WEAK FUBINI PROPERTY AND INFINITY HARMONIC FUNCTIONS IN RIEMANNIAN AND SUB-RIEMANNIAN MANIFOLDS

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ABSTRACT. We examine the relationship between infinity harmonic functions, absolutely minimizing Lipschitz extensions, strong absolutely minimizing Lipschitz extensions, and absolutely gradient minimizing extensions in Carnot-Carathéodory spaces. Using the weak Fubini property we show that absolutely minimizing Lipschitz extensions are infinity harmonic in any sub-Riemannian manifold.

1. INTRODUCTION

In this paper we study the relationships between absolutely minimizing Lipschitz extensions and infinity harmonic functions in Riemannian manifolds and Carnot-Carathéodory spaces. In the classic Euclidean setting, *infinity harmonic functions* are the viscosity solutions of the infinity Laplace equation

$$0 = \Delta_\infty u = \sum_{i,j=1}^n u_{x_i,x_j} u_{x_i} u_{x_j}.$$

On the other hand, given $Y \subset\subset X \subset \mathbb{R}^n$ and a Lipschitz function $F : Y \to \mathbb{R}$, an *absolutely minimizing Lipschitz extension* (AMLE) of F is a Lipschitz function $u : X \to \mathbb{R}$ such that $u = F$ on Y and the Lipschitz constant of u

$$\mathrm{Lip}(u, U) := \sup_{x,y \in U,\ x \neq y} \frac{|u(x) - u(y)|}{|x - y|}$$

is minimal for any open set $U \subset\subset X \backslash Y$.

We are particularly interested in the case where $Y = \partial\Omega$ and $X = \overline{\Omega}$ for some bounded domain $\Omega \subset \mathbb{R}^n$. If this is the case, the AMLE of g on $\overline{\Omega}$ is exactly the viscosity solution of

$$(1) \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \\ \qquad u = g & \text{on } \partial\Omega; \end{cases}$$

see e.g. [13].

The concept of absolutely minimizing Lipschitz extension makes sense in any metric space; such functions exist and are uniquely determined by their boundary values in any length space (see [22] and [14]). On the other hand, the definition of infinity harmonic functions can be considered in manifolds where we have identified a way to define second order derivatives. The two main examples of the manifolds under consideration are Riemannian spaces and Carnot-Carathéodory (also called sub-Riemannian) spaces, both of which are length spaces endowed with their natural metric.

We recall briefly what sub-Riemannian geometries are (see [4, 17]). Let $\{X_1(x), \ldots, X_m(x)\}$ be a family of smooth vector fields on $\mathbb{R}^n$ (C^1-regularity would be enough for our purposes) and set

$$\mathcal{X}_x := \mathrm{Span}\{X_1(x), \ldots, X_m(x)\} \quad \text{and} \quad \mathcal{X} := \{(x, v) \,|\, x \in \mathbb{R}^n, v \in \mathcal{X}_x\}.$$

Definition 1.1. A *sub-Riemannian structure* in $\mathbb{R}^n$ is a triple

$$(\mathbb{R}^n, \mathcal{X}, \langle \cdot, \cdot \rangle_g),$$

where $\langle \cdot, \cdot \rangle_g$ is a Riemannian metric defined on $\mathcal{X}$ by requiring that the vector fields $\{X_1, \ldots, X_m\}$ are orthonormal.

An absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ is called *horizontal* if $\dot\gamma(t) \in \mathcal{X}_{\gamma(t)}$, a. e. $t \in [0, T]$; i.e. if there exists $\alpha(t) = (\alpha_1(t), \ldots, \alpha_m(t))$ measurable functions such that

$$(2) \qquad \dot\gamma(t) = \sum_{i=1}^{m} \alpha_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in [0, T].$$

We set $|\dot\gamma(t)|_g = \langle \dot\gamma(t), \dot\gamma(t) \rangle_g^{\frac{1}{2}}$ and define the length-functional

$$(3) \qquad l(\gamma) = \int_0^T |\dot\gamma(t)|_g dt = \int_0^T \sqrt{\alpha_1^2(t) + \cdots + \alpha_m^2(t)} \, dt.$$

According to the previous notation we write

$$(4) \qquad |v|_{\mathcal{X}} = \left(\sum_{i=1}^{m} v_i^2 \right)^{1/2}$$

for any horizontal vector $v = \sum_{i=1}^{m} v_i X_i(x) \in \mathcal{X}_x$.

Once defined the length-functional, we can introduce the following distance

$$(5) \qquad d(x, y) := \inf\{l(\gamma) \,|\, \gamma \text{ horizontal curve joining } x \text{ to } y\}.$$

Whenever the Hörmander condition is satisfied (i.e. the Lie algebra associated to $\mathcal{X}$ generates at any point the whole of $\mathbb{R}^n$) this distance is finite, continuous with respect to the Euclidean topology, and minimizing geodesics exist but, in general, they are not unique, even locally.

Several approaches have been used to get the equivalence between infinity harmonic funtions and AMLEs in different settings. We present a metric approach, based on the notion of *strong absolutely minimizing*

Lipschitz extensions (SAMLEs) recently introduced by Juutinen and Shanmugalingam in [15]. In their paper the equivalence between AMLE and SAMLE is proved in spaces where a *weak Fubini property* holds. One of our main results is Theorem 3.4, where we prove that such a property holds in any Carnot-Carathéodory space, thus establishing the equivalence between AMLE and SAMLE in this setting.

The paper is organized as follows. In Section 2 we discuss AMLEs in a general metric setting. We recall the notion of SAMLEs, the weak Fubini property and various results proved in [15]. In Section 3 we prove the validity of the weak Fubini property in general sub-Riemannian manifolds. We first show the result in the particular case of Riemannian manifolds. Then we give the proof for Carnot groups and the general sub-Riemannian case. In Section 4 we recall various definitions and properties regarding the infinity Laplace equation and infinity harmonic functions in sub-Riemannian manifolds. We discuss the notion of absolutely gradient minimizing extensions (AGMEs), which is known to be equivalent to infinity harmonic functions in Riemannian manifolds and Carnot groups (see [3, 25]). However, in general sub-Riemannian manifolds only one implication is known: any AGME is infinity harmonic. We then show that AGMEs are equivalent to SAMLEs which allow us to derive our main result: in a general Carnot-Carathéodory space any AMLE is an infinity harmonic function. A first application of this result is related to the theory of the tug-of-war, introduced by Y. Peres, O. Schramm, S. Sheffield and D.B. Wilson in [22]. They prove that the limit of the values of suitable tug-of-war stochastic games is the unique AMLE of a given Lipschitz function g, in any length space. In particular, they deduce that such a value function is also infinity harmonic in the Euclidean case. By applying our result, we can conclude that this limit of value functions of tug-of-war games is infinity harmonic also in in the sub-Riemannian.

2. AMLEs, SAMLEs and the weak Fubini property.

The problem of finding a Lipschitz extension of some given Lipschitz function $F : Y \subset \mathbb{R}^n \to \mathbb{R}$, has been studied extensively. See for example [2, 8]. Given a metric space (X, d), a proper subset Y of X, and a Lipschitz function $F : Y \to \mathbb{R}$, we are interested in finding a "good" Lipschitz function, extending F to the whole X. Before giving the definitions, we recall that the Lipschitz constant of F in a subset Y is defined as

$$(6) \qquad \mathrm{Lip}(F, Y) = \sup_{\substack{x,y \in Y \\ x \neq y}} \frac{|F(x) - F(y)|}{d(x, y)}.$$

Given a metric space (X, d), a proper subset Y of X and a Lipschitz function $F : Y \to \mathbb{R}$, finding a *minimal Lipschitz extension* of F on X

requires finding a Lipschitz function $u : X \to \mathbb{R}$ such that

$$\mathrm{Lip}(F, Y) = \mathrm{Lip}(u, X).$$

Minimal Lipschitz extensions always exist. McShane and Whitney found the two following minimal extensions (see [16, 26])

$$u_1(x) = \inf_{y \in Y} \big\{ F(y) + \mathrm{Lip}(F, Y) d(x, y) \big\},$$

and

$$u_2(x) = \sup_{y \in Y} \big\{ F(y) - \mathrm{Lip}(F, Y) d(x, y) \big\}.$$

Hence minimal Lipschitz extensions are, in general, not unique.

Definition 2.1. Let (X, d) be a metric space and Y a proper subset of X. Given a Lipschitz function $F : Y \to \mathbb{R}$, we say that $u : X \to \mathbb{R}$ is an *absolutely minimizing Lipschitz extension* (AMLE) of F on X when

    (i) u is a minimal Lipschitz extension of F on X,
    (ii) for any open set $U \subset\subset X \backslash Y$

(7) $$\mathrm{Lip}(u, U) = \mathrm{Lip}(u, \partial U).$$

We say that a function u is AMLE in an open set $\Omega \subset \mathbb{R}^n$ if it is AMLE of its boundary datum, i.e. if $u \in \mathrm{Lip}(\overline{\Omega})$ and (7) holds, for any $U \subset\subset \Omega$.

In the Euclidean setting, it is well-known that it is possible to characterize AMLEs using comparison with cones ([2, 8]). Champion and De Pascale have shown in [7] how to extend this characterization to any length space (X, d), by using suitable metric cones.

Definition 2.2. Let (X, d) be a length space and Ω be a proper and open subset of X. We say that the function $u : \overline{\Omega} \to \mathbb{R}$ satisfies *comparison with metric cones from above* if, for any $z \in X$, $a \in \mathbb{R}$, $b \geq 0$ and $U \subset \Omega$ open, the implication

$$u(x) \leq a + b\, d(x, z) \text{ on } \partial(U \setminus \{z\}) \ \Rightarrow\ u(x) \leq a + b\, d(x, z) \text{ on } \overline{U}$$

holds. We say that the function u satisfies *comparison with metric cones from below* if, for any z, a, b and U as before, we have

$$u(x) \geq a - b\, d(x, z) \text{ on } \partial(U \setminus \{z\}) \ \Rightarrow\ u(x) \geq a - b\, d(x, z) \text{ on } \overline{U}.$$

The function u satisfies *comparison with metric cones* if u satisfies comparisons from above and below. Note that a function u satisfies comparison from below exactly when $-u$ satisfies comparison from above.

Remark 2.3. For any $z \in X$, $b \geq 0$ and $a \in \mathbb{R}$, the function $\varphi(x) = a \pm b\, d(x, z)$ in the above definition is called a *metric cone*.

Theorem 2.4 ([7]). *Let (X, d) be a length space and $\Omega \subset \mathbb{R}^n$ open and bounded, then $u : \overline{\Omega} \to \mathbb{R}$ is an AMLE if and only if u satisfies comparison with metric cones in Ω.*

Recall that a length space is a metric space where the distance between any pair of points is the infimum of the lengths of continuous arcs joining them. Riemannian and sub-Riemannian manifolds are length spaces.

The existence of AMLEs in general length spaces was first shown by Juutinen in [14], while the uniqueness was proved by Peres, Schramm, Sheffield and Wilson in [22], by using the fact that AMLEs arise as limits of value functions of random tug-of-war games when the step tends to zero. In particular, they prove the following

Theorem 2.5 ([22])**.** *Let (X, d) be a length space and Y be a proper subset of X. For any given Lipschitz function $F : Y \to \mathbb{R}$, there exists a unique AMLE of F on X.*

Next, we introduce the notion of strong absolutely minimizing Lipschitz extensions. Let U be an open subset in a metric measure space (X, d, μ) and $F : U \to \mathbb{R}$ be a function. For a fixed point $x \in U$, the local Lipschitz constant of F at the point x is given by

$$(8) \qquad \text{Lip } F(x) = \lim_{R \to 0^+} \sup_{y \in B_R(x) \setminus \{x\}} \frac{|F(x) - F(y)|}{d(x, y)},$$

where $B_R(x)$ is the ball of radius R and center x. The local Lipschitz constant of a function F on a open set $U \subset X$ is defined as

$$(9) \qquad \widehat{\text{Lip}}\,(F, U) = \mu\text{-ess} \sup_{x \in U} \text{Lip } F(x).$$

Here, μ denotes the "natural" measure on the space X. For Riemannian manifolds, this has to be understood as the volume measure induced by the Riemannian structure; when dealing with a sub-Riemannian structure defined on $\mathbb{R}^n$, μ will instead denote the Lebesgue measure $\mathcal{L}^n$. The measure μ will always be understood and frequently omitted.

Definition 2.6. Let (X, d, μ) be a metric measure space. Given a Lipschitz function $F \colon Y \to \mathbb{R}^n$ with $Y \subset X$, we say that a Lispchitz function $u \colon X \to \mathbb{R}^n$ is a *strong absolutely minimizing Lipschitz extension* (SAMLE) of F on X if $u = F$ on Y and, for any $U \subset\subset X \backslash Y$ and for any $v \in \text{Lip}(\overline{U})$ with $u = v$ on ∂U, we have

$$\widehat{\text{Lip}}(u, U) \leq \widehat{\text{Lip}}(v, U).$$

Note that we always have $\widehat{\text{Lip}}(u, U) \leq \text{Lip}(u, U)$. The opposite inequality depends on the structure of the set. In fact, if U is not geodesically convex, it is possible to give examples where $\widehat{\text{Lip}}(u, U) < \text{Lip}(u, U)$ (see Ex. 4.6 below).

Nevertheless, in many cases there is no difference between minimizing $\widehat{\text{Lip}}(u, U)$ or $\text{Lip}(u, U)$. To study when this is the case we need to

introduce the weak Fubini property, which is the main object under investigation in this paper.

Consider a metric measure space (X, d, μ). Given a family Γ of curves in X, the set of admissible metrics is

$$M(\Gamma) = \{\rho \colon \rho \geq 0 \text{ is Borel and } \int_\gamma \rho \geq 1 \text{ for all } \gamma \in \Gamma\}.$$

The modulus of the curve family Γ is given by

$$\text{Mod } \Gamma = \inf\left\{\int_X \rho \, d\mu \colon \rho \in M(\Gamma)\right\}.$$

Note that if two curve families satisfy $\Gamma_1 \subset \Gamma_2$, then their moduli satisfy $\text{Mod } \Gamma_1 \leq \text{Mod } \Gamma_2$.

Definition 2.7. Let (X, d, μ) be a metric measure space. We say that X satisfies the *weak Fubini property* if there exists $C > 0$ and $\tau_0 > 0$ such that, for any $0 < \tau \leq \tau_0$ and B_1, B_2 balls with

$$d(B_1, B_2) > \tau \max\{\text{diam}(B_1), \text{diam}(B_2)\},$$

we have

$$\text{Mod } \Gamma(B_1, B_2, \tau) > 0,$$

where the family $\Gamma = \Gamma(B_1, B_2, \tau)$ consists of all curves in X, joining B_1 and B_2 and such that $l(\gamma) \leq d(B_1, B_2) + C\tau$.

Remark 2.8. The property, given by Definition 2.7, is called weak Fubini since it can be related to the existence of a Fubini-type decomposition for the measure μ. Let us clarify this in the Euclidean space $\mathbb{R}^n$ endowed with the Lebesgue measure $\mathcal{L}^n$. For some fixed $\tau > 0$ let us consider a segment $\overline{\gamma} \in \Gamma := \Gamma(B_1, B_2, \tau)$ and let U be a neighbourhood of $\overline{\gamma}$ such that $U \subset \cup\{\gamma : \gamma \in \Gamma\}$. Let $\overline{x} \in B_1$ be the first endpoint of $\overline{\gamma}$ and Π be the affine plane through $\overline{x}$ orthogonal to $\overline{\gamma}$; set $V := B_1 \cap \Pi$. Then we can decompose $\mathcal{L}^n$ as $d\mathcal{L}^n\big|_U = d\mathcal{H}^{n-1}\big|_V \oplus dt$ and, by Fubini Theorem, for any $\rho \in M(\Gamma)$ it holds

$$\int_X \rho \, d\mathcal{L}^n \geq \int_U \rho \, d\mathcal{L}^n = \int_V \left(\int_{\gamma_x} \rho \, dt\right) d\mathcal{H}^{n-1}(x) \geq \mathcal{H}^{n-1}(V) > 0,$$

where γ_x denotes the segment (belonging to Γ) parallel to $\overline{\gamma}$ and with endpoint $x \in V$. Hence the weak Fubini property holds.

Using the weak Fubini property, Juutinen and Shanmugalingam proved the following equivalence.

Theorem 2.9 ([15])**.** *Let (X, d, μ) be a metric measure space. If (X, d) is a length space and the weak Fubini property holds, then u is an AMLE if and only if u is a SAMLE.*

Our goal is to prove that the weak Fubini property holds in any sub-Riemannian manifold, so that Theorem 2.9 applies to such structures.

3. The weak Fubini property in sub-Riemannian manifolds.

The proof of the validity of the weak Fubini property in general sub-Riemannian manifolds is somewhat technical. We show the result first in the particular cases of Riemannian manifolds and Carnot groups, where the proofs are easier.

Theorem 3.1. *Let (M, d, μ) be a n-dimensional Riemannian manifold endowed with the geodesic distance d and the natural volume measure μ. Then the weak Fubini property holds.*

Proof. Set $C = 1$ and fix $\tau > 0$. Consider B_1, B_2 as in Definition 2.7. Fix $x \in B_1$ and $y \in B_2$ such that $d(x, y) \leq d(B_1, B_2) + \tau/2$. Write $d = d(x, y)$ and consider a quasi-geodesic $\gamma : [0, d + \epsilon] \to M$, parametrized by arc length, with $\gamma(0) = x$ and $\gamma(d + \epsilon) = y$. We may assume that $\operatorname{Im} \gamma \subset M$ is an embedded sub-manifold and that $l(\gamma) = d + \epsilon \leq d(B_1, B_2) + \tau/2$.

Our strategy is to build a suitable subfamily A of all the admissible curves $\Gamma := \Gamma(B_1, B_2, \tau)$ and show that $\operatorname{Mod}(A) > 0$. Denote by $N\gamma$ the normal bundle (with rank $n - 1$) to the sub-manifold γ

$$N\gamma = \{(\gamma(t), w) \,|\, w \in N_{\gamma(t)}\gamma \subset T_{\gamma(t)}M, \ t \in [0, d + \epsilon]\}$$

and consider the exponential map

$$\begin{aligned} \exp \ : \quad N\gamma &\mapsto \mathbb{R}^n \\ (\gamma(t), w) &\mapsto \exp(w)(\gamma(t)) \,. \end{aligned}$$

It is well known (see e.g. [20, Ch. 7, Prop. 26]) that γ has a normal neighbourhood U in M; i.e. $\exp : Z \to U$ is a diffeomorphism between U and an open set $Z \subset N\gamma$ with the property that $(\gamma(t), 0) \in Z$ for any t. It is not restrictive to assume that Z is diffeomorphic to $[0, d + \epsilon] \times V$ for some open set $V \subset \mathbb{R}^{n-1}$. Denote by $I : [0, d + \epsilon] \times V \to Z$ such a diffeomorphism, which we can assume to satisfy $I(t, 0) = (\gamma(t), 0)$. The composition

$$F = \exp \circ I : [0, d + \epsilon] \times V \to U$$

is a diffeomorphism and satisfies $F(t, 0) = \exp(\gamma(t), 0) = \gamma(t)$. Without loss of generality, we may assume that

$$\inf \{|\det JF(t, v)| : (t, v) \in [0, d + \epsilon] \times V\} = k > 0,$$

where by JF we mean the Jacobian-matrix of F with respect to (t, v). For any $v \in V$ define the curve $\gamma_v(t) = F(t, v)$. Recall that $\gamma_0 = F(\cdot, 0) = \gamma$. The subfamily of all the admissible curves we are going to consider is $A = \{\gamma_v := F(\cdot, v) \,|\, v \in V\} \subset \Gamma$. In fact, possibly restricting V and U, we can assume that

$$l(\gamma_v) \leq l(\gamma) + \frac{\tau}{2}, \quad \gamma_v(0) \in B_1 \quad \text{and} \quad \gamma_v(d + \epsilon) \in B_2,$$

so that $\gamma_v \in \Gamma$ for any $v \in V$. Since $|\dot\gamma_0(t)| = |\dot\gamma(t)| = 1$, we can always choose V sufficiently small in order to have $|\dot\gamma_v(t)| \le 2$ for any $(t, v) \in [0, d + \epsilon] \times V$ (in fact $F(t, v)$ is a diffeomorphism, so $\dot\gamma_v \to \dot\gamma$, as $v \to 0$).

Consider a Borel positive function $\rho : M \to \mathbb{R}$ with $\int_\gamma \rho \ge 1$ for any $\gamma \in \Gamma$. A change of variable gives

$$\int_M \rho\, d\mu \ge \int_U \rho\, d\mu$$
$$= \int_{[0,d+\epsilon]\times V} \rho(F(t,v))|\det JF(t,v)|\, d\mathcal{L}^{n-1}(v)\, dt$$
$$\ge k \int_V \left(\int_0^{d+\epsilon} \rho(\gamma_v(t))dt \right) d\mathcal{L}^{n-1}(v).$$

Being $\gamma_v \in \Gamma$ we can observe that

$$1 \le \int_{\gamma_v} \rho = \int_0^{d+\epsilon} \rho(\gamma_v(t))|\dot\gamma_v(t)|dt \le 2 \int_0^{d+\epsilon} \rho(\gamma_v(t))\, dt$$

and conclude

$$\text{Mod } (A) = \inf \left\{ \int_M \rho\, d\mu \mid \rho \text{ admissible for } A \right\} \ge \frac{k}{2} \mathcal{L}^{n-1}(V) > 0,$$

since V is open and non empty in $\mathbb{R}^{n-1}$. Hence Mod $(\Gamma) > 0$, as desired. $\qquad\square$

To show that the weak Fubini property holds in Carnot groups we extend the previous argument. We briefly recall that a Carnot group $\mathbb{G}$ is a Carnot-Carathéodory space endowed with a group operation $\cdot$ and a one-parameter group of dilations $(\delta_s)_{s\ge0}$ such that

$$\|\delta_s(x)\|_{CC} = s \|x\|_{CC}, \quad \text{for any } s \ge 0,$$

where $\|x\|_{CC} = d(x, 0)$ is the Carnot-Carathéodory norm defined on $\mathbb{G}$. We recall also that a Carnot group of finite step k has a stratified Lie algebra g with grading $g = \oplus_{i=1}^k V_i$. Since k is finite, the exponential map between g and $\mathbb{G}$ is always assumed to be the identity (after possibly applying an appropriate diffeomorphism). Vector fields in V_i have order i. The full gradient of a function u therefore consists of derivatives of different orders. The horizontal gradient is the projection of the gradient onto V_1. We shall denote the second-order terms of the gradient by Tu and we note that it is the projection of the gradient onto V_2. For more complete details concerning Carnot groups, we address the interested reader to [12] and the references therein.

Next we show the validity of the weak Fubini property in Carnot groups. The proof will serve as the basis for the sub-Riemannian case, which is necessarily more technical.

We begin with a technical lemma, which will also be a keypoint in the proof of the general result.

Lemma 3.2. *Let $\mathcal{X} = \{X_1(x), \ldots, X_m(x)\}$ be a family of smooth vector fields in $\mathbb{R}^n$ satisfying the Hörmander condition and let B_1, B_2 be balls in $\mathbb{R}^n$ with positive distance. For any $\tau > 0$ there esists a smooth horizontal curve γ between $x \in B_1$ and $y \in B_2$ such that*

$$l(\gamma) \le d(B_1, B_2) + \frac{\tau}{2}.$$

Proof. By the Hörmander condition, there exists an horizontal curve $\gamma_0 : [0, T] \to \mathbb{R}^n$ between $\overline{B_1}$ and $\overline{B_2}$, which realizes the distance. Note that in general $\gamma_0(0) \in \partial B_1$ and $\gamma_0(T) \in \partial B_2$. Nevertheless, since the distance is continuous, we can always find a new horizontal curve $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) \in B_1$, $\gamma(T) \in B_2$ and $l(\gamma) \le d(B_1, B_2) + \frac{\tau}{4}$. Let us now consider such a quasi-geodesic γ between B_1 and B_2. Since γ is horizontal, then

$$\dot{\gamma}(t) = \sum_{i=1}^m \alpha_i(t) X_i(\gamma(t)),$$

for some measurable function $\alpha(t) = (\alpha_1(t), \ldots, \alpha_m(t)) \in \mathbb{R}^m$.
Up to a reparametrization, we can assume that $\alpha_i \in L^\infty([0, T])$. For any $\tau > 0$, we define a family of horizontal smooth curves γ^τ by

$$\dot{\gamma}^\tau(t) = \sum_{i=1}^m \alpha_i^\tau(t) X_i(\gamma^\tau(t)), \quad \gamma^\tau(0) = \gamma(0),$$

where $C^\infty([0, T]) \ni \alpha^\tau \to \alpha$ in $L^1([0, T])$ as $\tau \to 0$ (the esistence of such $\alpha^\tau(t)$ is standard by convolutions).
In order to conclude, it is enough to prove that $l(\gamma^\tau) \to l(\gamma)$ and $\gamma^\tau(T) \to \gamma(T)$ as $\tau \to 0$.
We first show the convergence of $l(\gamma^\tau)$. In fact,

$$\left| l(\gamma^\tau) - l(\gamma) \right| = \left| \int_0^T \|\alpha^\tau(t)\| \, dt - \int_0^T \|\alpha(t)\| \, dt \right|$$

$$\le \int_0^T \left| \|\alpha^\tau(t)\| - \|\alpha(t)\| \right| dt$$

$$\le \int_0^T \|\alpha^\tau(t) - \alpha(t)\| \, dt \to 0, \quad \text{as } \tau \to 0.$$

The pointwise convergence of the curves can be easily proved by Gronwall's Lemma. Recalling that $\gamma(0) = \gamma^\tau(0)$ for any $\tau > 0$, then

$$|\gamma^\tau(t) - \gamma(t)| \le \sum_{i=1}^m \int_0^t \left| \alpha_i^\tau(s) X_i(\gamma^\tau(s)) - \alpha_i(s) X_i(\gamma(s)) \right| ds$$

$$\le \sum_{i=1}^m \int_0^t \left(\left| \alpha_i^\tau(s) - \alpha_i(s) \right| |X_i(\gamma(s))| + |\alpha_i^\tau(s)| \left| X_i(\gamma^\tau(s)) - X_i(\gamma(s)) \right| \right) ds,$$

where we have added $\pm \sum_{i=1}^{m} \alpha_i^{\tau}(s) X_i(\gamma(s))$. Since α_i and X_i are smooth, they are Lipschitz and bounded in a compact set containing $\mathrm{Im}(\gamma)$. Moreover we may assume that $\|\alpha_i^{\tau}\|_{\infty} \leq \|\alpha_i\|_{\infty}$, bound independent on τ. Therefore we get the following estimates:

$$|\gamma^{\tau}(t) - \gamma(t)| \leq C_1 \sum_{i=1}^{m} \int_0^t |\alpha_i^{\tau}(s) - \alpha_i(s)| ds + C_2 \int_0^t |\gamma^{\tau}(s) - \gamma(s)| ds$$

$$\leq C_1 \sum_{i=1}^{m} \int_0^T |\alpha_i^{\tau}(s) - \alpha_i(s)| ds + C_2 \int_0^t |\gamma^{\tau}(s) - \gamma(s)| ds.$$

Note that $C_1 \sum_{i=1}^{m} \int_0^T |\alpha_i^{\tau}(s) - \alpha_i(s)| ds$ is constant in t, then, by Gronwall's Lemma, we can conclude

$$|\gamma^{\tau}(t) - \gamma(t)| \leq C_1 e^{C_2 T} \sum_{i=1}^{m} \int_0^T |\alpha_i^{\tau}(s) - \alpha_i(s)| ds \to 0, \quad \text{as } \tau \to 0.$$

In particular, we can apply the previous convergence to $t = T$ and deduce that $\gamma^{\tau}(T) \in B_2$ for sufficiently small $\tau > 0$.
This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We can now prove the weak Fubini property in Carnot groups.

Theorem 3.3. *In any Carnot group $\mathbb{G}$, endowed with the associated Carnot-Carathéodory distance $d(x, y)$ and the natural volume measure μ, the weak Fubini property holds.*

Proof. Let $C = 1, \tau > 0$ be fixed and B_1, B_2 as in Definition 2.7. We can consider a *quasi-geodesic* γ as in Lemma 3.2, i.e. γ smooth, horizontal and such that $l(\gamma) \leq d(B_1, B_2) + \frac{\tau}{2}$, with $x = \gamma(0) \in B_1$ and $y = \gamma(T) \in B_2$.

Since $\gamma(t)$ is horizontal and smooth, then there exists a smooth function $\alpha : [0, T] \to \mathbb{R}^m$ with $\alpha(t) = (\alpha_1(t), \ldots, \alpha_m(t))$ such that

$$\dot{\gamma}(t) = \sum_{j=1}^{m} \alpha_j(t) X_j(\gamma(t)), \quad \text{for any } t \in [0, T].$$

We can approximate in L^1 the function α by locally-constant functions, i.e. we can find $h : [0, T] \to \mathbb{R}^m$ such that

   i) There exist values

$$0 = T_0 < T_1 < T_2 < \cdots < T_q = T$$

      such that $h(t) = h^i = (h_1^i, \ldots, h_m^i) \in \mathbb{R}^m$, whenever $t \in (T_{i-1}, T_i]$, with $i = 1, \ldots, q$;

  ii) $h_1^i \neq 0$, for any $i = 1, \ldots, q$. In particular there exists $\delta > 0$ such that $|h_1^i| \geq \delta$, for any $i = 1, \ldots, q$;

iii) if $\eta : [0, T] \to \mathbb{G}$ is the curve defined by $\eta(0) = x$ and

$$\dot{\eta}(t) = \sum_{j=1}^{m} h_j(t) X_j(\eta(t)),$$

then $l(\eta) < l(\gamma) + \frac{\tau}{2} \leq d(B_1, B_2) + \tau$. Moreover, as we have seen in Lemma 3.2, $y' := \eta(T)$ is close to y, which implies $y' \in B_2$.

Recall that we identify $\mathbb{G}$ with $\mathbb{R}^n$ by means of exponential coordinates. It is well known that the set

$$\mathbb{W} := \{x \in \mathbb{R}^n \mid x_1 = 0\}$$

is a maximal subgroup of $\mathbb{G}$ which can be identified with $\mathbb{R}^{n-1}$ via the map $\mathbb{R}^{n-1} \ni w \mapsto (0, w) \in \mathbb{R}^n \equiv \mathbb{G}$. Given a (relatively) open neighborhood $U \subset \mathbb{W} \equiv \mathbb{R}^{n-1}$ of x, we define the map

$$\begin{aligned} \Phi \ : \ & U \times [0, T] \to \mathbb{G} \equiv \mathbb{R}^n \\ & (w, t) \longmapsto (0, w) \cdot \eta(t) \end{aligned}$$

The translated curve $\eta_w(t) := \Phi(w, \eta(t))$ is still smooth on $[0, T]$ and so, if U is sufficiently small, it joins $w \in B_1$ (which is a point close to x) to a point $w \cdot y' \in B_2$ and

$$l(\eta_w) = l(\eta) < d(B_1, B_2) + \tau,$$

for any $w \in \mathbb{W}$. In particular, $\eta_w \in \Gamma := \Gamma(B_1, B_2, \tau)$ for any $w \in U$.

Assume for a moment that

$$|\det J\Phi(w, t)| \geq \delta \text{ for } \mathcal{L}^n\text{-a.e.} (w, t) \in \mathbb{R}^n.$$

Consider a Borel function $\rho \geq 0$ on $\mathbb{G}$ such that $\int_c \rho \geq 1$ for any $c \in \Gamma$. In particular $\int_{\eta_w} \rho \geq 1$ for any $w \in U$ and therefore there exists a measurable function $i : U \to \{1, \ldots, q\}$ such that

$$\int_{\eta_w^i} \rho \geq \frac{1}{q}, \quad \text{for any } w \in U, \qquad \text{where } \eta_w^i := \eta_w\big|_{(T_{i(w)-1}, T_{i(w)}]}.$$

Fix $i_0 \in \{1, \ldots, q\}$ such that $\mathcal{L}^{n-1}(U_{i_0}) > 0$, where $U_{i_0} := \{w \in U \mid i(w) = i_0\}$. Therefore

$$\begin{aligned} \int_{\mathbb{G}} \rho \, d\mu \ &\geq \ \int_{\Phi(U_{i_0} \times (T_{i_0-1}, T_{i_0}))} \rho \, d\mu \\ &= \ \int_{U_{i_0}} \left(\int_{T_{i_0-1}}^{T_{i_0}} |\det J\Phi(w, t)| \rho(\eta_w(t)) \, dt \right) d\mathcal{L}^{n-1}(w) \\ &\geq \ \int_{U_{i_0}} \frac{\delta}{\|\dot{\eta}_w\|_\infty} \left(\int_{T_{i_0-1}}^{T_{i_0}} |\dot{\eta}_w(t)| \rho(\eta_w(t)) \, dt \right) d\mathcal{L}^{n-1}(w) \\ &\geq \ \frac{\delta}{\|\dot{\eta}\|_\infty} \frac{1}{q} \, \mathcal{L}^{n-1}(U_{i_0}) \end{aligned}$$

and this implies that

$$\mathrm{Mod}(\Gamma) \geq \frac{\delta \, \mathcal{L}^{n-1}(U_{i_0})}{q \, \|\dot{\eta}\|_\infty} > 0,$$

as desired.

We still need to prove that $|\det J\Phi(w,t)| \geq \delta$ for $\mathcal{L}^n$-a.e. $(w,t) \in \mathbb{R}^n$. Let $t \in [0,T]$ be fixed. By the definition of Φ, the first coordinates of η and η_w coincide: this is due to the the Baker-Campbell-Hausdorff formula (see e.g. [24]) which implies that, in exponential coordinates, the group law is commutative in the first m coordinates (i.e., the horizontal ones).

Therefore we can consider the map Φ_t defined by

$$\mathbb{R}^{n-1} \equiv \mathbb{W} \supset U \ni w \overset{\Phi_t}{\longmapsto} \Phi(w,t) \in \{\eta_1(t)\} \times \mathbb{R}^{n-1} \equiv \mathbb{R}^{n-1}\,.$$

Since $\frac{\partial}{\partial t}\Phi(w,t) = \dot{\eta}(t) = \sum_{j=1}^m h_j(t)X_j(\eta(t))$, the first component of $\frac{\partial}{\partial t}\Phi(w,t)$ is exactly h_1, because in exponential coordinates the vector fields look as

$$X_1 = (\underbrace{1,0,\ldots,0}_{\in \mathbb{R}^m}, *,\ldots, *) \quad \ldots \quad X_m = (\underbrace{0,\ldots,0,1}_{\in \mathbb{R}^m}, *,\ldots, *).$$

Therefore

$$J\Phi(w,t) = \begin{pmatrix} \mathbf{0} & h^1(t) \\ J\Phi_t(w) & v(t) \end{pmatrix},$$

where $\mathbf{0}$ is the null $1 \times (n-1)$-matrix and $v(t) \in \mathbb{R}^{n-1}$. This implies that $|\det J\Phi| = |h_1(t)| \, |\det J\Phi_t|$.

By assumption (ii), our claim will be proved once we show that

$$\det J\Phi_t = 1.$$

That is easy to verify. In fact, the differential of the right translation by $\eta(t)$ is an $n \times n$ lower triangular matrix M with elements on the diagonal equal to 1. Hence, it is sufficient to observe that $J\Phi_t$ is the $(n-1) \times (n-1)$ minor of M obtained by erasing its first row and its first column to conclude that $\det J\Phi_t = 1$. $\qquad \square$

We conclude this section by proving the weak Fubini property for general sub-Riemannian spaces.

Let $\mathcal{X}$ be a distribution on $\mathbb{R}^n$ satisfying the Hörmander condition. We assume that the $\dim \mathcal{X}_x = m$ is independent from x and that $\mathcal{X}$ is spanned by smooth vector fields $X_1(x), \ldots, X_m(x)$ satisfying the Hörmander condition. Using Lemma 3.2 it is not difficult to see that for open sets $E, F \subset \mathbb{R}^n$ it holds

$$(10) \quad \begin{aligned} d(E,F) = \inf \big\{ l(\gamma) \,\big|\, & \gamma : [0,T] \to \mathbb{R}^n \text{ horizontal and smooth,} \\ & \gamma(0) \in E, \gamma(T) \in F \big\} \end{aligned}$$

where the length $l(\gamma)$ is the given by (3) and d is the Carnot-Carathéodry distance defind in (5).

Theorem 3.4. *Let $\left(\mathbb{R}^n, \mathcal{X}, \langle\cdot,\cdot\rangle_g\right)$ be a sub-Riemannian space endowed with the Carnot-Carathéodory distance defined in* (5) *and the natural volume measure μ; then the weak Fubini property holds.*

Proof. Let $\tau > 0$ be fixed and consider balls B_1, B_2 as in Definition 2.7. By (10) there exists a smooth horizontal curve $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) \in B_1$, $\gamma(T) \in B_2$, with

$$l(\gamma) \le d(B_1, B_2) + \frac{\tau}{2}.$$

We may also assume that γ is parametrized by arclength, so that $T = l(\gamma)$. Since $\dot\gamma$ is smooth and $|\dot\gamma| = 1$, then γ is locally injective.

For the reader's convenience, we divide the proof into several steps. Our strategy is as follows: In Steps 0-3 we construct a family $A = \{\gamma_x\}_{x\in\Sigma_0}$ of horizontal Lipschitz curves $\gamma_x : [0, T] \to \mathbb{R}^n$ which are, in some sense, *parallel* to γ. The space of parameters Σ_0 is an open subset of $\mathbb{R}^{n-1}$ containing the origin. The family of curves A spans a neighbourhood U of γ in $\mathbb{R}^n$. They can be constructed in such a way that $\gamma_0 = \gamma$ and $\gamma_x \in \Gamma := \Gamma(B_1, B_2, \tau)$ for any $x \in \Sigma_0$, i.e. $A \subset \Gamma$ (see Step 4).

We claim that $\operatorname{Mod} A > 0$, which would imply $\operatorname{Mod}\Gamma > 0$. To this aim, let ρ be a positive Borel function on $\mathbb{R}^n$ such that $\int_c \rho \ge 1$ for any curve $c \in \Gamma$; in particular, $\int_{\gamma_x} \rho \ge 1$ for any $\gamma_x \in A$. Our explicit construction will show that the speed of any γ_x is bounded away from 0. This condition will imply that

$$\int_0^T \rho(\gamma_x(s))ds \ge C_1 > 0$$

and, in turn, that $\int_{[0,T]\times\Sigma_0} \rho(\gamma_x(s))d\mathcal{L}^n(s, x) \ge C_2 > 0$. From this information, and the fact that the map $(s, x) \mapsto \gamma_x(s)$ is locally a Lipschitz homeomorphism, it will follow that $\int_{\mathbb{R}^n} \rho \ge C_3 > 0$ for any admissible ρ, i.e. $\operatorname{Mod} A \ge C_3$: this will establish the claim.

Step 0: preliminary considerations.
Let $\varepsilon > 0$ be fixed and sufficiently small, in a sense we will specify later. We claim that there exist

$$0 = s_0 < s_1 < \cdots < s_q < s_{q+1} = T$$

and neighbourhoods $U_i \subset \mathbb{R}^n$ of $\gamma([s_i, s_{i+1}])$, $i = 0, \dots, q$, with the following properties:

- there exists a horizontal vector field $Z_i \in C^\infty(U_i, \mathbb{R}^n)$ such that

(11) $Z_i(\gamma(s)) = \dot\gamma(s)$ for any $s \in [s_i, s_{i+1}]$ and
     $0 < 1 - \varepsilon \le |Z_i|_{\mathcal{X}} \le 1 + \varepsilon$ on U_i,

    (recall that $|v|_{\mathcal{X}}$ is the norm defined in (4));

- there is a C^∞ change of coordinates $(u_1, \ldots, u_n)$ according to which U_i is an open connected subset of $\mathbb{R}^n_{u_1,\ldots,u_n}$ with $\gamma(s_i) = 0$ and $Z_i = \partial_{u_1}$.

We stress in particular that, according to such coordinates, we have

$$\gamma(s) = (s - s_i, 0, \ldots, 0) \qquad \text{for any } s \in [s_i, s_{i+1}].$$

The curves γ_x will be constructed as concatenation of integral lines of (multiples of) the vector fields Z_i (see Steps 1,2,3). We also point out that we do not know a priori that γ is simple, otherwise one could easily prove our claim with $q = 0$, which would make the construction of the curves γ_x much easier.

Let us prove our claim. Since γ is smooth and locally injective, for any $t \in [0, T]$ we can find

- $t' = t'(t), t'' = t''(t)$ such that $t' < t''$ and $0 \le t' \le t \le t'' \le T$
- a neighbourhood U_t of $\gamma(t)$

with the following properties:

- $\gamma_{|[t',t'']}$ is simple and $\gamma([t', t'']) \subset U_t$;
- there exists a horizontal vector field $Z_t \in C^\infty(U_t, \mathbb{R}^n)$ extending the horizontal C^∞ vector $\dot\gamma$, that is, $Z_t(\gamma(s)) = \dot\gamma(s)$ for any $s \in [t', t'']$;
- there exists $\varepsilon > 0$ such that $1 - \varepsilon < |Z_t|_{\mathcal{X}} < 1 + \varepsilon$ on U_t (since $|\dot\gamma| = 1$, i.e. $|Z_t|_{\mathcal{X}} = 1$ on $\gamma([t', t''])$, it will be enough to take U_t sufficiently small);
- there is a C^∞ change of coordinates according to which U_t is an open connected subset of $\mathbb{R}^n_{u_1,\ldots,u_n}$ with $\gamma(t') = 0$ and $Z_t = \partial_{u_1}$ (this follows from the Rectifiability Theorem for ODEs, see e.g. [1]).

In particular, according to such coordinates, we have

$$\gamma(s) = (s - t', 0, \ldots, 0) \qquad \text{for any } s \in [t', t''].$$

By compactness of $[0, T]$ and the fact that none of the intervals $[t', t'']$ has empty interior, there exists a finite set $\{t_i\}_{i=0}^q \subset [0, T]$ such that

$$[0, T] \subset \bigcup_{i=0}^q [t'_i, t''_i] \quad \text{where } t'_i := t'(t_i), t''_i := t''(t_i) \,.$$

We may assume that the covering $\{[t'_i, t''_i]\}_i$ is minimal, in the sense that there exists no couple i, j with $i \ne j$ and such that $[t'_i, t''_i] \subset [t'_j, t''_j]$. In particular $t'_i \ne t'_j$, $t''_i \ne t''_j$ whenever $i \ne j$; moreover, the implication

$$(12) \qquad\qquad t'_i < t'_j \Longrightarrow t''_i < t''_j$$

holds. Let our intervals be ordered so that $i < j \Rightarrow t'_i < t'_j$; one must have

$$(13) \qquad\qquad t'_{i+1} \le t''_i \qquad \text{for any } i,$$

otherwise
$$(t_i'', t_{i+1}') \cap [t_j', t_j''] = \emptyset \text{ for any } j = 0, \ldots, q$$
(it is sufficient to consider separately the cases $j \leq i$ and $j > i$ and to use (12)). This would lead to a contradiction since $(t_i'', t_{i+1}') \not\subset \cup_j [t_j', t_j'']$.

Let us set $s_i := t_i'$, $i = 0, \ldots, q$ and $s_{q+1} := T = t_q''$; we have by (13)
$$[s_i, s_{i+1}] = [t_i', t_{i+1}'] \subset [t_i', t_i'']$$
and our claim follows by setting $U_i := U_{t_i}$ and $Z_i := Z_{t_i}$.

Step 1: construction of the horizontal curves γ_x on $[0, s_1]$.
Let us consider $\gamma_{|[0,s_1]}$ in the coordinate chart $(u_1, \ldots, u_n)$ for U_0 given by Step 0: such curve appears as the path $s \mapsto (s, 0, \ldots, 0)$; moreover, $\gamma(0) = 0$ and $Z_0 = \partial_{u_1}$.
We can fix an open set $\Sigma_0 \subset \{u_1 = 0\} \equiv \mathbb{R}^{n-1}$ with $\gamma(0) = 0 \in \Sigma_0$; we may assume that $[0, s_1] \times \Sigma_0 \subset\subset U_0$. Therefore the map
$$\gamma_x(s) := \exp(s Z_0)(x) = (s, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$$
is well defined for $s \in [0, s_1]$, $x \in \Sigma_0$ and, since $\dot\gamma_x = \partial_{u_1} = Z_0$, it is a horizontal curve in U_0 with $1 - \varepsilon < |\dot\gamma_x|_{\mathcal{X}} < 1 + \varepsilon$. Define
$$\Sigma_1 := \{s_1\} \times \Sigma_0 = \bigcup_{x \in \Sigma_0} \gamma_x(s_1).$$

Step 2: extension of γ_x to $[s_1, s_2]$.
Possibly restricting Σ_0, the C^∞ hypersurface Σ_1 is contained in U_1. Σ_1 is transversal to $Z_1 = \partial_{u_1}$ at $\gamma(s_1) = 0 \in \Sigma_1$ (where $u_1, \ldots, u_n$ are now the coordinates chosen for U_1). Possibly considering a smaller Σ_0, Σ_1 is then a C^∞ graph over the hyperplane $\{u_1 = 0\}$, i.e.
$$\Sigma_1 = \{(\phi_1(y), y) : y \in V_1\}$$
for some C^∞ real map ϕ_1 defined on an open subset $V_1 \subset \{u_1 = 0\}$. We have $\phi_1(0) = 0$ and, after restricting Σ_0, we may also assume $|\phi_1| < \delta$, for some δ to be chosen later.
We are going to extend the curves $\{\gamma_x\}_{x \in \Sigma_0}$ to $[0, s_2]$. We have $\gamma_x(s_1) = (\phi_1(v_x^1), v_x^1) \in \Sigma_1$ for some $v_x^1 \in V_1$. We set
$$\gamma_x(s) := \left(\phi_1(v_x^1) + \frac{s_2 - s_1 - \phi_1(v_x^1)}{s_2 - s_1}(s - s_1), v_x^1 \right), \quad s \in [s_1, s_2].$$

In other words, $\gamma_{x|[s_1,s_2]}$ is a parametrization of the segment joining $(\phi(v_x^1), v_x^1)$ and $(s_2 - s_1, v_x^1)$; moreover, it is a horizontal curve because its derivative is a multiple of $\partial_{u_1} = Z_1$. It is clear that, possibly restricting Σ_0, $\gamma_x(s)$ is well defined for any $x \in \Sigma_0, s \in [0, s_2]$ and is a horizontal curve in U_1. Set
$$\Sigma_2 := \{s_2 - s_1\} \times V_1 = \bigcup_{x \in \Sigma_0} \gamma_x(s_2)$$
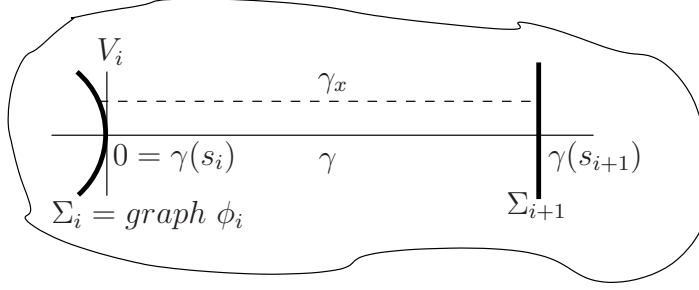
FIGURE 1. Construction of γ_x in the chart (U_i, ψ_i)
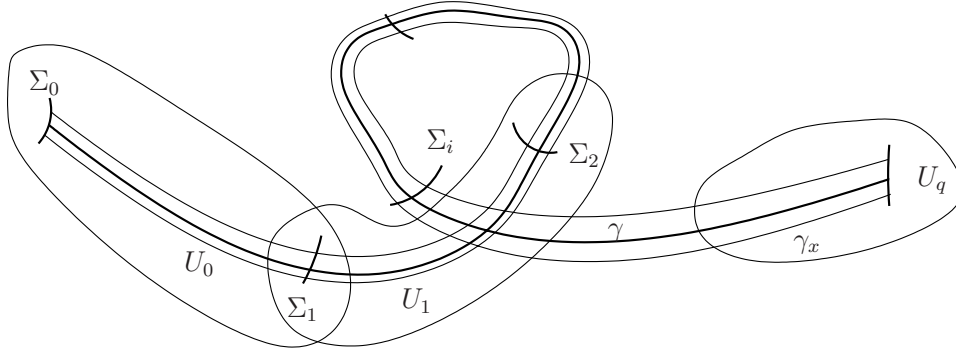
FIGURE 2. The curves γ_x span a tubolar neighbourhood of γ

to be the C^∞ surface consisting of the endpoints $\gamma_x(s_2)$: in the chart for U_1, Σ_2 appears as a portion of an hyperplane.

Step 3: extension of γ_x to the whole $[0, T]$.
As in Step 2 one can prove that, in the coordinate chart for U_2 given by Step 0, the surface Σ_2 is the graph of a C^∞ map $\phi_2 : V_2 \to \mathbb{R}$ with $|\phi_2| < \delta$; in doing so, it could be necessary to reduce Σ_0 and/or δ. We have $\gamma_x(s_2) = (\phi_2(v_x^2), v_x^2)$ for some $v_x^2 \in V_2$. We extend γ_x by

$$\gamma_x(s) := \left(\phi_2(v_x^2) + \frac{s_3 - s_2 - \phi_2(v_x^2)}{s_3 - s_2}(s - s_2), v_x^2 \right), \quad s \in [s_2, s_3]$$

and set $\Sigma_3 := \bigcup_{x \in \Sigma_0} \gamma_x(s_3) = \{s_3 - s_2\} \times V_2$.
By induction on i we end up with a horizontal curve $\gamma_x : [0, T] \to \mathbb{R}^n$ defined for $x \in \Sigma_0$. The endpoints of each γ_x are, respectively, a point of Σ_0 and a point lying on a C^∞ surface $\Sigma_{q+1} \subset U_q$.

Step 4: each γ_x belongs to Γ.
Again restricting Σ_0, we may assume that $\Sigma_0 \subset B_1$ and $\Sigma_{q+1} \subset B_2$, so that

$$\gamma_x(0) \in B_1 \text{ and } \gamma_x(T) \in B_2 \text{ for any } x \in \Sigma_0.$$

If δ is sufficiently small, for any $x \in \Sigma_0$ and $s \in [s_i, s_{i+1}]$ there exists $v_x \in V_i$ such that

$$|\dot{\gamma}_x(s)|_{\mathcal{X}} = \left| \frac{s_{i+1} - s_i - \phi_i(v_x)}{s_{i+1} - s_i} \right| |\partial_{u_1^i}|_{\mathcal{X}} \leq 1 + 2\varepsilon,$$

because, by (11), $|\partial_{u_i^i}|_{\mathcal{X}} = |Z_i| \leq 1 + \varepsilon$; here, $u_1^i, \ldots, u_n^i$ denote the coordinates for U_i given by Step 0. Therefore

$$l(\gamma_x) \leq (1 + 2\varepsilon)T \leq (1 + 2\varepsilon) \left(d(B_1, B_2) + \frac{\tau}{2} \right)$$

and then $l(\gamma_x) \leq d(B_1, B_2) + \tau$ provided ε is sufficiently small (depending only on $d(B_1, B_2)$ and τ). In particular, $\gamma_x \in \Gamma$ (with $C = 1$).

Step 5: the Jacobian of the map $(s, x) \mapsto \gamma_x(s)$.
By construction, the Lipschitz map

$$[0, T] \times \Sigma_0 \ni (s, x) \xmapsto{F} \gamma_x(s) \in \mathbb{R}^n$$

is locally a diffeomorphism; it is not necessarily injective because, for example, γ could have self-intersections. Nevertheless, for any i the map $F_{|[s_i, s_{i+1}] \times \Sigma_0}$ is a diffeomorphism (onto its image). Possibly restricting Σ_0 we can suppose that

$$C := \inf\{|\det JF(s, x)| : (s, x) \in [0, T] \times \Sigma_0\} > 0.$$

Step 6: conclusion.
Let ρ be a positive Borel function on $\mathbb{R}^n$ with $\int_c \rho \geq 1$ for any $c \in \Gamma$. Since $\int_{\gamma_x} \rho \geq 1$ for any $x \in \Sigma_0$, there exists a Borel map $I : \Sigma_0 \to \{0, \ldots, q\}$ such that

$$\int_{\gamma_x|[s_{I(x)}, s_{I(x)+1}]} \rho \geq \frac{1}{q+1}.$$

In particular there exists $j \in \{0, 1, \ldots, q\}$ such that

$$\mathcal{L}^{n-1}(\{x \in \Sigma_0 : I(x) = j\}) \geq \tfrac{1}{q+1} \mathcal{L}^{n-1}(\Sigma_0)$$

We have therefore

$$\int_{\mathbb{R}^n} \rho \, d\mu \geq \int_{F([s_j, s_{j+1}] \times \Sigma_0)} \rho \, d\mu$$

$$= \int_{[s_j, s_{j+1}] \times \Sigma_0} |\det JF(s, x)| \rho(F(s, x)) \, d\mathcal{L}^n(s, x),$$

where we have used a change of variable. It follows that

$$
\int_{\mathbb{R}^n} \rho \, d\mu \geq C \int_{\Sigma_0} \int_{s_j}^{s_{j+1}} \rho(\gamma_x(s)) \, ds \, dx
$$

$$
\geq \frac{C}{1+2\varepsilon} \int_{\Sigma_0} \int_{s_j}^{s_{j+1}} \rho(\gamma_x(s))|\dot{\gamma}_x(s)|_{\mathcal{X}} \, ds \, dx
$$

$$
\geq \frac{C}{1+2\varepsilon} \int_{\{x\in\Sigma_0 : I(x)=j\}} \left(\int_{\gamma_x} \rho \right) dx
$$

$$
\geq \frac{C}{1+2\varepsilon} \cdot \frac{1}{q+1} \cdot \mathcal{L}^{n-1}(\Sigma_0) \cdot \frac{1}{q+1} > 0
$$

and then $\operatorname{Mod} A > 0$ with $A = \{\gamma_x\}_{x\in\Sigma_0}$, as claimed. $\qquad\square$

Therefore, by Theorems 2.9 and the weak Fubini property, we can conclude the following equivalence.

Corollary 3.5. *In Riemannian and sub-Riemannian manifolds, endowed with their natural volume measure, AMLEs and SAMLEs are equivalent notions.*

We also state the following corollary, which will be of use later.

Corollary 3.6. *Let $\wp = \wp(x)$ be a property holding for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ and let $x_0, y_0 \in \mathbb{R}^n$ be fixed. Then, for any $\varepsilon > 0$ there exist two points $x_\varepsilon \in B_\varepsilon(x_0)$, $y_\varepsilon \in B_\varepsilon(y_0)$ and a horizontal curve $\gamma : [0, T] \to \mathbb{R}^n$ with $\gamma(0) = x_\varepsilon$ and $\gamma(T) = y_\varepsilon$ such that $l(\gamma) < d(x_0, y_0) + \varepsilon$ and $\wp$ holds at $\gamma(t)$ for $\mathcal{L}^1$-a.e. $t \in [0, T]$.*

Proof. Consider $B_1 := B_\varepsilon(x_0)$, $B_2 := B_\varepsilon(y_0)$. By repeating the construction given in Theorem 3.4 (with $\tau := \varepsilon$) we can find $T > 0$ and

- a C^∞ hypersurface $\Sigma_0 \subset B_\varepsilon(x_0)$;
- $0 = s_0 < s_1 < \cdots < s_q < s_{q+1} = T$;
- a Lipschitz continuous map $F : [0, T] \times \Sigma_0 \to \mathbb{R}^n$

such that

- for any i, F is a C^∞ diffeomorphism from $([s_i, s_{i+1}] \times \Sigma_0)$ onto its image;
- for any $x \in \Sigma_0$, the curve $\gamma_x := F(\cdot, x)$ is horizontal, $\gamma_x(0) \in B_\varepsilon(x_0)$ and $\gamma_x(T) \in B_\varepsilon(y_0)$;
- the length of each γ_x is less than $d(x_0, y_0) + \varepsilon$.

Let (U_i, ψ_i) be the charts utilized in the same construction and recall that, for any $x \in \Sigma_0$ and any i the curve $\gamma_{x|[s_i, s_{i+1}]}$ is a straigth horizontal segment according to the coordinates (U_i, ψ_i). By the classical Fubini theorem, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_0$ the property $\wp$ holds at $\gamma_x(t)$ for $\mathcal{L}^1$-a.e. $t \in [s_i, s_{i+1}]$. As a consequence, for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_0$ the property $\wp$ holds at $\gamma_x(t)$ for $\mathcal{L}^1$-a.e. $t \in [0, T]$, and this concludes the proof. $\qquad\square$

Remark 3.7. A better insight of the proof of Corollary 3.6 shows that there are *a lot* of horizontal curves γ as in the statement.

4. Equivalence between absolutely minimizing Lipschitz extensions and infinity harmonic functions.

As we have already pointed out in the introduction, it is known that, in Euclidean spaces, AMLEs and infinity harmonic functions are equivalent notions. We are going to introduce the corresponding horizontal operators and to investigate the same relation in Riemannian and sub-Riemannian manifolds. As usual, $\mathcal{X} = \{X_1(x), \ldots, X_m(x)\}$ denotes a family of smooth vector fields in $\mathbb{R}^n$ satisfying the Hörmander condition and d is the associated Carnot-Carathéodory distance.

Definition 4.1. We define the *horizontal gradient* of $u : \mathbb{R}^n \to \mathbb{R}$ as

$$D_{\mathcal{X}}u := \big(X_1 u, \ldots, X_m u\big)$$

and the *symmetrized matrix of horizontal second order derivatives* of u as

$$\big(D^2_{\mathcal{X}}u\big)^{\star}_{i,j} := \frac{1}{2}\left(X_i X_j u + X_j X_i u\right), \quad \text{for} \quad i,j = 1, \ldots, m.$$

Using the previous intrinsic differential operators, we introduce the (renormalized) infinity Laplace operator by

$$\Delta_{\infty,\mathcal{X}}u := \langle \big(D^2_{\mathcal{X}}u\big)^{\star} D_{\mathcal{X}}u, D_{\mathcal{X}}u \rangle.$$

The equation under consideration is the infinity Laplace equation given by

(14) $$\Delta_{\infty,\mathcal{X}}u = 0.$$

We say that a function u is of class $C^1_{\mathcal{X}}$ if u is continuous and $X_i u$ is continuous for $i = 1, \ldots, m$. Analogously, $u \in C^k_{\mathcal{X}}$ with $k \geq 1$ if u is continuous with its first k horizontal derivatives. Of course, we do not specify if such continuity is w.r.t. the metric d or the usual Euclidean one because, under the Hörmander condition, the topology induced by d is equivalent to the Euclidean one.

Definition 4.2. We call *(horizontal) infinity harmonic function* any viscosity solution of (14); that is, an infinity harmonic function u is a continuous function such that

(1) if $\phi \in C^2_{\mathcal{X}}$ is such that $u - \phi$ has a local maximum at x_0, then $-\Delta_{\infty,\mathcal{X}}\phi(x_0) \leq 0$ (subsolution property).
(2) if $\phi \in C^2_{\mathcal{X}}$ is such that $u - \phi$ has a local minimum at x_0, then $-\Delta_{\infty,\mathcal{X}}\psi(x_0) \geq 0$ (supersolution property).

Our aim is that of understanding whether, given a bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in \text{Lip}(\partial\Omega, d)$, a function u is the AMLE of g on $\overline{\Omega}$ if and only if it is infinity harmonic on Ω with $u = g$ on $\partial\Omega$. In order to obtain this equivalence, we first need to introduce the concept of absolutely

gradient minimizing extension. In fact, it is well know that, at least in the Euclidean case, the infinity Laplacian is the Aronsson equation of the L^∞-norm of the gradient; in particular, infinity harmonic functions are those functions minimizing such a norm.

As common in the literature, we hereafter denote by $W^{1,\infty}_{\mathcal{X}}(\Omega)$ the space of real functions on Ω whose first-order horizontal derivatives are essentially bounded.

Definition 4.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $g \in \mathrm{Lip}(\partial\Omega, d)$. We say that $u \in W^{1,\infty}_{\mathcal{X}}(\Omega)$ is an *absolutely gradient minimizing extension* (AGME) of g on $\overline{\Omega}$ if

(1) $u = g$ on $\partial\Omega$
(2) for any $U \subset\subset \Omega$ open, if $v \in W^{1,\infty}_0(U)$ with $v = u$ on ∂U, then

$$\|D_{\mathcal{X}}u\|_{\infty,U} \leq \|D_{\mathcal{X}}v\|_{\infty,U}.$$

The following result was proved by C. Wang [25].

Theorem 4.4. *If u is AGME in a Carnot-Carathéodory space, then u is a viscosity solution of* (14).

Still in [25], the author proved also that, in any Carnot group, the viscosity solutions of

$$(15) \qquad \begin{cases} \Delta_{\infty,\mathcal{X}}u = 0, & \text{on } \Omega \\ \quad\;\; u = g, & \text{on } \partial\Omega \end{cases}$$

are unique. The previous uniqueness result is also known in Riemannian manifolds ([3]) and in Grušin-type spaces ([5]). Uniqueness for the problem (15) implies that the reverse implication (i.e. that infinity harmonic functions are AGMEs) holds, thus establishing the full equivalence between AGMEs and infinity harmonic functions.

We can summarize the previous results in the following corollary.

Corollary 4.5 ([25], [3], [5]). *In Riemanian manifolds, Carnot groups and Grušin-type spaces, a function is AGME if and only if it is infinity harmonic.*

Unfortunately, in general Carnot-Carathéorory spaces the uniqueness for problem (15), as well as the implication "infinity harmonic $\Rightarrow$ AGME", are still open problems. Therefore the previous full equivalence is not yet established in the general setting.

We want to study also the relationships between the notions of AMLE and AGME. In the Euclidean setting it is known that $\mathrm{Lip}(u, U)$ equals $\|Du\|_{L^\infty(U)}$ for any Lipschitz function u on a convex set $U \subset \mathbb{R}^n$. If U is not convex only one inequality holds, namely $\mathrm{Lip}(u, U) \geq \|Du\|_{L^\infty(U)}$. The reverse inequality is in general false, as the following example shows.

Example 4.6. Define $\Omega = B_1(0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and let $\overline{B_r}(0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$, with $0 < r \ll 1$ fixed. Consider the open set

$$\widetilde{\Omega} = \Omega \setminus \left(\overline{B_r}(0) \cup \{(x, y) \in \mathbb{R}^2 \mid y = 0, x \leq 0\} \right),$$

then $\widetilde{\Omega}$ is connected and bounded but not convex. Let us consider the "angle" function $u : \widetilde{\Omega} \to \mathbb{R}$ defined by

$$u(x, y) = \begin{cases} \arctan \frac{y}{x} & \text{if } x > 0 \\ \pi + \arctan \frac{y}{x} & \text{if } x < 0 \text{ and } y > 0 \\ -\pi + \arctan \frac{y}{x} & \text{if } x < 0 \text{ and } y < 0 \\ \pi/2 & \text{if } x = 0 \text{ and } y > r \\ -\pi/2 & \text{if } x = 0 \text{ and } y < -r \end{cases}$$

It is easy to see that

$$|Du| = \frac{1}{\sqrt{x^2 + y^2}} < \frac{1}{r}, \quad \text{for any } (x, y) \in \widetilde{\Omega},$$

so that $\|Du\|_{\infty, \widetilde{\Omega}} \leq \frac{1}{r}$.

Now look at points $P = (x_P, y_P), Q = (x_Q, y_Q) \in \widetilde{\Omega}$ such that $x_P = x_Q \approx -1$ and $0 < y_P \ll 1$ and $-1 \ll y_Q < 0$, so that $d(P, Q) < \varepsilon \ll 1$. We have

$$u(P) \approx \pi \quad \text{while} \quad u(Q) \approx -\pi$$

Hence

$$\frac{|u(P) - u(Q)|}{|P - Q|} \approx \frac{2\pi}{\varepsilon}$$

and then $\text{Lip}(u, \widetilde{\Omega}) = +\infty$, whence $\text{Lip}(u, \widetilde{\Omega}) > \|Du\|_{\infty, \widetilde{\Omega}}$.

An analogue result holds for geodesically convex sets in Carnot-Carathéodory spaces; in this setting, however, the geometry of geodesically convex sets is not as well-behaved as in Euclidean spaces. For example, in the Heisenberg group the only geodesically convex sets are the whole space, the empty set and the geodesics themselves [18]. However, let us note that balls are starshaped (i.e. geodesically convex with respect to an interior fixed point) w.r.t. the center in any metric space. This observation motivated us to use the notion of SAMLEs in order to link AMLEs to infinity harmonic functions.

We are going to show that in any Carnot-Carathéodory space the local Lipschitz norm introduced in (9) and the L^∞-norm of the gradient are indeed the same object.

Theorem 4.7. *Let $\left(\mathbb{R}^n, \mathcal{X}, \langle\, , \,\rangle_g\right)$ be a Riemannian or sub-Riemannian manifold and $d(x, y)$ the associated Riemannian or, respectively, Carnot-Carathéodory distance. Let $U \subset \mathbb{R}^n$ be an open set and $u : U \to \mathbb{R}$ be*

a Lipschitz function. Then

$$\widehat{\mathrm{Lip}}\,(u, U) = \|D_{\mathcal{X}} u\|_{\infty, U}\,.$$

Proof. In [10] (see also [11]) it was proved that, under the Hörmander condition, for any open and bounded set $\Omega \subset U$ and for any Lipschitz function $u : U \to \mathbb{R}$ it holds

$$\|D_{\mathcal{X}} u\|_{\infty, \Omega} \leq \mathrm{Lip}(u, \Omega).$$

Using this result in $\Omega = B_R(x_0) \subset U$ and passing to the limit as $R \to 0^+$ we find

$$|D_{\mathcal{X}} u(x)| \leq \mathrm{Lip}\, u(x), \quad \text{a.e. } x \in U.$$

On considering the essential supremum we achieve

$$\|D_{\mathcal{X}} u\|_{\infty, U} \leq \operatorname*{ess\,sup}_{x \in U} \mathrm{Lip}\, u(x) = \widehat{\mathrm{Lip}}\,(u, U).$$

We are left to prove the reverse inequality.

Let $x \in U$ be fixed and $R > 0$ be such that $B_R(x) \subset U$ and fix $y \in B_R(x)$. By Corollary 3.6, for any $\varepsilon > 0$ there exists a horizontal curve $\gamma_\varepsilon : [0, T] \to \mathbb{R}^n$ such that $\gamma_\varepsilon(0) = x_\varepsilon, \gamma_\varepsilon(T) = y_\varepsilon$ and

$$d(x, x_\varepsilon) < \varepsilon, \ d(y, y_\varepsilon) < \varepsilon, \ l(\gamma_\varepsilon) < d(x, y) + 3\varepsilon$$

$$u \text{ is horizontally differentiable at } \gamma_\varepsilon(t) \text{ for } \mathcal{L}^1\text{-a.e. } t \in [0, T].$$

If ε is such that $d(x, y) + 3\varepsilon < R$, then $\mathrm{Im}\, \gamma_\varepsilon \subset U$ and

$$|u(x_\varepsilon) - u(y_\varepsilon)| \leq \left| \int_0^T \frac{d}{dt} u(\gamma_\varepsilon(t)) dt \right| = \left| \int_0^T D_{\mathcal{X}} u(\gamma_\varepsilon(t)) \cdot \dot{\gamma}_\varepsilon(t) dt \right|$$

$$\leq \|D_{\mathcal{X}} u\|_{\infty, B_R(x)} \int_0^T |\dot{\gamma}_\varepsilon(t)| dt = \|D_{\mathcal{X}} u\|_{\infty, B_R(x)}\, l(\gamma_\varepsilon)$$

$$= \|D_{\mathcal{X}} u\|_{\infty, B_R(x)} \big(d(x, y) + 3\varepsilon \big).$$

Taking into account the continuity of u, as $\varepsilon \to 0$ we obtain

$$|u(x) - u(y)| \leq \|D_{\mathcal{X}} u\|_{\infty, B_R(x)}\, d(x, y)$$

Taking the supremum among $y \in B_R(x) \setminus \{x\}$ and passing to the limit as $R \to 0^+$, we can conclude

$$\mathrm{Lip}\, u(x) \leq \|D_{\mathcal{X}} u\|_{\infty, U}, \quad \forall x \in U,$$

i.e.

$$\widehat{\mathrm{Lip}}\,(u, U) = \operatorname*{ess\,sup}_{x \in U} \mathrm{Lip}\, u(x) \leq \|D_{\mathcal{X}} u\|_{\infty, U}$$

which concludes the proof. $\qquad\square$

In the particular case of Carnot group, Taylor approximation allows to prove the identity between $\mathrm{Lip}\, u(x)$ and $|D_{\mathcal{X}} u(x)|$ for a. e x. This is not necessary in order to study the relation between AMLEs and infinity harmonic functions; nevertheless, we decided to include it since it can be useful in different contexts.

The intrinsic Taylor's approximation in Carnot groups was proved in [21]; more precisely, if u is a Lipschitz function on a Carnot group $\mathbb{G}$, then u is horizontally differentiable at a.e. $x \in \mathbb{G}$, i.e.

$$(16) \qquad u(y \cdot x) = u(x) + \langle D_{\mathcal{X}} u(x), \overline{y} \rangle + o(\|y\|_{CC})$$

where $\overline{y}$ denotes the horizontal projection of y onto the horizontal layer.

Proposition 4.8. *Let $\mathbb{G}$ be a Carnot group, $u : \mathbb{G} \to \mathbb{R}$ a Lipschitz function and $x \in \mathbb{G}$ a point such that u is horizontally differentiable at x. Then*

$$(17) \qquad \mathrm{Lip}\, u(x) = |D_{\mathcal{X}} u(x)|.$$

Proof. Let $R > 0$ be fixed and let h be a "horizontal" point (i.e., $\overline{h} = h$), to be chosen later, such that $\|h\|_{CC} = 1$. Let $s \in (0, R)$ and define $y := x \cdot \delta_s(h) \in B_R(x)$. By (16) we have

$$u(y) = u(x) + \langle D_{\mathcal{X}} u(x), \overline{x^{-1} \cdot y} \rangle + o\left(\left\| x^{-1} \cdot y \right\|_{CC} \right)$$
$$= \langle D_{\mathcal{X}} u(x), \overline{\delta_s(h)} \rangle + o\left(\|\delta_s(h)\|_{CC} \right)$$

so that

$$u(y) - u(x) = s \langle D_{\mathcal{X}} u(x), h \rangle + o(s).$$

Since $d(x, y) = \|\delta_s(h)\|_{CC} = s$ we get

$$\frac{u(y) - u(x)}{d(x, y)} = \langle D_{\mathcal{X}} u(x), h \rangle + o(1).$$

After choosing $h := \frac{D_{\mathcal{X}} u(x)}{|D_{\mathcal{X}} u(x)|}$, we obtain

$$\sup_{y \in B_R(x) \setminus \{x_0\}} \frac{u(y) - u(x)}{d(x, y)} \geq |D_{\mathcal{X}} u(x)| + o(1).$$

On passing to the limit as $R \to 0^+$ we conclude

$$(18) \qquad \mathrm{Lip}\, u(x) \geq |D_{\mathcal{X}} u(x)|.$$

In order to get the reverse inequality, we proceed in a similar way. We write the first order Taylor's approximation at a point $y = x \cdot \delta_s(h)$ with $\|h\|_{CC} = 1$ but h not necessarily horizontal. We then have

$$u(y) - u(x) = s \langle D_{\mathcal{X}} u(x), \overline{h} \rangle + o(s).$$

Using the Cauchy-Schwartz inequality we find

$$\frac{|u(y) - u(x)|}{d(y, x)} \leq |D_{\mathcal{X}} u(x)| |\overline{h}| + o(1).$$

Note that $|\overline{h}| \leq \|h\|_{CC} = 1$ and so

$$\frac{|u(y) - u(x)|}{d(y, x)} \leq |D_{\mathcal{X}} u(x)| + o(1),$$

for any $y \in B_R(x)$. Taking the supremum and then passing to the limit as $R \to 0^+$, we can conclude

$$(19) \qquad \qquad \text{Lip}\, u(x) \leq |D_{\mathcal{X}} u(x)|.$$

Inequalities (18) and (19) are enough to conclude the proof of the Theorem. $\qquad \square$

Remark 4.9. Recall that, whenever u is Lipschitz w.r.t. the Carnot-Carathédory distance in some bounded set $\Omega \subset \mathbb{R}^n$, then $D_{\mathcal{X}} u \in L^\infty(\Omega)$ (see [10] and [11]). In the particular case of a Carnot group the result is indeed stronger: the Lipschitz continuity w.r.t. the Carnot-Carathédory distance implies that the function is also horizontally differentiable a.e. (Pansu-Rademacher Theorem, e.g. in [17]).

Before stating the final result, i.e. the equivalence between AMLEs and infinity harmonic functions, we would like to summarize all the equivalences that have been obtained so far:
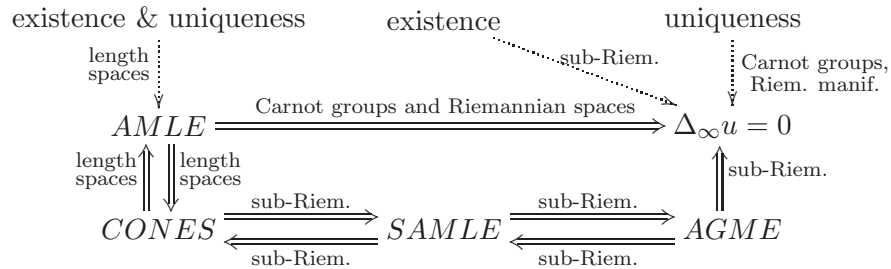
We can conclude the following.

Theorem 4.10. *Let $(\mathbb{R}^n, \mathcal{X}, \langle\ ,\ \rangle_g)$ be a Riemannian or sub-Riemannian manifold, $\Omega \subset \mathbb{R}^n$ open and bounded and let $g : \partial\Omega \to \mathbb{R}$ be a Lipschitz boundary datum. If $u : \overline{\Omega} \to \mathbb{R}$ is the AMLE of g, then u is infinity harmonic on Ω.*
Moreover, in the particular cases of Riemannian manifolds, Carnot groups and Grušin spaces, also the reverse implication holds.

Proof. By the weak Fubini property, we know that AMLEs and SAM-LEs are equivalent notions. Moreover, by Theorem 4.7, u is AMLE if and only if u is AGME. In order to conclude, it is sufficient to recall that AGMEs are infinity harmonic and, by the uniqueness results for the infinity Laplacian, the reverse implication holds in Riemaniann manifolds, Carnot groups and Grušin spaces. $\qquad \square$

Finally, we would like to apply our main theorem to a result recently proved by the first two named authors and Bieske in [6]. There, the authors have used the relation between AMLEs and infinity harmonic functions to give a geometric characterization of the set where the distance from a fixed point is infinity harmonic in Riemannian manifolds and Carnot groups. Actually, only one implication is necessary. Therefore that characterization can now be generalized to any

sub-Riemannian structure. Let us briefly recall this result.

Let us fix a base point $x_0 \in \mathbb{R}^n$ and define $d(x) := d(x, x_0)$ as, respectively, the Riemannian or Carnot-Carathéodory distance from x_0. Hereafter, we indicate by E' the set of all the points x such that any geodesic from x_0 to x cease to be length minimizing exactly at x.

Theorem 4.11 ([6]). *The function $d(x)$ is AMLE in the set $A :=$ $\mathbb{R}^n \setminus \big(E' \cup \{x_0\}\big)$, since there $d(x)$ satisfies comparison by metric cones by above and below.*

In Euclidean spaces $E' = \emptyset$, thus the Euclidean distance $d(x) = |x|$ is infinity harmonic in $\mathbb{R}^n \setminus \{0\}$, as we already knew. In the Riemannian sphere S^n we get that $d(x)$ is AMLE in $S^n \setminus \{x_0, x_1\}$, x_1 being the antipodal point to x_0. In the 1-dimensional Heisenberg group, the result implies that the Carnot-Carathéodory distance from the origin is AMLE outside the center of the group.

By applying Theorem 4.10 we can derive the following corollary.

Corollary 4.12. *If $d(x)$ is a Riemannian or Carnot-Carathéodory distance from a point, then $d(x)$ is infinity harmonic in A.*

Proof. By Theorem 4.11 we know that $d(x)$ is AMLE in A. By Theorem 4.10 this implies that $d(x)$ is also infinity harmonic, in any Riemannian or sub-Riemannian structure. Hence the corollary is proved. $\square$

In [6] the authors have shown explicitly that $d(x)$ is not infinity harmonic in the center of the Heisenberg group.

The relation between so-called bilateral solutions of a Hamilton-Jacobi equation and the corresponding Aronsson equation found by Soravia in [23] can be applied to the special case of the eikonal equation and the infinity Laplace equation. In this case, Soravia's result yields the same characterization by geodesics. In fact it is known that the Carnot-Carathéodory distance from a point is a viscosity solution of the eikonal equation in $\mathbb{R}^n$ except the point (see [9] and [19]) and by [23] it turns out to be a bilateral solution exactly in $A = \mathbb{R}^n \setminus \big(E' \cup \{x_0\}\big)$. Therefore, at least in Carnot groups, the set A is exactly the set where $d(x)$ is AMLE and infinity harmonic.

Proving the same characterization in general sub-Riemannian structures seems much harder, due in particular to the difficulties concerning comparison principles for the infinity Laplace equation in such a generality.

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