * R$ is smooth in $\mathbb{R}^d \setminus \text{spt } T$.*

Proof. As we have seen in (2.59), the current $T * R$ has finite mass. By Riesz's representation theorem, it follows that $T * R$ can be identified with a (regular, vector-valued) Borel measure. We claim that $T * R$ is absolutely continuous with respect to the Lebesgue measure; then, the Radon-Nikodym theorem will imply that $T * R$ is carried by an integrable $\Lambda_{m+1, \mathbb{R}^k}(\mathbb{R}^d)$ -field. Given a Borel set $F \subseteq \mathbb{R}^d$, we will denote by $T * R \llcorner F$ the restriction of $T * R$ to F , which is well-defined in the measure-theoretical sense, thanks to Riesz's representation theorem. Let $\varepsilon > 0$ be a small number. As R is integrable, there exists $\delta > 0$ such that

$$\int_E |R(x)| dx \leq \varepsilon \quad \text{for any measurable set } E \subseteq \mathbb{R}^d \text{ such that } |E| \leq \delta. \quad (2.62)$$

Let $F \subseteq \mathbb{R}^d$ be a Borel set such that $|F| = 0$. As the Lebesgue measure is regular, there exists an open set $U \supseteq F$ such that $|U| \leq \delta$. Let $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{m+1}(\mathbb{R}^d))$ be a test form such that $\text{spt } \omega \subseteq U$ and $\|\omega\|_{\text{com},p} \leq 1$. Then, for any index $i \in \{1, \dots, k\}$ and any $x \in \mathbb{R}^d$, we have

$$|\bar{R} * \omega_i(x)|_{\text{com}} \leq \int_{\mathbb{R}^d} |\omega_i(x-y)|_{\text{com}} |R(-y)| dy \leq \int_{-x-U} |R(z)| dz \stackrel{(2.62)}{\leq} \varepsilon$$

(where the notation $|\cdot|_{\text{com}}$ stands for the classical comass norm of a form). By reasoning as in (2.59), we deduce that

$$|T * R(\omega)| \leq k\sqrt{k} \varepsilon \mathbb{M}_p(T). \quad (2.63)$$

Now, for any positive integer j , let

$$F_j := \left\{ x \in F : \text{dist}(x, \partial U) \geq \frac{1}{j} \right\}$$

(where $\text{dist}(x, \partial U)$ denotes the Euclidean distance of x from ∂U) and let $\varphi_j \in C_c^\infty(U)$ be a smooth cut-off function, such that $\varphi_j = 1$ in F_j and $0 \leq \varphi_j \leq 1$ in U . Given any form $\tilde{\omega} \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{m+1}(\mathbb{R}^d))$ such that $\|\tilde{\omega}\|_{\text{com},p} \leq 1$, the inequality (2.63) (applied to $\omega := \varphi_j \tilde{\omega}$) implies

$$|(T * R \llcorner F_j)(\tilde{\omega})| = |(T * R \llcorner F_j)(\varphi_j \tilde{\omega})| \leq k\sqrt{k} \varepsilon \mathbb{M}_p(T).$$

By taking the supremum over $\tilde{\omega}$, we obtain $\mathbb{M}_p(T * R \llcorner F_j) \leq k\sqrt{k} \varepsilon \mathbb{M}_p(T)$ and, sending j to infinity, $\mathbb{M}_p(T * R \llcorner F) \leq k\sqrt{k} \varepsilon \mathbb{M}_p(T)$. As $\varepsilon > 0$ is arbitrary, it follows that $\mathbb{M}_p(T * R \llcorner F) = 0$

and hence, $T * R$ is absolutely continuous with respect to the Lebesgue measure. Therefore, $T * R$ is carried by an integrable vector field, which we still denote by $T * R$.

Suppose now that R is smooth in $\mathbb{R}^d \setminus \{0\}$. We claim that $T * R$ is smooth in $\mathbb{R}^d \setminus \text{spt } T$. Let $\delta > 0$, and let $U_\delta := \{x \in \mathbb{R}^d : \text{dist}(x, \text{spt } T) > \delta\}$. Let R_δ be a vector field that is smooth everywhere in \mathbb{R}^d and coincides with R in $\mathbb{R}^d \setminus \mathbb{B}_\delta^d(0)$. Finally, let $\omega \in C_c^\infty(\mathbb{R}^d, \Lambda_{\mathbb{R}^k}^{m+1}(\mathbb{R}^d))$ be a test form such that $\text{spt } \omega \subseteq U_\delta$. Then, for any index $i \in \{1, \dots, k\}$ and any $x \in \text{spt } T$, we have

$$\bar{R} * \omega_i(x) = \int_{x-U_\delta} \omega_i(x-y) \lrcorner R(-y) \, dy = \int_{x-U_\delta} \omega_i(x-y) \lrcorner R_\delta(-y) \, dy = \bar{R}_\delta * \omega_i(x),$$

because $x - U_\delta \subseteq \mathbb{R}^d \setminus \mathbb{B}_\delta^d(0)$. It follows that $T * R(\omega) = T * R_\delta(\omega)$ for any test form ω with $\text{spt } \omega \subseteq U_\delta$ and hence, $T * R = T * R_\delta$ in U_δ . As the convolution between a distribution and a smooth function (of compact support) is a smooth function, we deduce that $T * R$ is smooth in U_δ for any $\delta > 0$. The lemma follows. \square

2.3 The distributional Jacobian

We introduce the notion of distributional Jacobian for torus-valued maps. Let us recall first this notion in case of \mathbb{S}^1 -valued maps. Let $u = (u^1, u^2) \in W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{S}^1)$, we define the pre-jacobian 1-form as

$$j(u) := u^1 \, du^2 - u^2 \, du^1 \quad (2.64)$$

and the distributional Jacobian as the 2-form

$$J(u) := \frac{1}{2} dj(u), \quad (2.65)$$

where the differential is taken in the sense of distributions on \mathbb{R}^d . Moreover, we associate each $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{S}^1)$ a $(d-1)$ -dimensional (respectively, $(d-2)$ -dimensional) classical current by $\star j(u)$ (respectively, $\star J(u)$), where \star is the Hodge-star operator, as defined in (2.35).

Now, for each torus-valued map $\mathbf{u} = (u_1, \dots, u_k) \in W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{T}^k)$ i.e. each $u_i \in W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{S}^1)$, we define its pre-jacobian $\mathbf{j}\mathbf{u}$ and Jacobian $\mathbf{J}\mathbf{u}$ component-wise, i.e.

$$\mathbf{j}\mathbf{u} := (j(u_1), \dots, j(u_k)), \quad (2.66)$$

and

$$\mathbf{J}\mathbf{u} := (J(u_1), \dots, J(u_k)). \quad (2.67)$$

Moreover, we can associate to each $\mathbf{u} = (u_1, \dots, u_k) \in W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{T}^k)$ a $(d-1)$ -dimensional- \mathbb{R}^k current, given by

$$\star \mathbf{j}\mathbf{u} = (\star j(u_1), \dots, \star j(u_k)), \quad (2.68)$$

and a $(d-2)$ -dimensional- \mathbb{R}^k current, given by

$$\star \mathbf{J}\mathbf{u} = (\star J(u_1), \dots, \star J(u_k)). \quad (2.69)$$

Remark 2.15. Let S be a \mathbb{Z}^k -integral flat boundary. Let \mathcal{Q}_S be the set of maps $\mathbf{u} \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{C}^k)$ that are constant in a neighbourhood of infinity and satisfy $\star \mathbf{J}\mathbf{u} = \pi S$. For any $\mathbf{u} \in \mathcal{Q}_S$, we have

$$\partial(\star \mathbf{j}\mathbf{u}) = 2\pi S \quad (2.70)$$

thanks to (2.45).

3 Energy-minimizing maps with prescribed singularities and mass minimization for normal currents

In this section, we focus on the proof of Theorem A. Throughout this section, we always assume that S is a $(d - 2)$ -dimensional integral flat boundary S with compact support and finite mass that satisfies the condition (10). For the reader's convenience, we recall the functionals we consider. Let \mathcal{Q}_S be defined as in Remark 2.15. For $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S$ and any $p \in [1, \infty)$, we define

$$H_p(\mathbf{u}) := \int_{\mathbb{R}^d} (|\nabla u_1|^p + |\nabla u_2|^p + \dots + |\nabla u_k|^p)^{\frac{1}{p}} dx, \quad (3.1)$$

while for $p = \infty$

$$H_\infty(\mathbf{u}) := \int_{\mathbb{R}^d} \max(|\nabla u_1|, |\nabla u_2|, \dots, |\nabla u_k|) dx. \quad (3.2)$$

We set

$$\mathbb{H}_p(S) := \inf \{H_p(\mathbf{u}) : \mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S\}. \quad (3.3)$$

The second functional we consider is defined in terms of the so-called nuclear norm of the gradient. For $p \in [1, \infty]$, the p -nuclear norm of a linear map $A: \mathbb{R}^d \rightarrow \mathbb{C}^k$ is defined as follows:

$$|A|_{\text{nucl},p} := \inf \left\{ \sum_{i=1}^l \|z_i\|_p |v_i| : A = \sum_{i=1}^l z_i \otimes v_i, z_i \in \mathbb{C}^k, v_i \text{ is a 1-covector in } \mathbb{R}^d \right\}. \quad (3.4)$$

Remark 3.1. It may occasionally be convenient to consider A as a linear map $\mathbb{R}^d \rightarrow \mathbb{R}^{2k}$, by identifying \mathbb{C}^k with \mathbb{R}^{2k} , in the usual way. However, in the right-hand side of (3.4), $\|\cdot\|_p$ denotes the ℓ^p -norm in \mathbb{C}^k , *not* \mathbb{R}^{2k} . In other words, given a vector $\zeta = (\zeta_1, \dots, \zeta_k) \in \mathbb{C}^k$ and $p \in [1, \infty)$, we define $\|\zeta\|_p := \left(\sum_{j=1}^k |\zeta_j|^p \right)^{1/p}$, where $|\zeta_j|$ is the Euclidean norm of the j -th component $\zeta_j \in \mathbb{C} \simeq \mathbb{R}^2$. Similarly, we define $\|\zeta\|_\infty := \max_{1 \leq j \leq k} |\zeta_j|$.

The terminology ‘nuclear norm’ refers to the notion of nuclear linear operators between topological vector spaces, which has been introduced by Grothendieck [20] in much greater generality. Here, of course, the setting is much simpler because we only need to consider linear operators between the finite-dimensional Banach spaces $(\mathbb{R}^d, |\cdot|)$ and $(\mathbb{C}^k, \|\cdot\|_p)$. For a detailed discussion on the nuclear norm of operators between Banach spaces, see e.g. [23].

Remark 3.2. The choice of the nuclear norm can be explained by the fact that the nuclear norm $|\cdot|_{\text{nucl},p}$ is dual to the standard operator norm on the space of linear maps $(\mathbb{C}^k, \|\cdot\|_p) \rightarrow (\mathbb{R}^d, |\cdot|)$ (see e.g. [23, Proposition 1.11]). Heuristically, just as the operator norm ‘is analogous’ to the comass norm on $\Lambda_{\mathbb{R}^k}^1(\mathbb{R}^d)$, so does the nuclear norm ‘correspond’ to the mass norm on $\Lambda_{1,\mathbb{R}^k}(\mathbb{R}^d)$. (For a precise statement, see Lemma 3.7 below.) In case $p = 2$, the nuclear norm reduces to the so-called trace norm or 1-Schatten norm: for any linear map $A: \mathbb{R}^d \rightarrow \mathbb{C}^k$ we have

$$|A|_{\text{nucl},2} = \text{trace} \left((A^* A)^{1/2} \right) = \sum_{i=1}^s |\lambda_i|$$

where A^* is the adjoint of A , $s := \min(d, 2k)$ and $\lambda_1, \dots, \lambda_s$ are the singular values of A .

We set

$$E_p(\mathbf{u}) := \int_{\mathbb{R}^d} |\nabla \mathbf{u}(x)|_{\text{nucl},p} dx, \quad (3.5)$$

where $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S$, and

$$\mathbb{E}_p(S) := \inf \{E_p(\mathbf{u}) : \mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S\}. \quad (3.6)$$

Remark 3.3. When $k = 1$, the problems (3.3) and (3.6) reduce to the one studied in [12, 13]. When $d = 2$, our energy E_p belongs to the class of energies investigated in [5], where the authors make a connection with branched optimal transport theory. Therefore, in the planar case we provide information on the relationship between energy minimizing torus-valued maps with prescribed topological points singularities and the branched optimal transport problem or Gilbert-Steiner problem between those points. This is in the spirit of the work by Brezis, Coron and Lieb [12], which provides the connection between the energy of harmonic maps and the optimal transport Monge-Kantorovich problem.

We start with some auxiliary results.

Lemma 3.4. *Let R be a classical integer-multiplicity flat boundary in \mathbb{R}^d , of dimension $(d - 2)$, with compact support. Let T be a classical current of dimension $(d - 1)$, of finite mass, such that $\partial T = R$. Assume that T is represented by an integrable vector field (still denoted T) and that $T \in C^\infty(\mathbb{R}^d \setminus \text{spt } R, \Lambda_{d-1}(\mathbb{R}^d))$. Let $\gamma: \mathbb{S}^1 \rightarrow \mathbb{R}^d$ be a Lipschitz loop, such that $\gamma(\mathbb{S}^1) \subseteq \mathbb{R}^d \setminus \text{spt } R$. Then, the integral*

$$\int_{\gamma(\mathbb{S}^1)} \star T$$

is an integer number.

Proof. The Hodge dual $\star T$ is a differential 1-form and is smooth away from the support of R , so the integral of $\star T$ on $\gamma(\mathbb{S}^1)$ is well-defined. In order to prove that the integral is an integer number, we can assume without loss of generality that R is a polyhedral current. Indeed, let $\varepsilon > 0$ be a small number. Even if R is not polyhedral, by Federer's polyhedral approximation theorem [16, § 4.2.20 and 4.2.22] there exist an integer-multiplicity polyhedral current P_1 and finite-mass, integer-multiplicity currents Q_1, Q'_1 , all of them supported in an ε -neighbourhood of $\text{spt } R$, such that $R - P_1 = Q_1 + \partial Q'_1$. As $\partial R = 0$, we have $\partial Q_1 = -\partial P_1$ and in particular, the boundary of Q_1 is polyhedral. Therefore, by Federer and Fleming's deformation theorem [16, § 4.2.9], we can write $Q_1 = P_2 + \partial Q_2$, where P_2, Q_2 are integer-multiplicity currents of finite mass supported in an ε -neighbourhood of $\text{spt } Q_1$ and P_2 is polyhedral. Let $\tilde{T} := T - Q'_1 - Q_2$. Then, $\partial \tilde{T} = R - \partial(Q'_1 + Q_2) = P_1 + P_2$ is polyhedral and, since we have assumed that $\gamma(\mathbb{S}^1)$ is a compact subset of $\mathbb{R}^d \setminus \text{spt } R$, we can take ε small enough so that $\tilde{T} = T$ in a neighbourhood of $\gamma(\mathbb{S}^1)$. Therefore, up to replacing T with \tilde{T} , there is no loss of generality in assuming that $\partial T = R$ is polyhedral. Moreover, it is not restrictive to assume that $\gamma(\mathbb{S}^1)$ is the boundary of a 2-dimensional polyhedron Σ . Indeed, the form $\star T$ is closed in the complement of $\text{spt } R$ (due to Equation (2.46)) and $\gamma(\mathbb{S}^1)$ can be approximated by polyhedral boundaries, each of which is a finite sum of boundaries of individual polyhedra. Finally, up to a small perturbation, we can take Σ transverse to each simplex in R . Then, for each $(d - 2)$ -dimensional simplex K in R , the intersection $K \cap \Sigma$ contains at most one point.

Let us assign an orientation to each $(d - 2)$ -simplex K in R , by considering a unit $(d - 2)$ -vector τ_K that spans the plan of K . Let τ_Σ be a unit 2-vector that spans the plan of Σ . We define the intersection number $I(K, \Sigma)$ as follows: $I(K, \Sigma) := 0$ if $K \cap \Sigma$ is empty; $I(K, \Sigma) := 1$ if $K \cap \Sigma$ is non-empty and $\star(\tau_K \wedge \tau_\Sigma) > 0$; and $I(K, \Sigma) := -1$ otherwise. Let $\theta(K) \in \mathbb{Z}$ be the

multiplicity of R along the (oriented) simplex K . We claim that

$$\int_{\partial\Sigma} \star T = \sum_K \theta(K) I(K, \Sigma), \quad (3.7)$$

where the sum is taken over all $(d-2)$ -simplices K of R . Indeed, let $\varepsilon > 0$ be a small parameter and let ρ_ε be a mollifier, as in (2.47). Stokes' theorem implies

$$\int_{\partial\Sigma} \star(T * \rho_\varepsilon) = \int_\Sigma d(\star(T * \rho_\varepsilon)) \stackrel{(2.46)}{=} - \int_\Sigma \star\partial(T * \rho_\varepsilon)$$

and hence, by Lemma 2.13,

$$\int_{\partial\Sigma} \star(T * \rho_\varepsilon) = - \int_\Sigma \star(R * \rho_\varepsilon). \quad (3.8)$$

The right-hand side of (3.8) can be further evaluated by writing the convolution $R * \rho_\varepsilon$ as an integral (which is possible, because R is polyhedral). Then, (3.7) follows by taking the limit as $\varepsilon \rightarrow 0$ in both sides of (3.8), and (3.7) implies the conclusion of the lemma. \square

Lemma 3.5. *Let $T = (T_1, \dots, T_k)$ be an \mathbb{R}^k -normal current of dimension $(d-1)$ and having compact support, such that $\partial T = S$. Let $p \in [1, \infty]$ be fixed. Then, the following statements hold.*

(i) *For any $\eta > 0$, there exists an \mathbb{R}^k -normal current of dimension $(d-1)$, $\bar{T} = (\bar{T}_1, \dots, \bar{T}_k)$, such that $\bar{T}_i \in C^\infty(\mathbb{R}^d \setminus \text{spt } S_i; \Lambda_{d-1}(\mathbb{R}^d))$ for any index i , $\partial \bar{T} = S$ and $\mathbb{M}_p(\bar{T}) \leq \mathbb{M}_p(T) + \eta$.*

(ii) *Moreover, there exists $\mathbf{u} \in \mathcal{Q}_S$ such that*

$$\frac{1}{2\pi} \star \mathbf{j}(\mathbf{u}) = \bar{T}. \quad (3.9)$$

Proof.

Proof of (i). We prove this fact by using the properties of the convolution, introduced in Section 2.2.1, 2.2.2. Let ρ_ε be a symmetric mollifier kernel, exactly as in Section 2.2.1. We consider the convolution $T * \rho_\varepsilon$. Each component of $T * \rho_\varepsilon$ is a classical current carried by a smooth $(d-1)$ -vector-valued function with compact support in \mathbb{R}^d . Moreover, for ε small enough, the mass of $T * \rho_\varepsilon$ is arbitrarily close to the mass of T (see Property (iii) in Section 2.2.1). However, we cannot choose $T_\varepsilon = T * \rho_\varepsilon$, because the boundary of $T * \rho_\varepsilon$ is not necessarily equal to S — in fact, Property (ii) in Section 2.2.1 gives

$$\partial(T * \rho_\varepsilon) = S * \rho_\varepsilon. \quad (3.10)$$

Therefore, we need to modify $T * \rho_\varepsilon$ in a suitable way, so as to obtain a new current whose boundary is equal to S .

By Lemma A.1 in the appendix, there exists a vector-valued map $R_\varepsilon \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ that is smooth in $\mathbb{R}^d \setminus \{0\}$, satisfies

$$\text{div } R_\varepsilon = \delta_0 - \rho_\varepsilon \quad (3.11)$$

and $\|R_\varepsilon\|_{L^1(\mathbb{R}^d)} \rightarrow 0$, $\text{spt}(R_\varepsilon) \subset \bar{\mathbb{B}}_\varepsilon(0)$. We consider the convolution $S * R_\varepsilon$, defined in Section 2.2.2. This is a $(d-1)$ -dimensional normal \mathbb{R}^k -current, and the i -th component of $S * R_\varepsilon$ is carried by a vector field that is smooth in the complement of $\text{spt} S_i$, by Lemma 2.14. Moreover, (2.61) implies

$$\partial(S * R_\varepsilon) = (-1)^{d-2} R_\varepsilon * \partial S - S * \text{div} R_\varepsilon = \rho_\varepsilon * S - S. \quad (3.12)$$

Let $T_\varepsilon := T * \rho_\varepsilon - S * R_\varepsilon$. By (3.10) and (3.12), we have $\partial T_\varepsilon = S$. Moreover, each component T_ε^i of T_ε is carried by a vector-field in $C_c^\infty(\mathbb{R}^d \setminus \text{spt} S_i; \Lambda^{d-1}(\mathbb{R}^d))$ and, thanks to (2.53), (2.60), we have

$$\begin{aligned} \mathbb{M}_p(T_\varepsilon) &\leq \mathbb{M}_p(T * \rho_\varepsilon) + \mathbb{M}_p(S * R_\varepsilon) \\ &\leq \mathbb{M}_p(T) + \|R_\varepsilon\|_{L^1(\mathbb{R}^d)} \mathbb{M}_p(S) \\ &\leq \mathbb{M}_p(T) + o(1) \end{aligned} \quad (3.13)$$

as $\varepsilon \rightarrow 0$. Therefore, taking ε small enough (depending on η), the current $\bar{T} := T_\varepsilon$ has all the required properties.

Proof of (ii). We recall that G_i is the subset of $W_{\text{loc}}^{1,1}(\mathbb{R}^d; \mathbb{S}^1)$ consisting of the maps u_i that are constant outside an open ball of radius $r_i = r_i(u_i)$ and are such that $\star J(u_i) = \pi S_i$. We shall prove that, for each $i = 1, \dots, k$, there exists $u_i \in G_i$ such that

$$\frac{1}{2\pi} \star j(u_i) = T_{\varepsilon,i}. \quad (3.14)$$

Then, choosing $\mathbf{u} = (u_1, \dots, u_k)$, the lemma will follow. Let fix a point x_0 in $\mathbb{R}^d \setminus \text{spt} S_i$. The set $\mathbb{R}^d \setminus \text{spt} S_i$ is open and, by assumption, connected. Therefore, for any $x \in \mathbb{R}^d \setminus \text{spt} S_i$ there exists a path $\gamma_x^i: [0, 1]$, whose support is contained in $\mathbb{R}^d \setminus \text{spt} S_i$, that connects x and x_0 . Define

$$\theta_i(x) := (-1)^{d-1} \int_{\gamma_x^i} \star T_{\varepsilon,i}, \quad (3.15)$$

then let $u_i(x) := e^{2\pi i \theta_i(x)}$. The quantity $u_i(x)$ is well-defined and independent of the choice of γ_x^i . Indeed, if γ_1, γ_2 are two paths connecting x and x_0 , Lemma 3.4 implies

$$\int_{\gamma_1} \star T_{\varepsilon,i} - \int_{\gamma_2} \star T_{\varepsilon,i} \in \mathbb{Z}. \quad (3.16)$$

We now prove that the map $u_i: \mathbb{R}^d \setminus \text{spt} S_i \rightarrow \mathbb{S}^1$ is smooth in $\mathbb{R}^d \setminus \text{spt} S_i$. Take an arbitrary small open ball centered at y_0 with radius r , $\mathbb{B}_r^d(y_0) \subset \mathbb{R}^d \setminus \text{spt} S_i$. Thanks to (2.45), one has

$$(-1)^{d-1} d(\star T_{\varepsilon,i}) = \partial T_{\varepsilon,i} = S_i = 0 \quad \text{in } \mathbb{B}_r^d(y_0).$$

Therefore, there exists a smooth function $\varphi_i: \mathbb{B}_r^d(y_0) \rightarrow \mathbb{R}$ such that $d\varphi_i = (-1)^{d-1} \star T_{\varepsilon,i}$. Up to an additive constant, we can assume that $\varphi_i(y_0) = 0$. For any $x \in \mathbb{B}_r^d(y_0)$, let σ_x be the path connecting y_0 to x along the radius of the ball. By definition of u_i , one has

$$u_i(x) = u_i(y_0) e^{2\pi i \varphi_i(x)}, \quad (3.17)$$

where

$$\bar{\varphi}_i(x) := (-1)^{d-1} \int_{\sigma_x^i} \star T_{\varepsilon,i} = \int_{\sigma_x^i} d\varphi_i = \varphi_i(x). \quad (3.18)$$

Therefore, $u_i(x) = u_i(y_0) e^{2\pi i \varphi_i(x)}$. This proves that u_i is smooth in $\mathbb{B}_r^d(y_0)$, because φ_i is. Moreover, by explicit computation, we obtain

$$j(u_i) = 2\pi d\varphi_i = 2\pi_i(-1)^{d-1} \star T_{\varepsilon,i} \quad (3.19)$$

and (3.14) follows. Due to the special structure of the target manifold \mathbb{S}^1 , we have, for each $x \in \mathbb{R}^d$,

$$|\star j(u_i)(x)| = |j(u_i)(x)| = |\nabla u_i(x)| \quad (3.20)$$

(we recall that the notation $|\cdot|$, without any subscript, indicates the Euclidean norm; see Remark 2.5). From (3.14) and (3.20), combined with the fact that $T_{\varepsilon,i}$ has finite mass, it follows that $u_i \in W^{1,1}(\mathbb{R}^d \setminus \text{spt}(S_i), \mathbb{S}^1)$. On the other hand, a general property of Sobolev spaces is that $W^{1,1}(\mathbb{R}^d \setminus F) = W^{1,1}(\mathbb{R}^d)$ for any closed set $F \subseteq \mathbb{R}^d$ such that $\mathcal{H}^{d-1}(F) = 0$ (see e.g. [28, Theorem 1.2.5 p. 16]). Therefore, taking Assumption (10) into account, we deduce that $u_i \in W^{1,1}(\mathbb{R}^d, \mathbb{S}^1)$. Moreover, (3.14) and (2.45) imply

$$\star J(u_i) = \frac{1}{2} \star (dj(u_i)) = \frac{1}{2} \partial(\star j(u_i)) = \pi \partial T_{\varepsilon,i} = \pi S_i. \quad (3.21)$$

Finally, since $T_{\varepsilon,i}$ has compact support, it follows by construction that u_i is constant outside some open ball that contains $T_{\varepsilon,i}$ (see Equation 3.15). This shows that $u_i \in G_i$ and completes the proof. \square

Remark 3.6. We deduce from Lemma 3.5 that, if S has finite mass and the assumption (10) is satisfied, then the space \mathcal{Q}_S is non-empty.

Before we state our next lemma, we introduce a notation: for $\omega \in \Lambda_{\mathbb{R}^k}^1(\mathbb{R}^d)$, we define

$$\bar{N}(\omega) := \inf \left\{ \sum_{i=1}^l \|z_i\|_q^* |\omega_i| : \omega = \sum_{i=1}^l z_i \otimes \omega_i, z_i \in (\mathbb{R}^k)^*, \omega_i \text{ is a 1-covector in } \mathbb{R}^d \right\}, \quad (3.22)$$

where $q \in [1, \infty]$ is the conjugate exponent of p , i.e. $1/p + 1/q = 1$. (We recall that $\|\cdot\|_q^*$ denotes the dual of the q -norm on \mathbb{R}^k ; therefore, $\|\cdot\|_q^*$ is the p -norm on $(\mathbb{R}^k)^*$.) The right-hand side of (3.22) is formally analogous to the characterization of the mass norm of a vector, given by Equation (2.23); however, the function \bar{N} is defined on forms, not vectors. Finally, we recall that the mass norm $|\cdot|_{\text{mass},p}$ for vectors is defined by (2.22).

Lemma 3.7. *Let $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S$. Then, for a.e. $x \in \mathbb{R}^d$, there holds*

$$|\nabla \mathbf{u}(x)|_{\text{nucl},p} = \bar{N}(\mathbf{ju}(x)) = |\star \mathbf{ju}(x)|_{\text{mass},p}. \quad (3.23)$$

Proof. Let $x \in \mathbb{R}^d$ be a differentiability point for u . As $\mathbf{ju}(x)$ is an $(\mathbb{R}^k)^*$ -valued 1-covector, we have $\bar{N}(\mathbf{ju}(x)) = |\star \mathbf{ju}(x)|_{\text{mass},p}$ (for more details, see Lemma A.2 and Remark A.3 in the appendix). We shall prove that $|\nabla \mathbf{u}(x)|_{\text{nucl},p} = \bar{N}(\mathbf{ju}(x))$.

Step 1. We first prove that

$$|\nabla \mathbf{u}(x)|_{\text{nucl},p} \geq \bar{N}(\mathbf{ju}(x)). \quad (3.24)$$

Consider a decomposition of $\nabla \mathbf{u}(x)$ of the form

$$\nabla \mathbf{u}(x) = \sum_{h=1}^{\ell} z_h \otimes v_h, \quad (3.25)$$

where $z_h = (z_h^1, \dots, z_h^k) \in \mathbb{C}^k \simeq \mathbb{R}^{2k}$ and v_h is a 1-covector in \mathbb{R}^d for any $h = 1, \dots, \ell$. Let $m \in \{1, \dots, k\}$ be fixed. We identify $\nabla \mathbf{u}(x)$ with a real $(2k) \times d$ -matrix and multiply on the left both sides of the equality (3.25) by the (column) vector $(0, \dots, iu_m(x), \dots, 0)^\top \in \mathbb{C}^k \simeq \mathbb{R}^{2k}$, in accordance with the rule of multiplication of matrices. Recalling that Equation (2.64) can be written in the form $j(u_m) = \langle iu_m, du_m \rangle$ (where $\langle \cdot, \cdot \rangle$ stands for the inner product between complex numbers), we obtain that

$$j(u_m)(x) = \sum_{h=1}^{\ell} \langle iu_m(x), z_h^m \rangle v_h \quad (3.26)$$

for each $m = 1, \dots, k$. Therefore, it follows that

$$\mathbf{ju}(x) = \sum_{h=1}^{\ell} z_{\mathbf{u},h} \otimes v_h \quad \text{where } z_{\mathbf{u},h} := \langle iu_1(x), z_j^1 \rangle e_1^* + \dots + \langle iu_k(x), z_j^k \rangle e_k^* \in (\mathbb{R}^k)^*. \quad (3.27)$$

We observe that for any $h = 1, \dots, \ell$ and for any $m = 1, \dots, k$, by applying the Cauchy-Schwarz inequality we have

$$|\langle iu_m(x), z_h^m \rangle| \leq |iu_m(x)| |z_h^m| = |z_h^m|,$$

since $|iu_m(x)| = 1$. Then,

$$\sum_{h=1}^{\ell} \|z_h\|_p |v_h| \geq \sum_{h=1}^{\ell} \|z_{\mathbf{u},h}\|_q^* |v_h|. \quad (3.28)$$

From the equality (3.28), combined with the definition of \bar{N} (Equation (3.22)) and the definition of the nuclear norm of a matrix (3.4) (see also remark 3.1), we conclude that (3.24) holds.

Step 2. Now, we prove that

$$|\nabla \mathbf{u}(x)|_{\text{nucl},p} \leq \bar{N}(\mathbf{ju}(x)). \quad (3.29)$$

We first observe that, for any $v \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{S}^1)$, one has

$$\nabla v(x) = iv(x) \otimes j(v)(x) \quad (3.30)$$

for a.e. $x \in \mathbb{R}^d$. Indeed, by differentiating the constraint $|v|^2 = 1$ a.e., for any test vector $e \in \mathbb{R}^d$ one obtains $\langle \partial_e v(x), v(x) \rangle = 0$ for a.e. $x \in \mathbb{R}^d$ and hence, $\partial_e v(x)$ must be parallel to $iv(x)$ a.e. Then, taking (2.64) into account, (3.30) follows. Now suppose that

$$\mathbf{ju}(x) = \sum_{h=1}^{\ell} z_h \otimes v_h, \quad (3.31)$$

where $z_h = (z_h^1, \dots, z_h^k) \in (\mathbb{R}^k)^*$ and v_h is a 1-covector in \mathbb{R}^d , for $h = 1, \dots, \ell$. By applying (3.30), we deduce

$$\nabla \mathbf{u}(x) = \sum_{m=1}^k \nabla u_m(x) e_m^* = \sum_{m=1}^k iu_m(x) e_m^* \otimes j(u_m)(x)$$

and hence, thanks to (3.31),

$$\nabla \mathbf{u}(x) = \sum_{m=1}^k \sum_{h=1}^{\ell} iz_h^m u_m(x) e_m^* \otimes v_h = \sum_{h=1}^{\ell} \zeta_h \otimes v_h, \quad (3.32)$$

where $\zeta_h := \sum_{m=1}^k iz_h^m u_m(x) e_m^* \in \mathbb{C}^k$ (and, we recall, $z_h^m \in \mathbb{R}$ is the m -th component of z_h). Since $|u_m(x)| = 1$, the Cauchy-Schwarz inequality implies that $\|\zeta_h\|_p \leq \|z_h\|_q^*$ for any $h \in \{1, \dots, \ell\}$. Therefore, from (3.4), (3.22) and (3.32), we obtain (3.29). \square

Proof of Theorem A. We shall prove the equality (13) holds true for $1 \leq p < \infty$ first, and deal with the case $p = \infty$ later.

Step 1. Let $p \in [1, \infty)$ be fixed. In this step we shall prove that

$$\mathbb{E}_p(S) = 2\pi \mathbb{P}_{\mathbb{R}^k, p}(S). \quad (3.33)$$

Let $\mathbf{u} \in \mathcal{Q}_S$. From Lemma 3.7, one has

$$\int_{\mathbb{R}^d} |\nabla \mathbf{u}|_{\text{nucl}, p} \, dx = \int_{\mathbb{R}^d} |\star \mathbf{j}\mathbf{u}|_{\text{mass}, p} \, dx. \quad (3.34)$$

Moreover, $T := \frac{1}{2\pi} \star \mathbf{j}\mathbf{u}$ is an \mathbb{R}^k -normal current of dimension $(d-1)$ such that $\partial T = S$ (see Remark 2.15) and

$$\mathbb{M}_p(T) = \frac{1}{2\pi} \int_{\mathbb{R}^d} |\star \mathbf{j}\mathbf{u}|_{\text{mass}, p} \, dx \quad (3.35)$$

(see Remark 2.10). Therefore, in view of (3.34) and (3.35), we deduce that

$$\int_{\mathbb{R}^d} |\nabla \mathbf{u}|_{\text{nucl}, p} \, dx = \int_{\mathbb{R}^d} |\star \mathbf{j}\mathbf{u}|_{\text{mass}, p} \, dx \geq 2\pi \mathbb{P}_{\mathbb{R}^k, p}(S). \quad (3.36)$$

By taking the infimum over all $\mathbf{u} \in \mathcal{Q}_S$ in the inequality (3.36), we obtain that

$$\mathbb{E}_p(S) \geq 2\pi \mathbb{P}_{\mathbb{R}^k, p}(S). \quad (3.37)$$

Now, let T be an \mathbb{R}^k -normal current such that $\partial T = S$ and $\mathbb{M}_p(T) \leq \infty$. We are going to prove that for each $\eta > 0$, there exists $\mathbf{u} \in \mathcal{Q}_S$ such that

$$2\pi \mathbb{M}_p(T) + \eta \geq E_p(\mathbf{u}). \quad (3.38)$$

This will imply the opposite inequality

$$2\pi \mathbb{P}_{\mathbb{R}^k, p}(S) \geq \mathbb{E}_p(S), \quad (3.39)$$

and (3.33) will follow. Without loss of generality, we can assume that T has compact support. Indeed, since we have assumed that S has compact support, we can consider a closed ball $\bar{\mathbb{B}}_r^d(0)$ with center at the origin and radius $r > 0$ such that $\text{spt } S \subset \bar{\mathbb{B}}_r^d(0)$. Let $P_r: \mathbb{R}^d \rightarrow \bar{\mathbb{B}}_r^d(0)$ be the nearest-point projection onto $\bar{\mathbb{B}}_r^d(0)$. P_r is a Lipschitz map with Lipschitz constant 1. Then, the push-forward $P_{r\#}(T) = (P_{r\#}(T_1), \dots, P_{r\#}(T_k))$ is an \mathbb{R}^k -normal current of dimension $d-1$, with support contained in $\bar{\mathbb{B}}_r^d(0)$ and $\partial P_{r\#}(T) = P_{r\#}(\partial T) = S$. Moreover,

$$\mathbb{M}_p(P_{r\#}(T) - T) \leq \sum_{i=1}^k \mathbb{M}_p(P_{r\#}(T_i) - T_i) \leq \sum_{i=1}^k \mathbb{M}_p(T_i \llcorner (\mathbb{R}^d \setminus \mathbb{B}_r(0))) \quad (3.40)$$

(where \llcorner denotes the restriction). The right hand side of (3.40) tends to zero, and we obtain that $P_{r\#}(T)$ converges to T with respect to the mass norm, as $r \rightarrow \infty$. Therefore, there is no loss of generality in assuming T is a current of compact support; but then, (3.38) follows from Lemma 3.5. This completes the proof of (3.33).

Step 2. Again, let $p \in [1, \infty)$. In this step we shall prove that

$$\mathbb{H}_p(S) \leq \mathbb{E}_p(S) \leq \mathbb{H}_1(S) \leq k^{1-\frac{1}{p}} \mathbb{H}_p(S). \quad (3.41)$$

In fact, for any $\mathbf{u} \in \mathcal{Q}_S$ there holds

$$H_p(\mathbf{u}) \leq E_p(\mathbf{u}) \leq H_1(\mathbf{u}) \leq k^{1-\frac{1}{p}} H_p(\mathbf{u}). \quad (3.42)$$

Indeed, by using (2.24) (or see Lemma A.6), Remark 2.10 and Lemma 3.7 imply

$$\int_{\mathbb{R}^d} \left(\sum_{i=1}^k |\star j(u_i)|^p \right)^{\frac{1}{p}} dx \leq E_p(\mathbf{u}) = \int_{\mathbb{R}^d} |\star \mathbf{j}\mathbf{u}|_{\text{mass},p} dx \leq \int_{\mathbb{R}^d} \sum_{i=1}^k |\star j(u_i)|_{\text{mass}} dx \quad (3.43)$$

where $|\star j(u_i)|$ is the Euclidean norm of the classical $(d-1)$ -vector $j(u_i)$. However, all $(d-1)$ -vectors are simple, therefore $|\star j(u_i)| = |j(u_i)|$ a.e. in \mathbb{R}^d and a standard computation shows that $|j(u_i)| = |\nabla u_i|$ a.e. This proves

$$H_p(\mathbf{u}) \leq E_p(\mathbf{u}) \leq H_1(\mathbf{u}). \quad (3.44)$$

Furthermore, the inequality

$$H_1(\mathbf{u}) \leq k^{1-\frac{1}{p}} H_p(\mathbf{u}) \quad (3.45)$$

follows from Holder's inequality, as for any $z \in \mathbb{R}^k$, one has $\|z\|_1 \leq k^{1-\frac{1}{p}} \|z\|_p$.

Step 3. To complete the proof, we shall check that the equality (13) holds true in case $p = \infty$. To this aim, we will prove that $\mathbb{P}_{\mathbb{R}^k,p}(S)$ is a decreasing function of p and that

$$\lim_{p \rightarrow \infty} \mathbb{P}_{\mathbb{R}^k,p}(S) = \mathbb{P}_{\mathbb{R}^k,\infty}(S). \quad (3.46)$$

Similar properties hold for $\mathbb{H}_p(S)$, $\mathbb{E}_p(S)$. We begin to prove (3.46). Let T be a normal \mathbb{R}^k -current of dimension $d-1$. According to Lemma A.5 in the appendix, for any $1 \leq p_1 \leq p_2 \leq \infty$ there holds

$$\mathbb{M}_{p_2}(T) \leq \mathbb{M}_{p_1}(T) \leq k^{\frac{p_2-p_1}{p_1 p_2}} \mathbb{M}_{p_2}(T). \quad (3.47)$$

The inequality (3.47) implies that

$$\mathbb{P}_{\mathbb{R}^k,p_2}(S) \leq \mathbb{P}_{\mathbb{R}^k,p_1}(S) \leq k^{\frac{p_2-p_1}{p_1 p_2}} \mathbb{P}_{\mathbb{R}^k,p_2}(S). \quad (3.48)$$

Therefore, $\mathbb{P}_{\mathbb{R}^k,p}(S)$ is decreasing in p and bounded from below, so its limit as $p \rightarrow \infty$ exists and is unique. In particular, when $p_2 = \infty$, $p_1 = p \geq 1$, the inequality (3.48) becomes

$$\mathbb{P}_{\mathbb{R}^k,\infty}(S) \leq \mathbb{P}_{\mathbb{R}^k,p}(S) \leq k^{\frac{1}{p}} \mathbb{P}_{\mathbb{R}^k,\infty}(S). \quad (3.49)$$

By taking the limit as $p \rightarrow \infty$ in (3.49), we obtain that

$$\lim_{p \rightarrow \infty} \mathbb{P}_{\mathbb{R}^k,p}(S) = \mathbb{P}_{\mathbb{R}^k,\infty}(S). \quad (3.50)$$

By making use of the inequalities $\|z\|_{p_2} \leq \|z\|_{p_1} \leq k^{(p_2-p_1)/(p_1 p_2)} \|z\|_{p_1}$ for any $z \in \mathbb{R}^k$ and $1 \leq p_1 \leq p_2 < \infty$, in a similar way, we also can prove that $\mathbb{H}_p(S)$, $\mathbb{E}_p(S)$ are decreasing functions of p and that

$$\begin{aligned} \lim_{p \rightarrow \infty} \mathbb{H}_p(S) &= \mathbb{H}_\infty(S), \\ \lim_{p \rightarrow \infty} \mathbb{E}_p(S) &= \mathbb{E}_\infty(S). \end{aligned}$$

Finally, by taking the limit as $p \rightarrow \infty$ in both sides of (3.33) and (3.41), we obtain that

$$\mathbb{H}_\infty(S) \leq \mathbb{E}_\infty(S) = 2\pi \mathbb{P}_{\mathbb{R}^k,\infty}(S) \leq \mathbb{H}_1(S) \leq k \mathbb{H}_\infty(S), \quad (3.51)$$

which completes the proof. \square

Remark 3.8. By reasoning as in Step 3, we deduce that $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ is a decreasing function of p and that

$$\lim_{p \rightarrow \infty} \mathbb{P}_{\mathbb{Z}^k,p}(S) = \mathbb{P}_{\mathbb{Z}^k,\infty}(S). \quad (3.52)$$

Therefore, from Theorem A, we obtain the following inequality:

$$\mathbb{H}_p(S) \leq \mathbb{E}_p(S) = 2\pi \mathbb{P}_{\mathbb{R}^k,p}(S) \leq 2\pi \mathbb{P}_{\mathbb{Z}^k,p}(S) \leq 2\pi \mathbb{P}_{\mathbb{Z}^k,1}(S) = \mathbb{H}_1(S) \quad (3.53)$$

for any value of $p \in [1, \infty]$. The equality $2\pi \mathbb{P}_{\mathbb{Z}^k,1}(S) = \mathbb{H}_1(S)$ in (3.53) can be established by using the dipole construction and applying the coarea formula for each component-wise (see [5, Remark 18]); moreover, in Section 4 we provide an alternative proof of the same equality (see Remark 4.7). It is also natural to ask under which conditions Plateau's problems $\mathbb{P}_{\mathbb{Z}^k,p}(S)$, $\mathbb{P}_{\mathbb{R}^k,p}(S)$ are equivalent to $\mathbb{E}_p(S)$, $\mathbb{H}_p(S)$. This is the case, for instance, when $p = 1$, as all the inequalities in (3.53) reduce to equalities when $p = 1$. Moreover, if a minimizer of the problem $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ admits a calibration (see Definition 2.11), then $\mathbb{E}_p(S)$, $\mathbb{P}_{\mathbb{Z}^k,p}(S)$, and $\mathbb{P}_{\mathbb{R}^k,p}(S)$ are equal to each other. Indeed, the existence of a calibration implies that

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) = \mathbb{P}_{\mathbb{R}^k,p}(S). \quad (3.54)$$

From (3.54), with the aid of Theorem A, we obtain

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) = \mathbb{P}_{\mathbb{R}^k,p}(S) = \mathbb{E}_p(S). \quad (3.55)$$

Remark 3.9. In case $d = 2$, the analysis contained in this section is consistent with [5], although we allow for slightly more general boundary data S than in [5] and, most importantly, we do not assume the existence of a calibration for minimizers of $\mathbb{P}_{\mathbb{Z}^k,p}(S)$. However, we do not know if our results extend to maps that take their values in products of higher-dimensional spheres, $\mathbb{S}^m \times \dots \times \mathbb{S}^m$ with $m \geq 2$. Furthermore, from Theorem A we deduce that, in the absence of a calibration, the conclusion of [5, Theorem 1] (see Equation (5) in the introduction) may fail. Let us consider the following example, given in [10, Example 4.2]. Let $d = 2$, $k = 4$, $p = \infty$, and let S be the \mathbb{Z}^4 -current carried by the vertices of a regular pentagon $P_1P_2P_3P_4P_5$, where the vertex P_i has multiplicity $e_i = (0, \dots, 1, \dots, 0)$ for $i = 1, \dots, 4$ (the 1 is in the i -th position) and P_5 has multiplicity $(-1, -1, -1, -1)$. As shown in [10], we have

$$2\pi \mathbb{P}_{\mathbb{Z}^k,\infty}(S) > 2\pi \mathbb{P}_{\mathbb{R}^k,\infty}(S). \quad (3.56)$$

This inequality, combined with Theorem A, implies that

$$2\pi \mathbb{P}_{\mathbb{Z}^k,\infty}(S) > \mathbb{E}_\infty(S). \quad (3.57)$$

This provides a counterexample for (5), thus proving that the existence of a calibration is an assumption that cannot be removed from [5, Theorem 1]. Whether or not the same phenomenon happens also for other values of $p \in (1, \infty)$ is still an open question.

As Equation (3.56) shows, the inequality $\mathbb{P}_{\mathbb{R}^k,p}(S) \leq \mathbb{P}_{\mathbb{Z}^k,p}(S)$ may be strict, in general. Nevertheless, in the next section we will derive a bound for $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ in terms of $\mathbb{P}_{\mathbb{R}^k,p}(S)$. To this end, we will first write a characterization for $\mathbb{P}_{\mathbb{Z}^k,p}(S)$ in terms of torus-valued maps.

4 BV-liftings of torus-valued maps and mass minimization for integral currents

In this section, we address the lifting problem for torus-valued map of bounded variation. As we will see later in the section, lifting results will allow us to obtain a bound for $\mathbb{P}_{\mathbb{Z}^k, p}(S)$ in terms of $\mathbb{P}_{\mathbb{R}^k, p}(S)$.

In general terms, the lifting problem can be formulated as follows. Let \mathcal{N} be a smooth, compact, connected Riemannian manifold without boundary and let $p: \mathcal{C} \rightarrow \mathcal{N}$ be the smooth Riemannian universal covering of \mathcal{N} . Let Ω be a smooth, bounded domain in \mathbb{R}^d and $\mathbf{u}: \Omega \rightarrow \mathcal{N}$ a measurable map. A measurable map $\boldsymbol{\theta}: \Omega \rightarrow \mathcal{C}$ is called a *lifting* of \mathbf{u} if $p \circ \boldsymbol{\theta} = \mathbf{u}$ a.e. in Ω . We consider the following question: given a regular map $\mathbf{u}: \Omega \rightarrow \mathcal{N}$, is there a lifting $\boldsymbol{\theta}: \Omega \rightarrow \mathcal{C}$ of \mathbf{u} which has the same regularity of \mathbf{u} ? The answer depends on what kind of regularity we assume for \mathbf{u} . For instance, whenever Ω is a simple connected domain, a classical topological result guarantees that any C^k -map $\mathbf{u}: \Omega \rightarrow \mathcal{N}$ (with $k = 0, \dots, \infty$) has a C^k lifting. The same result holds true for the other functional spaces, for instance the Sobolev space $W^{1,p}$ with $p \geq 2$ (see e.g. [8, 6, 30]). However, the conclusion may fail for other functional spaces — for instance, the Sobolev space $W^{1,p}$ with $1 \leq p < 2$. (For example, when the domain $\Omega = \mathbb{B}_1^2(0)$ is the unit ball in \mathbb{R}^2 and $\mathcal{N} = \mathbb{S}^1$, the map $u(x) := \frac{x}{|x|}$ belongs to $W^{1,p}(\Omega, \mathbb{S}^1)$ but has no lifting in $W^{1,p}(\Omega, \mathbb{R})$.) Therefore, we are naturally led to relax the regularity requirements on the liftings and look for a lifting in a larger space. It was initially proved in [19] that for any map $u \in \text{BV}(\Omega; \mathbb{S}^1)$, there always exists a function $\theta \in \text{BV}(\Omega; \mathbb{R})$ of bounded variation such that $u = e^{i\theta}$ a.e. (here Ω does not need to be simply connected). A similar lifting result can be established when the target is a general closed manifold [14].

There are many contributions to the lifting problem (see for instance [15, 11, 29, 9, 7, 22, 31] and the references therein). In this paper, we consider as target space the manifold $\mathcal{N} = \mathbb{T}^k = \mathbb{S}^1 \times \dots \times \mathbb{S}^1 \subseteq \mathbb{C}^k$, with its covering space $\mathcal{C} = \mathbb{R}^k$ and the lifting map given by

$$\begin{aligned} p: \mathbb{R}^k &\longrightarrow \mathbb{T}^k = \mathbb{S}^1 \times \dots \times \mathbb{S}^1 \\ (\theta_1, \dots, \theta_k) &\longmapsto (e^{i\theta_1}, \dots, e^{i\theta_k}). \end{aligned} \tag{4.1}$$

We will also use the following notation for the covering map: given $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$, we define

$$\mathbf{e}^{i\boldsymbol{\theta}} := p(\boldsymbol{\theta}) = (e^{i\theta_1}, \dots, e^{i\theta_k}) \in \mathbb{T}^k. \tag{4.2}$$

Since we are interested in BV-liftings (i.e., liftings that belong to the space of *functions of Bounded Variation*), let us briefly recall the notion of BV-space (see e.g. [4] for more details). A function $\boldsymbol{\theta} \in L_{\text{loc}}^1(\mathbb{R}^d, \mathbb{R}^k)$ is a function of locally bounded variation, $\boldsymbol{\theta} \in \text{BV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$, if its distributional derivative $D\boldsymbol{\theta}$ is a vector-valued Radon measure on the σ -algebra of Borel sets of \mathbb{R}^d and the total variation of $|D\boldsymbol{\theta}|(\Omega)$ is finite for any *bounded*, open set $\Omega \subseteq \mathbb{R}^d$. The derivative $D\boldsymbol{\theta}$ can then be decomposed as:

$$D\boldsymbol{\theta} = D^a\boldsymbol{\theta} + D^j\boldsymbol{\theta} + D^c\boldsymbol{\theta}, \tag{4.3}$$

where

- $D^a\boldsymbol{\theta}$ is the absolutely continuous part of $D\boldsymbol{\theta}$ with respect to the Lebesgue measure, which can be written as $D^a\boldsymbol{\theta} = \nabla\boldsymbol{\theta} \mathcal{L}^d$, where $\nabla\boldsymbol{\theta}$ is the approximate differential;

- $D^j\boldsymbol{\theta}$ is the jump part, which can be written as $D^j\boldsymbol{\theta} = (\boldsymbol{\theta}^+ - \boldsymbol{\theta}^-) \otimes n \mathcal{H}^{d-1} \llcorner S(\boldsymbol{\theta})$, where $S(\boldsymbol{\theta})$ is the set of approximate discontinuities (or jump set), oriented by the unit normal vector field n , and $\boldsymbol{\theta}^+$, $\boldsymbol{\theta}^-$ are the one-sided approximate limits of $\boldsymbol{\theta}$ on either side of $S(\boldsymbol{\theta})$;
- $D^c\boldsymbol{\theta}$ is the Cantor part of $D\boldsymbol{\theta}$, which is mutually singular to both $D^a\boldsymbol{\theta}$ and $D^j\boldsymbol{\theta}$.

The total variation of $\boldsymbol{\theta} \in \text{BV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ on a bounded open set $\Omega \subseteq \mathbb{R}^d$ can be represented as

$$|D\boldsymbol{\theta}|(\Omega) = \int_{\Omega} |\nabla\boldsymbol{\theta}| \, dx + \int_{S(\boldsymbol{\theta}) \cap \Omega} |\boldsymbol{\theta}^+ - \boldsymbol{\theta}^-| \, d\mathcal{H}^{d-1} + |D^c\boldsymbol{\theta}|(\Omega). \quad (4.4)$$

We say that $\boldsymbol{\theta}$ is a locally SBV-function, and we write $\boldsymbol{\theta} \in \text{SBV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$, if $\boldsymbol{\theta} \in \text{BV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ and $D^c\boldsymbol{\theta} = 0$. For any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \text{SBV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$, and any $p \in [1, \infty]$, we define

$$|\boldsymbol{\theta}|_{\text{SBV},p} := \int_{\mathbb{R}^d} \|\nabla\boldsymbol{\theta}\|_p \, dx + \int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+ - \boldsymbol{\theta}^-\|_p \, d\mathcal{H}^{d-1}, \quad (4.5)$$

where

$$\int_{\mathbb{R}^d} \|\nabla\boldsymbol{\theta}\|_p \, dx := \int_{\mathbb{R}^d} \|(|\nabla\theta_1|, \dots, |\nabla\theta_k|)\|_p \, dx = \int_{\mathbb{R}^d} (|\nabla\theta_1|^p + \dots + |\nabla\theta_k|^p)^{\frac{1}{p}} \, dx,$$

if p is finite; the case $p = \infty$ is defined in a similar way. In general, the value of the quantity $|\boldsymbol{\theta}|_{\text{SBV},p}$ may be infinite.

Throughout this section, we consider again a $(d-1)$ -dimensional integer-multiplicity flat boundary S with compact support and finite mass. Unless otherwise stated, we assume that S satisfies the conditions (10) (however, the proof of Theorem C will not rely on the assumption (10)). We consider a map $\mathbf{u} = (u_1, \dots, u_k)$ that belongs to \mathcal{Q}_S — that is, $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^d, \mathbb{C}^k)$ is constant in a neighbourhood of infinity and satisfies $\star\mathbf{J}(\mathbf{u}) = \pi S$. A map $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ is said to be a BV-lifting of \mathbf{u} if $\boldsymbol{\theta}$ belongs to $\text{BV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^k)$ and $\mathbf{e}^{i\boldsymbol{\theta}} = \mathbf{u}$, where $\mathbf{e}^{i\boldsymbol{\theta}}$ is defined as in (4.2). Earlier results on the lifting problem for circle-valued maps [19, 15] imply that any map $\mathbf{u} \in \mathcal{Q}_S$ admits a lifting $\boldsymbol{\theta} \in \text{BV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^k)$ that satisfies a suitable estimate.

Theorem 4.1. *Let $\mathbf{u} = (u_1, \dots, u_k)$ be in \mathcal{Q}_S . Then, there exists a function $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in \text{SBV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^k)$ such that*

$$\mathbf{e}^{i\boldsymbol{\theta}} = \mathbf{u} \quad (4.6)$$

and

$$|\boldsymbol{\theta}|_{\text{SBV},p} \leq 2k^{1-\frac{1}{p}} H_p(\mathbf{u}). \quad (4.7)$$

Remark 4.2. The existence of a BV-lifting for torus-valued maps, and more generally for maps with values in a closed manifold, has been proved in [14, Theorem 1]. However, the results of [14] do not provide an explicit estimate for the constant in (4.7). Instead, we will apply a result by Dávila and Ignat [15], who proved the existence of BV-liftings for circle-valued maps and provided the optimal value of the constant in (4.7) when $k = 1$. We do not know whether the factor $2k^{1-\frac{1}{p}}$ is optimal for $k > 1$ (see Theorem 4.6).

Proof of Theorem 4.1. Applying the results of [15], for each $j = 1, \dots, k$ there exists a function $\theta_j \in \text{SBV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$ such that θ_j is a lifting of u_j , i.e.

$$e^{i\theta_j} = u_j, \quad (4.8)$$

and

$$|\theta_j|_{\text{BV}} = \int_{\mathbb{R}^d} |\nabla \theta_j| \, dx + \int_{S(\theta_j)} |\theta_j^+ - \theta_j^-| \, d\mathcal{H}^{d-1} \leq 2 \int_{\mathbb{R}^d} |\nabla u_j| \, dx. \quad (4.9)$$

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. By taking the sum over j in both sides of (4.9), we obtain

$$|\boldsymbol{\theta}|_{\text{SBV},1} \leq 2H_1(\mathbf{u}). \quad (4.10)$$

Taking into account that, for any $z \in \mathbb{R}^k$, the function $p \in [1, \infty] \mapsto \|z\|_p$ is decreasing (see Lemma A.32), we obtain

$$|\boldsymbol{\theta}|_{\text{SBV},p} \leq |\boldsymbol{\theta}|_{\text{SBV},1} \leq 2H_1(\mathbf{u}) \leq 2k^{1-1/p} H_p(\mathbf{u}), \quad (4.11)$$

we have made use of (4.10) and applied the Holder's inequality. \square

Remark 4.3. One has $|\nabla \theta_j(x)| = |\nabla u_j(x)|$ at points where θ_j is approximately differentiable. Indeed, let x be a point of approximate differentiability for θ . By differentiating the identity $e^{i\theta_j} = u_j$, we obtain for any $\ell = 1, \dots, d$

$$\frac{\partial \theta_j(x)}{\partial x_\ell} e^{i\theta_j(x)} = \frac{\partial u_j(x)}{\partial x_\ell}. \quad (4.12)$$

Thus,

$$\left| \frac{\partial \theta_j(x)}{\partial x_\ell} \right| = \left| \frac{\partial u_j(x)}{\partial x_\ell} \right|, \quad (4.13)$$

As a consequence, we deduce

$$|\nabla \theta_j(x)| = |\nabla u_j(x)| \quad (4.14)$$

for a.e. $x \in \mathbb{R}^d$ and hence,

$$\int_{\mathbb{R}^d} \|\nabla \boldsymbol{\theta}\|_p \, dx = H_p(\mathbf{u}) \quad (4.15)$$

for any $p \in [1, \infty]$.

We now address the relationship between the lifting problem and mass minimization among integral currents, and prove Theorem B. For convenience, we recall some notation. Given S as above, let us choose a map $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{Q}_S$. This is possible because, under our assumption (10) for S , the set \mathcal{Q}_S is non-empty (see Remark 3.6). We consider the following problem, for $p \in [1, \infty]$:

$$\mathbb{L}_p(S, \mathbf{u}) := \inf \left\{ \int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p \, d\mathcal{H}^{d-1}(x) : \boldsymbol{\theta} \text{ is a BV-lifting of } \mathbf{u} \right\}. \quad (\text{L})$$

First, we show that the infimum at the right-hand side of (L) is independent of the choice of \mathbf{u} .

Lemma 4.4. *If $\mathbf{u} \in \mathcal{Q}_S$, $\mathbf{v} \in \mathcal{Q}_S$ (in particular, $\star \mathbf{J}(\mathbf{u}) = \star \mathbf{J}(\mathbf{v}) = \pi S$), then $\mathbb{L}_p(S, \mathbf{u}) = \mathbb{L}_p(S, \mathbf{v})$ for any $p \in [1, \infty]$.*

Proof. Since $\star \mathbf{J}(\mathbf{u}) = \star \mathbf{J}(\mathbf{v}) = \pi S$, by an explicit computation we obtain

$$J(u_j \bar{v}_j) = J(u_j) - J(v_j) = 0$$

for any index $j = 1, \dots, k$. Therefore, there exists $w_j \in W^{1,1}(\mathbb{R}^d; \mathbb{R})$ such that

$$u_j \bar{v}_j = e^{iw_j} \quad (4.16)$$

(see for instance [13, Lemma 1.8]). On the other hand, we have $v_j \bar{v}_j = |v_j|^2 = 1$, which implies

$$u_j = e^{iw_j} v_j. \quad (4.17)$$

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ be a BV-lifting of \mathbf{u} . Then, $\bar{\boldsymbol{\theta}} = (\theta_1 - w_1, \dots, \theta_k - w_k)$ is a BV-lifting of \mathbf{v} with the same jump as $\boldsymbol{\theta}$, since each w_j is a Sobolev function. Conversely, for any BV-lifting of \mathbf{v} there exists a BV-lifting of \mathbf{u} with the same jump. Therefore, $\mathbb{L}_p(S, \mathbf{u}) = \mathbb{L}_p(S, \mathbf{v})$, as claimed. \square

In view of Lemma 4.4, from now on we write $\mathbb{L}_p(S)$ instead of $\mathbb{L}_p(S, \mathbf{u})$.

Remark 4.5. The infimum at the left-hand side of (L) is always attained. Indeed, if $(\boldsymbol{\theta}_j)_{j \in \mathbb{N}}$ is a minimizing sequence for $\mathbb{L}_p(S)$, then the chain rule implies that $\boldsymbol{\theta}_j \in \text{SBV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ (if $\boldsymbol{\theta}_j$ had a nonvanishing Cantor part, then $\mathbf{u} = \mathbf{e}^{i\boldsymbol{\theta}}$ would have a nonvanishing Cantor part, too, because the map $\theta \mapsto e^{i\theta}$ has injective differential at any point). Moreover, the absolutely continuous gradient $\nabla \boldsymbol{\theta}_j$ is bounded in $L^1(\mathbb{R}^d)$, by Remark (4.3). Therefore, the sequence $(\boldsymbol{\theta}_j)_{j \in \mathbb{N}}$ is bounded in $\text{BV}_{\text{loc}}(\mathbb{R}^d)$ and, up to extraction of a subsequence, it converges weakly in BV to a limit $\boldsymbol{\theta} \in \text{BV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$, which is a lifting of \mathbf{u} and a minimizer of (L).

Proof of Theorem B. Let $\mathbf{u} \in \mathcal{Q}_S$ be given. We need to prove that $\mathbb{L}(S) = 2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S)$. The proof relies on the properties of the jump set for liftings of torus-valued $W^{1,1}$ -maps. In case $k = 1$, this properties have been studied in, e.g., [19, 21]. However, for the reader's convenience, we will present the arguments in a (mostly) self-contained way.

Step 1. First, we prove the inequality

$$\mathbb{L}_p(S) \geq 2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S). \quad (4.18)$$

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ be a BV-lifting of \mathbf{u} . We can define a current $T(\boldsymbol{\theta})$, with coefficients in \mathbb{Z}^k , associated with the jump part of $\boldsymbol{\theta}$, in such a way that $\partial T(\boldsymbol{\theta}) = S$. More precisely, $T(\boldsymbol{\theta})$ is defined component-wise as $T(\boldsymbol{\theta}) = (T_1, \dots, T_k)$, where

$$T_i = \llbracket S(\theta_i), \tau_i, c_i \rrbracket$$

for each index i . Here, $S(\theta_i)$ is the jump set of θ_i , $c_i(x) := \frac{1}{2\pi}(\theta_i^+(x) - \theta_i^-(x))$ for \mathcal{H}^{d-1} a.e. $x \in S(\theta_i)$, and τ_i is a unit $(d-1)$ -vector orienting the jump set $S(\theta_i)$. We observe that the function c_i must take integer values (at least \mathcal{H}^{d-1} -almost everywhere), because $u_i = e^{i\theta_i}$ is a Sobolev function, with no jump.

We shall now prove that each component T_i has boundary S_i . The differential operator d satisfies the identity $d^2\theta_i = 0$ in the sense of distributions — that is, for any $v \in C_c^\infty(\mathbb{R}^d, \Lambda_2(\mathbb{R}^d))$ one has

$$\langle d\theta_i, d^*v \rangle = 0. \quad (4.19)$$

Here $\langle \cdot, \cdot \rangle$ is the duality between (form-valued) measures and (vector-valued) functions, while d^* is the codifferential, defined in (2.42). From Remark 4.3, at the point x where θ_i is approximately differentiable, one has

$$d\theta_i(x) = u_i(x) \wedge du_i(x) = j(u_i)(x). \quad (4.20)$$

In view of (4.20), Equation (4.19) can be written as

$$\int_{\mathbb{R}^d} \langle j(u_i), d^*v \rangle dx + \int_{S(\theta_i)} (\theta_i^+ - \theta_i^-)(n_i, d^*v) d\mathcal{H}^{d-1} = 0, \quad (4.21)$$

where n_i is the unit normal to $S(\theta_i)$. We observe that $(-1)^{d-1}n_i = (\star\tau_i)^\#$, because τ_i is the unit tangent $(d-1)$ -vector to $S(\theta_i)$. Let $\omega := \star v$. As \star is an isometry, we have

$$-(n_i, d^*v) = (-1)^d(n_i, \star d\omega) = -(\star n_i, d\omega) = (\tau_i, d\omega) \quad (4.22)$$

(where (\cdot, \cdot) denotes the Eulidean scalar product on both vectors and covectors). From (2.42), (4.21) and (4.22), we deduce that

$$\int_{\mathbb{R}^d} \langle dj(u_i), v \rangle dx - 2\pi \int_{S(\theta_i)} c_i(\tau_i, d\omega) d\mathcal{H}^{d-1} = 0 \quad (4.23)$$

or equivalently,

$$\star J(u_i)(\omega) - \pi T_i(d\omega) = 0 \quad \iff \quad (\star J(u_i) - \pi \partial T_i)(\omega) = 0.$$

As v is arbitrary, and hence ω is, we have proved that $\pi \partial T_i = \frac{1}{\pi} J(u_i) = S_i$, as claimed. Moreover, by definition of $T(\boldsymbol{\theta})$, one has

$$\int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p d\mathcal{H}^{d-1}(x) = 2\pi \mathbb{M}_p(T(\boldsymbol{\theta})), \quad (4.24)$$

so (4.18) follows.

Step 2. Now, we prove the opposite inequality,

$$\mathbb{L}_p(S) \leq 2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S). \quad (4.25)$$

Let $\bar{T} = (\bar{T}_1, \bar{T}_2, \dots, \bar{T}_k)$ be a \mathbb{Z}^k -rectifiable current such that $\partial \bar{T} = S$. Let $\mathbf{u} \in \mathcal{Q}_S$ (\mathcal{Q}_S is non-empty, see Remark 3.6): we shall prove that, for any index j , there exists a BV-lifting θ_j of u_j such that the jump part of θ_j is associated with the current \bar{T}_j — that is,

$$\bar{T}_j = \llbracket S(\bar{\theta}_j), \bar{\tau}_j, \bar{c}_j \rrbracket, \quad (4.26)$$

where $S(\bar{\theta}_j)$ is the jump set of $\bar{\theta}_j$, $\bar{\tau}_j$ is a unit $(d-1)$ -vector field that orients $S(\bar{\theta}_j)$ and $\bar{c}_j(x) = \frac{1}{2\pi} (\bar{\theta}_j^+(x) - \bar{\theta}_j^-(x))$ for \mathcal{H}^{d-1} -a.e. $x \in S_{\bar{\theta}_j}$. Indeed, let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ be a BV-lifting of \mathbf{u} (which exists, by Theorem 4.1), and let $T(\boldsymbol{\theta})$ be the current associated with the jump part of u_j , as constructed in Step 1. As we have seen, $\partial T(\boldsymbol{\theta}) = S_j$, which implies $\partial(\bar{T} - T(\boldsymbol{\theta})) = 0$. Therefore, for any index j there exists a d -dimensional integral current R_j , with locally finite mass, such that $\partial R_j = \bar{T}_j - T_j(\boldsymbol{\theta})$. Moreover, $\star R_i$ can be identified with a scalar function, which belongs to $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$ and takes its values in \mathbb{Z} , since R_j is integral. From (2.46), one also has

$$d(\star R_j) = -\star(\partial R_j) = \star(T_j(\boldsymbol{\theta}) - \bar{T}_j), \quad (4.27)$$

$$\bar{\theta}_j = \theta - 2\pi \star R_j. \quad (4.28)$$

Then, $\bar{\theta}_j$ is a BV-lifting of u_j and from (4.28), (4.27) we deduce that the current associated with the jump set of $\bar{\theta}_j$ is $T(\theta_j) - (T_j(\boldsymbol{\theta}) - \bar{T}_j) = \bar{T}_j$. (Indeed, by construction $T(\theta_j)$ coincides with the Hodge dual to the jump part of the distributional differential $d\theta_j$, up to a sign.) Therefore, the function $\bar{\boldsymbol{\theta}} = (\bar{\theta}_1, \dots, \bar{\theta}_k)$ is a BV-lifting of $\mathbf{u} = (u_1, \dots, u_k)$ which satisfies (4.26). Moreover,

$$\mathbb{L}_p(S) \leq \int_{S(\bar{\boldsymbol{\theta}})} \|\bar{\boldsymbol{\theta}}^+(x) - \bar{\boldsymbol{\theta}}^-(x)\|_p d\mathcal{H}^{d-1}(x) = 2\pi\mathbb{M}_p(\bar{T}). \quad (4.29)$$

Therefore, we can conclude that (4.25) holds. \square

From Theorem B, we can deduce a sufficient and necessary condition for the equality $\mathbb{H}_p(S) = \mathbb{P}_{\mathbb{Z}^k, p}(S)$ to hold, in terms of the lifting constant in (4.7).

Theorem 4.6. *Let $1 \leq p \leq \infty$ and let S be a \mathbb{Z}^k -boundary of compact support S and finite mass that satisfies the assumption (10). Then, there holds $2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{H}_p(S)$ if and only if for any $\mathbf{u} \in \mathcal{Q}_S$, there exists a lifting $\boldsymbol{\theta} \in \text{SBV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^k)$ of \mathbf{u} such that*

$$|\boldsymbol{\theta}|_{\text{SBV}, p} \leq 2H_p(\mathbf{u}). \quad (4.30)$$

Proof of Theorem 4.6. Suppose first that $2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{H}_p(S)$. Let $\mathbf{u} \in \mathcal{Q}_S$. By Theorem B and Remark 4.5, there exists a lifting $\boldsymbol{\theta} \in \text{SBV}(\mathbb{R}^d; \mathbb{R}^k)$ of \mathbf{u} such that

$$\int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p d\mathcal{H}^{d-1}(x) = \mathbb{L}_p(S) = \mathbb{H}_p(S) \leq H_p(\mathbf{u}). \quad (4.31)$$

From Remark 4.3 and the inequality (4.31), we deduce that

$$|\boldsymbol{\theta}|_{\text{SBV}, p} \leq 2H_p(\mathbf{u}).$$

Conversely, suppose that for any $\mathbf{u} \in \mathcal{Q}_S$ there exists a lifting $\boldsymbol{\theta} \in \text{SBV}(\mathbb{R}^d; \mathbb{R}^k)$ of \mathbf{u} such that

$$|\boldsymbol{\theta}|_{\text{SBV}, p} \leq 2H_p(\mathbf{u}).$$

Again, from Remark 4.3, this implies

$$\int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p d\mathcal{H}^{d-1}(x) \leq H_p(\mathbf{u}), \quad (4.32)$$

Combining (4.32) with Theorem B, we obtain that

$$2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) \leq H_p(\mathbf{u}).$$

As \mathbf{u} can be taken arbitrarily, we conclude that

$$2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) \leq \mathbb{H}_p(S). \quad \square$$

Remark 4.7. The equality $2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{H}_p(S)$ is true when $k = 1$ or $p = 1$, because in this case, we have the lifting property with factor 2, by Theorem 4.1. However, the equality $2\pi \mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{H}_p(S)$ may fail in general. One may give examples (see [10, Example 4.2] and Remark 3.9 above) where

$$\mathbb{P}_{\mathbb{Z}^k, p}(S) > \mathbb{P}_{\mathbb{R}^k, p}(S).$$

In this case, Theorem A implies

$$\mathbb{H}_p(S) < \mathbb{P}_{\mathbb{Z}^k, p}(S). \quad (4.33)$$

and, by Theorem 4.6, the optimal lifting constant (with respect to the p -norm) for maps $\mathbf{u} \in \mathcal{Q}_S$ will be strictly larger than 2.

Finally, from Theorem A, Theorem 4.1, and Theorem B we deduce the core result of our paper, namely Theorem C.

Proof of Theorem C. We need to prove that $\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq (2k^{1-1/p} - 1)\mathbb{P}_{\mathbb{R}^k,p}(S)$, for any \mathbb{Z}^k -boundary S of compact support.

Step 1. We first consider the case S is an integral polyhedral current (so that, in particular, S has finite mass and satisfies the assumption (10)). By Remark 3.6, we know that there exists $\mathbf{u} \in \mathcal{Q}_S$. According to Theorem 4.1, there exists a lifting $\boldsymbol{\theta} \in \text{SBV}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^k)$ of \mathbf{u} such that

$$|\boldsymbol{\theta}|_{\text{SBV},p} \leq 2k^{1-\frac{1}{p}}H_p(\mathbf{u}). \quad (4.34)$$

This inequality, together with Remark 4.3, implies

$$\int_{S(\boldsymbol{\theta})} \|\boldsymbol{\theta}^+(x) - \boldsymbol{\theta}^-(x)\|_p \, d\mathcal{H}^{d-1}(x) \leq (2k^{1-\frac{1}{p}} - 1)H_p(\mathbf{u}). \quad (4.35)$$

From (4.35), Theorem A and Theorem B, it follows that

$$2\pi\mathbb{P}_{\mathbb{Z}^k,p}(S) = \mathbb{L}(S) \leq (2k^{1-\frac{1}{p}} - 1)\mathbb{H}_p(S) \leq (2k^{1-\frac{1}{p}} - 1)2\pi\mathbb{P}_{\mathbb{R}^k,p}(S) \quad (4.36)$$

and hence

$$\mathbb{P}_{\mathbb{Z}^k,p}(S) \leq (2k^{1-\frac{1}{p}} - 1)\mathbb{P}_{\mathbb{R}^k,p}(S), \quad (4.37)$$

as claimed.

Step 2. Let S be a \mathbb{Z}^k -boundary of dimension $(d-2)$ in the ambient space \mathbb{R}^d . According to [16, § 4.2.21], for each index $i = 1, \dots, k$ there exists a sequence $(S_i^n)_{n \in \mathbb{N}}$ of integer-multiplicity, polyhedral $(d-2)$ -cycles (that is, currents such that $\partial S_i^n = 0$ — see Definition 2.9), as well as sequences of currents $(Q_i^n)_{n \in \mathbb{N}}$, $(R_i^n)_{n \in \mathbb{N}}$ (of dimension $d-2$, $d-1$ respectively) such that

$$S_i - S_i^n = \partial R_i^n + Q_i^n \quad (4.38)$$

and $\mathbb{M}(R_i^n) + \mathbb{M}(Q_i^n) \rightarrow 0$ as $n \rightarrow \infty$. Set $S^n := (S_1^n, \dots, S_k^n)$. We shall prove that, as $n \rightarrow \infty$, $\mathbb{P}_{\mathbb{Z}^k,p}(S^n)$, $\mathbb{P}_{\mathbb{R}^k,p}(S^n)$ converges to $\mathbb{P}_{\mathbb{Z}^k,p}(S)$, $\mathbb{P}_{\mathbb{R}^k,p}(S)$ respectively. By the previous step, each S^n satisfies the inequality (4.37), so the theorem will follow by passing to the limit as $n \rightarrow \infty$. We shall only prove that $\mathbb{P}_{\mathbb{R}^k,p}(S^n) \rightarrow \mathbb{P}_{\mathbb{R}^k,p}(S)$; the claim $\mathbb{P}_{\mathbb{Z}^k,p}(S^n) \rightarrow \mathbb{P}_{\mathbb{Z}^k,p}(S)$ follows by similar arguments. As both S_i and S_i^n are cycles, by taking the differential of both sides of (4.38) we obtain that $\partial Q_i^n = 0$ for any n and i . Then, by the isoperimetric inequality [16, § 4.2.10], for each n and i there exists an integral current \bar{R}_i^n such that $\partial \bar{R}_i^n = Q_i^n$ and

$$\mathbb{M}(\bar{R}_i^n) \leq C \mathbb{M}(Q_i^n)^{\frac{d-1}{d-2}}$$

for some constant C that depends on d only. Let $C_i^n := \bar{R}_i^n + R_i^n$. Then, we obtain

$$S_i - S_i^n = \partial C_i^n, \quad \mathbb{M}(C_i^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.39)$$

Let $T = (T_1, \dots, T_k)$ be a competitor for the problem $\mathbb{P}_{\mathbb{R}^k,p}(S)$. Define $C^n := (C_1^n, \dots, C_k^n)$ and $\bar{T}^n = T - C^n$. One has $\partial \bar{T}^n = S^n$ and $\mathbb{M}_p(T - \bar{T}^n) \rightarrow 0$ as $n \rightarrow \infty$, because

$$\mathbb{M}_p(T - \bar{T}^n) \leq \sum_{i=1}^k \mathbb{M}(C_i^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.40)$$

Therefore, we obtain $\limsup_{n \rightarrow \infty} \mathbb{P}_{\mathbb{R}^k, p}(S^n) \leq \mathbb{M}_p(T)$ and, since T is arbitrary,

$$\limsup_{n \rightarrow \infty} \mathbb{P}_{\mathbb{R}^k, p}(S^n) \leq \mathbb{P}_{\mathbb{R}^k, p}(S).$$

The opposite inequality, $\liminf_{n \rightarrow \infty} \mathbb{P}_{\mathbb{R}^k, p}(S^n) \geq \mathbb{P}_{\mathbb{R}^k, p}(S)$, is obtained by taking an arbitrary \mathbb{R}^k -normal current T^n such that $\partial T^n = S^n$ and defining $T := T^n + C^n$. \square

Remark 4.8. Assume that we have the following equality

$$\mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{H}_p(S) \quad (4.41)$$

(which is not always true, as we have seen in Remark 4.7). Then, Theorem A implies

$$\mathbb{P}_{\mathbb{Z}^k, p}(S) \leq \mathbb{P}_{\mathbb{R}^k, p}(S)$$

i.e.

$$\mathbb{P}_{\mathbb{Z}^k, p}(S) = \mathbb{P}_{\mathbb{R}^k, p}(S). \quad (4.42)$$

In particular, due to Theorem 4.6, a sufficient condition for (4.42) is that any $\mathbf{u} \in \mathcal{Q}_S$ admits a lifting $\boldsymbol{\theta} \in \text{SBV}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^k)$ with $|\boldsymbol{\theta}|_{\text{SBV}, p} \leq 2H_p(\mathbf{u})$.

A Appendix

In this appendix, we prove some technical results we used in the paper. First, we recall some notation from Section 2.13. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a radial function (i.e. $\rho(x) = \bar{\rho}(|x|)$ for any $x \in \mathbb{R}^d$, for some function $\bar{\rho}: \mathbb{R} \rightarrow \mathbb{R}$) such that $0 \leq \rho \leq 1$, $\text{spt}(\rho) \subseteq \bar{\mathbb{B}}_1^d(0)$ and $\int_{\mathbb{R}^d} \rho(x) dx = 1$. For $\varepsilon > 0$, we define

$$\rho_\varepsilon(x) := \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^d. \quad (A.1)$$

Lemma A.1. *For any $\varepsilon > 0$, there exists an integrable vector field $R_\varepsilon \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$ such that*

(i) $\text{div } R_\varepsilon = \delta_0 - \rho_\varepsilon$ in the sense of $D'(\mathbb{R}^d)$,

(ii) $\text{spt}(R_\varepsilon) \subset \bar{\mathbb{B}}_\varepsilon^d(0)$,

(iii) $\|R_\varepsilon\|_{L^1(\mathbb{R}^d)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. We shall prove that there exists an integrable vector field $R \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R}^d)$ such that

$$\text{div } R = \delta_0 - \rho \quad (A.2)$$

as distributions in \mathbb{R}^d and $\text{spt}(R) \subset \bar{\mathbb{B}}_1^d(0)$. Then, the vector field R_ε defined by

$$R_\varepsilon(x) := \varepsilon^{-d+1} R\left(\frac{x}{\varepsilon}\right) \quad (A.3)$$

will satisfy all the desired properties.

We define

$$\xi(t) := \frac{1}{t^d} \int_t^1 s^{d-1} \rho(s) ds \quad \text{for } t > 0 \quad (A.4)$$

and $R(x) := \xi(|x|)x$ for $x \in \mathbb{R}^d \setminus \{0\}$. We observe that, since $\text{spt } \rho \subset \bar{\mathbb{B}}_1^d(0)$, we have $R(x) = 0$ for any $x \in \mathbb{R}^k$ with $|x| \geq 1$. Moreover, for $|x| \leq 1$ and $x \neq 0$ we have

$$|R(x)| \leq \frac{1}{d|x|^d}(1 - |x|^d)|x| \leq \frac{1}{d|x|^{d-1}}. \quad (\text{A.5})$$

Therefore, $R \in L^1(\mathbb{R}^d; \mathbb{R}^d)$. One has,

$$\text{div } R(x) = -\rho(x) \quad \text{for any } x \in \mathbb{R}^d \setminus \{0\}. \quad (\text{A.6})$$

Indeed, for any $x \neq 0$ we have:

$$\frac{\partial R_i(x)}{\partial x_i} = \frac{\partial}{\partial x_i} (\xi(|x|)) x_i + \xi(|x|) = \xi'(|x|) \frac{x_i^2}{|x|} + \xi(|x|). \quad (\text{A.7})$$

By differentiating (A.4), we obtain

$$\xi'(t) = -\frac{\rho(t)}{t} - \frac{d}{t}\xi(t) \quad (\text{A.8})$$

for any $t > 0$. By substituting (A.8) into (A.7), we obtain

$$\frac{\partial R_i(x)}{\partial x_i} = -\frac{x_i^2}{|x|^2}\rho(|x|) - d\frac{x_i^2}{|x|^2}\xi(|x|) + \xi(|x|) \quad (\text{A.9})$$

and, by taking the sum over $i = 1, \dots, d$ in both sides of (A.9), (A.6) follows. We are left to prove that

$$\text{div } R = \delta_0 - \rho \quad (\text{A.10})$$

over \mathbb{R}^d in the sense of distributions, that is,

$$-\int_{\mathbb{R}^d} \nabla \varphi(x) \cdot R(x) dx = \varphi(0) - \int_{\mathbb{R}^d} \rho(x)\varphi(x) dx \quad (\text{A.11})$$

for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Since $R \in L^1(\mathbb{R}^d; \mathbb{R}^d)$, one has

$$\begin{aligned} -\int_{\mathbb{R}^d} \nabla \varphi(x) \cdot R(x) dx &= -\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d \setminus \mathbb{B}_\varepsilon^d(0)} \nabla \varphi(x) \cdot R(x) dx \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{R}^d \setminus \mathbb{B}_\varepsilon^d(0)} \text{div } R(x) \varphi(x) dx + \int_{\partial \mathbb{B}_\varepsilon^d(0)} R(x) \cdot \frac{x}{|x|} \varphi(x) d\mathcal{H}^{d-1}(x) \right) \end{aligned} \quad (\text{A.12})$$

by the divergence theorem. In the right hand side of (A.12), we have

$$\int_{\mathbb{R}^d \setminus \mathbb{B}_\varepsilon^d(0)} \text{div } R(x) \varphi(x) dx = -\int_{\mathbb{R}^d \setminus \mathbb{B}_\varepsilon^d(0)} \rho(x) \varphi(x) dx \rightarrow -\int_{\mathbb{R}^d} \rho(x) \varphi(x) dx \quad (\text{A.13})$$

as $\varepsilon \rightarrow 0$, because of (A.6). As for the second term, we apply the definition of R to deduce

$$\begin{aligned} \int_{\partial \mathbb{B}_\varepsilon^d(0)} R(x) \cdot \frac{x}{|x|} \varphi(x) d\mathcal{H}^{d-1}(x) &= \frac{1}{\varepsilon^{d-1}} \int_{\partial \mathbb{B}_\varepsilon^d(0)} \left(\int_\varepsilon^1 s^{d-1} \rho(s) ds \right) \varphi(x) d\mathcal{H}^{d-1}(x) \\ &= \mathcal{H}^{d-1}(\mathbb{S}^{d-1}) \cdot \int_\varepsilon^1 s^{d-1} \rho(s) ds \cdot \int_{\partial \mathbb{B}_\varepsilon^d(0)} \varphi(x) d\mathcal{H}^{d-1}(x), \end{aligned} \quad (\text{A.14})$$

where f denotes the integral average. Since ρ is radial and φ is continuous, we deduce

$$\int_{\partial\mathbb{B}_\varepsilon^d(0)} R(x) \cdot \frac{x}{|x|} \varphi(x) d\mathcal{H}^{d-1}(x) \rightarrow \int_{\mathbb{B}_1^d(0)} \rho(x) dx \cdot \varphi(0) = \varphi(0) \quad (\text{A.15})$$

as $\varepsilon \rightarrow 0$. Combining (A.12), (A.13) and (A.15), we obtain (A.10). \square

In the next lemma, we consider the p -mass norm of a vector $v \in \Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$, defined (as in Equation (2.22)) by

$$|v|_{\text{mass},p} := \sup \left\{ \langle \omega, v \rangle : \omega \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d), |\omega|_{\text{com},p} \leq 1 \right\}.$$

We will also consider the quantity

$$N(v) := \inf \left\{ \sum_{i=1}^l \|z_i\|_p |v_i| : v = \sum_{i=1}^l z_i \otimes v_i, z_i \in \mathbb{R}^k, v_i \text{ is a simple } m\text{-vector in } \mathbb{R}^d \right\}. \quad (\text{A.16})$$

The following lemma extends [16, § 1.8.1] to the case $k > 1$.

Lemma A.2. *Let v be an \mathbb{R}^k -valued m -vector in \mathbb{R}^d , i.e. $v \in \Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$. Then,*

$$|v|_{\text{mass},p} = N(v).$$

Proof. Let v be an \mathbb{R}^k -valued m -vector in \mathbb{R}^d . We first prove that $|v|_{\text{mass},p} \leq N(v)$. Suppose that $v = \sum_{i=1}^l z_i \otimes v_i$, where $z_i \in \mathbb{R}^k$ and each v_i is a simple m -vector in \mathbb{R}^d . Let $\omega \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ be such that $|\omega|_{\text{com},p} \leq 1$. Then, one has

$$\langle \omega, v \rangle = \sum_{i=1}^l \langle \omega, z_i \otimes v_i \rangle \leq \sum_{i=1}^l \|z_i\|_p |v_i|, \quad (\text{A.17})$$

where we have made use of the fact that $|\omega|_{\text{com},p} \leq 1$ and each $z_i \otimes v_i$ has rank 1. To prove the reverse inequality, we shall make use of the Hahn-Banach theorem. From the definition (A.16), it is not difficult to check that N is a seminorm on $\Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$. Then, by the Hahn-Banach theorem, there exists $\omega \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ such that

$$\langle \omega, v \rangle = N(v) \quad (\text{A.18})$$

and

$$\langle \omega, \tau \rangle \leq N(\tau) \quad (\text{A.19})$$

for any $\tau \in \Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$. We observe that the inequality (A.19) holds true, in particular, for any rank-1 vector $\tau = \theta \otimes t$, where $\theta \in \mathbb{R}^k$ and t is a simple m -vector in \mathbb{R}^d . Then, by definition of the comass norm (see Equation (2.21)), Equation (A.19) implies

$$|\omega|_{\text{com},p} \leq 1. \quad (\text{A.20})$$

Therefore,

$$N(v) = \langle \omega, v \rangle \leq |v|_{\text{mass},p} |\omega|_{\text{com},p} \leq |v|_{\text{mass},p} \quad (\text{A.21})$$

which concludes the proof. \square

Remark A.3. Let ω be a $(\mathbb{R}^k)^*$ -valued 1-covector in \mathbb{R}^d . We define

$$\bar{N}(\omega) := \inf \left\{ \sum_{i=1}^l \|z_i\|_q^* |\omega_i| : \omega = \sum_{i=1}^l z_i \otimes \omega_i, z_i \in (\mathbb{R}^k)^*, \omega_i \text{ is a 1-covector in } \mathbb{R}^d \right\},$$

where q is the conjugate exponent of p . (We recall that $\|\cdot\|_q^*$ denotes the dual of the q -norm on \mathbb{R}^k ; therefore, $\|\cdot\|_q^*$ is the p -norm on $(\mathbb{R}^k)^*$.) Then,

$$\bar{N}(\omega) = N(\star\omega) = |\star\omega|_{\text{mass},p}. \quad (\text{A.22})$$

To see this, we observe that if $\omega = \sum_{i=1}^l z_i \otimes \omega_i$, then $\star\omega = \sum_{i=1}^l z_i \otimes \star\omega_i$ (up to identifying each z_i with its image through the isomorphism $(\mathbb{R}^k)^* \rightarrow \mathbb{R}^k$ induced by the canonical basis). Moreover, in codimension 1, each $\star\omega_i$ is a $(d-1)$ -simple vector and $|\star\omega_i| = |\omega_i|$. Therefore, $\bar{N}(\omega) \geq N(\star\omega)$. The opposite inequality follows by similar arguments.

Lemma A.4. In \mathbb{R}^k , the norm $\|\cdot\|_p$ is decreasing in p — that is, for any $1 \leq p_1 \leq p_2 \leq \infty$ and any $z \in \mathbb{R}^k$ one has

$$\|z\|_{p_2} \leq \|z\|_{p_1}. \quad (\text{A.23})$$

Moreover, for any $z \in \mathbb{R}^k$, one has

$$\lim_{p \rightarrow \infty} \|z\|_p = \|z\|_\infty. \quad (\text{A.24})$$

Proof. It is easy to see that, when $p_2 = \infty$, the inequality (A.23) holds true. Let us assume that $p_2 < \infty$, and let $z \in \mathbb{R}^k$. If $z = 0$, (A.23) is obviously satisfied. If $z \neq 0$, let $y_j = \frac{|z_j|}{\|z\|_{p_2}}$ for any $j = 1, \dots, k$ and set $y = (y_1, \dots, y_k)$. One has $|y_j| \leq 1$ and, hence,

$$|y_j|^{p_1} \geq |y_j|^{p_2} \quad (\text{A.25})$$

for any index j . This implies

$$\|y\|_{p_1} \geq 1 \quad \iff \quad \|z\|_{p_1} \geq \|z\|_{p_2}, \quad (\text{A.26})$$

i.e. the norm $\|\cdot\|_p$ is decreasing in p . Moreover, by using Holder's inequality we have

$$\|z\|_p \leq k^{\frac{1}{p}} \|z\|_\infty. \quad (\text{A.27})$$

Therefore, from (A.26) and (A.27) we deduce that

$$\|z\|_\infty \leq \|z\|_p \leq k^{\frac{1}{p}} \|z\|_\infty. \quad (\text{A.28})$$

Let p tends to infinity in (A.28), we obtain that

$$\lim_{p \rightarrow \infty} \|z\|_p = \|z\|_\infty, \quad (\text{A.29})$$

which is what we needed to prove. \square

Lemma A.5. Let T be a normal \mathbb{R}^k -current in \mathbb{R}^d . For any $1 \leq p_1 \leq p_2 < \infty$, there holds

$$\mathbb{M}_{p_2}(T) \leq \mathbb{M}_{p_1}(T) \leq k^{\frac{p_2-p_1}{p_1 p_2}} \mathbb{M}_{p_2}(T) \quad (\text{A.30})$$

and for any $1 \leq p_1 \leq \infty$, there holds

$$\mathbb{M}_\infty(T) \leq \mathbb{M}_{p_1}(T) \leq k^{\frac{1}{p_1}} \mathbb{M}_\infty(T). \quad (\text{A.31})$$

Proof. For any $1 \leq p_1 \leq p_2 < \infty$ and any $z \in \mathbb{R}^k$, according to Lemma A.4, one has

$$\|z\|_{p_2} \leq \|z\|_{p_1}. \quad (\text{A.32})$$

On the other hand, by Holder's inequality we have

$$\|z\|_{p_1} \leq k^{\frac{p_2-p_1}{p_1 p_2}} \|z\|_{p_2}. \quad (\text{A.33})$$

Let $\omega \in C_c^\infty(\mathbb{R}^d; \Lambda_{\mathbb{R}^k}^{d-1}(\mathbb{R}^d))$. From (A.32), using the definition of the comass norm (see Equation (2.21)) and keeping into account that the Hölder conjugate exponents satisfy $p_1^* \geq p_2^*$, we deduce

$$\|\omega\|_{\text{com}, p_1} \leq \|\omega\|_{\text{com}, p_2}, \quad (\text{A.34})$$

This inequality implies, via a duality argument, that

$$\mathbb{M}_{p_2}(T) \leq \mathbb{M}_{p_1}(T). \quad (\text{A.35})$$

In a similar way, from (A.33) we deduce

$$\|\omega\|_{\text{com}, p_2} \leq k^{\frac{p_1^*-p_2^*}{p_1^* p_2^*}} \|\omega\|_{\text{com}, p_1} = k^{\frac{p_2-p_1}{p_1 p_2}} \|\omega\|_{\text{com}, p_1} \quad (\text{A.36})$$

and hence, by duality,

$$\mathbb{M}_{p_1}(T) \leq k^{\frac{p_2-p_1}{p_1 p_2}} \mathbb{M}_{p_2}(T).$$

When $p_2 = \infty$, similar arguments apply. \square

The next lemma summarizes a few results on the comparison between the comass norm of a $(\mathbb{R}^k)^*$ -form (respectively, the mass norm of a \mathbb{R}^k -valued vector) and the comass (respectively, mass) norm of its components. Given a (classical) form $\sigma \in \Lambda^m(\mathbb{R}^d)$ and a (classical) vector $\tau \in \Lambda_m(\mathbb{R}^d)$, we denote by $|\sigma|_{\text{com}}$ the (classical) comass norm of σ and by $|\tau|_{\text{mass}}$ the (classical) mass norm of τ .

Lemma A.6. *Let $w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ and let $w_i = \langle w, \cdot, e_i \rangle \in \Lambda^m(\mathbb{R}^d)$ be the i -th component of w , for $i = 1, \dots, k$. Let $p \in [1, \infty]$ and $q \in [1, \infty]$ be conjugate exponents, such that $1/p + 1/q = 1$. If $p \in (1, \infty]$, then there holds*

$$\max_{1 \leq i \leq k} |w_i|_{\text{com}} \leq |w|_{\text{com}, p} \leq \left(\sum_{i=1}^k |w_i|_{\text{com}}^q \right)^{\frac{1}{q}}, \quad (\text{A.37})$$

while for $p = 1$, there holds

$$|w|_{\text{com}, 1} = \max_{1 \leq i \leq k} |w_i|_{\text{com}}. \quad (\text{A.38})$$

In a similar way, let $v \in \Lambda_{m, \mathbb{R}^k}(\mathbb{R}^d)$ be a vector with components $v_1, \dots, v_k \in \Lambda_m(\mathbb{R}^d)$. If $1 \leq p < \infty$, then there holds

$$\left(\sum_{i=1}^k |v_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^k |v_i|_{\text{mass}}^p \right)^{\frac{1}{p}} \leq |v|_{\text{mass}, p} \leq \sum_{i=1}^k |v_i|_{\text{mass}}, \quad (\text{A.39})$$

while for $p = \infty$ we have

$$\max_{1 \leq i \leq k} |v_i| \leq \max_{1 \leq i \leq k} |v_i|_{\text{mass}} \leq |v|_{\text{mass}, \infty} \leq \sum_{i=1}^k |v_i|_{\text{mass}}. \quad (\text{A.40})$$

where $|\cdot|$ stands for the Euclidean norm.

Remark A.7. Equations (A.37) and (A.39) imply that

$$|w|_{\text{com},\infty} = \sum_{i=1}^k |w_i|_{\text{com}}, \quad |v|_{\text{mass},1} = \sum_{i=1}^k |v_i|_{\text{mass}}$$

for any $w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ and any $v \in \Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$.

Proof of Lemma A.6. We recall that $\|\cdot\|_p^*$ is the dual norm to $\|\cdot\|_p$, i.e. the q -norm on the dual space $(\mathbb{R}^k)^*$.

Proof of (A.37) and (A.38). Let $p > 1$. By the definition of comass norm, one has

$$\begin{aligned} |w|_{\text{com},p} &:= \sup \left\{ \|\langle w; \tau, \cdot \rangle\|_p^* : |\tau| \leq 1, \tau \text{ is a simple } m\text{-vector} \right\} \\ &= \sup \left\{ \left(\sum_{i=1}^k |\langle w_i, \tau \rangle|^q \right)^{\frac{1}{q}} : |\tau| \leq 1, \tau \text{ is a simple } m\text{-vector} \right\} \\ &\leq \left(\sum_{i=1}^k \sup \{ |\langle w_i, \tau \rangle| : |\tau| \leq 1, \tau \text{ is a simple } m\text{-vector} \}^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^k |w_i|_{\text{com}}^q \right)^{\frac{1}{q}}. \end{aligned} \tag{A.41}$$

A similar argument shows that

$$|w|_{\text{com},1} \leq \max_{1 \leq i \leq k} |w_i|_{\text{com}}. \tag{A.42}$$

Moreover, for each $i \in \{1, \dots, k\}$, there exists a simple vector $\tau_i \in \Lambda_m(\mathbb{R}^d)$ such that $|\tau_i| = 1$ and $|w_i|_{\text{com}} = \langle w_i; \tau_i \rangle$. Then, for any index i , we have

$$\begin{aligned} |w|_{\text{com},p} &= \sup \left\{ \left(\sum_{i=1}^k |\langle w_i, \tau \rangle|^q \right)^{\frac{1}{q}} : |\tau| \leq 1, \tau \text{ is a simple } m\text{-vector} \right\} \\ &\geq \left(\sum_{i=1}^k |\langle w_i, \tau_i \rangle|^q \right)^{\frac{1}{q}} \geq |\langle w_i, \tau_i \rangle| = |w_i|_{\text{com}}. \end{aligned} \tag{A.43}$$

Combining (A.43) with (A.41) and (A.42), we deduce (A.37) and (A.38).

Proof of (A.39). Let us take a vector $v \in \Lambda_{m,\mathbb{R}^k}(\mathbb{R}^d)$ with components $v_1, \dots, v_k \in \Lambda_m(\mathbb{R}^d)$. Let $p \in [1, \infty]$. We have

$$\begin{aligned} |v|_{\text{mass},p} &:= \sup \left\{ \langle w, v \rangle : w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d), |w|_{\text{com},p} \leq 1 \right\} \\ &\geq \sup \left\{ \sum_{i=1}^k \langle w_i, v_i \rangle : w \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d), \|w^p\|_q \leq 1 \right\}, \end{aligned} \tag{A.44}$$

where $w^p := (|w_1|_{\text{com}}, \dots, |w_k|_{\text{com}}) \in \mathbb{R}^k$. In (A.44), we have made use of the inequalities (A.37) and (A.38). By the Hahn-Banach Theorem, for each $i = 1, \dots, k$ there exists $w_i \in \Lambda^m(\mathbb{R}^d)$ such that $|w_i|_{\text{com}} \leq 1$ and

$$\langle w_i, v_i \rangle = |v_i|_{\text{mass}}. \tag{A.45}$$

Now, suppose for a moment that $p < \infty$. Let $\bar{w}_i := \lambda |v_i|_{\text{mass}}^{p-1} w_i$, where $\lambda := \left(\sum_{j=1}^k |v_j|_{\text{mass}}^p \right)^{\frac{1}{p}-1}$. Then, it is not difficult to check that the form $\bar{w} = (\bar{w}_1, \dots, \bar{w}_k)$ satisfies $\|\bar{w}^p\|_q \leq 1$. Moreover, the inequality (A.44) implies

$$\langle \bar{w}, v \rangle = \left(\sum_{j=1}^k |v_j|_{\text{mass}}^p \right)^{\frac{1}{p}} \leq |v|_{\text{mass},p}. \quad (\text{A.46})$$

Moreover, by the definition of mass norm, we obtain that for any $i = 1, \dots, k$

$$|v_i| \leq |v_i|_{\text{mass}}. \quad (\text{A.47})$$

This implies that

$$\left(\sum_{i=1}^k |v_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^k |v_i|_{\text{mass}}^p \right)^{\frac{1}{p}}. \quad (\text{A.48})$$

In case $p = \infty$ we proceed in a similar way. We fix an index $i \in \{1, \dots, k\}$, define $\bar{w}_i := w_i$, $\bar{w}_j := 0$ for $i \neq j$, and $\bar{w} := (\bar{w}_1, \dots, \bar{w}_k)$. From (A.44) and (A.47), we deduce

$$|v_i| \leq |v_i|_{\text{mass}} = \langle \bar{w}, v \rangle \leq |v|_{\text{mass},p} \quad \text{for any } i \in \{1, \dots, k\}. \quad (\text{A.49})$$

In order to complete the proof of (A.39) and (A.40), it only remains to prove the following inequality:

$$|v|_{\text{mass},p} \leq \sum_{i=1}^k |v_i|_{\text{mass}} \quad (\text{A.50})$$

for any $p \in [1, \infty]$. We observe that for any $w = (w_1, \dots, w_k) \in \Lambda_{\mathbb{R}^k}^m(\mathbb{R}^d)$ such that $|w|_{\text{com},p} \leq 1$, there holds

$$\langle w, v \rangle = \sum_{i=1}^k \langle w_i, v_i \rangle \leq \sum_{i=1}^k |v_i|_{\text{mass}}, \quad (\text{A.51})$$

because of (A.43). By taking the supremum over w , (A.50) follows. \square

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