THE FINE STRUCTURE OF THE SINGULAR SET OF AREA-MINIMIZING INTEGRAL CURRENTS III: FREQUENCY 1 FLAT SINGULAR POINTS AND $\mathcal{H}^{m-2}$-A.E UNIQUENESS OF TANGENT CONES

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Abstract. We consider an area-minimizing integral current T of codimension higher than 1 in a smooth Riemannian manifold Σ. We prove that T has a unique tangent cone, which is a superposition of planes, at $\mathcal{H}^{m-2}$-a.e. point in its support. In combination with works of the first and third authors, we conclude that the singular set of T is countably $(m-2)$-rectifiable. The techniques in the present work can be seen as a counterpart for area-minimizers, in arbitrary codimension, to those developed by Simon ([29]) for multiplicity one classes of minimal surfaces and Wickramasekera ([32]) for stable minimal hypersurfaces.

Contents

1. Introduction and main results

Let T be an m-dimensional integral current in a complete smooth $(m+\bar{n})$-dimensional Riemannian manifold Σ. We assume that T is area-minimizing in some (relatively) open $\Omega \subset \Sigma$, i.e.

$$\mathbf{M}(T) \leq \mathbf{M}(T + \partial S)$$

for any $(m+1)$-dimensional integral current S supported in Ω. The (*interior*) *regular set* $\mathrm{Reg}(T)$ is the set of points $p \in \mathrm{spt}(T) \cap \Omega \setminus \mathrm{spt}(\partial T)$ for which there is an open ball $\mathbf{B}$ containing p, a regular orientable minimal surface $\Lambda \subset \mathbf{B}$ without boundary in $\mathbf{B}$, and an integer $Q \in \mathbb{N}$

such that $T \llcorner \mathbf{B} = Q[\![\Lambda]\!]$. The (*interior*) *singular set* $\mathrm{Sing}(T)$ is then given by the complement in $\Omega \cap \mathrm{spt}\,(T)$ of $\mathrm{Reg}(T)$. This is the third of three papers (the others being [8, 9]) devoted to proving the following theorem.

Theorem 1.1. *Let T be an m-dimensional area-minimizing integral current in a C^{3,κ_0} complete Riemannian manifold of dimension $m + \bar{n} \geq m + 2$, with $\kappa_0 > 0$. Then,* $\mathrm{Sing}(T)$ *is $(m-2)$-rectifiable and T has a unique tangent cone at $\mathcal{H}^{m-2}$-a.e. $q \in \mathrm{Sing}(T)$.*

We refer to the first work [8] for the historical context and the motivation of our study, we just recall here that the regularity of $\mathrm{Sing}(T)$ given by Theorem 1.1 is close to optimal due to the recent work [23], which shows that $\mathrm{Sing}(T)$ can be precised to be any fractal subset of a smooth oriented $(m-2)$-dimensional manifold. The only foreseeable improvement of Theorem 1.1 is to show that the $(m-2)$-dimensional Hausdorff measure of $\mathrm{Sing}(T)$ is locally finite in $\mathrm{spt}(T) \setminus \mathrm{spt}(\partial T)$. We also recall that this is known in the special case $m = 2$ by [3] (cf. also [14–16]) and that the statement about the uniqueness of tangent cones is covered, in that case, by [31].

Recall that, following Almgren's stratification theorem, we can subdivide $\mathrm{Sing}(T)$ into the disjoint union of

- the subset $\mathcal{S}^{(m-2)}(T)$ of points p at which any tangent cone to T has at most $m-2$ linearly independent directions of translation invariance;
- the remaining set $\mathrm{Sing}(T) \setminus \mathcal{S}^{(m-2)}(T)$ of those singular points at which at least one tangent cone is a flat plane (counted with some integer multiplicity $Q \geq 1$).

Consistently with [8] we use the notation $\mathfrak{F}(T)$ for the latter set and we will call its elements *flat singular points*. As a consequence of the general theory of Naber and Valtorta ([27]) it follows that $\mathcal{S}^{(m-2)}(T)$ is $(m-2)$-rectifiable, therefore the main novelty of our work is to establish the rectifiability of $\mathfrak{F}(T)$. In fact the techniques of the present paper, which can be seen as a counterpart in the higher codimension case (and for area-minimizing integral currents) of the seminal works by L. Simon ([29]) and N. Wickramasekera ([32]), can be used to derive an independent proof that $\mathcal{S}^{(m-2)}(T)$ is rectifiable. However the ideas in [26, 27] are still a crucial ingredient in [9].

By the constancy theorem, the density $\Theta(T, p)$ at any $p \in \mathfrak{F}(T)$ is a positive integer Q, which moreover obeys $Q > 1$ by Allard's Regularity Theorem. We can therefore stratify $\mathfrak{F}(T)$ as $\bigcup_{Q=2}^{\infty} \mathfrak{F}_Q(T)$, where

$$\mathfrak{F}_Q(T) := \{p \in \mathfrak{F}(T) : \Theta(T, p) = Q\}\,.$$

According to [8] we can further subdivide each $\mathfrak{F}_Q(T)$ by introducing a suitable function

$$\mathfrak{F}(T) \ni p \mapsto \mathrm{I}(T, p) \in [1, \infty)\,.$$

Loosely speaking, $\mathrm{I}(T, p)$ detects the infinitesimal homogeneity of the "singular behavior of T" around p. A good illustration of this number is given in [8, Example 1.2] in the case of classical holomorphic curves of $\mathbb{C}^2$, which by Federer's theorem are area-minimizing integral 2-dimensional currents in $\mathbb{R}^4$. Consider $\Lambda := \{(w - h(z))^Q = z^p k(z) : (z, w) \in \mathbb{C}^2\}$ and require that

- $p > Q \geq 2$ are coprime integers;
- h and k are holomorphic functions;
- $k(0) \neq 0$.

If $T = [\![\Lambda]\!]$ in $\mathbb{R}^4 \cong \mathbb{C}^2$, then $\mathrm{I}(T, 0) = p/Q$. Given however the lack of precise information about the singular behavior of a general m-dimensional area-minimizing integral current, the actual definition of $\mathrm{I}(T, p)$ is rather involved: note for instance that we do not know that the tangent cone to T at $p \in \mathfrak{F}$ is unique, or even that all tangent cones are flat. In the papers [8, 9], the first and third authors prove that

Theorem 1.2. *Let T be an m-dimensional area-minimizing integral current in a C^{3,κ_0} complete Riemannian manifold of dimension $m + \bar{n} \geq m + 2$, with $\kappa_0 > 0$. Then $\mathfrak{F}_{Q,>1} := \mathfrak{F}_Q(T) \cap \{\mathrm{I}(T, \cdot) > 1\}$ is $(m-2)$-rectifiable and the tangent cone is unique at every $p \in \mathfrak{F}_{Q,>1}$.*

In the present paper, we handle the rectifiability question for the remaining part of $\mathfrak{F}_Q(T)$, as well as the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cone question in the remaining portion of the singular set. More precisely, we prove the following:

Theorem 1.3. *Let T be as in Theorem 1.2. The following holds:*

(i) *The set $\mathfrak{F}_{Q,1}(T) := \mathfrak{F}_Q(T) \cap \{\mathrm{I}(T,\cdot) = 1\}$ is $\mathcal{H}^{m-2}$-null;*
(ii) *T has a unique tangent cone at $\mathcal{H}^{m-2}$-a.e. $p \in \mathcal{S}^{(m-2)}(T)$.*

1.1. **Comparison with the works of Krummel and Wickramasekera.** While we were completing this and the two works [8,9] leading to our proof of Theorem 1.1, we have learned that in the works [20–22], Krummel and Wickramasekera arrived independently at a program that shows the same final result; we refer to the introduction of [8] for a more general comparison between the two programs.

The present paper and [20] are in fact the two works which are most similar. Both [20] and this paper rely in an essential way on an new height bound: we refer to Theorem 3.2 in the first part of this paper for our precise statement. Indeed, the only difference between the two bounds seems to be that ours is stated in the more general setting of an arbitrary ambient manifold which satisfies some mild regularity assumptions, a setting which we believe can be reached equally well in [20] at the price of some more technical work.

Building upon this height bound, both the present work and [20] rely in a fundamental way on the estimates of [29], which are in turn used to perform a suitable blow-up analysis to prove the decay theorem which is the subject of the second part of this work. To handle the situation in which the current is very close to a plane, but much closer to a cone with $m-2$ linearly independent directions of translational invariance, both [20] and the present work also rely on important ideas introduced by Wickramasekera in [32].

Finally, the ideas of [29] are also used in a substantial way in both papers to prove that the flat singular points at which the sheets of the surface meet with order of contact 1, namely $\mathfrak{F}_{Q,1}(T)$, is $\mathcal{H}^{m-2}$-negligible. The result achieved in [20,21] is in fact stronger: using our notation they actually show that $\mathfrak{F}_{Q,<1+\delta}$ is $\mathcal{H}^{m-2}$-negligible for a sufficiently small $\delta = \delta(Q,m,n)$, while $\mathfrak{F}_{Q,\geq 1+\delta}$ is (relatively) closed in any ball $\mathbf{B}_r(x)$ where the current is sufficiently close to a multiplicity-Q flat plane.

While we do not pursue such a finer result here, we also believe with some additional work we can deliver these very conclusions. In fact the closedness of $\mathfrak{F}_{Q,\geq 1+\delta}$ is a consequence, in [21], of the almost-monotonicity of the planar frequency function and a similar almost-monotonicity holds also for the Almgren frequency function relative to the center manifolds as used in [8,9]. As for the fact that $\mathfrak{F}_{Q,<1+\delta}$ is $\mathcal{H}^{m-2}$-negligible, we believe that it can be achieved with appropriate modifications of the arguments in the final part of this paper, given that at these points all the coarse blow-ups will be homogeneous with degree $d \in [1, 1+\delta]$, and thus, for a sufficiently small δ, close to 1-homogeneous Dir-minimizers. Indeed, it is an interesting question whether there is a frequency gap (depending on the number of values of the Dirichlet minimizer) for Dirichlet minimizers. Namely, if there exists $\delta = \delta(m,n,Q) > 0$ such that any homogeneous Q-valued Dirichlet minimizer of degree $\alpha \in [1, 1+\delta)$ is in fact homogeneous of degree 1. Such a result would then immediately imply that in fact the sets $\mathfrak{F}_{Q,1}$ and $\mathfrak{F}_{Q,<1+\delta}$ coincide.

2. Preliminaries and notation

2.1. **Notation.** Throughout this work, $C, C_0, C_1, \dots$ will denote constants which depend only on $m, n, \bar{n}, Q$. Constants depending on other parameters will typically be denoted by

$$\bar{C}, \bar{C}_1, \bar{C}_2, \dots,$$

with dependencies given. For $q \in \mathrm{spt}(T)$, the currents $T_{q,r}$ will denote the dilations $(\iota_{q,r})_\sharp T$, where $\iota_{q,r}(x) := \frac{x-q}{r}$. For $p \in \mathrm{spt}(T)$, $\mathbf{B}_r(p)$ denotes the open $(m+n)$-dimensional Euclidean ball of radius r centered at p, while $B_r(p,\pi)$ denotes the (open) m-dimensional disk $\mathbf{B}_r(p) \cap \pi$ of radius r centered at p in the m-dimensional plane $\pi \subset \mathbb{R}^{m+\bar{n}}$ passing through p. $\mathbf{C}_r(p,\pi)$ denotes the $(m+n)$-dimensional cylinder $B_r(p,\pi) \times \pi^\perp$ of radius r centered at p. We let

$\mathbf{p}_\pi : \mathbb{R}^{m+n} \to \pi$ denote the orthogonal projection onto π, while $\mathbf{p}_\pi^\perp$ denotes the orthogonal projection onto $\pi^\perp$. The plane π is omitted if clear from the context; if the center p is omitted, then it is assumed to be the origin. For example, $\mathbf{C}_r := \mathbf{C}_r(0,\pi)$ for $r > 0$ if the choice of π is clear from the context. $\|T\|$ denotes the mass measure induced by T, while ω_m denotes the m-dimensional Hausdorff measure of the m-dimensional unit disk $B_1(\pi)$. The Hausdorff distance between two subsets A and B of $\mathbb{R}^{m+\bar{n}}$ will be denoted by $\mathrm{dist}(A,B)$.

The scalar product between vectors $v, w \in \mathbb{R}^{m+n}$ is denoted by $v \cdot w$ and likewise for product of matrices we will use $A \cdot B$. $|\cdot|$ will denote the Euclidean norm of vectors and the Hilbert-Schmidt norm of matrices. $\mathcal{H}^s$ will denote the Hausdorff s-dimensional measure, while in the particular instance of subsets E of $\mathbb{R}^m$ we will use the shorthand $|E|$ for their Lebesgue measure. This convention will often also be used for the $\mathcal{H}^m$ measures of subsets E of affine m-dimensional subspaces of $\mathbb{R}^{m+n}$.

We collect here a table of additional symbols used repeatedly throughout the paper, for the reader's convenience.

r, ρ, s, t	typically denote radii
i, j, k	indices
α, β, π	m-dimensional planes
ϖ	$(m+\bar{n})$-dimensional plane
$\varepsilon, \delta, \eta$	small numbers, with ε the smallest in hierarchy
γ, κ, μ	exponents
$\varsigma, \sigma, \tau, \varkappa$	parameters
ϕ, θ, ϑ	angles
φ, ψ, χ	test functions
f, g, h, u, v, w	functions, with f, u, v and w typically denoting multi-valued approximations
Ψ, Σ	Ψ the parameterization of the ambient manifold Σ
$\Sigma_{p,r}$	the rescaled manifold $\iota_{p,r}(\Sigma)$
S, T	currents
p, q	points in $\mathbb{R}^{m+n}$
x, y, z, ξ, ζ	variables (typically in m-dimensional subspaces of $\mathbb{R}^{m+n}$)
$\mathbf{p}$, $\mathbf{p}^\perp$	orthogonal projection, projection to orthogonal complement, respectively
$\mathbf{1}_E$	indicator function of the set E
$\mathbf{A}$	the L^∞ norm of the second fundamental form
$\Theta(T,p)$	the m-dimensional Hausdorff density of T at a point p;
$\mathrm{Sing}(T), \mathfrak{F}(T)$	singular sets of T, with $\mathfrak{F}(T)$ the flat singularities
$\mathfrak{F}_Q(T)$	flat singularities of T where the density of T is Q
$\mathfrak{F}_{Q,1}(T)$	points in $\mathfrak{F}(T)$ with $\mathrm{I}(T,\cdot) = 1$
$L, \ell(L)$	L a cube, $\ell(L)$ half the side length
A, B	linear maps
M	balancing constant (cf. Definition 8.4)
X	vector field
$\mathbf{S}$	$(m-2)$-invariant cones that are superpositions of m-planes
N	natural number, typically denoting the number of planes in $\mathbf{S}$
V	spines of cones
$\mathscr{P}$	set of m-dimensional planes
$\mathscr{C}$	set of $(m-2)$-invariant cones that are superpositions of m-planes
$\mathbf{E}^p(T,\mathbf{B})$	planar excess of T in the $(m+n)$-dimensional ball $\mathbf{B}$
$\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}), \hat{\mathbf{E}}(\mathbf{S},T,\mathbf{B})$	one-sided L^2 conical excess in $\mathbf{B}$ (T close to $\mathbf{S}$, $\mathbf{S}$ close to T, resp.)
$\mathbb{E}(T,\mathbf{S},\mathbf{B})$	double-sided conical excess in $\mathbf{B}$
$B_a(V)$	fixed tubular neighbourhood of radius a of the spine V being removed from $\mathbf{B}_1$
$\boldsymbol{\sigma}(\mathbf{S})$	minimal pairwise Hausdorff distance between the planes in $\mathbf{S}$ in $\mathbf{B}_1$
$\boldsymbol{\mu}(\mathbf{S})$	maximal pairwise Hausdorff distance between the planes in $\mathbf{S}$ in $\mathbf{B}_1$
$\mathcal{A}_Q(\mathbb{R}^n)$	the space of Q-tuples of vectors in $\mathbb{R}^n$ (cf. [10])

Since our statements are local and invariant under dilations and translations, we will work with the following underlying assumption throughout.

Assumption 2.1. $m \geq 3$ and $n \geq \bar{n} \geq 2$ are integers. T is an m-dimensional integral current in $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$ with $\partial T \llcorner \mathbf{B}_{7\sqrt{m}} = 0$, where Σ is an $(m+\bar{n})$-dimensional embedded submanifold of $\mathbb{R}^{m+n} = \mathbb{R}^{m+\bar{n}+l}$ of class C^{3,κ_0} with $\kappa_0 > 0$. T is area-minimizing in $\Sigma \cap \mathbf{B}_{7\sqrt{m}}$. $\Sigma \cap \mathbf{B}_{7\sqrt{m}}(p)$ is the graph of a C^{3,κ_0} function $\Psi_p : \mathrm{T}_p\Sigma \cap \mathbf{B}_{7\sqrt{m}}(p) \to \mathrm{T}_p\Sigma^\perp$ for every $p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}$. Moreover

$$\boldsymbol{c}(\Sigma) := \sup_{p \in \Sigma \cap \mathbf{B}_{7\sqrt{m}}} \|D\Psi_p\|_{C^{2,\kappa_0}} \leq \bar{\varepsilon},$$

where $\bar{\varepsilon} \leq 1$ is a small positive constant which will be specified later.

Following the notation of [8], we let $\mathbf{A}_\Sigma$ denote the C^0 norm of the second fundamental form A_Σ of Σ in $\mathbf{B}_{7\sqrt{m}}$. In particular, under Assumption 2.1, we have

$$\mathbf{A}_\Sigma := \|A_\Sigma\|_{C^0(\Sigma \cap \mathbf{B}_{7\sqrt{m}})} \leq C_0 \boldsymbol{c}(\Sigma) \leq C_0 \bar{\varepsilon}.$$

We will often drop the subscript as the underlying ambient manifold will mostly be clear from the context.

We recall that the *oriented tilt-excess* of T in $\mathbf{C}_r(p, \pi_0)$ relative to an m-dimensional oriented plane π is defined by

$$\mathbf{E}(T, \mathbf{C}_r(p, \pi_0), \pi) := \frac{1}{2\omega_m r^m} \int_{\mathbf{C}_r(p,\pi_0)} |\vec{T}(x) - \vec{\pi}(x)|^2 \, d\|T\|(x),$$

while

$$\mathbf{E}(T, \mathbf{C}_r(p, \pi_0)) := \min_{\pi \subset T_p\Sigma} \mathbf{E}(T, \mathbf{C}_r(p, \pi_0), \pi)$$

where the minimum is taken over all m-dimensional oriented planes $\pi \subset T_p\Sigma$ (identified with their corresponding planes in $\mathbb{R}^{n+m}$). The oriented tilt-excess of T in $\mathbf{B}_r(p)$ relative to an m-dimensional oriented plane π and the optimal oriented tilt-excess in $\mathbf{B}_r(p)$, denoted respectively by $\mathbf{E}(T, \mathbf{B}_r(p), \pi)$ and $\mathbf{E}(T, \mathbf{B}_r(p))$, are defined analogously. The tilt-excess is morally a planar L^2 "gradient" excess. We will shortly also introduce a notion of planar L^2 "height" excess.

We also refer to [8, Section 3.1] for the notion of a *coarse blow-up* of T at p, which, roughly speaking, under the assumption that $r^2\mathbf{A}^2 \ll \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r}(p)) \ll 1$ along a given family of scales r, gives rise to a Dir-minimizing Q-valued function over $B_1(0, \pi_0)$ for some m-dimensional plane π_0 as a subsequential normalized limit. The graph of this Dir-minimizer approximates the rescaled current $T_{p,\rho}$ in the domain $\mathbf{C}_1(0, \pi_0) \cap \mathbf{B}_4$ with a mass error which is $o(\mathbf{E}(T, \mathbf{B}_{6\sqrt{m}}))$, along with other related error estimates which are detailed in [8, Section 3.1] and [11]. We additionally refer the reader to [8] for all other relevant notation and terminology relating to coarse blow-ups.

The properties of $\mathfrak{F}_{Q,1}(T)$ that will be most useful to us here are contained within the following proposition, which is a consequence of the analysis in [8] (more precisely, see (9), Proposition 4.1 and Corollary 4.3 therein).

Proposition 2.2. *Let T, Σ and $\mathbf{A}$ be as in Assumption 2.1. For every $p \in \mathfrak{F}_{Q,1}(T)$ and any sequence $r_k \downarrow 0$ with the property that $\mathbf{E}_k := \mathbf{E}(T, \mathbf{B}_{6\sqrt{m}r_k}(p)) \downarrow 0$ the following holds:*

- (i) *For $\mathbf{A}_k := \mathbf{A}_{\Sigma_{p,r_k}}$ we have $\lim_{k\to\infty} \mathbf{E}_k^{-1} r_k^{2-2\delta_2} \mathbf{A}_k^2 = 0$;*
- (ii) *Any coarse blow-up $\bar{f}$ generated by a subsequence of $\{r_k\}$ has positive Dirichlet energy, is 1-homogeneous, and satisfies $\boldsymbol{\eta} \circ \bar{f} \equiv 0$.*

2.2. **Main decay theorem.** To prove Theorem 1.1, we will show that a key excess decay theorem holds under the assumption that T is much closer to a cone with exactly $m-2$ directions of translation invariance than it is to any m-dimensional plane. Before coming to the statement of this theorem, let us first introduce suitable notions of L^2 height excess of T relative to such cones. We begin by defining the cones of interest.

Definition 2.3. For every fixed integer $Q \geq 2$ we denote by $\mathscr{C}(Q)$ those subsets of $\mathbb{R}^{m+n}$ which are unions of $N \leq Q$ m-dimensional planes $\pi_1, \dots, \pi_N$ satisfying the following properties:

(i) $\pi_i \cap \pi_j$ is the same $(m-2)$-dimensional plane V for every pair (i,j) with $i<j$;
(ii) Each plane π_i is contained in the same $(m+\bar{n})$-dimensional plane ϖ.

If $p \in \Sigma$, then $\mathscr{C}(Q,p)$ will denote the subset of $\mathscr{C}(Q)$ for which $\varpi = T_p\Sigma$.

$\mathscr{P}$ and $\mathscr{P}(p)$ will denote the subset of those elements of $\mathscr{C}(Q)$ and $\mathscr{C}(Q,p)$ respectively which consist of a single plane; namely, with $N=1$. For $\mathbf{S} \in \mathscr{C}(Q) \setminus \mathscr{P}$, the $(m-2)$-dimensional plane V described in (i) above is referred to as the *spine* of $\mathbf{S}$ and will often be denoted by $V(\mathbf{S})$.

We are now in a position to introduce the *conical L^2 height excess* between T and elements in $\mathscr{C}(Q)$.

Definition 2.4. Given a ball $\mathbf{B}_r(q) \subset \mathbb{R}^{m+n}$ and a cone $\mathbf{S} \in \mathscr{C}(Q)$, we define the *one-sided conical L^2 height excess of T relative to $\mathbf{S}$ in $\mathbf{B}_r(q)$*, denoted $\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_r(q))$, by

$$\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_r(q)) \coloneqq \frac{1}{r^{m+2}}\int_{\mathbf{B}_r(q)} \operatorname{dist}^2(p,\mathbf{S})\, d\|T\|(p).$$

At the risk of abusing notation, we further define the corresponding *reverse one-sided excess* as

$$\hat{\mathbf{E}}(\mathbf{S},T,\mathbf{B}_r(q)) \coloneqq \frac{1}{r^{m+2}}\int_{\mathbf{B}_r(q)\cap\mathbf{S}\setminus\mathbf{B}_{ar}(V(\mathbf{S}))} \operatorname{dist}^2(x,\operatorname{spt}(T))\, d\mathcal{H}^m(x)\,,$$

where $a=a(m)$ is a dimensional constant, to be specified later. We subsequently define the *two-sided conical L^2 height excess* as

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_r(q)) := \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_r(q)) + \hat{\mathbf{E}}(\mathbf{S},T,\mathbf{B}_r(q))\,.$$

We finally introduce the *planar L^2 height excess* which is given by

$$\mathbf{E}^p(T,\mathbf{B}_r(q)) = \min_{\pi\in\mathscr{P}(q)} \hat{\mathbf{E}}(T,\pi,\mathbf{B}_r(q))\,.$$

Let us now state our main excess decay theorem. This is similar in spirit to the excess decay theorem originally seen in [29, Lemma 1], however a crucial difference with the present setting is that in [29, Lemma 1] there is a built-in multiplicity one assumption which in particular rules out branch point singularities a priori. Our excess decay theorem in contrast is in a higher multiplicity setting, and therefore is closer – and indeed our proof follows a similar pattern – to the excess decay theorems first seen in [32, Section 13] (see also [19, Lemma 5.6 and Lemma 12.1] and [24, Theorem 2.1]).

Theorem 2.5 (Excess Decay Theorem)**.** *For every $Q,m,n,\bar{n}$, and $\varsigma>0$, there are positive constants $\varepsilon_0=\varepsilon_0(Q,m,n,\bar{n},\varsigma)\le\frac{1}{2}$, $r_0=r_0(Q,m,n,\bar{n},\varsigma)\le\frac{1}{2}$ and $C=C(Q,m,n,\bar{n})$ with the following property. Assume that*

(i) *T and Σ are as in Assumption 2.1;*
(ii) *$\|T\|(\mathbf{B}_1)\le(Q+\frac{1}{2})\omega_m$;*
(iii) *There is $\mathbf{S}\in\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ such that*

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\le\varepsilon_0^2\mathbf{E}^p(T,\mathbf{B}_1) \tag{2.1}$$

and

$$\mathbf{B}_{\varepsilon_0}(\xi)\cap\{p:\Theta(T,p)\ge Q\}\neq\emptyset \qquad \forall\xi\in V(\mathbf{S})\cap\mathbf{B}_{1/2}\,; \tag{2.2}$$

(iv) $\mathbf{A}^2\le\varepsilon_0^2\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1)$ *for* any $\mathbf{S}'\in\mathscr{C}(Q,0)$.

Then there is a $\mathbf{S}'\in\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ such that

(a) $\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})\le\varsigma\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$
(b) $\dfrac{\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})}{\mathbf{E}^p(T,\mathbf{B}_{r_0})}\le 2\varsigma\dfrac{\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)}{\mathbf{E}^p(T,\mathbf{B}_1)}$
(c) $\operatorname{dist}^2(\mathbf{S}'\cap\mathbf{B}_1,\mathbf{S}\cap\mathbf{B}_1)\le C\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$
(d) $\operatorname{dist}^2(V(\mathbf{S})\cap\mathbf{B}_1,V(\mathbf{S}')\cap\mathbf{B}_1)\le C\dfrac{\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)}{\mathbf{E}^p(T,\mathbf{B}_1)}$.

2.3. **Structure of the paper.** The majority of this paper will be dedicated to the proof of Theorem 2.5. We begin with a refined L^2-L^∞ height bound in Part 1, which will be a key tool for the estimates in the remainder of the paper. However, we believe that this height bound is of interest in itself, and would have a number of other applications.

Part 2 will contain the proof of the Excess Decay Theorem 2.5. The proof consists of several key parts, modeled off the foundational works of Simon ([29]) and Wickramasekera ([32]); the reader familiar with these works will find many of the arguments in this part similar in spirit to those seen in these, and we will make the effort to point out the similarities and differences throughout the work to aid the reader. Section 7 contains some key results regarding the relative positioning and angles between pairs of planes in cones in $\mathscr{C}(Q)\setminus\mathscr{P}$, followed by Section 8, which introduces effective graphical approximations for T over such cones. In Section 9, we then use the preceding two sections to demonstrate that we may replace the initial cone $\mathbf{S}$ in Theorem 2.5 with a *balanced* cone (cf. Definition 8.4). In Section 10 we reduce the proof of Theorem 2.5 to an a priori much weaker decay statement, in particular one where we can assume that the two-sided excess is much smaller than the *minimal* angle in the cone (as opposed to the maximal angle, which is morally what (2.1) says). Section 11 is then dedicated to the *Simon estimates*, including the non-concentration estimates, at the spine of the cone in $\mathscr{C}(Q)\setminus\mathscr{P}$, after which, in Section 12 we demonstrate the excess decay conclusion at the level of the *linearized problem* of multiple-valued Dirichlet minimizers. Part 2 is then concluded with Section 13, in which we put together everything from the previous sections to conclude the proof of Theorem 2.5.

In Part 3 we use Theorem 2.5 combined with a covering procedure, analogously to that done by Simon ([29]), to prove Theorem 1.1.

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Part 1. L^2-L^∞ Height Bound

The aim of this part is to prove a generalization of Allard's tilt-excess and L^∞ estimates (see [1, Section 8]). Allard's original work, in the context of stationary varifolds, bounds the L^∞ distance and L^2 tilt excess from a single plane with the L^2 distance to it. Here, we will restrict ourselves to the much smaller class of area-minimizing currents, with the additional benefit being that are able to control the L^∞ distance and L^2 tilt excess from a *finite* collection of mutually disjoint planes by the L^2 distance to the union of the planes.

3. Main statements

For the remainder of this part, we make the following additional assumption.

Assumption 3.1. T, Σ, and $\mathbf{A}$ are as in Assumption 2.1. For some oriented m-dimensional plane $\pi_0\subset\mathbb{R}^{m+\bar n}$ passing through the origin and some positive integer Q, we have

$$(\mathbf{p}_{\pi_0})_\sharp T\llcorner\mathbf{C}_2(0,\pi_0)=Q[\![B_2(\pi_0)]\!]\,,$$

and $\|T\|(\mathbf{C}_2)\le(Q+\frac12)\omega_m2^m$.

The main result of this part is the following.

Theorem 3.2 (L^∞ and tilt-excess estimates)**.** *For every $1\le r<2$, Q, and N, there is a positive constant $\bar C=\bar C(Q,m,n,\bar n,N,r)>0$ with the following property. Suppose that T, Σ, $\mathbf{A}$ and π_0 are as in Assumption 3.1, let $p_1,\dots,p_N\in\pi_0^\perp$ be distinct points, and set $\boldsymbol{\pi}:=\bigcup_i p_i+\pi_0$. Let*

$$E:=\int_{\mathbf{C}_2}\mathrm{dist}^2(p,\boldsymbol{\pi})\,d\|T\|(p)\,. \tag{3.1}$$

Then

$$\mathbf{E}(T,\mathbf{C}_r,\pi_0)\le\bar C(E+\mathbf{A}^2) \tag{3.2}$$

and, if $E\le1$,

$$\mathrm{spt}(T)\cap\mathbf{C}_r\subset\{p:\mathrm{dist}(p,\boldsymbol{\pi})\le\bar C(E^{1/2}+\mathbf{A})\}\,. \tag{3.3}$$

A translation followed by a simple scaling argument clearly gives corresponding estimates when 0 is replaced by an arbitrary center and the scales r and 2 are replaced by two arbitrary radii $\rho < R$. We additionally record the following consequence of Theorem 3.2, which will be used frequently in the rest of the paper.

Corollary 3.3. *For each pair of positive integers Q and N, there is a positive constant $\delta = \delta(Q, m, n, \bar{n}, N)$ with the following properties. Assume that:*

- (i) *T, Σ, and $\mathbf{A}$ are as in Assumption 2.1;*
- (ii) *T is area-minimizing in Σ and for some positive $r \leq \frac{1}{4}$ and $q \in \mathrm{spt}(T) \cap \mathbf{B}_1$ we have $\partial T \llcorner \mathbf{C}_{4r}(q) = 0$, $(\mathbf{p}_{\pi_0})_\sharp T = Q[\![B_{4r}(q)]\!]$, and $\|T\|(\mathbf{C}_{2r}(q)) \leq \omega_m(Q + \frac{1}{2})(2r)^m$;*
- (iii) *$p_1, \ldots, p_N \in \mathbb{R}^{m+n}$ are distinct points with $\mathbf{p}_{\pi_0}(p_i) = q$ and $\varkappa := \min\{|p_i - p_j| : i < j\}$;*
- (iv) *$\pi_1, \ldots, \pi_N$ are oriented planes passing through the origin with*
$$\tau := \max_i |\pi_i - \pi_0| \leq \delta \min\{1, r^{-1}\varkappa\}\,; \tag{3.4}$$
- (v) *Upon setting $\boldsymbol{\pi} = \bigcup_i (p_i + \pi_i)$, we have*
$$(r\mathbf{A})^2 + (2r)^{-m-2} \int_{\mathbf{C}_{2r}(q)} \mathrm{dist}^2(p, \boldsymbol{\pi}) d\|T\| \leq \delta^2 \min\{1, r^{-2}\varkappa^2\}\,. \tag{3.5}$$

Then $T \llcorner \mathbf{C}_r(q) = \sum_{i=1}^N T_i$ where

- (a) *Each T_i is an integral current with $\partial T_i \llcorner \mathbf{C}_r(q) = 0$;*
- (b) *$\mathrm{dist}(q, \boldsymbol{\pi}) = \mathrm{dist}(q, p_i + \pi_i)$ for each $q \in \mathrm{spt}(T_i)$;*
- (c) *$(\mathbf{p}_{\pi_0})_\sharp T_i = Q_i[\![B_r(q)]\!]$ for some non-negative integer Q_i.*

3.1. **Proof of Corollary 3.3.** First of all, by scaling we can assume that $r = 1$ and up to translation we can assume $q = 0$. We then introduce
$$E := \int_{\mathbf{C}_2} \mathrm{dist}^2(p, \boldsymbol{\pi}) d\|T\|(p)\,.$$
Next let $\bar{\boldsymbol{\pi}} := \bigcup_i (p_i + \pi_0)$ and observe that
$$\bar{E} := \int_{\mathbf{C}_2} \mathrm{dist}^2(p, \bar{\boldsymbol{\pi}}) d\|T\|(p) \leq CE + C\tau^2 \leq C\delta^2 \min\{1, \varkappa^2\}\,,$$
for a constant $C = C(m, n, \bar{n}, Q)$ which we stress is independent of δ and $\varkappa$. Since we also have $\mathbf{A}^2 \leq \delta^2\varkappa^2$, we can apply Theorem 3.2 to conclude that
$$\mathrm{spt}(T) \cap \mathbf{C}_{3/2} \subset \{p : \mathrm{dist}(p, \bar{\boldsymbol{\pi}}) \leq C_1\delta\varkappa\}\,,$$
where $C_1 = C_1(m, n, \bar{n}, Q, N)$ is again independent of δ and $\varkappa$. In particular if we choose $\delta = \delta(m, n, \bar{n}, Q, N) > 0$ so that $C_1\delta < 1/4$, we have
$$\mathrm{spt}(T) \cap \mathbf{C}_{3/2} \subset \{p : \mathrm{dist}(p, \bar{\boldsymbol{\pi}}) < 4^{-1}\varkappa\}\,.$$
Due to the definition of $\varkappa$, $\{p : \mathrm{dist}(p, \bar{\boldsymbol{\pi}}) < 4^{-1}\varkappa\}$ consists of N disjoint open sets $U_i := \{p : \mathrm{dist}(p - p_i, \pi_0) < \varkappa/4\}$. We then set $T_i := T \llcorner \mathbf{C}_{3/2} \cap U_i$; clearly each T_i is integral, has no boundary in $\mathbf{C}_{3/2}$, and $\sum_i T_i = T \llcorner \mathbf{C}_{3/2}$. It follows also that the T_i's have disjoint support and so each of them is area-minimizing. If we further take $\delta < 1/8$ such that $C_1\delta < 1/8$, we can in addition ensure that for each point $p \in U_i$,
$$\mathrm{dist}(p, p_i + \pi_i) \leq \frac{\varkappa}{4}, \tag{3.6}$$
$$\mathrm{dist}(p, p_j + \pi_j) \geq \frac{\varkappa}{2} \qquad \forall j \neq i\,. \tag{3.7}$$
and thus (b) is certainly satisfied. Moreover, by [13, Lemma 1.6], observe that $(\mathbf{p}_{\pi_0})_\sharp T_i = Q_i[\![B_1]\!]$ for some $Q_i \in \mathbb{Z}$; clearly we must have $\sum_i Q_i = Q$. Again applying Theorem 3.2 we have
$$\mathbf{E}(T, \mathbf{C}_1, \pi_0) \leq C\delta^2\,.$$
In particular, if δ is chosen small enough, by [11, Theorem 2.4] we can ensure the existence of a subset $K \subset B_1(\pi_0)$ of positive measure with the property that for each $x \in K$, the slice $\langle T, \mathbf{p}_{\pi_0}, x\rangle$ (see [28] for the definition and properties of the slicing map) given by $\sum_j k_j\delta_{\xi_j}$ for

some finite collection of positive integers k_j and points $\xi_j(x) \in \pi_0^\perp$. Fix any such point x and observe that

$$\langle T_i, \mathbf{p}_{\pi_0}, x\rangle = \sum_{\xi_j \in U_i} k_j \delta_{\xi_j}$$

while

$$Q_i = \sum_{\xi_j \in U_i} k_j \,.$$

It follows immediately that Q_i cannot be negative, which completes the proof. □

4. Preliminaries

In this section, we collect all the required preliminary results for the proof of Theorem 3.2.

4.1. **Oriented and non-oriented tilt-excess.** Given an m-dimensional plane π and a cylinder $\mathbf{C} = \mathbf{C}_r(q, \pi)$, recall that the *non-oriented tilt excess* is given by

$$\mathbf{E}^{no}(T, \mathbf{C}) := \frac{1}{2\omega_m r^m} \int_{\mathbf{C}} |\mathbf{p}_{T(x)} - \mathbf{p}_\pi|^2 \, d\|T\|(x)\,, \tag{4.1}$$

where $T(x)$ denotes the (approximate) tangent plane to T at x. This is more generally defined for integral varifolds and does not take into consideration the orientation of T and π, in contrast with the oriented tilt excess $\mathbf{E}(T, \mathbf{C})$ defined previously.

It is obvious that $|\mathbf{p}_\alpha - \mathbf{p}_\beta| \leq C|\vec{\alpha} - \vec{\beta}|$ for a geometric constant $C = C(m, n)$ and for every pair of oriented planes, so $\mathbf{E}^{no}(T, \mathbf{C}) \leq C\mathbf{E}(T, \mathbf{C})$. On the other hand, the opposite inequality is only true if $|\vec{\alpha} - \vec{\beta}|$ is sufficiently small, for instance, if it is no larger than 1. In particular $\vec{\alpha}$ and $\vec{\beta}$ could be opposite orientations for the same linear subspace: in that case $|\vec{\alpha} - \vec{\beta}| = 2$ while $|\mathbf{p}_\alpha - \mathbf{p}_\beta| = 0$.

Nonetheless, for area-minimizing currents as in Assumption 2.1, the nonoriented excess controls the oriented excess. The idea of the argument is borrowed from [7, Theorem 16.1], but we repeat it here for clarity. A simple scaling and translation argument, which is left to the reader, gives a corresponding estimate on any pair of parallel concentric cylinders.

Proposition 4.1. *For every $1 \leq r < 2$ there is a constant $\bar{C} = \bar{C}(Q, m, n, \bar{n}, 2 - r)$ such that, if T , Σ and $\mathbf{A}$ are as in Assumption 3.1, then*

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(\mathbf{E}^{no}(T, \mathbf{C}_2) + \mathbf{A}^2)\,. \tag{4.2}$$

Proof. Since $\mathbf{E}(T, \mathbf{C}_r) \leq \frac{1}{\omega_m r^m}\|T\|(\mathbf{C}_r)$, we can without loss of generality assume that

$$\mathbf{E}^{no}(T, \mathbf{C}_2) + \mathbf{A}^2 \leq \delta$$

for some fixed constant δ, as long as in the end we choose $\delta = \delta(Q, m, n, \bar{n}, 2 - r) > 0$.

Moreover, for any $\eta > 0$, if δ is chosen to be sufficiently small depending on η, we can also assume without loss of generality that $\mathbf{E}(T, \mathbf{C}_{1+r/2}) \leq \eta$ for another fixed constant η: indeed, arguing by contradiction, if this were not true for some $\eta > 0$, we could find a sequence of $\delta_k \downarrow 0$ and a sequence of T_k with $\mathbf{E}^{no}(T_k, \mathbf{C}_2) \leq \delta_k \downarrow 0$ yet $\mathbf{E}(T_k, \mathbf{C}_{1+r/2}) > \eta$ for all k. If the supports of the currents are equibounded, then this would give that the T_k converge, locally in mass in $\mathbf{C}_2$, to a union of planes which are parallel to π_0 (counted with a suitable multiplicity), which in turn would imply $\mathbf{E}(T_k, \mathbf{C}_{1+r/2}) \to 0$, giving the desired contradiction. If the supports are not equibounded, one can resort to the height bound [13, Theorem A.1] to decompose each T_k into a disjoint finite sum of area-minimizing currents T_k^j, each of which have equibounded supports (after translation). One can then argue as above for T_k^j, for each j.

We thus assume

$$\Lambda_0 := \mathbf{E}(T, \mathbf{C}_{1+r/2}) + \mathbf{A}^2 \leq \eta\,, \tag{4.3}$$

for some η whose choice will be given later (depending only on $Q, m, n, \bar{n}, 2 - r$). We also assume

$$\mathbf{A}^2 \leq \mathbf{E}(T, \mathbf{C}_{1+r/2})\,, \tag{4.4}$$

otherwise the estimate we are looking for is trivially true. In particular $\Lambda_0 \leq 2\mathbf{E}(T, \mathbf{C}_{1+r/2})$.

Set $r_0 := 1 + \frac{r}{2}$. Recalling Almgren's strong Lipschitz approximation ([11, Theorem 1.4]), provided η is sufficiently small (depending on $Q, m, n, \bar{n}, 2-r$), there is a set $K \subset B_{r_0/4}(\pi_0)$ and a Lipschitz Q-valued map $f : B_{r_0/4} \to \mathcal{A}_Q(\pi_0^\perp)$ such that

- $\|T\|((B_{r_0/4} \setminus K) \times \pi_0^\perp) \leq C\Lambda_0^{1+2\gamma}$;
- $\mathrm{Lip}\,(f) \leq C\Lambda_0^{2\gamma}$;
- $\mathbf{G}_f \llcorner (K \times \pi_0^\perp) = T \llcorner (K \times \pi_0^\perp)$.

γ and C are fixed positive constants depending on $Q, m, n, \bar{n}$. At the price of making C larger, we can achieve the same estimates with $B_{r_0 - C\Lambda_0^{2\gamma}}$ in place of $B_{r_0/4}$; this is achieved with a simple covering argument, applying the Lipschitz approximation theorem in a collection of $C\Lambda_0^{-m\beta}$ cylinders whose cross-sections are disks in π_0 of radius Λ_0^β which cover $B_{r_0 - C\Lambda_0^\gamma}$, for a suitable choice of $\beta = \beta(\gamma, m)$ (for a detailed argument, see [7, Proposition 16.2]).

Moreover, if we assume that η is small enough, by halving the exponents 2γ to γ we can assume that the constant C in all the above estimates (including in the radius of $B_{r_0 - C\Lambda_0^{2\gamma}}$) is at most $\frac{1}{4}$. More precisely, for that fixed $C = C(Q, m, n, \bar{n})$ we have $C\Lambda_0^{2\gamma} = C\Lambda_0^\gamma \cdot \Lambda_0^\gamma \leq (C\eta^\gamma)\Lambda_0^\gamma$, so if we choose η small enough so that $C\eta^{\gamma/2} \leq 1/4$, we guarantee this.

We now set

$$r_1 := r_0 - \Lambda_0^\gamma \,.$$

Summarizing we have an approximation on B_{r_1}, which we still denote by f, such that

- $\|T\|((B_{r_1} \setminus K) \times \pi_0^\perp) \leq \frac{1}{4}\Lambda_0^{1+\gamma}$;
- $\mathrm{Lip}\,(f) \leq \Lambda_0^\gamma$;
- $\mathbf{G}_f \llcorner (K \times \pi_0^\perp) = T \llcorner (K \times \pi_0^\perp)$.

If η is small enough (which guarantees smallness of $\mathrm{Lip}(f)$, and hence comparability between the oriented and non-oriented tilt-excess of $\mathbf{G}_f$), we can estimate

$$\mathbf{E}(\mathbf{G}_f \llcorner (K \times \pi_0^\perp), \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(\mathbf{G}_f \llcorner (K \times \pi_0^\perp), \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(T, \mathbf{C}_2)$$

so that, in particular

$$\mathbf{E}(T, \mathbf{C}_{r_1}) \leq C\mathbf{E}^{no}(T, \mathbf{C}_2) + \frac{1}{2}\Lambda_0^{1+\gamma} \,.$$

If $\Lambda_0^{1+\gamma} \leq \mathbf{E}(T, \mathbf{C}_{r_1})$ we are then done, provided that $r_1 > r$, which may be achieved by taking η sufficiently small. Otherwise we must have

$$\mathbf{E}(T, \mathbf{C}_{r_1}) \leq \Lambda_0^{1+\gamma} \,.$$

We now iterate the above argument, inductively setting

$$r_k := r_{k-1} - \Lambda_{k-1}^\gamma \tag{4.5}$$

$$\Lambda_k := \mathbf{E}(T, \mathbf{C}_{r_k}) \leq \Lambda_{k-1}^{1+\gamma} \,. \tag{4.6}$$

We stop at a certain step if we have the desired estimate and $r_k \geq r$.

Since $\Lambda_k \leq \Lambda_0^{(1+\gamma)^k} \leq \eta^{(1+\gamma)^k}$ and $r_0 - r > 0$, provided that η is small enough, the inequality $r_k \geq r$ is always guaranteed, no matter how large k is. Therefore, if the procedure never stops, we conclude that $\mathbf{E}(T, \mathbf{C}_r) = 0$. But of course in this case the sought-after inequality is trivially true. This completes the proof. □

4.2. **The case $N = 1$.** Observe that the difficulty of Theorem 3.2 lies in the case when $N \geq 2$. Indeed, when Q is arbitrary and $N = 1$, the L^2 tilt-excess estimate (3.2) is an immediate consequence of the work of Allard in [1] (see also [5, Proposition 4.1]), combined with Proposition 4.1. Regarding the L^∞ estimate (3.3), this is also contained within [1] when the ambient space is Euclidean (and thus $\mathbf{A} = 0$). More generally the general techniques in [1] would give a linear dependence of the estimate in $\mathbf{A}$: to see that in our case this can be improved to a quadratic the reader can consult, for instance, [30]. From now on we will therefore assume the following.

Proposition 4.2. *The conclusions of Theorem 3.2 holds for $N = 1$ (and Q arbitrary).*

4.3. **Combinatorial lemmas.** We will make use of the following combinatorial lemmas. The first one is essentially a consequence of [10, Lemma 3.8], but since we need additional information we provide a self-contained proof. All of the proofs are deferred to Appendix A.

Lemma 4.3. *Fix any positive $\bar\delta \le \frac{1}{2}$ and a natural number $N \ge 2$. There is a constant $\bar C = \bar C(\bar\delta, N)$ with the following property. Given a set (of distinct points) $P \subset \mathbb{R}^n$ with cardinality N, there is a subset $P' \subset P$ of cardinality at least two such that:*

(i) $\max\{|q-p| : q, p \in P'\} \le \bar C \min\{|q-p| : q, p \in P',\ q \ne p\}$;
(ii) $\mathrm{dist}(p, P') \le \bar\delta \min\{|q_1 - q_2| : q_1, q_2 \in P',\ q_1 \ne q_2\}$ *for every* $p \in P$;
(iii) *the points in $P \setminus P'$ can be ordered as $\{p_1, \dots, p_J\}$ so that, setting $P_0 = P$ and $P_j := P \setminus \{p_1, \dots, p_j\}$, the following property holds:*

$$\mathrm{dist}(p_j, P_j) = \min\{|p-q| : p, q \in P_{j-1},\ p \ne q\}\,.$$

Lemma 4.4. *Fix any $0 < \delta \le \frac{1}{2}$ and $\varepsilon > 0$. Given a set (of distinct points) $P \subset \mathbb{R}^n$ with cardinality $N \ge 2$, we can find a nonempty subset $\tilde P \subset P$ such that*

(i) $\mathrm{dist}(p, \tilde P) \le \delta^{-1}(1+\delta^{-1})^{N-2}\varepsilon$ *for every* $p \in P$;
(ii) *Either $\tilde P$ is a singleton, or* $\max\{\varepsilon, \mathrm{dist}(p, \tilde P)\} \le \delta \min\left\{ |q_1 - q_2| : q_1, q_2 \in \tilde P,\ q_1 \ne q_2 \right\}$ *for any* $p \in P$.

Lemma 4.5. *Consider a set $P = \{p_1, \dots, p_N\} \subset \mathbb{R}^n$ of $N \ge 2$ distinct points and set $M := \max_{i,j}\{|p_i - p_j|\}$. Then we can decompose $P = P_1 \cup P_2$ into two disjoint non-empty sets such that*

$$\min\{|p_1 - p_2| : p_1 \in P_1, p_2 \in P_2\} \ge \frac{M}{2^{N-2}}\,.$$

4.4. **Well-separated case.** Let us first demonstrate the validity of Theorem 3.2 under the assumption that the planes in $\boldsymbol{\pi}$ are well-separated.

Lemma 4.6. *For every $1 \le \bar r < 2$ there is a constant $\sigma_1 = \sigma_1(Q, m, n, \bar n, 2-\bar r) > 0$ with the following property. Let T be as in Assumption 3.1, while $p_1, \dots, p_N \in \pi_0^\perp$ are distinct points and $\boldsymbol{\pi} := \bigcup_i p_i + \pi_0$. Assume that*

$$E \le \sigma_1 \min\{H, 1\}^{m+2}\,, \tag{4.7}$$

where E is as in (3.1) and $H := \min\{|p_i - p_j| : i \ne j\}$. Then

$$\mathrm{spt}\,(T) \cap \mathbf{C}_{\bar r} \subset \{q : \mathrm{dist}(q, \boldsymbol{\pi}) \le \tfrac{H}{4}\}, \tag{4.8}$$

and, in particular, all the conclusions of Theorem 3.2 hold.

First of all, observe that it suffices to prove (4.8) in order to conclude all the conclusions of Theorem 3.2. Indeed, if we set $T_i := T \llcorner \mathbf{C}_{\bar r} \cap \{q : \mathrm{dist}(q, p_i + \pi_0) \le \frac{H}{4}\}$, we obtain the decomposition $T \llcorner \mathbf{C}_{\bar r} = T_1 + \cdots + T_N$ with

(i) $\partial T_i \llcorner \mathbf{C}_{\bar r} = 0$;
(ii) $\mathrm{spt}\,(T_i) \cap \mathrm{spt}\,(T_j) = \emptyset$ for $i \ne j$;
(iii) $\mathrm{dist}(q, \boldsymbol{\pi}) = \mathrm{dist}(q, p_i + \pi_0)$ for every $q \in \mathrm{spt}\,(T_i)$.

In particular, given $r \in [1, 2)$, for $\bar r \in (r, 2)$ the conclusions of Theorem 3.2 can be drawn by using the case $N = 1$ applied to each T_i.

The proof of Lemma 4.6 is based on the following yet simpler lemma.

Lemma 4.7. *For every $1 \le \bar r < 2$ there is a constant $\sigma_2 = \sigma_2(Q, m, n, \bar n, 2-\bar r) > 0$ with the following property. Let T, $\boldsymbol{\pi}$, E, and H be as in Lemma 4.6, but instead of (4.7) assume that*

$$E \le \sigma_2 \qquad \text{and} \qquad H \ge 1\,. \tag{4.9}$$

Then (4.8) and all the conclusions of Theorem 3.2 hold in $\mathbf{C}_{\bar r}$.

Proof. Observe that since $H \ge 1$, by Chebyshev's inequality we have

$$\|T\|(\mathbf{C}_{\bar r} \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \ge \tfrac{H}{8}\}) \le 64\sigma_2\,.$$

On the other hand, for $\rho := \min\{2-\bar r, \frac{H}{8}\}$, if $\mathrm{spt}\,(T) \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \geq \frac{H}{4}\}$ is not empty the monotonicity formula would give

$$\|T\|(\mathbf{C}_{\bar r} \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \geq \tfrac{H}{8}\}) \geq C^{-1}\rho^m$$

for some dimensional constant $C > 0$, which yields a contradiction for a sufficiently small choice of σ_2. □

Proof of Lemma 4.6. Recall that we just need to prove (4.8). If $H \geq 1$ the claim follows from Lemma 4.7. If $H \leq 1$, we just need to show that

$$\mathrm{spt}(T) \cap \mathbf{C}_{\bar r H}(q) \cap \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) > \tfrac{H}{4}\} = \emptyset$$

whenever the cylinder $\mathbf{C}_{2H}(q)$ is contained in the original cylinder $\mathbf{C}_2$. But then it suffices to consider the current $T_{q,H} := (\lambda_{q,H})_\sharp T$ for the map $\lambda_{q,H}(x) := \frac{x-q}{H}$ and to apply Lemma 4.7 to $T_{q,H}$ and $\lambda_{q,H}(\boldsymbol{\pi})$: the pair falls under the assumptions provided $\sigma_1 \leq \sigma_2$, after an obvious scaling argument. □

We record another two simple observations which will be useful in the sequel. They are both proven in exactly the same way as Lemma 4.7.

Lemma 4.8. *For every $1 \leq r < 2$, there is a constant $\sigma_3 = \sigma_3(Q, m, n, \bar n, 2-r) > 0$ with the following property. Let T and $\boldsymbol{\pi}$ be as in Lemma 4.7, but instead of (4.9) assume only that*

$$E \leq \sigma_3\,. \tag{4.10}$$

Then $\mathrm{spt}\,(T) \cap \mathbf{C}_r \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \leq 1\}$.

Lemma 4.9. *For every $1 \leq r < 2$, there is a constant $\bar\sigma = \bar\sigma(Q, m, n, \bar n, N, 2-r) > 0$ such that the following holds. Let T and $\boldsymbol{\pi}$ be as in Lemma 4.7, but instead of (4.9) assume that $M := \max_{i,j}\{|p_i - p_j|\} \geq 1$ and*

$$E \leq \bar\sigma M^2\,. \tag{4.11}$$

Then $\mathrm{spt}\,(T) \cap \mathbf{C}_r \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \leq 2^{-N} M\}$.

Combining Lemma 4.9 with the combinatorial Lemma 4.5, we get the following separation lemma.

Lemma 4.10. *Under the assumptions of Lemma 4.9, there is a decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}^1 \cup \boldsymbol{\pi}^2$ and $T \llcorner \mathbf{C}_r = T_1 + T_2$ such that*

(i) *The sets $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2$ are disjoint, non-empty, and unions of a subset of the planes in $\boldsymbol{\pi}$;*
(ii) $\partial T_i \llcorner \mathbf{C}_r = 0$ *for $i = 1, 2$;*
(iii) $\mathrm{spt}\,(T_1) \cap \mathrm{spt}\,(T_2) = \emptyset$*;*
(iv) $\mathrm{dist}(q, \boldsymbol{\pi}) = \mathrm{dist}(q, \boldsymbol{\pi}^i)$ *for every $q \in \mathrm{spt}\,(T_i)$ and each $i = 1, 2$.*

Arguing as in the proof of Lemma 4.6, we can draw the following further conclusion.

Lemma 4.11. *For every $1 \leq r < 2$, there is a constant $\tilde\sigma = \tilde\sigma(Q, m, n, \bar n, N, 2-r) > 0$ such that the following holds. Let T and $\boldsymbol{\pi}$ be as in Lemma 4.6, but instead of (4.7), for $M := \max_{i,j}\{|p_i - p_j|\}$ assume that*

$$E \leq \tilde\sigma \min\{M, 1\}^{m+2}\,.$$

Then there is a decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}^1 \cup \boldsymbol{\pi}^2$ and $T \llcorner \mathbf{C}_r = T_1 + T_2$, as in Lemma 4.10.

5. Proof of tilt-excess estimate

The aim of this section is to prove (3.2). This will be done independently to the proof of L^∞ estimate (3.3); we defer the latter to the next section. We begin with two "approximate" estimates on the oriented tilt-excess.

5.1. **First lemma.** In this lemma we derive a first approximate estimate.

Lemma 5.1. *For every pair of radii $1 \le r < R \le 2$ there are constants $\bar C = \bar C(Q, m, n, \bar n, R - r) > 0$ and $\gamma = \gamma(Q, m, n, \bar n) > 0$ such that the following holds. Let T, Σ and $\mathbf{A}$ be as in Assumption 3.1, suppose that $p_1, \dots, p_N \in \pi_0^\perp$ are distinct points, and let $\boldsymbol{\pi} := \bigcup_i p_i + \pi_0$. Assume that E is as in (3.1) and let $H := \min\{|p_i - p_j| : i \neq j\}$. Then*

$$\mathbf{E}(T, \mathbf{C}_r) \le \bar C(E + \mathbf{A}^2) + \bar C \left(\frac{E}{H^2}\right)^\gamma \mathbf{E}(T, \mathbf{C}_R) + \bar C \mathbf{E}(T, \mathbf{C}_R)^{1+\gamma} . \tag{5.1}$$

Proof. We start by defining a suitable vector field $X : \mathbb{R}^{m+n} \to \pi_0^\perp$. In fact, we will actually define $\bar X : \pi_0^\perp \to \pi_0^\perp$, and then set $X(x) := \bar X(\mathbf{p}_{\pi_0}^\perp(x))$. Firstly, define $\tilde X$ on the union of the disks $B_{H/4}(p_i, \pi_0^\perp)$ by taking $\tilde X(y) := (y - p_i)$ on each $B_{H/4}(p_i, \pi_0^\perp)$. It is simple to check that the Lipschitz constant of $\tilde X$ can be bounded by 3; indeed, if y_1, y_2 lie in disks $B_{H/4}(p_{i_1}, \pi_0^\perp)$ and $B_{H/4}(p_{i_2}, \pi_0^\perp)$ respectively, then

$$|p_{i_1} - p_{i_2}| \le |y_1 - y_2| + H/2 \le |y_1 - y_2| + |p_{i_1} - p_{i_2}|/2 \implies |p_{i_1} - p_{i_2}| \le 2|y_1 - y_2|.$$

The desired Lipschitz bound of 3 follows immediately. We can thus use the Kirszbraun theorem to extend $\tilde X$ to a vector field $\bar X : \pi_0^\perp \to \pi_0^\perp$ which has the same Lipschitz constant; this determines $\bar X$, and in turn determines X. In particular, we deduce that X has the following properties:

(i) X takes values in $\pi_0^\perp$;
(ii) $|X(x)| \le 3\,\mathrm{dist}(x, \boldsymbol{\pi})$ and $|\nabla_v X(x)| \le 3|v|$;
(iii) $\nabla_v X = 0$ for every $v \in \pi_0$;
(iv) $X(x) = \mathbf{p}_{\pi_0}^\perp(x - p_i)$ if $\mathrm{dist}(x, p_i + \pi_0) \le \frac{H}{4}$.

Note that by regularizing $\bar X$ via convolution, we can obtain a smooth vector field with the same properties, except that (iv) will hold in a slightly smaller tubular neighborhood of $\boldsymbol{\pi}$. We will thus ignore the regularity issues and use X as a test in the first variation formula for T.

In order to simplify our notation we introduce the set

$$G := \left\{x \in \mathbf{C}_2 : \mathrm{dist}(x, \boldsymbol{\pi}) \le \frac{H}{4}\right\} = \bigcup_i \left\{x \in \mathbf{C}_2 : \mathrm{dist}(x, p_i + \pi_0) \le \frac{H}{4}\right\}$$

and write G^c for its complement in $\mathbf{C}_2$. Because of (iv), (see e.g. [5, Lemma 4.2]) we have

$$\mathrm{div}_{\vec T(x)} X(x) = \frac{1}{2}|\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2 \qquad \forall x \in G \cap \mathrm{spt}\,(T), \tag{5.2}$$

while, because of (i), (iii), and the Lipschitz regularity of X,

$$|\mathrm{div}_{\vec T(x)} X(x)| \le C|\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2 \qquad \forall x \in G^c \cap \mathrm{spt}\,(T) . \tag{5.3}$$

To see the latter, let $e_1, \dots, e_m$ be an orthonormal base of the tangent plane to T at x and compute

$$\begin{aligned}
|\mathrm{div}_{\vec T(x)} X(x)| &\le \sum_i |\nabla_{e_i} X(x) \cdot e_i| \overset{\text{(i)}}{=} \sum_i |\nabla_{e_i} X(x) \cdot (e_i - \mathbf{p}_{\pi_0}(e_i))| \\
&\overset{\text{(iii)}}{=} \sum_i |(\nabla_{e_i} X(x) - \nabla_{\mathbf{p}_{\pi_0}(e_i)} X(x)) \cdot (e_i - \mathbf{p}_{\pi_0}(e_i))| \\
&\le \sum_i |e_i - \mathbf{p}_{\pi_0}(e_i)||\nabla_{e_i - \mathbf{p}_{\pi_0}(e_i)} X(x)| \\
&\overset{\text{(ii)}}{\le} 3 \sum_i |e_i - \mathbf{p}_{\pi_0}(e_i)|^2 \le C|\mathbf{p}_{\vec T(x)} - \mathbf{p}_{\pi_0}|^2 .
\end{aligned}$$

Let us now choose radii $r_1 < r_2$ so that $r_1 - r = r_2 - r_1 = R - r_2$ (i.e. $r_1 = (R + 2r)/3$, $r_2 = (2R + r)/3$). Let $\tilde\chi : \pi_0 \to \mathbb{R}$ be a smooth cut-off function which is identically 1 on $B_{r_1}(0, \pi_0)$, vanishes outside $\bar B_{r_2}(0, \pi_0)$, and satisfies $|D\tilde\chi| \le 2/(r_2 - r_1)$. Extend $\tilde\chi$ vertically to get a function $\chi : \mathbb{R}^{m+n} \to \mathbb{R}$, namely $\chi(x) = \tilde\chi(\mathbf{p}_{\pi_0}(x))$. We now want to take $\chi^2 X$ as a test function in the first variation formula for (the stationary varifold associated to) T; since $\chi^2 X$ need not have compact support in the directions orthogonal to π, to justify this simply note that since χ is supported in the cylinder $\bar{\mathbf{C}}_{r_2}$, and since the support of T is compact on any cylinder which is slightly smaller than $\mathbf{C}_2$, we can multiply $\chi^2 X$ by another test function which is translation invariant in the π_0 variables, is 1 in the ball of radius $2\sup\{\mathbf{p}_{\pi_0}^\perp(x) : x \in \mathrm{spt}\|T\| \cap \bar{\mathbf{C}}_{r_2}\}$ in the $\pi_0^\perp$ variables, and vanishes outside the ball of radius $3\sup\{\mathbf{p}_{\pi_0}^\perp(x) : x \in \mathrm{spt}\|T\| \cap \bar{\mathbf{C}}_{r_2}\}$; this then has compact support in all of $\mathbf{C}_2$ and agrees with $\chi^2 X$ on $\mathbf{C}_2 \cap \mathrm{spt}(T)$, therefore allowing us to take $\chi^2 X$ as a test function. Hence, we have

$$\begin{aligned}\int_{\mathbf{C}_2} \chi^2 \mathrm{div}_{\vec{T}(x)} X(x) d\|T\|(x) = &- \int_{\mathbf{C}_2} \chi^2 X(x) \cdot \vec{H}_T(x)\, d\|T\|(x)\\ &- 2\int_{\mathbf{C}_2} \chi(x) \nabla_{\vec{T}(x)} \chi(x) \cdot X(x)\, d\|T\|(x)\,,\end{aligned}$$

where $\vec{H}_T$ is the generalized mean curvature of T. Combining this with (5.2), (5.3), (ii), and the fact that $|\vec{H}_T(x)| \leq C\mathbf{A}$, we get

$$\begin{aligned}\int_{\mathbf{C}_2} \chi^2 |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x) \leq{}& C\int_{\mathbf{C}_{r_2}\cap G^c} |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x)\\ &+ C(E + \mathbf{A}^2) + 2\left|\int_{\mathbf{C}_2} \chi(x)\nabla_{\vec{T}(x)}\chi(x) \cdot X(x)\, d\|T\|(x)\right| .\end{aligned}$$

Observe next that in light of (i), $\nabla_{\vec{T}}\chi \cdot X = (\nabla_{\vec{T}}\chi - \nabla_{\vec{\pi}_0}\chi) \cdot X(x)$. We can thus estimate further

$$\begin{aligned}2\left|\int_{\mathbf{C}_2} \chi(x)\nabla_{\vec{T}(x)}\chi(x) \cdot X(x)\, d\|T\|(x)\right| &\leq \frac{C}{R-r}\int_{\mathbf{C}_{r_2}} \chi(x)|\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}||X(x)|\, d\|T\|(x)\\ &\leq \frac{CE}{(R-r)^2} + \frac{1}{4}\int_{\mathbf{C}_2} \chi^2(x)|\mathbf{p}_{\pi_0} - \mathbf{p}_{T(x)}|^2\, d\|T\|(x),\end{aligned}$$

where in the latter inequality we have used that $ab \leq \frac{a}{2\varepsilon} + \frac{\varepsilon b}{2}$ for any two non-negative numbers a and b and a suitably small choice of $\varepsilon > 0$, combined with (ii). In particular, we conclude that

$$\mathbf{E}^{no}(T, \mathbf{C}_{r_1}) \leq \bar{C}(E + \mathbf{A}^2) + C\int_{\mathbf{C}_{r_2}\cap G^c} |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x)\,,$$

for $\bar{C}$ now also dependent on $R - r$. When combined with Proposition 4.1, we arrive at

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(E + \mathbf{A}^2) + C\int_{\mathbf{C}_{r_2}\cap G^c} |\mathbf{p}_{T(x)} - \mathbf{p}_{\pi_0}|^2\, d\|T\|(x)\,.$$

where again $\bar{C} = \bar{C}(Q, m, n, \bar{n}, R - r) > 0$ (we recall that $r_1 - r = (R - r)/3$ here). We now use (a scaled version of) Almgren's estimate [11, Theorem 7.1] (the first version of this estimate is in [2, Sections 3.24–3.26, Section 3.30(8)]), applied within $\mathbf{C}_{r_2}$, with the choice $A = \mathbf{p}_{\pi_0}(\mathbf{C}_{r_2} \cap G^c \cap \mathrm{spt}(T)) \equiv B_{r_2}(\pi_0) \cap \mathbf{p}_{\pi_0}(G^c \cap \mathrm{spt}(T))$, to conclude that

$$\begin{aligned}\mathbf{E}(T, \mathbf{C}_r) &\leq \bar{C}(E + \mathbf{A}^2) + \bar{C}(\mathbf{E}(T, \mathbf{C}_R) + \mathbf{A}^2)^{1+\gamma} + \bar{C}|A|^\gamma(\mathbf{E}(T, \mathbf{C}_R) + \mathbf{A}^2)\\ &\leq \bar{C}(E + \mathbf{A}^2) + \bar{C}\mathbf{E}(T, \mathbf{C}_R)^{1+\gamma} + \bar{C}\|T\|(G^c \cap \mathbf{C}_R)^\gamma \mathbf{E}(T, \mathbf{C}_R)\,,\end{aligned}$$

where we have used that $\mathbf{A} \leq 1$ and $|A| \leq \|T\|(G^c \cap \mathbf{C}_R)$ (here, $|A|$ denotes the measure of the set A).

However, observe that $\mathrm{dist}(x, \boldsymbol{\pi}) \geq \frac{H}{4}$ for all $x \in G^c$ and thus by Chebyshev we get

$$\|T\|(G^c \cap \mathbf{C}_R) \leq \frac{CE}{H^2}\,,$$

which concludes the proof. □

5.2. **Second lemma.** Now, we iterate Lemma 5.1 to achieve a closer approximation of (3.2), but under the additional assumption that the height excess of T relative to the family of planes is much smaller than the minimal separation of the planes. We will remove this assumption in the next section.

Proposition 5.2. *For every pair of scales $1 \le r < r_0 < 2$, there are constants $\bar{C} = \bar{C}(Q, m, n, \bar{n}, N, r_0 - r, 2 - r_0) > 0$ and $\sigma_4 = \sigma_4(Q, m, n, \bar{n}, N, r_0 - r, 2 - r_0) > 0$ with the following properties. Assume T, $\boldsymbol{\pi}$ and H are as Lemma 5.1. If in addition we have*

$$E \le \sigma_4 \min\{H^2, 1\}, \tag{5.4}$$

then

$$\mathbf{E}(T, \mathbf{C}_r) \le \bar{C}(E + \mathbf{A}^2) + \bar{C}\left(\frac{E}{H^2}\right)\mathbf{E}(T, \mathbf{C}_{r_0}). \tag{5.5}$$

Proof. This estimate will be proved by a two-variable induction over Q and N. Our inductive assumption is that

(IH) *The estimate is valid for any pair $Q' \le Q$ and $N' \le N$ with $Q' + N' < Q + N$.*

By Proposition 4.2, the case $N = 1$ and arbitrary Q is always valid. Moreover, due to (5.4), we can assume without loss of generality that $H \le 1$, since otherwise we can apply Lemma 4.7, yielding the conclusions of Theorem 3.2 in $\mathbf{C}_r$, which in particular trivially implies (5.5).

We can similarly assume that $M := \max\{|p_i - p_j|\} \le 1$ and

$$\frac{E}{M^{m+2}} \ge \tilde{\sigma}. \tag{5.6}$$

Indeed, if $M > 1$ then we can apply Lemma 4.10 to decompose T into T_1, T_2 (as in the statement of the lemma) and appeal to the induction assumption (IH). Note that the validity of the hypothesis (5.4) remains unchanged under such a decomposition (as E can only decrease while H can only increase). We can apply (IH) since if both $T_1, T_2 \neq 0$, then necessarily $\Theta(T_i, \cdot) \le Q - 1$ (and so Q is decreased), while if, say, $T_2 = 0$, then Q remains the same for T_1, but N decreases by at least 1. Thus, we may indeed assume that $M \le 1$. Given this, one can additionally assume that (5.6) indeed holds via the same argument, except now invoking Lemma 4.11 instead of Lemma 4.10.

Now observe that for every given $\pi_i \in \boldsymbol{\pi}$ we can easily estimate

$$\int_{\mathbf{C}_2} \mathrm{dist}^2(x, \pi_i)\, d\|T\|(x) \le 2M^2 + 2E\,;$$

Thus, using Proposition 4.2 (for any fixed plane $\pi_i \in \boldsymbol{\pi}$), we have for any $1 \le R < 2$,

$$\mathbf{E}(T, \mathbf{C}_R) \le \bar{C}(2M^2 + 2E + \mathbf{A}^2)\,. \tag{5.7}$$

where $\bar{C}$ also depends on $2-R$. Under our current assumptions, the right-hand side of (5.7) will be at most $\bar{C}$, provided we take σ_4 sufficiently small (depending only on allowed parameters). In particular, combining this with (5.6), and the assumption $H \le 1$ (so $E \le 1$ also) gives

$$\mathbf{E}(T, \mathbf{C}_R) \le \bar{C}\mathbf{A}^2 + \bar{C}E^{\frac{2}{m+2}} \le \tilde{C}\mathbf{A}^2 + \bar{C}\left(\frac{E}{H^2}\right)^{\frac{2}{m+2}},$$

where $\bar{C}$ now also depends on $\tilde{\sigma}$. Now taking $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ as in Lemma 5.1, the above bound clearly implies

$$\mathbf{E}(T, \mathbf{C}_R)^\gamma \le \bar{C}\mathbf{A}^{2\gamma} + \bar{C}\left(\frac{E}{H^2}\right)^{\frac{2\gamma}{m+2}},$$

where now $\bar{C}$ also depends on γ. Using Lemma 5.1 and (5.4) we now infer that for any $1 \le \rho < R < 2$ we have

$$\begin{aligned}\mathbf{E}(T, \mathbf{C}_\rho) &\le \bar{C}(E + \mathbf{A}^2) + \tilde{C}\left[\left(\frac{E}{H^2}\right)^{\frac{2\gamma}{m+2}} + \mathbf{A}^{2\gamma}\right]\mathbf{E}(T, \mathbf{C}_R) \\ &\le \bar{C}_1(E + \mathbf{A}^2) + \bar{C}_1\left(\sigma_4^{\frac{\gamma}{m+2}} + \mathbf{A}^{\frac{(m+3)\gamma}{m+2}}\right)\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}}\mathbf{E}(T, \mathbf{C}_R)\end{aligned} \tag{5.8}$$

where $\bar{C}_1 = \bar{C}_1 C(Q, m, n, \bar{n}, \gamma, \tilde{\sigma}, R - \rho, 2 - R)$. By selecting σ_4 small enough we can ensure that

$$\bar{C}_1 \sigma_4^{\frac{\gamma}{m+2}} \leq \frac{1}{4}\,.$$

Next, if $\bar{C}_1 \mathbf{A}^{\frac{(m+3)\gamma}{m+2}} \leq \frac{1}{4}$ we infer that

$$\mathbf{E}(T, \mathbf{C}_\rho) \leq \bar{C}_1(E + \mathbf{A}^2) + \frac{1}{2}\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_R)\,. \tag{5.9}$$

Otherwise, if $\bar{C}_1 \mathbf{A}^{\frac{(m+3)\gamma}{m+2}} \geq \frac{1}{4}$, we can estimate $\mathbf{A}^{2\gamma} \leq \bar{C}_2 \mathbf{A}^2$ for some constant $\bar{C}_2$ with the same dependencies as $\bar{C}_1$. Then, noting that (5.7) in particular gives that $\mathbf{E}(T, \mathbf{C}_R) \leq \bar{C}$ (as remarked previously), from (5.8) we have (with the same choice of σ_4 mentioned previously)

$$\mathbf{E}(T, \mathbf{C}_\rho) \leq \bar{C}_3(E + \mathbf{A}^2) + \frac{1}{4}\left(\frac{E}{H^2}\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_R)$$

with a worse constant $\bar{C}_3$ (with the same dependencies as $\bar{C}$). We can therefore assume (5.9) to be valid irrespective of the value of $\mathbf{A}$.

We are now in a position to iterate (5.9). Fix the smallest natural number $J = J(Q, m, n, \bar{n})$ such that $\frac{J\gamma}{m+2} \geq 1$ and apply (5.9) with $\rho = r_i$, $R = r_{i-1}$ for a sequence of radii r_i, where $r_J = r$, $r_{i-1} = r_i + \frac{r_0 - r}{J}$. This, in particular, fixes the size of the difference between radii used in all the inequalities used above, and in particular it gives a uniform bound for the constants above, hence fixing the choice of $\sigma_4 = \sigma_4(Q, m, n, \bar{n}, r_0 - r, 2 - r_0) > 0$.

If for some i we have

$$\left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_{i-1}}) \leq \mathbf{E}(T, \mathbf{C}_{r_i})$$

we then can absorb the second term in the right hand side of (5.9) into the left hand side to conclude that

$$\mathbf{E}(T, \mathbf{C}_{r_i}) \leq \bar{C}(E + \mathbf{A}^2)\,.$$

Given that $1 \leq r \leq r_i < 2$ we achieve the desired estimate (5.5) in this case. Otherwise, we must have

$$\mathbf{E}(T, \mathbf{C}_{r_i}) \leq \left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_{i-1}})$$

for all i, which leads us to

$$\mathbf{E}(T, \mathbf{C}_r) = \mathbf{E}(T, \mathbf{C}_{r_J}) \leq \left(\frac{E}{H^2} + \mathbf{A}^2\right)^{\frac{J\gamma}{m+2}} \mathbf{E}(T, \mathbf{C}_{r_0})\,. \tag{5.10}$$

Given that $\frac{J\gamma}{m+2} \geq 1$ (so we may write $\frac{J\gamma}{m+2} = 1 + \tilde{\gamma}$ for some $\tilde{\gamma} \geq 0$), and since $\frac{E}{H^2} \leq \sigma_4 \leq 1$, we again arrive at the desired estimates (5.5) (using again that $\mathbf{E}(T, \mathbf{C}_{r_0}) \leq \bar{C}$ from (5.7) to absorb the $\mathbf{A}^2$ factor from the parenthesis into the first term on the right-hand side of (5.5)). □

5.3. **Proof of the tilt-excess estimate.** We are finally in a position to prove (3.2). The goal is to exploit the combinatorial results in Section 4.3 to remove the hypothesis (5.4) in Proposition 5.2 by possibly replacing $\boldsymbol{\pi}$ with a smaller, refined sub-collection of planes $\bar{\boldsymbol{\pi}}$, and in turn conclude the tilt-excess estimate for $\bar{\boldsymbol{\pi}}$.

Let $\boldsymbol{\pi} := \bigcup_i (p_i + \pi_0)$ be as in the statement of Theorem 3.2. Fix a positive parameter $0 < \delta \leq 1/2$ (whose choice will be determined later) and apply Lemma 4.4 to $P = \{p_1, \ldots, p_N\}$ with this choice of δ and $\varepsilon = E^{1/2}$, yielding a subset $\tilde{P} \subset P$ obeying the conclusions of Lemma 4.4. Let $\tilde{\boldsymbol{\pi}} = \bigcup_{\tilde{p} \in \tilde{P}} (\tilde{p} + \pi_0) \subset \boldsymbol{\pi}$ be the corresponding union of parallel planes. By property (i) of Lemma 4.4 we then have

$$\tilde{E} := \int_{\mathbf{C}_2} \mathrm{dist}^2(q, \tilde{\boldsymbol{\pi}})\, d\|T\|(q) \leq 2E + C \max_i \mathrm{dist}(p_i, \tilde{P})^2 \tag{5.11}$$

and hence

$$\tilde{E} \leq C(1 + \delta^{-2}(1 + \delta^{-1})^{2N-4})E\,. \tag{5.12}$$

If $\tilde{P}$ is a singleton, then we can apply Proposition 4.2 to T and $\bar{\boldsymbol{\pi}}$ to conclude (3.2) from (5.12), since we are then in the case $N = 1$ of Theorem 3.2.

We may thus henceforth assume that $\tilde{P}$ consists of at least two distinct points. In this case, property (ii) of Lemma 4.4 gives that $E^{1/2} \leq \delta\tilde{H}$ and $\max_i \operatorname{dist}(p_i, \tilde{P}) \leq \delta\tilde{H}$, where $\tilde{H} := \min\{|\tilde{p} - \tilde{q}| : \tilde{p}, \tilde{q} \in \tilde{P}, \tilde{p} \neq \tilde{q}\}$, and thus combining this with (5.11), we get

$$\tilde{E} \leq C\delta^2\tilde{H}^2\,, \tag{5.13}$$

where the constant $C = C(Q, m, n, \bar{n})$ is independent of δ.

Fix another constant $0 < \bar{\delta} \leq 1/2$ (to be determined later). We now apply Lemma 4.3 with this $\bar{\delta}$ to further refine $\tilde{\boldsymbol{\pi}}$, finding a second subset $P_* \subset \tilde{P}$ with the properties listed in Lemma 4.3. By property (i) of Lemma 4.3, if we denote by $M_* := \max\{|p - q| : p, q \in P'\}$ and $H_* := \min\{|p' - q'| : p', q' \in P',\ p' \neq q'\}$, we achieve that

$$M_* \leq \bar{C}(\bar{\delta}, N)H_* \tag{5.14}$$

and, combining (ii) of Lemma 4.3 with (5.13), we have

$$E_* := \int_{\mathbf{C}_2} \operatorname{dist}^2(q, \boldsymbol{\pi}_*)\, d\|T\|(q) \leq 2\tilde{E} + C\bar{\delta}^2H_*^2 \leq C(\delta^2 + \bar{\delta}^2)H_*^2\,, \tag{5.15}$$

since $\tilde{H} \leq H_*$. Here, $C = C(Q, m, n, \bar{n})$ is a new constant which we stress is independent of $\bar{\delta}$, and $\boldsymbol{\pi}_* := \bigcup_{p_* \in P_*}(p_* + \pi_0)$ is the union of parallel planes corresponding to P_*. Let $J := \#(\tilde{P} \setminus P_*)$ and let $P_j \subset \tilde{P}$ be the collections of points given by property (iii) of Lemma 4.3; in particular, $P_0 = \tilde{P}$, $P_J = P_*$. Define radii $(r_j)_{j=0}^{J+2}$ by $r_{J+2} = 2$, $r_0 = r$, and $r_j - r_{j-1} = \frac{2-r}{J+2}$. We will apply various estimates on the tilt-excess between the radii $r_j < r_{j-1}$; note that the parameter σ_4 in Proposition 5.2 when applied at such scales obeys $\sigma_4 = \sigma_4\big(Q, m, n, \bar{n}, N, (2-r)/(J+2)\big)$ and so is now fixed (independent of j). One should note however that the dependence of σ_4 in the radius variable, $(2-r)/(J+2)$, only actually depends on a lower bound on the radius, which here is $(2-r)/N$, meaning $\sigma_4 = \sigma_4(Q, m, n, \bar{n}, N, 2-r)$. Recalling (5.13) and (5.15), we now choose δ and $\bar{\delta}$, depending only on $Q, m, n, \bar{n}$, so that

$$E_* \leq \sigma_4 H_*^2 \qquad \text{and} \qquad \tilde{E} \leq \sigma_4\tilde{H}^2\,. \tag{5.16}$$

We may further assume that $H_*, \tilde{H} \leq 1$, since otherwise we may apply Lemma 4.7 to reach the desired conclusion. Having fixed all the parameters, we may henceforth treat all constants depending on them as just $\bar{C} = \bar{C}(Q, m, n, \bar{n}, N, 2-r)$.

Now, applying Proposition 4.2 (or, more precisely, the $N = 1$ case of (3.2)) to T with $r = r_{J+1}$ and $p + \pi_0$ in place of π_0 for any $p \in P_*$, we get

$$\mathbf{E}(T, \mathbf{C}_{r_{J+1}}) \leq \bar{C}(E_* + M_*^2 + \mathbf{A}^2) \leq \tilde{C}(E_* + H_*^2 + \mathbf{A}^2)$$

using (5.14) in the last inequality (we stress that this is the only time we need to apply Proposition 4.2, namely to $P_* = P_J$ as it is the only time we have comparability between the maximum and minimum distances between the planes in $\boldsymbol{\pi}_j$, as defined below). If we now apply Proposition 5.2 with $\boldsymbol{\pi}_*$, r_J, r_{J+1} in place of $\boldsymbol{\pi}$, r, r_0 respectively, we get

$$\begin{aligned}\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_* + \mathbf{A}^2) + \bar{C}\left(\frac{E_*}{H_*^2}\right)\mathbf{E}(T, \mathbf{C}_{r_{J+1}}) &\leq \bar{C}(E_* + \mathbf{A}^2) + \bar{C}\left(\frac{E_*}{H_*^2}\right)(E_* + H_*^2 + \mathbf{A}^2)\\ &\leq \bar{C}(E_* + \mathbf{A}^2)\end{aligned} \tag{5.17}$$

where in the last inequality we have used the inequality $E_* \leq \sigma_4 H_*^2$ from (5.16).

Next, for $j = 0, 1, \ldots, J$, define the collections $\boldsymbol{\pi}_j := \bigcup_{p \in P_j}(p + \pi_0)$ of parallel planes associated to the sets P_j; we remark that $P_J = P_*$. For each such J set

$$E_j := \int_{\mathbf{C}_2} \operatorname{dist}^2(q, \boldsymbol{\pi}_j)\ \|T\|(q)$$

and

$$H_j := \min\{|p - q| : p, q \in P_j,\ p \neq q\}\,.$$

Now observe that, for each $j = 0, 1, \ldots, J$, by property (iii) of Lemma 4.3, we have that

$$E_j \leq 2E_{j-1} + CH_{j-1}^2. \tag{5.18}$$

Now combining (5.17), (5.18) with $j = J$, and the fact that $E_* = E_J$, we have

$$\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + H_{J-1}^2 + \mathbf{A}^2).$$

We now distinguish two possibilities. If $E_{J-1} \leq \sigma_4 H_{J-1}^2$, then we can apply Proposition 5.2 with $\boldsymbol{\pi}_{J-1}$, r_{J-1}, r_J in place of $\boldsymbol{\pi}$, r, r_0 to get

$$\mathbf{E}(T, \mathbf{C}_{r_{J-1}}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2) + \bar{C}\left(\frac{E_{J-1}}{H_{J-1}^2}\right)\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2).$$

On the other hand, if the opposite inequality holds, namely $H_{J-1}^2 \leq \sigma_4^{-1} E_{J-1}$, then we can estimate directly and see that

$$\mathbf{E}(T, \mathbf{C}_{r_{J-1}}) \leq \bar{C}\mathbf{E}(T, \mathbf{C}_{r_J}) \leq \bar{C}(E_{J-1} + \mathbf{A}^2)$$

where we remark that we have used in the first inequality that $r \geq 1$ here. So, we see that in either situation, this inequality holds.

Now iterate this argument, namely the one beginning after (5.18), we see that we get

$$\mathbf{E}(T, \mathbf{C}_{r_j}) \leq \bar{C}(E_j + \mathbf{A}^2)$$

for each $j = J, J-1, \ldots, 0$. In particular, taking $j = 0$, we get

$$\mathbf{E}(T, \mathbf{C}_r) \leq \bar{C}(E_0 + \mathbf{A}^2).$$

However, since $E_0 = \tilde{E}$ and $\tilde{E} \leq CE$ from (5.12), we reach the desired conclusion. □

6. Proof of L^∞ height bound

It remains to prove (3.3) of Theorem 3.2, which will be achieved by induction on N. As already observed, if $N = 1$ we know that Theorem 3.2 holds (for all Q) by Proposition 4.2. The core of the inductive argument is the following proposition.

Proposition 6.1. *Fix $N \geq 2$ and assume that, under the assumptions of Theorem 3.2, (3.3) holds for any $N' < N$ and any $Q' \leq Q$. Then it holds for N and Q.*

Clearly once we have shown this, Theorem 3.2 follows by induction.

6.1. A decay lemma.

The crucial ingredient for Proposition 6.1 is the following L^2 height excess decay, which crucially relies on the tilt-excess estimate (3.2) that we have already established.

Lemma 6.2. *There are constants $\rho_0 = \rho_0(m, n, Q) > 0$ and $C = C(Q, m, n, \bar{n}) > 0$ such that, for every fixed $0 < \rho \leq \rho_0$, there are $\sigma_5 = \sigma_5(Q, m, n, \bar{n}, N, \rho) > 0$ and $0 < \beta_0 = \beta_0(Q, m, n, \bar{n}) < 1$ such that the following holds. Assume T, E, and $\boldsymbol{\pi}$ are as in Theorem 3.2 with $P = \{p_1, \ldots, p_N\}$ and that*

$$E + \mathbf{A}^2 \leq \sigma_5. \tag{6.1}$$

Then there is another set of points $P' := \{q_1, \ldots, q_{N'}\}$ with $N' \leq Q$ such that:

(A) $\mathrm{dist}(q_i, P) \leq C(E + \mathbf{A}^2)^{1/2}$ *for each i;*
(B) *If we set $\boldsymbol{\pi}' := \bigcup(q_i + \pi_0)$, then*

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) \leq \rho^{m+2\beta_0}(E + \mathbf{A}^2)\,. \tag{6.2}$$

Remark 6.3. Note that we need not have $N' \leq N$ in the conclusion of Lemma 6.2; the number of planes in the new collection $\boldsymbol{\pi}'$ may increase.

Proof. As usual constants denoted by C will depend only upon $Q, m, n, \bar{n}$ (their dependence on N can be reduced to a dependence on Q given that $N \in \{1, \ldots, Q\}$). Recall that we have just shown the validity of the tilt-excess estimate (3.2), and thus (taking $r = 1$ in (3.2)), we have

$$\mathbf{E} := \mathbf{E}(T, \mathbf{C}_1) \leq C(E + \mathbf{A}^2)\,. \tag{6.3}$$

By choosing $\sigma_5 = \sigma_5(Q, m, n, \bar{n}, N) > 0$ sufficiently small, we can therefore ensure that both $\mathbf{E}$ and $\mathbf{A}$ are as small as we wish. We now subdivide into two cases, depending on the relative sizes of $\mathbf{E}$ and $\mathbf{A}$.

Case 1: $\mathbf{A}^3 \leq \mathbf{E}$. Here we apply the strong Lipschitz approximation theorem [11, Theorem 2.4]. By translating, we may without loss of generality assume that the origin belongs to $\mathrm{spt}\,(T)$ (and hence to the manifold Σ), that $\pi_0 = \mathbb{R}^m \times \{0\}$ (by rotating) and that $\Psi \equiv \Psi_0 : \mathbb{R}^{m+\bar{n}} \to \mathbb{R}^{n-\bar{n}}$ is the map parametrizing Σ graphically over $\mathbb{R}^{m+\bar{n}}$ in $\mathbf{C}_2$, as in Assumption 2.1. By [11, Remark 2.5], we have the estimates

$$\Psi(0) = 0 \quad \text{and} \quad \|D\Psi\|_{C^2} \leq C(\mathbf{E}^{1/2} + \mathbf{A})\,; \tag{6.4}$$

here, $C = C(m,n,\bar{n})$. By [11, Theorem 2.4] there exists constants $\gamma = \gamma(Q,m,n,\bar{n}) > 0$ and $\varepsilon = \varepsilon(Q,m,n,\bar{n}) > 0$ such that if $\mathbf{E} < \varepsilon$ (which can be guaranteed provided σ_5 is sufficiently small), then there is a multi-valued map $u : B_{1/4}(0,\pi_0) \to \mathbb{R}^{\bar{n}}$ such that $f = (u, \Psi(x,u))$ (using the notation of [11]) is a good approximation of $T \llcorner \mathbf{C}_{1/4}$, in the following sense:

(i) $\mathrm{Lip}\,(f) \leq C(\mathbf{E} + \mathbf{A}^2)^\gamma \leq C(E + \mathbf{A}^2)^\gamma$;

(ii) There is a closed set $K \subset B_{1/4}$ of measure at least $\frac{1}{2}|B_{1/4}|$ such that $\mathbf{G}_f \llcorner (K \times \mathbb{R}^n) = T \llcorner (K \times \mathbb{R}^n)$ and

$$\|T\|((B_{1/4} \setminus K) \times \mathbb{R}^n) \leq C(\mathbf{E} + \mathbf{A}^2)^{1+\gamma} \leq C(E + \mathbf{A}^2)^{1+\gamma}\,;$$

where $C = C(Q,m,n,\bar{n})$. Notice that in the above estimates, we have used (6.3) to get the improved control in terms of the L^2 height excess. From the above properties, the estimate [11, Theorem 2.4(2.6)] and (6.4) we also see that

$$\int_{B_{1/4}} |Df|^2 \leq C \int_K |Df|^2 + C(\mathbf{E} + \mathbf{A}^2)^{1+\gamma} \leq C\mathbf{E} + C(\mathbf{E} + \mathbf{A}^2)^{1+\gamma} \overset{(6.1)}{\leq} C(\mathbf{E} + \mathbf{A}^2)\,. \tag{6.5}$$

Moreover, for every fixed η, if $\mathbf{E}$ is sufficiently small (depending on η), by [11, Theorem 2.6] there is a Dir-minimizing function $v : B_{1/4} \to \mathcal{A}_Q(\mathbb{R}^{\bar{n}})$ such that, if we set $g = (v, \Psi(x,v))$, then

$$\int_{B_{1/4}} \mathcal{G}(f,g)^2 \leq \eta \mathbf{E} \tag{6.6}$$

$$\int_{B_{1/4}} |Dg|^2 \leq \int_{B_{1/4}} |Df|^2 + \eta\mathbf{E} \overset{(6.5)}{\leq} C(\mathbf{E} + \mathbf{A}^2) \overset{(6.3)}{\leq} C(E + \mathbf{A}^2)\,. \tag{6.7}$$

Note that the condition $\mathbf{A} \leq \mathbf{E}^{\frac{1}{4}+\bar{\delta}}$ in [11, Theorem 2.6] is satisfied, with $\bar{\delta} = 1/12$ in this case, since we are assuming $\mathbf{A}^3 \leq \mathbf{E}$.

Observe now that, by a simple Chebyshev argument, for at least half of the points $x \in K \cap B_{1/8}$ we have the following property:

(a) if $f(x) = \sum_i Q_i [\![f_i(x)]\!]$ (with $f_i(x)$ distinct), for each i there is a $j(i)$ such that $|f_i(x) - p_{j(i)}| \leq C E^{1/2}$.

Hence, by another Chebyshev argument, combined with (6.6), we may find at least one such point $x \in K \cap B_{1/8}$ for which we have the corresponding property for $g(x)$ (in fact, we can find a set of positive measure on which this holds):

(b) if $g(x) = \sum_i \tilde{Q}_i [\![g_i(x)]\!]$ (with $g_i(x)$ distinct), for every i there is a $j(i)$ such that $|g_i(x) - p_{j(i)}| \leq C E^{1/2}$.

Now, combining (6.7) with the Hölder estimate [10, Theorem 3.9] for Dir-minimizing functions, we also conclude an analogous estimate at 0:

(c) if $g_i(0) = \sum_i Q_i^* [\![g_i(0)]\!]$ (with $g_i(0)$ distinct), for every i there is a $j(i)$ such that $|g_i(0) - p_{j(i)}| \leq C(E^{1/2} + \mathbf{A})$

Notice that since $x \in B_{1/8}$, $\mathrm{dist}(x, \partial B_{1/4}) \geq C^{-1}$ and so the choice of δ in [10, Theorem 3.9] only depends on n, m, Q). We now set $q_i := g_i(0)$; obviously the number of distinct points q_i is at most Q. Clearly, by (c) above, this choice of q_i obeys conclusion (A) of the lemma.

Observe that, by all the estimates carried over so far and by Lemma 4.8, for each $\rho \leq \frac{1}{8}$ and some $\alpha = \alpha(m,Q) \in]0,1[$ we have

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x, \boldsymbol{\pi}')\, d\|T\|(x) \overset{\text{(ii)}}{\leq} C(E + \mathbf{A}^2)^{1+\gamma} + C \int_{B_{2\rho}} \mathcal{G}(f(x), g(0))^2\, dx$$

$$\leq C\sigma_5^\gamma(E+\mathbf{A}^2) + C\eta(E+\mathbf{A}^2) + C\rho^{m+2\alpha}(E+\mathbf{A}^2)\,;$$

where Lemma 4.8 was used in the first inequality, while in the second inequality we use (6.1), (6.6), the α-Hölder continuity of g from [10, Theorem 3.9], and (6.7).

Now choose $\beta_0 \in]0,\alpha/2]$, and choose $\rho_0 > 0$ with $C\rho_0^\alpha \leq \frac{1}{3}$. Then choose $\eta = \eta(\rho)$ such that $C\eta \leq \frac{1}{3}\rho^{m+2\beta_0}$. Then, given this choice of η, provided that we choose σ_5 sufficiently small so that our application of [11, Theorem 2.6] was valid with this η, and so that $C\sigma_5^\gamma \leq \frac{1}{3}\rho^{m+2\beta_0}$, the above estimate clearly then gives (B) of the lemma. This completes the proof of Case 1.

Case 2: In this case we have $\mathbf{E} \leq \mathbf{A}^3$. As in the previous case, using the same notation, we introduce the Lipschitz approximation $f = (u, \Psi(x,u))$ on $B_{1/4}(0,\pi_0)$. Observe that, since the graph of f coincides with the current T on $K \times \mathbb{R}^n \subset B_{1/4} \times \mathbb{R}^n$, we again have the first two inequalities of (6.5), only now we further estimate this as follows:

$$\int_{B_{1/4}} |Df|^2 \leq C\mathbf{E} + C(\mathbf{E}+\mathbf{A}^2)^{1+\gamma} \leq C\mathbf{A}^{2+2\gamma} \leq C\sigma_5^\gamma \mathbf{A}^2\,.$$

We then use the Poincaré inequality for Q-valued functions (see e.g. [10, Proposition 2.12 & Proposition 4.9]) and Hölder's inequality to find a point $Y \in \mathcal{A}_Q$ such that

$$\int_{B_{1/4}} \mathcal{G}(f,Y)^2 \leq C\left(\int_{B_{1/4}} \mathcal{G}(f,Y)^{2^*}\right)^{2/2^*} \leq C\int_{B_{1/4}} |Df|^2\,.$$

In particular we reach

$$\int_{B_{1/4}} \mathcal{G}(f,Y)^2 \leq C\sigma_5^\gamma \mathbf{A}^2\,. \tag{6.8}$$

Write $Y = \sum_i Q_i[\![q_i]\!]$, where the q_i are distinct, and set $\boldsymbol{\pi}' := \bigcup_i (q_i + \pi_0)$. We then find that, for every $0 < \rho \leq \frac{1}{8}$, by Lemma 4.8, (6.8), and (ii), we have

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x,\boldsymbol{\pi}')\,d\|T\|(x) \leq C\sigma_5^\gamma\mathbf{A}^2 + \|T\|((B_{1/4}\setminus K)\times\mathbb{R}^n) \leq C\sigma_5^\gamma \mathbf{A}^2\,.$$

In particular, for every fixed ρ, we may fix β_0 to be an arbitrary positive dimensional constant and choose σ_5 small enough (dependent on β_0) to guarantee that $C\sigma_5^\gamma \leq \rho^{m+2\beta_0}$

$$\int_{\mathbf{C}_{2\rho}} \mathrm{dist}^2(x,\boldsymbol{\pi}')\,d\|T\|(x) \leq \rho^{m+2\beta_0}\mathbf{A}^2\,.$$

This proves conclusion (B) of the lemma, in this case.

We are left with proving the estimate (A). Recall that, in light of (ii), we have $|B_{1/4}\setminus K| \leq \frac{1}{2}|B_{1/4}|$, and hence, via a similar Chebyshev argument as in Case 1, now using (6.8), we may estimate

$$\mathrm{dist}^2(q_i, \mathrm{spt}\, f(x)) \leq C\mathbf{A}^2 \qquad \forall i\,,$$

for all x in a subset $K' \subset K$ whose measure is at least $\frac{1}{4}|B_{1/4}|$. On the other hand, on at least half of the set K' we also have by another Chebyshev argument

$$\max_{p\in \mathrm{spt}\, f(x)} \mathrm{dist}^2(p,\boldsymbol{\pi}) \leq CE\,.$$

Therefore, choosing a point $p \in \mathrm{spt} f(x)$ for x in this latter set and using the triangle inequality we get

$$\mathrm{dist}^2(q_i,\boldsymbol{\pi}) \leq C(E+\mathbf{A}^2) \qquad \forall i\,,$$

which proves (A) and thus completes Case 2, which also completes the proof. □

6.2. **Proof of Proposition 6.1.** Having proved the L^2 height excess decay lemma, we are now in a position to prove the inductive step given in Proposition 6.1. Fix $1 \le r < 2$ as in the statement of Theorem 3.2 and set

$$M := \max_{i,j}\{|p_i - p_j|\}\,.$$

We will show the existence of a constant $\sigma_6 = \sigma_6(Q, m, n, \bar n, N, r) > 0$ such that, if

$$E + \mathbf{A}^2 \le \sigma_6 M^2\,, \tag{6.9}$$

then there is decomposition $\boldsymbol{\pi} = \boldsymbol{\pi}_1 \cup \boldsymbol{\pi}_2$ and $T \llcorner \mathbf{C}_{1+r/2} = T_1 + T_2$ such that

(i) The sets $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ are disjoint and nonempty;
(ii) $\partial T_i \llcorner \mathbf{C}_{1+r/2} = 0$;
(iii) $\mathrm{spt}\,(T_1) \cap \mathrm{spt}\,(T_2) = \emptyset$;
(iv) $\mathrm{dist}(q, \boldsymbol{\pi}) = \mathrm{dist}(q, \boldsymbol{\pi}_i)$ for every $q \in \mathrm{spt}\,(T_i)$, for each $i = 1, 2$.

In particular, assuming such a decomposition, after rescaling $1 + r/2$ to scale 2 we can apply the inductive assumption to T_1 and T_2 (note that whilst we could have, e.g. $T_2 = 0$, i.e. Q remains fixed, however by (i) we then know that N strictly decreases) to conclude

$$\mathrm{spt}(T_i) \cap \mathbf{C}_r \subset \{\mathrm{dist}(\cdot, \boldsymbol{\pi}_i) \le \bar C (E^{(i)} + \mathbf{A}^2)^{1/2}\}$$

where $\bar C = \bar C(Q, m, n, \bar n, N, r)$ and

$$E^{(i)} := \int_{\mathbf{C}_{1+r/2}} \mathrm{dist}^2(x, \boldsymbol{\pi}_i)\, d\|T_i\|(x)\,.$$

Given that obviously $E^{(1)}, E^{(2)} \le CE$, we would have reached the desired estimate (3.3), provided that (6.9) holds. On the other hand, if the estimate (6.9) does not hold, i.e. $M^2 \le \sigma_6^{-1}(E + \mathbf{A}^2)$, we can take an arbitrary plane $p_i + \pi_0$ from $\boldsymbol{\pi}$ and conclude that

$$\int_{\mathbf{C}_2} \mathrm{dist}^2(x, p_i + \pi_0)\, d\|T\|(x) \le \bar C (E + \mathbf{A}^2)\,.$$

It would then follow from the case $N = 1$ of Theorem 3.2, namely Proposition 4.2, that

$$\mathrm{spt}\,(T) \cap \mathbf{C}_r \subset \{\mathrm{dist}(\cdot, p_i + \pi_0) \le \bar C (E + \mathbf{A}^2)^{1/2}\} \subset \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \le \bar C (E + \mathbf{A}^2)^{1/2}\}\,,$$

for any fixed plane $p_i + \pi_0 \in \boldsymbol{\pi}$, proving (3.3).

We are thus left to prove the existence of the decomposition satisfying (i)-(iv) under the assumption that (6.9) holds for a sufficiently small σ_6. We first use the combinatorial Lemma 4.5 to decompose $P \equiv \{p_1, \dots, p_N\} = P_1 \cup P_2$, and consequently $\boldsymbol{\pi} = \boldsymbol{\pi}_1^* \cup \boldsymbol{\pi}_2^*$ for $\boldsymbol{\pi}_i^* = \bigcup_{p \in P_i}(p + \pi_0)$, with the property that P_1 and P_2 are disjoint non-empty sets satisfying

$$\mathrm{sep}\,(P_1, P_2) \equiv \min\{|p_1 - p_2| : p_1 \in P_1,\, p_2 \in P_2\} \ge \frac{M}{2^{N-2}}\,. \tag{6.10}$$

Our claim is that the desired decomposition will be achieved with the latter choice. In light of (6.10), this will follow immediately if we have that

$$\mathrm{spt}\,(T) \cap \mathbf{C}_{1+r/2} \subset \{x : \mathrm{dist}(x, \boldsymbol{\pi}) \le 2^{-N} M\}\,. \tag{6.11}$$

Let us thus show that (6.11) indeed holds. Fix a point $x \in B_{1+r/2}(0, \pi_0)$ and the scaling factor $1 - r/2$. Let T^0 be the rescaled current $(\iota_{x,1-r/2})_\sharp T$. Likewise, introduce the sets $\boldsymbol{\pi}_0 := \iota_{x,1-r/2}(\boldsymbol{\pi})$, $\Sigma^0 = \iota_{x,1-r/2}(\Sigma)$, the shorthand notation $\mathbf{A}_0$ for the supremum norm of the second fundamental form of Σ^0, and

$$E_0 := \int_{\mathbf{C}_2} \mathrm{dist}^2(y, \boldsymbol{\pi}_0)\, d\|T^0\|(y)\,.$$

Observe that $E_0 \le (1 - r/2)^{-m-2} E$, $\mathbf{A}_0^2 = (1 - r/2)^2 \mathbf{A}^2$. Setting $M_0 := (1 - r/2)^{-1} M$, since $1 - \frac{r}{2} \le 1$ we see immediately that

$$E_0 + \mathbf{A}_0^2 \le (1 - r/2)^{-m} \sigma_6 M_0^2\,.$$

Let now $\bar\sigma$ be the threshold of Lemma 4.9. We start by choosing σ_6 small enough so that

$$\sigma_7 := (1 - r/2)^{-m} \sigma_6 \le \bar\sigma\,. \tag{6.12}$$

In particular, if $M_0 \geq 1$ we could apply Lemma 4.9 and achieve the estimate
$$\mathrm{spt}\,(T^0) \cap \mathbf{C}_1 \subset \left\{\mathrm{dist}(\cdot, \boldsymbol{\pi}_0) \leq 2^{-N} M_0\right\}\,.$$
Given that this holds for every $x \in B_{1+r/2}(0, \pi_0)$, it would then imply (6.11) and conclude the proof.

Otherwise, if $M_0 < 1$, we may fix $\rho = \rho_0 < 1$, with ρ_0 as in Lemma 6.2, and let σ_5 be the corresponding threshold from Lemma 6.2. We may further choose σ_7 (as defined in (6.12)) small enough such that $\sigma_7 \leq \sigma_5$. In particular, since $M_0 < 1$, we can apply Lemma 6.2 to T^0 and $\boldsymbol{\pi}_0$. Denote by $\boldsymbol{\pi}_0'$ the corresponding new group of planes and let

- $T^1 = (\iota_{0,\rho})_\sharp T^0$;
- $\Sigma^1 = \iota_{0,\rho}(\Sigma^0)$, with $\mathbf{A}_1$ the supremum norm of the second fundamental form of Σ^1;
- $\bar{\boldsymbol{\pi}}_1 = \iota_{0,\rho}(\boldsymbol{\pi}_0')$;
- $M_1 = \rho^{-1} M_0$;
- $\bar{E}_1 = \int_{\mathbf{C}_2} \mathrm{dist}^2(x, \bar{\boldsymbol{\pi}}_1)\, d\|T^1\|(x)$.

Observe that from (6.2) we have the inequality (with $\beta_0 < 1$ as in Lemma 6.2)
$$\bar{E}_1 + \mathbf{A}_1^2 \leq \rho^{2\beta_0 - 2}(E_0 + \mathbf{A}_0^2) \leq \rho^{2\beta_0-2}\sigma_7 M_0^2 = \rho^{2\beta_0}\sigma_7 M_1^2\,. \tag{6.13}$$
As $\rho < 1$, we can thus keep iterating this procedure with the same ρ, and along the iteration each $\bar{\boldsymbol{\pi}}_k$ consists of some number of planes, which could be larger than the initial number N, but it never exceeds Q. This in particular means that the parameter σ_7 remains fixed, dependent on only ρ, Q, N and dimensional constants. generating currents T^j, ambient manifolds Σ^j, families of parallel planes $\boldsymbol{\pi}_j'$ and $\bar{\boldsymbol{\pi}}_j = \iota_{0,\rho}(\boldsymbol{\pi}_j')$ until we reach the first index k such that
$$M_k = \rho^{-k} M_0 \geq 1.$$

Let us now see how this iteration affects our original current T. Setting
$$\bar{E}_j := \int_{\mathbf{C}_2} \mathrm{dist}^2(x, \bar{\boldsymbol{\pi}}_j)\, d\|T^j\|(x)$$
we may conclude, analogously to (6.13), that
$$\bar{E}_j + \mathbf{A}_j^2 \leq \rho^{2\beta_0 j}\sigma_7 M_j^2\,.$$
Now define $\boldsymbol{\pi}_j := \iota_{0,\rho^j}(\boldsymbol{\pi}_0) = \iota_{0,\rho^j}(\iota_{x,(1-r/2)}(\boldsymbol{\pi}))$ and set
$$E_j := \int_{\mathbf{C}_2} \mathrm{dist}^2(x, \boldsymbol{\pi}_j)\, d\|T^j\|(x)\,.$$
We wish to estimate E_j and the distance of the new family of planes $\boldsymbol{\pi}_j'$ from the one $\boldsymbol{\pi}_j$ obtained by rescaling $\boldsymbol{\pi}_0$. We therefore introduce
$$d(j) := \max\{\mathrm{dist}(\pi, \boldsymbol{\pi}_j) : \pi \in \boldsymbol{\pi}_j'\}$$
Note that from (A) from Lemma 6.2 we have
$$d(0) \leq C(E_0 + \mathbf{A}_0^2)^{1/2} \leq C\sigma_7^{1/2} M_0\,.$$
Subsequently, using the triangle inequality (to estimate the distance from $\boldsymbol{\pi}_1$ to $\bar{\boldsymbol{\pi}}_1$ and then from $\bar{\boldsymbol{\pi}}_1$ to $\boldsymbol{\pi}_1'$)
$$d(1) \leq \left(C\sigma_7^{1/2} M_0\right)\rho^{-1} + C(E_1 + \mathbf{A}_1^2)^{1/2} \leq C\sigma_7^{1/2} M_1 (1 + \rho^\beta)$$
where we have used our previous bounds and the fact $M_1 = \rho^{-1} M_0$. Inductively, we achieve
$$d(j) \leq C\sigma_7^{1/2} M_j \sum_{i=0}^j \rho^{j\beta}\,;$$
we stress that C is the same each time we apply the lemma and so it is constant in j. In particular,
$$d(k) \leq C\sigma_7^{1/2} M_k\,.$$
Since
$$E_k \leq C d(k)^2 + C\bar{E}_k\,,$$

we conclude that

$$E_k \le C\sigma_7 M_k^2\,,$$

where the constant C depends only on $m, n, \bar{n}$, and Q. In particular, due to the fact that $M_k \ge 1$ by construction (and M_k is the maximal distance between the planes in $\boldsymbol{\pi}_k$), we may choose σ_7 smaller than a suitable geometric constant in order to apply Lemma 4.9, concluding that

$$\mathrm{spt}\,(T^k) \cap \mathbf{C}_1 \subset \{\mathrm{dist}(\cdot, \boldsymbol{\pi}_k) \le 2^{-N} M_k\}\,.$$

Rescaling this information in the original system of coordinates gives

$$\mathrm{spt}\,(T) \cap \mathbf{C}_{\rho^k(1-r/2)}(x) \subset \{\mathrm{dist}(\cdot, \boldsymbol{\pi}) \le 2^{-N} M\}\,.$$

The arbitrariness of $x \in B_{1+r/2}(0, \pi_0)$ gives then (6.11) and hence concludes the proof. □

The proof of Theorem 3.2 is therefore complete.

Part 2. Proof of the Main Excess Decay Theorem

In this part, we prove Theorem 2.5. The overall structure is as follows. In Section 7, we begin with a few technical preliminaries regarding the relative positioning of planes, and some combinatorial preliminaries which will be needed for the graphical parameterizations in the following section. In Section 8, we set up efficient graphical parameterizations for T over *balanced* cones in $\mathscr{C}(Q)$ (as defined in the preceding section), away from the spines of the cones and when the two-sided excess of T relative to $\mathbf{S}$ is much smaller than the *minimal angle* of the cone (such an assumption being used in this context is equivalent to that used by Wickramasekera [32, Section 10, Hypothesis ($\star\star$)]). This construction heavily relies on the height bound from Part 1, and given this our construction is a suitable adaptation of those seen in the work of Simon ([29]) and Wickramasekera ([32]), with added technical complications from working in arbitrary codimension and with area-minimizing currents. Section 9 is then dedicated to showing that we may assume that the cone $\mathbf{S}$ in Theorem 2.5 is balanced, allowing us to approximate T with a multi-valued graph over this cone away from its spine, as demonstrated in the previous section. In Section 10 we reduce the proof of Theorem 2.5 to an a priori much weaker weaker decay theorem which has a much stronger assumption on the size of the L^2 height excess of T relative to $\mathbf{S}$; the idea is to use an excess decay statement with multiple scales, analogous to that first seen in [32, Section 13]. In Section 11, we then show the non-concentration of excess and the Simon estimates at the spine of the cone $\mathbf{S}$; these are the key estimates first used in Simon's work ([29]) and then adapted to a degenerate setting, as we similar face here, by Wickramasekera ([32, Section 10]). We then display the analogue of Theorem 2.5 at the linearized level (for multi-valued Dir-minimizers) in Section 12. Finally, in Section 13, we conclude with a blow-up procedure, collecting all the previous arguments to yield a limiting Dir-minimizer that contradicts the results in Section 12 if we assume the failure of Theorem 2.5.

7. Relative position of pairs of planes

We begin with some elementary results on the relative positions of pairs of planes. These will be useful for results appearing later in this section regarding comparability of angle parameters for cones in $\mathscr{C}(Q)$ and multi-valued Dir-minimizers formed from superpositions of linear maps.

We start with a simple lemma in linear algebra.

Lemma 7.1. *Consider two m-dimensional linear subspaces α, β of $\mathbb{R}^N$ and the two quadratic forms $Q_1 : \alpha \to \mathbb{R}$, $Q_2 : \beta \to \mathbb{R}$ given by $Q_1(v) := \mathrm{dist}^2(v, \beta)$ and $Q_2(w) := \mathrm{dist}^2(w, \alpha)$. Then:*

(a) *Q_1 and Q_2 have the same eigenvalues with the same multiplicities.*

(b) *For every orthonormal base $v_1, \ldots, v_m$ of eigenvectors of Q_1 there is a corresponding orthonormal base $w_1, \ldots, w_m$ of eigenvectors of Q_2 with the property that*

$$Q_1(v_i) = Q_2(w_i) = 1 - (w_i \cdot v_i)^2\,,$$

and $v_i \cdot w_j = 0$ for $i \neq j$.

(c) *If $\alpha \cap \beta^\perp = \{0\}$ a choice of $w_1, \ldots, w_m$ is given by $w_i = \frac{\mathbf{p}_\beta(v_i)}{|\mathbf{p}_\beta(v_i)|}$.*

Before proving this, let us note that this lemma motivates the following definition:

Definition 7.2. Given two m-dimensional linear subspaces α, β of $\mathbb{R}^N$ whose intersection has dimension $m-k$, we order the k positive eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ of Q_1 as in Lemma 7.1, with the convention that the number of occurrences of the same real λ in the list equals its multiplicity as eigenvalue of Q_1. The *Morgan angles* of the pair α and β are the numbers $\theta_i(\alpha,\beta) := \arcsin\sqrt{\lambda_i}$ for $i = 1, \dots, k$.

Of course we can define the Morgan angles between two intersecting affine planes of the same dimension by simply translating an intersection point to the origin.

The following is then an immediate corollary of Lemma 7.1 and basic linear algebra:

Corollary 7.3. *Suppose that α, β are two m-dimensional linear subspaces of $\mathbb{R}^N$. Denote by V their $(m-k)$-dimensional intersection and let $\theta_1 \leq \cdots \leq \theta_k$ be their Morgan angles. Then*

$$|v| \sin\theta_1 \leq \mathrm{dist}(v, \beta) \leq |v| \sin\theta_k \qquad \forall v \in V^\perp \cap \alpha\,, \tag{7.1}$$

and

$$\mathrm{dist}(\alpha \cap \mathbf{B}_1, \beta \cap \mathbf{B}_1) = \sup\{\mathrm{dist}(v,\beta) : v \in \alpha \cap \mathbf{B}_1\} = \sin\theta_k\,. \tag{7.2}$$

Indeed, this simply follows because $V^\perp \cap \alpha$ is the eigenspace of the quadratic form $Q_1 = \mathrm{dist}^2(\cdot, \beta)$ spanned by those eigenvectors which have positive eigenvalues, combined with the fact that the extremal eigenvalues of a quadratic form are realized through constrained optimization over the unit sphere. Note that the first equality in (7.2) is simply due to the fact that α and β are m-dimensional subspaces.

Proof of Lemma 7.1. We prove (b) and (a) at the same time. Moreover, since we give an explicit construction for the vectors w_i, the proof will also show (c). We start by observing that

$$Q_1(v) = |\mathbf{p}_\beta^\perp(v)|^2 = v \cdot (\mathbf{p}_\beta^\perp(v))\,. \tag{7.3}$$

Now pick an orthonormal basis $\{v_1, \dots, v_m\}$ of α which diagonalizes Q_1. Moreover, let us remark that we can extend Q_1 to a quadratic form $\bar{Q}_1$ on $\mathbb{R}^N$ by setting it identically 0 on $\alpha^\perp$. $\bar{Q}_1$ can be computed explicitly as

$$\bar{Q}_1(z) = z \cdot (\mathbf{p}_\alpha \circ \mathbf{p}_\beta^\perp \circ \mathbf{p}_\alpha(z)) \tag{7.4}$$

for a general $z \in \mathbb{R}^N$. In particular, each v_i must be an eigenvector of the symmetric matrix $\mathbf{p}_\alpha \circ \mathbf{p}_\beta^\perp \circ \mathbf{p}_\alpha$, which in turn implies that

$$\mathbf{p}_\alpha(\mathbf{p}_\beta^\perp(v_i)) = \lambda_i v_i \tag{7.5}$$

for some constant $0 \leq \lambda_i \leq 1$. Observe, moreover, that in case $\lambda_i = 1$, then necessarily

$$\mathbf{p}_\beta^\perp(v_i) = v_i\,.$$

Without loss of generality let us order the v_i to assume that $0 \leq Q_1(v_i) \leq Q_1(v_j) \leq 1$ for all $i \leq j$. Let us write k for the largest element of $\{1, \dots, m\}$ for which $Q_1(v_k) < 1$. Observe that $\mathbf{p}_\beta(v_j) = 0$ for all $j > k$. For $i \leq k$, define $w_i := \frac{\mathbf{p}_\beta(v_i)}{|\mathbf{p}_\beta(v_i)|}$, and write $W = \mathrm{span}\{w_1, \dots, w_k\}$. Select any orthonormal basis $\{w_{k+1}, \dots, w_m\}$ of $W^\perp \cap \beta$. Since $\mathbf{p}_\beta(v_j) = 0$ for $j > k$, it is obvious that $W = \mathbf{p}_\beta(\alpha)$. In particular, it follows that $w_j \perp \alpha$ for every $j > k$, and so for $j > k$ we have $1 = Q_1(v_j) = Q_2(w_j)$ and $w_j \cdot v_i = 0$ for all i.

To handle the case $j \leq k$, note that, if $i \neq j$, then

$$\begin{aligned} v_i \cdot w_j &= |\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\beta(v_j)) = -|\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\beta^\perp(v_j)) \\ &= -|\mathbf{p}_\beta(v_j)|^{-1}(v_i \cdot \mathbf{p}_\alpha(\mathbf{p}_\beta^\perp(v_j))) \overset{(7.5)}{=} -\lambda_j |\mathbf{p}_\beta(v_j)|^{-1} v_i \cdot v_j = 0\,. \end{aligned}$$

Next consider $v = \sum_{i=1}^k a_i v_i$ and compute

$$\begin{aligned} Q_1(v) &= |\mathbf{p}_\beta^\perp(v)|^2 = |v|^2 - |\mathbf{p}_\beta(v)|^2 \\ &= \sum_i a_i^2 - \left|\sum_{i=1}^k (w_i \cdot v) w_i\right|^2 = \sum_i a_i^2 - \left|\sum_{i=1}^k a_i (w_i \cdot v_i) w_i\right|^2 \end{aligned}$$

$$= \sum_i a_i^2 \left(1 - (w_i \cdot v_i)^2\right) - 2 \sum_{i<j\leq k} a_i a_j (w_i \cdot w_j)(w_i \cdot v_i)(w_j \cdot v_j)\,.$$

Given that $Q_1(v) = \sum_i a_i^2 Q_1(v_i) = \sum_i a_i^2 (1 - (v_i \cdot w_i)^2)$ and the coefficients a_i can be chosen arbitrarily, we must then have

$$(w_i \cdot w_j)(w_i \cdot v_i)(w_j \cdot v_j) = 0$$

for every $1 \leq i < j \leq k$. In particular, since $(w_i \cdot v_i)(w_j \cdot v_j) \neq 0$ when $i < j \leq k$ we immediately infer that $w_i \cdot w_j = 0$, hence concluding that $w_1, \ldots, w_m$ is an orthonormal basis of β. Moreover, we have already seen that $Q_1(v_i, v_i) = 1 - (v_i \cdot w_i)^2$.

Next, fix $j \leq k$ and observe that, by construction, w_j is orthogonal to v_ℓ when $\ell > k$. We also already know that for $i, j \leq k$, $i \neq j$, that w_j is orthogonal to v_i. In particular, this gives that $\mathbf{p}_\alpha(w_j) = (v_j \cdot w_j) v_j$ for every j. Hence, given any $w = \sum_i b_i w_i \in \beta$, we have

$$Q_2(w) = |w|^2 - |\mathbf{p}_\alpha(w)|^2 = \sum_i \mu_i^2 (1 - (w_i \cdot v_i)^2).$$

This proves that $Q_1(v_i) = Q_2(w_i)$ and that w_i is an orthonormal basis of eigenvectors of Q_2, which completes the proof of (b) and also implies at the same time the statement (a). $\square$

7.1. **Rotating planes.** In this section we use the material of the previous one to define "canonical rotations" which map equidimensional linear subspaces onto each other. These objects are not strictly necessary for our considerations, but they help streamlining some arguments.

Consider two linear subspaces α and β of $\mathbb{R}^{m+n}$ of the same dimension k and assume that

$$\alpha \cap \beta^\perp = \{0\}\,. \tag{7.6}$$

We then define a canonical element $R(\alpha, \beta) \in O(\mathbb{R}^{m+n})$ mapping α onto β in the following fashion. First of all consider the quadratic form $Q(v) = \mathrm{dist}^2(v, \beta)$ on α introduced in the previous section and let $v_1, \ldots, v_k$ be an orthonormal base which diagonalizes Q. Likewise consider the quadratic form $Q^\perp$ on $\alpha^\perp$ defined by $\mathrm{dist}^2(v, \beta^\perp)$ and let $v_{k+1}, \ldots, v_{m+n}$ be an orthonormal base which diagonalizes $Q^\perp$. We then define $w_i := \frac{\mathbf{p}_\beta(v_i)}{|\mathbf{p}_\beta(v_i)|}$ for $1 \leq i \leq k$ and $w_i := \frac{\mathbf{p}^\perp_\beta(v_i)}{|\mathbf{p}^\perp_\beta(v_i)|}$ for $k+1 \leq i \leq m+n$. By Lemma 7.1, $w_1, \ldots, w_{m+n}$ is an orthonormal base of $\mathbb{R}^{m+n}$ and if we define $R(v_i) = w_i$ and extend R by linearity we clearly have an element of $O(\mathbb{R}^{m+n})$ with the property that $O(\alpha) = \beta$.

The properties of this map is then given in the following

Lemma 7.4. *Let α and β be two equidimensional linear subspaces such that $\alpha \cap \beta^\perp = \{0\}$. Then*

(a) *$R = R(\alpha, \beta)$ is well-defined, i.e. its definition does not depend on the choice of the diagonalizing orthonormal bases of Q and $Q^\perp$;*
(b) *R is an element of $SO(m+n)$;*
(c) *$R(\alpha, \beta) = (R(\beta, \alpha))^{-1}$ and R is the identity on $\alpha \cap \beta$ and $\alpha^\perp \cap \beta^\perp$; in particular $R(\alpha, \alpha)$ is the identity;*
(d) *R depends continuously on α and β.*

Proof. Concerning (a), the only ambiguity in the definition stems from the fact that the vectors $v_1, \ldots, v_{m+n}$ are not uniquely defined. Assume thus $v_1', \ldots v_m'$ and $v_{m+1}', \ldots, v_{m+n}'$ form two other orthonormal bases of Q and $Q^\perp$. Define w_i' accordingly. Moreover, without loss of generality, assume that the eigenvalues are the same for v_i and v_i'. If the eigenvalues of Q are all distinct and those of $Q^\perp$ are also all distinct, then $v_i = \pm v_i'$ and hence $w_i = \pm w_i'$ and we see immediately that the definition of R does not depend on the choice of the bases. Assume otherwise that there are some eigenvalues with higher multiplicity. To fix ideas assume therefore that $v_j, \ldots, v_\ell$ form a base of a maximal eigenspace W for a fixed eigenvalue λ of Q. Then $v_j', \ldots, v_\ell'$ is also a base of the same eigenspace. Note however that, by the very definition of Q, $|\mathbf{p}_\beta(v)| = \sqrt{1-\lambda}|v|$ for every $v \in W$. In particular we see immediately that, computing v_k' as a linear combination $\sum_{j\leq i\leq \ell} a_{\ell i} v_i$, then $w_k' = \sum_{j \leq i \leq \ell} a_{\ell i} w_i$. Therefore R is well defined in this case as well.

Concerning (b), observe that $v_i \cdot w_i > 0$ for all i, and thus $v_1, \ldots, v_{m+n}$ and $w_1, \ldots, w_{m+n}$ have the same orientation. Concerning (c) observe first that $\alpha \cap \beta$ is the maximal eigenspace of Q for the eigenvalue 0, while $\alpha^\perp \cap \beta^\perp$ is the maximal eigenspace of Q for the eigenvalue 0. Thus $v_1, \ldots, v_{m+n}$ contains an orthonormal base of the former and an orthonormal base for the latter and R is by definition the identity on these elements. Moreover, it turns out that, by Lemma 7.1, when defining $R(\beta, \alpha)$, we can choose $w_1, \ldots w_k$ and $w_{k+1}, \ldots, w_{m+n}$ as orthonormal bases diagonalizing $\mathrm{dist}^2(\cdot, \alpha)$ and $\mathrm{dist}^2(\cdot, \alpha^\perp)$. But then Lemma 7.1(b) implies that $v_i = \frac{\mathbf{p}_\alpha(w_i)}{|\mathbf{p}_\alpha(w_i)|}$ for $1 \leq i \leq m+n$, from which it immediately follows that $R(\beta, \alpha)$ maps w_i in v_i, and it is thus the inverse of $R(\alpha, \beta)$.

As for the continuous dependence, observe that if $\alpha_k \to \alpha$ and $\beta_k \to \beta$ and we fix $v_1^k, \ldots, v_{m+n}^k$ with $v_1^k, \ldots, v_m^k$ diagonalizing $\mathrm{dist}^2(\cdot, \beta_k)$ on α_k and $v_{m+1}^k, \ldots, v_{m+n}^k$ diagonalizing $\mathrm{dist}^2(\cdot, \beta_k^\perp)$ on $\alpha_k^\perp$, then we can extract a subsequence for which v_i^k converge to v_i. It follows immediately that the orthonormal vectors $v_1, \ldots, v_m$ diagonalize Q and $v_{m+1}, \ldots, v_{m+n}$ diagonalize $Q^\perp$. Hence the algorithm given to determine $R(\alpha_k, \beta_k)$ shows immediately that the corresponding subsequence must converge to $R(\alpha, \beta)$. This completes the proof. □

7.2. **Area-minimizers and Dir-minimizers.** Of particular interest for us is the following consequence of F. Morgan's work [25, Theorem 2].

Lemma 7.5. *Let $\mathbf{S} \subset \mathbb{R}^{m+n}$ be the union of N distinct m-dimensional planes $\alpha_1, \ldots, \alpha_N$ with the property that, for every $i < j$, $\alpha_i \cap \alpha_j$ is the same $(m-2)$-dimensional plane V. If T is an m-dimensional integral area-minimizing current such that $\mathrm{spt}(T) = \mathbf{S}$, then the (two) Morgan angles of any pair α_i, α_j, $i \neq j$, coincide.*

This follows from [25], since Corollary 7.3 gives the equivalence of θ_1 and θ_2 with the angles in [25] once the planes are oriented accordingly.

We complement this with the following counterpart for Dir-minimizers:

Proposition 7.6. *Let $m, n \geq 2$. There are positive absolute constants c_1 and c_2 depending on n such that the following holds. Assume $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizing map such that $u(x) = \sum_i k_i [\![L_i(x)]\!]$ for distinct linear maps $L_i : \mathbb{R}^m \to \mathbb{R}^n$ and non-zero integers k_i with the property that $\max_i |L_i| \leq c_1$ and that, for every $i < j$, the kernel of $L_i - L_j$ is the same $(m-2)$-dimensional subspace V of $\mathbb{R}^m$ while all L_i vanish on V. Let π_i be the m-dimensional planes given by the graphs of the L_i's. Then, for every pair $i < j$ the Morgan angles $\theta_k = \theta_k(\pi_i, \pi_j)$, $k = 1, 2$, satisfy the inequality $\theta_2 \leq c_2 \theta_1$.*

We stress here that as the subspace V in Proposition 7.6 has dimension $m - 2$, there are exactly two Morgan angles for each pair of planes π_i, π_j, $i \neq j$, which in the above are denoted $\theta_1(\pi_i, \pi_j)$, $\theta_2(\pi_i, \pi_j)$, which by definition obey $\theta_1(\pi_i, \pi_j) \leq \theta_2(\pi_i, \pi_j)$. In particular, Proposition 7.6 tells us that they are comparable to each other, while Lemma 7.5 tells us in the case the planes are area-minimizing, these two angles must in fact coincide. It is possible that this stronger result could hold for Dir-minimizing unions of planes also, however we will not need this here and so do not pursue this.

Proposition 7.6 will be derived from the following special case and some elementary linear algebra.

Lemma 7.7. *There is a geometric constant $\mu \geq 1$ with the following property. Let $u : \mathbb{R}^2 \to \mathcal{A}_2(\mathbb{R}^2)$ be a Dir-minimizing map such that $u(x) = [\![B(x)]\!] + [\![-B(x)]\!]$ for some linear $B : \mathbb{R}^2 \to \mathbb{R}^2$. Then B is quasiconformal, in the sense that*

$$\max\{|B(x)| : |x| = 1\} \leq \mu \min\{|B(x)| : |x| = 1\}. \tag{7.7}$$

Proof. Without loss of generality we can assume $|B| = 1$, otherwise normalize B to ensure this. If the lemma were false, then there would be a sequence of Dir-minimizing maps $u_k(x) = [\![B_k(x)]\!] + [\![-B_k(x)]\!]$, for some sequence $B_k : \mathbb{R}^2 \to \mathbb{R}^2$ of linear maps with $|B_k| = 1$, for which there is a sequence e_k of unit vectors with $B(e_k) \to 0$. Extracting a subsequence, we can assume that $e_k \to e$ and that $B_k \to B$. B is thus a linear map with $|B| = 1$, and so in particular if we set $u(x) := [\![B(x)]\!] + [\![-B(x)]\!]$, then u does not vanish identically. However, the line $\{\lambda e : \lambda \in \mathbb{R}\}$ would be a line of singularities, while from [10, Proposition 3.20] we know that u is Dir-minimizing. In particular we would contradict Almgren's partial regularity theorem for Dir-minimizers, cf. [10, Theorem 0.11]. This contradiction proves the result. $\square$

Given Lemma 7.7, the elementary linear algebra needed to prove Proposition 7.6 is the following:

Lemma 7.8. *There is a geometric constant $c = c(n) > 0$ with the following property. Consider two linear maps $L_1, L_2 : \mathbb{R}^2 \to \mathbb{R}^n$ with $|L_i| \leq c(n)$ and* rank $(L_1 - L_2) = 2$. *If*

$$\sigma_1 = \min\{|B(e)| : |e| = 1\} \leq \sigma_2 = \max\{|B(e)| : |e| = 1\}$$

denote the two singular values of $B = \frac{1}{2}(L_1 - L_2)$, and π_1, π_2 are the two planes in $\mathbb{R}^{2+n}$ given by the graphs of L_1 and L_2 respectively, then

$$\frac{\sigma_2}{4} \leq \theta_2(\pi_1, \pi_2) \leq 4\sigma_2$$
$$\frac{\sigma_1}{4} \leq \theta_1(\pi_1, \pi_2) \leq 4\sigma_1$$

where $\theta_1(\pi_1, \pi_2) \leq \theta_2(\pi_1, \pi_2)$ are the Morgan angles of the planes π_1, π_2.

Proof. Let $A := \frac{1}{2}(L_1 + L_2)$ and $B := \frac{1}{2}(L_1 - L_2)$. Fix $p \in \partial \mathbf{B}_1 \cap \pi_1$. Since in particular $p \in \pi_1$, we may find $x \in \mathbb{R}^2$ such that $p = (x, (A + B)(x))$. Note that $|x| \leq 1$ as $|p| = 1$. Since $(x, (A - B)(x)) \in \pi_2$ we clearly have

$$\operatorname{dist}(p, \pi_2) \leq |2B(x)| \leq 2 \max\{|B(e)| : |e| = 1\}\,. \tag{7.8}$$

Next, let $q \in \pi_2$ be the point of minimum distance in π_2 from p and let $\xi \in \mathbb{R}^2$ be so that $q = (\xi, L_2(\xi)) = (\xi, (A - B)(\xi))$. Then,

$$\operatorname{dist}^2(p, \pi_2) = |p - q|^2 = |x - \xi|^2 + |(A + B)(x) - (A - B)(\xi)|^2\,. \tag{7.9}$$

Set $\eta := x - \xi$ and rewrite the expression in (7.9) as

$$|\eta|^2 + |(A + B)(\eta + \xi) - (A - B)(\xi)|^2 = |\eta|^2 + |2B(\xi) + (A + B)(\eta)|^2.$$

Since η is the minimum point of the latter quadratic expression, we can differentiate it to find that η obeys

$$2\eta + 4(A + B)^T B(\xi) + 2(A + B)^T (A + B)(\eta) = 0.$$

In turn, we can solve this as

$$\eta = -2(\mathrm{Id} + (A + B)^T (A + B))^{-1} (A + B)^T (B(\xi)) = -2(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T (B(\xi))\,.$$

Insert the latter in (7.9) to find

$$\operatorname{dist}^2(p, \pi_2) = 4|(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T (B(\xi))|^2 + 4|\left(\mathrm{Id} - L_1(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T\right)(B(\xi))|^2\,. \tag{7.10}$$

Observe now that the operator norm of $(\mathrm{Id} + L_1^T L_1)^{-1}$ is always at most 1: indeed, since $\mathrm{Id} + L_1^T L_1$ is self-adjoint and its smallest eigenvalue is at least 1, its inverse is self-adjoint and has positive eigenvalues no larger than 1. We can therefore estimate

$$\left|\left(\mathrm{Id} - L_1(\mathrm{Id} + L_1^T L_1)^{-1} L_1^T\right)(B(\xi))\right| \geq |B(\xi)| - |L_1|^2 |B(\xi)|\,.$$

In particular, if we choose the constant c sufficiently small, we conclude

$$\operatorname{dist}^2(p, \pi_2) \geq \frac{1}{2}|B(\xi)|^2\,. \tag{7.11}$$

But, again under the assumption that c is sufficiently small, and since $|q| \leq 1$ (as $\operatorname{dist}(p, 0) = 1$ and $0 \in \pi_2$), we have that $|\xi| \geq \frac{\sqrt{2}}{2}$. Hence we get

$$\operatorname{dist}(p, \pi_2) \geq \frac{1}{2} \min\{|B(e)| : |e| = 1\}\,. \tag{7.12}$$

Combining (7.8) and (7.12) and recalling that $\sigma_2 = \max\{|B(e)| : |e| = 1\}$ and $\sigma_1 = \min\{|B(e)| : |e| = 1\}$ are the two singular values of B, we conclude that

$$\frac{\sigma_1}{2} \leq \min\{\operatorname{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq \max\{\operatorname{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq 2\sigma_2\,. \tag{7.13}$$

Next let $e \in \mathbb{R}^2$ be a unit vector with $|B(e)| = \sigma_2$ and consider $v = (e, L_1(e))$ and $v' := \frac{v}{|v|} \in \pi_1 \cap \partial \mathbf{B}_1$. Decreasing c further if necessary, we have

$$\mathrm{dist}(v, \pi_2) \leq \frac{3\sqrt{2}}{4} \mathrm{dist}(v', \pi_2) \leq \frac{3\sqrt{2}}{4} \max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} . \tag{7.14}$$

Consider $q = \mathbf{p}_{\pi_2}(v)$ and let $\xi \in \mathbb{R}^2$ be such that $(\xi, L_2(\xi)) = q$. Note that

$$|\xi - e| \leq |v - q| \leq |\mathbf{p}_{\pi_2} - \mathbf{p}_{\pi_1}||v| .$$

In particular $|\xi - e| \leq C(|L_1| + |L_2|) \leq Cc$. On the other hand, recall that we have the inequality (7.11) but for v in place of p (since we did not use the fact that $p \in \partial \mathbf{B}_1$ to achieve this). In particular, provided c is sufficiently small we can write

$$\mathrm{dist}(v, \pi_2) \geq \frac{1}{\sqrt{2}}|B(e)| - \frac{1}{\sqrt{2}}|B(e - \xi)| \geq \frac{\sigma_2}{\sqrt{2}}(1 - |\xi - e|) \geq \frac{3\sigma_2}{4\sqrt{2}} . \tag{7.15}$$

We can now combine (7.14) and (7.15), yielding

$$\sigma_2 \leq 2 \max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} . \tag{7.16}$$

In a similar fashion, fix now e such that $|B(e)| = \sigma_1$ and consider v and v' as above, for this choice of e. Arguing analogously to (7.8), it follows that

$$\min\{\mathrm{dist}(p, \pi_2) : p \in \pi_1 \cap \partial \mathbf{B}_1\} \leq \mathrm{dist}(v', \pi_2) \leq \mathrm{dist}(v, \pi_2) \leq 2|B(e)| , \tag{7.17}$$

because $(e, L_2(e)) \in \pi_2$. In particular we deduce that

$$\min\{\mathrm{dist}(p, \pi_2) : p \in \pi_1 \cap \partial \mathbf{B}_1\} \leq 2\sigma_1 . \tag{7.18}$$

Summarizing, if we combine (7.13), (7.16), and (7.18), we arrive at

$$\frac{\sigma_1}{2} \leq \min\{\mathrm{dist}(p, \pi_2) : p \in \pi_1 \cap \partial \mathbf{B}_1\} \leq 2\sigma_1 \tag{7.19}$$
$$\frac{\sigma_2}{2} \leq \max\{\mathrm{dist}(p, \pi_2) : p \in \pi_1 \cap \partial \mathbf{B}_1\} \leq 2\sigma_2 . \tag{7.20}$$

Recall now that, since we are dealing with two-dimensional planes, the variational definition of the eigenvalues of quadratic forms gives

$$\theta_1 = \arcsin(\min\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\})$$
$$\theta_2 = \arcsin(\max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}) .$$

On the other hand $\max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$ is controlled by $|L_1| + |L_2|$, hence if we choose the latter sufficiently small we get

$$\frac{1}{2} \min\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq \theta_1 \leq 2 \min\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$$
$$\frac{1}{2} \max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\} \leq \theta_2 \leq 2 \max\{\mathrm{dist}(p, \pi_2) : p \in \partial \mathbf{B}_1 \cap \pi_1\}$$

Combining the latter inequalities with (7.19) and (7.20) we conclude the proof. □

We record the following corollary; it is not needed for now, but will be convenient for us to use later on.

Corollary 7.9. *There is a geometric constant $c = c(n) > 0$ with the following property. Consider two linear maps $L_1, L_2 : \mathbb{R}^m \to \mathbb{R}^n$ with the property that $\ker(L_1 - L_2)$ is $(m-2)$-dimensional, let $\lambda > 0$, and denote by α_i and β_i the m-dimensional planes given by the graphs of L_i and λL_i respectively, for $i = 1, 2$. If $\max\{\lambda, 1\}|L_i| \leq c(n)$, then*

$$\frac{\theta_1(\alpha_1, \alpha_2)}{\theta_2(\alpha_1, \alpha_2)} \leq 16^2 \frac{\theta_1(\beta_1, \beta_2)}{\theta_2(\beta_1, \beta_2)} .$$

Proof. For $i = 1, 2$, letting σ_i and $\sigma_i^{(\lambda)}$ denote the singular values of $\frac{1}{2}(L_1 - L_2)$ and $\frac{1}{2}(\lambda L_1 - \lambda L_2)$ respectively, we have $\sigma_i^{(\lambda)} = \lambda \sigma_i$, and so the inequality is an immediate corollary of Lemma 7.8. □

We can now prove Proposition 7.6.

Proof of Proposition 7.6. First of all observe that, for every choice of i and j, the map $v(x) = [\![L_i(x)]\!] + [\![L_j(x)]\!]$ is Dir-minimizing. It therefore suffices to prove the proposition when $Q = 2$ and we have two distinct linear maps L_1 and L_2.

Next consider the $(m-2)$-dimensional subspace W which is the kernel of $L_1 - L_2$, and let $A = \frac{1}{2}(L_1 + L_2)$ and $B = \frac{1}{2}(L_1 - L_2)$. Observe that since A is linear, the function

$$\tilde{v}(x) := v(x) \ominus A(x) \equiv [\![L_1(x) - A(x)]\!] + [\![L_2(x) - A(x)]\!] = [\![B(x)]\!] + [\![-B(x)]\!]$$

is also Dir-minimizing (see [10, Lemma 3.23]). Now, for any point $z \in \mathbb{R}^m$, let us write $z = (x, y) \in W^\perp \times W$. Since the image of B on $W^\perp$ lies in a 2-dimensional subspace, and the kernel of B is $(m-2)$-dimensional, we may quotient out the kernel of B and consider it as a function $W^\perp \to \mathbb{R}^2$. As the domain of B is then a 2-dimensional subspace, we can then apply Lemma 7.7 to conclude that (7.7) holds for B, and thus we can control the ratio of the two singular values of B by a geometric constant, μ. But then observe that the two Morgan angles of the planes π_1 and π_2 coincide with the Morgan angles of the 2-dimensional planes of $\mathbb{R}^4$ constructed above. We can therefore apply Lemma 7.8 to conclude the proof of Proposition 7.6. □

7.3. **Separated regions, Alignment, and Shifting.** Here we collect three lemmas of a different flavor, which all have to do with the geometry of a collection of planes which all intersect in a common $(m-2)$-dimensional subspace; they will be used later on at different stages, when proving Theorem 2.5.

The following lemma shows that, even in the absence of a comparison estimate between the Morgan angles for a given finite collection of planes, it is possible to find a sizeable region of one of the planes α where the minimum distance of a point in that region to the other planes is comparable to the minimum of the Hausdorff distances of these planes to α.

Lemma 7.10 (Separated region)**.** *There is a constant $0 < c = c(m, N) < \frac{1}{2}$ with the following property. Suppose that $\alpha, \beta_1, \ldots, \beta_N$ are distinct m-dimensional subspaces of $\mathbb{R}^{m+n}$. Then there is a point $\xi \in \alpha \cap \partial \mathbf{B}_{1/2}$ with the property that*

$$\min_i \inf\{\operatorname{dist}(\zeta, \beta_i) : \zeta \in \mathbf{B}_c(\xi) \cap \alpha\} \geq c \min_i \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)\,. \tag{7.21}$$

Proof. Following the notation of Lemma 7.1 and Definition 7.2, let us denote by $Q_i : \alpha \to \mathbb{R}$ the quadratic forms $Q_i(v) := \operatorname{dist}^2(v, \beta_i)$ and let e_i be an eigenvector corresponding to the its maximal eigenvalue λ^i and hence also to the largest Morgan angle θ^i of the pair (α, β_i). Complete e_i to an orthonormal basis which diagonalizes Q_i. For any $\zeta \in \alpha$ we thus have

$$\operatorname{dist}^2(\zeta, \beta_i) \geq (e_i \cdot \zeta)^2 \lambda^i = (e_i \cdot \zeta)^2 \sin^2(\theta^i)$$

Recalling that, by Corollary 7.3, $\sin(\theta^i) = \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)$, we conclude that

$$\operatorname{dist}(\zeta, \beta_i) \geq |e_i \cdot \zeta| \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1)\,.$$

Therefore, we just need to find a vector $\xi \in \partial \mathbf{B}_{1/2} \cap \alpha$ with the property that

$$|\zeta \cdot e_i| \geq c \qquad \forall i, \forall \zeta \in \mathbf{B}_c(\xi) \cap \alpha$$

for $c = c(m, N)$ to be determined. Given the elementary estimate

$$|\zeta \cdot e_i| \geq |\xi \cdot e_i| - c\,,$$

for any such ζ and ξ, it therefore suffices to find $\xi \in \partial \mathbf{B}_{1/2} \cap \alpha$ such that

$$|\xi \cdot e_i| \geq 2c \qquad \forall i\,.$$

Now for each $i = 1, \ldots, N$, let $S_i := \{v \in \partial \mathbf{B}_{1/2} \cap \alpha : |v \cdot e_i| \leq 2c\}$ and observe that $\mathcal{H}^{m-1}(S_i) \leq C(m)c$ for some constant $C(m)$. In particular, $\mathcal{H}^m(\bigcup_i S_i) \leq NC(m)c$, and thus if $c = c(m, N)$ is chosen sufficiently small, $\partial \mathbf{B}_{1/2} \setminus \bigcup_i S_i$ must have positive measure, and hence it contains at least one point ξ which obeys the desired conditions. □

As an immediate corollary of Lemma 7.10 above we have the following.

Corollary 7.11. *Let $\alpha, \beta_1, \ldots, \beta_N$ be a collection of m-dimensional distinct planes of $\mathbb{R}^{m+n}$ and set $\mathbf{S} = \bigcup_i \beta_i$. Then there is a constant $\bar{C} = \bar{C}(N, m) > 0$ with*

$$\min_i \operatorname{dist}(\alpha \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\alpha \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1)\,.$$

The following lemma concerns itself with the alignment of spines of pairs of cones in $\mathscr{C}(Q)$, cf. Definition 2.3, and will be of fundamental importance.

Lemma 7.12. *For every $M > 0$ and natural numbers m, n, and Q, there is a constant $\bar{C} = \bar{C}(M, m, n, Q) > 0$ with the following property. Assume that*

(i) *$\mathbf{S}$ and $\mathbf{S}'$ consist of $2 \leq N, N' \leq Q$ m-dimensional distinct planes $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_{N'}$, respectively;*

(ii) *The intersection of every pair $\alpha_i \neq \alpha_j$ is a single $(m-2)$-dimensional subspace $V(\mathbf{S})$ and the intersection of every pair $\beta_i \neq \beta_j$ is a single $(m-2)$-dimensional subspace $V(\mathbf{S}')$;*

(iii) *For every pair $\alpha_i \neq \alpha_j$ the two Morgan angles $\theta_1(\alpha_i, \alpha_j) \leq \theta_2(\alpha_i, \alpha_j)$ satisfy $\theta_2 \leq M\theta_1$.*

Then

$$\operatorname{dist}(V(\mathbf{S}) \cap \mathbf{B}_1, V(\mathbf{S}') \cap \mathbf{B}_1) \leq \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)}\,. \tag{7.22}$$

Proof. We may assume without loss of generality that the planes α_i are ordered such that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) = \max_{i \neq j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,,$$

and observe that

$$\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) \leq \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)\,. \tag{7.23}$$

On the other hand, because of Corollary 7.11 and since $\alpha_i \subset \mathbf{S}$ for all i, we can select β_i and β_j such that

$$\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \beta_i \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1) \tag{7.24}$$

$$\operatorname{dist}(\alpha_2 \cap \mathbf{B}_1, \beta_j \cap \mathbf{B}_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,. \tag{7.25}$$

To simplify our notation we use V and V' in place of $V(\mathbf{S})$ and $V(\mathbf{S}')$. Because of the condition (iii) on the Morgan angles $\theta_k(\alpha_1, \alpha_2)$, using (7.1) and (7.2) we have

$$\operatorname{dist}(w, \alpha_2) \geq \bar{C}^{-1} \operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) |w - \mathbf{p}_V(w)| \qquad \forall w \in \alpha_1\,. \tag{7.26}$$

Fix now $v' \in V' \cap \mathbf{B}_1$ and observe that, since v' belongs to both β_i and β_j, due to (7.24) and (7.25) we must have

$$|\mathbf{p}_{\alpha_1}(v') - v'| = \operatorname{dist}(v', \alpha_1) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)$$
$$|\mathbf{p}_{\alpha_2}(v') - v'| = \operatorname{dist}(v', \alpha_2) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,.$$

In particular, from the triangle inequality and the fact $|\mathbf{p}_{\alpha_2}(v')| \leq 1$ one can then deduce

$$\operatorname{dist}(\mathbf{p}_{\alpha_1}(v'), \alpha_2) \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)\,. \tag{7.27}$$

Now, let $w = \mathbf{p}_{\alpha_1}(v')$ and observe that

$$\operatorname{dist}(v', V) \leq |v' - w| + |w - \mathbf{p}_V(w)| \leq \bar{C} \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1) + |w - \mathbf{p}_V(w)|\,, \tag{7.28}$$

while, by (7.26) and (7.27),

$$|w - \mathbf{p}_V(w)| \leq \bar{C} \frac{\operatorname{dist}(w, \alpha_2)}{\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)} \leq \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1)}\,. \tag{7.29}$$

Thus, given (7.23) and $\operatorname{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) \leq 1$, (7.28) and (7.29) lead to

$$\operatorname{dist}(v', V) \leq \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)}\,.$$

Observe however that v' is an arbitrary element of $V' \cap \mathbf{B}_1$, and we have thus reached

$$\sup\{\operatorname{dist}(v', V) : v' \in V' \cap \mathbf{B}_1\} \leq \bar{C} \frac{\operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)}{\min_i \operatorname{dist}(\mathbf{S} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)}\,.$$

On the other hand, since V and V' have the same dimension, by Corollary 7.3 we infer that

$$\sup\{\mathrm{dist}(v', V) : v' \in V' \cap \mathbf{B}_1)\} = \mathrm{dist}(V \cap \mathbf{B}_1, V' \cap \mathbf{B}_1)\,,$$

hence concluding the desired statement. □

Finally, the following lemma gives a lower bound for the distance of z to $q + \mathbf{S}$ for many points $z \in \mathbf{S}$. Before stating it we introduce some useful terminology.

Definition 7.13. A set $\Omega \subset \mathbb{R}^{m+n}$ is said to be *invariant under rotation around* a linear subspace V if $R(\Omega) = \Omega$ for any rotation R of $\mathbb{R}^{m+n}$ which fixes V.

Lemma 7.14 (Shifting lemma)**.** *For every $M \geq 1$ and every open set $U \subset \mathbf{B}_1$ that is invariant under rotations around V, there is a constant $\bar{C} = \bar{C}(M, m, n, N, U) > 0$ with the following properties. Assume that $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ satisfies (i), (ii), and (iii) in Lemma 7.12, let $q \in \mathbf{B}_{1/2}$, and let $\boldsymbol{\mu}(\mathbf{S}) = \max_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$. Then there is an index $j \in \{1, \ldots, N\}$ and a subset $\Omega \subset \alpha_j \cap U$ such that $\mathcal{H}^m(\Omega) \geq \bar{C}^{-1}$ and*

$$|\mathbf{p}^\perp_{\alpha_1}(q)| + \boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_1}(q)| \leq \bar{C}\,\mathrm{dist}(z, q + \mathbf{S}) \qquad \forall z \in \Omega\,. \tag{7.30}$$

Remark 7.15. Observe that Lemma 7.14 can be scaled. Under the assumption that $\mathbf{B}_r$ replaces $\mathbf{B}_1$, $U_{0,r} \equiv \iota^{-1}_{0,r}(U)$ replaces U, and $q \in \mathbf{B}_{r/2}$, we can conclude the existence of a subset $\Omega \subset U_{0,r}$ with measure larger than $\bar{C}^{-1} r^m$ with the property that (7.30) holds. Under these assumptions the constant $\bar{C}$ can be taken to be the same as the one in Lemma 7.14.

In order to prove Lemma 7.14, we will need the following elementary result.

Lemma 7.16. *There is a dimensional constant $C_0 = C_0(m, n) > 0$ with the following property. Let $\mathbf{S}$, M, and q be as in Lemma 7.14. Then*

$$|\mathbf{p}^\perp_{\alpha_1}(q)| + \boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_1}(q)| \leq C_0 M \max_i |\mathbf{p}^\perp_{\alpha_i}(q)|\,. \tag{7.31}$$

Proof. Let k be such that

$$|\mathbf{p}^\perp_{\alpha_k}(q)| = \max_i |\mathbf{p}^\perp_{\alpha_i}(q)|\,.$$

Thus, in particular, $|\mathbf{p}^\perp_{\alpha_1}(q)| \leq |\mathbf{p}^\perp_{\alpha_k}(q)|$, and so it remains to show that

$$\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_1}(q)| \leq C_0 M \max_i |\mathbf{p}^\perp_{\alpha_i}(q)|\,. \tag{7.32}$$

First of all pick j which maximizes $\mathrm{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1)$, so that in particular

$$\mathrm{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_l \cap \mathbf{B}_1) \leq \mathrm{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1) \qquad \forall l\,.$$

Thus, the triangle inequality yields

$$\mathrm{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_k \cap \mathbf{B}_1) \geq \frac{1}{2}\boldsymbol{\mu}(\mathbf{S})\,.$$

Next, recalling the definition of Morgan angles, observe that for every $w \in \alpha_j \cap V^\perp$ we have (using the above line, assumption (iii) in Lemma 7.12 and (7.2))

$$\boldsymbol{\mu}(\mathbf{S})|w| \leq C_0 M |\mathbf{p}^\perp_{\alpha_k}(w)|\,.$$

Now choose $w = \mathbf{p}_{V^\perp \cap \alpha_j}(q) = \mathbf{p}^\perp_V(q) - \mathbf{p}^\perp_{\alpha_j}(q)$. Since $\mathbf{p}^\perp_{\alpha_k} \circ \mathbf{p}^\perp_V = \mathbf{p}^\perp_{\alpha_k}$ (because $V \subset \alpha_k$) we may estimate

$$\begin{aligned}\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_j}(q)| &\leq C_0 M |\mathbf{p}^\perp_{\alpha_k}(\mathbf{p}^\perp_V(q))| + C_0 M |\mathbf{p}^\perp_{\alpha_k}(\mathbf{p}^\perp_{\alpha_j}(q))| \\ &\leq C_0 M |\mathbf{p}^\perp_{\alpha_k}(q)| + C_0 M |\mathbf{p}^\perp_{\alpha_j}(q)|\,.\end{aligned}$$

Using the maximality of k, we then reach

$$\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp \cap \alpha_j}(q)| \leq C_0 M |\mathbf{p}^\perp_{\alpha_k}(q)|\,. \tag{7.33}$$

It remains to replace α_j with α_1 in the projection on the left-hand side of the above inequality. Since $\mathbf{p}_{V^\perp \cap \alpha_1} \circ \mathbf{p}_{\alpha_j} = \mathbf{p}_{V^\perp \cap \alpha_1} \circ \mathbf{p}_{V^\perp \cap \alpha_j}$, we have

$$\begin{aligned}|\mathbf{p}_{V^\perp \cap \alpha_1}(q)| &\leq |\mathbf{p}_{V^\perp \cap \alpha_1}(\mathbf{p}_{V^\perp \cap \alpha_j}(q))| + |\mathbf{p}_{V^\perp \cap \alpha_1}((\mathbf{p}^\perp_{\alpha_j}(q))| \\ &\leq |\mathbf{p}_{V^\perp \cap \alpha_j}(q)| + |\mathbf{p}^\perp_{\alpha_j}(q)|\end{aligned}$$

$$\leq |\mathbf{p}_{V^\perp\cap\alpha_j}(q)| + |\mathbf{p}^\perp_{\alpha_k}(q)|\,.$$

Combining the latter inequality with (7.33) and using that $\boldsymbol{\mu}(\mathbf{S}) \leq 1$, we reach (7.32) and hence complete the proof of the lemma. □

We can now prove Lemma 7.14.

Proof of Lemma 7.14. We choose j such that $|\mathbf{p}^\perp_{\alpha_j}(q)| = \max_i |\mathbf{p}^\perp_{\alpha_i}(q)|$ and using Lemma 7.16 we aim at proving that

$$|\mathbf{p}^\perp_{\alpha_j}(q)| \leq \bar C_1 \operatorname{dist}(z, q+\mathbf{S}) \qquad \forall z\in\Omega\,, \tag{7.34}$$

for some set $\Omega\subset U$ with $\mathcal{H}^m(\Omega)\geq \bar C_1^{-1}$, where the constant $\bar C_1$ is allowed to depend on U. In fact it turns out that the constant $\bar C_1$ depends on

$$\gamma := \inf\{\mathcal{H}^m(U\cap\alpha) : V\subset\alpha \text{ and } \alpha \text{ is an } m\text{-dimensional subspace}\}\,. \tag{7.35}$$

It is not entirely obvious that γ is necessarily positive. Note, however, that if a point p belongs to U and another point q satisfies $\mathbf{p}_V(q) = \mathbf{p}_V(p)$ and $\operatorname{dist}(q,V) = \operatorname{dist}(p,V)$, then necessarily $q\in U$, in light of the rotational invariance of U. In particular there is an open subset $U'\subset V\times\mathbb{R}^+$ such that $U = \{p : (\mathbf{p}_V(p), \operatorname{dist}(p,V))\in U'\}$. From this we conclude that, not only is γ positive, but in fact that $\mathcal{H}^m(U\cap\alpha)$ is exactly the same number for every m-dimensional plane which contains V.

Observe first that (7.34) is equivalent to

$$\min_i |\mathbf{p}^\perp_{\alpha_i}(z-q)| \geq \bar C_1^{-1} |\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z\in\Omega\,. \tag{7.36}$$

Now assume that the claim is false, namely that (7.36) fails for all $z\in U\cap\alpha_j$ with the exception of a set E of Hausdorff measure smaller than $\bar C_1^{-1}$. In particular by choosing $\bar C_1$ large enough, we can ensure that, for some $i\in\{1,\ldots,N\}$ the set $F_i\subset\alpha_j\cap U$ where

$$|\mathbf{p}^\perp_{\alpha_i}(z-q)|\ \leq \bar C_1^{-1}|\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z\in F_i \tag{7.37}$$

has measure at least $\frac{\gamma}{N+1}$. We then claim that

$$|\mathbf{p}^\perp_{\alpha_i}(q)| \leq C\gamma^{-4}\bar C_1^{-1}|\mathbf{p}^\perp_{\alpha_j}(q)|\,, \tag{7.38}$$

$$|\mathbf{p}^\perp_{\alpha_i}(z)| \leq C\gamma^{-4}\bar C_1^{-1}|\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z\in\mathbf{B}_1\cap\alpha_j\,, \tag{7.39}$$

where the constant C depends only on m and N. Note that i is the index chosen such that $\mathcal{H}^m(F_i)\geq\frac{\gamma}{N+1}$, but the second estimate is claimed for every $z\in\mathbf{B}_1\cap\alpha_j$, and the latter will in the end be used to get a contradiction.

In order to prove (7.38) and (7.39) consider first the set $F_i' := \mathbf{p}_{V^\perp}(F_i)$. The latter belongs to the 2-dimensional subspace $V^\perp\cap\alpha_j$ and the coarea formula implies immediately that

$$\mathcal{H}^2(F_i') \geq C^{-1}\mathcal{H}^m(F_i) \geq C^{-1}\gamma \tag{7.40}$$

for a positive dimensional $C = C(m,N)$. Choose next a vector $e_1\in F_i'$ such that $|e_1|\geq \frac{1}{2}\sup\{|x| : x\in F_i'\}$ and observe that $|e_1|\geq C\sqrt{\gamma}$ because $4\pi|e_1|^2\geq\mathcal{H}^2(F_i')$. Hence choose $e_2\in F_i'$ such that

$$|e_2 - |e_1|^{-2}(e_2\cdot e_1)e_1| \geq \frac{1}{2}\sup\{|x-|e_1|^{-2}(x\cdot e_1)e_1| : x\in F_i'\}$$

and observe that $\mathcal{H}^2(F_i')\leq 4|e_2-|e_1|^{-2}(e_1\cdot e_2)e_1|$, so that

$$||e_1|e_2 - |e_1|^{-1}(e_1\cdot e_2)e_1| \geq C^{-1}\gamma^{3/2}\,.$$

We next define the linear map $\Phi:\mathbb{R}^2\to V^\perp\cap\alpha_j$ by $(\lambda_1,\lambda_2)\mapsto\lambda_1e_1+\lambda_2e_2$ and observe, by elementary geometry that $|\det\Phi| = ||e_1|e_2-|e_1|^{-1}(e_1\cdot e_2)e_1|$. In particular, $|\det\Phi^{-1}|\leq C\gamma^{-3/2}$ and, given that $|\Phi|\leq C$, we also have $|\Phi^{-1}|\leq C\gamma^{-3/2}$. Therefore $F_i'' := \Phi^{-1}(F_i')$ is contained in a disk of radius at most $C\gamma^{-3/2}$. Consider now the number

$$\mu := \sup\{|\lambda_1+\lambda_2-1| : (\lambda_1,\lambda_2)\in F_i''\}$$

and notice that $\mathcal{H}^2(F_i'') \leq C\gamma^{-3/2}\mu$. Since $|\det\Phi| \leq 1$ we get $\mathcal{H}^2(F_i') \leq \mathcal{H}^2(F_i'')$, and thus, when combined with (7.40), we infer $\mu \geq C^{-1}\gamma^{5/2}$, and thus the existence of $(\lambda_1, \lambda_2) \in F_i''$ such that

$$|\lambda_1 + \lambda_2 - 1| \geq C^{-1}\gamma^{5/2}\,. \tag{7.41}$$

Observe now that, by the very definition of F_i', there are $v_1, v_2, v_3 \in V$ such that

$$\begin{aligned}
&e_1 + v_1 \in F_i\\
&e_2 + v_2 \in F_i\\
&\lambda_1 e_1 + \lambda_2 e_2 + v_3 \in F_i\,.
\end{aligned}$$

Since $V \subset \alpha_i$ we can write

$$\begin{aligned}
&(\lambda_1 + \lambda_2 - 1)\mathbf{p}^\perp_{\alpha_i}(q) = \lambda_1\mathbf{p}^\perp_{\alpha_i}(q - e_1) + \lambda_2\mathbf{p}^\perp_{\alpha_i}(q - e_2) - \mathbf{p}^\perp_{\alpha_i}(q - (\lambda_1 e_1 + \lambda_2 e_2))\\
=&\lambda_1\mathbf{p}^\perp_{\alpha_i}(q - e_1 - v_1) + \lambda_2\mathbf{p}^\perp_{\alpha_i}(q - e_2 - v_2) - \mathbf{p}^\perp_{\alpha_i}(q - (\lambda_1 e_1 + \lambda_2 e_2 + v_3))\,.
\end{aligned}$$

In particular we conclude

$$|\mathbf{p}^\perp_{\alpha_i}(q)| \leq C\bar{C}_1^{-1}\gamma^{-5/2}|\mathbf{p}^\perp_{\alpha_j}(q)|\,, \tag{7.42}$$

which is in fact stronger than (7.38). On the other hand we can also write

$$|\mathbf{p}^\perp_{\alpha_i}(e_k)| \leq |\mathbf{p}^\perp_{\alpha_i}(e_k - q)| + |\mathbf{p}^\perp_{\alpha_i}(q)| \qquad k = 1, 2$$

and hence we immediately conclude

$$|\mathbf{p}^\perp_{\alpha_i}(e_k)| \leq C\bar{C}_1^{-1}\gamma^{-5/2}|\mathbf{p}^\perp_{\alpha_j}(q)|\,. \tag{7.43}$$

In turn, using again the map Φ we can express any $z \in \mathbf{B}_1 \cap \alpha_j$ as $z = \lambda_1 e_1 + \lambda_2 e_2 + v$ for some coefficients $|\lambda_i| \leq C\gamma^{-3/2}$ and some vector $v \in V$. In particular we achieve (7.39) from (7.43).

Observe however that (7.39) can be equivalently written as

$$\operatorname{dist}(z, \alpha_i) \leq C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}^\perp_{\alpha_j}(q)| \qquad \forall z \in \mathbf{B}_1 \cap \alpha_j\,, \tag{7.44}$$

and thus also as

$$\operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}^\perp_{\alpha_j}(q)| \tag{7.45}$$

Since γ is fixed, $|q| \leq \frac{1}{2}$, and α_i and α_j have the same dimension, the latter estimate implies that, by choosing $\bar{C}_1$ appropriately large, we can assume that the linear subspaces α_i and α_j are sufficiently close. In particular, given that $|q| \leq \frac{1}{2}$, for an appropriate large choice of $\bar{C}_1$ the affine subspace $q + \alpha_i^\perp$ must intersect $\mathbf{B}_1 \cap \alpha_j$ at some point z. But then at that point z we would have

$$\mathbf{p}^\perp_{\alpha_i}(q - z) = q - z\,. \tag{7.46}$$

Since $z \in \alpha_j$ we must have $|q - z| \geq \operatorname{dist}(q, \alpha_j) = |\mathbf{p}^\perp_{\alpha_j}(q)|$. Hence

$$|\mathbf{p}^\perp_{\alpha_i}(q - z)| \geq |\mathbf{p}^\perp_{\alpha_j}(q)|\,.$$

On the other hand, combining (7.38) and (7.39) we get

$$|\mathbf{p}^\perp_{\alpha_i}(q - z)| \leq 2C\bar{C}_1^{-1}\gamma^{-4}|\mathbf{p}^\perp_{\alpha_j}(q)|\,.$$

Since C is a constant which depends only on m and N we can now choose $\bar{C}_1$ large enough, depending only on γ, m, and N so that $C\bar{C}_1^{-1}\gamma^{-4} < \frac{1}{2}$. But then the last two inequalities would be in contradiction, unless $\mathbf{p}^\perp_{\alpha_j}(q) = \mathbf{p}^\perp_{\alpha_i}(q - z) = 0$. In particular we conclude that q belongs to α_j. In this case, however, (7.34) holds trivially. $\square$

8. Graphical approximations

In this section we start facing two of the geometric issues that complicate the proof of Theorem 2.5. In an ideal situation the cone $\mathbf{S}$ in the statement of the theorem consists of N m-dimensional planes $\alpha_1, \ldots, \alpha_N$ with the additional properties that, for any pair $\alpha_i \neq \alpha_j$,

(i) The Hausdorff distance between α_i and α_j is relatively larger than $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$;
(ii) The two Morgan angles $\theta_1(\alpha_i, \alpha_j)$, $\theta_2(\alpha_i, \alpha_j)$ formed by them are comparable.

Under these two additional assumptions we can hope to use the bounds of Theorem 3.2 and Corollary 3.3 to give a good approximation of T by graphs over the collection of planes α_i, after removing a small neighborhood of the spine $V(\mathbf{S})$.

Although there is no reason to assume (i) and (ii) for $\mathbf{S}$ a priori, we can hope to achieve them for a different cone $\mathbf{S}'$ without increasing the excess $\mathbb{E}(T, \mathbf{S}', \mathbf{B}_1)$ too much relative to the excess $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$. In light of this we first specify an algorithm that allows us to gain control on how much the excess increases when we discard planes of $\mathbf{S}$ until we achieve (i). This algorithm is summarized in the Pruning Lemma 8.2. For later use we want to iteratively apply Lemma 8.2 and keep track of the structure of the planes which have been discarded during this process; this is accomplished in Lemma 8.3. As for (ii), we will work under the assumption that it holds for now. Later, in Section 9.1, we will demonstrate that the new cone $\mathbf{S}'$ achieving (i) indeed additionally satisfies (ii).

In the remainder of this Section, we show how to gain a graphical approximation under a suitable quantification of (i) and (ii). We begin with a "crude approximation" in Section 8.3, followed by a more intricate "refined approximation" in Section 8.5; the latter will be needed later.

8.1. Pruning Lemma and Layer Subdivision.

The main purpose of this section is to introduce two very useful elementary combinatorial lemmas with the aim discussed above.

We will always work under the following assumption. We will often use the terminology "plane" when referring to a linear subspace of $\mathbb{R}^{m+n}$.

Assumption 8.1. $\mathbf{S} \subset \mathbb{R}^{m+n}$ is an m-dimensional cone such that

(i) $\mathbf{S}$ is a union of N distinct m-dimensional planes $\alpha_1, \ldots, \alpha_N$;
(ii) for each pair $i \neq j$ the intersection $\alpha_i \cap \alpha_j$ is the same $(m-2)$-dimensional plane V, which we refer to as the *spine* of $\mathbf{S}$.

Note in particular that the cones in the class of $\mathscr{C}(p, Q)$ of Definition 2.3 fall under the latter assumption.

Our first technical lemma we call the *Pruning Lemma*. It has two main uses. One is to prove the second technical lemma (Lemma 8.3); we will explain the meaning behind that lemma when we get to it. The other use will be to "prune" a cone, throwing away some of its planes, and ultimately get that the excess relative to the pruned cone is sufficiently small relative to the minimal angle of the pruned cone, which is a crucical assumption for our graphical approximation results later.

Lemma 8.2 (Pruning lemma)**.** *Let $N \geq 2$, $D > 0$, and $0 < \delta \leq 1$. Set $\Gamma := \delta^{2-N}(N-1)!$ and $\varepsilon := (1+\Gamma)^{-1}\delta$. If*

(i) *$\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is as in Assumption 8.1;*
(ii) *$D \leq \varepsilon \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$;*

then there is a subcollection $I \subset \{1, \ldots, N\}$ consisting of at least 2 elements and satisfying the following requirements:

$$\max_j \min_{i \in I} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma D \tag{8.1}$$

$$D + \max_j \min_{i \in I} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \delta \min_{j,\ell \in I : j < \ell} \operatorname{dist}(\alpha_j \cap \mathbf{B}_1, \alpha_\ell \cap \mathbf{B}_1) . \tag{8.2}$$

$$\max_{i,j \in I : i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) . \tag{8.3}$$

Proof. Set $I(0) = \{1, \dots, N\}$. If either (a) $N = 2$ or (b) $N \geq 3$ and

$$D \leq \delta \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{8.4}$$

then we select $I = I(0)$ and the proof is complete, since the left hand side of (8.1) is zero (and hence the left hand side of (8.2) equals D), while (8.3) is obvious. Observe also that, since $\varepsilon < \delta$, the condition (8.4) is implied by (ii) when $N = 2$.

Otherwise, we select indices ℓ_1 and ℓ_2 such that

$$\min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{\ell_1} \cap \mathbf{B}_1, \alpha_{\ell_2} \cap \mathbf{B}_1)$$

and indices j_1 and j_2 such that

$$\max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{j_1} \cap \mathbf{B}_1, \alpha_{j_2} \cap \mathbf{B}_1)\,.$$

Since $N \geq 3$ we can choose them so that $\{j_1, j_2\} \neq \{\ell_1, \ell_2\}$. In particular, we can pick $\ell(0) \in \{\ell_1, \ell_2\} \setminus \{j_1, j_2\}$. We then set $I(1) := I(0) \setminus \{\ell(0)\}$. Notice that

$$\max_{i<j\in I(1)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j\in I(0)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,,$$

while, by the assumption that (8.4) fails,

$$\min_{j\in I(1)} \operatorname{dist}(\alpha_{\ell(0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) < \delta^{-1} D\,.$$

Assuming that we have inductively selected sets $I(0), I(1), \dots I(s)$, we use the same procedure above to select a new subset $I(s+1) \subset I(s)$ by removing one element $\ell(s)$, provided that the cardinality of $I(s)$ is at least 3 and

$$D + \max_j \min_{i\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) > \delta \min_{i<j\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,. \tag{8.5}$$

Otherwise, we stop; clearly this process must terminate in finitely many steps. We denote by σ the index of the stopping step; note that $\sigma \leq N-2$. We claim that $I = I(\sigma)$ satisfies the requirements of the lemma.

First of all we prove the inequality

$$\min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s')} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq (s-s') \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \quad \forall s' < s \leq \sigma\,. \tag{8.6}$$

Note that if $s = s'+1$ the inequality is in fact an obvious equality (just from how it is written). In particular, the claim holds when $s' = \sigma - 1$. We now assume that the claim holds for all $s' > s_0$ and will proceed to show it when $s' = s_0$, by induction. Fix $s > s_0$ and let $j_* \in I(s_0+1)$ be such that

$$\min_{j\in I(s_0+1)} \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1)\,.$$

We first observe that, by the very definition of $\ell(s_0)$ and j_* we have

$$\operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) \leq \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,. \tag{8.7}$$

In particular if $j_* \in I(s)$, the inequality (8.6) is obvious.

Otherwise $j_* \notin I(s)$ and so $j_* = \ell(s_*)$ for some $s_0 < s_* < s$. In this case, using (8.7) and the fact that, by the inductive assumption,(8.6) holds for $s = s_*$, we write

$$\begin{aligned}
&\min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\
&\leq \operatorname{dist}(\alpha_{\ell(s_0)} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) + \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s_*)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\
&\leq \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) + (s - s_*) \min_{j\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,.
\end{aligned}$$

In particular, since $s_* - s_0 \geq 1$, we have shown (8.6) for $s = s_0$. We thus conclude that (8.6) indeed holds by induction over s.

Next, note that (8.6) implies that

$$\max_j \min_{i\in I(s)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq s \min_{i\in I(s)} \operatorname{dist}(\alpha_{\ell(s-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) \tag{8.8}$$

for all s, by simply maximizing over all $s' < s$ on the left-hand side of (8.6), since $s - s' \leq s$. In particular, combined with (8.5), we must have

$$D + t \min_{i \in I(t)} \mathrm{dist}(\alpha_{\ell(t-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1) > \delta \min_{i \in I(t+1)} \mathrm{dist}(\alpha_{\ell(t)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)$$

for $t = 1, \ldots, s-1$, with $s \leq \sigma$ fixed arbitrarily (note that the right-hand side of this expression equals that of (8.6), by definition of $\ell(t)$). Setting $d(t) := \min_{i \in I(t)} \mathrm{dist}(\alpha_{\ell(t-1)} \cap \mathbf{B}_1, \alpha_i \cap \mathbf{B}_1)$, we rewrite the above as the recursive inequality

$$\delta^{-1}(D + td(t)) > d(t+1)\,,$$

which, setting the convention $d(0) = 0$, can be assumed valid for $j = 0$ as well. We can thus iterate this to get

$$\begin{aligned}
d(s) &\leq \delta^{-1} D \left(1 + \delta^{-1}(s-1) + \delta^{-2}(s-1)(s-2) + \cdots + \delta^{-(s-1)} \cdot (s-1)!\right) \\
&\leq \delta^{-1} D \left(s \cdot \delta^{-(s-1)} \cdot (s-1)!\right) \\
&= \delta^{-s} s! D\,.
\end{aligned}$$

Since $s \leq N-2$, we get $d(s) \leq \delta^{2-N}(N-2)!D$, so combining with (8.8), for any $s \leq \sigma$ we have

$$\max_j \min_{i \in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq sd(s) \leq (N-2)d(s) \leq \delta^{2-N}(N-1)!D\,.$$

It is therefore the case that (8.1) holds with $I = I(\sigma)$.

Next, (8.2) certainly holds with $I = I(\sigma)$ by construction if $|I(\sigma)| \geq 3$, since then the procedure stopped due to the fact that (8.5) fails. We thus have to show that (8.2) holds when $|I(\sigma)| = 2$. In this case observe that our procedure guarantees that

$$\max_{i<j \in I(\sigma)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{8.9}$$

which shows that (8.3) holds in general. But then, as $|I(\sigma)| = 2$ we certainly have

$$\min_{i<j \in I(\sigma)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) = \max_{i<j \in I(\sigma)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

and so combining this with (8.9), using assumption (ii) and the fact that we have already proved (8.1), we have

$$\begin{aligned}
D + \max_j \min_{i \in I(\sigma)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq D + \Gamma D &= (1+\Gamma)D \\
&\leq (1+\Gamma)\varepsilon \max_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \\
&= \delta \min_{i<j \in I(\sigma)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,,
\end{aligned}$$

proving (8.2) in this case also (we have used that $\varepsilon = (1+\Gamma)^{-1}\delta$ here). □

We now iteratively apply Lemma 8.2 to a cone S comprised of planes $\{\alpha_1, \ldots, \alpha_N\}$ as in Assumption 8.1, getting a finite family of subcollections of the integers $\{1, \ldots, N\}$ leading to a family of simpler cones, in the following fashion. The collections of planes after each application of Lemma 8.2 can be thought of as "layers", which are subcollections of a starting one. Moreover, when we fix plane in a certain layer, the closest one among those of the previous layers is much closer than the minimum distance between any pair of planes belonging to these previous layers. What we end up with is that each plane we throw away when moving from one layer to the next is a distance comparable to the minimal distance in the original layer, whilst for the last layer we have comparability between the maximum and minimum distances of the planes.

Lemma 8.3 (Layer subdivision)**.** *For every integer $N \geq 2$ and every $0 < \delta \leq 1$, there is $\eta = \eta(\delta, N) > 0$ with the following properties. Let $\mathbf{S}$ and $\alpha_1, \ldots, \alpha_N$ be as in Assumption 8.1. Then, there is $\kappa \in \mathbb{N}$ and subcollections $I(0) \supsetneq I(1) \supsetneq \cdots \supsetneq I(\kappa)$ of the integers $\{1, \ldots, N\}$, each of cardinality at least 2 and with $I(0) = \{1, \ldots, N\}$, so that the numbers*

$$m(s) := \min_{i<j \in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{8.10}$$

$$d(s) := \max_{i\in I(0)} \min_{j\in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{8.11}$$

$$M(s) := \max_{i<j\in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \tag{8.12}$$

satisfy the following requirements:

(i) $M(\kappa) = M(0)$;
(ii) $\eta M(\kappa) \leq m(\kappa)$;
(iii) $d(s) \leq \delta m(s)$ *and* $\eta d(s) \leq m(s-1)$ *for every* $1 \leq s \leq \kappa$;
(iv) $m(s-1) \leq \delta m(s)$ *for every* $1 \leq s \leq \kappa$.

Proof. Let $0 < \delta \leq 1$. We fix $\eta > 0$ such that $\eta \leq \varepsilon$, where ε is the constant in Lemma 8.2 corresponding to δ/N in place of δ. In particular, $N\eta \leq \Gamma^{-1}$, where Γ is again as in Lemma 8.2 with δ replaced by δ/N.

If $\eta M(0) \leq m(0)$, we set $\kappa = 0$ and obviously the Lemma holds. Otherwise, we inductively apply Lemma 8.2 with $D = m(s-1)$ to produce $I(s)$ from $I(s-1)$, as long as $\eta M(s-1) > m(s-1)$. We wish to check that the conclusions of the lemma hold for this particular choice of subcollections. The fact that the sequence of sets is strictly decreasing and each set has cardinality at least two are obvious. Property (i) is immediate from (8.3) of Lemma 8.2, and (iv) is immediate from (8.2), since we are taking $D = m(s-1)$. Property (ii) holds by construction, as the process must terminate in finite time; at worst, when $\kappa = N-2$ and $|I(\kappa)| = 2$. Moreover, by (8.2) of Lemma 8.2 we have the inequality

$$\max_{i\in I(s-1)} \min_{j\in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \frac{\delta}{N} m(s)\,.$$

for every $s = 1, \ldots, \kappa$ (recall that when generating $I(s)$, we are applying Lemma 8.2 with $I(s-1)$ in place of $\{1, \ldots, N\}$). But the triangle inequality then gives

$$d(s) \leq \frac{\delta}{N}(m(1) + \cdots + m(s))\,.$$

Since $m(t) \leq m(s)$ for all $1 \leq t < s$, and $\kappa \leq N-2$, we therefore must have $d(s) \leq \delta m(s)$ for every s, proving the first inequality in (iii). Finally, observe that by (8.1) of Lemma 8.2, we also have

$$\max_{i\in I(s-1)} \min_{j\in I(s)} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma m(s-1)$$

for $s = 1, \ldots, \kappa$, and thus by the same triangle inequality argument as above, we achieve $d(s) \leq N\Gamma m(s-1)$ for every s. Hence, recalling that $N\eta \leq \Gamma^{-1}$, we achieve the second inequality of (iii). □

8.2. **Graphical parameterizations for T over $\mathbf{S}$.** The aim of this part is to efficiently parameterize area-minimizing currents over cones $\mathbf{S}$ as in Definition 2.3 satisfying the additional pairwise Morgan angle comparability condition (ii) outlined in the introduction of Section 8. We recall that we are working under the Assumption 2.1 throughout. Moreover, recall the sets $\mathscr{C}(Q)$ and $\mathscr{P}$ as in Definition 2.3 and the L^2 based excesses of Definition 2.4.

The important concept of a "balanced cone" is given in the following definition.

Definition 8.4. Let $\mathbf{S} \in \mathscr{C}(Q)$, $M \geq 1$, and let $\alpha_1, \ldots, \alpha_N$ be the N distinct m-dimensional planes forming $\mathbf{S}$. We say that $\mathbf{S}$ is M*-balanced* if for every $i \neq j$ the inequality

$$\theta_2(\alpha_i, \alpha_j) \leq M\theta_1(\alpha_i, \alpha_j) \tag{8.13}$$

holds for the two Morgan angles of the pair α_i, α_j.

Observe that the condition is empty when $\mathbf{S} \in \mathscr{P}$. Moreover, if we say that a cone $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N \in \mathscr{C}(Q)$ is M-balanced for some $M > 0$, we will implicitly be assuming that all the α_i are distinct. Loosely speaking, we are interested in balanced cones because two planes within a balanced cone can only degenerate to a single plane, and not to two planes meeting along an $(n-1)$-dimensional axis.

We will give two different graphical approximation results: a series of "crude approximation" results, followed by a series of "refined approximation" results.

8.3. **Crude approximation statements.** Here we will give the statements of the crude approximation results; in the next section, we will prove all of them. The key starting point is the following splitting lemma. We recall the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

Lemma 8.5 (Crude splitting). *For every $Q, m, n, \bar{n} \in \mathbb{N}$ and every $M > 0$, there are constants $\delta = \delta(Q, m, n, \bar{n}, M) > 0$ and $\varrho = \varrho(Q, m, n, \bar{n}, M) > 0$ with the following property. Let T be as in Assumption 2.1 with $\|T\|(\mathbf{B}_4) \leq (Q + \frac{1}{2})4^m\omega_m$. Assume that $2 \leq N \leq Q$, that $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N \in \mathscr{C}(Q)$ is M-balanced, and*

$$\int_{\mathbf{B}_4 \setminus B_{1/32}(V)} \mathrm{dist}^2(p, \mathbf{S})\, d\|T\|(p) + \mathbf{A}^2 \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S})^2 =: \delta^2\sigma^2\,, \tag{8.14}$$

where $V = V(\mathbf{S})$ is the spine of $\mathbf{S}$. Then the following properties hold:

(a) *The sets $W_i := (\mathbf{B}_4 \setminus B_{1/32}(V)) \cap \{\mathrm{dist}(\cdot, \alpha_i) < \varrho\sigma\}$ are pairwise disjoint;*
(b) $\mathrm{spt}(T) \cap \mathbf{B}_3 \setminus B_{1/16}(V) \subset \bigcup_i W_i$.

From the above lemma, the tilt-excess bound from Theorem 3.2, and Almgren's strong Lipschitz approximation over planes ([11, Theorem 1.4]), we can then conclude the following, where we use heavily the notation of [11, Theorem 1.4] (in particular, if u is a Lipschitz multi-valued map, $\mathrm{gr}\,(u)$ will denote its set-theoretic graph and $\mathbf{G}_u$ the current naturally induced by it).

Proposition 8.6. *Let T, W_i, and all associated notation be as in Lemma 8.5. Consider for each $i \in \{1, \ldots, N\}$ the regions $\Omega_i := (\mathbf{B}_2 \cap \alpha_i) \setminus \overline{B}_{1/16}(V)$ and $\boldsymbol{\Omega}_i := \mathbf{B}_3 \cap \mathbf{p}_{\alpha_i}^{-1}(\Omega_i)$. Set $T_i := T \llcorner (W_i \cap \boldsymbol{\Omega}_i)$ and*

$$E_i := \int_{\mathbf{B}_3 \setminus B_{1/32}(V)} \mathrm{dist}^2(p, \alpha_i)\, d\|T_i\|(p)$$

Then, there are non-negative integers $Q_1, \ldots, Q_N$ with $\sum_i Q_i \leq Q$ satisfying the following properties:

(a) $\partial T_i \llcorner \boldsymbol{\Omega}_i = 0$;
(b) *α_i can be appropriately oriented so that $(\mathbf{p}_{\alpha_i})_\sharp T_i = Q_i [\![\Omega_i]\!]$;*
(c) *The following estimate holds*

$$\mathrm{dist}^2(q, \alpha_i) \equiv |\mathbf{p}_{\alpha_i}^\perp(q)|^2 \leq CE_i + C\mathbf{A}^2 \qquad \forall q \in \mathrm{spt}(T_i) \cap \boldsymbol{\Omega}_i\,; \tag{8.15}$$

(d) *For all i with $Q_i \geq 1$, there are Lipschitz multi-valued maps $u_i : \Omega_i \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ and closed sets $K_i \subset \Omega_i$ such that $\mathrm{gr}(u_i) \subset \Sigma$, $T_i \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i)$, and the following estimates hold:*

$$\|u_i\|_\infty^2 + \|Du_i\|_{L^2}^2 \leq C(E_i + \mathbf{A}^2) \tag{8.16}$$

$$\mathrm{Lip}\,(u_i) \leq C(E_i + \mathbf{A}^2)^\gamma \tag{8.17}$$

$$|\Omega_i \setminus K_i| + \|T\|(\boldsymbol{\Omega}_i \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i)) \leq C(E_i + \mathbf{A}^2)^{1+\gamma}\,; \tag{8.18}$$

(e) *$Q_i = 0$ if and only if $T_i = 0$;*
(f) *Finally, if in addition we have the "reverse excess" estimate*

$$\int_{\mathbf{S} \cap \mathbf{B}_2 \setminus B_{1/4}(V)} \mathrm{dist}^2(p, \mathrm{spt}(T))\, d\mathcal{H}^m(p) \leq \delta^2\sigma^2\,, \tag{8.19}$$

then $Q_i \geq 1$ for every i.

Here, $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ and $C = C(Q, m, n, \bar{n}) > 0$;

A simple consequence of the above statements, combined with an elementary covering argument is that, if we fix $\rho > 0$ and $\eta > 0$, the same conclusions hold if we were to remove the neighborhood $B_\rho(V)$ of the spine and instead take $\Omega_i = (\mathbf{B}_{4-2\eta} \cap \alpha_i) \setminus B_{\rho+\eta}(V)$, as long as the constants in the estimates are allowed to depend on ρ and η also. Since this will prove useful in several instances, we record the exact statements here.

Corollary 8.7. *For every $Q, m, n, \bar{n} \in \mathbb{N}$ and $M > 0$, $\rho > 0$, and $\eta > 0$, there are constants $\delta = \delta(Q, m, n, \bar{n}, M, \rho, \eta) > 0$ and $\varrho = \varrho(Q, m, n, \bar{n}, M, \rho, \eta) > 0$ with the following property. Assume T, $\mathbf{S}$, and V are as in Lemma 8.5. Then, all the conclusions of the Lemma 8.5 apply if we replace $\mathbf{B}_3 \setminus B_{1/16}(V)$ with $\mathbf{B}_{4-\eta} \setminus B_{\rho+\eta}(V)$. Moreover, the conclusions of Proposition 8.6 apply for the regions Ω_i defined instead as $(\mathbf{B}_{4-2\eta} \cap \alpha_i) \setminus B_{\rho+\eta}(V)$, for the domain of integration in the definition of E_i taken to be $\mathbf{B}_{4-\eta} \setminus B_\rho(V)$, and with the constant C in the estimates depending also on η and ρ.*

In several situations we will need to ensure that $\sum_i Q_i = Q$ in Proposition 8.6 or Corollary 8.7. This conclusion needs however an additional assumption. Depending on the situation we will either be able to use (1) the existence of a point of large density; or (2) the existence of a region with sufficiently large mass; to guarantee this.

Lemma 8.8. *Assume that T satisfies the assumptions of Lemma 8.5 and either:*

(a) *$\{\Theta(T, \cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$ for a sufficiently small $\varepsilon = \varepsilon(Q, m, n, \bar{n})$; or*
(b) *there is a closed set Ω with non-empty interior that is invariant under rotation around V (cf. Definition 7.13), for which $\|T\|(\Omega) \geq (Q - \frac{1}{2})\mathcal{H}^m(\alpha_1 \cap \Omega)$.*

Then, if Q_i is as in Proposition 8.6 and δ is sufficiently small, we have $\sum_i Q_i = Q$.

Finally, we remark that all the statements above can be suitably scaled and translated to analogous statements where the initial domain $\mathbf{B}_4$ in Lemma 8.5 is replaced by an arbitrary ball $\mathbf{B}_{4r}(q)$.

An approximation statement analogous to Proposition 8.6 also holds in the much simpler setting in which $\mathbf{S}$ consists of a single plane. This case is somewhat special because we cannot identify a unique spine V and at the same time the number σ in Lemma 8.5 is ∞, hence the smallness condition (8.14) would be empty. For this reason we state the proposition separately even though it could be embedded as a special case of Proposition 8.6.

Proposition 8.9 (Crude approximation on a single plane)**.** *For every $Q, m, n, \bar{n} \in \mathbb{N}$, there exist $\delta = \delta(Q, m, n, \bar{n}) > 0$ and $C = C(Q, m, n, \bar{n})$ with the following property. Let T be as in Assumption 2.1 with $\|T\|(\mathbf{B}_4) \leq (Q + \frac{1}{2})4^m\omega_m$. Assume $\mathbf{S} = \alpha_1 \in \mathscr{P}$, $V \subset \alpha_1$ is an $(m-2)$-dimensional subspace, and*

$$E_1 + \mathbf{A}^2 := \int_{\mathbf{B}_4 \setminus B_{1/32}(V)} \operatorname{dist}(p, \mathbf{S})^2 \, d\|T\|(p) + \mathbf{A}^2 \leq \delta^2 \,. \tag{8.20}$$

Set $\Omega_1 := (\mathbf{B}_2 \cap \alpha_1) \setminus B_{1/16}(V)$ and $\mathbf{\Omega}_1 := \mathbf{B}_3 \cap \mathbf{p}_{\alpha_1}^{-1}(\Omega_1)$ and $T_1 := T \llcorner \mathbf{\Omega}_1$ Then, there is non-negative integer $Q_1 \leq Q$ such that the following holds:

(a) $\partial T_1 \llcorner \mathbf{\Omega}_1 = 0$
(a) *α_1 can be appropriately oriented so that $(\mathbf{p}_{\alpha_1})_\sharp T_1 = Q_1[\![\Omega_1]\!]$;*
(c) *The following estimate holds*

$$\operatorname{dist}^2(q, \alpha_1) \equiv |q - \mathbf{p}_{\alpha_1}(q)|^2 \leq CE_1 + C\mathbf{A}^2 \qquad \forall q \in \operatorname{spt}(T_1) \cap \mathbf{\Omega}_1 \,; \tag{8.21}$$

(d) *There is a Lipschitz multi-valued map $u_1 : \Omega_1 \to \mathcal{A}_{Q_1}(\alpha_1^\perp)$ and a closed set $K_1 \subset \Omega_1$ such that $\operatorname{gr}(u_1) \subset \Sigma$, $T_1 \llcorner \mathbf{p}_{\alpha_1}^{-1}(K_1) = \mathbf{G}_u \llcorner \mathbf{p}_{\alpha_1}^{-1}(K_1)$ and the following estimates hold:*

$$\|u_1\|_\infty^2 + \|Du_1\|_{L^2}^2 \leq C(E_1 + \mathbf{A}^2) \tag{8.22}$$

$$\operatorname{Lip}(u_1) \leq C(E_1 + \mathbf{A}^2)^\gamma \tag{8.23}$$

$$|\Omega_1 \setminus K_1| + \|T\|(\mathbf{\Omega}_1 \setminus \mathbf{p}_{\alpha_1}^{-1}(K_1)) \leq C(E_1 + \mathbf{A}^2)^{1+\gamma} \,; \tag{8.24}$$

(e) *$Q_1 = 0$ if and only if $T_1 = 0$;*
(f) *If one of conditions (a) and (b) in Lemma 8.8 holds then $Q_1 = Q$.*

Here, $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ and $C = C(Q, m, n, \bar{n}) > 0$.

Likewise, we have an analogous version of Corollary 8.7 where we can allow for more general domains in the assumptions and conclusions and, through an obvious scaling argument, versions in which $\mathbf{B}_4(0)$ is replaced by an arbitrary ball $\mathbf{B}_{4r}(p)$.

We now prove all of the above results.

8.4. **Proofs of Crude Approximation Results.** As previously explained, Corollary 8.7 follows from Proposition 8.6 and Lemma 8.5 and an elementary covering argument; we omit the specific details. We will therefore prove Lemma 8.5, Proposition 8.6, and Lemma 8.8, all at once. Proposition 8.9 then follows by the same arguments, and so we will again leave the details to the reader.

Proof of Lemma 8.5. We remark that statement (a) of Lemma 8.5 holds as soon as ϱ is smaller than an suitable constant which depends only on M (and the number of planes as well as the dimensions). We fix any such ϱ, and claim that as soon as δ is small enough (b) holds as well.

Indeed, to this end we may argue by contradiction: consider a sequence T_k of integral currents and a sequence Σ_k of Riemannian submanifolds of $\mathbb{R}^{m+n}$ satisfying Assumption 2.1, together with cones $\mathbf{S}_k \in \mathscr{C}(Q)$ such that

(i) $\|T_k\|(\mathbf{B}_4) \leq (Q+\frac{1}{2})\omega_m 4^m$;
(ii) $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_{N(k)}^k \in \mathscr{C}(Q)$ is M-balanced, where $N(k) \leq Q$;
(iii) If we write $\sigma_k := \boldsymbol{\sigma}(\mathbf{S}_k)$ and

$$\mathbf{E}_k := \int_{\mathbf{B}_4 \setminus B_{1/32}(V(\mathbf{S}_k))} \text{dist}^2(p, \mathbf{S}_k) d\|T_k\|(p)\,,$$

we have $\sigma_k^{-2}(E_k + \mathbf{A}_k^2) \to 0$, where $\mathbf{A}_k$ corresponds to Σ_k;
(iv) There are points $p_k \in \text{spt}(T_k) \cap \mathbf{B}_3 \setminus B_{1/16}(V(\mathbf{S}_k))$ such that $\text{dist}(p_k, \mathbf{S}_k) \geq \varrho\sigma_k$.

Observe that σ_k is a bounded sequence since $0 \leq \sigma_k \leq 1$. Hence, up to the extraction of a subsequence and after applying suitable rotations, we can assume that:

(v) $V(\mathbf{S}_k)$ is a fixed $(m-2)$-dimensional plane V and $N \equiv N(k) \leq Q$ is a fixed integer;
(vi) $\mathbf{S}_k$ converges, locally in Hausdorff distance, to $\mathbf{S} \in \mathscr{C}(Q)$ which is the union of $N' \leq N$ distinct planes α_i such that $\alpha_i \cap \alpha_j = V$ for all pairs $i < j$ (note that we could have $N' = 1$, in which case the latter condition here is vacuous);
(vii) T_k converges to an integral current T which is area-minimizing in $\mathbf{B}_4$ and obeys $\partial T = 0$;
(viii) $\text{spt}(T) \cap \mathbf{B}_4 \setminus B_{1/32}(V) \subset \mathbf{S}$.

Now, since $\text{Sing}(T)$ has Hausdorff dimension at most $m-2$ ([2]), the unique continuation principle for the minimal surface system gives that $\text{spt}(T) \subset \mathbf{S}$. By the constancy theorem and the fact that $\partial T = 0$, it then follows that $T = \sum_i \bar{Q}_i[\![\alpha_i]\!]$ where the $\bar{Q}_i$ are integers. Orienting the α_i suitably, we can assume that $\bar{Q}_i \geq 0$. In particular, $\sum_i \bar{Q}_i = \Theta(T,0) \leq Q + \frac{1}{2}$, and hence $\sum_i \bar{Q}_i \leq Q$.

We also know that $\text{spt}(T_k)$ converges locally in Hausdorff distance to $\text{spt}(T)$. In particular, passing to a subsequence, the points p_k converge to some point p which must lie in one of the planes α_j which form $\mathbf{S}$. We denote by π_0 this latter plane and observe that clearly $|p| \leq 3$ whilst $\text{dist}(p, V) \geq \frac{1}{16}$. For each fixed i, the sequence of planes α_i^k must converge (locally in Hausdorff distance) to some plane of $\mathbf{S}$ and, again upon extraction of a suitable subsequence and relabelling, we may assume that there is an $N_0 \leq N$ with the property that α_i^k converges to π_0 when $i \leq N_0$, whilst it converges to some other plane of $\mathbf{S}$ when $i > N_0$. Clearly we know that $N_0 \geq 1$ by construction.

Now for a fixed parameter $\eta > 0$, consider the currents $T_k' := T_k \llcorner ((\mathbf{B}_{7/2} \setminus B_{1/16}(V)) \cap \{\text{dist}(\cdot, \pi_0) < \eta\})$ and the cones $\mathbf{S}_k' := \alpha_1^k \cup \cdots \cup \alpha_{N_0}^k$. Observe that, if we choose η sufficiently small, the convergence properties outlined above imply that:

- $\partial T_k' = 0$ in $\mathbf{B}_{7/2} \setminus B_{1/16}(V)$ for all k sufficiently large;
- $\text{dist}(q, \mathbf{S}_k) = \text{dist}(q, \mathbf{S}_k')$ for all $q \in \text{spt}(T_k')$ and all k sufficiently large;
- T_k' converges to $\bar{Q}_j[\![\pi_0 \cap \mathbf{B}_{7/2} \setminus B_{1/16}(V)]\!]$, where $1 \leq \bar{Q}_j \leq Q$ is as above.

Now consider the cylinder $\mathbf{C}_{4r}(p, \pi_0)$, where p is the limit point as above and r is a geometric constant. Observe that, upon choosing r suitably, in light of (iii), for all k sufficiently large we can apply Corollary 3.3 to T_k' in the cylinder $\mathbf{C}_{4r}(p, \pi_0)$ (which we may ensure is disjoint from $B_{1/32}(V)$ provided that we take $4r \leq 1/64$). We can therefore decompose $T_k' \llcorner \mathbf{C}_r(p, \pi_0)$ as $T_{k,1}' + \cdots + T_{k,N_0}'$ with the property that

$$\text{dist}(q, \alpha_i^k) = \text{dist}(q, \mathbf{S}_k') = \text{dist}(q, \mathbf{S}_k) \qquad \forall q \in \text{spt}(T_{k,i}')\,. \tag{8.25}$$

Moreover, for each fixed i, the currents $T'_{k,i}$ converge to $Q_i[\![B_r(p,\pi_0)]\!]$, for some non-negative integers Q_i which obey $\sum_i Q_i = \bar{Q}_j$. Consider now the points $p_{k,i} := \alpha_i^k \cap \mathbf{p}_{\pi_0}^{-1}(p)$ (which exist for all k sufficiently large) and the cylinders $\mathbf{C}^{k,i} := \mathbf{C}_{r/2}(p_{k,i},\alpha_i^k)$. The latter cylinder is converging to $\mathbf{C}_{r/2}(p,\pi_0)$ and hence in particular, for k large enough, we know that $\mathbf{C}^{k,i} \cap \mathbf{B}_1 \subset \mathbf{C}_r(p,\pi_0)$. Hence, $\partial T'_{k,i} \llcorner \mathbf{C}^{k,i} = 0$ for k large enough because $\partial T'_{k,i} \llcorner \mathbf{C}_r(p,\pi_0) = 0$ and $\mathrm{spt}(T'_{k,i})$ are converging locally in Hausdorff distance to $\bar{B}_r(p,\pi_0)$. In particular, it follows that $(\mathbf{p}_{\alpha_i^k})_\sharp T'_{k,i} = Q'_i[\![B_{r/2}(p_{k,i},\alpha_i^k)]\!]$ for some integers $Q'_i \geq 0$ and that $\|T'_{k,i}\|(\mathbf{C}^{k,i}) \to Q_i\omega_m 2^{-m}r^m$.

However, because of (8.25) we also have

$$\int_{\mathbf{C}^i} \mathrm{dist}^2(q,\alpha_i^k)\, d\|T'_{k,i}\|(q) \leq \mathbf{E}_k\,.$$

We can now apply (a scaled version of) the L^∞ estimate (3.3) from Theorem 3.2 to conclude that

$$\mathrm{dist}^2(q,\alpha_i^k) \leq Cr^{-m-2}\mathbf{E}_k + Cr^2\mathbf{A}^2 \leq C(r^{-m-2}+r^2)\delta^2\sigma_k^2$$

for all $q \in \mathrm{spt}(T'_{k,i}) \cap \mathbf{C}_{r/4}(p_{k,i},\alpha_i^k)$. But the contradiction point p_k must be contained in one of the cylinders $\mathbf{C}_{r/4}(p_{k,i},\alpha_i^k)$ for k sufficiently large, and it is also contained in the support of $T'_k = \sum_i T'_{k,i}$; we may pass to a subsequence to fix the i for which this is true. Hence, for k large enough we must have the estimate

$$\mathrm{dist}^2(p_k,\alpha_i^k) \leq C(r^{-m-2}+r^2)\delta^2\sigma_k^2$$

where i is now fixed. But since r is fixed, it suffices to choose δ small enough to ensure $C(r^{-m-2}+r^2)\delta^2 \leq \varrho^2/4$ and contradict (iv) above; notice that δ indeed has the correct dependencies. □

Proof of Proposition 8.6. First of all notice that if we enlarge the regions Ω_i to $\Omega'_i := \mathbf{B}_{5/2} \setminus B_{1/16}(V)$ and define $\mathbf{\Omega}'_i := \mathbf{B}_2 \cap \mathbf{p}_{\alpha_i}^{-1}(\Omega'_i)$, the conclusions (a), (b), and the estimate (c) follow all from the arguments in the proof of Lemma 8.5 by covering $\mathbf{\Omega}_i$ with cylinders of radius r, basis parallel to α_i and centers in Ω_i. One only needs to check that the constant Q'_i is not changing with the cylinders of the covering, but this follows from the simple observation that the constant must be the same whenever two cylinders have non-empty intersection. The estimates in point (d) of Proposition 8.6 then follow from the Lipschitz approximation of [11] and (3.2) of Theorem 3.2.

As for the conclusion (e), because for each i one has the identity

$$\|T_i\|(\mathbf{\Omega}'_i) = Q_i|\Omega'_i| + \frac{1}{2}\int_{\Omega'_i} |\vec{T}-\vec{\alpha}_i|^2\, d\|T_i\|\,,$$

if we again apply the tilt-excess estimate (3.2) from Theorem 3.2, when $Q_i = 0$ we must have

$$\|T_i\|(\mathbf{\Omega}'_i) \leq C\delta^2\sigma^2\,.$$

In particular, if r is the radius of the cylinders considered in the proof of Lemma 8.5, once δ is sufficiently small, the monotonicity formula guarantees that $\|T_i\|(\mathbf{B}_r(q)) = 0$ for every ball $\mathbf{B}_r(q)$ which is contained in $\mathbf{\Omega}'_i$ (indeed, as soon as the mass ratio of T_i falls below 1, we get a contradiction if T_i is not zero as $\Theta(T_i,q) \geq 1$ at every point $q \in \mathrm{spt}(T_i)$). This then implies conclusion (e).

As for (f), we easily conclude from the argument above that, if $Q_i = 0$, then the distance of any point $q \in \Omega_i$ to $\mathrm{spt}(T)$ must be at least $\min\{\varrho\sigma, r\}$. We thus infer

$$\int_{\mathbf{S}\cap\mathbf{B}_3\setminus B_{1/32}(V)} \mathrm{dist}^2(p,\mathrm{spt}(T))\, d\mathcal{H}^m(p) \geq c\min\{\varrho^2\sigma^2, r^2\}$$

for some geometric constant c. This obviously contradicts (8.19) if δ is small enough. This completes the proof of Proposition 8.6. □

Proof of Lemma 8.8. If we can show using upper semi-continuity of the density that the limiting current T in the contradiction argument within the proof of Lemma 8.5 satisfies $\Theta(T,0) \geq Q$, then we are done (as necessarily, using the notation contained therein, $\Theta(T,0) = \sum_j \bar{Q}_j \leq Q$ and thus must equal Q). Note that $\sum_j \bar{Q}_j = \sum_i Q_i$, for Q_i as in Proposition 8.6.

In the first alternative of Lemma 8.8 we would have $\Theta(T_k, p_k) \geq Q$, for T_k as in the proof of Lemma 8.5 and some $p_k \to 0$ (as we can take the $\varepsilon = \varepsilon_k$ in condition (i) to be a vanishing sequence), and hence the desired property follows from the upper semi-continuity of the density. In the second alternative, observe that $\|T_k\|$ converges in the weak$-*$ topology to $\|T\|$ and thus, since Ω is closed, the assumption from this alternative gives $\|T\|(\Omega) \geq (Q - \frac{1}{2})\mathcal{H}^m(\alpha_1 \cap \Omega)$. On the other hand, since Ω is invariant under rotation around V, the particular structure of T (being a union of planes) implies that $\|T\|(\Omega) = \Theta(T, 0)\mathcal{H}^m(\alpha_1 \cap \Omega)$. Noting again that $\Theta(T, 0)$ is necessarily an integer (as T is a union of planes of integer multiplicity) and that $\mathcal{H}^m(\Omega) > 0$, we conclude $\Theta(T, 0) \geq Q$. This then completes the proof. □

8.5. **Refined approximation.** We now come to the second graphical approximation, which follows a much more refined procedure, using the layers introduced in Lemma 8.3. For the reader acquainted with [32], this should be compared to [32, Remark (3) in Section 8, and Section 10].

Assumption 8.10 (Assumptions for the refined approximation)**.** Suppose T and Σ are as in Assumption 2.1 and $\|T\|(\mathbf{B}_4) \leq 4^m(Q + \frac{1}{2})\omega_m$. Suppose $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is a cone in $\mathscr{C}(Q) \setminus \mathscr{P}$ which is M-balanced, where $M > 0$ is a given fixed constant, and V is the spine of $\mathbf{S}$. For a sufficiently small constant $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ whose choice will be fixed in Assumption 8.12 below, suppose that $\{\Theta(T, \cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$ and (cf. Proposition 8.6) suppose that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 . \tag{8.26}$$

Recall that $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4)$ is a two-sided L^2 excess between T and $\mathbf{S}$.

8.5.1. *Whitney decomposition.* Let L_0 be the closed cube in V with side-length $\frac{2}{\sqrt{m-2}}$ centered at 0 (if we make the identification $V = \mathbb{R}^{m-2}$, then we can write explicitly $L_0 = [-\frac{1}{\sqrt{m-2}}, \frac{1}{\sqrt{m-2}}]^{m-2}$) and consider the set

$$R := \{p : \mathbf{p}_V(p) \in L_0 \text{ and } 0 < |\mathbf{p}_{V^\perp}(p)| \leq 1\} . \tag{8.27}$$

We recall here that we are assuming $m \geq 3$ (cf. Assumption 2.1).

Obviously R is invariant under rotations around V. We next decompose R into a countable family of closed sets which are also invariant under rotations around V. Firstly, for every $\ell \in \mathbb{N}$ denote by $\mathcal{G}_\ell$ the collection of $(m-2)$-dimensional cubes in the spine V obtained by subdividing L_0 into $2^{\ell(m-2)}$ cubes of side-length $\frac{2^{1-\ell}}{\sqrt{m-2}}$, and we let $\mathcal{G} = \bigcup_\ell \mathcal{G}_\ell$. Note that we can generate $\mathcal{G}_{\ell+1}$ from $\mathcal{G}_\ell$ by bisecting every face of every cube in $\mathcal{G}_\ell$. We write L for a cube in $\mathcal{G}$, so $L \in \mathcal{G}_\ell$ for some $\ell \in \mathbb{N}$. When we want to emphasize the dependence of the integer ℓ on L we will write $\ell(L)$ and we will call it the *generation of* L. If $L \subset L'$ and $\ell(L') = \ell(L) + 1$, we then call L' the *parent* of L, and L a *child* of L', while more generally, when $\ell(L') > \ell(L)$, we say that L' is an *ancestor* of L and L a *descendant* of L'.

For every $L \in \mathcal{G}_\ell$ we let

$$R(L) := \{p : \mathbf{p}_V(p) \in L \quad \text{and} \quad 2^{-\ell-1} \leq |\mathbf{p}_{V^\perp}(p)| \leq 2^{-\ell}\} .$$

Observe that $R(L_0)$ is not the region R (as $L_0 \in \mathcal{G}_0$ so $\ell(L_0) = 0$), but rather $R = \bigcup_{L \in \mathcal{G}} R(L)$.

For each $L \in \mathcal{G}_\ell$ we let $y_L \in V$ be its center and denote by $\mathbf{B}(L)$ the ball $\mathbf{B}_{2^{2-\ell(L)}}(y_L)$ (in $\mathbb{R}^{m+n}$) and by $\mathbf{B}^h(L)$ the set $\mathbf{B}(L) \setminus B_{2^{-5-\ell(L)}}(V)$. It will be convenient to consider slight enlargements of the sets $R(L)$. More precisely, given a positive number $1 \leq \lambda \leq \frac{3}{2}$ and $L \in \mathcal{G}_\ell$ we will denote by λL the cube concentric to L in V with side-length $\frac{\lambda 2^{1-\ell}}{\sqrt{m-2}}$ and $\lambda R(L)$ the set

$$\lambda R(L) := \{p : \mathbf{p}_V(p) \in \lambda L \quad \text{and} \quad \lambda^{-1} 2^{-\ell-1} \leq |\mathbf{p}_{V^\perp}(p)| \leq \lambda 2^{-\ell}\} .$$

In fact, in the rest of the paper we will use some fixed choices of λ which depend only on the dimension and which are rather close to 1. Finally, an important role will be played by the "planar cross sections" of the sets $R(L)$ and $\lambda R(L)$, namely the intersections of these sets with the planes α_i forming the cone $\mathbf{S}$; these intersections will be denoted by L_i and λL_i, respectively.

The following elementary lemma, whose proof is left to the reader, summarizes some important geometric properties of the sets just introduced, and verifies that $\mathcal{G}$ is indeed a Whitney decomposition towards V. With a slight abuse of terminology we will talk about the interior of L and L_i meaning their interiors in the relative topology of V and α_i, respectively. In order to help the reader visualize the content of Lemma 8.11 we refer to Figure 1 below.

Lemma 8.11. *Consider the collection of cubes $\mathcal{G}$ introduced above and its elements L. Then the following properties hold:*

(i) *Given any pair of distinct $L, L' \in \mathcal{G}$ the interiors of $R(L)$ and $R(L')$ are pairwise disjoint and $R(L) \cap R(L') \neq \emptyset$ if and only if $L \cap L' \neq \emptyset$ and $|\ell(L) - \ell(L')| \leq 1$, while the interiors of L and L' are disjoint if $\ell(L) \leq \ell(L')$ and L' is not an ancestor of L.*

(ii) *The union of $R(L)$ ranging over all $L \in \mathcal{G}$ is the whole set R.*

(iii) *The diameters of the sets L, $R(L)$, λL, $\lambda R(L)$, L_i, λL_i, and $\mathbf{B}^h(L)$ are all comparable to $2^{-\ell(L)}$ and, with the exception of $L, \lambda L$, all comparable to the distance between an arbitrarily element within them and V; more precisely, any such diameter and distance is bounded above by $C2^{-\ell(L)}$ and bounded below by $C^{-1}2^{-\ell(L)}$ for some constant C which depends only on m and n.*

(iv) *There is a constant $C = C(m,n)$ such that, if $\mathbf{B}^h(L) \cap \mathbf{B}^h(L') \neq \emptyset$, then $|\ell(L) - \ell(L')| \leq C$ and $\operatorname{dist}(L, L') \leq C2^{-\ell(L)}$. In particular, for every $L \in \mathcal{G}$, the subset of $L' \in \mathcal{G}$ for which $\mathbf{B}^h(L)$ and $\mathbf{B}^h(L')$ have nonempty intersection is bounded by a constant.*

(v) *$\sum_{L \in \mathcal{G}_\ell} \mathcal{H}^{m-2}(L) = C(m)$ for any ℓ and therefore, for any $\kappa > 0$,*

$$\sum_{L \in \mathcal{G}} 2^{-(m-2+\kappa)\ell(L)} \leq C(\kappa, m)\,. \tag{8.28}$$

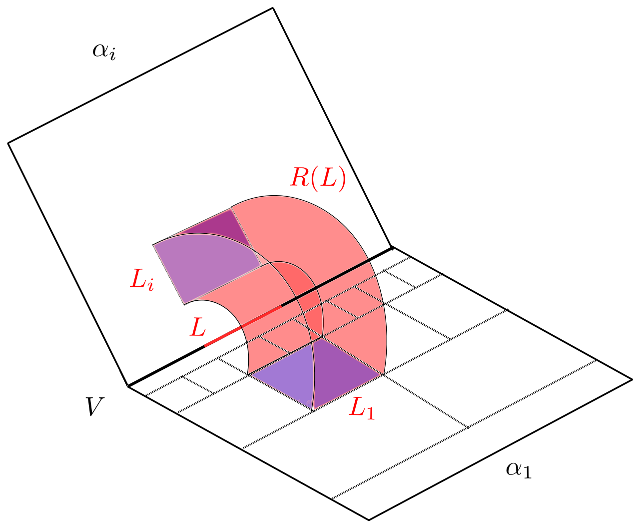

FIGURE 1. An illustration of the Whitney decomposition of R (illustrated on α_1). The subspace V is represented by the thick line joining the two planes, with a representation cube L showing, with the corresponding sets $R(L)$ (in red) and L_i (in blue). The set $R(L)$ is obtained by rotating the given cube L_1 around V in the ambient $(m+n)$-dimensional space, a portion of which is shown.

8.5.2. *Layering and choice of the parameters.* We now fix a $\bar{\delta} > 0$ (whose choice will specified below in Assumption 8.12) and apply the layering subdivision Lemma 8.3 with this $\bar{\delta}$ in place of δ therein, to identify a family of sub-cones $\mathbf{S} = \mathbf{S}_0 \supsetneq \mathbf{S}_1 \supsetneq \cdots \supsetneq \mathbf{S}_\kappa$ where $\mathbf{S}_k$ consists of the union of the planes α_i with $i \in I(k)$ for the set of indices $I(k)$ given by Lemma 8.3. We then distinguish two cases:

(a) if $\max_{i<j \in I(\kappa)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) < \bar{\delta}$, we define an additional cone $\mathbf{S}_{\kappa+1}$ consisting of a single plane, given by the smallest index in $I(\kappa)$ and we set $\bar{\kappa} := \kappa + 1$ and $I(\bar{\kappa}) := \{\min I(\kappa)\}$;

(b) otherwise, we select no smaller cone and set $\bar\kappa := \kappa$.

We next detail the choice of the various parameters involved in our discussion.

Assumption 8.12 (Selection of the parameters)**.** Firstly, we denote by δ^* the minimum of the parameters δ needed to ensure that Proposition 8.6, Lemma 8.8, and Proposition 8.9 are applicable to all the cones $\mathbf{S}_k$, $k \in \{0, 1, \dots, \bar\kappa\}$: note that all the $\mathbf{S}_k$ are M-balanced by construction and that therefore $\delta^* = \delta^*(m, n, \bar n, Q, M) > 0$, and in particular its choice does not depend on $\bar\delta$. Subsequently, we fix a parameter $\tau = \tau(m, n, \bar n, Q, M) > 0$ smaller than $\frac{\delta^*}{C}$ for some large constant $C = C(m, n, \bar n, Q) > 0$. The parameter $\bar\delta$ leading to the layering $\mathbf{S}_0 \supsetneq \mathbf{S}_1 \supsetneq \dots \supsetneq \mathbf{S}_{\bar\kappa}$ is then chosen to be much smaller than τ; so $\bar\delta = \bar\delta(m, n, \bar n, Q, M) > 0$. In particular, $\bar\delta \le \delta^*$. Finally, $\varepsilon = \varepsilon(m, n, \bar n, Q, \delta^*, \bar\delta, \tau) > 0$ will be chosen even smaller than $\bar\delta$.

8.5.3. *Outer, central, and inner regions.* We will next subdivide the cubes in $\mathcal{G}$ using the following criterion. In order to simplify our notation we introduce the shorthand:

$$\mathbf{E}(L, k) := 2^{(m+2)\ell(L)} \int_{\mathbf{B}^h(L)} \operatorname{dist}^2(q, \mathbf{S}_k)\, d\|T\|(q)\,.$$

For every $k \in \{0, 1, \dots, \bar\kappa\}$, recall that $I(k)$ gives the subset of $\{1, \dots, N\}$ such that $\mathbf{S}_k = \bigcup_{i \in I(k)} \alpha_i$. If $I(k)$ consists of more than one element, we set

$$\mathbf{s}(k) := \min_{i < j \in I(k)} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)\,, \tag{8.29}$$

while we set $\mathbf{s}(k) := \bar\delta$ if $I(k)$ is a singleton.

Definition 8.13. Let $L \in \mathcal{G}$. We say that:

(i) L is an *outer cube* if $\mathbf{E}(L', 0) \le \tau^2 \mathbf{s}(0)^2$ for every ancestor L' of L (including L).
(ii) L is a *central cube* if it is not an outer cube and if $\min_k \mathbf{E}(L', k)/\mathbf{s}(k)^2 \le \tau^2$ for every ancestor L' of L (including L).
(iii) L is an *inner cube* if it is neither an outer nor a central cube, but its parent is an outer or a central cube.

The corresponding families of cubes will be denoted by $\mathcal{G}^o$, $\mathcal{G}^c$, and $\mathcal{G}^{in}$, respectively. Observe that any cube $L \in \mathcal{G}$ is either an outer cube, or a central cube, or an inner cube, or a descendant of inner cube.

We correspondingly define three subregions of R:

- The *outer region*, denoted R^o, is the union of $R(L)$ for L varying over elements of $\mathcal{G}^o$.
- The *central region*, denoted R^c, is the union of $R(L)$ for L varying over elements of $\mathcal{G}^c$.
- Finally, the *inner region*, denoted R^{in}, is the union of $R(L)$ for L ranging over the elements of $\mathcal{G}$ which are neither outer nor central cubes.

Alternatively, the inner region can be defined as the union of $R(L)$ for L ranging over the inner cubes and their descendants. For a visual illustration of the subdivision and the corresponding cubes we refer to Figure 2.

The following lemma will be pivotal to define our refined approximation.

Lemma 8.14. *Let T and $\mathbf{S}$ be as in Assumption 8.10 and assume the parameters $\bar\delta$, δ^*, τ, and ε satisfy Assumption 8.12 and the ratios $\frac{\varepsilon}{\bar\delta}$, $\frac{\bar\delta}{\tau}$, and $\frac{\tau}{\delta^*}$ are smaller than a constant $c = c(Q, m, n, \bar n) > 0$. Then*

(i) *$L_0 \in \mathcal{G}^o$ and moreover, for every choice of τ and $\ell \in \mathbb{N}$ there is a constant $\bar c = \bar c(Q, m, n, \bar n, \tau, \ell) > 0$ such that, if $\varepsilon < \bar c$, then $\mathcal{G}_\ell \subset \mathcal{G}^o$.*
(ii) *For every $L \in \mathcal{G}^c$ there is an index $k(L) \in \{0, \dots, \bar\kappa\}$ such that*
$$\mathbf{E}(L, k) \le \tau^2 \mathbf{s}(k)^2 \tag{8.30}$$
for all $k \ge k(L)$ while for all $k < k(L)$ we have
$$\mathbf{E}(L, k) > \tau^2 \mathbf{s}(k)^2\,. \tag{8.31}$$
(iii) *For every $L \in \mathcal{G}^o$ we have that (8.30) holds for $k = 0$ (and thus for every k), whilst for every $L \in \mathcal{G}^{in}$ we have that (8.31) holds for every k.*

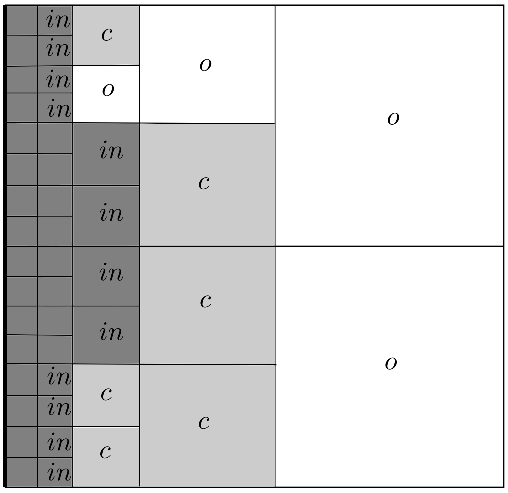

FIGURE 2. An example of a possible labeling of the cubes and of a corresponding subdivision of R. The outer region is white, while the central region is lightly shadowed and the inner region is shadowed. The labels o, c, and in identify rotationally invariant sets $R(L)$ corresponding to cubes L which are, respectively, outer, central, and inner cubes. Note that descendants of inner cubes *are not* inner cubes, even though the corresponding rotationally invariant regions are included in the inner region.

(iv) *There is a constant* $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \bar{\delta}, \delta^*, \tau) > 0$ *such that*

$$\mathbf{E}(L, k(L)) \leq \bar{C}\mathbf{E}(L, 0) \qquad \forall L \in \mathcal{G}^c \tag{8.32}$$

$$1 \leq \bar{C}\mathbf{E}(L, 0) \qquad \forall L \in \mathcal{G}^{in}. \tag{8.33}$$

(v) *For every* $L \in \mathcal{G}^o$*, Proposition 8.6 is applicable to the current* $T_{y_L, 2^{-\ell(L)}}$ *and the cone* $\mathbf{S}_0$*, whilst for every* $L \in \mathcal{G}^c$ *either Proposition 8.6 or Proposition 8.9 is applicable to the current* $T_{y_L, 2^{-\ell(L)}}$ *and the cone* $\mathbf{S}_{k(L)}$ *(depending on whether* $\mathbf{S}_{k(L)}$ *consists of at least two planes or is a single plane).*

Given Lemma 8.14 it is convenient to introduce the convention $k(L) = 0$ if $L \in \mathcal{G}^o$ and to set $\mathbf{E}(L) := \mathbf{E}(L, k(L))$. Note however that $k(L)$ might be indeed equal to 0 for several elements of $\mathcal{G}^c$ as well.

Proof. First of all, for any $L \in \mathcal{G}$ we have, since $\mathbf{B}^h(L) \subset \mathbf{B}_4$, that

$$\mathbf{E}(L, 0) \leq C 2^{(m+2)\ell(L)} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) \leq C 2^{(m+2)\ell(L)} \varepsilon^2 \mathbf{s}(0)^2 \,,$$

where the second inequality comes from (8.26), and where C depends only on m. In particular statement (i) in the Lemma follows immediately, as for any $L \in \mathcal{G}_\ell$ we have an upper bound on $\ell(L)$ and so choosing $C 2^{(m+2)\ell} \varepsilon^2 < \tau^2$ we get the desired inequality.

Regarding statement (ii), notice that

$$\mathbf{E}(L, k) \leq C \operatorname{dist}^2(\mathbf{S}_{k-1}, \mathbf{S}_k) + C\mathbf{E}(L, k-1) \tag{8.34}$$

for each $k = 0, \ldots, \bar{\kappa}$ and for some constant $C = C(m, n)$. Recalling that $\operatorname{dist}(\mathbf{S}_{k-1}, \mathbf{S}_k) \leq \bar{\delta}\mathbf{s}(k)$ and that $\mathbf{s}(k-1) \leq \bar{\delta}\mathbf{s}(k)$ (by Lemma 8.3 (iii) and (iv)), we immediately get

$$\mathbf{E}(L, k) \leq C \left(1 + \frac{\mathbf{E}(L, k-1)}{\mathbf{s}(k-1)^2}\right) \bar{\delta}^2 \mathbf{s}(k)^2 \,.$$

In particular, if $\frac{\bar{\delta}}{\tau}$ is smaller than an appropriate geometric constant, we infer that the inequality $\mathbf{E}(L, k-1) \leq \tau^2 \mathbf{s}(k-1)^2$ implies $\mathbf{E}(L, k) \leq \tau^2 \mathbf{s}(k)^2$. But then statement (ii) holds if we let $k(L)$ be the smallest integer k for which $\mathbf{E}(L, k) \leq \tau^2 \mathbf{s}(k)^2$, whose existence is guaranteed by the assumption that $L \in \mathcal{G}^c$. The same argument implies immediately the statement (iii) (as if $L \in \mathcal{G}^o$ then we know the minimum integer $k(L)$ above is $k(L) = 0$, whilst if $L \in \mathcal{G}^{in}$ the minimum over all k of the ratio $\mathbf{E}(L,k)/\mathbf{s}(k)^2$ is larger than τ^2).

Now recalling the second half of Lemma 8.3(iii), we have

$$\mathrm{dist}(\mathbf{S}_{k-1}, \mathbf{S}_k) \leq \eta^{-1}\mathbf{s}(k-1)\,,$$

where η depends on N and $\bar\delta$. In particular, recalling (8.34), if $\mathbf{E}(L,k-1) > \tau^2\mathbf{s}(k-1)^2$ we conclude the inequality

$$\mathbf{E}(L,k) \leq \frac{C}{\eta^2}\mathbf{s}(k-1)^2 + C\mathbf{E}(L,k-1) \leq C(1+(\eta\tau)^{-2})\mathbf{E}(L,k-1)\,.$$

If $L \in \mathcal{G}^c$ we can apply the latter for $k = 1,\ldots,k(L)$ to conclude (8.32). If $L \in \mathcal{G}^{in}$ we can apply it for every k, combined with (8.31), to conclude that

$$\mathbf{s}(\bar\kappa)^2 \leq C\mathbf{E}(L,0)\,.$$

We now distinguish two cases:

- $I(\bar\kappa)$ consists of a single element; in this case $\mathbf{s}(\bar\kappa) = \bar\delta$.
- $I(\bar\kappa)$ consists of more than one element; in this case (from Lemma 8.3(ii) and the defining property (b) of $\bar\kappa$ in Section 8.5.2)

$$\mathbf{s}(\bar\kappa) \geq \eta \max_{i<j\in I(\bar\kappa)} d(\alpha_i\cap\mathbf{B}_1, \alpha_j\cap\mathbf{B}_1) \geq \eta\bar\delta\,.$$

In both cases we conclude that $\mathbf{s}(\bar\kappa)$ is bounded away from 0 and thus (8.33) holds, proving (iv).

As for (v), observe that for $L \in \mathcal{G}^o \cup \mathcal{G}^c$,

$$\int_{\mathbf{B}_4\setminus B_{1/32}(V)} \mathrm{dist}^2(q, \mathbf{S}_{k(L)})\, d\|T_{y_L,2^{-\ell(L)}}\|(q) = \mathbf{E}(L,k(L)) \leq \tau^2\mathbf{s}(k(L))^2\,,$$

where we have used (iii). Moreover, $\mathbf{s}(0) \leq \mathbf{s}(k)$ for all k and $\mathbf{A}^2 \leq \varepsilon^2\mathbf{s}(0)^2$ by (8.26). Hence, if we denote by $\mathbf{A}_L$ the supremum norm of the second fundamental form of the rescaled manifold $\Sigma_{y_L,2^{-\ell(L)}}$, we then have

$$\int_{\mathbf{B}_4\setminus B_{1/32}(V)} \mathrm{dist}^2(q, \mathbf{S}_{k(L)})\, d\|T_{y_L,2^{-\ell(L)}}\|(q) + \mathbf{A}_L^2 \leq C(\tau^2 + 2^{-2\ell(L)}\varepsilon^2)\mathbf{s}(k)^2\,.$$

Recalling the definition of $\mathbf{s}(k)$ it suffices to guarantee that $C(\tau^2+\varepsilon^2) \leq (\delta^*)^2$ to guarantee that Proposition 8.6 is applicable in case $I(k)$ consists of more than one element. Otherwise we know that $\mathbf{s}(k) = \bar\delta$ and the same smallness condition guarantees the applicability of Proposition 8.9. This completes the proof. □

8.5.4. *Local approximations.* For each outer and central cube L, from Lemma 8.14(v) we apply Proposition 8.6 or Proposition 8.9 to the current $T_{y_L,2^{-\ell(L)}}$ and the cone $\mathbf{S}_{k(L)}$ (the choice of which proposition to apply depends only on whether $|I(k(L))| > 1$ or not). We thus gain a corresponding Lipschitz approximation for $T_{y_L,2^{-\ell(L)}}$ on the planar domains $(\mathbf{B}_2 \setminus \overline{B}_{1/16}(V)) \cap \mathbf{S}_{k(L)}$. By translating and scaling back we gain corresponding Lipschitz approximations of the current T defined over the open domains $\Omega(L) := (\mathbf{B}_{2^{1-\ell(L)}}(y_L) \setminus \overline{B}_{2^{-4-\ell(L)}}(V)) \cap \mathbf{S}_{k(L)}$ of $\mathbf{S}_k$.

We set $\Omega_i(L) := \Omega(L)\cap\alpha_i$ and denote by $u_{L,i}$ the corresponding multi-valued function given above and by $Q_{L,i}$ the number of values $u_{L,i}$ takes. We moreover denote by $\mathbf{\Omega}_i(L)$ the sets $2^{-\ell(L)}\mathbf{\Omega}_i + y_L$ and by $K_i(L)$ the sets $2^{-\ell(L)}K_i + y_L$, with $\mathbf{\Omega}_i$ and K_i given by Proposition 8.6 (or Proposition 8.9). These objects are defined only for the indices i belonging to the collection $I(k(L))$, but we may extend our notation to allow also the case $Q_{L,i} = 0$, in which the map $u_{L,i}$ does not exist (these will be the indices of planes not contained in $\mathbf{S}_{k(L)}$); this results in a slight abuse of our terminology and notation. The collection of corresponding multi-valued functions (ranging over all i) corresponding to a given L will be denoted by u_L, and we will call them "local approximations of T related to L". The following is the key proposition detailing the refined approximation.

Proposition 8.15 (Refined approximation)**.** *Let T, Σ and $\mathbf{S}$ be as in Assumption 8.10, and suppose that the assumptions of Lemma 8.14 hold. There exists $1 < \lambda = \lambda(m) \leq \frac{3}{2}$ and $\bar{C} = \bar{C}(Q, m, n, \bar{n}, \delta^*) > 0$, not depending on $\bar{\delta}$, τ, and ε, such that the local approximations u_L satisfy the following properties.*

(i) *For every fixed $i \in \{1, \ldots, N\}$, $Q_{L,i}$ is the same* positive *integer for every $L \in \mathcal{G}^o$.*
(ii) *$\sum_i Q_{L,i} = Q$ for every $L \in \mathcal{G}^o \cup \mathcal{G}^c$.*
(iii) *For every $L \in \mathcal{G}^o \cup \mathcal{G}^c$ we have $\mathrm{spt}(T) \cap \lambda R(L) \subset \bigcup_i \mathbf{\Omega}_i(L)$ and*

$$2^{2\ell(L)}|q - \mathbf{p}_{\alpha_i}(q)|^2 \leq \bar{C}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \qquad \forall q \in \mathrm{spt}(T) \cap \mathbf{\Omega}_i(L)\,; \tag{8.35}$$

(iv) *For $L \in \mathcal{G}^o \cup \mathcal{G}^c$, if we set*

$$T_{L,i} := T \llcorner \mathbf{\Omega}_i(L) \cap \left\{ \mathrm{dist}(\cdot, \alpha_i) < \bar{C} 2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2} \right\}$$

(with $\bar{C}$ larger than the constant in the estimate (8.35)), then, for $K_i(L) = 2^{-\ell(L)}K_i + y_L \subset \Omega_i(L)$ as above,

$$T_{L,i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i(L)) = \mathbf{G}_{u_{L,i}} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i(L))\,,$$

$\mathrm{gr}\,(u_{L,i}) \subset \Sigma$*, and the following estimates hold:*

$$2^{2\ell(L)}\|u_{L,i}\|_\infty^2 + 2^{m\ell(L)}\|Du_{L,i}\|_{L^2}^2 \leq \bar{C}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \tag{8.36}$$

$$\mathrm{Lip}\,(u_{L,i}) \leq C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma \tag{8.37}$$

$$|\Omega_i(L) \setminus K_i(L)| + \|T_{L,i}\|(\mathbf{\Omega}_i(L) \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i(L))) \leq \bar{C} 2^{-m\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma}\,. \tag{8.38}$$

(v) *For every $L \in \mathcal{G}^o \cup \mathcal{G}^c$, $\Theta(T, \cdot) \leq \max_i Q_{L,i} + \frac{1}{2}$ on $R(L)$. In particular, $\Theta(T, \cdot) \leq Q - \frac{1}{2}$ on $R(L)$ if $L \in \mathcal{G}^o$.*

We will prove Proposition 8.15 in tandem with the next lemma. Consider a domain $U \subset \lambda R(L)$ which is invariant under rotations around V and set $U_i := \alpha_i \cap U$. We wish to replace $T \llcorner U$ with the union of the portions of the graphs of the functions $u_{L,i}$ lying over U_i. This will generate errors of two types. One type is due to the fact that $\mathrm{spt}(T) \cap U$ is not completely contained in the union of the graphs of the functions $u_{L,i}$: this will be taken care of by Proposition 8.15 above. A second type of error is due to the fact that, even though the estimate (8.35) holds, if we define

$$\tilde{U} := \bigcup_i \{q : \mathbf{p}_{\alpha_i}(q) \in U_i \quad \text{and} \quad |q - \mathbf{p}_{\alpha_i}(q)| \leq \bar{C} 2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2}\} \tag{8.39}$$

(where $\bar{C}$ is assumed to be the constant of estimate (8.35)), there still is a difference between $\mathrm{spt}(T) \cap \tilde{U}$ and $\mathrm{spt}(T) \cap U$. The purpose of the following lemma is to estimate errors of this second type:

Lemma 8.16. *Under the assumptions of Lemma 8.14, consider $L \in \mathcal{G}^o \cup \mathcal{G}^c$, let $U \subset \lambda R(L)$ be a set invariant under rotations around V whose cross-sections $U_i = U \cap \alpha_i$ are Lipschitz open sets or the closures of Lipschitz open sets, and define $\tilde{U}$ as in (8.39). Then*

$$\begin{aligned}\|T\|(U \setminus \tilde{U}) + \|T\|(\tilde{U} \setminus U) \leq \bar{C} \left(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2\right) 2^{-\ell(L)}\mathcal{H}^{m-1}(\partial U_i)&\\ + \bar{C} 2^{-m\ell(L)} \left(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2\right)^{1+\gamma}&\end{aligned} \tag{8.40}$$

where the constant $\bar{C}$ depends on the parameters $m, n, \bar{n}, Q, \bar{\delta}$, and the Lipschitz regularity of the boundary of the rescaled cross-section $2^{\ell(L)}U_i$.

Remark 8.17. We will in fact only apply this Lemma to sets U whose cross-sections $U_i = U \cap \alpha_i$ have a limited number of shapes, up to the rescaling factor $2^{-\ell(L)}$. In particular, in these cases the corresponding constant $\bar{C}$ in the estimate depends only upon $m, n, \bar{n}, Q$, and $\bar{\delta}$.

Proof of Proposition 8.15 and of Lemma 8.16. We will prove the two statements at the same time. We start by showing that the estimates of the statements (iii) and (iv) of Proposition 8.15 hold. Fix $L \in \mathcal{G}^o \cup \mathcal{G}^c$, let $k = k(L)$, and $\ell = \ell(L)$. In order to simplify our notation we write $T' := T_{y_L, 2^{-\ell}}$ and $\Sigma' := \Sigma_{y_L, 2^{-\ell}}$. The supremum norm of the second fundamental form of Σ' is clearly $2^{-\ell}\mathbf{A}$. On the other hand, by scaling invariance,

$$\int_{\mathbf{B}_4 \setminus B_{1/32}(V)} \mathrm{dist}^2(q, \mathbf{S}_k)\, d\|T'\|(q) = \mathbf{E}(L)\,. \tag{8.41}$$

Let us assume that $I(k)$ consists of more than one element; in particular, by Lemma 8.14(v), we can apply Proposition 8.6 to T', Σ' and $\mathbf{S}_k$. The argument is entirely analogous when $I(k)$ consists of one element except instead we apply Proposition 8.9, and so we will just focus on the former case. We consider the domains $\mathbf{\Omega}_i$ given by Proposition 8.6 and recall that from Lemma 8.5 it follows that

$$\mathrm{spt}(T') \cap \mathbf{B}_3 \setminus B_{1/16}(V) \subset \bigcup_i \mathbf{\Omega}_i\,,$$

In turn this rescales to the statement

$$\mathrm{spt}(T) \cap \mathbf{B}_{3\cdot 2^{-\ell}}(y_L) \setminus B_{2^{-\ell-4}}(V) \subset \bigcup_i \mathbf{\Omega}_i(L)\,.$$

For an appropriate choice of the constant λ (i.e. sufficiently close to 1),

$$\lambda R(L) \subset \mathbf{B}_{3\cdot 2^{-\ell}}(y_L) \setminus B_{2^{-\ell-4}}(V)$$

and so the first claim of Proposition 8.15(iii) follows. As for the height estimate in (iii), Proposition 8.6(c) and (8.41) gives

$$|q - \mathbf{p}_{\alpha_i}(q)|^2 \le C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2) \qquad \forall q \in \mathbf{\Omega}_i \cap \mathrm{spt}(T')$$

which scales to

$$|q - \mathbf{p}_{\alpha_i}(q)|^2 \le C 2^{-2\ell}(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2) \qquad \forall q \in \mathbf{\Omega}_i(L) \cap \mathrm{spt}(T)\,,$$

i.e. (8.35). This proves (iii).

The first two claims of Proposition 8.15(iv) follow directly from the corresponding claims in Proposition 8.6. As for the estimates, if we denote by u_i and K_i the multi-valued functions approximating $T'_i := T' \llcorner \mathbf{\Omega}_i \cap W_i$ and the corresponding "coincidence sets" given by Proposition 8.6, then, taking into account (8.41), we have the estimates

$$\begin{aligned}
\|u_i\|_\infty^2 + \|Du_i\|_{L^2}^2 &\le C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2) \\
\mathrm{Lip}\,(u_i) &\le C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2)^\gamma \\
|\Omega_i \setminus K_i| + \|T'_i\|(\mathbf{\Omega}_i \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i)) &\le C(\mathbf{E}(L) + 2^{-2\ell}\mathbf{A}^2)^{1+\gamma}\,.
\end{aligned}$$

The estimates in Proposition 8.15(iv) therefore follow from the obvious scaling relations

$$\begin{aligned}
\|u_i\|_\infty &= 2^\ell \|u_{L,i}\|_\infty \\
\|Du_i\|_\infty &= \|Du_{L,i}\|_\infty \\
\|Du_i\|_{L^2} &= 2^{m\ell/2}\|Du_{L,i}\|_{L^2} \\
|\Omega_i \setminus K_i| &= 2^{m\ell}|\Omega_i(L) \setminus K_i(L)| \\
\|T'_i\|(\mathbf{p}_{\alpha_i}^{-1}(\Omega_i \setminus K_i)) &= 2^{m\ell}\|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(\Omega_i(L) \setminus K_i(L)))
\end{aligned}$$

which completes the proof of (iv).

Next, we will prove Proposition 8.15(i). We claim that $Q_{L,i} = Q_{L',i}$ if $L, L' \in \mathcal{G}^o$ and L' is the parent of L; notice that since $L_0 \in \mathcal{G}^o$ by Lemma 8.14(i), this will hold for any $i \in I(0)$, and thus for every $i \in \{1, \ldots, N\}$. First, notice that because of (8.35), it is easy to see that $T_{L,i}$ and $T_{L',i}$ coincide over $\mathbf{\Omega}_i(L) \cap \mathbf{\Omega}_i(L')$. In particular, the two currents $\mathbf{G}_{u_{L,i}}$ and $\mathbf{G}_{u_{L',i}}$ coincide over $\mathbf{p}_{\alpha_i}^{-1}(K_i(L) \cap K_i(L'))$. Clearly by (iv), namely (8.38), $K_i(L) \cap K_i(L')$ has positive measure provided $\mathbf{E}(L)$, $\mathbf{E}(L')$, and $\mathbf{A}$ are smaller than a geometric constant, and all these conditions can be ensured by choosing τ and ε smaller than a geometric constant (here it is crucial that the constant C in the estimates of statement (iv) does not depend on τ and ε). But once we know that $K_i(L) \cap K_i(L')$ has positive measure and the current graphs coincide over this set we clearly conclude that $Q_{L,i} = Q_{L',i}$.

Given this, since every parent of an element in $\mathcal{G}^o$ belongs to $\mathcal{G}^o$ (by definition of $\mathcal{G}^o$) we conclude from the above that $Q_{L,i} = Q_{L_0,i}$, which obviously implies the first claim of Proposition 8.15(i). The fact that all of them are positive follows from Proposition 8.6(f) once we assume ε is sufficiently small, because it will force $Q_{L_0,i} \geq 1$ for every i. Thus (i) is proved.

Next, because of (i), the conclusion of point (ii) holds for $L \in \mathcal{G}^o$ once we show $\sum_i Q_{L_0,i} = Q$. For the latter we can use Lemma 8.8 with assumption (a), since $y_{L_0} = 0$ and $\{\Theta(T,\cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$.

In order to prove (ii) when $L \in \mathcal{G}^c$ we are again going to argue inductively, showing that:

(A) If $\sum_i Q_{L',i} = Q$ for $L' \in \mathcal{G}^c \cup \mathcal{G}^o$, then $\sum_i Q_{L,i} = Q$ for every child $L \in \mathcal{G}^c \cup \mathcal{G}^o$ of L'.

In order to show (A) we will in fact use the result of Lemma 8.16, which we shall prove now.

Fix $L \in \mathcal{G}^c \cup \mathcal{G}^o$ and a set $U \subset \lambda R(L)$ which is invariant under rotations around V. Firstly, from what has been proved so far of Proposition 8.15, namely (iii) and (iv), we can conclude that $T \llcorner (U \cup \tilde{U}) = \sum_i T_{L,i} \llcorner (U \cup \tilde{U})$ and $\mathrm{spt}(T_{L,i}) \cap \mathrm{spt}(T_{L,j}) = \emptyset$ for all $i < j$ (which follows from (iv) provided τ and ε are sufficiently small, as then the neighbourhoods of the α_i in (iv) are all disjoint, and from the choice of λ, which ensures that $U \cap \tilde{U} \cap \mathrm{spt}(T)$ is contained in the union of $\mathbf{\Omega}_i(L)$). In particular, we have

$$\|T\|(U \Delta \tilde{U}) = \sum_i \|T_{L,i}\|(U \Delta \tilde{U})$$

where $U \Delta \tilde{U} := (U \setminus \tilde{U}) \cup (\tilde{U} \setminus U)$ is the symmetric difference of U and $\tilde{U}$. Thus we can again use the conclusion (iv) in Proposition 8.15 to estimate further

$$\|T\|(U \Delta \tilde{U}) \leq \sum_i \|\mathbf{G}_{u_{L,i}}\|(U \Delta \tilde{U}) + C 2^{-m\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)^{1+\gamma}.$$

Consider now the set

$$\Delta_i := \mathbf{p}_{\alpha_i}(\mathrm{gr}\,(u_{L,i}) \cap (U \Delta \tilde{U}))$$

Since by choosing τ and ε small enough we can assume $\mathrm{Lip}(u_{L,i}) \leq 1$ (by (8.37)), it follows immediately from this this Lipschitz regularity that

$$\|\mathbf{G}_{u_{L,i}}\|(U \Delta \tilde{U}) \leq C Q_{L,i} |\Delta_i|$$

for some geometric constant C. Since $\sum_i Q_{L,i} \leq Q$ (note in particular that we do not need the equality $\sum_i Q_{L,i} = Q$ in this argument), in order to reach the estimate of Lemma 8.16 it suffices to show that

$$|\Delta_i| \leq C(\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) 2^{-\ell(L)} \mathcal{H}^{m-1}(\partial U_i)\,, \tag{8.42}$$

where C has the dependencies in the statement of the lemma (recall that $U_i = U \cap \alpha_i$). We will indeed show that

$$\mathrm{dist}(q, \partial U_i) \leq C 2^{-\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \qquad \forall q \in \Delta_i\,.$$

In particular, using the Lipschitz regularity of ∂U_i, (8.42) follows immediately (by taking a suitable cover of ∂U_i and taking the tubular neighbourhood of each set in this cover with radius 2 times the above distance bound to cover Δ_i). In order to show (8.42), fix a point $x \in \Delta_i$ and observe that by definition there must be a point $p \in \mathrm{gr}\,(u_{L,i}) \cap (U \Delta \tilde{U})$ such that $x = \mathbf{p}_{\alpha_i}(p)$. Let $v = \mathbf{p}_V(p) = \mathbf{p}_V(x)$ and denote by σ the half-line in α_i which originates at v and contains x. On this half-line we denote by y the point such that

$$\mathrm{dist}(y, V) = |y - v| = \mathrm{dist}(p, V)\,, \tag{8.43}$$

and moreover as all points on this half-line project to v, we have

$$\mathbf{p}_V(y) = \mathbf{p}_V(p)\,. \tag{8.44}$$

Observe that $p \in U$ if and only if $y \in U$ (using that U is invariant under rotations about V and (8.43), (8.44)), which happens if and only if $y \in U_i$. On the other hand $p \in \tilde{U}$ if and only if $x = \mathbf{p}_{\alpha_i}(p) \in U_i$. Therefore one of the following two alternatives hold:

- $p \in U \setminus \tilde{U}$, and hence $y \in U_i$ but $x \notin U_i$
- $p \in \tilde{U} \setminus U$, and hence $x \in U_i$ but $y \notin U_i$.

In both cases the segment σ joining x and y must contain a point of ∂U_i and thus $\mathrm{dist}(x, \partial U_i) \leq |x-y|$. We therefore wish to estimate the latter.

To that end consider the triangle with vertices v, p, and x and let θ be the angle at v. We then have

$$|x-y| = |p-v|(1-\cos\theta) \leq C2^{-\ell(L)}(1-\cos\theta) \tag{8.45}$$

$$\sin\theta = \frac{|p-x|}{|p-v|} \leq C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2}\,. \tag{8.46}$$

Here, the upper bound on $|p-v|$ comes from (8.36) and the upper bound on the distance of U_i from V, while the lower bound on $|p-v|$ also comes from the lower bound on the distance of U_i to V, and the upper bound on $|p-x|$ comes from (8.36) (this is using that $p \in \mathrm{gr}(u_{L,i})$, but also holds more generally when $p \in \mathrm{spt}(T_{L,i})$). In particular as (8.46) tells us that θ is small (e.g. $\theta < \pi/3$ suffices), we get $\sin(\theta) \geq \theta/2$; as $1 - \cos(\theta) \leq \theta^2/2$ is always true, combining these with (8.45) and (8.46) we establish the desired bound on $|x-y|$, hence completing the proof of (8.42) and so also the proof of Lemma 8.16.

Having proved Lemma 8.16 we now come to the proof of (A). Consider $L, L' \in \mathcal{G}^c \cup \mathcal{G}^o$, with L a child of L'. Consider the set $U := \lambda R(L) \cap \lambda R(L')$; clearly U is invariant under rotations around V. Under the assumption that $\sum_i Q_{L',i} = Q$, we can use Lemma 8.16 to prove

$$\|T\|(U) \geq (Q - \tfrac{1}{2})\mathcal{H}^m(U_1)\,. \tag{8.47}$$

for $U_1 := U \cap \alpha_1$ as before. Indeed, introduce the set $\tilde{U}$ for this choice of U as in Lemma 8.16 and note that

$$\|T\|(\tilde{U}) = \sum_i \|T_{L',i}\|(\tilde{U}) \geq \sum_i Q_{L',i}\mathcal{H}^m(U_i) = Q\mathcal{H}^m(U_1)$$

(observe that the set $\tilde{U}$ is formed by cylindrical domains with cross-sections U_i, and in particular $(\mathbf{p}_{\alpha_i})_\sharp(T_{L',i} \llcorner \tilde{U}) = Q_{L',i}[\![U_i]\!]$.

We can now use Lemma 8.16 to estimate

$$\|T\|(U) \geq \|T\|(\tilde{U}) - \|T\|(\tilde{U} \setminus U) \geq Q\mathcal{H}^m(U_1) - C2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\mathcal{H}^{m-1}(\partial U_1)\,.$$

Note that in principle the constant C appearing in the estimate depends on the region U (which in turn depends on L and L'), however the latter is determined by the cross-section which, after rescaling by $2^{\ell(L)}$ and translating, is the intersection of two shapes ranging in a finite number of possibilities (the number of which depends on λ). In particular, by Remark 8.17, we can assume that the constant C depends only on the constants m, n, $\bar{n}$, Q, $\bar{\delta}$ and λ (note however that we have already fixed $\lambda = \lambda(m)$). On the other hand we also get $2^{-\ell(L)}\mathcal{H}^{m-1}(\partial U_1) \leq C\mathcal{H}^m(U_1)$, with a constant C depending on m and λ. We thus conclude

$$\|T\|(U) \geq (Q - C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2))\mathcal{H}^m(U_1) \geq (Q - C(\tau^2 + \varepsilon^2))\mathcal{H}^m(U_1)\,.$$

Since the constant C is independent of both τ and ε, an appropriate smallness condition on these two parameters guarantees the validity of (8.47).

If we now consider the current $T_{y_L, 2^{-\ell(L)}}$, the corresponding rescaled set $\Omega = 2^{\ell(L)}(U - y_L)$ satisfies requirement (b) of Lemma 8.8, which implies that $\sum_i Q_{L,i} = Q$, if τ is sufficiently small. While it is true that Lemma 8.8 imposes a smallness condition on τ which depends on the set Ω, it is easy to see that the latter varies among a fixed number of sets (since L is a child of L'), which depend on the relative position of L compared to L' because once λ is fixed, $\lambda R(L) \cap \lambda R(L')$ overlap for a fixed number of cubes L, L' always. In particular, there is a choice of smallness condition on τ which ensures the applicability of the lemma for every pair of cubes L and L' as in (A). Therefore to finish the proof, note that as (ii) holds for $\mathcal{G}^o$, it follows by induction from (A) that (ii) holds for every $L \in \mathcal{G}^c \cup \mathcal{G}^o$, as every cube L in this set is a descendent of a cube in $\mathcal{G}^o$, while their ancestors are all in $\mathcal{G}^c \cup \mathcal{G}^o$. So we are done with this part.

We finally prove (v). Fix a point $p \in R(L) \cap \mathrm{spt}(T)$ and note that it must belong to $\mathrm{spt}(T_{L,i})$ for some i. For $\rho = (\lambda - 1)\frac{2^{-\ell(L)-1}}{\sqrt{m-2}}$ we estimate

$$\|T_i\|(\mathbf{B}_\rho(p)) \le \|T_i\|(\mathbf{C}_\rho(p, \pi_i)) \le Q_{L,i}\rho^m + C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma} 2^{-m\ell(L)}$$
$$\le (Q_{L,i} + C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma})\rho^m .$$

Then if ε and τ are small enough, we conclude the claim from (8.30) and the monotonicity formula. □

8.6. **Coherent approximation on the outer region and first blow-up.** For technical reasons, it is useful to have a single multi-valued approximation defined over the union of all L_i for L varying among the elements of $\mathcal{G}^o$. Proposition 8.19 below gives a precise statement of this, but we first introduce some notation.

Definition 8.18. Let T, Σ, and $\mathbf{S}$ be as in Proposition 8.15. Fix $L \in \mathcal{G}^o$. We denote by $\mathscr{N}(L)$ the set of $L' \in \mathcal{G}^o$ such that $R(L) \cap R(L') \neq \emptyset$, and let

$$\bar{\mathbf{E}}(L) := \max\{\mathbf{E}(L') : L' \in \mathscr{N}(L)\} .$$

Proposition 8.19 (Coherent outer approximation)**.** *Let T and $\mathbf{S}$ be as in Proposition 8.15. For every $i \in \{1, \dots, N\}$ we define*

$$R_i^o := \bigcup_{L \in \mathcal{G}^o} L_i \equiv \alpha_i \cap \bigcup_{L \in \mathcal{G}^o} R(L)$$

and let $Q_i := Q_{L_0,i}$. Then, there are Lipschitz multi-valued maps $u_i : R_i^o \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ and closed subsets $\bar{K}_i(L) \subset L_i$ satisfying the following properties.

(i) $\mathrm{gr}\,(u_i) \subset \Sigma$ *and* $T_{L,i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(\bar{K}_i(L)) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(\bar{K}_i(L))$ *for every* $L \in \mathcal{G}^o$, *where* $T_{L,i}$ *are defined as in Proposition 8.15;*

(ii) *The following estimates hold*

$$2^{2\ell(L)}\|u_i\|^2_{L^\infty(L_i)} + 2^{m\ell(L)}\|Du_i\|^2_{L^2(L_i)} \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \tag{8.48}$$

$$\|Du_i\|_{L^\infty(L_i)} \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma \tag{8.49}$$

$$|L_i \setminus \bar{K}_i(L)| + \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus \bar{K}_i(L))) \le C 2^{-m\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma} , \tag{8.50}$$

where the constant C depends only upon $Q, m, n, \bar{n}$, and $\bar{\delta}$.

Proof. The ideas of the proof are borrowed from [10, Section 1.2.2] and [13, Section 6.2]. On the one hand there is a slight complication compared to the arguments borrowed from [10, Section 1.2.2] due to the fact that the regions $R(L)$ are not cubes; on other hand compared to the arguments borrowed from [13, Section 6.2] our situation is considerably simpler.

First observe that, since $L_i = R(L) \cap \alpha_i$, the set $\mathscr{N}(L)$ is equivalently described as those cubes $L' \in \mathcal{G}^o$ such that $L_i \cap L'_i \neq \emptyset$ for some i. In fact, given the invariance of $R(L)$ under rotations around V, if this is true for some i then it is true for all i, and so we can fix an arbitrary i. Notice that

(a) the cardinality of $\mathscr{N}(L)$ is bounded by a dimensional constant;
(b) $|\ell(L') - \ell(L)| \le 1$ for every $L' \in \mathscr{N}(L)$.

Indeed, (a) follows by the construction of the collection of cubes $\mathcal{G}$ and their associated sets $R(L)$, while (b) is a consequence of Lemma 8.11(i). For each cube $L \in \mathcal{G}^o$, consider the approximations $u_{L,i}$ and the coincidence sets $K_i(L)$ given by Proposition 8.15 and define

$$\bar{K}_i(L) := \bigcap_{L' \in \mathscr{N}(L)} K_i(L') .$$

Observe that for every $L' \in \mathscr{N}(L)$ we have $u_{L,i} = u_{L',i}$ on $\bar{K}_i(L)$. Note that

$$|L_i \setminus \bar{K}_i(L)| \le \sum_{L' \in \mathscr{N}(L)} |\Omega_i(L') \setminus K_i(L')|$$

and

$$
\begin{aligned}
&\|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i\setminus\bar{K}_i(L)))\\
&\qquad\le \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L))) + \sum_{L'\in\mathscr{N}(L)\setminus\{L\}} \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L')))\\
&\qquad\le \|T_{L,i}\|(\mathbf{p}_{\alpha_i}^{-1}(L_i \setminus K_i(L))) + \sum_{L'\in\mathscr{N}(L)\setminus\{L\}} \|T_{L',i}\|(\mathbf{\Omega}_i(L') \setminus \mathbf{p}_{\alpha_i}^{-1}(K_i(L'))),
\end{aligned}
$$

and thus (8.50) follows from Proposition 8.15(iv) and (a) and (b) above.

Note that we may define $u_i := u_{L,i}$ on $\bar{K}_i(L)$, which is clearly well-defined. To complete the proof we need to extend u_i from $\bigcup_{L\in\mathcal{G}^o} \bar{K}_i(L)$ to a Lipschitz map on $R_i^o := \bigcup_{L\in\mathcal{G}^o} L_i$ which satisfies $\mathrm{gr}\,(u_i) \subset \Sigma$ and the estimates (8.48) and (8.49). Recall that we have the estimates

$$
\|u_i\|_{L^\infty(\bar{K}_i(L))} \le C2^{-\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2} \tag{8.51}
$$
$$
\mathrm{Lip}(u_i|_{\bar{K}_i(L)}) \le C(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma\,, \tag{8.52}
$$

by Proposition 8.15(iv), just because on the set $\bar{K}_i(L)$ the map u_i coincides with $u_{L,i}$.

Observe moreover that, if we consider the larger domains $\tilde{K}_i(L) := \bigcup_{L'\in\mathscr{N}(L)} \bar{K}_i(L)$, we still have estimates analogous to (8.51), (8.52), with $\bar{\mathbf{E}}(L)$ replacing $\mathbf{E}(L)$, namely

$$
\|u_i\|_{L^\infty(\tilde{K}_i(L))} \le C2^{-\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2} \tag{8.53}
$$
$$
\mathrm{Lip}(u_i|_{\tilde{K}_i(L)}) \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma\,, \tag{8.54}
$$

Our aim is to show that we can find an extension u_i to $\bigcup_{L\in\mathcal{G}^o} L_i$ and use (8.53), (8.54) to show that for each $L \in \mathcal{G}^o$ this extension satisfies

$$
\|u_i\|_{L^\infty(L_i)} \le C2^{-\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1/2} \tag{8.55}
$$
$$
\mathrm{Lip}(u_i|_{L_i}) \le C(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma\,, \tag{8.56}
$$

The remaining claim, namely the L^2 bound on Du_i over L_i claimed in point (ii) of the proposition, is then an obvious consequence of (8.52), the bound on $|L_i \setminus \bar{K}_i(L)|$, and the fact that

$$
\|Du_i\|^2_{L^2}(\bar{K}_i(L)) \le C2^{-m\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\,,
$$

the latter being again a consequence of $u_i|_{\bar{K}_i(L)} = u_{L,i}$.

In order to accomplish the latter task we first observe that we can ignore the requirement that $\mathrm{gr}\,(u_i) \subset \Sigma$. Indeed, fix the $\bar{n}$-dimensional subspace $\pi = \alpha_i^\perp \cap T_0\Sigma$ which is the orthogonal complement of α_i in $T_0\Sigma$ and let $\Psi : \mathbf{B}_7 \cap T_0\Sigma \to T_0\Sigma^\perp$ be the map whose graph describes Σ. Recall that $\|D^2\Psi\|_{C^0} \le C\mathbf{A}$ and, since $D\Psi(0) = 0$, we conclude $\|D\Psi\|_{C^0} \le C\mathbf{A}$ as well. It thus suffices to find an extension of the π-component u_i^π of the map u_i with the desired estimate and compose it with Ψ in the remaining components of $\alpha_i^\perp$ to find the desired extension of u_i (the formula for the latter map would then be $x \mapsto (u_i^\pi(x), \Psi(x, u_i^\pi(x))) \in \pi \times T_0\Sigma^\perp$).

Once we are allowed to ignore the above issue, we consider a cellular decomposition of the L_i's into 0-cells (the 0-skeleton), 1-cells attached to the 0-cells (the 1-skeleton), 2-cells, and so on, with the final m-dimensional cells being the interiors of the L_i's. This can be done canonically across all L, note indeed that each L_i is the product, in $V \times (V^\perp \cap \alpha_i)$, of the cube $L \subset V$ with the annulus $\{y \in V^\perp \cap \alpha_i : 2^{-\ell(L)-1} \le |y| \le 2^{-\ell(L)}\}$.

We denote by $\mathcal{S}_i$ the collection of i-cells in the above. We also slightly fatten each 0-cell p to open neighborhoods U^p so that the separation between any U^p and U^q with $p, q \in L_i$ is at least $c2^{-\ell(L)}$ for some dimensional constant $c > 0$. Now, for every 1-cell σ with endpoints $p, q \in \mathcal{S}_0$, we slightly fatten $\sigma \setminus (U^p \cup U^q)$ to an open set U^σ and again we take care that the separation between two fattenings U^σ and U^τ for distinct 1-cells σ and τ contained in the same L_i is at least $c2^{-\ell(L)}$. We proceed in this way over all skeleta. Figure 3 gives a graphical illustration of this on some specific 0, 1, and 2-cells.

We then find a Lipschitz extension of the maps u_i to the points 0-skeleton in the following fashion:

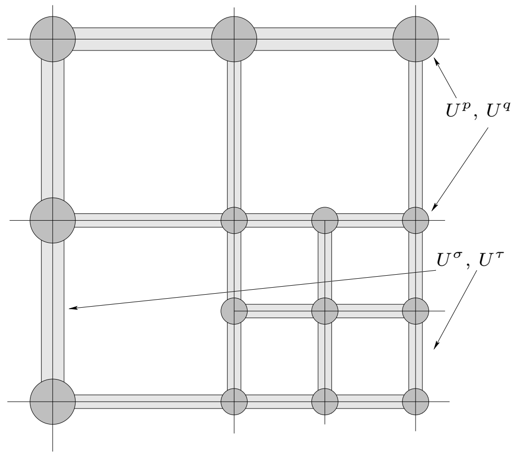

FIGURE 3. An illustration of the fattening of some 0-cells, 1-cells and 2-cells, used in the extension algorithm to find a coherent approximation.

For each point P in the 0-skeleton, we let $N(P)$ be the union of L'_i for those cubes $L' \in \mathcal{G}^o$ which contain P. The extension is then done using [10, Theorem 1.7], which ensures that, after setting $K := \bigcup_{L\in\mathcal{G}^o} \bar{K}_i(L)$

$$\mathrm{Lip}(u_i|_{(K\cap N(P))\cup U^p}) \le C\mathrm{Lip}(u_i|_{K\cap N(P)}) \tag{8.57}$$

$$\|u_i\|_{L^\infty((K\cap N(P))\cup U^p)} \le C\|u_i\|_{L^\infty(K\cap N(P))} \,. \tag{8.58}$$

To extend to the 1-cells fix any $\sigma \in \mathcal{S}_1$ contained in some L_i with endpoints p and q, and let $N(\sigma)$ be the union of all the L'_i which intersect σ. Observe that, because U^p and U^q are far enough apart from one another, using (8.57) and (8.58) we get the estimates

$$\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q)}) \le C\mathrm{Lip}(u_i|_{K\cap N(\sigma)}) + C2^{-\ell(L)}\|u_i\|_{L^\infty(K\cap N(\sigma))}$$
$$\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^p\cup U^q))} \le C\|u_i\|_{L^\infty(K\cap N(\sigma))} \,.$$

We then proceed and define u_i on each U^σ *separately* so that (again, using [10, Theorem 1.7])

$$\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q\cup U^\sigma)}) \le C\mathrm{Lip}(u_i|_{(K\cap N(\sigma))\cup(U^p\cup U^q)})$$
$$\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^p\cup U^q\cup U^\sigma))} \le C\|u_i\|_{L^\infty((K\cap N(\sigma))\cup(U^q\cup U^p))} \,.$$

We proceed inductively in this fashion and after going over all the skeleta we finally arrive at an extension which, defining $N(L_i)$ as the union of the L'_i over $L' \in \mathscr{N}(L)$, satisfies

$$\mathrm{Lip}(u|_{(K\cap N(L_i))\cup L_i}) \le C\mathrm{Lip}(u|_{K\cap N(L_i)}) + C2^{-\ell(L)}\|u_i\|_{L^\infty(K\cap N(L_i))} \,.$$
$$\|u_i\|_{L^\infty((K\cap N(L_i))\cup L_i)} \le C\|u_i\|_{L^\infty(K\cap N(L_i))} \,.$$

Given that $K \cap N(L_i) = \tilde{K}_i(L)$, we conclude the desired bounds from (8.53) and (8.54). □

8.7. **Blow-up.** We have thus constructed a collection of N multi-valued approximations on the outer region. From the estimates of the previous proposition and a suitable covering argument it is easy to see that, away from the spine V, their Dirichlet energies are controlled by the conical excess $\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)$. Our next main goal is to show that, after we normalize them by $(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4))^{1/2}$, they are close to a Dir-minimizing function. The control of the Dirichlet energy is still too crude for our final purpose, as it degenerates when we get closer to the spine: in order to gain a uniform control outside the spine we will need to use Simon's estimates; Section 11 will be devoted to this task. However, this "first blow-up" will be useful for two reasons: it is sufficient for the purpose of Section 9.1, where we "balance" the best approximating cone (recall the discussion at the beginning of Section 8), and it will ultimately be used in the final blow-up argument to prove Dir-minimality of the blow-up limit away from the spine. In the final blow-up argument, the better control on the Dirichlet energy will then be used to prove Dir-minimality across the spine, using the fact that $(m-2)$-dimensional subspaces have vanishing $W^{1,2}$-capacity.

Proposition 8.20 (First blow-up). *Let T and $\mathbf{S}$ be as in Proposition 8.15 and assume the parameters δ^*, τ, and $\bar\delta$ are fixed. Then, for every $\sigma, \varsigma > 0$ there are constants $C = C(m,n,Q,\delta^*,\tau,\bar\delta) > 0$ and $\varepsilon = \varepsilon(m,n,Q,\delta^*,\tau,\bar\delta,\sigma,\varsigma) > 0$ such that the following holds:*

(i) *$R \setminus B_\sigma(V)$ is contained in the outer region R^o (recall R is as in (8.27));*

(ii) *If u_i are the maps of Proposition 8.19 and $R_i := (R\setminus B_\sigma(V)) \cap \alpha_i$ then*

$$\int_{R_i} |Du_i|^2 \le C\sigma^{-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + C\mathbf{A}^2 \,. \tag{8.59}$$

(iii) *If additionally $\mathbf{A}^2 \le \varepsilon^2 \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)$ and we set $v_i := \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)^{-1/2} u_i$, then there is a map $w_i : R_i \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ which is Dir-minimizing and such that*

$$d_{W^{1,2}}(v_i, w_i) \le \varsigma \,, \tag{8.60}$$

where $d_{W^{1,2}}$ is the $W^{1,2}$ distance defined in [10].

Note that, crucially, while the parameter ε must be chosen suitably small depending on ς, the constant C in (8.59) is instead independent of it. As already mentioned, the control (8.59) is indeed sub-optimal and we will later be able to remove the factor σ^{-2}.

Remark 8.21. We wish to spent a few words on a procedure which will be used often in the rest of the paper. Assume we have a sequence of currents T_k, manifolds Σ_k, and cones $\mathbf{S}_k$ satisfying the assumptions of Proposition 8.20 for some fixed choice of the parameters δ^*, τ, and $\bar\delta$. We assume that the cones $\mathbf{S}_k$ are converging locally in the sense of Hausdorff distance to some limiting $\mathbf{S}_\infty$, and fix some sequence of m-dimensional planes $\alpha_i^k \subset \mathbf{S}_k$ converging to a plane $\alpha_i^\infty \subset \mathbf{S}_\infty$. Assume moreover that $\hat{\mathbf{E}}_k = \hat{\mathbf{E}}(T_k,\mathbf{S}_k,\mathbf{B}_1)$ and $\hat{\mathbf{E}}(T_k,\mathbf{S}_k,\mathbf{B}_1)^{-1}\mathbf{A}_k^2$ are both converging to zero. Let u_i^k be the coherent outer approximations over the domains $R_{i,k}^o$ which consist of the outer regions R_k^o for each T_k intersected with the planes α_i^k. We would like to use Proposition 8.20 to extract a Dir-minimizing limit of suitable normalizations of the maps u_i^k, namely $v_i^k := E_k^{-1/2} u_i^k$ for some choice of normalization constants E_k satisfying $\hat{\mathbf{E}}_k \le E_k$. One technical point is that the maps v_i^k are not defined on the same plane. In order to deal with issue apply a rotation and map α_i^k onto α_i^∞. In fact, even though this is not needed, it is convenient to choose the canonical rotations $R_k = R(\alpha_i^k, \alpha_i^\infty)$ of Lemma 7.4. We can then extract, up to extracting a subsequence, a limit v_i of $v_i^k \circ R_k^{-1}$; note that the latter maps are defined over (subdomains of) the same plane. Observe that the rotations R_k do depend on i and thus we do not have a single canonical rotation which works for every i. However, we can also see from Lemma 7.4 that, if we consider the graphs of the maps v_i^k as subsets of $\mathbb{R}^{m+n}$, the latter are indeed converging to the graph of v_i. Our limiting object is, in that sense, canonical.

In the sequel, when we are referring to "the blow-up v_i of the maps v_i^k" we will assume that we have followed the above algorithm.

Proof. First of all we let σ be given and fixed. We then select ℓ so that $\frac{\sigma}{2} \le 2^{-\ell} \le \sigma$ and define

$$\mathcal{G}_{\le \ell} := \bigcup_{j=0}^{\ell} \mathcal{G}_j \,.$$

We can then appeal to Lemma 8.14(i) to get that, if ε is small enough, then $\mathcal{G}_{\ell+1} \subset \mathcal{G}^o$. But by definition of outer cubes the latter implies in fact $\mathcal{G}_{\le \ell+1} \subset \mathcal{G}^o$, which implies the first claim of the proposition.

Let $L \in \mathcal{G}_{\le \ell}$, and observe that (b) in the proof of Proposition 8.19 tells us that for any $L' \in \mathscr{N}(L)$, we have $L' \in \mathcal{G}_{\le \ell+1}$. Appealing to Proposition 8.19(ii) (namely, (8.48)) we have

$$\begin{aligned}\int_{L_i} |Du_i|^2 &\le C 2^{-m\ell(L)}(\bar{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\\ &\le C 2^{-(m+2)\ell(L)}\mathbf{A}^2 + C\sum_{L'\in\mathscr{N}(L)} 2^{-m\ell(L')}\mathbf{E}(L',0)\,,\end{aligned}$$

where $\mathscr{N}(L)$ and $\bar{\mathbf{E}}(L)$ are as in Definition 8.18. Since the cardinality of $\mathscr{N}(L)$ is bounded by a geometric constant $C = C(m,n)$, we immediately conclude

$$\begin{aligned}\int_{R_i} |Du_i|^2 &\le C \sum_{L\in\mathcal{G}_{\le \ell+1}} 2^{-m\ell(L)}(\mathbf{E}(L,0) + 2^{-2\ell(L)}\mathbf{A}^2)\\ &= C \sum_{L\in\mathcal{G}_{\le \ell+1}} \left(2^{2\ell(L)} \int_{\mathbf{B}^h(L)} \mathrm{dist}^2(q,\mathbf{S})\, d\|T\|(q) + 2^{-(m+2)\ell(L)}\mathbf{A}^2\right)\\ &\le C\sigma^{-2} \sum_{L\in\mathcal{G}^o} \int_{\mathbf{B}^h(L)} \mathrm{dist}^2(q,\mathbf{S})\, d\|T\|(q) + C\mathbf{A}^2\,,\end{aligned}$$

where we have used (8.28) (with $\kappa = 4$). We next observe that $\mathbf{B}^h(L) \subset \mathbf{B}_4$ for every $L \in \mathcal{G}$ and that, by Lemma 8.11(iv), for given any point $q \in \mathbf{B}_4$ the cardinality of elements $L \in \mathcal{G}$ for which $q \in \mathbf{B}^h(L)$ is bounded by a constant $C = C(m,n)$. We thus conclude (8.59).

All that remains to be proven is (iii). By the definition of $T_{L,i}$ as the restriction of T on a suitable open set, for each i there is an integral current T_i with the following properties:

- $\partial T_i = 0$ on $\mathbf{p}_{\alpha_i}^{-1}\left(\bigcup_{L\in\mathcal{G}^o} L_i\right)$;
- T_i is area-minimizing;
- $T_i \llcorner \mathbf{p}_{\alpha_i}^{-1}(L_i) = T_{L,i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(L_i)$.

In particular, if we define $K_i := \bigcup_{L\in\mathcal{G}^o} \bar{K}_i(L)$, the argument leading to (8.59) and the estimates in Proposition 8.19 lead to the following:

$$T_i \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i \cap R_i) = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_i}^{-1}(K_i \cap R_i) \tag{8.61}$$
$$\|u_i\|^2_{L^\infty(R_i)} \le C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2) \tag{8.62}$$
$$\|Du_i\|_{L^\infty} \le C(\sigma^{-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2)^\gamma \tag{8.63}$$
$$|R_i \setminus K_i| + \|T_i\|(\mathbf{p}_{\alpha_i}^{-1}(R_i\setminus K_i)) \le C(\sigma^{-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2)^{1+\gamma}\,. \tag{8.64}$$

We are thus in the same position to apply the arguments of [11] leading to [11, Theorem 2.6] in order to conclude point (iii) of Proposition 8.20. The reader will notice that the only obstruction to applying [11, Theorem 2.6] is that the domain of the map u_i given above is not a ball. However, the arguments only use the regularity of the boundary of the domain, and since the boundary of R_i is Lipschitz, those arguments apply here as well. □

9. Cone balancing

In this section we introduce a suitable procedure which allows us to pass from a possibly unbalanced cone to a balanced cone whilst only changing the excess by a constant. This procedure is done under the assumption that the two-sided L^2 height excess of T relative to a cone $\mathbf{S} \in \mathscr{C}(Q)$ is significantly smaller than the planar L^2 height excess of T. To make our exposition cleaner, we recall the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$
$$\boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \mathrm{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1)$$

where i,j in the minimum and maximum range over the indices of the planes in $\mathbf{S}$.

Proposition 9.1 (Cone balancing). *Assume that T and Σ are as in Assumption 2.1, $\mathbf{B}_1 = \mathbf{B}_1(0) \subset \Omega$, $\mathbf{S} \in \mathscr{C}(Q)$, and $\alpha_1,\dots,\alpha_N$ are as in Definition 2.3. Then, there are constants $C = C(Q,m,n,\bar{n}) > 0$ and $\varepsilon_0 = \varepsilon_0(Q,m,n,\bar{n}) > 0$ with the following property. Assume that*

$$\mathbf{A}^2 \le \varepsilon_0^2 \mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \le \varepsilon_0^4 \mathbf{E}^p(T,\mathbf{B}_1)\,. \tag{9.1}$$

Then there is a subset $\{i_1,\dots,i_k\} \subset \{1,\dots,N\}$ with $k \ge 2$ such that, upon setting $\mathbf{S}' = \alpha_{i_1} \cup \dots \cup \alpha_{i_k}$, the following holds:

(a) *$\mathbf{S}'$ is C-balanced;*
(b) *$\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1) \le C\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$;*
(c) *$\mathrm{dist}^2(\mathbf{S}\cap\mathbf{B}_1, \mathbf{S}'\cap\mathbf{B}_1) \le C\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$;*

(d) $C^{-1}\mathbf{E}^p(T,\mathbf{B}_1) \le \boldsymbol{\mu}(\mathbf{S})^2 = \boldsymbol{\mu}(\mathbf{S}')^2 \le C\mathbf{E}^p(T,\mathbf{B}_1)$.

Using the Pruning Lemma (Lemma 8.2), the proof of Proposition 9.1 can be reduced to showing the following proposition, which roughly says that if the two-sided L^2 height excess of T relative to $\mathbf{S}$ is significantly smaller than the minimal angle in the cone $\mathbf{S}$, then in fact $\mathbf{S}$ is already C-balanced for some constant C. Indeed, intuitively, since T is area-minimizing, if it is very close to $\mathbf{S}$ then we would expect the union of planes in $\mathbf{S}$ to roughly behave like an area-minimizer, and from Morgan's result (Lemma 7.5) we would expect $\mathbf{S}$ to be balanced.

Proposition 9.2. *Assume that T, Σ, and $\mathbf{S}$ are as in Proposition 9.1. Then there are constants $C = C(m,n,\bar{n},N)$ and $\varepsilon = \varepsilon(m,n,\bar{n},N)$ with the following property. If we additionally have that*

$$N \ge 2 \quad \text{and} \quad \mathbf{A}^2 + \mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \le \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2\,, \tag{9.2}$$

then $\mathbf{S}$ is C-balanced.

Let us first show that Proposition 9.2 implies Proposition 9.1.

Proof of Proposition 9.1. Suppose that Proposition 9.2 holds, and let ε_* be the minimum over $N \le Q$ of all the constants $\varepsilon = \varepsilon(m,n,\bar{n},N)$ from Proposition 9.2. Then fix $\varepsilon_0 \le \varepsilon_*$ to be determined later.

The hypotheses of Proposition 9.1 in particular give that $\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \le \varepsilon_0^2 \mathbf{E}^p(T,\mathbf{B}_1)$. Next we estimate

$$\begin{aligned}\mathbf{E}^p(T,\mathbf{B}_1) &\le \int_{\mathbf{B}_1} \operatorname{dist}^2(x,\alpha_1)\, d\|T\|(x) \le C\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + C\max_{i<j}\operatorname{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)\\ &\le C\varepsilon_0^2\mathbf{E}^p(T,\mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2\,,\end{aligned}$$

where $C = C(m,n)$. In particular for ε_0 smaller than a geometric constant we get

$$\mathbf{E}^p(T,\mathbf{B}_1) \le C\boldsymbol{\mu}(\mathbf{S})^2\,, \tag{9.3}$$

where $C = C(m,n)$. Thus we also conclude

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \le \underbrace{C\varepsilon_0^2}_{=:\eta}\boldsymbol{\mu}(\mathbf{S})^2\,.$$

Let us now fix $\delta > 0$ (which will be determined later), and let Γ be as in Lemma 8.2 for this choice of δ and N. If we take $\eta^{1/2} = (1+\Gamma)^{-1}\delta$ in the above, we see that if $\varepsilon_0 = \varepsilon_0(m,n,N,\delta) > 0$ is sufficiently small, and $D := \mathbb{E}(T,\mathbf{S},\mathbf{B}_1)^{1/2}$, then

$$D \le (1+\Gamma)^{-1}\delta\boldsymbol{\mu}(\mathbf{S})\,.$$

Thus we are in the situation to apply Lemma 8.2 with this choice of D. This yields a subset $I = \{i_1,\ldots,i_k\} \subset \{1,\ldots,N\}$, with $k \ge 2$, such that the corresponding planes $\{\alpha_{i_1},\ldots,\alpha_{i_k}\}$ satisfy (8.1), (8.2), (8.3) of the Pruning Lemma. Now set $\mathbf{S}' := \alpha_{i_1}\cup\cdots\cup\alpha_{i_k}$; we claim that Proposition 9.1 holds with this $\mathbf{S}'$. We start by observing that, since $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\mu}(\mathbf{S}')$ (by (8.3)) and (9.3) holds, we just need

$$\boldsymbol{\mu}(\mathbf{S}')^2 \le C\mathbf{E}^p(T,\mathbf{B}_1) \tag{9.4}$$

to complete the proof of (d). However, let us first prove (a)–(c).

Observe that conclusion (8.1) of the Pruning Lemma gives

$$\max_{j=1,\ldots,N}\min_{i\in I}\operatorname{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1) \le \Gamma^2\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

In particular, this will give condition (c) once we have chosen δ appropriately. Furthermore, observe the following consequence of this: for $q \in \operatorname{spt}(T)\cap\mathbf{B}_1$, suppose $\operatorname{dist}(q,\mathbf{S}') = \operatorname{dist}(q,\alpha_{i_j})$ for some $i_j \in I$. If $\operatorname{dist}(q,\mathbf{S}) = \operatorname{dist}(q,\alpha_{i_j})$, then we clearly have $\operatorname{dist}(q,\mathbf{S}') = \operatorname{dist}(q,\mathbf{S})$. Otherwise, we must have $\operatorname{dist}(q,\mathbf{S}) = \operatorname{dist}(q,\alpha_\ell)$ for some $\ell \notin I$. By the above consequence of the Pruning Lemma however, there is some $i_{j_*} \in I$ for which

$$\operatorname{dist}^2(\alpha_\ell\cap\mathbf{B}_1,\alpha_{i_{j_*}}\cap\mathbf{B}_1) \le \Gamma^2\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

In particular, we must have

$$\operatorname{dist}^2(q,\mathbf{S}') \le \operatorname{dist}^2(q,\alpha_{i_{j_*}}\cap\mathbf{B}_1) \le 4\operatorname{dist}^2(q,\alpha_\ell) + 4\operatorname{dist}^2(\alpha_\ell\cap\mathbf{B}_1,\alpha_{i_{j_*}}\cap\mathbf{B}_1)$$

$$\leq 4\,\mathrm{dist}^2(q,\mathbf{S}) + 4\Gamma^2\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

In either case, we see that for any $q\in \mathrm{spt}(T)\cap\mathbf{B}_1$,

$$\mathrm{dist}^2(q,\mathbf{S}') \leq 4\,\mathrm{dist}^2(q,\mathbf{S}) + 4\Gamma^2\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$$

which evidently gives

$$\hat{\mathbf{E}}(T,\mathbf{S}',\mathbf{B}_1) \leq 4(1+\Gamma^2(Q+1)\omega_m)\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

This deals with controlling one term of the two-sided height excess $\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1)$. However, controlling the other term is simple as $\mathbf{S}'\subset\mathbf{S}$, and so $\hat{\mathbf{E}}(\mathbf{S}',T,\mathbf{B}_1)\leq\hat{\mathbf{E}}(\mathbf{S},T,\mathbf{B}_1)$. Combining we therefore get $\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1)\leq C\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$ where $C=C(Q,m,n,\delta)$; this proves conclusion (b) provided we choose $\delta=\delta(Q,m,n,\bar{n})>0$ in the end.

To show that conclusion (a) holds we will apply Proposition 9.2 to $\mathbf{S}'$. For this we must verify that the hypothesis (9.2) holds in this situation. Note that from (8.2) of the Pruning Lemma we have

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) + \max_{j=1,\dots,N}\min_{i\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1) \leq 2\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1).$$

In particular, if we perform exactly the same bounds as above when we proved (b), except replacing the estimate from (8.1) by the above, we would end up with

$$\begin{aligned}\hat{\mathbf{E}}(T,\mathbf{S}',\mathbf{B}_1) &\leq 4\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + 8(Q+1)\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)\\ &\leq (8Q+12)\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)\end{aligned}$$

where in the second inequality we have used the fact $\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\leq 2\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)$ again from the statement of (8.2) above. But again, since for the other half of the excess we have

$$\hat{\mathbf{E}}(\mathbf{S}',T,\mathbf{B}_1)\leq\hat{\mathbf{E}}(\mathbf{S},T,\mathbf{B}_1)\leq\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\leq 2\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)$$

we see that

$$\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1)\leq(8Q+14)\delta^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1).$$

Hence, if we choose $\delta=\delta(Q,m,n,\bar{n})>0$ obeying $(8Q+14)\delta^2<\epsilon_*/2$, we get that one part of the inequality in (9.2) holds for T and $\mathbf{S}'$. However, the other part of the inequality in (9.2) evidently follows, since by assumption we have

$$\mathbf{A}^2\leq\varepsilon_0^2\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\leq 2\delta^2\varepsilon_0^2\min_{i<j\in I}\mathrm{dist}^2(\alpha_i\cap\mathbf{B}_1,\alpha_j\cap\mathbf{B}_1)$$

and thus we have that (9.2) holds for suitably chosen $\delta=\delta(Q,m,n,\bar{n})>0$. Hence, with this choice of δ we can apply Proposition 9.2 to see that $\mathbf{S}'$ is C-balanced for some $C=C(Q,m,n,\bar{n})>0$, which completes the proof of (a).

It remains to prove (d), namely, as already observed, (9.4). Fix any plane π and observe that there is one plane α_{i_1} from $\mathbf{S}'$ with $i_1\in I$ (which without loss of generality by relabeling we can assume to be $i_1=1$) such that

$$\mathrm{dist}(\alpha_1\cap\mathbf{B}_1,\pi\cap\mathbf{B}_1)\geq\frac{1}{2}\boldsymbol{\mu}(\mathbf{S}')\,.$$

Indeed, if not we would get a contradiction to the definition of $\boldsymbol{\mu}(\mathbf{S}')$ by the triangle inequality.

We next show that there is an element $p\in\alpha_1$, a radius $r(m,n)>0$, and constant $C(m,n)>0$ with the property that $\mathbf{B}_r(p)\subset\mathbf{B}_{3/4}\setminus B_{1/4}(V)$ and

$$\mathrm{dist}(q,\pi)\geq C^{-1}\boldsymbol{\mu}(\mathbf{S}')\qquad\forall q\in B_r(p,\alpha_1)\,. \tag{9.5}$$

Recall first that $\mathrm{dist}(p,\pi)=|\mathbf{p}_\pi^\perp(p)|$ and that, by linearity of the map $\mathbf{p}_\pi^\perp$ we have $|\mathbf{p}_\pi^\perp(p)|=|p|\cdot|\mathbf{p}_\pi^\perp(p/|p|)|=|p|\,\mathrm{dist}(p/|p|,\pi)$, and so

$$|\mathbf{p}_\pi^\perp(p)|\leq\mathrm{dist}(\alpha_1\cap\mathbf{B}_1,\pi\cap\mathbf{B}_1)|p|\qquad\forall p\in\alpha_1\,. \tag{9.6}$$

Next choose any $v\in V^\perp\cap\alpha_1$ with $|v|=\frac{1}{2}$. Consider then the disk $B_{1/8}(v,\alpha_1)$ and inside this disk select a base $e_1,\dots e_m$ of α_1 with the property that any element $x\in\alpha_1\cap\overline{\mathbf{B}}_1$ can be written as a linear combination $\sum_i \lambda_i e_i$ with $|\lambda_i| \leq C = C(m)$. It follows that, for some element e_i we must necessarily have

$$|\mathbf{p}_\pi^\perp(e_i)| \geq \frac{1}{mC}\,\mathrm{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1)\,.$$

Indeed by Corollary 7.3 there is a vector $e \in \overline{\mathbf{B}}_1 \cap \alpha$ with $|\mathbf{p}_\pi^\perp(e)| = \mathrm{dist}(\alpha \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1)$. We can thus use the property above to write $e = \sum_i \lambda_i e_i$ and estimate

$$\mathrm{dist}(\alpha \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) = |\mathbf{p}_\pi^\perp(e)| \leq \sum_i |\lambda_i||\mathbf{p}_\pi^\perp(e_i)| \leq C \sum_i |\mathbf{p}_\pi^\perp(e_i)|\,.$$

Set then $p = e_i$ and choose the radius r to equal $\min\{\frac{1}{2mC}, \frac{1}{8}\}$. We can then use the last inequality and (9.6) to show that for all $q \in B_r(p, \alpha_1)$,

$$\mathrm{dist}(q,\pi) = |\mathbf{p}_\pi^\perp(q)| \geq |\mathbf{p}_\pi^\perp(p)| - |\mathbf{p}_\pi^\perp(q-p)| \geq \frac{1}{2mC}\,\mathrm{dist}(\alpha_1 \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) \geq \frac{1}{4mC}\boldsymbol{\mu}(\mathbf{S})\,.$$

This establishes (9.5).

In particular, if the parameter ε_0 is small enough, we can apply Lemma 8.5, Proposition 8.6, and Lemma 8.8 to $T_{0,4}$ and $\mathbf{S}'$ (note that we have already established that $\mathbf{S}'$ is C-balanced by the above and that the hypotheses of these results do hold here). In particular, let ϱ be the parameter in Lemma 8.5. If we consider the current

$$T' := T \llcorner \mathbf{B}_r(p) \cap \{\mathrm{dist}(\cdot, \alpha_1) \leq \varrho \boldsymbol{\sigma}(\mathbf{S}')\}$$

we will have that $\partial T' = 0$ in $\mathbf{B}_r(p)$ and that $\|T'\|(\mathbf{B}_r(p)) \geq C^{-1} r^m$ for some constant $C(m,n)$. Moreover, from (8.15) in Proposition (8.6), we have

$$\mathrm{dist}(q,\alpha_1) \leq C\hat{\mathbf{E}}(T,\mathbf{S}',\mathbf{B}_1)^{1/2} + C\mathbf{A} \qquad \forall q \in \mathbf{B}_r(p) \cap \mathrm{spt}(T')$$

for some constant $C = C(Q,m,n,\bar{n})$. Since $\hat{\mathbf{E}}^2(T,\mathbf{S}',\mathbf{B}_1) + \mathbf{A}^2 \leq C\varepsilon_0^2 \boldsymbol{\mu}(\mathbf{S}')^2$, if ε_0 is chosen sufficiently small we then easily conclude from the above and (9.5) that

$$\mathrm{dist}(q,\pi) \geq C^{-1}\boldsymbol{\mu}(\mathbf{S}') \qquad \forall q \in \mathbf{B}_r(p) \cap \mathrm{spt}(T')\,.$$

Squaring and integrating the latter inequality with respect to $d\|T'\|$, and using the lower bound $\|T'\|(\mathbf{B}_r(p)) \geq C^{-1} r^m$ we reach

$$\boldsymbol{\mu}(\mathbf{S}')^2 \leq C\hat{\mathbf{E}}(T,\pi,\mathbf{B}_1)\,.$$

However, since π is an arbitrary m-dimensional plane, this completes the proof of (9.4). □

We now come to the proof of Proposition 9.2. We first prove a key special case of it when $\boldsymbol{\mu}(\mathbf{S})$ and $\boldsymbol{\sigma}(\mathbf{S})$ are comparable.

Proposition 9.3. *Assume that T, Σ, and $\mathbf{S}$ are as in Proposition 9.1. Fix $\eta > 0$. Then, there exist constants $C = C(m,n,\bar{n},N,\eta)$ and $\varepsilon = \varepsilon(m,n,\bar{n},N,\eta) > 0$ with the following property. If (9.2) and the assumptions of Proposition 9.1 hold with this choice of ε, and moreover if*

$$\eta\boldsymbol{\mu}(\mathbf{S}) \leq \boldsymbol{\sigma}(\mathbf{S})\,, \tag{9.7}$$

then $\mathbf{S}$ is C-balanced.

Notice in particular that this implies the $N = 2$ case of Proposition 9.2, as in that situation we have $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$.

Proof. We argue by contradiction. If the proposition were false, then we could find sequences $T_k, \Sigma_k, \mathbf{A}_k$ and $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_N^k$ as in Proposition 9.2 such that

$$\mathbf{A}_k^2 + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2$$

and for which (9.7) holds for each k, but (after relabelling the planes) the Morgan angles $\theta_1(\alpha_1^k, \alpha_2^k)$ and $\theta_2(\alpha_1^k, \alpha_2^k)$ obey

$$\frac{\theta_1(\alpha_1^k,\alpha_2^k)}{\theta_2(\alpha_1^k,\alpha_2^k)} \to 0. \tag{9.8}$$

We have two possibilities: either (a) $\limsup_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k) > 0$ or (b) $\limsup_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k) = 0$. In the case of (a) the situation is simple: we simply pass to a subsequence (which we do not relabel) for which $\lim_{k\to\infty}\boldsymbol{\sigma}(\mathbf{S}_k)>0$, $T_k\to T_\infty$ as currents, where T_∞ is some m-dimensional area-minimizing integral current in $\mathbf{B}_1$, and $\mathbf{S}_k\to\mathbf{S}_\infty=\alpha_1^\infty\cup\cdots\cup\alpha_N^\infty$ for a collection of N *distinct* planes α_i^∞, locally in Hausdorff distance, with $\alpha_i^k\to\alpha_i^\infty$ for each $i=1,\dots,N$. By the assumption that $\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)\to 0$, we must have $\operatorname{spt}(T_\infty)\cap\mathbf{B}_1=\mathbf{S}_\infty\cap\mathbf{B}_1$. Indeed, we first see that $\operatorname{spt}(T_\infty)\cap\mathbf{B}_1\subset\mathbf{S}_\infty\cap\mathbf{B}_1$ and then the constancy theorem implies $T_\infty\llcorner\mathbf{B}_1=\sum_i k_i[\![\alpha_i^\infty]\!]$ for some $k_i\in\mathbb{Z}$. The only obstruction to the equality $\operatorname{spt}(T_\infty)\cap\mathbf{B}_1=\mathbf{S}_\infty\cap\mathbf{B}_1$ is then the vanishing of some of the coefficients k_i, which would come from orientation cancellation in the limit of the T_k; however, this would contradict the T_k being area-minimizing.

But then by Lemma 7.5, we would necessarily have $\theta_1(\alpha_i^\infty,\alpha_j^\infty)=\theta_2(\alpha_i^\infty,\alpha_j^\infty)>0$ for each $i<j\in\{1,\dots,N\}$, where the fact that these angles are non-zero follows from the assumption that $\liminf_k\boldsymbol{\sigma}(\mathbf{S}_k)>0$. But we clearly have $\theta_1(\alpha_1^k,\alpha_2^k)\to\theta_1(\alpha_1^\infty,\alpha_2^\infty)$ and $\theta_2(\alpha_1^k,\alpha_2^k)\to\theta_2(\alpha_1^\infty,\alpha_2^\infty)$ from the local Hausdorff distance convergence in $\mathbf{B}_1$, and hence we would have $\theta_1(\alpha_1^k,\alpha_2^k)/\theta_2(\alpha_1^k,\alpha_2^k)\to 1$, contradicting (9.8).

Now let us handle the case (b). Here, we can pass to a further subsequence to ensure that $\alpha_i^k\to\alpha_\infty$ for all $i=1,\dots,N$, locally in Hausdorff distance, for some m-dimensional plane α_∞. Since $\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)\to 0$ we therefore also have that $\hat{\mathbf{E}}(T_k,\alpha_\infty,\mathbf{B}_1)\to 0$. Note that we may assume without loss of generality that $\alpha_1^k=\alpha_\infty$ for all k: indeed, we can choose a sequence of rotations $q_k:\mathbb{R}^{m+n}\to\mathbb{R}^{m+n}$ with $q_k\to\mathrm{id}_{\mathbb{R}^{m+n}}$ and obeying $q_k(\alpha_1^k)=\alpha_\infty$, and then consider $q_k(\mathbf{S}_k)$ and $(q_k)_\sharp T_k$; note that $\boldsymbol{\sigma}(\mathbf{S}_k)=\boldsymbol{\sigma}(q_k(\mathbf{S}_k))$ (see the discussion in Section 7.1). Hence, for k sufficiently large, we can consider the (strong) Lipschitz approximations $f_k:B_{3/4}(0,\alpha_\infty)\to\mathcal{A}_Q(\alpha_\infty^\perp)$ of Almgren (see [11, Theorem 2.4]) for the T_k relative to α_∞ (here, the assumption $\hat{\mathbf{E}}(T_k,\alpha_\infty,\mathbf{B}_1)\to 0$ is sufficient in light of Allard's tilt-excess inequality, see for example [1, Proposition 4.1]). Now set

$$\bar{f}_k:=\frac{f_k}{\boldsymbol{\sigma}(\mathbf{S}_k)}.$$

Note that because $\alpha_1^k=\alpha_\infty$ for all k, we have

$$\hat{\mathbf{E}}(T_k,\alpha_\infty,\mathbf{B}_1)\le 4\hat{\mathbf{E}}(T_k,\mathbf{S}_k,\mathbf{B}_1)+C\boldsymbol{\mu}(\mathbf{S}_k)^2\le(4\varepsilon_k^2+C\eta^{-1})\boldsymbol{\sigma}(\mathbf{S}_k)^2$$

and so, again using Allard's tilt-excess inequality, the estimates from Almgren's strong Lipschitz approximation (see again [11, Theorem 2.4]) give that $\bar{f}_k\to\bar{f}_\infty$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{B}_{3/4})\cap L^2(\mathbf{B}_{3/4})$ for some Dir-minimizer $\bar{f}_\infty$. Note also that, if for each i, we denote by L_i^k the linear maps parameterizing the planes α_i^k over α_∞ (in particular, $L_1^k=0$ by construction, although this will not play any role in the proof), then we necessarily have that $|L_i^k|\le C\boldsymbol{\mu}(\mathbf{S}_k)\le C\eta^{-1}\boldsymbol{\sigma}(\mathbf{S}_k)$ for some dimensional constant C, and so if we write

$$\bar{L}_i^k:=\frac{L_i^k}{\boldsymbol{\sigma}(\mathbf{S}_k)}\,,$$

then we can pass to a further subsequence to ensure also that $\bar{L}_i^k\to\bar{L}_i$ for some linear map $\bar{L}_i$ over α_∞. Clearly, for each $i\neq j$ we have $|\bar{L}_i-\bar{L}_j|\ge c>0$ by definition of $\boldsymbol{\sigma}(\mathbf{S}_k)$, and so $\bar{L}_i\neq\bar{L}_j$ whenever $i\neq j$. Noting that

$$\int\operatorname{dist}^2(x,\mathbf{S}_k)\,d\|\mathbf{G}_{f_k}\|(x)\le C\hat{\mathbf{E}}(T_k,\mathbf{S}_k,\mathbf{B}_1)+C\left(\mathbf{A}_k^2+\hat{\mathbf{E}}(T_k,\alpha_\infty,\mathbf{B}_1)\right)^{1+\gamma}=o(\boldsymbol{\sigma}(\mathbf{S}_k)^2)$$

where $o(\boldsymbol{\sigma}(\mathbf{S}_k)^2)$ denotes a term which converges to zero as $k\to\infty$ when divided by $\boldsymbol{\sigma}(\mathbf{S}_k)^2$. Dividing both sides of the above inequality by $\boldsymbol{\sigma}(\mathbf{S}_k)^2$ and taking $k\to\infty$, we see that necessarily the graph of $\bar{f}_\infty$ is supported in the union of the graphs of $\bar{L}_i$, $i=1,\dots,N$, and so in particular we have

$$\bar{f}_\infty=\sum_i Q_i[\![\bar{L}_i]\!]$$

for some non-negative integers Q_i such that $\sum_i Q_i=Q$. We claim that necessarily both Q_1,Q_2 (and in fact all Q_i, but we won't need this to see the contradiction) are strictly positive. Indeed, once we have this we can complete the proof of the claim in the following manner: for every constant $\lambda>0$ the map $\lambda\bar{f}_\infty$ is Dir-minimizing, and so if $Q_1,Q_2>0$ then, by Proposition 7.6, the planes $\bar{\alpha}_i$ that are the graphs of $\lambda\bar{L}_i$ are c_2-balanced for all λ sufficiently small, where c_2 is an absolute constant (note that necessarily the intersection of the $\bar{\alpha}_i$ is a $(m-2)$-dimensional subspace, as the intersection of the α_i^k is an $(m-2)$-dimensional subspace for all k, and so up to passing to a subsequence this subspace will converge to an $(m-2)$-dimensional subspace which is necessarily the intersection of the planes $\bar{\alpha}_i$). In particular, we have

$$\frac{\theta_1(\bar{\alpha}_1, \bar{\alpha}_2)}{\theta_2(\bar{\alpha}_1, \bar{\alpha}_2)} \geq c_2 > 0.$$

Fix such a constant λ for which the above is true. Now, let $\bar{\alpha}_i^k$ be the planes which are the graphs of $\lambda \bar{L}_i^k$, $i = 1, 2$. Shrinking λ if necessary (which can be done independently of k, as $|\bar{L}_i^k| \leq C$ for all k and some geometric constant C), we can infer from Corollary 7.9 that we have

$$\frac{\theta_1(\bar{\alpha}_1^k, \bar{\alpha}_2^k)}{\theta_2(\bar{\alpha}_1^k, \bar{\alpha}_2^k)} \leq 16^2 \frac{\theta_1(\alpha_1^k, \alpha_2^k)}{\theta_2(\alpha_1^k, \alpha_2^k)}.$$

But then our assumption (9.8) implies from this that

$$\frac{\theta_1(\bar{\alpha}_1^k, \bar{\alpha}_2^k)}{\theta_2(\bar{\alpha}_1^k, \bar{\alpha}_2^k)} \to 0\,.$$

Noting that

$$\frac{\theta_1(\bar{\alpha}_1^k, \bar{\alpha}_2^k)}{\theta_2(\bar{\alpha}_1^k, \bar{\alpha}_2^k)} \to \frac{\theta_1(\bar{\alpha}_1, \bar{\alpha}_2)}{\theta_2(\bar{\alpha}_1, \bar{\alpha}_2)} \geq c_2 > 0$$

this gives the desired contradiction.

Therefore, all that remains to show is that $Q_1, Q_2 > 0$. In order to prove this, we use Lemma 7.10 to find points $\xi_k \in \alpha_1^k \cap \partial \mathbf{B}_{1/2}$ and a positive constant $c_0 = c_0(m, N)$ with the property that

$$\min_{j>1} \inf\{\operatorname{dist}(\zeta, \alpha_j^k) : \zeta \in \mathbf{B}_{2c_0}(\xi_k) \cap \alpha_1^k\} \geq 2c_0 \min_{j>1} \operatorname{dist}(\alpha_1^k \cap \mathbf{B}_1, \alpha_j^k \cap \mathbf{B}_1)\,. \tag{9.9}$$

In particular, for $\delta = \delta(Q, m, n, \bar{n}, N)$ as in the Splitting Corollary (Corollary 3.3) and for $r = r(\delta, \eta)$ sufficiently small, the assumptions of Corollary 3.3 are satisfied in $\mathbf{C}_{4rc_0}(\xi_k)$.

Indeed, conditions (i), (ii), (iii) are clear. (iv) follows from (9.9), as it gives

$$\min_{j>1} \inf\{\operatorname{dist}(\zeta, \alpha_j^k) : \zeta \in \mathbf{B}_{2c_0}(\xi_k) \cap \alpha_1^k\} \geq 2c_0 \boldsymbol{\sigma}(\mathbf{S}_k) \geq 2c_0 \eta \boldsymbol{\mu}(\mathbf{S}_k)$$

which in turn gives (iv) if we chose $r = 2\delta c_0 \eta$ with a possibly smaller constant c_0. Note indeed that in the left hand side of (iv) we are using the distance between oriented planes, however it is just a matter of choosing the right orientation for the planes to be able to conclude that left hand side of (iv) is bounded above by $C\boldsymbol{\mu}(\mathbf{S})$. Finally, condition (v) follows from the fact $\mathbf{A}_k^2 + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2$, combined with the bound $\boldsymbol{\mu}(\mathbf{S}_k) \leq \delta \min\{1, r^{-1}\varkappa\}$ just established for the above choice of r.

From Corollary 3.3 we may conclude that $T \llcorner \mathbf{B}_{rc_0}(\xi_k)$ splits into currents T_i, $i = 1, \ldots, N$, with the T_i supported in small disjoint neighborhoods of $\alpha_i^k \cap \mathbf{C}_{rc_0}(\xi_k)$; in particular, T_1 and T_2 are supported in two small disjoint neighbourhoods of $\alpha_1^k \cap \mathbf{C}_{rc_0}(\xi_k)$ and $\alpha_2^k \cap \mathbf{C}_{rc_0}(\xi_k)$ respectively, with the property that $(\mathbf{p}_{\alpha_1^k})_\sharp T_i = Q_i(k) [\![B_{rc_0}(\xi_k)]\!]$ for *positive* integers $Q_i(k) \geq 1$. Upon extraction of a subsequence we can assume the $Q_i(k)$ equal positive integers Q_i' independent of k and it follows easily that $Q_i' = Q_i$. This concludes the proof. □

We now come to the proof of Proposition 9.2. At one point in the proof we will need the following proposition, which also plays a key role later on in our work. Its proof will be given in Section 12, where we collect several important facts regarding Dir-minimizers.

Proposition 9.4. *Assume $\Omega \subset \mathbb{R}^m$ is a Lipschitz domain, $V \subset \mathbb{R}^m$ is an $(m-2)$-dimensional plane, and $v \in W^{1,2}(\Omega; \mathcal{A}_Q(\mathbb{R}^n))$ is a map with the property that the restriction of v to $\Omega_\varepsilon := \Omega \setminus B_\varepsilon(V)$ is Dir-minimizing for every $\varepsilon > 0$. Then v is Dir-minimizing in Ω.*

Proof of Proposition 9.2. We argue by induction on N. The case $N = 2$ is already established by Proposition 9.3, since in that situation we have $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$.

So fix $N \geq 3$. We may assume inductively the validity of Proposition 9.2 for all $N' < N$. If $C(m,n,\bar{n},N')$ and $\varepsilon = \varepsilon(m,n,\bar{n},N')$ denote the corresponding constants, set

$$C_1 := \max_{N' \leq N-1} C(m,n,\bar{n},N') \quad \text{and} \quad \varepsilon_1^* := \min_{N' \leq N-1} \varepsilon(m,n,\bar{n},N').$$

Suppose however that the conclusion of the proposition fails for N. We may therefore find sequences T_k, Σ_k, $\mathbf{A}_k$, $\mathbf{S}_k = \alpha_1^k \cup \cdots \cup \alpha_N^k$ and $\varepsilon_k \downarrow 0$ with

$$\mathbf{A}_k^2 \leq \varepsilon_k^2 \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq \varepsilon_k^4 \boldsymbol{\sigma}(\mathbf{S}_k)^2, \tag{9.10}$$

such that, up to relabelling the planes in $\mathbf{S}_k$, we have

$$\frac{\theta_1(\alpha_1^k, \alpha_2^k)}{\theta_2(\alpha_1^k, \alpha_2^k)} \to 0. \tag{9.11}$$

Now, we may assume that

$$\lim_{k\to\infty} \boldsymbol{\sigma}(\mathbf{S}_k)/\boldsymbol{\mu}(\mathbf{S}_k) = 0\,. \tag{9.12}$$

Indeed, if this were not true, then up to passing to a subsequence we may assume that $\boldsymbol{\sigma}(\mathbf{S}_k) \geq \eta^* \boldsymbol{\mu}(\mathbf{S}_k)$ for all k and some $\eta^* > 0$, at which point we may apply Proposition 9.3 to get that for all k sufficiently large, the cones $\mathbf{S}_k$ are C-balanced for some $C > 0$ independent of k. But then we would have $\theta_1(\alpha_1^k, \alpha_2^k)/\theta_2(\alpha_1^k, \alpha_2^k) \geq C^{-1} > 0$ for all k, in direct contradiction to (9.10).

Now apply the layer subdivision lemma (Lemma 8.3) with $\delta = 1$ to each $\mathbf{S}_k$ (the exact choice of δ is unimportant). This provides subcollections $\{1,\ldots,N\} = I_k(0) \supsetneq I_k(1) \supsetneq \cdots \supsetneq I_k(\kappa(k))$ satisfying properties (i)–(iv) of Lemma 8.3. Up to extracting a further subsequence (which we again do not relabel) we can assume that $\kappa(k)$ is independent of k, and moreover that all the sets of indices $I_k(l)$, $\ell = 0,1,\ldots,\kappa$ are independent of k; we therefore label them $I(l)$. Since we are assuming now that $\boldsymbol{\sigma}(\mathbf{S}_k)/\boldsymbol{\mu}(\mathbf{S}_k) \to 0$, we necessarily have that $\kappa \geq 1$, as otherwise conclusion (ii) of Lemma 8.3 would give that $\eta\boldsymbol{\mu}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k)$ for all k and for some fixed $\eta = \eta(N) > 0$ independent of k, giving a contradiction.

For each k and $l \in \{0,1,\ldots,\kappa\}$, write $\mathbf{S}_k^{(l)} := \bigcup_{i\in I(\ell)} \alpha_i^k$ for the cone at the l^{th} layer of $\mathbf{S}_k$. Also write

$$\mathbf{d}_k^{(l)} := \max_{i\in I(0)} \min_{j\in I(l)} \operatorname{dist}(\alpha_i^k \cap \mathbf{B}_1, \alpha_j^k \cap \mathbf{B}_1).$$

From (iii) of Lemma 8.3, we know that

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell-1)})}{\mathbf{d}_k^{(l)}} \geq \eta > 0 \tag{9.13}$$

for all k and $\ell \in \{1,\ldots,\kappa\}$, where again $\eta = \eta(N) > 0$ is a fixed constant.

Suppose first that we have

$$\lim_{k\to\infty} \frac{\mathbf{d}_k^{(1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})} = 0\,. \tag{9.14}$$

Then we have $\mathbf{d}_k^{(1)} \leq \delta_k \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$ for some non-negative sequence $\delta_k \to 0$. We can thus estimate

$$\begin{aligned}\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1) &\leq \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C(\mathbf{d}_k^{(1)})^2 \\ &\leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 + C\delta_k^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2 = (\varepsilon_k^2 + C\delta_k^2)\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2\,,\end{aligned} \tag{9.15}$$

where $C = C(m,n,Q)$ and we have used that $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$, since $\mathbf{S}_k^{(1)} \subset \mathbf{S}_k$; this inclusion also gives

$$\hat{\mathbf{E}}(\mathbf{S}_k^{(1)}, T_k, \mathbf{B}_1) \leq \hat{\mathbf{E}}(\mathbf{S}_k, T_k, \mathbf{B}_1) \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})^2\,.$$

Thus, for all k sufficiently large, we will have that hypothesis (9.2) holds with the smallness threshold ε_1^*. This allows us to conclude, by the induction hypothesis, that $\mathbf{S}_k^{(1)}$ is C-balanced for some $C = C(m,n,\bar{n},N)$ (in fact, $C = C_1$).

Now, we know from (9.14) that $\mathbf{d}_k^{(1)} \to 0$. Analogously to case (b) in the proof of Proposition 9.3, we perform a blow-up. This time, however, we will be normalizing by $\mathbf{d}_k$ and building graphical approximations over the (balanced) cones $\mathbf{S}_k^{(1)}$. Indeed, (9.15) tells us that the hypotheses of Proposition 8.15 (and therefore Propositions 8.19 and 8.20) hold for T_k relative to the balanced cone $\mathbf{S}_k^{(1)}$ for all k sufficiently large. Thus, T_k splits into a sum of disjoint currents, denoted by T_k^j, near each plane $\alpha_j^k \subset \mathbf{S}_k^{(1)}$.

Now choose any sequences $\varsigma_k \downarrow 0$ and $\sigma_k \downarrow 0$. Passing to a subsequence, we may apply Proposition 8.20 with σ_k, ς_k in place of σ, ς to T_k and $\mathbf{S}_k^{(1)}$ to find multi-valued Lipschitz maps u_k^j over the sets $R_i^k := (R \setminus B_{\sigma_k}(V)) \cap \alpha_i^k$ for $i \in I(1)$ with the property that, if $v_k^i := \hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1)^{-1/2} u_k^i$, then

$$d_{W^{1,2}}(v_k^i, w_k^i) \leq \varsigma_k \tag{9.16}$$

where w_k^i are Dir-minimizing maps on R_i^k. So, if we set $\bar{u}_i^k := (\mathbf{d}_k^{(1)})^{-1} u_i^k$, then estimating similarly to (9.15), since $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \mathbf{d}_k^{(1)}$, we have

$$\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1) \leq (\varepsilon_k^2 + C)(\mathbf{d}_k^{(1)})^2$$

meaning that we can blow-up via rescaling by $\mathbf{d}_k^{(1)}$ instead of $\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(1)}, \mathbf{B}_1)^{-1/2}$ (see the remark after Proposition 8.20). Indeed, the estimates (8.16)–(8.18), combined with the conclusion (ii) of Proposition 8.20, (9.16), and a diagonal argument give that $\bar{u}_i^k \to \bar{u}_j^\infty$ locally strongly in $W^{1,2}(\mathbf{B}_1(0) \setminus V)$ and locally uniformly in $\mathbf{B}_1(0) \setminus V$, for some $\bar{u}_j^\infty$ which is Dir-minimizing in $\mathbf{B}_{1-\varepsilon}(0) \setminus B_\rho(V)$ for every $0 < \varepsilon, \rho < 1/2$. Note also that, if for each i, we denote by L_i^k the linear maps parameterizing the planes $\alpha_i^k \in \mathbf{S}_k$ over the closest plane in $\mathbf{S}_k^{(1)}$, then we necessarily have that $|L_i^k| \leq C\mathbf{d}_k^{(1)}$ for some dimensional constant C. So, if we write

$$\bar{L}_i^k := (\mathbf{d}_k^{(1)})^{-1} L_i^k$$

then we can pass to a further subsequence to ensure also that $\bar{L}_i^k \to \bar{L}_i$ for some linear map $\bar{L}_i$ over some plane α_i^∞ in $\mathbf{S}_\infty^{(1)} := \lim_{k\to\infty} \mathbf{S}_k^{(1)}$ (note that $\mathbf{S}_\infty^{(1)}$ need not be a single plane). However, as $|L_i^k - L_j^k| \geq c\boldsymbol{\sigma}(\mathbf{S}_k)$ for some dimensional constant c (by definition of $\boldsymbol{\sigma}(\mathbf{S}_k)$) whenever L_i^k, L_j^k are defined over the same plane, from (9.13) we know that $|\bar{L}_i - \bar{L}_j| \geq c\eta > 0$, i.e. $\bar{L}_i \neq \bar{L}_j$ for $i \neq j \in I(0)$. Note now that

$$\int \mathrm{dist}^2(x, \mathbf{S}_k)\, d\|\mathbf{G}_{u_j^k}\|(x) \leq C\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C\left(\mathbf{A}_k^2 + \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)\right)^{1+\gamma} = o\left((\mathbf{d}_k^{(1)})^2\right)$$

where $\gamma = \gamma(Q, m, n, \bar{n}) > 0$ is as in Proposition 8.6. Thus, dividing both sides by $(\mathbf{d}_k^{(1)})^2$ and taking $k \to \infty$, we see that necessarily the graph of $\bar{u}_j^\infty$ is supported on the union of all the graphs of the maps $\bar{L}_i$ associated to the plane α_j. Letting J_j denote this collection of indices i for $j \in I(1)$, on $\Omega = (\mathbf{B}_1 \cap \alpha_j) \setminus V$ we have

$$\bar{u}_j^\infty = \sum_{i \in J_j} Q_i^j [\![\bar{L}_i]\!]$$

for some non-negative integers Q_i^j such that $\sum_{i,j} Q_i^j = Q$. Following a similar argument to the proof in Proposition 9.3, we can in fact show that $Q_i^j \geq 1$ for each i, j. Now, since $\bar{u}_j^\infty$ is Dir-minimizing in $\mathbf{B}_{1-\varepsilon} \setminus B_\rho(V)$ for every $\varepsilon, \rho > 0$, we may now apply Proposition 9.4 to conclude that $\bar{u}_j^\infty$ extends to a Dir-minimizer (which we still denote by $\bar{u}_j^\infty$) on $\mathbf{B}_{1/2} \cap \alpha_j$. If the unbalancing assumption from (9.11) has $1, 2 \in J_j$ for some j, then we can argue as in case (b) of Proposition 9.3 to arrive at a contradiction. If we have $1 \in J_j$ and $2 \in J_{j^*}$ for some $j \neq j^*$, we can again argue in the same way, except now using the fact that $\mathbf{S}_k^{(1)}$ is balanced, and so the planes α_j^k, $\alpha_{j^*}^k$ are balanced. Thus, in this situation we arrive at a contradiction.

To summarize, we have now shown that (9.14) cannot hold. From Lemma 8.3 (as $\delta = 1$) we know that $\mathbf{d}_k^{(1)} \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(1)})$, and so we may therefore pass to a subsequence to ensure that

$$\mathbf{d}_k^{(1)} \geq c_1 \boldsymbol{\sigma}(\mathbf{S}_k^{(1)}) \tag{9.17}$$

for all k and for some constant $c_1 > 0$.

To progress, we now move to the next layer, namely the cones $\mathbf{S}_k^{(2)}$. We know from (9.13) that $\boldsymbol{\sigma}(\mathbf{S}_k^{(1)}) \geq \eta \mathbf{d}_k^{(2)}$. Now, if we have

$$\lim_{k\to\infty} \frac{\mathbf{d}_k^{(2)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(2)})} = 0 \tag{9.18}$$

we argue that we can follow an analogous argument to that above when (9.14) held, except now performing a blow-up of T_k relative to $\mathbf{S}_k^{(2)}$. Indeed, if (9.18) holds, then we have $\mathbf{d}_k^{(2)} \leq \delta_k' \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})$, for some non-negative sequence $\delta_k' \to 0$. We can estimate as in (9.15) to find

$$\begin{aligned}\hat{\mathbf{E}}(T_k, \mathbf{S}_k^{(2)}, \mathbf{B}_1) &\leq \hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1) + C(\mathbf{d}_k^{(2)})^2 \leq \varepsilon_k^2 \boldsymbol{\sigma}(\mathbf{S}_k)^2 + C(\delta_k')^2 \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})^2 \\ &\leq (\varepsilon_k^2 + C(\delta_k')^2)\boldsymbol{\sigma}(\mathbf{S}_k^{(2)})^2\end{aligned}$$

where again we have used $\boldsymbol{\sigma}(\mathbf{S}_k) \leq \boldsymbol{\sigma}(\mathbf{S}_k^{(2)})$ since $\mathbf{S}_k^{(2)} \subset \mathbf{S}_k$. Therefore, once again we get that the hypothesis (9.2) holds with the smallness threshold ε_1^* for all k sufficiently large, and so by the induction hypothesis we have that $\mathbf{S}_k^{(2)}$ is C-balanced for some $C = C(m, n, \bar{n}, N)$. The only aspect of the above blow-up procedure which we need to check in this case is the separation of all the rescaled linear functions of the planes in $\mathbf{S}_k$ over $\mathbf{S}_k^{(2)}$. As we are rescaling our approximations by $\mathbf{d}_k^{(2)}$, we therefore need to show that $\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(2)}}$ has a uniform lower bound. Indeed, combining (9.13) and (9.17), we have

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(2)}} = \frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\mathbf{d}_k^{(1)}} \cdot \frac{\mathbf{d}_k^{(1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})} \cdot \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(1)})}{\mathbf{d}_k^{(2)}} \geq \eta \cdot c_1 \cdot \eta = \eta^2 c_1 > 0\,. \tag{9.19}$$

This tells us that we may repeat the blow-up argument as above, with $\mathbf{d}_k^{(1)}$ replaced by $\mathbf{d}_k^{(2)}$ and $\mathbf{S}_k^{(1)}$ replaced with $\mathbf{S}_k^{(2)}$, to contradict the lack of balancing (9.11) of $\mathbf{S}_k$ when (9.18) holds.

Proceeding now inductively, we see that we must be able to find a subsequence with

$$\mathbf{d}_k^{(l)} \geq c_l \boldsymbol{\sigma}(\mathbf{S}_k^{(l)})$$

for all $l = 1, \dots, \kappa$ and some constants $c_l > 0$. However, we can now show that this is in direct contradiction to the assumption (9.12). Indeed, an identical calculation from (9.19) gives

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}_k)} = \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\kappa)})}{\boldsymbol{\mu}(\mathbf{S}_k)} \cdot \prod_{\ell=0}^{\kappa-1} \left(\frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell)})}{\mathbf{d}_k^{(\ell+1)}} \cdot \frac{\mathbf{d}_k^{(\ell+1)}}{\boldsymbol{\sigma}(\mathbf{S}_k^{(\ell+1)})} \right) \geq \frac{\boldsymbol{\sigma}(\mathbf{S}_k^{(\kappa)})}{\boldsymbol{\mu}(\mathbf{S}_k)} \cdot \eta^\kappa \prod_{l=1}^{\kappa} c_l\,,$$

where we recall $\mathbf{S}_k^{(0)} = \mathbf{S}_k$. But we know from Lemma 8.3(i) and (ii) that $\boldsymbol{\mu}(\mathbf{S}_k) = \boldsymbol{\mu}(\mathbf{S}_k^{(\kappa)})$ and $\eta \boldsymbol{\mu}(\mathbf{S}_k^{(\kappa)}) \leq \boldsymbol{\sigma}(\mathbf{S}_k^\kappa)$. Hence, the above gives

$$\frac{\boldsymbol{\sigma}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}_k)} \geq \eta^{\kappa+1} \prod_{l=1}^{\kappa} c_l > 0\,,$$

thus reaching the desired contradiction to (9.12), and completing the proof. □

10. Reduction of the main decay theorem

In this section, we reduce the proof of Theorem 2.5 to an a priori much weaker decay statement; Theorem 10.2 below. For this we will utilize the balancing result from Proposition 9.1. Recall that for a cone $\mathbf{S} \in \mathscr{C}(Q)$, we write

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1), \qquad \boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1).$$

From now on, we make the following assumption regarding the (balancing) constant M:

Assumption 10.1. The constant $M \geq 1$ is chosen so that the application Proposition 9.1 yields an M-balanced cone. Thus, $M = M(Q, m, n, \bar{n})$.

Theorem 10.2 (Weak Excess Decay Theorem)**.** *Fix $Q, m, n, \bar{n}$ as before, and let $M \geq 1$ be as in Assumption 10.1. Fix also $\varsigma_1 > 0$. Then, there are constants $\varepsilon_1 = \varepsilon_1(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$, $r_1^1 = r_1^1(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$, $r_1^2 = r_1^2(Q, m, n, \bar{n}, \varsigma_1) \in (0, 1/2]$, and such that the following holds. Suppose that*

(i) *T and Σ are as in Assumption 2.1;*
(ii) *$\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$;*
(iii) *There is $\mathbf{S} \in \mathscr{C}(Q, 0)$ which is M-balanced, such that*

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_1^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{10.1}$$

and

$$\mathbf{B}_{\varepsilon_1}(\xi) \cap \{p : \Theta(T, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}) \cap \mathbf{B}_{1/2}\,; \tag{10.2}$$

(iv) *$\mathbf{A}^2 \leq \varepsilon_1^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.*

Then, there is $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that for some $i \in \{1, 2\}$ we have

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_1^i}) \leq \varsigma_1 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{10.3}$$

The above theorem would appear to be considerably weaker than Theorem 2.5; not only are we assuming the cone $\mathbf{S}$ is balanced, but there is also a significant difference between the smallness assumption (2.1) and (10.1). In fact, (10.1) implies the second inequality in (2.1) for ε_1 suitably small (indeed, Proposition 8.6 then gives $\boldsymbol{\sigma}(\mathbf{S})^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1)$), whilst when the cone $\mathbf{S}$ arises from Proposition 9.1 (as it will) the second inequality in (2.1) is equivalent, up to constants, to (from Proposition 9.1(d))

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_0^2 \boldsymbol{\mu}(\mathbf{S})^2\,. \tag{10.4}$$

In order to show that in fact the above seemingly weaker statement implies Theorem 2.5, the idea is to first show that Theorem 10.2 implies a multiple radii decay version of Theorem 2.5, by first removing the necessary planes in the cone $\mathbf{S}$ to reach a balanced cone, and then inductively removing additional planes to obtain a cone (that is still balanced) for which assumption (10.1) holds. In doing so we reach an intermediate statement, namely an excess decay with finitely many scales, from which we then derive Theorem 2.5. This is the following, the idea of which is similar to that seen in [32, Section 13]:

Proposition 10.3 (Multiple Radii Excess Decay)**.** *Fix $Q, m, n, \bar{n}$ as before, and let $M \geq 1$ be as in Assumption 10.1. Let $\bar{N} := Q(Q-1)$. Fix $\varsigma_2 > 0$. Then, there exist positive constants $\varepsilon_2, r_1, \ldots, r_{\bar{N}} \leq \frac{1}{2}$, depending only on $Q, m, n, \bar{n}$, and ς_2, such that the following holds. Suppose that $T, \Sigma, \mathbf{S}$ are as in Theorem 2.5, i.e.:*

(i) *T and Σ are as in Assumption 2.1 and $\mathbf{B}_1 \subset \Omega$;*
(ii) *$\|T\|(\mathbf{B}_1) \leq (Q + \frac{1}{2})\omega_m$;*
(iii) *There is $\mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that*

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon_2^2 \mathbf{E}^p(T, \mathbf{B}_1),$$

and that

$$\mathbf{B}_{\varepsilon_2}(\xi) \cap \{p : \Theta(T, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}) \cap \mathbf{B}_{1/2}\,;$$

(iv) *$\mathbf{A}^2 \leq \varepsilon_2^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.*

Then, there is a $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ and $i \in \{1, \ldots, \bar{N}\}$ such that:

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_i}) \leq \varsigma_2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,. \tag{10.5}$$

We will then be able to show that Proposition 10.3 implies Theorem 2.5, which therefore demonstrates that the weak excess decay from Theorem 10.2 implies the stronger excess decay statement from Theorem 2.5. Let us begin by showing that Theorem 10.2 implies Proposition 10.3.

Proof of Proposition 10.3 from Theorem 10.2. Fix $T, \Sigma, \mathbf{S}$, and ς_2 as in the statement of Proposition 10.3.

Firstly, because of Proposition 9.1, if we take $\varepsilon_2 \leq \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(Q,m,n,\bar{n}) > 0$ is as in Proposition 9.1, then we can find a cone $\tilde{\mathbf{S}}$ which is M-balanced and obeys $\mathbb{E}(T,\tilde{\mathbf{S}},\mathbf{B}_1) \leq M\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$, where $M = M(Q,m,n,\bar{n})$ is as in Assumption 10.1. Thus, we have

$$\mathbb{E}(T,\tilde{\mathbf{S}},\mathbf{B}_1) \leq M\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq M\varepsilon_2^2\mathbf{E}^p(T,\mathbf{B}_1)$$

In particular, this coupled with the other estimates in Proposition 9.1 gives that if we can prove the result for $\tilde{\mathbf{S}}$, then the result follows for $\mathbf{S}$, up to changing ς_2 by a factor of M. Thus, we may without loss of generality assume that $\mathbf{S}$ is M-balanced and $M^{-1}\mathbf{E}^p(T,\mathbf{B}_1) \leq \boldsymbol{\mu}(\mathbf{S}) \leq M\mathbf{E}^p(T,\mathbf{B}_1)$.

We would now like to apply Theorem 10.2. However, a priori, it may be the case that (10.1) does not hold for $\mathbf{S}$, as we are merely assuming that $\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq \varepsilon_2^2\mathbf{E}^p(T,\mathbf{B}_1)$. The remaining part of the argument deals with this difficulty; the price to pay is that the decay might occur at one of finitely many scales, but nonetheless the number of possible scales which are needed belong to a set of controlled cardinality.

We start by introducing, for every integer $k \in \{2,\dots,Q\}$, functions $\varepsilon_{(k)}(s)$, $r^1_{(k)}(s)$, and $r^2_{(k)}(s)$ of a parameter $s > 0$ as follows:

- If one takes $Q = k$ and $\varsigma_1 = s$ in Theorem 10.2, then $\varepsilon_{(k)}(s) := \varepsilon_1(k,m,n,\bar{n},s)$, where the ε_1 is the constant from Theorem 10.2 with this choice of Q, ς_1, and $r^1_{(k)}(s) := r^1_1(k,m,n,\bar{n},s)$, $r^2_{(k)}(s) := r^2_1(k,m,n,\bar{n},s)$ are the corresponding radii, where r^1_1 and r^2_1 are as in Theorem 10.2.

In particular, k here is the number of planes forming the cone $\mathbf{S}$ to which Theorem 10.2 is applied, whilst s is the specified decay factor.

We now denote by N the number of planes forming $\mathbf{S}$. If $N = 2$, then $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\sigma}(\mathbf{S})$, and so from (10.4) we get that the assumptions of Theorem 10.2 hold (up to ε_2 changing by a constant), and so if we take ε_2 smaller than $\varepsilon_{(2)}(\varsigma_2)$, we see that we can apply Theorem 10.2. The corresponding radii from this application of Theorem 10.2 will be denoted by $r^1_{2,2}$ and $r^2_{2,2}$ (the first subscript here denotes the number of planes, N, in the cone $\mathbf{S}$, whilst the second subscript denotes the number of planes left at the point of application of Theorem 10.2 – see below), which are simply $r^1_{(2)}(\varsigma_2)$ and $r^2_{(2)}(\varsigma_2)$. In particular, the proof is complete in the case $Q = 2$.

From now on we therefore assume $N \geq 3$. We may take $\varepsilon_2 \leq \varepsilon_{(N)}(\varsigma_2)$. We then consider two cases. If we have

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq \varepsilon_{(N)}(\varsigma_2)^2\boldsymbol{\sigma}(\mathbf{S})^2$$

then we can simply apply Theorem 10.2 to get the desired decay at one of the two radii $r^1_{N,N} := r^1_{(N)}(\varsigma_2)$, $r^2_{N,N} := r^2_{(N)}(\varsigma_2)$. We can therefore assume that

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) > \varepsilon_{(N)}(\varsigma_2)^2\boldsymbol{\sigma}(\mathbf{S})^2. \tag{10.6}$$

We now follow the same idea in the argument for the Pruning Lemma (Lemma 8.2). Let $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$, where the α_i are distinct m-dimensional planes. We can assume, upon relabelling, that

$$\mathrm{dist}(\alpha_1 \cap \mathbf{B}_1, \alpha_2 \cap \mathbf{B}_1) = \boldsymbol{\sigma}(\mathbf{S})$$

and that there are two indices $i_*, j_* \neq 1$ with

$$\mathrm{dist}(\alpha_{i_*} \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) = \boldsymbol{\mu}(\mathbf{S}).$$

We now remove the plane α_1 and consider $\mathbf{S}_{N-1} := \alpha_2 \cup \cdots \cup \alpha_N$; note that $\mathbf{S}_{N-1}$ is still M-balanced, $V(\mathbf{S}_{N-1}) = V(\mathbf{S})$, $\boldsymbol{\mu}(\mathbf{S}) = \boldsymbol{\mu}(\mathbf{S}_{N-1})$, and

$$\mathrm{dist}^2(\mathbf{S}\cap\mathbf{B}_1, \mathbf{S}_{N-1}\cap\mathbf{B}_1) \leq \boldsymbol{\sigma}(\mathbf{S})^2 \leq \varepsilon_{(N)}(\varsigma_2)^{-2}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1);$$

these conditions imply that if we can show the result for $\mathbf{S}_{N-1}$, the corresponding statement for $\mathbf{S}$ follows.

Now observe that

$$\mathbb{E}(T,\mathbf{S}_{N-1},\mathbf{B}_1) \leq C_0\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) + C_0\boldsymbol{\sigma}(\mathbf{S})^2 \leq C_0\left(1 + \varepsilon_{(N)}(\varsigma_2)^{-2}\right)\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

for some constant $C_0 = C_0(Q,m,n,\bar{n}) \geq 1$. Let us write $C_1^* := C_1(1+\varepsilon_{(N)}(\varsigma_2)^{-2})$, and so $C_1^* = C_1^*(N,m,n,\bar{n},\varsigma_2) > 1$, and so the above inequality is just

$$\mathbb{E}(T,\mathbf{S}_{N-1},\mathbf{B}_1) \leq C_1^*\mathbb{E}(T,\mathbf{S},\mathbf{B}_1).$$

Therefore, if

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq \varepsilon_{(N-1)}(\varsigma_2/C_1^*)^2\sigma(\mathbf{S}_{N-1})^2 \tag{10.7}$$

and if we take $\varepsilon_2 \leq \varepsilon_{(N-1)}(\varsigma_2/C_1^*)/\sqrt{C_1^*}$ (which of course we may), then the above gives that we can apply Theorem 10.2 to the triple T,Σ, and $\mathbf{S}_{N-1}$ to conclude the desired decay statements in Proposition 10.3, for one of the radii

$$r^i_{N,N-1} := r^i_{(N-1)}(\varsigma_2/C_1^*), \quad i \in \{1,2\}.$$

If (10.7) is not true, i.e. if we cannot apply Theorem 10.2 as above, then we must have

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) > \varepsilon_{(N-1)}(\varsigma_2/C_1^*)^2\boldsymbol{\sigma}(\mathbf{S}_{N-1})^2. \tag{10.8}$$

We now wish to apply the above procedure inductively; indeed, suppose we have performed the above procedure K times, and at each stage we have not been able to apply Theorem 10.2. Thus, upon relabelling the planes in $\mathbf{S}$, for $k=0,1,\dots,K$ we have cones $\mathbf{S}_{N-k} = \alpha_{k+1}\cup\cdots\cup\alpha_N$ which are all M-balanced and obey $V(\mathbf{S}_{N-k}) = V(\mathbf{S})$, $\boldsymbol{\mu}(\mathbf{S}_{N-k}) = \boldsymbol{\mu}(\mathbf{S})$. We also know that $\mathbf{S}_{N-(k+1)}$ is formed from $\mathbf{S}_{N-k}$ by removing a single plane, namely α_{k+1}, which is a plane in $\mathbf{S}_{N-k}$ achieving the minimal separation $\boldsymbol{\sigma}(\mathbf{S}_{N-k})$ and such that there are two other planes in $\mathbf{S}_{N-k}$ which achieve the maximal separation $\boldsymbol{\mu}(\mathbf{S}_{N-k})$ (this can be done as long as $N-k\geq 3$, i.e. $0\leq k\leq N-3$). The criterion along this sequence is that

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) > \varepsilon_{(N-k)}(\varsigma_2/C_k^*)^2\boldsymbol{\sigma}(\mathbf{S}_{N-k})^2$$

where $C_k^* = C_0(C_{k-1}^* + \varepsilon_{(N-(k-1))}(\varsigma_2/C_{k-1}^*)^{-2})$ is defined inductively for $k\geq 1$, with $C_0^* = 1$ and $C_0 = C_0(Q,m,n,\bar{n})$ is as above, and

$$\mathbb{E}(T,\mathbf{S}_{N-k},\mathbf{B}_1) \leq C_k^*\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\,;$$

for this procedure we need to assume $\varepsilon_2 \leq \varepsilon_{(N-k)}(\varsigma/C_k^*)/\sqrt{C_k^*}$ for each $k=0,1,\dots,K$ (which of course we may). We also therefore have

$$\mathrm{dist}^2(\mathbf{S}_{N-(k+1)}\cap\mathbf{B}_1,\mathbf{S}_{N-k}\cap\mathbf{B}_1) \leq \boldsymbol{\sigma}(\mathbf{S}_{N-k})^2 \leq \varepsilon_{(N-k)}(\varsigma_2/C_k^*)^{-2}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)$$

and thus

$$\begin{aligned}\mathbb{E}(T,\mathbf{S}_{N-(k+1)},\mathbf{B}_1) &\leq C\mathbb{E}(T,\mathbf{S}_{N-k},\mathbf{B}_1) + C\boldsymbol{\sigma}(\mathbf{S}_{N-k})^2\\ &\leq C_0\mathbb{E}(T,\mathbf{S}_{N-k},\mathbf{B}_1) + C_0\varepsilon_{(N-k)}(\varsigma_2/C_k^*)^{-2}\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)\end{aligned}$$

which in particular gives the inductive definition of C_k^*. Note that this procedure can occur at most $N-2$ times, and thus all the constants and smallness assumptions here only depend on $N,m,n,\bar{n}$, and ς_2, and so certainly we can choose $\varepsilon_2 = \varepsilon_2(N,m,n,\bar{n},\varsigma_2)$ small enough so that the above procedure is guaranteed whenever we cannot apply Theorem 10.2.

Hence, for this choice of ε_2, if we are unable to apply Theorem 10.2 at all steps to conclude, the above process must eventually terminate when the cone is formed of exactly two planes, i.e. at $\mathbf{S}_2$. Since we are unable to apply Theorem 10.2 to $\mathbf{S}_2$, we must have

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) > \varepsilon_{(2)}(\varsigma_2/C_{N-2}^*)^2\boldsymbol{\sigma}(\mathbf{S}_2)^2. \tag{10.9}$$

However, as $\mathbf{S}_2$ consists of two planes we know that $\boldsymbol{\sigma}(\mathbf{S}_2) = \boldsymbol{\mu}(\mathbf{S}_2)$, and moreover since by construction we know that $\boldsymbol{\mu}(\mathbf{S}_2) = \boldsymbol{\mu}(\mathbf{S})$, we see that (10.9) is equivalent to

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) > \varepsilon_{(2)}(\varsigma_2/C_{N-2}^*)^2\boldsymbol{\mu}(\mathbf{S})^2. \tag{10.10}$$

However, we are assuming (see Proposition 10.3(iii)) that $\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq \varepsilon_2^2\mathbf{E}^p(T,\mathbf{B}_1)$, which from (10.4) we know implies

$$\mathbb{E}(T,\mathbf{S},\mathbf{B}_1) \leq C\varepsilon_2^2\boldsymbol{\mu}(\mathbf{S})^2$$

for some $C = C(Q,m,n,\bar{n}) > 0$. Thus, if we ensure that $C\varepsilon_2^2 < \frac{1}{2}\varepsilon_{(2)}(\varsigma_2/C_{N-2}^*)^2$ also, then this would give a contradiction to (10.10); hence, for such a choice of ε_2 we see that once we reach $\mathbf{S}_2$, we in fact *must* be able to apply Theorem 10.2 in the above procedure.

We hence conclude that, for any fixed $N \leq Q$, if the threshold ε_2 satisfies all the (finitely many) smallness conditions above, then we have that Proposition 10.3 holds for one of the radii $r^1_{N,N}$, $r^2_{N,N}$, $r^1_{N,N-1}, r^2_{N,N-1}, \dots, r^1_{N,2}, r^1_{N,2}$, of which there are $2(N-1)$ radii. In particular, if ε_2 satisfies all of these smallness conditions as we range N over $\{2, \dots, Q\}$, then the above argument is valid for any cone $\mathbf{S} \in \mathscr{C}(Q,0)$ comprised of $N \in \{2, \dots, Q\}$ planes, i.e. for every $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}$. Thus, Proposition 10.3 holds for every $\mathbf{S} \in \mathcal{C}(Q,0) \setminus \mathscr{P}$ for one of the radii in the collection $\{r^i_{j,k}\}$, where $i \in \{1,2\}$ and $2 \leq k \leq j \leq Q$. Thus, the number of possible decay scales is

$$\sum_{N=2}^{Q} 2(N-1) = Q(Q-1) = \bar{N}$$

as claimed in the statement of Proposition 10.3. This completes the proof. □

Now we will prove Theorem 2.5 from Proposition 10.3, thus reducing the proof of Theorem 2.5 to proving the weaker version, Theorem 10.2. First, we observe a corollary of the Proposition 9.1 and Proposition 8.15, the purpose of which is to control the excess of rescalings; it will be a useful tool in the proof of the reduction.

Corollary 10.4. *Suppose $\tilde{T}$ and $\tilde{\Sigma}$ satisfy Assumption 2.1, let $\tilde{\mathbf{S}} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$, and write $\tilde{\mathbf{A}}^2$ for the supremum norm of the second fundamental form of $\tilde{\Sigma}$. Fix also a radius $\bar{r} \in (0,1]$. Then, there are constants $\tilde{\varepsilon} = \tilde{\varepsilon}(Q,m,n,\bar{n},\bar{r}) > 0$ and $\tilde{C} = \tilde{C}(Q,m,n,\bar{n},\bar{r}) > 0$ such that the following holds. If we have*

$$\mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1) \leq \tilde{\varepsilon}^2 \mathbf{E}^p(\tilde{T}, \mathbf{B}_1)$$

and

$$\tilde{\mathbf{A}}^2 \leq \tilde{\varepsilon}^2 \mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1) \qquad \forall \tilde{\mathbf{S}} \in \mathscr{C}(Q,0)\,,$$

then there is $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_r) \leq \tilde{C} \mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1) \qquad \forall r \in [\bar{r}, 1]. \tag{10.11}$$

Proof. Observe that for any $r \in [\bar{r}, 1]$ we clearly have

$$\hat{\mathbf{E}}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_r) = r^{-m-2} \int_{\mathbf{B}_r} \operatorname{dist}^2(x, \tilde{\mathbf{S}})\, d\|\tilde{T}\|(x) \leq \bar{r}^{-m-2} \hat{\mathbf{E}}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1).$$

On the other hand, there is no obvious reason to have a bound of the form

$$\hat{\mathbf{E}}(\tilde{\mathbf{S}}, \tilde{T}, \mathbf{B}_r) \leq C(\bar{r}) \mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)$$

because in the integral defining the reverse excess $\hat{\mathbf{E}}(\tilde{\mathbf{S}}, \tilde{T}, \mathbf{B}_r)$ we omit a tubular neighbourhood of the spine $V(\tilde{\mathbf{S}})$ of radius ar, and thus there is no inclusion property of one domain of integration into the other when we are comparing two different scales.

To get around this problem, first note that, provided $\tilde{\varepsilon} \leq \varepsilon_0$ where $\varepsilon_0 = \varepsilon_0(Q,m,n,\bar{n})$ is as in Proposition 9.1, we can apply Proposition 9.1 to assume without loss of generality that $\tilde{\mathbf{S}}$ is M-balanced (M as in Assumption 10.1; all that changes is the two-sided excess increases by a factor of $M = M(Q,m,n,\bar{n})$).

Fix $\delta > 0$ (to be determined later). Now, apply the Pruning Lemma (Lemma 8.2) with this choice of δ and with $D = \mathbf{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)^{1/2}$ to the (balanced) cone $\tilde{\mathbf{S}}$, in exactly the same way as was done in the proof of Proposition 9.1, to yield a new cone $\mathbf{S}'$. The latter is the union of some subset of the planes in $\tilde{\mathbf{S}}$, in particular it is M-balanced and obeys $V(\mathbf{S}') = V(\tilde{\mathbf{S}})$. Moreover it satisfies

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C_\delta \mathbb{E}(\tilde{T}, \tilde{\mathbf{S}}, \mathbf{B}_1)$$

and

$$\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C\delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2$$

where here $C_\delta = C_\delta(Q,m,n,\bar{n},\delta)$ and $C = C(Q,m,n,\bar{n})$. In particular, we have

$$\tilde{\mathbf{A}}^2 + \mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1) \leq C(\tilde{\varepsilon}^2 + \delta^2) \boldsymbol{\sigma}(\mathbf{S}')^2.$$

Thus, we are in a situation to apply the refined approximation (Proposition 8.15) to $\tilde{T}$, $\mathbf{S}'$, provided we take $\tilde{\varepsilon}, \delta$ sufficiently small. The region we are interested in building the refined approximation over is $\mathbf{B}_1 \setminus B_{a\bar{r}/2}(V(\tilde{\mathbf{S}}))$, and so in particular the radius of this neighbourhood of the spine is fixed only depending on $\bar{r}$ and a. Thus, by Lemma 8.14, if we take $\tilde{\varepsilon}$ and δ smaller than a constant depending only on $Q, m, n, \bar{n}, \bar{r}$, we can ensure that $\mathbf{B}_1 \setminus B_{a\bar{r}/2}(V(\tilde{\mathbf{S}})) \subset R^o$, (recall that R^o is the outer region of the refined approximation). In particular, one can apply Proposition 8.15; the estimates therein (summed over the cubes $L \in \mathcal{G}^o \cup \mathcal{G}^c$ that intersect $\mathbf{B}_r \setminus B_{ar}(V(\tilde{\mathbf{S}}))$ give that

$$\hat{\mathbf{E}}(\mathbf{S}', \tilde{T}, \mathbf{B}_r) \leq \tilde{C}\mathbb{E}(\tilde{T}, \mathbf{S}', \mathbf{B}_1)$$

for any $r \in [\bar{r}, 1]$, where $\tilde{C} = \tilde{C}(Q, m, n, \bar{n}, \bar{r})$. Hence by choosing such an appropriate $\delta = \delta(Q, m, n, \bar{n}, \bar{r})$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(Q, m, n, \bar{n}, \bar{r})$, we see that the proof is complete. □

The next lemma gives the final ingredient for our proof that Proposition 10.3 implies Theorem 2.5(a). The lemma provides a condition for which the rescalings of Corollary 10.4 also satisfy the conditions of Theorem 2.5.

Lemma 10.5. *Fix a radius $\bar{r} \in (0, 1]$ and $\varepsilon_2 \in (0, 1)$. Then, there exists a positive number $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar{n}, \bar{r}, \varepsilon_2)$ such that the following holds. Suppose that T, Σ, and $\mathbf{S}$ satisfy the assumptions in Theorem 2.5 with this choice of ε_0. Then, for every radius $r \in [\bar{r}, 1]$ we have the following. If there is a cone $\mathbf{S}_r \in \mathscr{C}(Q, 0)$ which obeys*

$$\mathbb{E}(T, \mathbf{S}_r, \mathbf{B}_r) \leq \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1), \tag{10.12}$$

and moreover

$$\mathbf{A}^2 r^2 \leq \varepsilon_0^2 \inf\{\mathbb{E}(T, \mathbf{S}', \mathbf{B}_r) : \mathbf{S}' \in \mathscr{C}(Q, 0)\}\,, \tag{10.13}$$

then the rescalings $T_{0,r}$, $\Sigma_{0,r}$, and $\mathbf{S}_r$ satisfy the assumptions (i)–(iii) of Theorem 2.5 (and so also Proposition 10.3 and Corollary 10.4) with ε_2 in place of ε_0, i.e.

$$\mathbb{E}(T_{0,r}, \mathbf{S}_r, \mathbf{B}_1) \leq \varepsilon_2^2 \mathbf{E}^p(T_{0,r}, \mathbf{B}_1)$$

and

$$\mathbf{B}_{\varepsilon_2}(\xi) \cap \{p : \Theta(T_{0,r}, p) \geq Q\} \neq \emptyset \qquad \forall \xi \in V(\mathbf{S}_r) \cap \mathbf{B}_{1/2}.$$

Proof. We argue by contradiction. Fix ε_2 and $\bar{r}$. If the conclusion of the lemma is false, then we can find a sequence of currents T_k, manifolds Σ_k, cones $\mathbf{S}_k$, and $\varepsilon_0^k \downarrow 0$ as in Theorem 2.5 with ε_0^k in place of ε_0 for which the present lemma fails. Namely, there exist radii $r_k \in [\bar{r}, 1]$ and cones $\mathbf{S}_{r_k} \equiv (\mathbf{S}_k)_{r_k}$ which obey

$$(\varepsilon_0^k)^{-2} r_k^2 \mathbf{A}_k^2 \leq \mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \leq \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$$

yet either we have (as the inequality $\mathbf{A}_{0,r_k}^2 \leq \varepsilon_2^2 \mathbb{E}((T_k)_{0,r_k}, \mathbf{S}_{r_k}, \mathbf{B}_1)$ holds for all k sufficiently large by the above assumption)

$$\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) > \varepsilon_2^2 \mathbf{E}^p(T_k, \mathbf{B}_{r_k})$$

or

$$\mathbf{B}_{r_k \varepsilon_2}(\xi) \cap \{p : \Theta(T_k, p) \geq Q\} = \emptyset \quad \text{ for some } \xi \in V(\mathbf{S}_{r_k}) \cap \mathbf{B}_{r_k/2}.$$

In particular, since the assumptions of Theorem 2.5 hold for T_k and $\mathbf{S}_k$, points of density at least Q in T_k accumulate on the spines $V(\mathbf{S}_k)$. Thus, if we pass to a subsequence for which $V(\mathbf{S}_k)$ and $V(\mathbf{S}_{r_k})$ converge, the second condition above in fact gives that we must have

$$\lim_{k\to\infty} \operatorname{dist}(V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1, V(\mathbf{S}_k) \cap \mathbf{B}_1) \geq \bar{r}\varepsilon_2/2 > 0 \tag{10.14}$$

whilst the first condition above gives

$$\liminf_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_{r_k})} \geq \varepsilon_2^2 > 0. \tag{10.15}$$

Without loss of generality, we can select planes π_k realizing $\mathbf{E}^p(T_k, \mathbf{B}_1)$, and, up to performing a rotation, we can assume they all coincide with the same fixed plane, π_0. We can also pass to a subsequence to ensure that the number N_k of planes forming $\mathbf{S}_k$ is a constant N (obeying $N \leq Q$), and similarly the number of planes forming $\mathbf{S}_{r_k}$ is a constant $\bar{N}$.

Let us first contradict (10.15). Note that we have

$$\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \leq \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1) \to 0\,.$$

and thus (10.15) tells us that we must have $\mathbf{E}^p(T_k, \mathbf{B}_{r_k}) \to 0$. In particular, up to subsequences, we know that the current T_k is converging to a limiting current T_∞ which coincides with an integer multiple of some plane π_∞ in $\mathbf{B}_{r_k}$. On the other hand, if we extract a converging subsequence, not relabelled, we have $\mathbf{S}_k \to \mathbf{S}_\infty$ in the local Hausdorff topology for some $\mathbf{S}_\infty \in \mathscr{C}(Q, 0)$. Hence we also conclude that $\mathrm{spt}(T) \cap \mathbf{B}_1 \equiv \mathbf{S}_\infty \cap \mathbf{B}_1$. This implies that the cone $\mathbf{S}_\infty \in \mathscr{C}(Q, 0)$ is in fact the plane π_∞. In particular we conclude $\mathbf{E}^p(T_k, \mathbf{B}_1) \to 0$.

We can therefore apply Almgren's (strong) Lipschitz approximation to the sequence T_k and perform a blow-up procedure. We normalize the Lipschitz approximations by

$$E_k^{1/2} = \mathbf{E}^p(T_k, \mathbf{B}_1)^{1/2}$$

and, upon possibly applying suitable rotations (see Remark 8.21), extract a Dir-minimizing function $f_\infty : B_1(\pi_\infty) \to \mathcal{A}_Q(\pi_\infty^\perp)$ in the blow-up limit. Moreover, since $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \le (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, we see that if we describe the cones $\mathbf{S}_k$ as a union of graphs of linear maps $L_1^k, \ldots, L_N^k : \pi_\infty \to \pi_\infty^\perp$ then, as we have previously seen, upon extraction of a subsequence we will have that ${E_k}^{-1/2} L_i^k$ converges to some linear function L_i^∞ for each $i = 1, \ldots, N$. At least two of these linear functions are distinct, given that by Proposition 9.1(d) $\boldsymbol{\mu}(\mathbf{S})$ is comparable to $E_k^{1/2}$. But now this would imply that $C^{-1} \le \mathbf{E}^p(T_k, \mathbf{B}_{r_k})/\mathbf{E}^p(T_k, \mathbf{B}_1) \le C$ for some geometric constant $C > 0$, and hence we would have

$$\frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_{r_k})} \le C \frac{\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k})}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \le C \frac{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \le C(\varepsilon_0^k)^2 \to 0$$

which is in direct contradiction to (10.15).

Now we contradict (10.14). We distinguish two cases here, depending on whether $\mathbf{E}^p(T, \mathbf{B}_1)$ converges to 0 or not. In the latter case we can assume, after extracting a suitable subsequence, that $\mathbf{S}_k$ converges to a non-planar cone $\mathbf{S}_\infty \in \mathscr{C}(Q, 0) \setminus \mathscr{P}$ in the local Hausdorff topology which is the support of an area-minimizing current. But then it would follow that $\mathbf{S}_{r_k}$ converges to the same cone. In particular $V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1$ and $V(\mathbf{S}_k) \cap \mathbf{B}_1$ both converge to $V(\mathbf{S}_\infty) \cap \mathbf{B}_1$, contracting (10.14).

In the other case note that, since $\mathbb{E}(T_k, \mathbf{S}_{r_k}, \mathbf{B}_{r_k}) \le (\varepsilon_0^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, if we assume that the cones $\mathbf{S}_{r_k}$ are the union of $\bar{N}$ distinct linear maps $\bar{L}_i^k$, we see that ${E_k}^{-1/2} \bar{L}_i^k$ converge to a linear map $\bar{L}_i^\infty$ for each $i = 1, \ldots, \bar{N}$. Furthermore the union of the graphs of these limiting linear maps must coincide with the support of the map f_∞ found previously (indeed, we get from the above inequality that they must coincide on $\mathbf{B}_{\bar{r}}$, and hence on all of $\mathbf{B}_1$ as they are linear). But this is only possible if

$$\mathrm{dist}(V(\mathbf{S}_{r_k}) \cap \mathbf{B}_1, V(\mathbf{S}_k) \cap \mathbf{B}_1) \to 0$$

which contradicts (10.14). This completes the proof. □

We are now ready to complete the proof of Theorem 2.5 from Proposition 10.3. We first address the most important conclusion, which is the decay estimate in (a). We will see after how (b), (c), and (d) can be derived from this.

Proof of Theorem 2.5(a) from Proposition 10.3. We therefore fix T, Σ, $\mathbf{S}$, and ς as in the statement of Theorem 2.5, and fix parameters ε_0 and $r_0 < 1/2$ to be specified later.

Choose $\varsigma_2 = \frac{1}{2}$ in Proposition 10.3, and denote by $\varepsilon_2 = \varepsilon_2(Q, m, n, \bar{n})$ the corresponding constant from Proposition 10.3 for this choice of ς_2. Also denote by $r_1, \ldots, r_{\bar{N}} \le \frac{1}{2}$ the corresponding radii from Proposition 10.3 with this choice of ς_2. Write $\bar{r} := \min\{r_i : i = 1, \ldots, \bar{N}\}$, which depends only on $Q, m, n, \bar{n}$.

Define

$$\mathbb{E}(T, \mathbf{B}_r) := \inf\{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r) : \mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)\}$$

and

$$\mathcal{A} := \{r \in [r_0, 1/2] : \varepsilon_0^{-2} \mathbf{A}^2 r^2 \ge \mathbb{E}(T, \mathbf{B}_r)\}.$$

We now distinguish three possibilities.

Case 1: $\mathcal{A} = \emptyset$. We then have $\varepsilon_0^{-2} r^2 \mathbf{A}^2 \le \mathbb{E}(T, \mathbf{B}_r)$ for all $r \in [r_0, 1/2]$. If $\varepsilon_0 \le \varepsilon_2$, then at scale 1 the assumptions of Theorem 2.5 give that we can apply Proposition 10.3 to $T, \Sigma, \mathbf{S}$

to get that there exists $\rho_1 \in [\bar{r}, 1/2]$ (indeed, $\rho_1 = r_i$ for some $i = 1, \ldots, \bar{N}$, but this is not relevant) and a cone $\mathbf{S}_1 \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_{\rho_1}) \leq \frac{1}{2}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Let us assume (as we may) that $r_0 < \bar{r}$. Since $\mathcal{A} = \emptyset$, we therefore fall into the realm of Lemma 10.5, taking ε_2 and $\bar{r}$ to be the above constants. If ε_0 is smaller than the corresponding constant from Lemma 10.5 with this choice of ε_2 and $\bar{r}$ (and thus only depends on $Q, m, n, \bar{n}$), we see that Lemma 10.5 gives that T_{0,ρ_1}, Σ_{0,ρ_1}, and some cone $\mathbf{S}_1'$ (namely the cone achieving the infimum in $\mathbb{E}(T, \mathbf{B}_{\rho_1})$) obey the necessary conditions to apply Proposition 10.3 again. Thus, we find a second radius ρ_2 obeying $\rho_2/\rho_1 \in [\bar{r}, 1/2]$ and a cone $\mathbf{S}_2 \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ obeying

$$\mathbb{E}(T, \mathbf{S}_2, \mathbf{B}_{\rho_2}) \leq \frac{1}{2}\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_{\rho_1}) \leq \frac{1}{4}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

We now keep iterating this procedure, until we arrive at a radius ρ_k so that the next radius ρ_{k+1} is smaller than r_0 (in particular we no longer necessarily have that $\varepsilon_0^{-2}\mathbf{A}^2\rho_{k+1}^2 \leq \mathbb{E}(T, \mathbf{B}_{\rho_{k+1}})$); as $\rho_{k+1}/\rho_k \in [\bar{r}, 1/2]$, when this happens we have $\bar{r} \leq \rho_{k+1}/\rho_k < r_0/\rho_k$. Since we can apply Corollary 10.4 to T_{0,ρ_k}, Σ_{0,ρ_k}, and $\mathbf{S}_k$ (again, using the fact that $\rho_k \notin \mathcal{A} = \emptyset$ and Lemma 10.5), provided ε_2 is smaller than the corresponding constant from Corollary 10.4 with this choice of $\bar{r}$ (which can of course be arranged), we therefore get the existence of a cone $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that (choosing $r^* = r_0/\rho_k \in [\bar{r}, 1]$ in Corollary 10.4, so that $r^*\rho_k = r_0$)

$$\mathbb{E}(T_{0,\rho_k}, \mathbf{S}', \mathbf{B}_{r^*}) \leq C\mathbb{E}(T_{0,\rho_k}, \mathbf{S}_k, \mathbf{B}_1)$$

where $C = C(Q, m, n, \bar{n})$ (indeed the constant C depends on $\bar{r}$, but at this stage the latter has been fixed as depending only upon Q, m, n, and $\bar{n}$). In particular, combined with the above iteration, the last inequality yields

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq C\mathbb{E}(T, \mathbf{S}_k, \mathbf{B}_{\rho_k}) \leq C \cdot 2^{-k}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Observe however that as $\bar{r} < r_0/\rho_k < r_0/\bar{r}^k$, i.e. $r_0 > \bar{r}^{k+1}$, we have $k + 1 \geq \log(r_0)/\log(\bar{r})$. Hence, choosing $r_0 = r_0(Q, m, n, \bar{n})$ sufficiently small we can ensure that $k \geq \frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor$, and hence

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq C2^{-\frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1). \tag{10.16}$$

Now we may further take r_0 small enough to make sure that $C2^{-\frac{1}{2}\left\lfloor \frac{\log(r_0)}{\log(\bar{r})} \right\rfloor} \leq \varsigma$ to deduce the conclusion (a) of Theorem 2.5.

Case 2: $\inf \mathcal{A} = r_0$. Note that $\mathcal{A}$ is certainly a closed set, and so in this case we have $r_0 \in \mathcal{A}$. Hence, we have

$$\mathbb{E}(T, \mathbf{B}_{r_0}) \leq \varepsilon_0^{-2}\mathbf{A}^2 r_0^2 \leq r_0^2\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

where the second inequality comes from our assumption in Theorem 2.5. In particular, we can find $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ with

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq 2\mathbb{E}(T, \mathbf{B}_{r_0}) \leq 2r_0^2\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$$

and so we just need to take $2r_0^2 < \varsigma$ to deduce conclusion (a) of Theorem 2.5.

Case 3: $\mathcal{A} \neq \emptyset$ yet $\inf \mathcal{A} > r_0$. In this case, let $r_I = \inf \mathcal{A} \in \mathcal{A}$. Let $\mathbf{S}_I \in \mathscr{C}(Q, 0) \setminus \mathcal{P}(0)$ be such that

$$\mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}) \leq 2\mathbb{E}(T, \mathbf{B}_{r_I}) \leq 2\varepsilon_0^{-2}\mathbf{A}^2 r_I^2. \tag{10.17}$$

Now observe that

$$\lim_{r \uparrow r_I} \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_r) = \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I})$$

Moreover, as $[r_0, r_I) \neq \emptyset$ and $[r_0, r_I) \cap \mathcal{A} = \emptyset$, we see that for all $r \in [r_0, r_I)$,

$$\varepsilon_0^{-2} r^2 \mathbf{A}^2 \leq \mathbb{E}(T, \mathbf{B}_r) \leq \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_r).$$

Taking $r \uparrow r_I$ in this expression, we deduce

$$\varepsilon_0^{-2} r_I^2 \mathbf{A}^2 \leq \mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}).$$

Moreover, from (10.17) and our assumption (2.1) we know

$$\mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}) \le 2\varepsilon_0^{-2}\mathbf{A}^2 r_I^2 \le 2r_I^2\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1). \tag{10.18}$$

Hence, we have all the necessary assumptions to apply Lemma 10.5, which guarantees that we can apply Proposition 10.3 to T_{0,r_I}, Σ_{0,r_I}, and $\mathbf{S}_I$. We can now argue as in Case 1, but with r_I replacing 1 as a starting point of the iteration and with r_0/r_I as the lower endpoint. We therefore get the existence of an $\mathbf{S}' \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ with the property that (see (10.16))

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \le C2^{-\frac{1}{2}\frac{\log(r_0/r_I)}{\log(\bar r)}}\mathbb{E}(T, \mathbf{S}_I, \mathbf{B}_{r_I}).$$

Combining this with (10.18) we arrive at

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \le \left(2r_I^2 C 2^{\frac{1}{2}\frac{\log(r_I)}{\log(\bar r)}}\right) 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar r)}}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Noting that $2^{\frac{1}{2}\frac{\log(r_I)}{\log(\bar r)}} = \left(\frac{1}{r_I}\right)^{\frac{\log(2)}{2\log(1/\bar r)}} \le r_I^{-\frac{1}{2}}$, the above becomes

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \le \left(2Cr_I^{3/2}\right) 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar r)}}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \le 2C \cdot 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar r)}}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1).$$

Thus choosing r_0 small enough so that $2C \cdot 2^{-\frac{1}{2}\frac{\log(r_0)}{\log(\bar r)}} \le \varsigma$, we deduce (a) from Theorem 2.5. As the above three situations exhaust all possibilities, the proof is completed. ☐

To summarise: we have now reduced the proof of point (a) in Theorem 2.5 to proving the a priori much weaker result of Theorem 10.2. We will next address how points (b), (c), and (d) follow from (a). In fact, since it will prove to be useful also later when we get to the rectifiability statement in Theorem 1.1, we state here a more general lemma.

Lemma 10.6. *There is a constant $C = C(Q, m, n, \bar n) > 0$ with the following property. For every $r_0 > 0$, there exists $\varepsilon = \varepsilon(Q, m, n, \bar n, r_0) > 0$ such that if*

(i) *T and Σ are as in Assumption 2.1;*
(ii) *$\|T\|(\mathbf{B}_1) \le (Q + \frac{1}{2})\omega_m$; and*
(iii) *there is $\mathbf{S} \in \mathscr{C}(Q,0) \setminus \mathscr{P}(0)$ such that*

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \le \varepsilon^2\mathbf{E}^p(T, \mathbf{B}_1)\ ; \tag{10.19}$$

then

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \le \mathbf{E}^p(T, \mathbf{B}_{r_0}) \le C\mathbf{E}^p(T, \mathbf{B}_1)\,. \tag{10.20}$$

Moreover, there is a constant $\bar C = \bar C(Q, m, n, \bar n, r_0)$ such that, provided ε is chosen possibly smaller, if there is another cone $\mathbf{S}' \in \mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ that also obeys

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \le \varepsilon^2\mathbf{E}^p(T, \mathbf{B}_1)\,, \tag{10.21}$$

then

$$\operatorname{dist}^2(\mathbf{S} \cap \mathbf{B}_1, \mathbf{S}' \cap \mathbf{B}_1)) \le \bar C(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0})) \tag{10.22}$$

$$\operatorname{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_1, V(\mathbf{S}') \cap \mathbf{B}_1) \le \bar C\mathbf{E}^p(T, \mathbf{B}_1)^{-1}(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}))\,. \tag{10.23}$$

Proof. First of all we argue for (10.20). We start by demonstrating that

$$\boldsymbol{\mu}(\mathbf{S})^2 \le C\mathbf{E}^p(T, \mathbf{B}_1)\,. \tag{10.24}$$

In fact, fix π such that $\mathbf{E}^p(T, \mathbf{B}_1) = \hat{\mathbf{E}}(T, \pi, \mathbf{B}_1)$ and use Allard's L^2–L^∞ height bound to infer that

$$\operatorname{dist}^2(q, \pi) \le C\mathbf{E}^p(T, \mathbf{B}_1) + C\mathbf{A}^2 \qquad \forall q \in \operatorname{spt}(T) \cap \mathbf{B}_{1/2}\,. \tag{10.25}$$

Next, let α_i be one of the planes which form $\mathbf{S}$ and recall that

$$\int_{\alpha_i \cap \mathbf{B}_1 \setminus B_a(V)} \operatorname{dist}^2(q, \operatorname{spt}(T))\, d\mathcal{H}^m(q) \le \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,.$$

Using Chebyshev's inequality (applied to the measure $\mathcal{H}^m \llcorner \alpha_i$), we have that the set $F_i \subset \alpha_i \cap \mathbf{B}_{1/4} \setminus \mathbf{B}_a(V)$ of points $q \in \alpha_i \cap \mathbf{B}_{1/4} \setminus B_a(V)$ obeying

$$\mathrm{dist}^2(q, \mathrm{spt}(T)) \leq \frac{C}{\mathcal{H}^m(\alpha_i \cap \mathbf{B}_{1/4} \setminus \mathbf{B}_a(V))} \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \tag{10.26}$$

satisfies $\mathcal{H}^m(F_i) \geq \left(1 - \frac{1}{C}\right) \mathcal{H}^m(\alpha_i \cap \mathbf{B}_{1/4} \setminus \mathbf{B}_a(V))$ and so for a choice of $C = C(m) > 0$ sufficiently large, there exist m linearly independent vectors $e_1, \ldots, e_m \in F_i$ with the property that every $v \in \alpha_i \cap \mathbf{B}_1$ can be written as $v = \sum_j \lambda_j e_j$ with $|\lambda_j| \leq C$.

Combining the above distance inequalities, for each $j \in \{1, \ldots, m\}$ we have

$$\mathrm{dist}^2(e_j, \pi) \leq C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2 + \mathbf{E}^p(T, \mathbf{B}_1)) \leq C\mathbf{E}^p(T, \mathbf{B}_1) .$$

For each $v \in \alpha_i \cap \mathbf{B}_1$, using the identity $\mathrm{dist}(v, \pi) = |\mathbf{p}_\pi^\perp(v)|$, this in turn implies

$$\mathrm{dist}^2(v, \pi) \leq C\mathbf{E}^p(T, \mathbf{B}_1) .$$

We thus can use (7.2) of Corollary 7.3 to infer that

$$\mathrm{dist}^2(\alpha_i \cap \mathbf{B}_1, \pi \cap \mathbf{B}_1) \leq C\mathbf{E}^p(T, \mathbf{B}_1) , \tag{10.27}$$

which in turn implies (10.24).

Observe also that we have the inequality

$$\mathbf{E}^p(T, \mathbf{B}_1) \leq \hat{\mathbf{E}}(T, \alpha_1, \mathbf{B}_1) \leq C\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2 \leq C\varepsilon^2\mathbf{E}^p(T, \mathbf{B}_1) + C\boldsymbol{\mu}(\mathbf{S})^2 ,$$

and so in particular we conclude

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \leq \boldsymbol{\mu}(\mathbf{S})^2 \leq C\mathbf{E}^p(T, \mathbf{B}_1) , \tag{10.28}$$

if ε is sufficiently small. Moreover, since we have not used any information other than the smallness of $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2$ with respect to $\mathbf{E}^p(T, \mathbf{B}_1)$, the same argument applies to $\mathbf{S}'$ and T_{0,r_0}, which allows us to conclude

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_{r_0}) \leq \boldsymbol{\mu}(\mathbf{S}')^2 \leq C\mathbf{E}^p(T, \mathbf{B}_{r_0}) . \tag{10.29}$$

We next observe that from (10.24) (see also (10.27)) we immediately get the inequality on the right-hand side of (10.20). Indeed, we can write

$$\begin{aligned}\mathbf{E}^p(T, \mathbf{B}_{r_0}) &\leq \hat{\mathbf{E}}(T, \pi, \mathbf{B}_{r_0}) \leq Cr_0^{-2} \max_i \mathrm{dist}^2(\alpha_i \cap \mathbf{B}_{r_0}, \pi \cap \mathbf{B}_{r_0}) + C\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_{r_0}) \\ &\leq C\mathbf{E}^p(T, \mathbf{B}_1) + Cr_0^{-m-2}\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1) \leq C(1 + \varepsilon r_0^{-m-2})\mathbf{E}^p(T, \mathbf{B}_1)\end{aligned}$$

and so to conclude the right-hand inequality in (10.20) it suffices to choose ε small compared to r_0.

We next argue by contradiction for the left-hand inequality in (10.20). If the conclusion is false, then we could find a sequence of area-minimizing currents T_k, ambient manifolds Σ_k, cones $\mathbf{S}_k \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$, and parameters $\varepsilon^k \downarrow 0$ for which the left-hand inequality in (10.20) fails for some fixed r_0, namely

$$\lim_{k\to\infty} \frac{\mathbf{E}^p(T_k, \mathbf{B}_{r_0})}{\mathbf{E}^p(T_k, \mathbf{B}_1)} < C_*^{-1} , \tag{10.30}$$

for some constant constant C_* which will be specified only later and which will turn out to be independent of r_0. Without loss of generality, we can select planes π_k with $\mathbf{E}^p(T_k, \mathbf{B}_1) = \hat{\mathbf{E}}(T_k, \pi_k, \mathbf{B}_1)$, and, up to performing a rotation, we can assume they all coincide with the same fixed plane, π_∞. We can also pass to a subsequence to ensure that the number N_k of planes forming $\mathbf{S}_k$ is a constant N (obeying $N \leq Q$).

Observe that we must necessarily have

$$\mathbf{E}^p(T_k, \mathbf{B}_1) \to 0 .$$

Indeed, we can assume $\mathbf{S}_k \to \mathbf{S}_\infty \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ locally in the Hausdorff topology, while T_k converges weakly to some area-minimizing current T_∞ and it follows easily that $\mathrm{spt}(T_\infty) \cap \mathbf{B}_1 =$

$\mathbf{S}_\infty \cap \mathbf{B}_1$. If $\mathbf{E}^p(T_k, \mathbf{B}_1)$ does not converge to 0, then T_∞ is not supported in a plane, so it must be an area-minimizing integral cone which is not planar. But then we see immediately that

$$\lim_{k\to\infty} \frac{\mathbf{E}^p(T_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_{r_0})} = 1\,.$$

In particular this would be a contradiction if we impose $C_* \geq 1$ in (10.30).

We conclude then that $\mathbf{S}_k$ converges indeed to π_∞. We can therefore apply Almgren's (strong) Lipschitz approximation [11, Theorem 2.4] to the sequence T_k relative to the plane π_∞ and perform a blow-up procedure. We normalize the Lipschitz approximations by

$$E_k^{1/2} = \mathbf{E}^p(T_k, \mathbf{B}_1)^{1/2}$$

and, upon possibly applying suitable rotations (see Remark 8.21), extract a Dir-minimizing function $f_\infty : B_1(\pi_\infty) \to \mathcal{A}_Q(\pi_\infty^\perp)$ in the blow-up limit (the Dir-minimizing property follows from [11, Theorem 2.6]). Moreover, since $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1) \leq (\varepsilon^k)^2 \mathbf{E}^p(T_k, \mathbf{B}_1)$, we see that if we describe the cones $\mathbf{S}_k$ as a union of graphs of linear maps $L_1^k, \dots, L_N^k : \pi_\infty \to \pi_\infty^\perp$ then upon extraction of a subsequence we will have that $E_k^{-1/2} L_i^k$ converges to some linear function L_i^∞ for each $i = 1, \dots, N$, because of (10.24).

Observe that:

- L_i^∞ and L_j^∞ are not necessarily distinct for every $i \neq j$;
- However there is *at least one* pair of indices $i \neq j$ for which they are indeed distinct.

The second fact is a simple consequence of (10.28).

Up to reordering, we can assume that there are exactly $\bar N \geq 2$ distinct linear maps $L_1^\infty, \dots, L_{\bar N}^\infty$ in the collection $\{L_i^\infty\}$. Consider then the cones $\mathbf{S}'_k \subset \mathbf{S}_k$ given by the graph of the linear maps $L_1^k, \dots, L_{\bar N}^k$. It follows from the arguments outlined so far that

$$\lim_{k\to\infty} \frac{\hat{\mathbf{E}}(T_k, \mathbf{S}'_k, \mathbf{B}_1)}{\mathbf{E}^p(T_k, \mathbf{B}_1)} \to 0\,,$$

$$\liminf_{k\to\infty} \frac{\boldsymbol{\sigma}(\mathbf{S}'_k)}{\boldsymbol{\mu}(\mathbf{S}'_k)} > 0\,,$$

and

$$\lim_{k\to\infty} \frac{\boldsymbol{\mu}(\mathbf{S}_k)}{\boldsymbol{\mu}(\mathbf{S}'_k)} = 1\,.$$

Together with the fact that $C^{-1} \leq \frac{\mathbf{E}^p(T_k, \mathbf{B}_1)}{\boldsymbol{\mu}(\mathbf{S}_k)^2} \leq C$, we conclude that

$$\max_{1\leq i<j\leq \bar N} \int_{B_1} |L_i^\infty - L_j^\infty|^2 \geq 2C_*^{-1}$$

for a suitable choice of $C_* = C_*(Q, m, n, \bar n) > 0$.

We can now argue as in the proof of Proposition 9.3 to conclude that

$$f_\infty = \sum_{i=1}^{\bar N} Q_i [\![L_i^\infty]\!]$$

for some *positive* integers $Q_i \geq 1$.

Consider now for each k a plane π_k^1 such that $\hat{\mathbf{E}}(T_k, \pi_k^1, \mathbf{B}_{r_0}) = \mathbf{E}^p(T_k, \mathbf{B}_{r_0})$. By Allard's L^∞-L^2 bound, for each $v \in \pi_k^1 \cap \mathbf{B}_{r_0/2}$ there is a point $q \in \mathrm{spt}(T_k) \cap \mathbf{B}_{r_0}$ such that $\mathbf{p}_{\pi_k^1}(q) = v$ and

$$|v - p|^2 \leq C r_0^2 (\mathbf{E}^p(T_k, \mathbf{B}_{r_0}) + r_0^2 \mathbf{A}_k^2) \leq C r_0^{-m} \mathbf{E}^p(T_k, \mathbf{B}_1)\,.$$

However, by (10.25), $\mathrm{dist}^2(p, \pi_\infty) \leq C(\mathbf{E}^p(T_k, \mathbf{B}_1) + \mathbf{A}_k^2) \leq C\mathbf{E}^p(T_k, \mathbf{B}_1)$. It follows immediately that

$$\mathrm{dist}^2(\pi_k^1 \cap \mathbf{B}_{r_0}, \pi_\infty \cap \mathbf{B}_{r_0}) \leq C r_0^{-m} \mathbf{E}^p(T_k, \mathbf{B}_1)$$

which in turn implies

$$\mathrm{dist}^2(\pi_k^1 \cap \mathbf{B}_1, \pi_\infty \cap \mathbf{B}_1) \leq C r_0^{-m-2} \mathbf{E}^p(T_k, \mathbf{B}_1)\,.$$

Let $A_k : \pi_\infty \to \pi_\infty^\perp$ be the linear map whose graph gives π_k^1 and note that, from the above discussion, $\|A_k\|_{L^\infty(B_1(0,\pi_\infty))} \leq C r_0^{-1-m/2} E_k^{1/2}$. In particular we can extract an L^2 limit A_∞ of the maps $E_k^{-1/2} A_k$, up to subsequences. Now the estimates in [11, Theorem 2.4] imply

$$\lim_{k\to\infty} \frac{\mathbf{E}^p(T_k, \mathbf{B}_{r_0})}{\mathbf{E}^p(T_k, \mathbf{B}_1)} = r_0^{-m-2} \int_{B_{r_0}} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 = \int_{B_1} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 ,$$

where in the last equality we have used the 1-homogeneity of both f_∞ and A_∞. On the other hand by the triangle inequality,

$$\int_{B_1} \mathcal{G}(f_\infty, Q[\![A_\infty]\!])^2 \geq \frac{1}{2} \max_{1\leq i<j\leq \bar N} \int_{B_1} |L_i^\infty - L_j^\infty|^2 \geq C_*^{-1} .$$

Thus, we are in contradiction with (10.30), which concludes the proof of (10.20).

Step 1: Reduction via enlarging, balancing, and pruning. We will first show that one can without loss of generality assume also the following on $\mathbf{S}$ and $\mathbf{S}'$:

(B) $\mathbf{S}$ and $\mathbf{S}'$ are both M-balanced (with M the constant of Assumption 10.1);

(P) For a suitably small constant $\delta = \delta(Q, m, n, \bar n, r_0) > 0$ we have

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{10.31}$$

$$r_0^2 \mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_0}) \leq \delta^2 \boldsymbol{\sigma}(\mathbf{S}')^2 . \tag{10.32}$$

For the present reduction step, we will show the existence of another cone $\mathbf{S}_1 \in \mathscr{C}(Q, 0)$ such that:

(i) (B) and (10.31) above hold for $\mathbf{S}_1$ in place of $\mathbf{S}$;
(ii) $V(\mathbf{S}) = V(\mathbf{S}_1)$;
(iii) $\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) \leq \bar C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$, for a constant $\bar C = \bar C(Q, m, n, \bar n, \delta)$;
(iv) $\mathrm{dist}^2(\mathbf{S}_1 \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1)^2 \leq \bar C(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$;

We can then use the same argument to find a cone $\mathbf{S}_1'$ satisfying (i)–(iv) for T_{0,r_0} and $\mathbf{S}'$. Given this, it is easy to check that the assumptions of the lemma hold with $T, \mathbf{S}_1, \mathbf{S}_1'$ in place of $T, \mathbf{S}, \mathbf{S}'$ (up to controlled constant factors), and that if one can prove the result for $T, \mathbf{S}_1, \mathbf{S}_1'$, then the result follows for $T, \mathbf{S}, \mathbf{S}'$. Thus, after Step 1 we will have reduced the proof to the case where we can also assume the validity of (B) and (P).

In order to accomplish the task of this step, we observe that, by choosing ε in (10.19) small enough, T, Σ, and $\mathbf{S}$ satisfy all the requirements of Proposition 9.1, except possibly $\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$, for the parameter $\varepsilon_0 = \varepsilon_0(Q, m, n, \bar n)$ in Proposition 9.1. Let us assume for the moment that this does not hold, i.e.

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) < \varepsilon_0^{-2} \mathbf{A}^2 . \tag{10.33}$$

In this case we wish to deform $\mathbf{S}$ to another cone $\mathbf{S}_e \in \mathscr{C}(Q, 0)$ such that $\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) = \varepsilon_0^{-2} \mathbf{A}^2$ and $V(\mathbf{S}_e) = V(\mathbf{S}) =: V$. This can be accomplished as follows. We consider a plane $\pi \in \mathscr{P}(0)$ containing V and observe that, because of (10.19), provided that we take $\varepsilon < \sqrt{N}\varepsilon_0$, we have

$$N \hat{\mathbf{E}}(T, \pi, \mathbf{B}_1) > \varepsilon_0^{-2} \mathbf{A}^2 . \tag{10.34}$$

Next enumerate the (distinct) planes forming $\mathbf{S}$ as $\alpha_1, \ldots, \alpha_N$, and for each $i = 1, \ldots, N$ fix a continuous path of planes $\alpha_i(t)$ in $T_0\Sigma$ connecting $\alpha_i(0) = \alpha_i$ and $\alpha_i(1) = \pi$, with the property that $V \subset \alpha_i(t)$ for all $t \in [0, 1]$. The existence of the path can be reduced to the existence of a continuous path for the cross sections $V^\perp \cap \alpha_i(t) \subset V^\perp \cap T_0\Sigma$ and the existence of the latter is a consequence of the path connectedness of the Grassmannian of 2-planes in $\mathbb{R}^{2+\bar n}$).

Next we want to avoid that along the path we have $\alpha_i(t) = \alpha_j(t)$ for some $i \neq j$ and some $0 < t < 1$. To that end, denote by X the set of m-dimensional subspaces of $T_0\Sigma$ which contain V, and consider its N-fold product $X \times \cdots \times X$. Let $Z \subset X \times \cdots \times X$ be the set of elements $(\beta_1, \beta_2, \ldots, \beta_N)$ such that $\beta_i = \beta_j$ for some $i \neq j$. We then need to show that $(X \times \cdots \times X \setminus Z) \cup \{(\pi, \ldots, \pi)\}$ is path connected. Note that Z can be written as the union of $Z_{ij} := \{(\beta_1, \ldots, \beta_N) : \beta_i = \beta_j\}$. On the other hand the dimension of Z_{ij} equals the dimension of Z_{12}, while the latter can be written as $\Delta \times X \times \cdots \times X$ where $\Delta \subset X \times X$ is the diagonal

$\{(\beta,\beta) : \beta \in X\}$. Hence the codimension of Z in $X \times \cdots \times X$ equals the codimension of the diagonal Δ in $X \times X$. Since the codimension of Δ in $X \times X$ is the dimension of X, which is strictly larger than 1, we conclude that the codimension of Z in $X \times \cdots \times X$ is at least 2. This in particular implies the path connectedness of the desired set, meaning we can assume our path obeys $\alpha_i(t) \neq \alpha_j(t)$ for all $i \neq j$ and all $t \in (0,1)$.

Now let $t \mapsto \mathbf{S}(t) = \alpha_1(t) \cup \cdots \cup \alpha_N(t)$, $t \in [0,1]$, denote the above continuous path from $\mathbf{S}$ to π. Observe that by definition,

$$t \mapsto \hat{\mathbf{E}}(T, \mathbf{S}(t), \mathbf{B}_1) \quad \text{is continuous for } 0 \leq t \leq 1, \tag{10.35}$$

$$t \mapsto \hat{\mathbf{E}}(\mathbf{S}(t), T, \mathbf{B}_1) \quad \text{is continuous for } 0 \leq t < 1, \tag{10.36}$$

and

$$\lim_{t \uparrow 1} \hat{\mathbf{E}}(\mathbf{S}(t), T, \mathbf{B}_1) = N \hat{\mathbf{E}}(\pi, T, \mathbf{B}_1)\,.$$

These properties, combined with (10.33) and (10.34), are enough to claim the existence of a $t_e \in [0,1)$ such that for $\mathbf{S}_e := \mathbf{S}(t_e)$ we have

$$\frac{1}{2}\varepsilon_0^{-2}\mathbf{A}^2 < \mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2\,.$$

However, it might be that $\mathbf{S}_e$ is not an element of $\mathscr{C}(Q,0)$; this can only happen if the intersection of two of the planes forming $\mathbf{S}_e$ is strictly bigger than V (and so is $(m-1)$-dimensional as they do not coincide). Denote by $Z' \subset X \times \cdots \times X$ the set of elements $(\beta_1, \ldots, \beta_N)$ such that $\dim(\beta_i \cap \beta_j) \geq m-1$ for some $i \neq j$. Z' is closed and has non-empty interior in $X \times \cdots \times X$. In particular, if we perturb the planes forming $\mathbf{S}_e$ slightly (within the N-fold product space $X \times \cdots \times X$), in light of the continuity of the map $\tilde{\mathbf{S}} \mapsto \mathbf{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$, we can ensure that the inequality above holds and that at the same time the new cone, which we will abuse of notation and still denote by $\mathbf{S}_e$, is an element of $\mathscr{C}(Q,0)$.

We have now found a suitable cone $\mathbf{S}_e$ when the unfavourable assumption (10.33) holds. Notice that for such $\mathbf{S}_e$ we also have

$$\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2 \leq \varepsilon_0^{-2}\varepsilon^2 \hat{\mathbf{E}}(T, \mathbf{B}_1)$$

by (10.19), and thus if $\varepsilon^2 < \varepsilon_0^4$, we have that the assumptions of Proposition (9.1) hold for T and $\mathbf{S}_e$. If we are in the favourable situation where (10.33) does not hold, then we have

$$\mathbf{A}^2 \leq \varepsilon_0^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,,$$

and so we can simply set $\mathbf{S}_e := \mathbf{S}$, and again all the assumptions of Proposition 9.1 hold for T and $\mathbf{S}_e$.

Thus, we are in a position to apply Proposition 9.1 to T and $\mathbf{S}_e$ to find a cone $\mathbf{S}_1$ which satisfies (ii) and is M-balanced. However, it must be noticed that indeed Proposition 9.1 is proved by showing that $\mathbf{S}_1$ satisfies the smallness assumption (9.2) in Proposition 9.2, with the choice of $\varepsilon = \varepsilon(m,n,\bar{n},N)$ therein. In particular, since the argument of Proposition 9.1 allows one to ensure that $\mathbf{S}_1$ satisfies (9.2) with ε replaced by any $\delta \leq \varepsilon$, up to further decreasing ε_0 (dependent on δ), we get exactly (10.31), and moreover we are still left with the freedom of choosing δ.

By Proposition 9.1 we also know that

$$\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) \leq \bar{C}\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1)\,,$$

where $\bar{C} = \bar{C}(Q,m,n,\bar{n},\delta)$. On the other hand, by construction either $\mathbf{S}_e = \mathbf{S}$, or

$$\mathbb{E}(T, \mathbf{S}_e, \mathbf{B}_1) \leq \varepsilon_0^{-2}\mathbf{A}^2\,,$$

and in particular we infer (iii) (we stress that now $\varepsilon_0 = \varepsilon_0(Q,m,n,\bar{n},\delta)$).

So far we have proved that (i), (ii), and (iii) hold. Property (iv) would follow from Proposition 9.1 in the case $\mathbf{S}_e = \mathbf{S}$, but unfortunately this may not be the case and so we will have to provide a more subtle argument. In fact, we will use the validity of (B) and (P) for $\mathbf{S}_1$ to prove (iv) even when $\mathbf{S}_e \neq \mathbf{S}$ (for δ chosen sufficiently small).

We first assume that $\delta = \delta(Q, m, n, \bar{n}) > 0$ is sufficiently small so that Lemma 8.5 and Proposition 8.6 apply to $T_{0,1/4}$ and $\mathbf{S}_1$. We will show that this implies

$$\text{dist}^2(\mathbf{S}_1 \cap \mathbf{B}_1, \mathbf{S} \cap \mathbf{B}_1)^2 \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\,. \tag{10.37}$$

As observed, we would need to show this when $\mathbf{E}(T, \mathbf{S}, \mathbf{B}_1) < \varepsilon_0^{-2}\mathbf{A}^2$, but the argument is in fact more general and does not use the latter information. Combined with (iii), (10.37) proves (iv).

We enumerate the (distinct) planes $\beta_1, \ldots, \beta_{N'}$ forming $\mathbf{S}_1$, while we also recall the enumeration $\alpha_1, \ldots, \alpha_N$ for $\mathbf{S}$. Let $W_i \subset \mathbf{B}_4$ be the disjoint neighbourhoods of the planes β_i as in Lemma 8.5, and consider their homothetic rescalings $\tilde{W}_i := (W_i)_{0,4}$. Observe that $T \llcorner \mathbf{B}_{3/4} \setminus B_{1/32}(V)$ is supported in the union of the $\tilde{W}_i$ and let $T_i := T \llcorner \tilde{W}_i$. By Proposition 8.6(c) we know that

$$\text{dist}(p, \beta_i) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbf{A}^2)^{1/2} \qquad \text{for all } p \in \text{spt}(T_i)\,. \tag{10.38}$$

For each fixed $i = 1, \ldots, N'$, consider a unit vector $e_1 \in V^\perp \cap \beta_i$, let $\xi_1 := \frac{e_1}{4}$, and define the disk $B_i := B_{1/32}(\xi_1, \beta_i)$. By Proposition 8.6(f) we know that $((\mathbf{p}_{\beta_i})_\sharp T_i) \llcorner (B_{1/2}(0, \beta_i) \setminus B_{1/16}(V)) = Q_i [\![(B_{1/2}(0, \beta_i) \setminus B_{1/16}(V)]\!]$ for some integer $1 \leq Q_i \leq Q$, and thus

$$\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i)) \geq c_0 > 0 \tag{10.39}$$

for some geometric constant $c_0(m, n, \bar{n})$. Since

$$\int \text{dist}^2(p, \mathbf{S})\, d\|T_i\|(p) \leq \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,,$$

for a fixed constant $\tilde{C} = \frac{C_*}{c_0}$, with $C_* > 0$ to be determined, by Chebyshev's inequality and (10.39) we have

$$\|T_i\|(\{p \in \mathbf{p}_{\beta_i}^{-1}(B_i) : \text{dist}^2(p, \mathbf{S}) > \tilde{C}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\}) \leq C_*^{-1}\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i))\,.$$

Thus, the set $E_i \subset \mathbf{p}_{\beta_i}^{-1}(B_i) \cap \text{spt}\|T_i\|$ of points $p \in \mathbf{p}_{\beta_i}^{-1}(B_i)$ which obey

$$\text{dist}^2(p, \mathbf{S}) \leq \tilde{C}\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)\,, \tag{10.40}$$

has $\|T_i\|(E_i) \geq (1 - 1/C_*)\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i))$.

We then use Proposition 8.6(iv) to estimate

$$\begin{aligned}
Q_i \mathcal{H}^m(B_i \setminus \mathbf{p}_{\beta_i}(E_i)) &\leq C_0 \|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i \setminus \mathbf{p}_{\beta_i}(E_i))) + C_0(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1))^{1+\gamma} \\
&\leq C_0 \|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i) \setminus E_i) + C_0(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1))^{1+\gamma} \\
&\leq \frac{C_0}{C_*}\|T_i\|(\mathbf{p}_{\beta_i}^{-1}(B_i)) + C_0(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1))^{1+\gamma} \\
&\leq \frac{C_0}{C_*} Q_i \mathcal{H}^m(B_i) + C_0(\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1))^{1+\gamma}
\end{aligned}$$

where $C_0 = C_0(Q, m, n, \bar{n})$. Note however that $\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) \leq C_1(\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)$ for another constant $C_1 = C_1(Q, m, n, \bar{n}, \delta)$. Hence, if $\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$ is sufficiently small and C_* sufficiently large compared to C_0, we get

$$\mathcal{H}^m(\mathbf{p}_{\beta_i}(E_i)) \geq \frac{1}{2}\mathcal{H}^m(B_i)\,.$$

For each point $p \in E_i$ we let $j(p) \in \{1, \ldots, N\}$ be an index such that $\text{dist}(p, \alpha_{j(p)}) = \text{dist}(p, \mathbf{S})$. We then define $F_{i,j} := \mathbf{p}_{\beta_i}(\{p \in E_i : j(p) = j\})$. Clearly from the above lower bound on $\mathcal{H}^m(\mathbf{p}_{\beta_i}(E_i))$, there must be an index $j_* \in \{1, \ldots, N\}$ for which $\mathcal{H}^m(F_{i,j_*}) \geq \frac{1}{2N}\mathcal{H}^m(B_i)$. Fix now an arbitrary point w_1 in $F_{i,j*}$ and recall that, since it belongs to $B_i = B_{1/32}(\xi_1, \beta_i)$, $\frac{3}{8} \geq |w_1| \geq \frac{1}{8}$. Next,we let $r_* > 0$ and consider the open set

$$\Lambda(r_*) := \{w \in \beta_i : |w \cdot w_1|^2 > (1 - r_*^2)|w_1|^2|w|^2\}\,.$$

By the lower bound on the Hausdorff measure of F_{i,j_*} the set $F_{i,j*} \setminus \Lambda(r_*)$ must contain at least one point w_2 if we choose $r_* = r_*(m, N)$ appropriately.

Clearly w_2 enjoys as well the bound $\frac{3}{8} \geq |w_2| \geq \frac{1}{8}$. Moreover, since e_1 is orthogonal to V, the angles formed by the w_k and V must both be larger than a geometric constant. Finally, the sine of the angle formed by w_1 and w_2 is at least r_*. In particular we can complete the pair w_1, w_2 to a basis of β_i, via an orthonormal basis $v_1, \ldots, v_{m-2}$ of V. Thus, any vector $v \in \mathbf{B}_1 \cap \beta_i$ can be written as a linear combination

$$v = \sum_i \lambda_i v_i + \lambda_{m-1} w_1 + \lambda_m w_2$$

for a choice of λ_i which satisfy $|\lambda_i| \leq C$ for some constant $C = C(m, N)$.

For the points w_1, w_2, if we let p_1, p_2 denote points in E_i which obey $j(p) = j_*$ and $p_{\beta_i}(p_k) = w_k$ for $k = 1, 2$, it follows from (10.40) and (10.38) that, for $k = 1, 2$,

$$\begin{aligned}|\mathbf{p}^\perp_{\alpha_{j_*}}(w_k)|^2 &= \operatorname{dist}^2(w_k, \alpha_{j_*}) \leq 2\operatorname{dist}^2(w_k, p_k) + 2\operatorname{dist}^2(p_k, \alpha_{j_*}) \\ &= 2\operatorname{dist}^2(p_k, \beta_i) + 2\operatorname{dist}^2(p_k, \mathbf{S}) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2)\end{aligned}$$

where $C = C(Q, m, n, \bar{n}, \delta)$. In particular, using the linearity of $\mathbf{p}^\perp_{\alpha_{j_*}}$ and the fact that $V \subset \alpha_{j_*}$, this shows that

$$\operatorname{dist}^2(v, \alpha_{j_*}) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) \qquad \forall v \in \beta_i \cap \mathbf{B}_1 .$$

Summarizing our argument so far, for every plane β_i in $\mathbf{S}_1$ we can show the existence of a plane α_{j_*} in $\mathbf{S}$ such that

$$\operatorname{dist}^2(\beta_i \cap \mathbf{B}_1, \alpha_{j_*} \cap \mathbf{B}_1) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) . \tag{10.41}$$

We now would like to prove the converse; namely, if we fix any plane α_j forming $\mathbf{S}$, we look for a plane β_{i_*} in $\mathbf{S}_1$ satisfying (10.41) with i_*, j in place of i, j_*. This time, we choose a unit vector $f_1 \in V^\perp \cap \alpha_j$. We let $\zeta_1 = f_1/4$ and set $B_j := B_{1/32}(\zeta_1, \alpha_j)$. We then know that

$$\int_{B_j} \operatorname{dist}^2(q, \operatorname{spt}(T)) d\mathcal{H}^m(q) \leq \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) .$$

Once again applying Chebyshev's inequality, this time with $\mathcal{H}^m$, we get that if $F_j \subset B_j$ is the set of points $q \in B_j$ which obey

$$\operatorname{dist}^2(q, \operatorname{spt}(T)) \leq (2/\mathcal{H}^m(B_j))\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) , \tag{10.42}$$

then we have $\mathcal{H}^m(F_j) \geq \frac{1}{2}\mathcal{H}^m(B_j)$. Now for each $q \in F_j$ let $p(q) \in \operatorname{spt}(T)$ be such that $\operatorname{dist}(q, \operatorname{spt}(T)) = |q - p(q)|$. Thus, (10.42) tells us that if ε is chosen small enough, we can guarantee that $|q - p(q)| < 1/16$ and so in particular $p(q) \in \mathbf{B}_{3/4} \setminus B_{1/32}(V)$. It thus follows that $p(q) \in \tilde{W}_i$ for some $i \in \{1, \ldots, N'\}$. For each such i, let $F_{j,i}$ be the set of points $q \in F_j$ for which at least one such $p(q)$ belongs to $\tilde{W}_i$. Arguing as before, it follows that for some i_* we have $\mathcal{H}^m(F_{j,i_*}) \geq \frac{1}{2N'}\mathcal{H}^m(B_j)$. We can now argue as above that F_{j,i_*} contains two appropriate points w_1, w_2, complete the pair with vectors in V to form an appropriate base of α_j and use this base to prove

$$\operatorname{dist}^2(\alpha_j \cap \mathbf{B}_1, \beta_{i_*} \cap \mathbf{B}_1) \leq C(\mathbb{E}(T, \mathbf{S}_1, \mathbf{B}_1) + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) + \mathbf{A}^2) .$$

Combined with (10.41), this therefore completes the proof of (10.37) and hence the first step.

Step 2. Concluding from the reduction. Now that we have reduced to the setting where we may assume the validity of (B) and (P), we conclude the proof of the proposition by showing that the claimed conclusions hold under these additional assumptions, if we choose δ small enough. First of all we observe that, by (iv) in Step 1, (10.20), (10.28), and (10.29) we have

$$C^{-1}\mathbf{E}^p(T, \mathbf{B}_1) \leq \min\{\boldsymbol{\mu}(\mathbf{S})^2, \boldsymbol{\mu}(\mathbf{S}')^2\} \leq \max\{\boldsymbol{\mu}(\mathbf{S})^2, \boldsymbol{\mu}(\mathbf{S}')^2\} \leq C\mathbf{E}^p(T, \mathbf{B}_1) . \tag{10.43}$$

So we can appeal to Lemma 7.12 to conclude (10.23) from (10.22). We will thus focus on proving (10.22).

The latter task is similar to the one accomplished at the end of Step 1; while it is more complicated by the fact that $V(\mathbf{S})$ and $V(\mathbf{S}')$ might be different (unlike $V(\mathbf{S})$ and $V(\mathbf{S}_1)$ in Step 1), we can now take advantage of (B) and (P) for both (whereas previously, we only knew that $\mathbf{S}_1$ was balanced). Indeed, by choosing δ small enough we can ensure, using Lemma 8.5, Proposition 8.6 and Corollary 8.7 for $\mathbf{S}'$, $T_{0,r_0/4}$, and $\mathbf{S}$, $T_{0,1/4}$, respectively, that the following situation holds:

(i) If we enumerate the planes $\beta_1,\ldots,\beta_{N'}$ forming $\mathbf{S}'$ and set $V' := V(\mathbf{S}')$, then there are pairwise disjoint neighborhoods W_j' of $\beta_j \cap \mathbf{B}_{r_0/2} \setminus B_{r_0/16}(V')$ such that

$$\operatorname{spt}(T)\cap \mathbf{B}_{r_0/2}\setminus B_{r_0/16}(V') \subset \bigcup_j W_j'\,; \tag{10.44}$$

$$\operatorname{dist}^2(p,\beta_j) \le C(\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}) + \mathbf{A}^2 r_0^2)r_0^2 \qquad \forall p\in \operatorname{spt}(T)\cap W_j'\,; \tag{10.45}$$

$$\begin{aligned}(\mathbf{p}_{\beta_j})_\sharp(T\llcorner W_j'\cap \mathbf{p}_{\beta_j}^{-1}(B_{r_0/2}(0,\beta_j)\setminus B_{r_0/16}(V)))&\\ = Q_i[\![B_{r_0/2}(0,\beta_j)\setminus B_{r_0/16}(V)]\!]\,.&\end{aligned} \tag{10.46}$$

(ii) If we enumerate the planes $\alpha_1,\ldots,\alpha_N$ forming $\mathbf{S}$ and set $V = V(\mathbf{S})$, then there are pairwise disjoint neighborhoods W_i of $\alpha_i\cap\mathbf{B}_{1/2}\setminus B_{r_0/16}(V)$ such that

$$\operatorname{spt}(T)\cap\mathbf{B}_{1/2}\setminus B_{r_0/16}(V)\subset \bigcup_i W_i \tag{10.47}$$

$$\operatorname{dist}^2(p,\alpha_i)\le \bar C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2)r_0^2 \qquad \forall p\in\operatorname{spt}(T)\cap W_i\cap \mathbf{B}_{r_0} \tag{10.48}$$

$$\begin{aligned}(\mathbf{p}_{\alpha_i})_\sharp(T\llcorner W_i\cap \mathbf{p}_{\alpha_i}^{-1}(B_{r_0}(0,\alpha_i)\setminus B_{r_0/16}(V)))&\\ = Q_i[\![B_{r_0}(0,\alpha_i)\setminus B_{r_0/16}(V)]\!]\,.&\end{aligned} \tag{10.49}$$

We will now proceed to show that all these estimates imply, for each plane α_i in $\mathbf{S}$, the existence of a plane β_j in $\mathbf{S}'$ such that

$$\operatorname{dist}^2(\alpha_i\cap\mathbf{B}_1,\beta_j\cap\mathbf{B}_1)^2\le \bar C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_1)+\mathbf{A}^2)\,. \tag{10.50}$$

Since the argument can be symmetrized to switch the roles of α_i and β_j, this would conclude the proof of (10.22).

Without loss of generality we assume that $i=1$ and we start by fixing a vector $e\in\alpha_1$ with $|e| = \frac{r_0}{4}$ such that $\mathbf{B}_{2c_0r_0}(e)\subset \mathbf{B}_{r_0/2}\setminus(B_{r_0/16}(V)\cup B_{r_0/16}(V'))$, where $c_0=c_0(m)$ is a positive dimensional constant. For each point $q\in B_{c_0r_0}(e,\alpha_1)$, by (10.48) and (10.49) above we can find a point $p=p(q)\in\operatorname{spt}(T)\cap W_1$ such that

$$|p-q|^2\le C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2)r_0^2$$

and then by (10.44) we find W_j' (for some $j=j(q)\in\{1,\ldots,N'\}$) such that $p\in W_j'$. In particular, setting $q_j := \mathbf{p}_{\beta_j}(p)$, we conclude from (10.45) that

$$|q-q_j|^2\le C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,.$$

So, for each point $q\in B_{c_0r_0}(e,\alpha_1)$ we conclude that there is a plane $\beta_{j(q)}$ in $\mathbf{S}'$ such that

$$\operatorname{dist}(q,\beta_{j(q)})^2\le C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,. \tag{10.51}$$

Select now m linearly independent vectors $v_1,\ldots,v_m\in B_{c_0r_0}(e,\alpha_1)$ with the property that, if we set $e_i := \frac{v_i}{r_0}$, then every vector $v\in\mathbf{B}_1\cap\alpha_1$ is a linear combination

$$v=\sum_i \lambda_i e_i$$

with $|\lambda_i|\le C$, for some constant C which depends only on c_0 and therefore only on m.

In light of the above argument, for each vector v_i we may find a plane $\beta_{j(i)}$ in $\mathbf{S}'$ such that

$$\operatorname{dist}^2(v_i,\beta_{j(i)})\le C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,. \tag{10.52}$$

Now, we would like to achieve (10.52) but with *the same plane*, say $\beta_{j(1)}$, for each v_i, up to increasing the constant by a fixed amount. Suppose that $j(i)\neq j(1)$ for some $i>1$. Consider the line segment $[v_1,v_i]$ and parameterize it with a constant speed curve $\gamma:[0,1]\to[v_1,v_i]$ with $\gamma(0)=v_1$ and $\gamma(1)=v_i$. By continuity of $t\mapsto\operatorname{dist}(\gamma(t),\beta_{j(1)})$, for some $\sigma\in(0,1]$ we have

$$\operatorname{dist}^2(\gamma(t),\beta_{j(1)})\le 2C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2 \qquad \forall t\in[0,\sigma]\,.$$

Now let τ be the maximal number in $[0,1]$ such that

$$\mathrm{dist}^2(\gamma(\tau),\beta_{j(1)}) \leq 2C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,. \tag{10.53}$$

If $\tau=1$ then we indeed arrive at (10.52) with $\beta_{j(i)}$ replaced by $\beta_{j(1)}$ and C replaced by $2C$. Otherwise, if $\tau<1$, then we must have equality in (10.53). Since $\gamma(\tau)\in B_{c_0r_0}(e,\alpha_1)$, (10.51) implies that there must be another index $j'=j'(\tau)\neq j(1)$ such that

$$\mathrm{dist}^2(\gamma(\tau),\beta_{j'}) \leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,. \tag{10.54}$$

If $p_{j(1)}=\mathbf{p}_{\beta_{j(1)}}(\gamma(\tau))$ and $p_{j'}=\mathbf{p}_{\beta_{j'}}(\gamma(\tau))$ are the respective nearest point projections of $\gamma(\tau)$, and if we ensure that ε is taken to be small enough, we can guarantee that both points belong to $\mathbf{B}_{2c_0r_0}(e)$. Recall that our choice of the vector e guarantees that this ball does not intersect $B_{r_0/16}(V')$. In particular, if we set $q_{j(1)}=p_{j(1)}/r_0$ and $q_{j'}=p_{j'}/r_0$ we find $q_{j(1)}\in\beta_{j(1)}\cap\mathbf{B}_1\setminus B_{1/16}(V')$ and $q_{j'}\in\beta_{j'}\cap\mathbf{B}_1\setminus B_{1/16}(V')$ are such that

$$|q_{j(1)}-q_{j'}|^2\leq 6C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,.$$

Since the pair of planes have V' as common spine and are M-balanced, we can combine the above with Corollary 7.3 (namely, first use (7.2), then M-balanced, then the first inequality in (7.1), then the above) to conclude that

$$\mathrm{dist}^2(\beta_{j(1)}\cap\mathbf{B}_1,\beta_{j'}\cap\mathbf{B}_1)\leq C'(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})) \tag{10.55}$$

for some constant C' which depends on M and the previous constant C.

We now look at the point τ' which is the maximum in the set of $t\in[0,1]$ for which

$$\mathrm{dist}^2(\gamma(t)),\beta_{j'})^2\leq 2C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,, \tag{10.56}$$

where C is again the constant in (10.51). Now τ' must be strictly larger than τ, by (10.54). Moreover, if $\tau'<1$ then

$$\mathrm{dist}^2(\gamma(\tau'),\beta_{j'}) = 2C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,. \tag{10.57}$$

In this case we can look at the closest plane $\beta_{j''}$ to $\gamma(\tau')$, which satisfies

$$\mathrm{dist}^2(\gamma(\tau'),\beta_{j''}) \leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))r_0^2\,, \tag{10.58}$$

where C is the constant in (10.51). Obviously by construction $j''\notin\{j(1),j'\}$. But then we can repeat the argument above with j' replacing $j(1)$ and j'' replacing j' and estimate

$$\mathrm{dist}^2(\beta_{j'}\cap\mathbf{B}_1,\beta_{j''}\cap\mathbf{B}_1)\leq C'(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0})) \tag{10.59}$$

with the same constant C' of (10.54). We can iterate this procedure until we reach the right endpoint $t=1$. Any time that we do not stop, we find a *new* plane in the collection forming $\mathbf{S}'$. Since there is a finite number N' of such planes, we conclude that the procedure stops after at most $N'-1$ steps. The last plane we find is a plane β_{κ_i} which satisfies (10.52) with $2C$ in place of C and β_{κ_i} in place of $\beta_{j(i)}$. On the other hand we have a chain of at most $N'-1$ inequalities of the form (10.59) between each pair of consecutive planes found during the procedure. In particular for every i we have

$$\mathrm{dist}^2(\beta_{j(1)}\cap\mathbf{B}_1,\beta_{\kappa_i}\cap\mathbf{B}_1)\leq (N'-1)^2C'(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,,$$

which combined with (10.52) gives that (10.52) holds with $\beta_{j(1)}$ in place of $\beta_{j(i)}$ with a larger constant C, for every i, which is what we wanted. Summarizing the discussion above, we have found a larger constant C such that, if we set $j=j(1)$, then

$$\mathrm{dist}^2(e_i,\beta_j)=|\mathbf{p}^\perp_{\beta_j}(e_i)|^2\leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))$$

for every i. Writing an arbitrary vector in $\mathbf{B}_1\cap\alpha_1$ in terms of the basis e_i as described above, we then conclude that

$$\mathrm{dist}^2(\alpha_1\cap\mathbf{B}_1,\beta_j\cap\mathbf{B}_1)^2\leq C(\mathbb{E}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2r_0^2+\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_0}))\,.$$

This completes the proof of (10.50) and thus completes the proof of the lemma. □

Thus, to prove Theorem 2.5, now we just need to prove Theorem 10.2.

11. Estimates at the spine

In this section we address the pivotal estimates needed at the spine of the cone for the proof of Theorem 2.5, which are a suitable adaptation of the groundbreaking work of Simon in [29]. They will be crucial for demonstrating that in the end, after a blow-up procedure under the assumption that Q-points accumulate across the spine, the graphical approximations constructed in Proposition 8.20 will remain controlled as one approaches the spine, and will converge to a Dir-minimizer that has an $(m-2)$-dimensional subspace of Q-points. We start by detailing the assumptions which will be used through this section. Note that, in contrast to [6], we denote the estimates in Theorem 11.2, Corollary 11.3 and Proposition 11.4 collectively as *Simon's estimates*. Indeed, although the first occurrence of an estimate analogous to (11.2) below first appeared in the work of Hardt and Simon in [18], the framework therein is significantly simpler. The first appearance of the estimates (11.2), (11.3), (11.4) and (11.5), albeit in a multiplicity one setting, is in [29]; the corresponding estimates in a setting of higher multiplicity, as is the case in the present work, first appeared in [32]. We note that (11.6) is a more refined version of an estimate in [29], but in this form it appeared first in [32]. One important difference between our work and [29, 32] is that, even when we are at a fixed distance from the spine of the cone $\mathbf{S}$ and the excess of the current from $\mathbf{S}$ is very small, the current is not necessarily a graph, nor a multigraph (it is merely approximated suitably by the latter), and therefore we need to deal with appropriate additional error estimates coming from regions of non-graphicality.

Throughout this section, we will work under an analogous assumption to Assumption 8.10, but with possibly smaller parameters:

Assumption 11.1 (Assumptions for Simon's estimates)**.** Suppose T and Σ are as in Assumption 2.1 and $\|T\|(\mathbf{B}_4) \leq 4^m(Q+\frac{1}{2})\omega_m$. Suppose $\mathbf{S} = \alpha_1 \cup \cdots \cup \alpha_N$ is a cone in $\mathscr{C}(Q)\setminus\mathscr{P}$ which is M-balanced, where $M > 0$ is a given fixed constant, and V is the spine of $\mathbf{S}$. For a sufficiently small constant $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ smaller than the ε-threshold in Assumption 8.10, whose choice will be fixed by the statements of Theorem 11.2, Corollary 11.3, and Proposition 11.4 below, suppose that

$$\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 . \tag{11.1}$$

We recall once again the notation

$$\boldsymbol{\sigma}(\mathbf{S}) := \min_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) , \qquad \text{and} \qquad \boldsymbol{\mu}(\mathbf{S}) := \max_{i<j} \operatorname{dist}(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) .$$

Before stating the inequalities that we need, let us introduce the following short-hand notation for points $q \in \operatorname{spt}(T)$ at which the m-rectifiable set $\operatorname{spt}(T)$ has an approximate tangent plane $\pi(q)$ oriented by the simple m-vector $\vec{T}(q)$:

- $\mathbf{p}_{\vec{T}}$ and $\mathbf{p}_{\vec{T}}^{\perp}$ will denote the orthogonal projections onto $\pi(q)$ and its orthogonal complement $(\pi(q))^{\perp}$ respectively;
- $q_{\parallel}$ and $q^{\perp}$ will denote the vectors $\mathbf{p}_{\vec{T}}(q)$ and $\mathbf{p}_{\vec{T}}^{\perp}(q)$ respectively.

We now split the key estimates into three separate statements and we will proceed with their proofs afterwards.

Theorem 11.2 (Simon's error and gradient estimates)**.** *Assume T, Σ, and $\mathbf{S}$ are as in Assumption 11.1, suppose in addition that $\Theta(T, 0) \geq Q$ and set $r = \frac{1}{3\sqrt{m-2}}$. Then there is a constant $C = C(Q, m, n, \bar{n}, M) > 0$ and a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M) > 0$ in Assumption 11.1 sufficiently small such that*

$$\int_{\mathbf{B}_r} \frac{|q^{\perp}|^2}{|q|^{m+2}} \, d\|T\|(q) \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{11.2}$$

$$\int_{\mathbf{B}_r} |\mathbf{p}_V \circ \mathbf{p}_{\vec{T}}^{\perp}|^2 \, d\|T\| \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) . \tag{11.3}$$

We refer to the first estimate (11.2) as "Simon's error estimate", since it is estimating the error in the monotonicity formula. Meanwhile, we refer to the second estimate (11.3) as "Simon's gradient estimate", since if T were the graph of a function, this would be an L^2 bound on the derivative of the function in the directions parallel to V.

An important corollary which will not require much additional work is the following.

Corollary 11.3 (Simon's non-concentration estimate)**.** *Assume T, Σ, $\mathbf{S}$ and r are as in Theorem 11.2. Then, there is a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ in Assumption 11.1, possibly smaller than that in Theorem 11.2, such that for every $\kappa \in (0, m+2)$,*

$$\int_{\mathbf{B}_r} \frac{\operatorname{dist}^2(q, \mathbf{S})}{|q|^{m+2-\kappa}} \, d\|T\|(q) \leq C_\kappa (\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,, \tag{11.4}$$

where here $C_\kappa = C_\kappa(Q, m, n, \bar{n}, M, \kappa)$.

Finally, this part will be concluded by deriving the following consequence of Corollary 11.3, which will require a subtle geometric consideration.

Proposition 11.4 (Simon's shift inequality)**.** *Assume T, Σ, and $\mathbf{S}$ are as in Assumption 11.1 and in addition $\{\Theta(T, \cdot) \geq Q\} \cap \mathbf{B}_\varepsilon(0) \neq \emptyset$. Then there is a radius $r = r(Q, m, n, \bar{n})$ and a choice of $\varepsilon = \varepsilon(Q, m, n, \bar{n}, M)$ in Assumption 11.1, possibly smaller than those in Theorem 11.2 and Corollary 11.3 such that for each $\kappa \in (0, m+2)$, there are constants $\bar{C}_\kappa = \bar{C}_\kappa(Q, m, n, \bar{n}, M, \kappa) > 0$ and $C = C(Q, m, n, \bar{n}, M)$ such that the following holds. If $q_0 \in \mathbf{B}_r(0)$ and $\Theta(T, q_0) \geq Q$, then*

$$\int_{\mathbf{B}_{4r}(q_0)} \frac{\operatorname{dist}^2(q, q_0 + \mathbf{S})}{|q - q_0|^{m+2-\kappa}} \, d\|T\|(q) \leq \bar{C}_\kappa (\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,. \tag{11.5}$$

$$|\mathbf{p}_{\alpha_1}^\perp(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2 |\mathbf{p}_{V^\perp \cap \alpha_1}(q_0)|^2 \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \,. \tag{11.6}$$

Remark 11.5. Observe that $\mathbf{p}_V^\perp = \mathbf{p}_{\alpha_1}^\perp + \mathbf{p}_{V^\perp \cap \alpha_1}$. In particular, when we know that $\mathbf{S}$ is at a fixed positive distance from any plane (namely, $\boldsymbol{\mu}(\mathbf{S})$ is bounded from below away from 0), then (11.6) gives a control on how far q_0 can be from the spine V. This particular case corresponds to what Simon proves in [29], while, as already mentioned, (11.6) is a refinement which appears first in the work [32] of Wickramasekera.

11.1. Proof of the Simon's error and gradient estimates (Theorem 11.2).

11.1.1. *Monotonicity formula.* We begin by recalling the monotonicity formula for mass ratios, along with some consequences. For T as in Assumption 2.1 and $\rho \in (0, 4]$, the monotonicity formula reads

$$\int_{\mathbf{B}_\rho} \frac{|q^\perp|^2}{|q|^{m+2}} \, d\|T\|(q) = \frac{\|T\|(\mathbf{B}_\rho)}{\rho^m} - \omega_m \Theta(T, 0) - \frac{1}{m} \int_{\mathbf{B}_\rho} (q^\perp \cdot \vec{H}_T(q))(|q|^{-m} - \rho^{-m}) \, d\|T\|(q) \,.$$

The mean curvature vector $\vec{H}_T(q)$ is defined as

$$\vec{H}_T(q) := \sum_{i=1}^m A_\Sigma(e_i, e_i) \,,$$

where A_Σ is the second fundamental form of Σ and $e_1, \ldots, e_m$ is an orthonormal base of the approximate tangent $\pi(q)$ to $\operatorname{spt}(T)$ at q.

We next assume that $\Theta(T, 0) \geq Q$, fix the multiplicities $Q_1, \ldots, Q_N$ given by Proposition 8.19 and observe that for ρ as above, we have

$$\sum_i Q_i \mathcal{H}^m(\alpha_i \cap \mathbf{B}_\rho) = Q \omega_m \rho^m \leq \omega_m \Theta(T, 0) \rho^m \,. \tag{11.7}$$

Fix $r = \frac{1}{3\sqrt{m-2}}$ as in the statement and note that any constants depending on r become in turn dimensional constants. We moreover fix a smooth, monotone non-increasing function $\chi : [0, \infty) \to \mathbb{R}$ with $\chi \equiv 1$ on $[0, r]$ and $\chi \equiv 0$ on $[2r, \infty)$ and we introduce the function

$$\Gamma(t) := -\int_t^\infty \frac{d}{ds}(\chi(s)^2) s^m \, ds \,.$$

Recall the elementary equalities

$$\int_{\mathbf{B}_R} \chi(|q|)^2 \, d\mu(q) = \int_0^R \chi(t)^2 \frac{d}{dt}(\mu(\mathbf{B}_t)) \, dt \tag{11.8}$$

$$\int_{\mathbf{B}_R} \Gamma(|q|)\, d\mu(q) = \int_0^R \chi(t)^2 \frac{d}{dt}(t^m \mu(\mathbf{B}_t))\, dt\,, \tag{11.9}$$

which are valid for any Radon measure μ with $\mu(\{0\}) = 0$ via Fubini's Theorem and the definition of the distributional derivative of the BV function $t \mapsto \mu(\mathbf{B}_t)$. We then multiply both sides of the monotonicity formula by ρ^m, differentiate the resulting identity in ρ, multiply by $\chi(\rho)^2$, integrate over $\rho \in [0, 2r]$ and use (11.8), (11.9), and (11.7) (treating, for example, $\frac{|q^\perp|^2}{|q|^{m+2}} d\|T\|(q)$ as a Radon measure $d\mu$ as above). Since $\Gamma \geq C^{-1}\mathbf{1}_{[0,r]}$ for some constant $C(m,r) > 0$, we get

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) \leq C\left[\int \chi^2(|q|)\, d\|T\|(q) - \sum_i Q_i \int_{\alpha_i} \chi^2(|q|)\, d\mathcal{H}^m(q)\right] + C \underbrace{\int \frac{|\Gamma(|q|) q^\perp \cdot \vec{H}_T(q)|}{|q|^m}\, d\|T\|(q)}_{=:\,(\mathrm{A})}\,. \tag{11.10}$$

We first demonstrate that the error term (A) coming from the mean curvature of T may be controlled by $\mathbf{A}^2$ and to that end we observe that

$$|\vec{H}_T(q)| \leq m\mathbf{A}\,, \tag{11.11}$$

$$|T_q\Sigma - T_0\Sigma| \leq C\mathbf{A}|q| \tag{11.12}$$

$$|\mathbf{p}^\perp_{T_q\Sigma}(q)| \leq C\mathbf{A}|q| \qquad \forall q \in \Sigma\,. \tag{11.13}$$

Indeed the first inequality is obvious by the very definition of $\vec{H}_T$, while the second and the third are a simple exercise in differential geometry and we leave them to the reader.

Observe that since T is area-minimizing in Σ, $\vec{H}_T(q)$ is orthogonal to $T_q\Sigma$, while $\pi(q) \subset T_q\Sigma$, so

$$q^\perp \cdot \vec{H}_T(q) = q \cdot \vec{H}_T(q) = \mathbf{p}^\perp_{T_q\Sigma}(q) \cdot \vec{H}_T(q)\,. \tag{11.14}$$

In particular, since $\mathrm{spt}(T) \subset \Sigma$, (11.11) and (11.13) imply

$$(\mathrm{A}) \leq C\mathbf{A}^2 \int_{\mathbf{B}_r} |q|^{1-m}\, d\|T\|(q) \leq C\mathbf{A}^2$$

where again $C = C(m,r) > 0$. We can therefore write

$$\int_{\mathbf{B}_r} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) \leq C\left[\int \chi^2(|q|)\, d\|T\|(q) - \sum_i Q_i \int_{\alpha_i} \chi^2(|q|)\, d\mathcal{H}^m(q)\right] + C\mathbf{A}^2. \tag{11.15}$$

We now wish to estimate the first term on the right hand side of (11.15) in terms of the excess $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_4)$; to achieve this, we will test the first variation formula for both T and the current whose support is $\mathbf{S}$ with the vector field

$$X(q) = \chi(|q|)^2 \mathbf{p}^\perp_V(q)\,.$$

The first variation formula for T reads

$$\int \mathrm{div}_{\vec{T}}\, X(q)\, d\|T\|(q) = -\underbrace{\int X^\perp(q) \cdot \vec{H}_T(q)\, d\|T\|(q)}_{=:\,(\mathrm{B})}\,, \tag{11.16}$$

where we use the shorthand notation $X^\perp(q) = \mathbf{p}^\perp_{\vec{T}}(X(q))$ and

$$\mathrm{div}_{\vec{T}}\, X(q) = \sum_i \partial_{e_i} X(q) \cdot e_i \tag{11.17}$$

for any orthonormal base $e_1, \ldots, e_m$ of the approximate tangent $\pi(q)$ of $\mathrm{spt}(T)$.

Writing $x = \mathbf{p}^\perp_V(q)$, we can now use (11.11) to write

$$|X^\perp(q) \cdot \vec{H}_T(q)| = \chi(|q|)^2 |\mathbf{p}^\perp_{T_q\Sigma}(x) \cdot \vec{H}_T(q)| \leq m\mathbf{A}|\mathbf{p}^\perp_{T_q\Sigma}(x)|\,.$$

Observe next that $x = \mathbf{p}_V^\perp(q) = q - \mathbf{p}_V(q)$ and $V \subset T_0\Sigma$. Therefore for $q \in \mathrm{spt}(T)$ we can use (11.12) and (11.13) to estimate

$$|\mathbf{p}_{T_q\Sigma}^\perp(x)| \le |\mathbf{p}_{T_q\Sigma}^\perp(q)| + |\mathbf{p}_{T_q\Sigma}^\perp(\mathbf{p}_V(q))| \le C\mathbf{A}|q| + |\mathbf{p}_{T_q\Sigma}^\perp - \mathbf{p}_{T_0\Sigma}^\perp||\mathbf{p}_V(q)| \le C\mathbf{A}|q|\,.$$

In particular

$$|(\mathrm{B})| \le C\mathbf{A}^2 \int \chi(|q|)^2|q|\,d\|T\|(q) \le C\mathbf{A}^2\,,$$

which in turn leads to

$$\int \mathrm{div}_{\vec{T}} X(q)\,d\|T\|(q) = O(\mathbf{A}^2)\,. \tag{11.18}$$

For $x = \mathbf{p}_V^\perp(q)$, $q \in \mathrm{spt}(T)$, let $\mathbf{p}_{\vec{T}}(x) \equiv \mathbf{p}_{\pi(q)}(x)$ denote the orthogonal projection of x onto the approximate tangent plane $\pi(q)$ at q. We next compute $\mathrm{div}_{\vec{T}}\, X$:

$$\mathrm{div}_{\vec{T}}\, X(q) = 2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|) + \chi(|q)^2\mathrm{div}_{\vec{T}}x\,.$$

Let us now compute $\mathrm{div}_{\vec{T}}x$. Complete $e_1,\dots,e_m$ to an orthonormal basis $e_1,\dots,e_m,\nu_1\ \dots,\nu_n$ of $\mathbb{R}^{m+n}$, and compute

$$\begin{aligned}\mathrm{div}_{\vec{T}}x &= \sum_i e_i\cdot\partial_{e_i}x = \sum_i e_i\cdot\mathbf{p}_V^\perp(e_i)\\ &= m - \sum_i e_i\cdot\mathbf{p}_V(e_i) = m - \mathrm{tr}\,(\mathbf{p}_V) + \sum_j \nu_j\cdot\mathbf{p}_V(\nu_j) = 2 + \sum_j \nu_j\cdot\mathbf{p}_V(\nu_j)\,,\end{aligned}$$

where the last equality follows from the fact that V is $(m-2)$-dimensional. Let us now rewrite the sum on the right-hand side in a more convenient form. Observe that

$$\mathrm{tr}\,(\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp) = \mathrm{tr}\,(\mathbf{p}_{\vec{T}}^\perp\circ\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)\,,$$

and also, since $\mathbf{p}_V^T = \mathbf{p}_V$, $\mathbf{p}_{\vec{T}}^T = \mathbf{p}_{\vec{T}}$, and $\mathbf{p}_V\circ\mathbf{p}_V = \mathbf{p}_V$, we have

$$\begin{aligned}|\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2 &= \mathrm{tr}\,((\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)^T\circ(\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)) = \mathrm{tr}\,((\mathbf{p}_{\vec{T}}^\perp)^T\circ\mathbf{p}_V^T\circ\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)\\ &= \mathrm{tr}\,(\mathbf{p}_{\vec{T}}^\perp\circ\mathbf{p}_V\circ\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp) = \mathrm{tr}\,(\mathbf{p}_{\vec{T}}^\perp\circ\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)\,,\end{aligned}$$

Thus, we deduce that

$$\sum_j \nu_j\cdot\mathbf{p}_V(\nu_j) = \mathrm{tr}\,(\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp) = |\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2\,,$$

so in summary,

$$\mathrm{div}_{\vec{T}}\, X(q) = 2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|) + \chi(|q|)^2(2 + |\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2)\,. \tag{11.19}$$

Plugging this into (11.18) we then conclude that

$$\int \chi(|q|)^2(2 + |\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2)\,d\|T\|(q) \le C\mathbf{A}^2 - \int 2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|)\,d\|T\|(q)\,. \tag{11.20}$$

Since $\mathrm{id} = \mathbf{p}_V + \mathbf{p}_V^\perp = \mathbf{p}_{\vec{T}} + \mathbf{p}_{\vec{T}}^\perp$ and $x = \mathbf{p}_V^\perp(q)$, we have

$$\begin{aligned}\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|) &= \mathbf{p}_{\vec{T}}(x)\cdot\mathbf{p}_V(\nabla\chi(|q|)) + \mathbf{p}_{\vec{T}}(x)\cdot\mathbf{p}_{V^\perp}(\nabla\chi(|q|))\\ &= -\mathbf{p}_{\vec{T}}^\perp(x)\cdot\mathbf{p}_V(\nabla\chi(|q|)) + \mathbf{p}_{\vec{T}}(x)\cdot\mathbf{p}_{V^\perp}(\nabla\chi(|q|))\end{aligned} \tag{11.21}$$

On the other hand, once again using that $\mathbf{p}_V^T = \mathbf{p}_V$, we have

$$|\mathbf{p}_{\vec{T}}^\perp(x)\cdot\mathbf{p}_V(\nabla\chi(|q|))| = |(\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp)(\mathbf{p}_{\vec{T}}^\perp(x))\cdot\nabla\chi(|q|)| \le C|\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp||\mathbf{p}_{\vec{T}}^\perp(x)|\,.$$

Introducing the short-hand notation $x^\perp$ for $\mathbf{p}_{\vec{T}}^\perp(x)$, as well as $\nabla_V\chi(|q|)$ and $\nabla_{V^\perp}\chi(|q|)$ for $\mathbf{p}_V(\nabla\chi(|q|))$ and $\mathbf{p}_{V^\perp}(\nabla\chi(|q|))$ respectively, we arrive at

$$\begin{aligned}-2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla\chi(|q|) &\le -2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|) + C|\chi(|q|)||\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp||x^\perp|\\ &\le -2\chi(|q|)\,\mathbf{p}_{\vec{T}}(x)\cdot\nabla_{V^\perp}\chi(|q|) + \frac{1}{2}\chi(|q|)^2|\mathbf{p}_V\circ\mathbf{p}_{\vec{T}}^\perp|^2 + C|x^\perp|^2\mathbf{1}_{\mathbf{B}_{2r}}\,.\end{aligned} \tag{11.22}$$

Inserting the latter inequality in (11.20) we arrive at

$$\int \chi^2(|q|)\left(1+\tfrac{1}{4}|\mathbf{p}_V\circ\mathbf{p}_{\vec T}^\perp|^2\right)d\|T\|(q)\le C\mathbf{A}^2+C\int_{\mathbf{B}_{2r}}|x^\perp|^2d\|T\| \\ -\int \chi(|q|)\,\mathbf{p}_{\vec T}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\,. \tag{11.23}$$

Consider next the current $S:=\sum_i Q_i[\![\alpha_i]\!]$, where we recall that the Q_i are the multiplicities on the outer region. Observe that, since $X(q)=\chi(|q|)^2x$, if $\{\Phi_t\}_t$ denotes the one-parameter family of diffeomorphisms generated by X, then $(\Phi_t)_\sharp S=S$ for each t, since $\mathbf{S}$ is invariant under rescalings in any direction in $V^\perp$. In particular, the first variation formula tells us that we must have

$$\int \mathrm{div}_{\vec S}X\,d\|S\|=0\,.$$

Now we may repeat the computation above leading to (11.19), but with $\vec S$ in place of $\vec T$; notice however that (11.19) is in fact simpler because $\mathbf{p}_V\circ\mathbf{p}_{\vec S}^\perp=0$. Also notice that (11.21) is simpler, because for $q\in\mathbf{S}=\mathrm{spt}(S)$ we have that x is tangent to S and therefore $\mathbf{p}_{\vec S}^\perp=0$ whilst $\mathbf{p}_{\vec S}(x)=x$, meaning the first term on the right hand side of (11.21) vanishes and the second is simply $x\cdot\mathbf{p}_{V^\perp}(\nabla\chi(|q|))$. Thus, in place of (11.23) we get

$$\int\chi^2(|q|)\,d\|S\|(q)=-\int\chi(|q|)x\cdot\nabla_{V^\perp}\chi(|q|)\,d\|S\|(q)\,.$$

Subtracting this from (11.23) and rearranging, we arrive at

$$\begin{aligned}\int\chi^2(|q|)\,d\|T\|(q)&-\int\chi^2(|q|)\,d\|S\|(q)\\ &\le\int\chi^2(|q|)\,d\|T\|(q)-\int\chi^2(|q|)\,d\|S\|(q)+\frac14\int\chi^2(|q|)|\mathbf{p}_V\circ\mathbf{p}_{\vec T}^\perp|^2\,d\|T\|(q)\\ &\le C\mathbf{A}^2+C\int_{\mathbf{B}_{2r}}|x^\perp|^2d\|T\|+\int\chi(|q|)x\cdot\nabla_{V^\perp}\chi(|q|)\,d\|S\|(q)\\ &\quad-\int\chi(|q|)\,\mathbf{p}_{\vec T}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\,.\end{aligned}\tag{11.24}$$

On the other hand, observe that for any function f on $\mathbb{R}^{m+n}$,

$$\int f(q)\,d\|S\|(q)=\sum_i Q_i\int_{\alpha_i}f(q)\,d\mathcal{H}^m(q)\,.$$

Therefore, combining (11.24) with our monotonicity formula estimate (11.15), we arrive at

$$\int_{\mathbf{B}_r}\frac{|q^\perp|^2}{|q|^{m+2}}\,d\|T\|(q)\le C\mathbf{A}^2+C\int_{\mathbf{B}_{2r}}|x^\perp|^2d\|T\|+\sum_i Q_i\int_{\alpha_i}\chi(|q|)\,x\cdot\nabla_{V^\perp}\chi(|q|)\,d\mathcal{H}^m(q) \\ -\int\chi(|q|)\,\mathbf{p}_{\vec T}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\,. \tag{11.25}$$

Observe however that (11.15) gives

$$\int\chi^2(|q|)d\|T\|(q)-\int\chi^2(|q|)d\|S\|(q)\ge -C\mathbf{A}^2\,.$$

Thus, from (11.24) we may further infer that

$$\int_{\mathbf{B}_r}|\mathbf{p}_V\circ\mathbf{p}_{\vec T}^\perp|^2\,d\|T\|\le C\mathbf{A}^2+C\int_{\mathbf{B}_{2r}}|x^\perp|^2d\|T\|+\sum_i Q_i\int_{\alpha_i}\chi(|q|)\,x\cdot\nabla_{V^\perp}\chi(|q|)\,d\mathcal{H}^m(q) \\ -\int\chi(|q|)\,\mathbf{p}_{\vec T}(x)\cdot\nabla_{V^\perp}\chi(|q|)\,d\|T\|(q)\,. \tag{11.26}$$

11.1.2. *Key estimates.* The rest of the proof is dedicated to estimating the terms in the right hand side of (11.25) and, equivalently, (11.26). To that end we will use the refined graphical approximation of Proposition 8.15 and observe that r has been chosen so that $\mathbf{B}_{2r} \subset R \cup V$. Observe moreover that V is a negligible set in all the integrals appearing in the right and side of (11.25) and we will therefore ignore it. Recalling the inner, central, and outer regions (cf. Definition 8.13), we will split our task into four estimates , depending on whether we are integrating over the inner region (near the spine), or central or outer regions (the latter two will be coupled together):

$$\underbrace{\int_{R^{in}} |x^\perp|^2 d\|T\|}_{=:\,(\mathrm{C})} \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{11.27}$$

$$\underbrace{\sum_j Q_j \int_{\alpha_j \cap R^{in}} \chi(|q|)\, |x \cdot \nabla_{V^\perp} \chi(|q|)|\, d\mathcal{H}^m(q) + \int_{R^{in}} \chi(|q|)\, |\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|)|\, d\|T\|(q)}_{=:\,(\mathrm{D})}$$
$$\leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{11.28}$$

$$\underbrace{\int_{R^o \cup R^c} |x^\perp|^2 d\|T\|}_{=:\,(\mathrm{E})} \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4)) \tag{11.29}$$

$$\underbrace{\left| \sum_j Q_j \int_{\alpha_j \cap (R^o \cup R^c)} \chi(|q|)\, x \cdot \nabla_{V^\perp} \chi(|q|)\, d\mathcal{H}^m(q) - \int_{R^o \cup R^c} \chi(|q|)\, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|)\, d\|T\|(q) \right|}_{=:\,(\mathrm{F})}$$
$$\leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4))\,. \tag{11.30}$$

Once we have established these four estimates, the result follows from (11.25) and (11.26). We stress that, whilst in (11.28) we do not care about subtracting the two terms as in (11.25), (11.26) (indeed, our estimates on the inner region will suffice there), it is important in the outer and central regions that we are subtracting the two terms, as in (11.30); the argument will exploit in a crucial way a cancellation effect due to the fact that this is a difference between two nearly equal quantities (so, we do not wish to crudely estimate (F) by the sum of the two terms therein at any point, unlike in (D)).

11.1.3. *Estimates in the inner region.* This section is dedicated to prove the first two inequalities, (11.27) and (11.28). First of all observe that

$$|x^\perp|^2 \leq |x|^2 = \mathrm{dist}(q, V)^2 \tag{11.31}$$

Note that,

$$\chi(|q|)|x \cdot \nabla_{V^\perp} \chi(|q|)| = \chi(|q|)|x|^2 \frac{|\chi'(|q|)|}{|q|}\,. \tag{11.32}$$

On the other hand $\chi'(|q|) = 0$ if $|q| \leq r$ and $|\chi'(|q|)| \leq \frac{C}{r}$ otherwise; in particular we conclude that for $C = C(m) > 0$ we have

$$\chi(|q|)|x \cdot \nabla_{V^\perp} \chi(|q|)| \leq C\, \mathrm{dist}(q, V)^2\,. \tag{11.33}$$

On the other hand

$$\chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|)| \leq \chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot x| \frac{|\chi'(|q|)|}{|q|} = \chi(|q|)|\mathbf{p}_{\vec{T}}(x)|^2 \frac{|\chi'(|q|)|}{|q|}$$

and since $|\mathbf{p}_{\vec{T}}(x)| \leq |x|$, we can combine it with the second inequality of (11.31) to estimate

$$\chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|)| \leq C\, \mathrm{dist}(q, V)^2\,. \tag{11.34}$$

Using the monotonicity formula for $y \in [-\frac{1}{\sqrt{m-2}}, \frac{1}{\sqrt{m-2}}]^{m-2} \subset V$ and $\rho \leq 2$ (which in particular tells us that $\|T\|(\mathbf{B}_\rho(y)) \leq C\rho^m$), we easily conclude that

$$\int_{\mathbf{B}_\rho(y)} (|x^\perp|^2 + \chi(|q|)|\mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp}\chi(|q|)|)\, d\|T\|(q) \leq C\rho^{m+2}\,, \tag{11.35}$$

$$\sum_j Q_j \int_{\alpha_j \cap \mathbf{B}_\rho(y)} \chi(|q|)|x \cdot \nabla_{V^\perp}\chi(|q|)|\, d\mathcal{H}^m(q) \leq C\rho^{m+2}\,. \tag{11.36}$$

Consider now the cubes $L \in \mathcal{G}^{in}$ as in Definition 8.13 and enumerate them as $\{L_k\}_k$. Let $y_k := y_{L_k} \in V$ be the corresponding centers and let $\rho_k = 2^{2-\ell(L_k)}$ be the radii of the balls $\mathbf{B}(L_k)$ as defined in Section 8.5.1. Clearly $\{\mathbf{B}_{\rho_k}(y_k)\}$ is a covering of R^{in}. We now appeal to (8.33) in Lemma 8.14 and use (11.35), (11.36) with $y = y_k$ and $\rho = \rho_k$ to estimate

$$\text{(C)} + \text{(D)} \leq C \sum_k \rho_k^{m+2}\mathbf{E}(L_k, 0) = C\sum_k \int_{\mathbf{B}^h(L_k)} \operatorname{dist}^2(q, \mathbf{S})\, d\|T\|(q)\,. \tag{11.37}$$

Recall that, by Lemma 8.11(iv), the collection of sets $\mathbf{B}^h(L_k)$ has a control on their overlaps (each point in R belongs to at most $C(m)$ such sets). Therefore

$$\text{(C)} + \text{(D)} \leq C\int_{\mathbf{B}_4} \operatorname{dist}^2(q, \mathbf{S})\, d\|T\|(q)\,,$$

which gives (11.27) and (11.28).

11.1.4. *Estimates in the central and outer regions.* In this section we prove (11.29) and (11.30) and hence conclude the proof of Theorem 11.2. We first observe that, because of Lemma 8.14(iii), (iv) and Lemma 8.11(iv), (v) we have

$$\sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} 2^{-(m+2)\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \leq C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2)\,. \tag{11.38}$$

For each $L \in \mathcal{G}^c \cup \mathcal{G}^o$ we consider the approximations u_L given by Proposition 8.15. These approximations consist of $Q_{L,i}$-valued functions defined over the regions λL_i for some subcollection of the planes $\{\alpha_1, \ldots, \alpha_N\}$ for each L. In order to simplify our notation, we do not keep track of these collections for different L.

The estimate (11.29) is less laborious than (11.30), thanks to the fact that we do not need to exploit any cancellation effect. We can in particular write

$$\text{(E)} \leq \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \int_{R(L)} |x^\perp|^2\, d\|T\|\,. \tag{11.39}$$

Now, for each region $R(L)$ with $L \in \mathcal{G}^c \cup \mathcal{G}^o$, we can use that $|x^\perp|^2 \leq C2^{-2\ell(L)}$ on $\operatorname{spt}(T) \cap R(L)$, together with Proposition 8.15(iii) and (8.38), to estimate

$$\int_{R(L)} |x^\perp|^2 d\|T\| \leq \sum_i \int_{\Omega(L)} |x^\perp|^2\, d\|\mathbf{G}_{u_{L,i}}\| + C2^{-(m+2)\ell(L)}(\mathbf{E}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\,.$$

Using (11.38) we then conclude that

$$\text{(E)} \leq \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \sum_i \int_{\Omega(L)} |x^\perp|^2\, d\|\mathbf{G}_{u_{L,i}}\| + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2)\,. \tag{11.40}$$

Consider now coordinates z on α_i and recall that, for a point $q = (z, (u_{L,i})_j(z))$, for some $j = 1, \ldots, Q_{L,i}$, in the graph of the $Q_{L,i}$-valued function $u_{L,i} = \sum_j [\![(u_{L,i})_j]\!]$, the vector $x^\perp$ is the projection of $x = \mathbf{p}_V^\perp(q)$ onto the orthogonal complement of the tangent plane $\vec{\mathbf{G}}_{u_{L,i}}$ to the graph of $(u_{L,i})_j$ at q. In particular, since at such a point q we have $\left|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}} - \mathbf{p}^\perp_{\alpha_i}\right| \leq C|\nabla(u_{L,i})_j(z)|$, and moreover because $\mathbf{p}_{\alpha_i}^\perp \circ \mathbf{p}_V^\perp = \mathbf{p}_{\alpha_i}^\perp$, this yields (noting that $|x| \leq |q| \leq C2^{-\ell(L)}$)

$$\begin{aligned}|x^\perp| = \left|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}}(x)\right| &\leq \left|\mathbf{p}^\perp_{\vec{\mathbf{G}}_{u_{L,i}}} - \mathbf{p}^\perp_{\alpha_i}\right| \cdot |x| + |\mathbf{p}^\perp_{\alpha_i}(\mathbf{p}^\perp_V(q))| \\ &\leq C|\nabla(u_{L,i})_j(z)| \cdot 2^{-\ell(L)} + |(u_{L,i})_j(z)| \end{aligned} \tag{11.41}$$

Now if we square this expression and integrate it, we get from the two bounds in (8.36) (to control the integrand) as well as the Lipschitz bound (8.37) (to control the Jacobian) that

$$\sum_i \int_{\Omega(L)} |x^\perp|^2 \, d\|\mathbf{G}_{u_{L,i}}\| \le C \sum_{i,j} \int_{\Omega_i(L)} |\nabla (u_{L,i})_j(z)|^2 2^{-2\ell(L)} + |(u_{L,i})_j(z)|^2 d\|\mathbf{G}_{u_{L,i}}\|$$
$$\le C 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2).$$

In particular, plugging the latter estimate in (11.40) and using again (11.38) we reach (11.29).

We now come to the proof of (11.30), which is more laborious. First of all, because $\|\mathbf{G}_{u_{L,i}}\|(\partial R(L)) = 0$ for each $L \in \mathcal{G}^c \cup \mathcal{G}^o$, we can use Proposition 8.15(iii), the estimate (8.38) and the estimate (11.34) to deduce that

$$\left| \int_{\partial R(L)} \chi(|q|) \, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|T\|(q) \right| \le C 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \, .$$

Thus, letting $(R(L))^\circ$ denote the interior of $R(L)$, we can once again use (11.38) to estimate

$$\begin{aligned}(\mathrm{F}) \le \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \Bigg| \sum_j Q_j \int_{\alpha_j \cap (R(L))^\circ} \chi(|q|) \, x \cdot \nabla_{V^\perp} \chi(|q|) \, d\mathcal{H}^m \\ - \int_{(R(L))^\circ} \chi(|q|) \, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|T\| \Bigg| + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2) \, .\end{aligned} \tag{11.42}$$

We remove the boundary as above to ensure that our regions are disjoint. Now we want to straighten out the curved regions (using Lemma 8.16) to cylinders over disjoint cubes in our planes where we can pass to graphs. So, next, again taking into account (11.34), and then (recalling the notation $L_j = \alpha_j \cap R(L)$) using Lemma 8.16 with $U = (R(L))^\circ$ and $\tilde{U} = \bigcup_j \mathbf{p}_{\alpha_j}^{-1}(L_j)$, since $\mathcal{H}^{m-1}(\partial L_j) \le C 2^{-(m-1)\ell(L)}$, we can further estimate

$$\left| \int_{(R(L))^\circ} \chi(|q|) \, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|T\| - \sum_j \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|) \, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|T_{L,j}\| \right|$$
$$\le C 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2) \, .$$

We can now yet again use (11.34), as well as the estimate, (8.38) of Proposition 8.15 to further estimate

$$\Bigg| \sum_j \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|) \, \mathbf{p}_{\vec{T}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|T_{L,j}\|$$
$$- \sum_j \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|) \, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|\mathbf{G}_{u_{L,j}}\| \Bigg|$$
$$\le C 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)$$

Combining the above estimates with (11.42) and again making use of (11.38), we arrive at

$$\begin{aligned}(\mathrm{F}) \le \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \Bigg| \sum_j \Bigg[Q_j \int_{\alpha_j \cap (R(L))^\circ} \chi(|q|) \, x \cdot \nabla_{V^\perp} \chi(|q|) \, d\mathcal{H}^m \\ - \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|) \, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x) \cdot \nabla_{V^\perp} \chi(|q|) \, d\|\mathbf{G}_{u_{L,j}}\| \Bigg] \Bigg| \\ + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2) \, .\end{aligned} \tag{11.43}$$

Now note that the multiplicities Q_j are the ones from the outer region, and so they do not necessarily match the multiplicities $Q_{L,j}$ of the multi-valued functions $u_{L,j}$ when $L \in \mathcal{G}^c$. However recall the computation (see (11.32)),

$$\chi(|q|)x \cdot \nabla_{V^\perp}\chi(|q|) = \frac{\chi(|q|)\chi'(|q|)}{|q|}|x|^2\,, \tag{11.44}$$

which since $x \in V^\perp$ shows that the integrand $\chi(|q|)x \cdot \nabla_{V^\perp}\chi(|q|)$ is invariant under rotations which keep the spine V fixed. Since for every $j, k \in \{1, \dots, N\}$ there is a rotation which maps α_j onto α_k and fixes V, the integral

$$\int_{\alpha_j \cap (R(L))^\circ} \chi(|q|)\, x \cdot \nabla_{V^\perp}\chi(|q|)\, d\mathcal{H}^m$$

is independent of the plane α_j. In particular, given that $\sum_j Q_{L,j} = \sum_j Q_j = Q$, we can in fact write from (11.43) (using $L_j = \alpha_j \cap R(L)$

$$\begin{aligned}(\mathrm{F}) \le \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \sum_j \Bigg| Q_{L,j} \int_{L_j} \chi(|q|)\, x \cdot \nabla_{V^\perp}\chi(|q|)\, d\mathcal{H}^m \\ - \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|)\, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x) \cdot \nabla_{V^\perp}\chi(|q|)\, d\|\mathbf{G}_{u_{L,j}}\| \Bigg| \\ + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2)\end{aligned} \tag{11.45}$$

Next, following the same computation as for (11.34) we get

$$|\chi(|q|)\, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x) \cdot \nabla_{V^\perp}\chi(|q|)| \le C \operatorname{dist}(q, V)^2 \le C \cdot 2^{-2\ell(L)}\,,$$

and hence, through the usual Taylor expansion of the area functional for a multi-valued graph, for $k = 1, \dots, Q_{L,j}$ letting $q_k := (z, (u_{L,j})_k(z)) \equiv z + (u_{L,j})_k(z) \in \alpha_j \times \alpha_j^\perp$, we get

$$\begin{aligned}\Bigg| \int_{\mathbf{p}_{\alpha_j}^{-1}(L_j)} \chi(|q|)\, \mathbf{p}_{\vec{\mathbf{G}}_{u_{L,j}}}(x) \cdot \nabla_{V^\perp}\chi(|q|)\, d\|\mathbf{G}_{u_{L,j}}\| \\ - \int_{L_j} \sum_k \chi(|q_k|)\, \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \nabla_{V^\perp}\chi(|q_k|)\, d\mathcal{H}^m(z) \Bigg| \\ \le C 2^{-2\ell(L)} \int_{L_j} |Du_{L,j}|^2\, d\mathcal{H}^m\,.\end{aligned}$$

We can in particular use (11.45), (8.36), and (11.38) to obtain

$$\begin{aligned}(\mathrm{F}) \le \sum_{L \in \mathcal{G}^c \cup \mathcal{G}^o} \sum_j \Bigg| \int_{L_j} [Q_{L,j}\chi(|z|)\, \mathbf{p}_V^\perp(z) \cdot \nabla_{V^\perp}\chi(|z|) \\ - \sum_k \chi(|q_k|)\, \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \nabla_{V^\perp}\chi(|q_k|)]\, d\mathcal{H}^m(z) \Bigg| \\ + C(\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2)\end{aligned} \tag{11.46}$$

Now use the coordinates $z = (\zeta, \xi) \in V \times V^\perp$ for points $z \in \alpha_j$. Recalling (11.44), the first integrand above is then given by

$$h(z) := Q_{L,j} \frac{\chi(|z|)\chi'(|z|)}{|z|}|\xi|^2\,.$$

Meanwhile, we write the second integrand as

$$g(z) := \sum_{k=1}^{Q_{L,j}} \frac{\chi(|q_k|)\chi'(|q_k|)}{|q_k|} \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k)\,.$$

Considering that $\chi'(|q|) = 0$ when $|q| \le r$, for each $k = 1, \dots, Q_{L,j}$ we can estimate (by Taylor expansion)

$$\left| \frac{\chi(|q_k|)\chi'(|q_k|)}{|q_k|} - \frac{\chi(|z|)\chi'(|z|)}{|z|} \right| \le C 2^{2\ell(L)} |(u_{L,j})_k(z)|^2 .$$

On the other hand we have

$$\left| \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) \right| \le C 2^{-2\ell(L)} .$$

In particular, if we define

$$\bar{g}(z) := \sum_{k=1}^{Q_{L,j}} \frac{\chi(|z|)\chi'(|z|)}{|z|} \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) ,$$

we then get

$$|g(z) - \bar{g}(z)| \le C |u_{L,j}(z)|^2 . \tag{11.47}$$

Next, recalling the definition of q_k, notice that

$$\mathbf{p}_V^\perp(q_k) = \xi + (u_{L,j})_k(z) ,$$

On the other hand,

$$\begin{aligned}
&|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \mathbf{p}_V^\perp(q_k) - |\xi|^2| \\
&\qquad = \left| |\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi + (u_{L,j})_k)|^2 - |\xi|^2 \right| \\
&\qquad = \left| |\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)|^2 - |\xi|^2 \right| + |\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}((u_{L,j})_k)|^2 \\
&\qquad\qquad + 2 \left| \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi) \cdot \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}((u_{L,j})_k) \right| \\
&\qquad \le |\mathbf{p}^\perp_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)|^2 + |(u_{L,j})_k|^2 + 2|\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\xi)| \left| (\mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}} - \mathbf{p}_{\alpha_j})((u_{L,j})_k) \right| \\
&\qquad \le |\mathbf{p}^\perp_{\vec{\mathbf{G}}_{(u_{L,j})_k}} - \mathbf{p}^\perp_{\alpha_j}|^2 |\xi|^2 + |(u_{L,j})_k|^2 + 2|\xi| \cdot |D(u_{L,j})_k| |(u_{L,j})_k| \\
&\qquad \le |D(u_{L,j})_k|^2 2^{-2\ell(L)} + |(u_{L,j})_k|^2 + 2^{1-\ell(L)} |D(u_{L,j})_k| |(u_{L,j})_k| \\
&\qquad \le C 2^{-2\ell(L)} |D(u_{L,j})_k|^2 + C|(u_{L,j})_k|^2
\end{aligned}$$

where we have used that $\mathbf{p}_{\alpha_j}((u_{L,j})_k) = 0$ and $\mathbf{p}^\perp_{\alpha_j}(\xi) = 0$. In particular, we arrive at

$$|\bar{g}(z) - h(z)| \le C |u_{L,j}(z)|^2 + 2^{-2\ell(L)} |Du_{L,j}(z)|^2 .$$

Combining this last inequality and (11.47) with the bound (8.36), for each $L \in \mathcal{G}^c \cup \mathcal{G}^o$ and each index j enumerating the planes for the corresponding cone associated to L, we thus achieve

$$\begin{aligned}
&\left| \int_{L_j} Q_{L,j} \chi(|z|)\, \mathbf{p}_{V^\perp}(z) \cdot \nabla_{V^\perp} \chi(|z|)\, d\mathcal{H}^m(z) \right. \\
&\qquad\qquad \left. - \sum_k \int_{L_j} \chi(|q_k|)\, \mathbf{p}_{\vec{\mathbf{G}}_{(u_{L,j})_k}}(\mathbf{p}_V^\perp(q_k)) \cdot \nabla_{V^\perp} \chi(|q_k|)\, d\mathcal{H}^m(z) \right| \\
&\qquad\qquad\qquad \le C 2^{-(m+2)\ell(L)} (\mathbf{E}(L) + 2^{-2\ell(L)} \mathbf{A}^2)
\end{aligned}$$

Together with (11.46) and (11.38), this completes the proof of (11.30) and thus the proof of the theorem. □

11.2. **Proof of Simon's non-concentration estimate (Corollary 11.3).** We observe that Corollary 11.3 is a direct consequence of Theorem 11.2 and of the following lemma (after an appropriate rescaling to adjust the radius).

Lemma 11.6. *Let T, Σ and $\mathbf{S}$ be as in Assumption 11.1 with $\mathbf{B}_1 \subset \Omega$ and $\Theta(T, 0) \ge Q$. Then we may choose ε sufficiently small in Assumption 11.1 such that for each $\kappa > 0$ we have*

$$\int_{\mathbf{B}_1} \frac{\operatorname{dist}^2(q, \mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q) \le C_\kappa \int_{\mathbf{B}_1} \frac{|q^\perp|^2}{|q|^{m+2}}\, d\|T\|(q) + C_\kappa (\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2) , \tag{11.48}$$

where $C_\kappa = C_\kappa(Q, m, n, \bar n, M, \kappa) > 0$.

Proof of Lemma 11.6. Fix $\kappa \in (0, m+2)$. We test the first variation identity (11.16) for T with the vector field

$$X(q) := \operatorname{dist}^2(q, \mathbf{S}) \underbrace{(\max\{r, |q|\}^{-m-2+\kappa} - 1)_+}_{=: f(q)} q\,,$$

where $r > 0$ and for a function g we define $g_+(q) := \max\{g(q), 0\}$. Note that f is supported in $\mathbf{B}_1 \setminus \mathbf{B}_r$. In order to estimate the integrand in (B) of (11.16), we first observe that

$$f(q) \operatorname{dist}^2(q, \mathbf{S}) \le |q|^{\kappa - m}\,.$$

Recalling (11.14), (11.11), and (11.13), we have

$$|\vec H_T \cdot X^\perp(q)| \le C\mathbf{A}^2 |q|^{1+\kappa-m}\,.$$

Hence we can estimate

$$|(\mathrm{B})| \le C\mathbf{A}^2 \int_{\mathbf{B}_1} |q|^{1+\kappa-m}\, d\|T\|(q) \le C\mathbf{A}^2\,.$$

In particular we conclude that

$$\int \operatorname{div}_{\vec T} X \; d\|T\| = O(\mathbf{A}^2)\,. \tag{11.49}$$

We next compute

$$\operatorname{div}_{\vec T} X(q) = m \operatorname{dist}^2(q, \mathbf{S}) f(q) + f(q) \nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q_\parallel + \operatorname{dist}^2(q, \mathbf{S}) \nabla f(q) \cdot q_\parallel\,. \tag{11.50}$$

Moreover, we can explicitly compute

$$\begin{aligned}\nabla f(q) \cdot q_\parallel &= -(m+2-\kappa)|q|^{-m-4+\kappa} |q_\parallel|^2 \mathbf{1}_{\mathbf{B}_1 \setminus \mathbf{B}_r}(q)\\ &= -(m+2-\kappa)|q|^{-m-2+\kappa} \mathbf{1}_{\mathbf{B}_1 \setminus \mathbf{B}_r}(q) + (m+2-\kappa)|q|^{-m-4+\kappa}|q^\perp|^2 \mathbf{1}_{\mathbf{B}_1 \setminus \mathbf{B}_r}(q)\,.\end{aligned} \tag{11.51}$$

On the other hand, using the 2-homogeneity of $\operatorname{dist}^2(q, \mathbf{S})$ we can likewise compute

$$\begin{aligned}\nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q_\parallel &= \nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q - \nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q^\perp\\ &= 2 \operatorname{dist}^2(q, \mathbf{S}) - \nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q^\perp\,.\end{aligned}$$

Inserting this and (11.51) into (11.50) we reach

$$\begin{aligned}\operatorname{div}_{\vec T} X(q) &= \kappa \operatorname{dist}(q, \mathbf{S})^2 |q|^{-m-2+\kappa} \mathbf{1}_{\mathbf{B}_1 \setminus \mathbf{B}_r}(q)\\ &\quad + (m+2) r^{-m-2+\kappa} \operatorname{dist}^2(q, \mathbf{S}) \mathbf{1}_{\mathbf{B}_r}(q) - (m+2) \operatorname{dist}^2(q, \mathbf{S}) \mathbf{1}_{\mathbf{B}_1}(q)\\ &\quad + (m+2-\kappa) \operatorname{dist}^2(q, \mathbf{S}) |q^\perp|^2 |q|^{-m-4+\kappa} \mathbf{1}_{\mathbf{B}_1 \setminus \mathbf{B}_r}(q) - \nabla \operatorname{dist}^2(q, \mathbf{S}) \cdot q^\perp f(q)\,.\end{aligned} \tag{11.52}$$

In turn, inserting this in (11.49) we achieve

$$\begin{aligned}\kappa \int_{\mathbf{B}_1 \setminus \mathbf{B}_r} \frac{\operatorname{dist}^2(q, \mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q) &\le C\mathbf{A}^2 + (m+2)\hat{\mathbf{E}}(T, \mathbf{S}, \mathbf{B}_1)\\ &\quad + \underbrace{\int \nabla \operatorname{dist}(\mathbf{S}, q)^2 \cdot q^\perp f(q)\, d\|T\|(q)}_{=: (\mathrm{C})}\,.\end{aligned} \tag{11.53}$$

On the other hand, using that $\operatorname{dist}(q, \mathbf{S})$ is 1-Lipschitz, we can estimate

$$\begin{aligned}|(\mathrm{C})| &\le 2 \int \operatorname{dist}(q, \mathbf{S}) |q^\perp| f(q)\, d\|T\|(q)\\ &\le \kappa \int \operatorname{dist}^2(q, \mathbf{S}) f(q)\, d\|T\|(q) + \frac{1}{\kappa} \int |q^\perp|^2 f(q)\, d\|T\|(q)\\ &\le \kappa \int_{\mathbf{B}_1 \setminus \mathbf{B}_r} \frac{\operatorname{dist}^2(q, \mathbf{S})}{|q|^{m+2-\kappa}}\, d\|T\|(q)\end{aligned}$$

$$+\kappa r^{-m-2+\kappa}\int_{\mathbf{B}_r}\mathrm{dist}^2(q,\mathbf{S})\,d\|T\|(q)+\frac{1}{\kappa}\int_{\mathbf{B}_1}\frac{|q^\perp|^2}{|q|^{m+2}}\,d\|T\|(q)\,. \tag{11.54}$$

Inserting (11.54) into (11.53) we then reach

$$\int_{\mathbf{B}_1\setminus\mathbf{B}_r}\frac{\mathrm{dist}^2(q,\mathbf{S})}{|q|^{m+2-\kappa}}\,d\|T\|(q)\le C_\kappa\int_{\mathbf{B}_1}\frac{|q^\perp|^2}{|q|^{m+2}}\,d\|T\|(q)+C_\kappa(\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1))+C\frac{\|T\|(\mathbf{B}_r)}{r^{m-\kappa}}\,. \tag{11.55}$$

Letting $r\downarrow 0$ and using $\|T\|(\mathbf{B}_r)\le Cr^m$ we reach (11.48). □

11.3. **Proof of Simon's shift inequality (Proposition 11.4).** In Lemma 11.7 below we will show the following inequality for each $\kappa\in(0,m+2)$, under the assumption that ε is chosen sufficiently small and that $\rho=\rho(m)$ is a dimensional constant:

$$\int_{\mathbf{B}_\rho(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+2-\kappa}}\,d\|T\|(q)\le C^\star_\kappa(\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2)\,, \tag{11.56}$$

for $C^\star_\kappa=C^\star_\kappa(Q,m,n,\bar n)$. Assuming the validity of this, our aim is therefore to show that we also have

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le C(\mathbf{A}^2+\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\,, \tag{11.57}$$

for $C=C(Q,m,n,\bar n)$. Observe that (11.56) and (11.57) together yield (11.5).

Fix now a scale $\bar r\le\frac{\rho}{2}$ (for ρ fixed as in (11.56)), whose choice will be specified later. Observe that, by assuming ε is small enough depending on $\bar r$, Proposition 8.15(v) guarantees that $\mathrm{dist}(q_0,V)\le\frac{\bar r}{2}$. We can now apply a scaled version of Lemma 7.14 to find an index j and a subset $\Omega\subset\alpha_j\cap\mathbf{B}_{\bar r}(\mathbf{p}_V(q_0))\setminus B_{\bar r/2}(V)$ with the property that

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le\bar C\,\mathrm{dist}(z,q_0+\mathbf{S})^2\qquad\forall z\in\Omega \tag{11.58}$$

and $\mathcal{H}^m(\Omega)\ge\bar C^{-1}\bar r^m$ for some geometric constant $\bar C>0$ (from the proof of Lemma 7.14, we know $\bar C=\bar C(Q,m,n,\bar n)$ due to the specific choice of $U=\mathbf{B}_{\bar r}\setminus B_{\bar r/2}(V)$ here in Lemma 7.14; see also Remark 7.15). If we choose ε sufficiently small, depending on $\bar r$, by Proposition 8.15 we can find a further subset $\Omega'\subset\Omega$ with measure larger than $(2\bar C)^{-1}\bar r^m$ and such that over each $z\in\Omega'$ we can find a point p in the outer graphical approximation lying in $z+\alpha_j^\perp$ and in the support of the current T with the property that

$$\mathrm{dist}(z,q_0+\mathbf{S})\le\mathrm{dist}(p,q_0+\mathbf{S})+C\bar r^{-m/2}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar r^2\mathbf{A}^2)^{1/2}\,.$$

In particular we achieve

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le\bar C\,\mathrm{dist}^2(p,q_0+\mathbf{S})+C\bar r^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar r^2\mathbf{A}^2) \tag{11.59}$$

for all p in the subset Ω'' of points $p\in\mathrm{spt}(T)$ which coincide with the outer graphical approximation of Proposition 8.15 restricted to the subset Ω'. For this set we clearly have $\|T\|(\Omega'')\ge\mathcal{H}^m(\Omega')\ge(2\bar C)^{-1}\bar r^m$. Observe moreover that, since $\Omega'\subset\mathbf{B}_{\bar r}(\mathbf{p}_V(q_0))\subset\mathbf{B}_{3\bar r/2}(q_0)$, if ε is sufficiently small (depending on $\bar r$), then $\Omega''\subset\mathbf{B}_{2\bar r}(q_0)$. We now integrate (11.59) over Ω'' with respect to $d\|T\|$ to find

$$\begin{aligned}|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le C\bar r^{-m}&\int_{\mathbf{B}_{2\bar r}(q_0)}\mathrm{dist}^2(p,q_0+\mathbf{S})\,d\|T\|(p)\\&+C\bar r^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar r^2\mathbf{A}^2)\,.\end{aligned}$$

We can in particular write

$$\begin{aligned}|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le C\bar r^{\frac{7}{4}}&\int_{\mathbf{B}_{2\bar r}(q_0)}\frac{\mathrm{dist}^2(q,q_0+\mathbf{S})}{|q-q_0|^{m+7/4}}\,d\|T\|(q)\\&+C\bar r^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)+\bar r^2\mathbf{A}^2)\,.\end{aligned} \tag{11.60}$$

Given that the constant C is independent of the radius $\bar r$, for any fixed $\delta>0$ if we choose $\bar r=\bar r(Q,m,n,\bar n,\delta)$ sufficiently small we achieve

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2\le\delta\int_{\mathbf{B}_{2\bar r}(q_0)}\frac{\mathrm{dist}(q,q_0+\mathbf{S})^2}{|q-q_0|^{m+7/4}}\,d\|T\|(q)$$

$$+ C\bar{r}^{-m}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \bar{r}^2\mathbf{A}^2)\,.$$

We can now insert the latter in (11.56) (with $\kappa = \frac{1}{4}$) and, upon fixing $\delta = \delta(Q, m, n, \bar{n})$ small enough (recalling that $2\bar{r} \leq \rho$), conclude that

$$\begin{aligned}\int_{\mathbf{B}_{2\bar{r}}(q_0)} \frac{\mathrm{dist}(q, q_0 + \mathbf{S})^2}{|q-q_0|^{m+7/4}}\, d\|T\|(q) &\leq \int_{\mathbf{B}_\rho(q_0)} \frac{\mathrm{dist}(q, q_0 + \mathbf{S})^2}{|q-q_0|^{m+7/4}}\, d\|T\|(q)\\ &\leq C\bar{r}^{-m}(\bar{r}^2\mathbf{A}^2 + C\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\\ &\qquad + \frac{1}{2}\int_{\mathbf{B}_{2\bar{r}}(q_0)} \frac{\mathrm{dist}(q, q_0 + \mathbf{S})^2}{|q-q_0|^{m+7/4}}\, d\|T\|(q)\,. \end{aligned} \tag{11.61}$$

In particular we conclude

$$\int_{\mathbf{B}_{2\bar{r}}(q_0)} \frac{\mathrm{dist}(q, q_0 + \mathbf{S})^2}{|q-q_0|^{m+7/4}}\, d\|T\|(q) \leq C\bar{r}^{-m}(\bar{r}^2\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4))\,,$$

for this fixed choice of $\bar{r}$ and inserting the latter into (11.60) we achieve (11.57).

We are left with the task of showing that (11.56) holds. This is accomplished in the following:

Lemma 11.7. *If T, Σ, and $\mathbf{S}$ are as in Assumption 11.1 and $\rho = \frac{1}{12(m-2)}$, then (11.56) holds for every point $q_0 \in \mathbf{B}_\rho$ with the property that $\Theta(T, q_0) \geq Q$.*

Proof. First of all, observe the following elementary fact for all $q, q_0, \mathbf{S}$:

$$\mathrm{dist}(q, q_0 + \mathbf{S}) \leq |\mathbf{p}^\perp_{\alpha_1}(q_0)| + C\boldsymbol{\mu}(\mathbf{S})|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)| + \mathrm{dist}(q, \mathbf{S})\,. \tag{11.62}$$

Fix $\bar{\rho} = \frac{1}{4\sqrt{m-2}}$. Next note that, assuming ε is sufficiently small, we can assume that $\mathbf{B}_{4\bar{\rho}} \setminus B_{a\bar{\rho}/8}(V)$ is in the outer region R^o and moreover that $\mathrm{dist}(q_0, V) < a\bar{\rho}/8$, by Proposition 8.15(v). Thus, using (11.62) with q_0 as in the statement of the lemma, we gain the inequality

$$\mathbb{E}(T_{q_0,\bar{\rho}}, \mathbf{S}, \mathbf{B}_4) + \mathbf{A}^2 \leq \underbrace{C_0(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2 + |\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2)}_{=:D^2}\,, \tag{11.63}$$

for some constant C_0 which is now independent of ε.

Note that, if $D^2 \leq \bar{\varepsilon}\boldsymbol{\sigma}(\mathbf{S})^2$ for $\bar{\varepsilon} = \bar{\varepsilon}(Q, m, n, \bar{n}, M)$ which is the threshold needed to apply Corollary 11.3, then we could apply Corollary 11.3 with $T_{q_0,\bar{\rho}}$ in place of T and the desired inequality (11.56) would then follow.

To handle the general case we fix a suitable $\delta > 0$, which will be chosen depending on $\bar{\varepsilon}$. We wish to apply the Pruning Lemma 8.2 with this choice of δ and D. Let $\varepsilon_* = \varepsilon_*(\delta, N)$ be the threshold needed for the applicability of Lemma 8.2 and observe that we need to prove that

$$D^2 \leq \varepsilon_*^2\boldsymbol{\mu}(\mathbf{S})^2\,. \tag{11.64}$$

Note that we can assume

$$C_0(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + \mathbf{A}^2) \leq \frac{\varepsilon_*^2}{2}\boldsymbol{\mu}(\mathbf{S})^2 \tag{11.65}$$

by choosing ε sufficiently small, depending on ε_* in addition to existing dependencies.

Next observe that for every fixed $\lambda > 0$, by choosing ε sufficiently small depending on λ, the set $\mathbf{B}_\rho \setminus B_\lambda(V)$ will be contained in the outer region R^o. In particular, applying Proposition 8.15(v) as before, it follows that $\mathrm{dist}(q_0, V) \leq \lambda$ and thus for $\lambda \leq \varepsilon_*/(2\sqrt{C_0})$ we gain

$$C_0\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(q_0)|^2 \leq \frac{\varepsilon_*^2}{4}\boldsymbol{\mu}(\mathbf{S})^2\,. \tag{11.66}$$

Consider moreover the ball $\mathbf{B} := \mathbf{B}_{4\lambda}(\mathbf{p}_V(q_0))$ and estimate

$$\hat{\mathbf{E}}(T, \alpha_1, \mathbf{B}) \leq C_1\boldsymbol{\mu}(\mathbf{S})^2 + C_1\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}) \leq C_1\boldsymbol{\mu}(\mathbf{S})^2 + C_1\lambda^{-m-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4)\,,$$

where $C_1 = C_1(m, n, \bar{n}, Q)$. In particular applying Allard's L^2–L^∞ bound for α_1 we conclude that

$$|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 \leq C_1\lambda^2\boldsymbol{\mu}(\mathbf{S})^2 + C_1\lambda^{-m}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + C_1\lambda^2\mathbf{A}^2\,.$$

We can now select λ so that $C_0C_1\lambda^2 \leq \frac{\varepsilon_*^2}{16}$, yielding

$$C_0|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 \leq \frac{\varepsilon_*^2}{16}\boldsymbol{\mu}(\mathbf{S})^2 + C\varepsilon_*^{-m-2}\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_4) + C\varepsilon_*^2\mathbf{A}^2 .$$

We are now in the position to choose ε sufficiently small (depending on ε_*) so that

$$C_0|\mathbf{p}^\perp_{\alpha_1}(q_0)|^2 \leq \frac{\varepsilon_*^2}{8}\boldsymbol{\mu}(\mathbf{S})^2 . \tag{11.67}$$

Summing (11.65), (11.66), and (11.67) we obtain (11.64).

So we can now indeed apply Lemma 8.2 to find a cone $\mathbf{S}' \subset \mathbf{S}$, indexed as $\mathbf{S}' = \bigcup_{i\in I}\alpha_i$, with the properties that

$$D^2 + \max_j \min_{i\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \delta^2\boldsymbol{\sigma}(\mathbf{S}')^2 ,$$

and

$$\max_j \min_{i\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq \Gamma^2 D^2 \tag{11.68}$$

for $\Gamma = \Gamma(\delta)$ given by Lemma 8.2. Now, since by (11.63) and the triangle inequality

$$\hat{\mathbf{E}}(T_{q_0,\bar\rho},\mathbf{S}',\mathbf{B}_4) \leq CD^2 + C\max_j \min_{i\in I} \operatorname{dist}^2(\alpha_i \cap \mathbf{B}_1, \alpha_j \cap \mathbf{B}_1) \leq C\delta^2\boldsymbol{\sigma}(\mathbf{S}')^2 , \tag{11.69}$$

where C is a constant independent of δ, and since $\mathbf{S}' \subset \mathbf{S}$ we have by (11.63) again that $\hat{\mathbf{E}}(\mathbf{S}',T_{q_0,\bar\rho},\mathbf{B}_4) + \mathbf{A}^2 \leq C_0D^2 \leq C\delta^2\boldsymbol{\sigma}(\mathbf{S}')^2$, we can choose $\delta = \delta(Q,m,n,\bar n, M)$ small to achieve the applicability of Corollary 11.3 with $\mathbf{S}'$ replacing $\mathbf{S}$ and $T_{q_0,\bar\rho}$ replacing T. Given that $\rho = \frac{\bar\rho}{3\sqrt{m-2}}$ the estimate (11.4), together with (11.69) and (11.68), give

$$\int_{\mathbf{B}_\rho(q_0)} \frac{\operatorname{dist}^2(q, q_0 + \mathbf{S}')}{|q-q_0|^{m+2-\kappa}}\, d\|T\|(q) \leq C(\mathbf{A}^2 + \Gamma^2 D^2) . \tag{11.70}$$

Since however we have chosen δ and thus we have Γ fixed, we can treat the latter as a constant depending only on Q, m, n, $\bar n$, and M. Since $\mathbf{S} \supset \mathbf{S}'$, we trivially have $\operatorname{dist}(q, q_0 + \mathbf{S}) \leq \operatorname{dist}(q, q_0 + \mathbf{S}')$, and hence, given our definition of D^2, (11.70) implies (11.56). □

12. Linearization

The aim of this section is to collect some results on Dir-minimizing functions which will be pivotal to close the proof of Theorem 2.5. We follow the notation of [10].

Definition 12.1. Let $Q, m,$ and n be positive integers. We denote by:

- $\mathscr{H}_1$ the space of 1-homogeneous locally Dir-minimizing functions $u : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$;
- $\mathscr{L}_1$ the subspace of $u \in \mathscr{H}^1$ which are invariant by translations along at least $(m-2)$-independent directions;
- $\mathscr{H}_1^0$ and $\mathscr{L}_1^0$ the subsets of $\mathscr{H}_1$ and $\mathscr{L}_1$ consisting of maps u such that $\boldsymbol{\eta}\circ u \equiv 0$.

Observe that all these spaces are locally compact in the L^2_{loc} topology. In particular, all the minima appearing in the inequalities below (e.g. see the right hand side of (12.1)) are attained. We begin with a suitable decay lemma.

Theorem 12.2. *For every $Q, m, n,$ and $\varepsilon > 0$, there is $\rho = \rho(Q,m,n,\varepsilon) \in (0, \frac{1}{2})$ with the following property. Assume that $u : \mathbb{R}^m \supset B_1 \to \mathcal{A}_Q(\mathbb{R}^n)$ is Dir-minimizing, that $u(0) = Q[\![0]\!]$, and $I_{0,u}(0) := \lim_{r\to 0} I_{0,u}(r) \geq 1$. Then*

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \varepsilon r^{m+2}\int_{B_1}|u|^2 \qquad \forall r \leq \rho . \tag{12.1}$$

If additionally for some constant $\vartheta > 0$ and for every $r \leq \frac{1}{2}$ there are $m-2$ points $z_1, \dots, z_{m-2} \in B_r$ such that $u(z_i) = Q[\![0]\!]$, $I_{z_i,u}(0) \geq 1$ and $\det((z_i\cdot z_j)_{i,j}) \geq \vartheta r^{m-2}$, then

$$\min_{v\in\mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \varepsilon r^{m+2}\int_{B_1}|u|^2 \qquad \forall r \leq \rho \tag{12.2}$$

(where ρ will depend also on ϑ). Finally, when $\boldsymbol{\eta}\circ u \equiv 0$ we have the equalities

$$\min_{v\in\mathscr{H}_1}\int_{B_r}\mathcal{G}(u,v)^2 = \min_{v\in\mathscr{H}_1^0}\int_{B_r}\mathcal{G}(u,v)^2\,, \tag{12.3}$$

$$\min_{v\in\mathscr{L}_1}\int_{B_r}\mathcal{G}(u,v)^2 = \min_{v\in\mathscr{L}_1^0}\int_{B_r}\mathcal{G}(u,v)^2\,. \tag{12.4}$$

Let us first remark that the identities (12.3) and (12.4) are simple consequences of the following elementary algebraic fact. Consider any pair $T=\sum_i[\![T_i]\!]$, $S=\sum_i[\![S_i]\!]\in\mathcal{A}_Q(\mathbb{R}^n)$ and define

$$T'=\sum_i[\![T_i-\boldsymbol{\eta}(T)]\!]\,,\qquad\text{and}\qquad S'=\sum_i[\![S_i-\boldsymbol{\eta}(S)]\!]\,,$$

then

$$\mathcal{G}(T,S)^2 = Q|\boldsymbol{\eta}(T)-\boldsymbol{\eta}(S)|^2+\mathcal{G}(T',S')^2\,.$$

Given this then, for example, in (12.3), we would have for $v\in\mathscr{H}_1$,

$$\mathcal{G}(u,v)^2 = Q|\boldsymbol{\eta}\circ v|^2+\mathcal{G}(u,v')^2$$

where $v'=v-\boldsymbol{\eta}\circ v$, meaning $\min_{v\in\mathscr{H}_1}\int_{B_r}\mathcal{G}(u,v)^2\geq\min_{v\in\mathscr{H}_1^0}\int_{B_r}\mathcal{G}(u,v)^2$; the other inequality is of course trivial as we are minimizing over a larger set. Thus, to prove Theorem 12.2 we just need to prove (12.1) and (12.2). We will first show, using a compactness argument, that if the frequency of a Dir-minimizer is pinched between 1 and $1+\eta$ then the function is very close to be 1-homogeneous. We begin with the following intermediate lemma.

Lemma 12.3. *For every $Q,m,n,\bar\varepsilon>0$, there is a constant $\eta=\eta(Q,m,n,\bar\varepsilon)>0$ with the following property. Fix $r>0$. Then if $u:B_r\to\mathcal{A}_Q(\mathbb{R}^n)$ is a Dir-minimizing function such that $u(0)=Q[\![0]\!]$, $I_{0,u}(0)\geq 1$ and $I_{0,u}(r)\leq 1+\eta$, then*

$$\min_{v\in\mathscr{H}_1}\int_{B_r}\mathcal{G}(u,v)^2\leq\bar\varepsilon r\int_{\partial B_r}|u|^2\,. \tag{12.5}$$

If additionally for some constant $\vartheta>0$ there are $m-2$ points $z_1,\dots,z_{m-2}\in B_{r/2}$ such that $u(z_i)=Q[\![0]\!]$, $I_{z_i,u}(0)\geq 1$ and $\det((z_i\cdot z_j)_{i,j})\geq\vartheta r^{m-2}$, then, under the assumption that the parameter η is small enough, depending also on the value of ϑ, we have

$$\min_{v\in\mathscr{L}_1}\int_{B_r}\mathcal{G}(u,v)^2\leq\bar\varepsilon r\int_{\partial B_r}|u|^2\,. \tag{12.6}$$

Proof. By scaling the domain and u, we can assume without loss of generality that $r=1$ and $H_u(1):=\int_{\partial B_1}|u|^2=1$. We then argue by contradiction and assume the statements to be false for some fixed $\bar\varepsilon$ (and ϑ) no matter how small η is. In particular we can set $\eta=\frac{1}{k}$ and select corresponding maps u_k satisfying $u_k(0)=Q[\![0]\!]$, $1\leq I_{0,u_k}(0)\leq I_{0,u_k}(1)\leq 1+\frac{1}{k}$, $H_{u_k}(1)=1$ and

$$\min_{v\in\mathscr{H}_1}\int_{B_1}\mathcal{G}(u_k,v)^2\geq\bar\varepsilon\,. \tag{12.7}$$

As for the second statement of the lemma, we know additionally

($\star$) the existence of points $z_1^k,\dots,z_{m-2}^k\in B_{1/2}$ such that $u_k(z_i^k)=Q[\![0]\!]$, $I_{z_i^k,u_k}(0)\geq 1$ and $\det((z_i^k\cdot z_j^k)_{i,j})\geq\vartheta$,

while

$$\min_{v\in\mathscr{L}_1}\int_{B_1}\mathcal{G}(u_k,v)^2\geq\bar\varepsilon\,. \tag{12.8}$$

As $\int_{B_1}|Du_k|^2=I_{0,u_k}(1)\cdot H_{u_k}(1)\leq 1+\frac{1}{k}$, we can appeal to the compact embedding of the $W^{1,2}(B_1;\mathcal{A}_Q(\mathbb{R}^n))$ in $L^2(B_1;\mathcal{A}_Q(\mathbb{R}^n))$ (cf. [10, Proposition 2.11]) to extract (in both cases) a subsequence, not relabeled, which converges strongly in L^2 to some u. In fact (up to extraction of another subsequence), we can also assume that $u_k|_{\partial B_1}\to u|_{\partial B_1}$ by the trace property of $W^{1,2}$ (cf. [10, Proposition 2.10]), while u is Dir-minimizing (cf. [10, Proposition 3.20]) and $u_k\to u$ strongly in $W^{1,2}_{\rm loc}(B_1)$. It also follows, from Hölder estimates (cf. [10, Theorem 3.9] that $u(0)=[\![0]\!]$. Also, from upper semi-continuity of the frequency, $I_{0,u}(0)\geq\limsup_{k\to\infty}I_{0,u_k}(0)\geq 1$, while $I_{0,u}(1) \leq \liminf_{k\to\infty} I_{0,u_k}(1) = 1$. Hence, by the monotonicity of the frequency function (cf. [10, Theorem 3.15]) $I_{0,u}(r) \equiv 1$, which in turn implies that u is 1-homogeneous (cf. [10, Corollary 3.16]). In particular $u \in \mathscr{H}_1$ and so the L^2 convergence of u_k to u contradicts (12.7).

Under the additional assumption ($\star$) we can assume, up to extraction of a further subsequence, that $z_i^k \to z_i \in B_{1/2}$ and obviously $\det((z_i \cdot z_j)_{i,j})\rangle \geq \vartheta$. In particular the vectors v_i span an $(m-2)$-dimensional subspace V. Again by the Hölder continuity of u_k and upper semi-continuity of the frequency function we conclude $u(z_i) = Q[\![0]\!]$ and $I_{z_i,u}(0) \geq 1$. But because of the 1-homogeneity of u we have the same properties for $u(\sigma z_i)$ for every $\sigma \in (0,1]$, in particular, again by upper semi-continuity of the frequency, $I_{z_i,u}(0) \leq 1$. Arguing as in [10, Proof of Lemma 3.4], it follows that $u(x+\lambda z_i) = u(x)$ for every x, every i, and every $\lambda \in \mathbb{R}$. In particular $u \in \mathscr{L}_1$. By the strong L^2-convergence of u_k to u this contradicts (12.8). □

Proof of Theorem 12.2. We fix ε and ϑ as in the statement of the Theorem. Fix $\bar\varepsilon > 0$ (to be determined, possibly smaller than ε), and let $\eta = \eta(Q,m,n,\bar\varepsilon)$ be the parameter given by Lemma 12.3 with this choice of $\bar\varepsilon$. We then define

$$\varrho := \inf\{0 \leq r \leq \tfrac{1}{2} : I_{0,u}(r) \geq 1+\eta\}\,.$$

In particular, we can apply Lemma 12.3 to infer, respectively in each case ((12.1) or (12.2)) that

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \bar\varepsilon r \int_{\partial B_r} |u|^2 \qquad \forall r \leq \varrho \tag{12.9}$$

$$\min_{v\in\mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq \bar\varepsilon r \int_{\partial B_r} |u|^2 \qquad \forall r \leq \varrho\,. \tag{12.10}$$

We know by the decay of the L^2 height in terms of the frequency (cf. [10, Corollary 3.18]) that

$$\int_{\partial B_r} |u|^2 \leq \left(\frac{r}{s}\right)^{m-1+2I_{0,u}(r)} \int_{\partial B_s} |u|^2 \qquad \forall r \leq s \leq 1\,. \tag{12.11}$$

Moreover, we have

$$\min_{t\in[1/2,1]} \int_{\partial B_t} |u|^2 \leq C \int_{B_1} |u|^2\,, \tag{12.12}$$

for some constant $C = C(Q,m,n)$. Thus, we infer from (12.9) (respectively (12.10)), combined with (12.11) with s chosen to be $t \in [1/2,1]$ realizing the minimum and the fact that $I_{0,u}(r) \geq 1$ for all $r > 0$, that

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq C\bar\varepsilon r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \varrho \tag{12.13}$$

$$\min_{v\in\mathscr{L}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq C\bar\varepsilon r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \varrho\,. \tag{12.14}$$

This proves the result for $r \leq \varrho$; however, ϱ is not a geometric constant (it depends on u), so we are not done. Now, by monotonicity of the frequency and the definition of ϱ we know that $I_{0,u}(r) \geq 1+\eta$ for every $r > \varrho$. So, again by (12.11) and (12.12) (choosing appropriate $t \in [1/2,1]$ again), for every $r \in (\varrho, 1]$ we have

$$\int_{\partial B_s} |u|^2 \leq \left(\frac{s}{r}\right)^{m+1} \int_{\partial B_r} |u|^2 \leq s^{m+1}\left(\frac{r}{t}\right)^{2\eta} \int_{\partial B_t} |u|^2 \leq C s^{m+1} r^{2\eta} \int_{B_1} |u|^2 \qquad \forall s \leq r\,.$$

Integrating the latter over $s \in [0,r]$ we get

$$\int_{B_r} |u|^2 \leq C r^{m+2+2\eta} \int_{B_1} |u|^2 \qquad \forall \varrho < r \leq 1\,.$$

Since, however, the function $v \equiv Q[\![0]\!]$ belongs to $\mathscr{L}_1 \subset \mathscr{H}_1$, we can combine the latter inequality with (12.13) (resp. (12.14)) to get

$$\min_{v\in\mathscr{H}_1} \int_{B_r} \mathcal{G}(u,v)^2 \leq C\max\{\bar\varepsilon, r^{2\eta}\} r^{m+2} \int_{B_1} |u|^2 \qquad \forall r \leq \frac{1}{2}\,, \tag{12.15}$$

and respectively

$$\min_{v\in\mathscr{L}_1}\int_{B_r}\mathcal{G}(u,v)^2 \le C\max\{\bar\varepsilon, r^{2\eta}\}r^{m+2}\int_{B_1}|u|^2 \qquad \forall r\le \frac{1}{2}, . \tag{12.16}$$

We now choose first $\bar\varepsilon$ so that $C\bar\varepsilon \le \varepsilon$, which in turn fixes the value of $\eta > 0$ determined by Lemma 12.3. Hence we can choose $\rho = \rho(Q,m,n,\varepsilon) > 0$ such that $C\rho^{2\eta} \le \varepsilon$. Then the desired estimates (12.1) and (12.2) for $r \le \rho$ follow. □

The second result we will need is a suitable "removability result" which we stated previously in Proposition 9.4. We recall the statement for the convenience of the reader.

Proposition 12.4. *Assume $\Omega \subset \mathbb{R}^m$ is a Lipschitz domain, $V \subset \mathbb{R}^m$ is an $(m-2)$-dimensional plane, and $v \in W^{1,2}(\Omega;\mathcal{A}_Q(\mathbb{R}^n))$ is a map with the property that the restriction of v to $\Omega_\varepsilon := \Omega\setminus B_\varepsilon(V)$ is Dir-minimizing for every $\varepsilon > 0$. Then, v is Dir-minimizing in Ω.*

We remark an important subtlety: the set which is removed from Ω to get to Ω_ε is *not* compactly contained in Ω when $m \ge 3$.

We will in fact prove the following lemma, where we use the notion of p-capacity, $\mathrm{Cap}^m_p(A)$, of a set $A \subset \mathbb{R}^m$. We refer the reader to e.g. [17] for the definition and preliminaries.

Lemma 12.5. *Let $\Omega \subset \mathbb{R}^m$ be any bounded Lipschitz domain and $A \subset \mathbb{R}^m$ any set with $\mathrm{Cap}^m_2(A) = 0$. Then, for any two maps $v,u \in W^{1,2}(\Omega,\mathcal{A}_Q(\mathbb{R}^n))$ such that $v = u$ on $\partial\Omega$ and for any $\delta > 0$, we can find a third map $w \in W^{1,2}(\Omega,\mathcal{A}_Q(\mathbb{R}^n))$ which coincides with v on $\partial\Omega\cup(\Omega\cap U)$ for some neighborhood U of A and such that*

$$\int_\Omega |Dw|^2 \le \int_\Omega |Du|^2 + \delta\,.$$

Let us first prove Proposition 12.4 from Lemma 12.5.

Proof of Proposition 12.4. First note that since V is an $(m-2)$-dimensional plane, it obeys $\mathrm{Cap}^m_2(V) = 0$ (note here $m > 2$, and so this follows because sets with finite $\mathcal{H}^{m-2}$-measure have vanishing 2-capacity, see [17]; the result still holds for $m = 2$ by using the classical logarithmic cut-off trick).

Suppose for contradiction that v is not Dir-minimizing in Ω. Then we can find a competitor $u \in W^{1,2}(\Omega;\mathcal{A}_Q(\mathbb{R}^n))$ with $v = u$ on $\partial\Omega$ and

$$\int_\Omega |Du|^2 < \int_\Omega |Dv|^2 - \delta$$

for some $\delta > 0$. Now apply Lemma 12.5 with this choice of v, u and δ to find a map $w \in W^{1,2}(\Omega;\mathcal{A}_Q(\mathbb{R}^n))$ which coincides with v on $\partial\Omega\cup(\Omega\cap B_{\varepsilon_*}(V))$ for some $\varepsilon_* > 0$. In particular, we have

$$\int_\Omega |Dw|^2 \le \int_\Omega |Du|^2 + \delta < \int_\Omega |Dv|^2.$$

However, this contradicts v being Dir-minimizing on Ω_{ε_*} (since $v = w$ on $B_{\varepsilon_*}(V)$). □

All that remains is to prove Lemma 12.5.

Proof of Lemma 12.5. We fix Ω a Lipschitz domain and we let $v, u : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be any pair of $W^{1,2}(\Omega;\mathcal{A}_Q(\mathbb{R}^n))$ functions with $v|_{\partial\Omega} = u|_{\partial\Omega}$, where the boundary data is defined in the Sobolev trace sense (see [10, Section 2.2.2]). Consider now $\tilde u := \boldsymbol{\xi}_{BW}\circ u$ and $\tilde v := \boldsymbol{\xi}_{BW}\circ v$, where $\boldsymbol{\xi}_{BW} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ is the special bi-Lipschitz embedding of White, which is constructed via a modification of the original embedding of Almgren (see [3] or [10, Corollary 2.2]).

Let $z := \tilde u - \tilde v$ and extend it to be identically equal to 0 on $\mathbb{R}^m\setminus\Omega$. We claim that there is a sequence $\{\zeta_k\} \subset W^{1,2}(\mathbb{R}^m,\mathbb{R}^N)$ with the following properties:

- ζ_k vanishes identically on $(\mathbb{R}^m\setminus\Omega)\cup U_k(A)$, where $U_k(A)$ is some open neighborhood of A (which depends on k);
- $\|\zeta_k - z\|_{W^{1,2}} \to 0$ as $k\uparrow\infty$.

Assuming the existence of such a sequence $\{\zeta_k\}$ for now, we then let $w_k := \boldsymbol{\xi}_{BW}^{-1}\circ(\boldsymbol{\rho}\circ(\tilde v+\zeta_k))$, where $\boldsymbol{\rho}$ is Almgren's Lipschitz retraction of $\mathbb{R}^N$ onto $\boldsymbol{\xi}_{BW}(\mathcal{A}_Q(\mathbb{R}^N))$ (cf. [10, Chapter 2]). Observe that $\tilde v+\zeta_k=\tilde v$ on $\partial\Omega\cup(\Omega\cap U_k(A))$ and so $w_k=v$ on $\partial\Omega\cup(\Omega\cap U_k(A))$. Moreover

$$\int_\Omega |Dw_k|^2=\int_\Omega |D(\boldsymbol{\rho}\circ(\tilde v+\zeta_k))|^2$$

and

$$\int_\Omega |Du|^2=\int_\Omega |D\tilde u|^2 .$$

On the other hand $\tilde v+\zeta_k$ converges to $\tilde v+z=\tilde v+(\tilde u-\tilde v)=\tilde u$ strongly in $W^{1,2}(\mathbb{R}^m;\mathbb{R}^N)$ and, because $\boldsymbol{\rho}$ is Lipschitz and equal to the identity on $\boldsymbol{\xi}_{BW}(\mathcal{A}_Q(\mathbb{R}^n))$, $\boldsymbol{\rho}\circ(\tilde v+\zeta_k)$ converges strongly in $W^{1,2}$ to $\boldsymbol{\rho}\circ\tilde u=\tilde u$. In particular we conclude from this and the above two identities that the Dirichlet energy of w_k converges to the Dirichlet energy of u, and thus to conclude the result of the lemma we just need to take $w=w_k$ for k sufficiently large (depending on δ).

So now we just need to show the existence of ζ_k satisfying the desired properties. By the definition of Cap_2^m, given $\varepsilon>0$, we may choose $\rho_\varepsilon\in C_c^\infty(B_\varepsilon(A))$ with $\rho_\varepsilon\geq \mathbf{1}_{B_{\varepsilon/2}(A)}$ and such that

$$\int |D\rho_\varepsilon|^2\leq \mathrm{Cap}_2^m(A)+\varepsilon=\varepsilon.$$

In particular we can assume, by truncating ρ_ε, that it takes values between 0 and 1; clearly it also follows that $\rho_\varepsilon\to 0$ pointwise almost everywhere as $\varepsilon\downarrow 0$.

Next we choose M sufficiently large and consider the truncation function $z_M=(z_M^1,\dots,z_M^N)$ given by

$$z_M^i:=\min\{z^i,M\}\mathbf{1}_{\{z^i>0\}}+\max\{z^i,-M\}\mathbf{1}_{\{z^i\leq 0\}},$$

where we have written $z=(z^1,\dots,z^N)$. It is simple to see that $z_M\to z$ in $W^{1,2}$ as $M\to\infty$, and so for each $k\in\{1,2,\dots\}$ we can select $M=M(k)$ so that $\|z_M-z\|_{W^{1,2}}\leq\frac{1}{2k}$. Observe also that z_M always vanishes outside of Ω. Next we take $\zeta_k:=z_M(1-\rho_{\varepsilon_k})$ for a sufficiently small ε_k to be chosen. Note indeed that for fixed M, $\|z_M\rho_\varepsilon\|_{L^2}\to 0$ as $\varepsilon\downarrow 0$ by dominated convergence. On the other hand

$$D(z_M(1-\rho_\varepsilon))-Dz_M=-\rho_\varepsilon Dz_M-z_MD\rho_\varepsilon$$

and $\|Dz_M\rho_\varepsilon\|_{L^2}\to 0$ as $\varepsilon\downarrow 0$, again by dominated convergence, while we have $\|z_MD\rho_\varepsilon\|_{L^2}\leq\sqrt{N}M\|D\rho_\varepsilon\|_{L^2}$ converges to 0 as well as long as we keep M fixed while $\varepsilon\downarrow 0$. Thus, $z_M(1-\rho_\varepsilon)\to z_M$ in $W^{1,2}$, and so we can choose $\varepsilon=\varepsilon_k$ sufficiently small so that $\|z_M(1-\rho_{\varepsilon_k})-z_M\|_{W^{1,2}}\leq\frac{1}{2k}$. This is the choice of ε_k used to define ζ_k. Note then that $\|\zeta_k-z\|_{W^{1,2}}\leq 1/k\to 0$, and also that ζ_k vanishes identically on $(\mathbb{R}^m\setminus\Omega)\cup B_{\varepsilon_k/2}(A)$; this completes the proof. □

13. Final blow-up argument

In this part we complete the proof of Theorem 10.2, which we recall in turn implies the validity of Theorem 2.5, as demonstrated in Section 10.

13.1. Two regimes.

The proof of Theorem 10.2 will be split into two cases, which will both be proved via a blow-up argument. We start by giving the detailed statements.

Proposition 13.1 (Collapsed decay). *For every $Q,m,n,\bar n$, and $\varsigma_1>0$ there are positive constants $\varepsilon_c=\varepsilon_c(Q,m,n,\bar n,\varsigma_1)\leq 1/2$ and $r_c=r_c(Q,m,n,\bar n,\varsigma_1)\leq 1/2$ with the following property. Assume that*

(i) *T and Σ are as in Assumption 2.1, and $\|T\|(\mathbf{B}_1)\leq\omega_m(Q+\frac12)$;*
(ii) *There is a cone $\mathbf{S}\in\mathscr{C}(Q,0)$ which is M-balanced (with M as in Assumption 10.1), such that (10.1) and (10.2) hold with ε_c in place of ε_1, and in addition $\boldsymbol{\mu}(\mathbf{S})\leq\varepsilon_c$;*
(iii) *$\mathbf{A}^2\leq\varepsilon_c^2\mathbb{E}(T,\tilde{\mathbf{S}},\mathbf{B}_1)$ for every $\tilde{\mathbf{S}}\in\mathscr{C}(Q,0)$.*

Then, there is another cone $\mathbf{S}'\in\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ such that

$$\mathbb{E}(T,\mathbf{S}',\mathbf{B}_{r_c})\leq\varsigma_1\mathbb{E}(T,\mathbf{S},\mathbf{B}_1). \tag{13.1}$$

Proposition 13.2 (Non-collapsed decay)**.** *For every $Q, m, n, \bar{n}$, $\varepsilon_c^\star > 0$, and $\varsigma_1 > 0$, there are positive constants $\varepsilon_{nc} = \varepsilon_{nc}(Q, m, n, \bar{n}, \varepsilon_c^\star, \varsigma_1) \leq 1/2$ and $r_{nc} = r_{nc}(Q, m, n, \bar{n}, \varepsilon_c^\star, \varsigma_1) \leq \frac{1}{2}$ with the following property. Assume that*

(i) *T and Σ are as in Assumption 2.1 and $\|T\|(\mathbf{B}_1) \leq \omega_m(Q + \frac{1}{2})$;*
(ii) *There is $\mathbf{S} \in \mathscr{C}(Q, 0)$ which is M-balanced (with M as in Assumption 10.1), such that (10.1) and (10.2) hold with ε_{nc} in place of ε_1 and in addition $\boldsymbol{\mu}(\mathbf{S}) \geq \varepsilon_c^\star$;*
(iii) *$\mathbf{A}^2 \leq \varepsilon_{nc}^2 \mathbb{E}(T, \tilde{\mathbf{S}}, \mathbf{B}_1)$ for every $\tilde{\mathbf{S}} \in \mathscr{C}(Q, 0)$.*

Then, there is another cone $\mathbf{S}' \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ such that

$$\mathbb{E}(T, \mathbf{S}', \mathbf{B}_{r_{nc}}) \leq \varsigma_1 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) . \tag{13.2}$$

Remark 13.3. In fact, for the proof of Proposition 13.1 and Proposition 13.2, instead of condition (iii) all we will need to assume is that $\mathbf{A}^2 \leq \varepsilon_c^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$ or $\mathbf{A}^2 \leq \varepsilon_{nc}^2 \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)$, respectively, for the cone $\mathbf{S}$ as in (ii) in each proposition, respectively.

Theorem 10.2 obviously follows from the two propositions above. In fact, being given $Q, m, n, \bar{n}$, and ς_1 as in Theorem 10.2, we first apply Proposition 13.1 and get ε_c and r_c. We then apply Proposition 13.2 with the same choice of $Q, m, n, \bar{n}$, and ς_1, and with $\varepsilon_c^\star = \varepsilon_c$, to get ε_{nc} and r_{nc}. It is then clear that Theorem 10.2 holds if we set $\varepsilon_1 := \min\{\varepsilon_c, \varepsilon_{nc}\}$, $r_2^1 := r_c$ and $r_2^2 := r_{nc}$.

Both propositions will be proved by a blow-up procedure, which will assume, by seeking a contradiction, that the statements fail. In the proof of Proposition 13.2 this means that $\varepsilon_c^\star$ is fixed while ε_{nc} will be taken to be arbitrarily small and so in particular $\boldsymbol{\mu}(\mathbf{S})$ will stay away from 0 while the ratio

$$\frac{\mathbb{E}(T, \mathbf{S}, \mathbf{B}_1)}{\boldsymbol{\sigma}(\mathbf{S})^2} \tag{13.3}$$

is arbitrarily small. We will call the latter a *Simon blow-up* (since in the work [29] the cones are uniformly non-collapsed, analogously to this case), and in this case we will take limits of suitable rescalings of the coherent outer approximations of Proposition 8.19.

In the proof of Proposition 13.1 we will instead assume that ε_c is arbitrarily small. This will mean that *both* the ratio in (13.3) and $\boldsymbol{\mu}(\mathbf{S})$ are arbitrarily small. In this case we will approximate the current in the outer region by reparameterizing the graphs of the coherent outer approximations over a single plane, which we can fix to be one of the planes forming $\mathbf{S}$, say α_1. We will call these the *transversal coherent approximations*, as opposed to the ones of the Simon blow-up, which will be called the *orthogonal coherent approximations*. By construction the transversal approximations are superpositions of multi-valued maps, each of them close to the linear map describing α_j as a graph over α_1. We will then subtract the latter linear map from the sheet of the corresponding multi-valued approximation and study the limits of appropriate rescalings. We will call this procedure a *Wickramasekera blow-up* (since the work [32] is the first appearance of such a blow-up where the cones are collapsing to a high-multiplicity flat plane).

13.2. **Transversal coherent approximation.** In this section we work under the assumption of Proposition 13.1 and recast the coherent outer approximation of Proposition 8.19 as an approximation through a Lipschitz multi-valued function over a single plane (the latter function representing the cone $\mathbf{S}$). The choice of the plane is not really important as long as its Hausdorff distance from $\mathbf{S}$ in $\mathbf{B}_1$ is comparable to $\boldsymbol{\mu}(\mathbf{S})$; we may without loss of generality take it to be the first indexed plane α_1 forming $\mathbf{S}$. Given a multi-valued function $g = \sum_i [\![g_i]\!]$ and a single-valued function f with the same domain, we also use the notational shorthand $g \ominus f$ for

$$g \ominus f(x) := \sum_i [\![g_i(x) - f(x)]\!] ,$$

and similarly we define $g \oplus f$. We moreover recall the set $\mathscr{N}(L)$ of "neighbouring cubes" to L and the quantity $\bar{\mathbf{E}}(L)$ introduced in Definition 8.18 and introduce

$$\widetilde{\mathbf{E}}(L) := \max\{\bar{\mathbf{E}}(L') : L' \in \mathscr{N}(L)\} = \max\{\mathbf{E}(L'') : L'' \in \mathscr{N}(L'),\, L' \in \mathscr{N}(L)\}$$

i.e. we maximize the excess of neighbours of neighbours.

Proposition 13.4 (Transversal coherent approximation)**.** *Let T, Σ, and $\mathbf{S}$ be as in Proposition 8.15, let $\ell \in \mathbb{N}$ be the maximal number such that $\mathcal{G}_{\ell+2} \subset \mathcal{G}^o$, and consider the regions $\widetilde{R}_i^o := \alpha_i \cap \bigcup_{L\in\mathcal{G}_{\leq \ell}} R(L)$. Let $u_i : R_i^o \to \mathcal{A}_{Q_i}(\alpha_i^\perp)$ be the Q_i-valued maps in Proposition 8.19. Under the additional assumption that $\boldsymbol{\mu}(\mathbf{S}) \leq c_0$ for a geometric constant $c_0 = c_0(m, n, \bar{n}) > 0$, the following holds.*

(a) *Each plane α_i is the graph over α_1 of a linear map $A_i : \alpha_1 \to \alpha_1^\perp$ with $|A_i| \leq C\boldsymbol{\mu}(\mathbf{S})$ and $\ker(A_i) = V$.*

(b) *For each i there is a map $v_i : \widetilde{R}_1^o \to \mathcal{A}_{Q_i}(\alpha_1^\perp)$ with the property that*

$$\mathbf{G}_{v_i} = \mathbf{G}_{u_i} \llcorner \mathbf{p}_{\alpha_1}^{-1}(\widetilde{R}_1^o)\,.$$

(c) *If we let $v : \widetilde{R}_1^o \to \mathcal{A}_Q(\alpha_1^\perp)$ be the Q-valued function $v := \sum_i [\![v_i]\!]$ (note that $Q = \sum_{i=1}^N Q_i$), then*

$$\|v\|_{L^\infty} + \|Dv\|_{L^2} \leq C\boldsymbol{\mu}(\mathbf{S}) \tag{13.4}$$

(d) *If we let*

$$K := \mathbf{p}_{\alpha_1}((\mathrm{spt}(T) \cap \mathbf{B}_{1/2} \cap \mathbf{p}_{\alpha_1}^{-1}(\widetilde{R}_1^o)) \setminus \mathrm{gr}\,(v))$$

then, for all $L \in \mathcal{G}_{\leq\ell}$,

$$|L_1 \setminus K| + \|T\|(\mathbf{p}_{\alpha_1}^{-1}(L_1 \setminus K)) \leq C 2^{-m\ell(L)}(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^{1+\gamma}\,, \tag{13.5}$$

where L_1 is as in Proposition 8.19.

(e) *If we let $w_i := v_i \ominus A_i : \tilde{R}_1^o \to \mathcal{A}_{Q_i}(\alpha_1^\perp)$, then*

$$2^{2\ell(L)}\|w_i\|^2_{L^\infty(L_1)} + 2^{m\ell(L)}\|Dw_i\|^2_{L^2(L_1)} \leq C(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2) \tag{13.6}$$

$$\|Dw_i\|_{L^\infty(L_1)} \leq C(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)^\gamma \tag{13.7}$$

Here, $C = C(Q, m, n, \bar{n})$.

Proof. Claim (a) is obvious. For the rest, all we are essentially doing is changing the coordinates of u_i, to give a parameterization over α_1 rather than α_i. Claims (b) and and the L^∞-estimate in (c) follow immediately from [12, Proposition 5.2]. The L^2-gradient estimate in (c) follows immediately from the L^2 bound in (13.6) and (a). Conclusion (d) follows immediately from the corresponding estimate in Proposition 8.19. As for (13.7), observe that, for a.e. $x \in \tilde{R}_1^o$, $|Dw_i|(x)$ equals, up to a geometric constant, the following quantity:

$$\sup\{|\vec{\mathbf{G}}_{v_i}(x,y) - \vec{\alpha}_i| : (x,y) \in \mathrm{gr}\,(v_i)\}$$

while an analogous formula holds for $|Du_i|(x)$, with $x \in R_i^o$: since the graph of v_i equals that of u_i in a different coordinate system, the estimate (13.7) follows from the analogous one for u_i in Proposition 8.19. Likewise, $\|Dw_i\|^2_{L^2(L_1)}$ is equivalent, up to constants, to

$$\int_{\mathbf{p}_{\alpha_1}^{-1}(L_1)} |\vec{\mathbf{G}}_{v_i}(p) - \vec{\alpha}_i|^2\, d\|\mathbf{G}_{v_i}\|(p)\,.$$

Thus from the corresponding inequality for u_i, we obtain the inequality

$$\|Dw_i\|^2_{L^2(L_1)} \leq C2^{-m\ell(L)}(\widetilde{\mathbf{E}}(L) + 2^{-2\ell(L)}\mathbf{A}^2)\,.$$

As for $\|w_i\|_{L^\infty(L_1)}$ it is easy to see that it is controlled by $\|u_i\|_{L^\infty(\lambda L_i)}$ for a choice of λ slightly larger than 1 (cf. the arguments in [12, Section 5]). This establishes (13.6) and so completes the proof. □

13.3. **Non-concentration estimates.** In this section we draw some important conclusions from Simon's estimates in Section 11. We will work under the following assumptions throughout this section.

Assumption 13.5. T and Σ are as in Assumption 2.1. $\mathbf{S} \in \mathscr{C}(Q, 0) \setminus \mathscr{P}(0)$ is such that

$$\mathbf{A}^2 + \mathbb{E}(T, \mathbf{S}, \mathbf{B}_1) \leq \varepsilon^2 \boldsymbol{\sigma}(\mathbf{S})^2 \tag{13.8}$$

and

$$\mathbf{B}_\varepsilon(y) \cap \{\Theta(T, \cdot) \geq Q\} \neq \emptyset \quad \text{for all } y \in V(\mathbf{S}) \cap \mathbf{B}_1\,, \tag{13.9}$$

for some $\varepsilon > 0$, to be determined.

Next, consider recall the family of cubes $\mathcal{G}$, where $L \in \mathcal{G}$ has center point $y_L \in V$, as defined in Section 8.5.1.

Definition 13.6. Let T, Σ, and $\mathbf{S}$ be as in Assumption 13.5. For every $L \in \mathcal{G}$ we let $\beta(L)$ be a point p in $\{\Theta(T,\cdot) \geq Q\}$ which minimizes the distance to y_L. This point will be called the *nail* of L.

Of course, there might be more than one candidate for the nail of L, but under Assumption 13.5, one nail always exists as the set $\{\Theta(T,\cdot) \geq Q\}$ is closed by upper semi-continuity of density. In the rest of the paper we will just assume that some arbitrary choice for each given nail has been made. The main conclusion of this section is then the following.

Proposition 13.7. *There is a constant $C = C(m,n,\bar{n},Q)$ such that, for every $\varrho > 0$ there exists $\varepsilon = \varepsilon(Q,m,n,\bar{n},\varrho) > 0$ with the following property. Assume T, Σ, and $\mathbf{S}$ are as in Assumption 13.5 with this choice of ε, let r_* denote the radius of Proposition 11.4 and let $r = \frac{r_*}{4}$. Then*

$$\int_{\mathbf{B}_r} \frac{\operatorname{dist}^2(q,\mathbf{S})}{\max\{\varrho, \operatorname{dist}(q,V)\}^{3/2}}\, d\|T\|(q) \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)\,. \tag{13.10}$$

Moreover, if $u_1,\dots,u_N$ are the coherent approximations of Proposition 8.19, upon suitably modifying them (without relabelling) we can assume that they satisfy the following stronger estimates for each $j \in \{1,\dots,N\}$:

$$\int_{(\mathbf{B}_r\cap\alpha_j)\setminus B_\varrho(V)} \frac{|Du_j(z)|^2}{\operatorname{dist}(z,V)^{3/2}}\, dz \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2) \tag{13.11}$$

$$\sum_{i:\, 2^{-i-1}\geq\varrho}\ \sum_{L\in\mathcal{G}^i} \int_{L_j} \frac{|u_j(z) \ominus (\mathbf{p}^\perp_{\alpha_j}(\beta(L)))|^2}{\operatorname{dist}(z,V)^{7/2}}\, dz \leq C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1) + \mathbf{A}^2)\,. \tag{13.12}$$

Finally, under the additional assumption that $\boldsymbol{\mu}(\mathbf{S}) \leq c_0$ for the constant $c_0 = c_0(m,n,\bar{n})$ of Proposition 13.4, the maps $w_1,\dots,w_N$ and $A_1,\dots,A_N$ defined therein satisfy the corresponding estimates

$$\int_{(\mathbf{B}_r\cap\alpha_1)\setminus B_\varrho(V)} \frac{|Dw_j(z)|^2}{\operatorname{dist}(z,V)^{3/2}}\, dz \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)) \tag{13.13}$$

$$\sum_{2^{-i-1}\geq\varrho}\ \sum_{L\in\mathcal{G}^i} \int_{L_1} \frac{|w_j(z) \ominus (\mathbf{p}^\perp_{\alpha_1}(\beta(L)) + A_j(\mathbf{p}_{V^\perp\cap\alpha_1}(\beta(L))))|^2}{\operatorname{dist}(z,V)^{7/2}}\, dz \leq C(\mathbf{A}^2 + \hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1))\,. \tag{13.14}$$

Proof. We begin with the first estimate (13.10); we may assume that $\varrho < 1$ is smaller than half the radius in Proposition 11.4, as if not the inequality follows trivially. We begin by estimating the part of the integral on the left-hand side that is over $\mathbf{B}_r \cap B_\varrho(V)$. Cover $V \cap \mathbf{B}_r$ with $C\varrho^{-(m-2)}$ balls $\mathbf{B}_\varrho(y_i)$ of radius ϱ (of course, we can assume $\mathbf{B}_\varrho(y_i) \cap V \neq \emptyset$ for each i) and observe that $\{\mathbf{B}_{2\varrho}(y_i)\}_i$ covers $B_\varrho(V) \cap \mathbf{B}_r$. If (13.9) holds with $\varepsilon < \varrho$, then for each i we can find a point $p_i \in \mathbf{B}_{2\varrho}(y_i)$ such that $\Theta(T,p_i) \geq Q$. In particular we can estimate, using Proposition 11.4 with $\kappa = 1/4$, centered at each p_i,

$$\begin{aligned}
\varrho^{-3/2}\int_{\mathbf{B}_r\cap B_\varrho(V)} \operatorname{dist}^2(q,\mathbf{S})\, d\|T\|(q) &\leq \varrho^{-3/2}\sum_i \int_{\mathbf{B}_r\cap B_{2\varrho}(y_i)} \operatorname{dist}^2(q,\mathbf{S})\, d\|T\|(q) \\
&\leq C\varrho^{-3/2}\sum_i \int_{\mathbf{B}_r\cap B_{2\varrho}(y_i)} \operatorname{dist}^2(q,p_i+\mathbf{S})\, d\|T\|(q) \\
&\qquad + C\varrho^{-3/2}\sum_i \varrho^m\left(|\mathbf{p}^\perp_{\alpha_1}(p_i)|^2 + \boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(p_i)|^2\right) \\
&\leq C\varrho^{m+7/4-3/2}\sum_i \int_{\mathbf{B}_r\cap B_{4\varrho}(p_i)} \frac{\operatorname{dist}^2(q,p_i+\mathbf{S})}{|q-p_i|^{m+7/4}}\, d\|T\|(q)
\end{aligned}$$

$$
\begin{aligned}
&+ C\varrho^{-(m-2)}\varrho^{m-3/2}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\\
&\le C\varrho^{-(m-2)}\varrho^{m+1/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) + C\varrho^{1/2}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\\
&\le C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,. \qquad (13.15)
\end{aligned}
$$

We next consider the region $\mathbf{B}_r\setminus B_\varrho(V)$ and note that we can we cover it with the regions $R(L)$ for $L\in\mathcal{G}$ such that $2^{-\ell(L)-1}\ge\varrho$. However, we only include in the cover the cubes L for which $R(L)$ have a nonempty intersection with $\mathbf{B}_r$: denote the latter family of cubes by $\mathscr{F}$. For each such cube we denote by $\beta(L)$ its nail as in Definition 13.6. Note in particular that, if $\varepsilon<\frac{\varrho}{4}$ in (13.9), then

$$
C^{-2}2^{-\ell(L)}\le C^{-1}|q-\beta(L)|\le \operatorname{dist}(q,V)\le C|q-\beta(L)|\le C^2 2^{-\ell(L)} \qquad \forall q\in R(L)\,,
$$

where C is a geometric constant. Again combining with Proposition 11.4 with $\kappa=\frac14$, this allows us to estimate

$$
\begin{aligned}
&\int_{\mathbf{B}_r\setminus B_\varrho(V)}\frac{\operatorname{dist}^2(q,\mathbf{S})}{\operatorname{dist}(q,V)^{3/2}}\,d\|T\|(q)\\
&\quad\le C\sum_{i\le -1-\log_2(\varrho)}\sum_{L\in\mathcal{G}_i\cap\mathscr{F}}2^{3i/2}\int_{R(L)}\operatorname{dist}^2(q,\mathbf{S})\,d\|T\|(q)\\
&\quad\le C\sum_{i\le -1-\log_2(\varrho)}\sum_{L\in\mathcal{G}_i\cap\mathscr{F}}2^{3i/2}\int_{R(L)}\operatorname{dist}^2(q,\beta(L)+\mathbf{S})\,d\|T\|(q)\\
&\qquad + C\sum_{i\le -1-\log_2(\varrho)}\sum_{L\in\mathcal{G}_i\cap\mathscr{F}}2^{3i/2}2^{-mi}\big(|\mathbf{p}_{\alpha_1}^\perp(\beta(L))|^2+\boldsymbol{\mu}(\mathbf{S})^2|\mathbf{p}_{V^\perp\cap\alpha_1}(\beta(L))|^2\big)\\
&\quad\le C\sum_{i\le -1-\log_2(\varrho)}\sum_{L\in\mathcal{G}_i\cap\mathscr{F}}2^{3i/2-mi-7i/4}\int_{R(L)}\frac{\operatorname{dist}^2(q,\beta(L)+\mathbf{S})}{|q-\beta(L)|^{m+7/4}}\,d\|T\|(q)\\
&\qquad + C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\sum_{i\le -1-\log_2(\varrho)}2^{3i/2-mi}\#\mathcal{G}_i\\
&\quad\le C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\sum_{i\le -1-\log_2(\varrho)}(2^{-i/4-mi}+2^{3i/2-mi})\#\mathcal{G}_i\,.
\end{aligned}
$$

Since the cardinality $\#\mathcal{G}_i$ of $\mathcal{G}_i$ is bounded by $C2^{(m-2)i}$, it follows that

$$
\int_{\mathbf{B}_r\setminus B_\varrho(V)}\frac{\operatorname{dist}^2(q,\mathbf{S})}{\operatorname{dist}(q,V)^{3/2}}\,d\|T\|(q)\le C(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,. \qquad (13.16)
$$

Clearly (13.15) and (13.16) imply (13.10).

We next use the same family $\mathscr{F}$ of cubes to prove the remaining estimates. Fix any cube $L\in\mathscr{F}$ and recall Proposition 8.15, but rather than approximating over $\mathbf{S}$, approximate the current over the shifted cone $\mathbf{S}+\beta(L)$ (which we can do by Proposition 11.4, namely (11.6)). We then gain an approximation $\tilde{u}_{L,i}:\mathbf{p}_{V^\perp\cap\alpha_i}(\beta(L))+\lambda L_i\to\mathcal{A}_{Q_i}(\alpha_i^\perp)$ which satisfies the estimate

$$
\|D\tilde{u}_{L,i}\|_{L^2}^2\le C2^{-m\ell(L)}(\hat{\mathbf{E}}(T,\mathbf{S}+\beta(L),\mathbf{B}_{C2^{-\ell(L)}}(\beta(L)))+2^{-2\ell(L)}\mathbf{A}^2)\,.
$$

Observe that, by Proposition 11.4, namely (11.5), we get

$$
\hat{\mathbf{E}}(T,\mathbf{S}+\beta(L),\mathbf{B}_{C2^{-\ell(L)}}(\beta(L)))\le C2^{\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,.
$$

(note that we can apply (11.5) provided $C2^{-\ell(L)}\le r$, where $r=r(Q,m,n,\bar n)$ is the radius from Proposition 11.4, i.e. provided $\ell(L)$ is larger than a constant depending on $Q,m,n,\bar n$. For the cubes L with $\ell(L)$ smaller than this constant, the inequality above follows more easily as we have an upper bound on $\ell(L)$). In particular we have

$$
\|D\tilde{u}_{L,i}\|_{L^2}^2\le C2^{-m\ell(L)+\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,.
$$

Arguing as above, Proposition 8.15 and (11.5) also give the estimate

$$
\|\tilde{u}_{L,i}\ominus(\mathbf{p}_{\alpha_i}^\perp(\beta(L)))\|_{L^\infty}\le C2^{-7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)
$$

We can now consider the map $\hat{u}_{L,i} := \tilde{u}_{L,i} \ominus (\mathbf{p}^{\perp}_{\alpha_i}(\beta(L)))$ over the domain $\mathbf{p}_{V^\perp\cap\alpha_i}(\beta(L)) + \lambda L_i$, which we can assume contains L_i, provided that ε is sufficiently small. Observe that we can use this new map in the algorithm within Proposition 8.19 in place of $u_{L,i}$ to construct a coherent approximation from $\hat{u}_{L,i}$, since this new map satisfies the same estimates as above. We can in turn use this new coherent approximation to construct a coherent transversal approximation as in Proposition 13.4. In particular, if we keep denoting these improved approximations by u_i and w_i without relabeling, they will satisfy the estimates

$$\int_{L_i} \frac{|Du_i|^2}{\mathrm{dist}(z,V)^{3/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{13.17}$$

$$\int_{L_1} \frac{|Dw_i|^2}{\mathrm{dist}(z,V)^{3/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{13.18}$$

$$\int_{L_i} \frac{|u_i \ominus (\mathbf{p}^{\perp}_{\alpha_i}(\beta(L)))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2) \tag{13.19}$$

$$\int_{L_1} \frac{|w_i \ominus (\mathbf{p}^{\perp}_{\alpha_1}(\beta(L)) + A_i(\mathbf{p}_{V^\perp\cap\alpha_1}(\beta(L))))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz \le C2^{-m\ell(L)+7\ell(L)/4}(\hat{\mathbf{E}}(T,\mathbf{S},\mathbf{B}_1)+\mathbf{A}^2)\,. \tag{13.20}$$

We now sum the latter estimates over $L \in \mathscr{F} \cap \mathcal{G}_i$ for $i \le -\log_2(\varrho) - 1$. Considering that the cardinality of $\mathcal{G}_i$ is $C2^{(m-2)i}$ we get (13.11)–(13.14). □

13.4. **Blow-ups.** We are now ready to define blow-up sequences for the contradiction argument in the two cases of Proposition 13.1 and Proposition 13.2. In both cases we have a sequence of currents T_k, cones $\mathbf{S}_k$, and manifolds Σ_k for which the assumption of their respective proposition fails, with a sequence of thresholds $\varepsilon_c = 1/k$ in the case of Proposition 13.1 and with $\varepsilon_{nc} = 1/k$ in the case of Proposition 13.2 (we stress that $\varepsilon_c^\star$ is a fixed number in the case of Proposition 13.2 and currently unrelated to ε_c!). In particular, we know that (cf. Remark 13.3)

$$\lim_{k\uparrow\infty}\left(\frac{\mathbf{A}_k^2}{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)} + \frac{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)}{\boldsymbol{\sigma}(\mathbf{S}_k)^2}\right) = 0 \tag{13.21}$$

while, denoting by $\beta^k(L)$ the nails of Definition 13.6 for T_k, we have

$$\lim_{k\uparrow\infty} |\beta^k(L) - y_L| = 0 \qquad \forall L \in \mathcal{G}\,. \tag{13.22}$$

Moreover, upon extracting a subsequence, we can assume that $\mathbf{S}_k$ consists of a fixed number $N(k) = N \in \{2,\dots,Q\}$ of distinct planes, which we denote by $\alpha_1^k,\dots,\alpha_N^k$. Applying suitable rotations, we can assume that $T_0\Sigma_k$, $V(\mathbf{S}_k)$, and α_1^k are independent of k. In particular, we will often use V in place of $V(\mathbf{S}_k)$ and α_1 in place of α_1^k.

The distinction between the case of Proposition 13.1 and Proposition 13.2 is that

- (C) In the collapsed case (Proposition 13.1) we have $\boldsymbol{\mu}(\mathbf{S}_k) \to 0$;
- (NC) In the non-collapsed case (Proposition 13.2) we have $\liminf_k \boldsymbol{\mu}(\mathbf{S}_k) \ge \varepsilon_c^\star > 0$, for some $\varepsilon_c^\star > 0$.

In particular, in the non-collapsed case we can assume in addition that:

- (NC1) Up to relabelling, $\mathrm{dist}(\alpha_1^k \cap \mathbf{B}_1, \alpha_N^k \cap \mathbf{B}_1) = \boldsymbol{\mu}(\mathbf{S}_k)$;
- (NC2) Each sequence α_i^k converges (in Hausdorff distance) to a plane α_i, and α_N is necessarily distinct from α_1 (but some of the other planes α_i could coincide).

In the collapsed case we consider the transversal coherent approximations w_i^k, while in the non-collapsed case we consider the orthogonal coherent approximations u_i^k. In both cases we assume that their domains of definition $\mathrm{Dom}\,(w_i^k)$ and $\mathrm{Dom}\,(u_i^k)$ contain, respectively, $(\mathbf{B}_r \cap \alpha_1) \setminus B_{1/k}(V)$ and $(\mathbf{B}_r \cap \alpha_i^k) \setminus B_{1/k}(V)$. We further consider the rescaled functions

$$\bar{u}_i^k := \frac{u_i^k}{\sqrt{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)}} \tag{13.23}$$

$$\bar{w}_i^k := \frac{w_i^k}{\sqrt{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)}}\,. \tag{13.24}$$

Observe that we have uniform estimates for the $W^{1,2}$ norms of $\bar{u}_i^k$ and $\bar{w}_k^i$ (from Proposition 13.7, which are even stronger as they have a weight). The maps w_i^k are also defined over the same plane α_1. For the maps u_i^k there is the annoying technicality that they are not. In the latter case we refer to Remark 8.21 and assume they are, up to composition with a small rotation, defined on the same planes α_i.

In particular, up to extraction of a subsequence we can assume that the $\bar{u}_i^k$ and $\bar{w}_i^k$ converge strongly in $W^{1,2}$ locally away from V and strongly in L^2 on the entirety of B_r (due to the non-concentration estimates from Proposition 13.7) to $W^{1,2}$ maps $\bar{u}_i$ and $\bar{w}_i$ defined on $(\mathbf{B}_r \cap \alpha_i)\setminus V$ and $(\mathbf{B}_r \cap \alpha_1)\setminus V$ respectively (and taking values in $\mathcal{A}_Q(\alpha_i^\perp)$ and $\mathcal{A}_Q(\alpha_1^\perp)$ respectively). However, since V is an $(m-2)$-dimensional subspace, which has zero 2-capacity, the maps $\bar{u}_i$ and $\bar{w}_i$ belong indeed to $W^{1,2}(\alpha_i \cap \mathbf{B}_r)$ and $W^{1,2}(\alpha_1 \cap \mathbf{B}_r)$ respectively. In the collapsed case we also introduce the linear maps $A_i^k : \alpha_1 \to \alpha_1^\perp$ whose graphs describe the planes α_i^k, and can consider their rescalings

$$\bar{A}_i^k := \frac{A_i^k}{\boldsymbol{\mu}(\mathbf{S}_k})\,. \tag{13.25}$$

Moreover, up to extraction of a further subsequence, we assume that $\bar{A}_i^k$ converges to some linear map $\bar{A}_i$.

We are now ready to state our main proposition concerning the properties of the limiting maps $\bar{u}_i$ and $\bar{w}_i$.

Proposition 13.8. *Let $\bar{u}_i^k$, $\bar{w}_i^k$ and their respective limits $\bar{u}_i$, $\bar{w}_i$ be as described above. The following holds in the non-collapsed case:*

- (a) *Each $\bar{u}_i$ is Dir-minimizing on $\mathbf{B}_r \cap \alpha_i$, for r as in Proposition 13.7.*
- (b) *$\bar{u}_i = Q_i[\![\boldsymbol{\eta} \circ \bar{u}_i]\!]$ on V.*
- (c) *$\bar{u}_i(0) = \boldsymbol{\eta}\circ\bar{u}_i(0) = 0$ and the frequency obeys $I_{0,\bar{u}_i}(0) \geq 1$. Moreover, $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$.*
- (d) *There is a smooth function $\beta : V \to V^\perp$ such that $\boldsymbol{\eta} \circ \bar{u}_i = \mathbf{p}_{\alpha_i}^\perp(\beta)$ on V for every i. (Note that with a slight abuse of notation we are using the same letter identifying the nails of Definition 13.6. In fact this function β is the trace of the limit of suitable normalizations of the nails.)*

The following holds in the collapsed case:

- (e) *Each $\bar{w}_i$ is Dir-minimizing on $\mathbf{B}_r \cap \alpha_1$.*
- (f) *$\bar{w}_i = Q_i[\![\boldsymbol{\eta} \circ \bar{w}_i]\!]$ on V.*
- (g) *$\bar{w}_i(0) = \boldsymbol{\eta} \circ \bar{w}_i(0) = 0$ and the frequency $I_{0,\bar{w}_i}(0) \geq 1$. Moreover, $I_{y,\bar{w}_i\ominus\boldsymbol{\eta}\circ\bar{w}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$;*
- (h) *There are smooth function $\beta^\perp : V \to \alpha_1^\perp$ and $\beta_\parallel : V \to V^\perp \cap \alpha_1$ such that $\boldsymbol{\eta} \circ \bar{w}_i = \beta^\perp + \bar{A}_i(\beta_\parallel)$ on V for all i.*

The functions $\bar{u}_i$ and β also enjoy the estimates

$$\|\bar{u}_i\|_{W^{1,2}} + \|\beta\|_{C^2} \leq C \tag{13.26}$$

for some constant C which depends only on $Q, m, n, \bar{n}$ and $\varepsilon_c^\star$. The functions $\bar{w}_i$, $\beta^\perp$, and $\beta_\parallel$ enjoy the estimates

$$\|\bar{w}_i\|_{W^{1,2}} + \|\beta^\perp\|_{C^2} + \|\beta_\parallel\|_{C^2} \leq C \tag{13.27}$$

for some constant C which depends only on $Q, m, n, \bar{n}$.

Proof. We start with (a). Here we first appeal to Proposition 8.20 to prove the Dir-minimality of $\bar{u}_i$ away from the spine V. Observe that, if, along the sequence $\bar{u}_i^k$, the double-sided excess $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ stays comparable to $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)$, this claim follows from Proposition 8.20(iii) because $\mathbf{A}^2$ is infinitesimal compared to $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$. The alternative left is that (up to subsequence) $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)/\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ is infinitesimal. In this case, by Proposition 8.20(ii), $D\bar{u}_i$ vanishes. Having established the Dir-minimality away from the spine, Proposition 12.4 allows us to conclude that it is Dir-minimizing on any $\Omega \subset\subset \mathbf{B}_r$; this proves (a).

Now let us show (b). We next observe that, by (11.6) (in which one can clearly replace α_1 by any other plane of $\mathbf{S}$), $|\mathbf{p}^\perp_{\alpha_i^k}(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$ and, because of the lower bound on $\mu(\mathbf{S}_k)$, $|\mathbf{p}_{V^\perp\cap\alpha_i^k}(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$, for each $i = 1, \ldots, N$. In particular we easily conclude that

$$|\mathbf{p}^\perp_V(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}$$

for a constant $C = C(Q, m, n, \bar{n}, \varepsilon_c^\star)$. Thus we can assume that for every L, we can find $\bar{\beta}(L)$ such that

$$\lim_{k\to\infty} \frac{\mathbf{p}^\perp_V(\beta^k(L))}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}} = \bar{\beta}(L)\,.$$

It is notationally convenient to think of $\bar{\beta}$ as a piecewise constant function which is defined over $\mathbf{B}_r$ as being identically equal to $\bar{\beta}(L)$ on the set $R(L)$ (note that this is well-defined away from V and the overlaps of the regions $R(L)$, but this set has measure 0). In that way we can divide by $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ in (13.12) and pass to the limit to conclude that

$$\int_{\mathbf{B}_r\cap\alpha_i} \frac{|\bar{u}_i(z) \ominus \mathbf{p}^\perp_{\alpha_i}(\bar{\beta})(z)|^2}{\mathrm{dist}(z, V)^{7/2}}\, dz < \infty\,. \tag{13.28}$$

But in particular, since $|\bar{u}_i(z) \ominus (\boldsymbol{\eta}\circ\bar{u}_i)(z)| \leq |\bar{u}_i(z) \ominus \mathbf{p}^\perp_{\alpha_i}(\bar{\beta})(z)|$ (this is just the trivial fact that the barycenter minimizes this quantity), we get

$$\int_{\mathbf{B}_r\cap\alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta}\circ\bar{u}_i(z)|^2}{\mathrm{dist}(z, V)^{7/2}}\, dz < \infty\,. \tag{13.29}$$

Given that the numerator is continuous, it must be identically 0 for $z \in V \cap \mathbf{B}_r$, otherwise the integral would diverge (recall that V has codimension 2 and thus the integral of $\mathrm{dist}(z, V)^{-s}$ diverges for every $s \geq 2$). In particular we have shown (b).

As for (c), recall first that we have the estimate of Corollary 11.3. Once again choosing $\kappa = \frac{1}{4}$, this is easily seen to imply the estimate

$$\int_{\mathbf{B}_r\cap\alpha_i} \frac{|\bar{u}_i(z)|^2}{|z|^{m+7/4}}\, dz < \infty\,.$$

Arguing as above, this clearly implies that $|\bar{u}_i(0)| = 0$ and hence the first claim of point (c). We next argue for the second part of the claim; here we resort to the full version of Corollary 11.3 for any $\kappa \in (0, m+2)$. Observe that it suffices to prove the latter claim $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \geq 1$ for all $y \in V \cap \mathbf{B}_r$. Indeed, the fact that $I_{0,\bar{u}_i}(0) \geq 1$ follows immediately from this at $y = 0$, combined with the property $\boldsymbol{\eta}\circ\bar{u}_i(0) = 0$ that we have just proved. Fix a point $y \in V \cap \mathbf{B}_r$ and for each $k \in \mathbb{N}$, consider a point $q_k \in \mathbf{B}_{\varepsilon_k}(y)$ with $\Theta(T_k, q_k) \geq Q$ and fix any $\rho \in (0, r)$. It follows from Proposition 11.4 that, if $p \in \mathrm{gr}\,(u_i^k) \cap \mathrm{spt}(T_k)$ and $|\mathbf{p}_{\alpha_i}(p)| \geq \rho$, then $\mathrm{dist}(p, q_k) + \mathbf{S}_k) = \mathrm{dist}(p, q_k + \alpha_i^k)$ for all k sufficiently large (depending on ρ and κ) and in particular, if $K_{k,i} \subset \alpha_i^k$ denotes the set over which the graph of u_i^k coincides with the current T, then Proposition 11.4

$$\int_{(\mathbf{B}_r(y)\setminus\mathbf{B}_\rho(y))\cap K_{k,i}} \frac{|u_i^k(z) \ominus \mathbf{p}^\perp_{\alpha_i^k}(q_k)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa)\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)\,,$$

for every positive κ, and for k sufficiently large, depending on ρ, κ.

Again recall that $|u_i^k \ominus \boldsymbol{\eta}\circ u_i^k| \leq |u_i^k \ominus \mathbf{p}^\perp_{\alpha_i^k}(q_k)|$, and so in particular we conclude that

$$\int_{(\mathbf{B}_r(y)\setminus\mathbf{B}_\rho(y))\cap K_{k,i}} \frac{|u_i^k(z) \ominus \boldsymbol{\eta}\circ u_i^k(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa)\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)\,.$$

Dividing by $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$, taking the limiting in k and noting that the measure of $(\mathbf{B}_r \setminus \mathbf{B}_\rho) \cap (\alpha_i^k \setminus K_{k,i})$ converges to 0 (cf. (8.50)), we arrive at

$$\int_{(\mathbf{B}_r(y)\setminus\mathbf{B}_\rho(y))\cap\alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta}\circ\bar{u}_i(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \leq C(\kappa)\,.$$

On the other hand letting $\rho \downarrow 0$ we then get

$$\int_{\mathbf{B}_r(y)\cap\alpha_i} \frac{|\bar{u}_i(z) \ominus \boldsymbol{\eta}\circ\bar{u}_i(z)|^2}{|z-y|^{m+2-\kappa}}\, dz \le C(\kappa) \tag{13.30}$$

Now let $\gamma := I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0)$. It follows from [10, Corollary 3.18] that for every $\gamma' > \gamma$ there is a positive constant C (depending on γ' and $\bar{u}_i$) such that

$$\int_{\partial\mathbf{B}_\sigma(y)\cap\alpha_1} |\bar{u}_i \ominus \boldsymbol{\eta}\circ\bar{u}_i|^2 \ge C\sigma^{m-1+2\gamma'}\,,$$

for every $\sigma \in (0,r)$. Combined with (13.30), we conclude that

$$\int_0^r \sigma^{2\gamma'+\kappa-3}\, d\sigma < \infty$$

which implies that $2\gamma'+\kappa > 2$. Letting $\kappa \downarrow 0$ and $\gamma' \downarrow \gamma = I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0)$ we then conclude that $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \ge 1$. Note that one could alternatively conclude that $I_{y,\bar{u}_i\ominus\boldsymbol{\eta}\circ\bar{u}_i}(0) \ge 1$ via the Hardt-Simon inequality, as done in [29].

We next come to point (d). Observe that if ζ is in $V^\perp$ then

$$|\zeta| \le C(|\mathbf{p}^\perp_{\alpha_1}(\zeta)| + |\mathbf{p}^\perp_{\alpha_N}(\zeta)|)\,, \tag{13.31}$$

(where the constant C in particular depends on $\varepsilon_c^\star$). We next observe that we have (from (13.28), (13.29))

$$\int_{\mathbf{B}_r\cap\alpha_i} \frac{|\mathbf{p}^\perp_{\alpha_i}(\bar\beta) - \boldsymbol{\eta}\circ\bar{u}_i|^2}{\operatorname{dist}(z,V)^{7/2}}\, dz < \infty\,. \tag{13.32}$$

For each $\rho \in (0,r)$ consider the functions defined on V which result averaging $\bar\beta$ and $\boldsymbol{\eta}\circ\bar{u}_i$ respectively, over 2-dimensional disks in α_i (for any $i \in \{1,\dots,N\}$) of radius ρ perpendicular to V and centered at $y \in V$:

$$\bar\beta_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_i} \bar\beta\,,$$

$$g_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_i} \boldsymbol{\eta}\circ\bar{u}_i\,.$$

We clearly have, from (13.32),

$$\int_{V\cap\mathbf{B}_{\sqrt{r^2-\rho^2}}} |\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho) - g_\rho|^2 \le C\rho^{3/2}\,.$$

In particular $\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho)$ converges, as $\rho\downarrow 0$, in $L^2(V\cap\mathbf{B}_r)$ to $\lim_{\rho\downarrow 0} g_\rho = \boldsymbol{\eta}\circ\bar{u}_i$. But then for any given $\rho\in(0,r)$,

$$\int_{V\cap\mathbf{B}_{\sqrt{r^2-\rho^2}}} |\mathbf{p}^\perp_{\alpha_i}(\bar\beta_\rho) - \mathbf{p}^\perp_{\alpha_i}(\bar\beta_{\rho'})|^2 \le C\rho^{3/2} + \int_{V\cap\mathbf{B}_{\sqrt{r^2-\rho^2}}} |g_\rho - g_{\rho'}|^2 \qquad \forall \rho' < \rho$$

Hence, recalling that $\bar\beta$ takes values in $V^\perp$, applying (13.31) and the above inequality with $i = 1, N$, we conclude

$$\int_{V\cap\mathbf{B}_{\sqrt{r^2-\rho^2}}} |\bar\beta_\rho - \bar\beta_{\rho'}|^2 \le C\rho^{3/2} + \int_{V\cap\mathbf{B}_{\sqrt{r^2-\rho^2}}} |g_\rho - g_{\rho'}|^2 \qquad \forall\rho' < \rho\,.$$

In particular $\{\bar\beta_\rho\}_{\rho>0}$ converges strongly in $L^2(V\cap\mathbf{B}_r)$ as $\rho\downarrow 0$ to some function $\beta \in L^2(V\cap\mathbf{B}_r)$ and clearly $\mathbf{p}^\perp_{\alpha_i}(\beta) = \boldsymbol{\eta}\circ\bar{u}_i$ for each $i = 1,\dots,N$. The smoothness of β follows easily from the smoothness of the $\boldsymbol{\eta}\circ\bar{u}_i$, which in turn is obvious from the fact that they are the traces on V of classical harmonic functions. This proves (d).

Coming to the estimate (13.26) in the non-collapsed case, by construction (namely, the L^2 non-concentration estimates (13.11), (13.12)), $\|\bar{u}_i\|_{W^{1,2}} \le C$ for some constant $C = C(Q,m,n,\bar{n})$, which in fact does not depend on $\varepsilon_c^\star$. Next, as $\mathbf{p}^\perp_{\alpha_i}(\beta) = \boldsymbol{\eta}\circ\bar{u}_i$ and the cone $\mathbf{S}$ is well-separated (cf. (13.31)), β is determined as a linear combination of the $\mathbf{p}^\perp_{\alpha_i}(\beta)$ for $i = 1, N$, which in turn are the traces of $\boldsymbol{\eta}\circ\bar{u}_i$. The latter are harmonic functions and they enjoy the same $W^{1,2}$ bound on $\bar{u}_i$. Hence, the C^2 estimate follows from the classical theory of harmonic functions.

The conclusion (e) is more subtle than (a). As above, we wish to argue that $\bar{w}_i$ is Dir-minimizing on every $\Omega \subset\subset \mathbf{B}_r \cap \alpha_1 \setminus V$ and invoke Proposition 12.4 to argue that therefore it is Dir-minimizing on $\mathbf{B}_r \cap \alpha_1$. First of all we assume, without loss of generality, that the double-sided excess $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ and the one sided excess $\hat{\mathbf{E}}(T_k, \mathbf{S}_k, \mathbf{B}_1)$ are comparable, since the alternative would be that the latter is infinitesimal compared to the former, and in that case $\bar{w}_i$ would vanish. Then, we notice that in order to argue for the minimality in $\Omega \subset\subset \mathbf{B}_r \cap \alpha_1 \setminus V$ we cannot invoke directly the argument of [11, Theorem 2.6]. However, since the mismatch in mass between the current T and the graphs of the multi-valued maps $\sum_i w_i^k \oplus A_i^k$ is controlled by $E_k^{1+\gamma} := \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1+\gamma}$, the area-minimizing property of T and the arguments in [11, Section 5] shows the following minimality property for the map $w_i^k \oplus A_i^k$:

(Min) If $\tilde{w}_i^k$ is a Lipschitz map which coincides with w_i^k outside of Ω and its Lipschitz constant is controlled by 1, then

$$\|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) \geq \|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) - C E_k^{1+\gamma} . \tag{13.33}$$

Our aim is to show that, if $\bar{w}_i$ is not Dir-minimizing, then (13.33) is violated for k sufficiently large. Assuming it is not Dir-minimizing, the Lipschitz truncation and “cut-and-paste” arguments of [11, Theorem 2.6] show the existence of a sequence $\tilde{w}_i^k$ of multi-valued maps such that

(i) $\tilde{w}_i^k = w_i^k$ outside Ω;
(ii) $\mathrm{Lip}(\tilde{w}_i^k) \leq C E_k^\gamma$;
(iii) The Dirichlet energy of $\tilde{w}_i^k$ has the following gain:

$$\int_\Omega |D\tilde{w}_i^k|^2 \leq \int_\Omega |Dw_i^k|^2 - \vartheta E_k , \tag{13.34}$$

for some positive $\vartheta > 0$ independent of k.

We now wish to show that (i)-(iii) contradict (13.33). We let $\mathcal{A}(Du)$ be the area integrand for the graph of a single-valued function u. More precisely, if we denote by $\mathcal{M}(B)$ the set of all $k \times k$ minors of the matrix B with $k \geq 2$, then[1]

$$\mathcal{A}(B) = \sqrt{1 + |B|^2 + \sum_{M \in \mathcal{M}(B)} \det(M)^2} .$$

We next use the notation $w_i^k = \sum_j [\![w_{i,j}^k]\!]$ and $\tilde{w}_i^k = \sum_j [\![\tilde{w}_{i,j}^k]\!]$. Using the area formula in [12, Corollary 1.11] we can compute

$$\|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) = \int_\Omega \sum_j \mathcal{A}(A_i^k + D\tilde{w}_{i,j}^k)$$

and

$$\|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) = \int_\Omega \sum_j \mathcal{A}(A_i^k + Dw_{i,j}^k) .$$

In particular, defining

$$\Delta := \|\mathbf{G}_{\tilde{w}_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) - \|\mathbf{G}_{w_i^k \oplus A_i^k}\|(\Omega \times \alpha_1^\perp) ,$$

we arrive at the expression

$$\Delta = \int_\Omega \left(\sum_j \mathcal{A}(A_i^k + D\tilde{w}_{i,j}^k) - \sum_j \mathcal{A}(A_i^k + Dw_{i,j}^k) \right) . \tag{13.35}$$

[1]Note that, when u is a scalar function, Du only has 1×1 minors, namely $\mathcal{M}(Du)$ is empty, and we would use the convention that the formula reduces to $\mathcal{A}(B) = \sqrt{1 + |B|^2}$. However in our case u is necessarily vector-valued.

We now wish to make a Taylor expansion of the function $\mathcal{A}$ at the constant A_i^k. In particular we can write

$$\mathcal{A}(A_i^k + B) = \mathcal{A}(A_i^k) + \mathcal{L}_{k,i}(B) + \mathcal{Q}_{k,i}(B) + O(|B|^3)$$

where $\mathcal{L}_{k,i}$ are appropriate linear functions and $\mathcal{Q}_{k,i}$ are appropriate quadratic forms. However note that at $A_i^k = 0$ the quadratic form would be $\frac{|B|^2}{2}$, and given that $|A_i^k| \leq \boldsymbol{\mu}(\mathbf{S}_k)$ we can further write

$$\mathcal{A}(A_i^k + B) = \mathcal{A}(A_i^k) + \mathcal{L}_i^k(B) + \frac{|B|^2}{2} + O((\boldsymbol{\mu}(\mathbf{S}_k) + |B|)|B|^2)\,.$$

Now, inserting the latter expansion in our expression (13.35) and using that $\|D\tilde{w}_i^k\|_\infty + \|Dw_i^k\|_\infty \leq CE_k^\gamma$ and $\|D\tilde{w}_i^k\|_{L^2} + \|Dw_i^k\|_{L^2} \leq CE_k$ we arrive at

$$\begin{aligned}\Delta &= \int_\Omega \left(\sum_j \left(\mathcal{L}_i^k(D\tilde{w}_{i,j}^k) + \frac{|D\tilde{w}_{i,j}^k|^2}{2}\right) - \sum_j \left(\mathcal{L}_i^k(Dw_{i,j}^k) + \frac{|Dw_{i,j}^k|^2}{2}\right)\right)\\ &\quad + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k)\\ &= \underbrace{Q_i \int_\Omega \mathcal{L}_i^k(D(\boldsymbol{\eta}\circ\tilde{w}_i^k - \boldsymbol{\eta}\circ w_i^k))}_{=:\,(\mathrm{L})} + \underbrace{\frac{1}{2}\int_\Omega (|D\tilde{w}_i^k|^2 - |Dw_i^k|^2)}_{=:(\mathscr{Q})} + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k)\,,\end{aligned}$$

where we have used the linearity of $\mathcal{L}_i^k$. Now observe that the function

$$f_k := \boldsymbol{\eta}\circ\tilde{w}_i^k - \boldsymbol{\eta}\circ w_i^k$$

is a single-valued Lipschitz function that vanishes on $\partial\Omega$. On the other hand $\mathcal{L}_i^k$ is a linear function. In particular, if we write $f_k = (f_k^1, \ldots, f_k^n)$ for the components of the functions f_k, there are vectors $v_k^1, \ldots v_k^n \in \mathbb{R}^n$ (determined by $\mathcal{L}_i^k$) such that

$$(\mathrm{L}) = \sum_l \int_\Omega v_k^l \cdot \nabla f_k^l = \sum_l \int_\Omega \mathrm{div}\,(v_k^l f_k^l) = \sum_l \int_{\partial\Omega} f_k^l \nu \cdot v_k^l = 0\,,$$

where ν denotes the unit normal determined by the Stokes' orientation of $\partial\Omega$. Moreover, by (13.34) we have

$$(\mathscr{Q}) \leq -\vartheta E_k\,.$$

We can thus write

$$\Delta \leq -\frac{\vartheta}{2}E_k + O((\boldsymbol{\mu}(\mathbf{S}_k) + E_k^\gamma)E_k)\,.$$

Since $E_k + \boldsymbol{\mu}(\mathbf{S}_k) \to 0$, the latter clearly contradicts (13.33). This concludes the proof of (e).

The argument for (f) is entirely analogous to the argument for (b), where we use (13.14) in place of (13.12): indeed (13.14) leads to the conclusion that

$$\int_{\mathbf{B}_r\cap\alpha_1} \frac{|\bar{w}_i \ominus \boldsymbol{\eta}\circ\bar{w}_i|^2}{\mathrm{dist}(z,V)^{7/2}} < \infty\,,$$

which in turn clearly implies (f) using the continuity of $\bar{w}_i$. The argument for (g) is entirely analogous to the argument for (c): using Corollary 11.3 with $\kappa = \frac{1}{4}$ we derive

$$\int_{\mathbf{B}_r\cap\alpha_1} \frac{|\bar{w}_i(z)|^2}{|z|^{m+7/4}}\,dz < \infty\,,$$

which yields $\bar{w}_i(0) = \boldsymbol{\eta}\circ\bar{w}_i(0) = 0$, while Proposition 11.4 gives

$$\int_{\mathbf{B}_r(y)\cap\alpha_1} \frac{|\bar{w}_i \ominus \boldsymbol{\eta}\circ\bar{w}_i(z)|^2}{|z-y|^{m+2-\kappa}}\,dz \leq C(\kappa) \qquad (13.36)$$

for $y \in V \cap \mathbf{B}_r$. We then use this in place of (13.30) to conclude the desired lower bound on the frequency.

We finally come to (h). Recall the notation $\beta^k(L)$ for the nail of L when the current is T_k. Recall that from (11.6) that

$$|\mathbf{p}_{\alpha_1}^\perp(\beta^k(L))| \leq C\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}\,,$$

$$\boldsymbol{\mu}(\mathbf{S}_k)|\mathbf{p}_{V^\perp\cap\alpha_1}(\beta^k(L))| \leq C\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)^{1/2}\,.$$

Arguing as in (d), we assume, upon extraction of a subsequence, that there are $\bar\beta^\perp$ and $\bar\beta^\parallel$ such that

$$\frac{\mathbf{p}^\perp_{\alpha_1}(\beta^k(L))}{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)^{1/2}} \to \bar\beta^\perp(L)$$
$$\frac{\boldsymbol{\mu}(\mathbf{S}_k)\mathbf{p}_{V^\perp\cap\alpha_1}(\beta^k(L))}{\mathbb{E}(T_k,\mathbf{S}_k,\mathbf{B}_1)^{1/2}} \to \bar\beta_\parallel(L)\,.$$

In analogy with the argument for (d) we consider both functions as defined over $\mathbf{B}_r\cap\alpha_1$. Since $\bar A^k_i = \boldsymbol{\mu}(\mathbf{S}_k)^{-1}A^k_i$, by passing into the limit in (13.14) we get to, for each $i=1,\dots,N$

$$\int_{\mathbf{B}_r\cap\alpha_1} \frac{|\bar w_i \ominus (\bar\beta^\perp + \bar A_i(\bar\beta_\parallel))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz < \infty\,. \tag{13.37}$$

Combining with (13.36), in turn these estimates imply

$$\int_{\mathbf{B}_r\cap\alpha_1} \frac{|\boldsymbol{\eta}\circ\bar w_i - (\bar\beta^\perp + \bar A_i(\bar\beta_\parallel))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz < \infty\,. \tag{13.38}$$

As $\bar A_1 = 0$, we immediately conclude that $\bar\beta^\perp = \boldsymbol{\eta}\circ\bar w_1$ on $V\cap\mathbf{B}_r$, and by the triangle inequality that

$$\int_{\mathbf{B}_r\cap\alpha_1} \frac{|\boldsymbol{\eta}\circ\bar w_i - (\boldsymbol{\eta}\circ\bar w_1 + \bar A_i(\bar\beta_\parallel))|^2}{\mathrm{dist}(z,V)^{7/2}}\,dz < \infty\,. \tag{13.39}$$

Next, notice that $\sum_i[\![\bar A_i]\!]$ is Dir-minimizing (indeed, this will simply be the usual blow-up of T_k relative to α_1). Moreover, because $\max_i|A^k_i| \geq C^{-1}\boldsymbol{\mu}(\mathbf{S}_k)$, we have that $\max_i|\bar A_i| \geq C^{-1} > 0$ for some constant $C = C(Q,m,n,\bar n)$; we also know that this maximum is achieved by $\bar A_N$. Since $\bar A_1 = 0$, the map $[\![\bar A_N]\!] + [\![0]\!]$ is Dir minimizing and hence, by subtracting the average and rescaling by a factor 2, so is $[\![\bar A_N]\!] + [\![-\bar A_N]\!]$. In particular we can apply Lemma 7.7 and conclude that, if we let W be the image of $\bar A_N$, $\bar A_N : V^\perp \to W$ is invertible and its inverse B satisfies $|B| \leq C$ for some $C = C(Q,m,n,\bar n)$.

We therefore have the identity $\bar\beta_\parallel = B(\bar A_N(\bar\beta_\parallel))$. For $y\in V\cap\mathbf{B}_r$, define now the functions $(\bar\beta_\parallel)_\rho$ and h_ρ as follows, analogously to those the proof of (d): for $y\in V\cap\mathbf{B}_r$,

$$(\bar\beta_\parallel)_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_1} \bar\beta_\parallel\,, \tag{13.40}$$

$$h_\rho(y) = \frac{1}{\pi\rho^2}\int_{\mathbf{B}_\rho(y)\cap(y+V^\perp)\cap\alpha_1} (\boldsymbol{\eta}\circ\bar w_i - \boldsymbol{\eta}\circ\bar w_1)\,. \tag{13.41}$$

Arguing as in the proof of (d) we use (13.39) to conclude that $\bar A_i((\bar\beta_\parallel)_\rho)$ converges in $L^2(V\cap\mathbf{B}_r)$ to the function $\boldsymbol{\eta}\circ\bar w_i - \boldsymbol{\eta}\circ\bar w_1 = \lim_{\rho\downarrow 0} h_\rho$, which in particular gives us the conclusion that $\boldsymbol{\eta}\circ\bar w_N - \boldsymbol{\eta}\circ\bar w_1$ is in the image of $\bar A_N$. Hence, if we define $\beta_\parallel := B(\boldsymbol{\eta}\circ\bar w_N - \boldsymbol{\eta}\circ\bar w_1)$ and $\beta^\perp := \boldsymbol{\eta}\circ\bar w_1$, we see immediately that we have the identity

$$\boldsymbol{\eta}\circ\bar w_i = \boldsymbol{\eta}\circ\bar w_1 + \bar A_i(\beta_\parallel) = \beta^\perp + \bar A_i(\beta_\parallel)\,.$$

In order to conclude the proof we need to show the desired estimates (13.27) on $\|\bar w_i\|_{W^{1,2}}$, $\|\beta^\perp\|_{C^2}$, and $\|\beta_\parallel\|_{C^2}$. The first is obvious because $\|\bar w^k_i\|_{W^{1,2}}$ is bounded by a universal constant. The second is also obvious because $\beta^\perp$ is the restriction to V of the harmonic function $\boldsymbol{\eta}\circ\bar w_1$ whose $W^{1,2}$ norm is controlled by $\|\bar w_1\|_{W^{1,2}}$. Finally, the estimate on $\|\beta_\parallel\|_{C^2}$ follows from the same argument because $\beta_\parallel = B(\boldsymbol{\eta}\circ\bar w_N - \boldsymbol{\eta}\circ\bar w_1)$ and the norm of the linear map B is bounded by a universal constant. Thus, the proof of Proposition 13.8 is complete. □

13.5. **Final argument.** In this section we are going to show Proposition 13.1 and Proposition 13.2. We fix the decay scale ς_1 and we will show that this will be reached at a certain radius, r_c or r_{nc}, whether we are in the collapsed or non-collapsed setting, respectively, via a contradiction argument. We start with Proposition 13.1; we fix a contradiction sequence T_k, $\mathbf{S}_k$, and Σ_k as in the previous section and use Proposition 13.8(e)-(h) to extract the blow-up limits $\bar A_i$, $\bar w_i$, $\beta^\perp$, and $\beta_\parallel$. As before, without loss of generality we have rotated so that the planes α^k_1 all coincide with the same plane α_1. We next observe that, by (e)-(h), the functions $\beta^\perp$ and $\beta_\|$ both equal 0 at the origin. We therefore linearize them and let $\gamma^\perp$ and $\gamma_\|$ be their respective linearizations at 0. Observe that the C^2-regularity of $\beta^\perp$ and $\beta_\|$ guarantees that

$$|\beta^\perp(y) - \gamma^\perp(y)| \le C|y|^2 \tag{13.42}$$

$$|\beta_\|(y) - \gamma_\|(y)| \le C|y|^2 \,. \tag{13.43}$$

The constant C depends only on the C^2 norm of $\beta^\perp$ and $\beta_\|$, which in turn is bounded by a constant depending only upon on $Q, m, n, \bar{n}$, by Proposition 13.8. Observe that $\gamma_\| : V \to V^\perp \cap \alpha_1$ and let $\gamma_\|^T : V^\perp \cap \alpha_1 \to V$ be its transpose. We build a skew-symmetric map of α_1 onto itself by mapping

$$V \oplus (V^\perp \cap \alpha_1) \ni y + x \mapsto \gamma_\|(y) - \gamma_\|^T(x)\,.$$

This skew-symmetric map generates a one-parameter family $R[t]$ of rotations of α_1, which we may extend to all of $\mathbb{R}^{m+\bar{n}}$ by setting it to be the identity on $\alpha_1^\perp$ and extended linearly. We next define the rotations

$$R_k := R\left[\frac{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}}{\boldsymbol{\mu}(\mathbf{S}_k)}\right]$$

and observe that these rotations map α_1 and $\alpha_1^\perp$ onto themselves.

The rotated cones $\mathbf{S}'_k := R_k(\mathbf{S}_k)$ are thus a first step towards the cones which will have the desired decay at the radius r_c. Next consider the Dir-minimizing map $\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i$ and the (single-valued) harmonic function $\zeta_i := \boldsymbol{\eta} \circ \bar{w}_i - \gamma^\perp - \bar{A}_i(\gamma_\|)$, where the latter two linear maps are extended in the $V^\perp \cap \alpha_1$ directions as constant (in particular, being linear, they are harmonic). To the map $\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i$ we apply Theorem 12.2 (namely (12.2), which we can do by Proposition 13.8(g)): for a fixed δ, which will be chosen later, we find a radius $\bar{r} = \bar{r}(Q, m, n, \bar{n}, \delta)$ such that for every $\rho < \bar{r}$ we can find a 1-homogeneous Dir-minimizer $h_{i,\rho} \in \mathscr{L}_1$ with the property that

$$\int_{\mathbf{B}_\rho \cap \alpha_1} \mathcal{G}(\bar{w}_i \ominus \boldsymbol{\eta} \circ \bar{w}_i, h_{i,\rho})^2 \le \delta \rho^{m+2}$$

(indeed, $\int_{\mathbf{B}_1 \cap \alpha_1} |\bar{w}_i|^2 \le 1$ by construction). As for the classical harmonic part ζ_i, since $D\zeta_i(0) = 0$, we find a linear map ξ_i (namely, the linearization of ζ_i) which vanishes on V such that

$$|\zeta_i(z) - \xi_i(z)| \le C|z|^2\,.$$

We fix now a radius $r_c < \bar{r}$. We are now ready to define a new sequence of cones $\mathbf{S}''_k$. We take the linear functions A_i^k whose graphs over α_1 give the planes α_i^k, hence the linear functions ξ_i, and construct the maps

$$A_i^k + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2} \xi_i\,,$$

and then we add to them the multi-valued functions $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2} h_{i,\rho}$, and compose the resulting Q-valued function with R_k^{-1}. The formula for this Q-valued function over α_1 is thus given by

$$\sum_i (\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2} h_{i,\rho} \oplus (A_i^k + \mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(\xi_i + \gamma^\perp))) \circ R_k^{-1}\,.$$

By construction, since the support of the graph of any element $h \in \mathscr{L}_1$ with $h(0) = Q[\![0]\!]$ lies in $\mathscr{C}(Q, 0)$, the graphs of these functions give new cones $\mathbf{S}''_k$ which belong to $\mathscr{C}(Q, 0)$. From the estimates that we have, we can check that, if $r_c = \rho \le \bar{r}$

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_c})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \le C\delta + Cr_c^2\,,$$

where C is just a geometric constant. In particular, we choose δ sufficiently small so that $C\delta \le \frac{\varsigma_1}{2}$, which in turn fixes $\bar{r}$, and hence we fix $r_c \le \bar{r}$ so that $Cr_c^2 \le \frac{\varsigma_1}{4}$. With this choice we conclude that

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_c})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \le \frac{3\varsigma_1}{4}\,.$$

Now, with this particular choice of r_c, which depends only on ς_1, we get for k large enough a contradiction to the absence of decay with factor ς_1. This proves Proposition 13.1.

The proof of Proposition 13.2 works in a very similar way. We again assume that ς_1 and $\varepsilon_c^\star$ are given, that r_{nc} is fixed, and that there is absence of decay by ς_1 for sequences T_k, Σ_k, and $\mathbf{S}_k$. We then apply Proposition 13.8 and get the maps $\bar{u}_i$ and β. Arguing as above, we linearize the map β (which vanishes at 0) to a map γ. Again we note that

$$|\beta(y) - \gamma(y)| \leq C|y|^2 .$$

However, this time the constant C depends on $\varepsilon_c^\star$ as well as the C^2 norm of β. First of all we consider γ as a map from V to $V^\perp$, we let $\gamma^T : V^\perp \to V$ be its transpose, we again build the skew-symmetric map

$$V \oplus V^\perp \ni y + x \mapsto \gamma(y) - \gamma^T(x)$$

and hence we consider the one-parameter family of rotations $R[t]$ generated by it. We then introduce the multi-valued functions $\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i$ and the harmonic functions $\boldsymbol{\eta} \circ \bar{u}_i - \mathbf{p}^\perp_{\alpha_i}(\gamma)$, where we assume that γ is extended in the $V^\perp$ directions as a constant map.

As above, we fix $\delta > 0$ (whose choice will be specified later) and we appeal to Theorem 12.2 to find a threshold $\bar{r} = \bar{r}(Q, m, n, \bar{n}, \delta) > 0$ with the property that, for every $\rho < \bar{r}$ we can find a 1-homogeneous map $h_{i,\rho} \in \mathscr{L}_1$ (again, this map lies in $\mathscr{L}_1$ due to (12.2), which is applicable due to Proposition 13.8(c)) with the property that

$$\int_{\mathbf{B}_\rho \cap \alpha_i} \mathcal{G}(\bar{u}_i \ominus \boldsymbol{\eta} \circ \bar{u}_i, h_{i,\rho})^2 \leq \delta \rho^{m+2} .$$

Likewise we find a linear map ξ_i which vanishes on V and such that

$$|(\boldsymbol{\eta} \circ \bar{u}_i - \mathbf{p}^\perp_{\alpha_i}(\gamma))(z) - \xi_i(z)| \leq C|z|^2 ,$$

where the constant C depends again on $\varepsilon_c^\star$.

We are now ready to find the desired new cones. First of all we define

$$R_k := R[\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}]$$

and we consider the first adjustment to the cones as

$$\mathbf{S}'_k := R_k(\mathbf{S}_k) .$$

Next recall that along the sequence we are fixing α_1^k to be always the same plane (by applying a suitable rotation), and we are assuming that α_i^k converges to α_i. Moreover, for each k and $i \neq 1$, we fix a rotation $O_{k,i}$ which maps the approximating planes α_i^k onto α_i (where we use Lemma 7.4 to determine the $O_{k,i}$). In particular, we now consider the maps $\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(h_{i,\rho} \oplus \xi_i)$ over α_i, compose them with $O_{k,i}^{-1} \circ R_k^{-1}$ on the right and with $R_k \circ O_{k,i}$ on the left.

We thus find the following multi-valued maps over $(\alpha_i^k)' = R_k(\alpha_i^k)$, namely

$$R_k \circ O_{k,i} \circ (\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)^{1/2}(h_{i,\rho} \oplus \xi_i)) \circ O_{k,i}^{-1} \circ R_k^{-1} .$$

The graphs of these maps give the new cone $\mathbf{S}''_k$, which are easily seen to belong to $\mathscr{C}(Q, 0)$. We now arrive at the same conclusion of the argument for Proposition 13.1, namely, under the assumption that $\rho = r_{nc} \leq \bar{r}$,

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_{nc}})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq C\delta + Cr_{nc}^2 .$$

The only difference is that in this case the constants C depend upon $\varepsilon_c^\star$ as well. However, since the $\varepsilon_c^\star$ and ς_1 are both fixed, the constants above are also fixed, and we just choose δ so that $C\delta \leq \frac{\varsigma_1}{2}$. This in turn fixes $\bar{r}$, and we can further choose $r_{nc} \leq \bar{r}$ so that $Cr_{nc}^2 \leq \frac{\varsigma_1}{4}$. As in the proof of Proposition 13.1 we conclude that

$$\lim_{k\to\infty} \frac{\mathbb{E}(T_k, \mathbf{S}''_k, \mathbf{B}_{r_{nc}})}{\mathbb{E}(T_k, \mathbf{S}_k, \mathbf{B}_1)} \leq \frac{3\varsigma_1}{4} .$$

Since r_{nc} is now fixed, this now contradicts the assumption that there was no decay by a factor ς_1 at that radius for any of the currents T_k. Thus the proof of Proposition 13.2 is complete. □

To summarize, we have now established Theorem 2.5.

Part 3. Uniqueness of Tangent Cones and Rectifiability

14. Uniqueness and Rectifiability

14.1. **Rectifiability.** In this section we will use Theorem 2.5 to prove Theorem 1.3(i), namely that the set $\mathfrak{F}_{Q,1}$ has $\mathcal{H}^{m-2}$-measure zero. Before proceeding with the proof that Theorem 2.5 implies Theorem 1.3(i), we begin with some preliminaries.

First of all, we would like to show that the cones in the class $\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$ are the "prevalent" fine blow-ups appearing in the compactness procedure in [8]. This can be thought of as the analogue of [29, Lemma 2.4] and [19, Lemma 4.3] for the present setting, and is the key (and, in fact, the only) ingredient needed from the analysis in [8].

Lemma 14.1. *For each $\varepsilon\in(0,1]$, the following holds. Suppose that T and Σ are as in Assumption 2.1. Then, for each $p\in\mathfrak{F}_{Q,1}(T)\cap\mathbf{B}_1$ there exists $\bar\rho=\bar\rho(p,\varepsilon)>0$ such that the following dichotomy holds for each $\rho\in(0,\bar\rho]$:*

(a) *There exists $\mathbf{S}\in\mathscr{C}(Q,p)\setminus\mathscr{P}(p)$ with*
$$(\rho\mathbf{A})^2+\mathbb{E}(T,\mathbf{S},\mathbf{B}_\rho(p))\le\varepsilon^2\mathbf{E}^p(T,\mathbf{B}_\rho(p));$$
(b) *There exists an $(m-3)$-dimensional affine subspace $V\subset T_p\Sigma$ (depending on ρ) such that*
$$\mathfrak{F}_{Q,1}\cap\mathbf{B}_\rho(p)\subset\{q:\operatorname{dist}(q,V)<\varepsilon\rho\}.$$

Remark 14.2. The subspace V in alternative (b) of Lemma 14.1 arises as either

- the spine of a non-flat area-minimizing cone;
- the set of multiplicity Q points of a 1-homogeneous Q-valued Dir-minimizer arising as a coarse blow-up at p (cf. Proposition 2.2).

Proof of Lemma 14.1. We argue by contradiction. Namely, suppose that there exists some $\varepsilon\in(0,1]$ for which the following holds. There exist T, Σ, a point $p\in\mathfrak{F}_{Q,1}(T)\cap\mathbf{B}_1$, and a corresponding sequence of scales $\bar\rho_k\downarrow0$ such that both (a) and (b) fail on balls $\mathbf{B}_{\bar\rho_k}(p)$, for this choice of ε.

Up to extracting a further subsequence, we have two possibilities. Either $T_{p,\bar\rho_k}$ converges to a flat tangent cone in $\mathbf{B}_{6\sqrt{m}}$, namely $Q[\![\pi]\!]$ for some m-dimensional plane π, or it converges to a non-flat tangent cone T_∞. In the latter case we let V be the spine of T_∞, and note that either V is $(m-2)$-dimensional, and hence its support is a cone $\mathbf{S}\in\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$, or the dimension of V is at most $m-3$. On the other hand, in the latter case $\{\Theta(T_k,\cdot)\ge Q\}\cap\overline{\mathbf{B}}_1$ converges in the sense of Hausdorff to a subset of $\{\Theta(T_\infty,\cdot)\ge Q\}$ (by upper semi-continuity of the density) and the latter set is in fact the spine V. Hence, in the latter case, alternative (b) of the lemma holds for all sufficiently large k, giving a contradiction. In the former case clearly alternative (a) holds for all sufficiently large k, again giving the desired contradiction (note that since $\lim_{k\to\infty}\mathbf{E}^p(T_{p,\bar\rho_k},\mathbf{B}_{6\sqrt{m}})>0$ in this case, we clearly have $(\bar\rho_k\mathbf{A})^2\le\varepsilon^2\mathbf{E}^p(T_{p,\bar\rho_k},\mathbf{B}_{6\sqrt{m}})$ for all k sufficiently large).

So we are left with contradicting the case where $T_{p,\bar\rho_k}$ converges to a flat tangent cone; in this situation $\mathbf{E}^p(T,\mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))\downarrow0$. Proposition 2.2 then tells us that
$$\mathbf{E}^p(T,\mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))^{-1}\cdot\rho_k^2\mathbf{A}_k^2\to0. \tag{14.1}$$
Now let π_k be a plane such that $\mathbf{E}^p(T,\mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))=\hat{\mathbf{E}}(T,\pi_k,\mathbf{B}_{6\sqrt{m}\bar\rho_k}(p))$ and, without loss of generality, we can rotate to assume that π_k is a fixed plane π for all k. We now consider a coarse blow-up f as defined in [8]; we know that f is non-trivial and 1-homogeneous by Proposition 2.2. We then have two possibilities: either f is translation invariant in $m-2$ independent directions, or its spine has dimension at most $m-3$. In the former case, notice that in combination with the fact that $I_{0,f}(0)=1$ (as f is 1-homogeneous), the support of the (multi-valued) graph of f is a superposition of planes $\mathbf{S}\in\mathscr{C}(Q,0)\setminus\mathscr{P}(0)$. In light of the estimates in [11, Theorem 2.4], combined with (14.1) and the strong L^2-convergence of the normalizations of the Lipschitz approximations of $T_{p,\bar\rho_k}\llcorner\mathbf{B}_{6\sqrt{m}}$ to f, we contradict (a).

Otherwise, consider the set Z of points z which are limits of $\mathbf{p}_\pi(p_k)$ with $\Theta(T_{p,\bar\rho_k},p_k)=Q$ and $|p_k|\le1$. By [11, Theorem 2.7] $f(z)=Q[\![\boldsymbol{\eta}\circ f(z)]\!]$ for any such point z (this also follows by the Hardt–Simon inequality). Moreover, since by Proposition 2.2 we know that $\boldsymbol{\eta}\circ f\equiv 0$, we in fact have $f(z)=Q[\![0]\!]$ at such z. By the 1-homogeneity of f and the upper semi-continuity of the frequency function, we know that $I_{z,f}(0)\le 1$. On the other hand by the Hardt-Simon inequality, as in [8, Section 3], $I_{z,f}(0)\ge 1$. Therefore, $I_{z,f}(0)=1$ and hence f is translation invariant along any $z\in Z$. In particular $Z\subset V$, which, being at most $(m-3)$-dimensional, would imply that (b) would hold for $\rho=\bar\rho_k$ when k is large enough. This contradiction therefore proves the result. □

It will become convenient to subdivide $\mathfrak{F}_{Q,1}(T)\cap\mathbf{B}_1$ as follows.

Definition 14.3. Fix $\varepsilon^\dagger>0$. Suppose that T, Σ are as in Assumption 2.1. For $r>0$, we let $\mathfrak{F}_{Q,1,r}(T)$ denote the set of all points $p\in\mathfrak{F}_{Q,1}(T)\cap\mathbf{B}_1$ for which the conclusions of Lemma 14.1 hold for all scales $\rho\in(0,r]$ with $\varepsilon^\dagger$ in place of ε, and moreover $\|T\|(\mathbf{B}_\rho(p))\le(Q+\frac{1}{4})\omega_m\rho^m$ for all $\rho\in(0,r]$.

Notice that by Lemma 14.1 we may write

$$\mathfrak{F}_{Q,1}(T)\cap\mathbf{B}_1=\bigcup_k\mathfrak{F}_{Q,1,2^{-k}}(T)\cap\mathbf{B}_1\,. \tag{14.2}$$

The fact that $\iota_{0,r}(\mathfrak{F}_{Q,1,r}(T))\subset\mathfrak{F}_{Q,1,1}(T_{0,r})$ for any $r>0$, combined with a translation, together imply that it suffices to prove Theorem 1.3(i) for $\mathfrak{F}_{Q,1,1}(T)$.

We will now proceed to show that a set satisfying a number of properties at every scale around every point is, up to a $\mathcal{H}^k$-negligible set, a countable union of k-dimensional $C^{1,\alpha}$ graphs. This is only one possible abstract formulation of what the arguments used by Simon in the proof of [29, Theorem 1] imply for general sets.

Assumption 14.4. $k\in\mathbb{N}$ and $\delta,\alpha,\varepsilon>0$ are fixed. E is a Borel subset of $\mathbf{B}_2\subset\mathbb{R}^{m+n}$ such that for every $p\in E\cap\mathbf{B}_1$ and every $r\in(0,1-|q|]$ there is a choice of a k-dimensional affine subspace $V(p,r)$ satisfying the following properties:

(1) $E\cap\mathbf{B}_r(p)\subset B_{\varepsilon r}(V(p,r))$.
(2) if $0\le a<b\le 1-|p|$ are radii for which the condition
$$V(p,r)\cap\mathbf{B}_{r/2}\subset B_{\delta r}(E) \tag{14.3}$$
holds for every $r\in(a,b]$, then the following estimate holds for every $a<s<r\le b$:
$$|\mathbf{p}_{V(p,r)}-\mathbf{p}_{V(p,s)}|\le\varepsilon\left(\frac{r}{b}\right)^\alpha\,. \tag{14.4}$$

We now state the result concerning how a set satisfying Assumption 14.4 decomposes into a union of a $\mathcal{H}^k$-negligible set and a countable union of k-dimensional $C^{1,\alpha}$ submanifolds:

Theorem 14.5. *Let $k\in\mathbb{N}$ and $\alpha,\delta>0$ be fixed numbers. There exists $\varepsilon_s=\varepsilon_s(k,m,n,\delta)>0$ such that, if E satisfies Assumption 14.4 with $\varepsilon\le\varepsilon_s$, then $E\cap\mathbf{B}_{1/2}$ can be decomposed as a disjoint union $\tilde E\cup R$ where*

(i) $\mathcal{H}^k(\tilde E)=0$;
(ii) *$\tilde E$ consists of all points $p\in E\cap\mathbf{B}_{1/2}$ for which there is a sequence $\rho_k\downarrow 0$ violating condition (14.3);*
(iii) *R is contained in a countable union of $C^{1,\alpha}$ graphs (each defined on an open cube in a k-dimensional affine subspace of $\mathbb{R}^{m+n}$).*

We omit the proof of Theorem 14.5, since it is a simple consequence of the arguments given in [29]. We next demonstrate how Theorem 1.3(i) follows from it. In order to do this, we require the following lemma (in place of [29, Lemma 1]), which is a simple consequence of Theorem 2.5. It verifies that not only does the ratio of double-sided excess and planar excess under the assumptions of Theorem 2.5, but in fact so does the maximum of this ratio and the ratio between $r^2\mathbf{A}^2$ and the planar excess.

Lemma 14.6. *There are positive constants $\varepsilon_{\mathrm f}=\varepsilon_{\mathrm f}(Q,m,n,\bar n)$, $\theta=\theta(Q,m,n,\bar n)\le\frac12$, and $C=C(Q,m,n,\bar n)$ with the following property. Let T and Σ be as in Assumption 2.1 and $p\in\mathfrak{F}_{Q,1,1}(T)$. Assume $0<r\le 2-|p|$ and*

- $\mathbf{S} \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ *satisfies* $\mathbb{E}(T,\mathbf{S},\mathbf{B}_r(p)) = \inf\{\mathbb{E}(T,\bar{\mathbf{S}},\mathbf{B}_r(p)) : \bar{\mathbf{S}} \in \mathscr{C}(Q,p)\}$;
- $\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T,\mathbf{S},\mathbf{B}_r(p))\} \le \varepsilon_{\mathrm{f}}^2 \mathbf{E}^p(T,\mathbf{B}_r(p))$;
- $\mathbf{B}_{\varepsilon_{\mathrm{f}} r}(q) \cap \mathfrak{F}_{Q,1,1}(T) \neq \emptyset$ *for every* $q \in V(\mathbf{S}) \cap \mathbf{B}_{r/4}(p)$.

Then there is a cone $\mathbf{S}' \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ *such that*

$$\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2\theta^2 r^2, \mathbb{E}(T,\mathbf{S}',\mathbf{B}_{\theta r}(p))\}}{\mathbf{E}^p(T,\mathbf{B}_{\theta r}(p))} \le \frac{1}{4}\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T,\mathbf{S},\mathbf{B}_r(p))\}}{\mathbf{E}^p(T,\mathbf{B}_r(p))}, \tag{14.5}$$

and

$$\mathrm{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_r(p), V(\mathbf{S}') \cap \mathbf{B}_r(p)) \le C\frac{\max\{\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2, \mathbb{E}(T,\mathbf{S},\mathbf{B}_r(p))}{\mathbf{E}^p(T,\mathbf{B}_r(p))}. \tag{14.6}$$

Proof. Let $\varsigma = \frac{1}{8}$ in Theorem 2.5, and let $\varepsilon_{\mathrm{f}} := \varepsilon_0$ and $\theta := r_0$ be as in Theorem 2.5 for this choice of ς. The decay (14.5) is obvious by Theorem 2.5 if $\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2 \le \mathbb{E}(T,\mathbf{S},\mathbf{B}_r)$. In the other case we apply the continuity argument in Step 1 of Lemma 10.6 to find a cone $\mathbf{S}_e$ such that $\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}r^2 \le \mathbb{E}(T,\mathbf{S}_e,\mathbf{B}_r) \le 2\varepsilon_{\mathrm{f}}^{-2}\mathbf{A}^2 r^2$ and obeying the conditions therein, at which point we may once again apply Theorem 2.5, now with $\varsigma = \frac{1}{16}$, and again choose $\varepsilon_{\mathrm{f}} := \varepsilon_0$ and $\theta := r_0$ to be as in Theorem 2.5 for this choice of ς. The conclusion (14.6) is an immediate consequence of Theorem 2.5(d), (14.5), and the reasoning above. □

We will also require the following lemma, which gives control on the tilting between different k-dimensional subspaces that a given set E satisfying Assumption 14.4 is bilaterally close to. It is analogous to [4, Lemma 5.13]. We thus refer the reader to the proof therein, and do not include it here.

Lemma 14.7. *Let $k \in \mathbb{N}$ and $\alpha, \delta > 0$ be fixed numbers. There are positive constants $C_0 = C_0(m,n,k)$, $\delta_0 = \delta_0(m,n,k)$, and $\varepsilon_0 = \varepsilon_0(m,n,k)$ such that the following holds. Assume $E \subset \mathbb{R}^{m+n}$, $p \in E$, and $r > 0$ are radii such that Assumption 14.4(1) and (14.3) hold for some $V = V(p,r)$ and $\varepsilon \le \varepsilon_0$ and $\delta \le \delta_0$. Assume moreover that V' is another k-dimensional affine subspace for which*

$$E \cap \mathbf{B}_r(p) \subset B_{\varepsilon r}(V').$$

Then

$$|\mathbf{p}_V - \mathbf{p}_{V'}| \le C_0\varepsilon \tag{14.7}$$

and, if we set $\delta' := 2\delta + C_0\varepsilon$,

$$V' \cap \mathbf{B}_{r/2}(p) \subset B_{\delta' r}(E). \tag{14.8}$$

We are now in a position to conclude the proof of the fact that $\mathfrak{F}_{Q,1}(T)$ is $\mathcal{H}^{m-2}$-null.

Proof of Theorem 1.3(i). Let $E = \mathfrak{F}_{Q,1,1}(T)$, let $k = m-2$, and in Definition 14.3 and fix $\varepsilon = \min\{\varepsilon_s, \varepsilon_{\mathrm{f}}\}$, where ε_{f} is the constant of Lemma 14.6 and ε_s is the constant of Theorem 14.5 Choose $\varepsilon^\dagger = \varepsilon$. We will proceed to verify that Assumption 14.4 holds for this choice of E and k, and with a choice of $\delta = \delta(\varepsilon)$ sufficiently small. First of all, notice that Lemma 14.1 implies that property (1) of Assumption 14.4 holds for each $p \in E \cap \mathbf{B}_1$, with $V(p,r)$ defined to be the spine of $\mathbf{S}$ (defined therein, which implicitly depends on p and r) if alternative (a) of Lemma 14.1 holds in $\mathbf{B}_r(p)$, and any $(m-2)$-dimensional affine subspace containing V of alternative (b) of Lemma 14.1 if that holds in $\mathbf{B}_r(p)$ instead. To see that (2) of Assumption 14.4 holds, we proceed as follows. Suppose that (14.3) holds for a given point $p \in E \cap \mathbf{B}_1$ and for all scales $r \in (a,b]$, for a given pair of radii $a < b$ as in (2). Then for all such scales r, the alternative (a) of Lemma 14.1 must hold in $\mathbf{B}_r(p)$, and in addition, Lemma 14.7 tells us that provided that δ is chosen sufficiently small (depending on ε_{f}), all of the hypotheses of Lemma 14.6 hold with the ball $\mathbf{B}_{r/2}(p)$ replacing $\mathbf{B}_r(p)$. Now fix any such r and find $j \in \mathbb{N}$ such that $r \in (\theta^j b, \theta^{j-1} b]$. Applying Lemma 14.6 j times successively, starting from $\mathbf{B}_b(p)$, yields

$$\mathrm{dist}(V(\mathbf{S}_{j-1}) \cap \mathbf{B}_b(p), V(\mathbf{S}_j) \cap \mathbf{B}_b(p)) \le C2^{-j}\varepsilon_{\mathrm{f}} \le \varepsilon\left(\frac{r}{b}\right)^\alpha,$$

for an appropriate choice of $\alpha = \alpha(Q,m,n,\bar{n},\theta)$, where $\mathbf{S}_j, \mathbf{S}_{j-1}$ are the cones $\mathbf{S}, \mathbf{S}'$ in Lemma 14.6 when r is replaced by $\theta^{j-1}r$. This verifies that property (2) of Assumption 14.4 holds for this choice of E, and thus allows us to apply Theorem 14.5 with the above choice of parameters.

As observed above for each point $p \in R$ all of the hypotheses of Lemma 14.6 are satisfied in $\mathbf{B}_r(p)$ for each sufficiently small scale, so one may iteratively apply this lemma to deduce that any tangent cone at each such point p will be supported in a unique element $\mathbf{S} \in \mathscr{C}(Q,p)\setminus\mathscr{P}(p)$. This, however, is in contradiction with the fact that $p \in \mathfrak{F}_{Q,1,1}(T)$. Therefore, $R = \emptyset$ here. Combined with (14.2), this completes the proof. $\square$

14.2. **Uniqueness of tangent cones.** Having proved the rectifiability, we now turn to the conclusion of Theorem 1.3, namely the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones. Recalling [8, Theorem 2.10], we know that the tangent cone is unique at every flat singular point in $\mathfrak{F}_Q(T) \setminus \mathfrak{F}_{Q,1}(T)$, and given the previous section, it is therefore unique at $\mathcal{H}^{m-2}$-a.e. point in $\mathfrak{F}_Q(T)$. Since Q is arbitrary, it remains to show that the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathcal{S}^{(m-2)}$. Although countable $(m-2)$-rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ follows from [27], we will achieve this independently as an additional consequence of the arguments in this section, together with the $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones, following an argument analogous to that in the previous section. First of all recall that every point $p \in \mathcal{S}^{(m-2)}\setminus\mathcal{S}^{(m-3)}$ has integer density and that $\mathcal{S}^{(m-3)}$ has Hausdorff dimension at most $m-3$. Then we have the following counterpart of Lemma 14.1 above.

Lemma 14.8. *For each $\varepsilon \in (0,1]$, the following holds. Suppose that T and Σ are as in Assumption 2.1. For each $p \in \mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)} \cap \mathbf{B}_1$ with $\Theta(T,p) = Q$, there exists $\bar\rho = \bar\rho(p,\varepsilon) > 0$ such that the following dichotomy holds for each $\rho \in (0,\bar\rho]$:*

(a) *There exists $\mathbf{S} \in \mathscr{C}(Q,p) \setminus \mathscr{P}(p)$ and*
$$(\rho\mathbf{A})^2 + \mathbb{E}(T,\mathbf{S},\mathbf{B}_\rho(p)) \le \varepsilon^2\mathbf{E}^p(T,\mathbf{B}_\rho(p));$$
(b) *There exists an $(m-3)$-dimensional affine subspace $V \subset T_p\Sigma$ (depending on ρ) such that*
$$\{\Theta(T,\cdot) \ge Q\} \cap \bar{\mathbf{B}}_\rho(p) \subset \{q : \operatorname{dist}(q,V) < \varepsilon\rho\}.$$

Proof. The proof is by contradiction and is much simpler than the one of Lemma 14.1 since in this case
$$\liminf_{\rho\downarrow 0} \mathbf{E}^p(T,\mathbf{B}_\rho(p)) > 0\,.$$
Therefore, any tangent cone to T at p either has an $(m-2)$-dimensional spine, and hence its support is an element of $\mathscr{C}(Q,p) \setminus \mathscr{P}(p)$, or it has an $(m-3)$-dimensional spine, leading to the desired contradiction in either case. $\square$

We now subdivide $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ into countably many pieces (analogously to the subdivision of $\mathfrak{F}_{Q,1}(T)$ carried out in Definition 14.3) in the following way:

Definition 14.9. Let $\varepsilon^\dagger > 0$ and $\delta > 0$ be fixed. Suppose that T, Σ are as in Assumption 2.1. For $r > 0$ we let $\mathcal{S}_{Q,\delta,r}(T)$ be the set of points $q \in (\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}) \cap \mathbf{B}_1$ of density $\Theta(q,T) = Q$ such that

- The dichotomy of Lemma 14.8 applies at every scale $\rho \in (0,r]$ with $\varepsilon = \varepsilon^\dagger$;
- $\mathbf{E}^p(T,\mathbf{B}_\rho) \ge \delta$ for every $\rho \in (0,r]$.

First observe that we have the decomposition
$$\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)} = \bigcup_{Q,j,\ell} \mathcal{S}_{Q,1/j,1/\ell}\,.$$
Thus, in order to conclude the $(m-2)$-rectifiability and $\mathcal{H}^{m-2}$-a.e. uniqueness of tangent cones, it suffices to prove that each piece $\mathcal{S}_{Q,1/j,1/\ell}$ in the above decomposition is rectifiable and the tangent cone is unique at $\mathcal{H}^{m-2}$-a.e. point $q \in \mathcal{S}_{Q,1/j,1/\ell}$. By scaling we assume $\ell = 1$, and we may without loss of generality further assume that $j = 1$, in order to simplify notation.

The proof of the $(m-2)$-rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$ and uniqueness of tangent cones at $\mathcal{H}^{m-2}$-a.e. point $p \in \mathcal{S}^{(m-2)}$ will once again be concluded from Theorem 14.5. In order to do this, we first require the following analogue of Lemma 14.6, which is proved in exactly the same way.

Lemma 14.10. *There are positive constants $\varepsilon_{\rm nf} = \varepsilon_{\rm nf}(Q, m, n, \bar n)$, $\theta = \theta(Q, m, n, \bar n) \le \frac{1}{2}$ with the following property. Let T and Σ be as in Assumption 2.1 and let $p \in \mathcal{S}_{Q,1,1}(T)$. Assume $0 < r \le 2 - |p|$ and*

- *$\mathbf{S} \in \mathscr{C}(Q, p) \setminus \mathscr{P}(p)$ satisfies $\mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p)) = \inf\{\mathbb{E}(T, \bar{\mathbf{S}}, \mathbf{B}_r(p)) : \bar{\mathbf{S}} \in \mathscr{C}(Q, p)\}$;*
- *$\max\{\varepsilon_{\rm nf}^{-2} r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))\} \le \varepsilon_{\rm nf}^2 \mathbf{E}^p(T, \mathbf{B}_r(p)))$;*
- *$\mathbf{B}_\varepsilon(q) \cap \mathcal{S}_{Q,1,1}(T) \ne \emptyset$ for every $q \in V(\mathbf{S}) \cap \mathbf{B}_{r/4}(p)$.*

Then there is a $\mathbf{S}' \in \mathscr{C}(Q, 0)$ such that

$$\frac{\max\{\varepsilon_{\rm nf}^{-2}\theta^2 r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}', \mathbf{B}_{\theta r})\}}{\mathbf{E}^p(T, \mathbf{B}_{\theta r})} \le \frac{1}{4} \frac{\max\{\varepsilon_{\rm nf}^{-2} r^2 \mathbf{A}^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r)\}}{\mathbf{E}^p(T, \mathbf{B}_r)} ,$$

and

$$\operatorname{dist}^2(V(\mathbf{S}) \cap \mathbf{B}_r(p), V(\mathbf{S}') \cap \mathbf{B}_r(p)) \le C \frac{\max\{\varepsilon_{\rm f}^{-2} \mathbf{A}^2 r^2, \mathbb{E}(T, \mathbf{S}, \mathbf{B}_r(p))}{\mathbf{E}^p(T, \mathbf{B}_r(p))} .$$

Proof of Theorem 1.3(ii) and rectifiability of $\mathcal{S}^{(m-2)} \setminus \mathcal{S}^{(m-3)}$. Let $E = \mathcal{S}_{Q,1,1}(T)$, let $k = m-2$, and in Definition 14.3 and fix $\varepsilon = \min\{\varepsilon_s, \varepsilon_{\rm nf}\}$, where $\varepsilon_{\rm nf}$ is the constant in Lemma 14.10 and ε_s is the constant in Theorem 14.5. Choose $\varepsilon^\dagger = \varepsilon$. Arguing exactly as in the proof of Theorem 1.3(i), with Lemma 14.8 and Lemma 14.10 applied in place of Lemma 14.1 and 14.6 respectively, we verify that Assumption 14.4 holds for this choice of E and k, and with a choice of $\delta = \delta(\varepsilon)$ sufficiently small. We may thus apply Theorem 14.5 with the above choice of parameters. At each point $p \in R$, all of the hypotheses of Lemma 14.10 are satisfied in $\mathbf{B}_r(p)$ for each sufficiently small scale $r > 0$. This in turn implies that the tangent cone is supported in a unique element $\mathbf{S} \in \mathscr{C}(Q, p) \setminus \mathscr{P}(p)$ at each such point, and the supports of the rescalings $T_{p,r}$ decay towards $\mathbf{S}$ with a power law for all r sufficiently small. But, because of this decay and the graphical approximation in Proposition 8.15 applied to $T_{p,r}$ and this unique $\mathbf{S}$, we can select r sufficiently small to see that the origin is in fact in the closure of the outer region. Conclusion (i) of Proposition 8.15 ensures that the multiplicities assigned to the planes in $\mathbf{S} \cap \mathbf{B}_\rho(0)$ with $\rho \le r$ do not change with the radius. Thus, not only is the support of the tangent cone unique, but so is the tangent cone itself. Combined with (14.2), this completes the proof. □

Appendix A. Proofs of combinatorial results in Section 4.3

A.1. **Proof of Lemma 4.3.** First of all, note that if $N = 2$ the conclusion of the lemma is trivially true for $P' = P$, so we may assume $N \ge 3$. Then we define inductively the sets P_j, with $j = \{0, \dots, N-2\}$ and the elements p_j in the following way. $P_0 = P$ and, for every $j \ge 1$, we select $p_j \in P_{j-1}$ with the property that

$$\begin{aligned} M_{j-1} &:= \max\{|p-q| : p, q \in P_{j-1}\} = \max\{|p-q| : p, q \in P_{j-1} \setminus \{p_j\}\} \\ m_{j-1} &:= \min\{|p-q| : p, q \in P_{j-1},\ p \ne q\} \le \min\{|p-q| : p, q \in P_{j-1} \setminus \{p_j\},\ p \ne q\} \end{aligned}$$

Since the cardinality of P_{j-1} is always at least three it is clear that such an element exists: we first select p, q such that $|p-q| = \min\{|a-b| : a \ne b \in P_{j-1}\}$ and then p', q' such that $|p'-q'| = \max\{|a-b| : a, b \in P_{j-1}\}$. If the pairs (p, q) and (p', q') are the same, then obviously all points in P_{j-1} are equidistant and we can define p_j to be any of them. Otherwise the two pairs have at most one element in common, and we define p_j to be an element in $\{q, p\} \setminus \{q', p'\}$. We then let $P_j := P_{j-1} \setminus \{p_j\} = P \setminus \{p_1, \dots, p_j\}$. By construction, we have

$$m_{j-1} = \operatorname{dist}(p_j, P_{j-1} \setminus \{p_j\}) = \operatorname{dist}(p_j, P_j) , \tag{A.1}$$

while at the same time $M_j = M_0$ and $m_j \ge m_{j-1}$. Observe in particular that the requirements (iii) is satisfied if we terminate this process at any stage P_J.

Now let $\lambda = \lambda(\bar\delta) \ge 1$ be a large number whose choice will be specified later and let J be the smallest number j such that $M_j \le \lambda^{N-2-j} m_j$. Since P_{N-2} consists of two elements, $M_{N-2} = m_{N-2}$ and thus such a number J exists. Obviously $P' = P_J$ satisfies (i) with $C = \lambda^{N-2}$. We claim that for (ii) to hold we just need λ to be large enough. Note that

$$\min\{|p-q| : p, q \in P',\ p \ne q\} = m_J \ge \lambda^{-(N-2-J)} M_J ,$$

while, for every $p \in \{p_1, \dots, p_J\}$, by our definition of J and by (A.1), we have

$$\begin{aligned}\operatorname{dist}(p, P') &\le \sum_{i=1}^{J} \operatorname{dist}(p_i, P_i) = \sum_{i=1}^{J} m_{i-1} \le M_J \lambda^{-(N-2-J)} \sum_{k\ge 0} \lambda^{-k} \\ &= \frac{M_J \lambda^{-(N-2-J)}}{\lambda - 1} \le \frac{m_J}{\lambda - 1}\,.\end{aligned}$$

In particular, it suffices to choose $\frac{1}{\lambda-1} = \bar{\delta}$.

A.2. **Proof of Lemma 4.4.** If $\varepsilon \le \delta \min\{|q_1 - q_2| : q_1, q_2 \in P,\ q_1 \neq q_2\}$, we then set $\tilde{P} = P$. Otherwise, pick a pair of points $p_0, p_1 \in P$ not satisfying this property and set $P_1 := P \setminus \{p_1\}$. By construction, $\operatorname{dist}(p_1, P_1) \le \delta^{-1}\varepsilon$. If $N = 2$, $\tilde{P} := P_1$ contains a single element and hence (ii) holds. On the other hand, if $N \ge 3$ and $\delta^{-1}\varepsilon \le \delta \min\{|q_1 - q_2| : q_1, q_2 \in P_1,\ q_1 \neq q_2\}$, we define $\tilde{P} = P_1$, which satisfies (ii) because $\operatorname{dist}(p_1, P_1) \le \delta^{-1}\varepsilon$. Otherwise we discard yet another point p_2 from P in the same way as above, with P_1 in place of P, and set $P_2 := P_1 \setminus \{p_2\}$. This time, we notice that

$$\operatorname{dist}(p_2, P_2) \le \delta^{-2}\varepsilon\,,$$

while

$$\operatorname{dist}(p_1, P_2) \le \operatorname{dist}(p_1, P_1) + \operatorname{dist}(p_2, P_2) \le (\delta^{-1} + \delta^{-2})\varepsilon\,.$$

Iterating this procedure, we generate a family of sets P_j, $j = 1, \dots, J$. We stop if either P_J is a singleton, or if

$$\delta^{-1}(1 + \delta^{-1})^{J-1}\varepsilon \le \delta \min\{|q_1 - q_2| : q_1, q_2 \in P_J,\ q_1 \neq q_2\}\,.$$

For every point $p \in P$, since $\operatorname{dist}(p_j, P_j) \le \delta^{-2}(1 + \delta^{-1})^{j-2}$ for each $j \ge 2$, we therefore inductively have

$$\operatorname{dist}(p, P_J) \le \sum_{j=1}^{J} \operatorname{dist}(p_j, P_j) \le \delta^{-1}(\delta^{-1} + 1)^{J-1}\varepsilon\,.$$

In particular, when we stop the set $\tilde{P} = P_J$ satisfies (ii). Since P_{N-1} would necessarily be a singleton, we must stop at a $J \le N = 1$, which makes the estimate (i) trivially true.

A.3. **Proof of Lemma 4.5.** When $N = 2$ and $P = \{p_1, p_2\}$, we can clearly just let $P_j := \{p_j\}$ for $j = 1, 2$. We now argue by induction on N. Fix $N \ge 3$, and suppose the conclusion of the lemma holds for $N' \le N - 1$. Select $p \in P$ such that $\max\{|q - q'| : q, q' \in P \setminus \{p\}\} = M = \max\{|q - q'| : q, q' \in P\}$. By our inductive hypothesis, we can then decompose $P \setminus \{p\}$ into $P'_1 \cup P'_2$ such that

$$\min\{|p_1 - p_2| : p_1 \in P'_1, p_2 \in P'_2\} \ge \frac{M}{2^{N-3}}\,.$$

Let $p_i \in P'_i$ be such that $\operatorname{dist}(p, p_i) = \operatorname{dist}(p, P'_i)$ for $i = 1, 2$. Since $\operatorname{dist}(p_1, p_2) \ge \frac{M}{2^{N-3}}$, at least one among $\operatorname{dist}(p_i, p)$ has to be $\ge \frac{M}{2^{N-2}}$. Assuming, upon relabeling, that the latter happens for $i = 1$, we set $P_1 = P'_1$ and $P_2 = P'_2 \cup \{p\}$, proving the conclusion of the Lemma.

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