EXISTENCE THEOREM FOR A DIRICHLET PROBLEM WITH FREE DISCONTINUITY SET

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Abstract. We study the **free discontinuity problem**

$$\min\left\{\int_{\Omega\setminus K}|\nabla u|^2dx+\lambda\mathcal{H}_{n-1}(K)\right\}$$

where the minimum is taken over all the closed sets $K\subset\overline{\Omega}$ and the functions $u\in C^1(\Omega\setminus K)\cap C^0(\overline{\Omega}\setminus(M\cup K))$ with $u=w$ on $\partial\Omega\setminus(M\cup K)$; here Ω is a bounded domain in $\mathbf{R}^n$, $n\geq 2$, such that $\mathcal{H}_{n-1}(\partial\Omega)<+\infty$ and $\partial\Omega$ is a C^1 surface up to an $\mathcal{H}_{n-1}$ negligible closed set M, $w\in C^1(\partial\Omega\setminus M)\cap L^\infty(\partial\Omega\setminus M)$, $0<\lambda<+\infty$ and $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

1. Introduction.

In this paper we prove the following existence theorem of a solution for a *free discontinuity problem with Dirichlet type boundary conditions.*

THEOREM 1.1. *Let $n\in\mathbf{N}$, $n\geq 2$, let $\Omega\subset\mathbf{R}^n$ be a bounded domain with $\mathcal{H}_{n-1}(\partial\Omega)<+\infty$; assume that a closed set M exists such that $\mathcal{H}_{n-1}(M)=0$ and $\partial\Omega\setminus M$ is a C^1 surface; let $w\in C^1(\partial\Omega\setminus M)\cap L^\infty(\partial\Omega\setminus M)$ and let $0<\lambda<+\infty$.Then there exists at least one pair (K,u) minimizing the functional $\mathcal{G}$ defined for every closed set $K\subset\overline{\Omega}$ and for every $u\in C^1(\Omega\setminus K)\cap C^0(\overline{\Omega}\setminus(M\cup K))$ with $u=w$ on $\partial\Omega\setminus(M\cup K)$ by*

$$\mathcal{G}(K,u)=\int_{\Omega\setminus K}|\nabla u|^2dx+\lambda\mathcal{H}_{n-1}(K),$$

where $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

An existence theorem for a free discontinuity minimum problem with Neumann type boundary conditions has been recently proved in [14]. In the case $n=2$ Theorem 1.1 of [14] has provided the beginning of a positive answer to a problem of image segmentation in Computer Vision Theory posed by D. Mumford and J. Shah in [24] (see also [6], [8], [10], [11], [23]). We refer to [12] and [16] for very interesting conjectures on further regularity properties of the set K, which, if proved, shall provide also a complete and positive answer to the image segmentation problem mentioned above.

Following [12], these minimum problems fall into the class of the *variational problems*, in a given open set $\Omega \subset \mathbf{R}^n$, *with free discontinuities*, since a solution is a pair (K, u), where K is a closed set, u is a smooth function in $\Omega \setminus K$ and K is not necessarily the union of essential boundaries, unlike the situation with free boundary problems (see [5], [1]). We remark that free discontinuity problems more general than the ones until now considered should be regarded as a possible schematization for various problems in Mathematical Physics where are present both volume forces and surface tensions (see [7], [9], [12], [17], [20], [25]).

Beside the existence theorem of a pair (K, u) minimizing the functional $\mathcal{G}$, in this paper we prove also some regularity properties for the closed set K. In particular we prove the following proposition.

PROPOSITION 1.2. *If (K, u) is a minimizing pair given by Theorem 1.1, then*

(i) *K is $(\mathcal{H}_{n-1}, n-1)$ rectifiable, i.e. (as in [18]) there exists a sequence of C^1 surfaces (S_h) such that*

$$\mathcal{H}_{n-1}(K \setminus \bigcup_h S_h) = 0;$$

(ii) *there exists a unique minimizing pair (K', u') such that $K' \subseteq K$, $\mathcal{H}_{n-1}(K \setminus K') = 0$, $u = u'$ in $\overline{\Omega} \setminus (M \cup K)$ and for every $x \in K' \setminus M$*

$$\liminf_{\rho \to 0} \rho^{1-n} \mathcal{H}_{n-1}(K' \cap \overline{B_\rho(x)}) > 0.$$

Taking into account the idea of the so-called direct methods in Calculus of Variations, we join to the functional $\mathcal{G}$ a new functional $\mathcal{F}$, defined on a class of special bounded variation functions (the class $SBV(\mathbf{R}^n)$ recently introduced in [13]), where a topology can be suitably found such that $\mathcal{F}$ is at the same time lower semicontinuous and coercive.

Theorem 1.1 and Proposition 1.2 are established (see section 4) by proving first the existence of a solution u for a minimum problem for $\mathcal{F}$ over all the competing $SBV(\mathbf{R}^n)$ functions with given Dirichlet boundary conditions (see Lemma 4.1), and then by using some partial regularity properties of the singular set of the function u (see Theorem 3.12 and Remark 3.13).

We prove that the minimum of $\mathcal{F}$ is also the minimum of $\mathcal{G}$ and moreover we show that by a minimizer of $\mathcal{F}$ one can obtain a minimizing pair of $\mathcal{G}$ and viceversa (see Remark 4.3).

In order to prove the previous results we use both interior estimates for u already proved in [14] and new estimates on the behavior of u near a boundary point (see section 3).

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2. Preliminary results for functions in $SBV(\Omega)$.

In this section, given an open set $\Omega \subseteq \mathbf{R}^n$, we define, following [13], the class of special bounded variation functions $SBV(\Omega)$ and we point out a few of its properties with new results gained in [14].

For a given set $E \subset \mathbf{R}^n$ we denote by χ_E its characteristic function, by $\overline{E}$ its topological closure and by ∂E its topological boundary; moreover we denote by $\mathcal{H}_{n-1}(E)$ its $(n-1)$-dimensional Hausdorff measure and by $|E|$ its Lebesgue outer measure. If Ω , Ω' are open subsets in $\mathbf{R}^n$, with

$\Omega \subset\subset \Omega'$ we mean that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset \Omega'$. The word *domain* is used to mean an open set $\Omega \subseteq \mathbf{R}^n$ such that $\partial\Omega = \partial(\mathbf{R}^n \setminus \overline{\Omega})$.

We indicate by $B_\rho(x)$ the ball $\{y \in \mathbf{R}^n; |y-x| < \rho\}$, and we set $B_\rho = B_\rho(0)$, $\omega_n = |B_1|$.

Let $u : \Omega \to \mathbf{R}$ be a Borel function; for $x \in \Omega$ and $z \in \tilde{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ we say (following [13]) that z is the approximate limit of u at x, and we write

$$z = \operatorname{ap}\lim_{y \to x} u(y),$$

if

$$g(z) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} g(u(y))dy}{|B_\rho|}$$

for every $g \in C^0(\tilde{\mathbf{R}})$; if $z \in \mathbf{R}$ this definition is equivalent to 2.9.12 in [18].

The set

$$S_u = \{x \in \Omega;\ \operatorname{ap}\lim_{y \to x} u(y) \quad \text{does not exist }\}$$

is a Borel set, negligible with respect to the Lebesgue measure; for brevity's sake we denote by $\tilde{u} : \Omega \setminus S_u \to \tilde{\mathbf{R}}$ the function

$$\tilde{u}(x) = \operatorname{ap}\lim_{y \to x} u(y).$$

Let $x \in \Omega \setminus S_u$ be such that $\tilde{u}(x) \in \mathbf{R}$; we say that u is approximately differentiable at x if there exists a vector $\nabla u(x) \in \mathbf{R}^n$ (approximate gradient of u at x) such that

$$\operatorname{ap}\lim_{y \to x} \frac{|u(y) - \tilde{u}(x) - \nabla u(x) \cdot (y-x)|}{|y-x|} = 0.$$

For every $u \in L^1_{\text{loc}}(\Omega)$ we define (see [19])

$$\int_\Omega |Du| = \sup\left\{ \int_\Omega u \operatorname{div}\phi\ dx; \phi \in C^1_0(\Omega; \mathbf{R}^n), |\phi| \le 1 \right\}.$$

By $BV(\Omega)$ we denote the Banach space of all functions u of $L^1(\Omega)$ with $\int_\Omega |Du| < +\infty$.

It is well-known that $u \in BV(\Omega)$ iff $u \in L^1(\Omega)$ and its distributional derivative Du is a bounded vector measure. For the main properties of the functions of bounded variation we refer e.g. to [15], [18], [19], [22].

Here we recall only that for every $u \in BV(\Omega)$ the following properties hold:

S_u is $(\mathcal{H}_{n-1}, n-1)$ rectifiable (see [15], or [18], 4.5.9(16));

$\mathcal{H}_{n-1}(\{x \in \Omega; \tilde{u}(x) = \infty\}) = 0$ (see [18], 4.5.9(3));

∇u exists a.e. on Ω and coincides with the Radon-Nikodym derivative of Du with respect to the Lebesgue measure (see [18], 4.5.9(26));

for $\mathcal{H}_{n-1}$ almost all $x \in \mathbf{R}^n$ there exist $\nu = \nu(x) \in \partial B_1$, $\operatorname{tr}^+(x,u,\nu) \in \mathbf{R}$ and $\operatorname{tr}^-(x,u,\nu) \in \mathbf{R}$ (outer and inner trace, respectively, of u at x in the direction ν) such that

$$\lim_{\rho \to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y \cdot \nu > 0\}} |u(y) - \operatorname{tr}^+(x,u,\nu)| dy = 0,$$

$$\lim_{\rho \to 0} \rho^{-n} \int_{\{y \in B_\rho(x); y \cdot \nu < 0\}} |u(y) - \operatorname{tr}^-(x,u,\nu)| dy = 0,$$

and

$$\int_\Omega |Du| \geq \int_\Omega |\nabla u| dx + \int_{S_u \cap \Omega} |\mathrm{tr}^+(x,u,\nu) - \mathrm{tr}^-(x,u,\nu)| d\mathcal{H}_{n-1} \tag{2.1}$$

(see [18], 4.5.9(17),(22),(15)).

Following [13], we define a class of special bounded variation functions which are characterized by a property stronger than (2.1).

DEFINITION 2.1. We define $SBV(\Omega)$ as the class of all functions $u \in BV(\Omega)$ such that

$$\int_\Omega |Du| = \int_\Omega |\nabla u| dx + \int_{S_u \cap \Omega} |\mathrm{tr}^+(x,u,\nu) - \mathrm{tr}^-(x,u,\nu)| d\mathcal{H}_{n-1}. \tag{2.2}$$

We remark that the well-known Cantor-Vitali function has bounded variation, but it does not satisfy (2.2).

REMARK 2.2. Let $u \in BV(\Omega)$ and set $u_a = (u \wedge a) \vee (-a)$ for $0 < a < +\infty$. The following properties hold:

$|\nabla u_a| \leq |\nabla u|$ a.e. on Ω;

$\mathcal{H}_{n-1}((S_{u_a} \setminus S_u) \cap \Omega) = 0$;

$\int_\Omega |Du_a| \leq \int_\Omega |Du|$;

$\int_\Omega |\nabla u| dx = \lim_{a \to +\infty} \int_\Omega |\nabla u_a| dx$;

$\mathcal{H}_{n-1}(S_u \cap \Omega) = \lim_{a \to +\infty} \mathcal{H}_{n-1}(S_{u_a} \cap \Omega)$;

$\int_\Omega |Du| = \lim_{a \to +\infty} \int_\Omega |Du_a|$.

Moreover, for $u \in BV(\Omega)$, it holds:

$u \in SBV(\Omega)$ iff $u_a \in SBV(\Omega)$ for every $0 < a < +\infty$;

and more generally:

$u \in SBV(\Omega)$ iff $\phi(u) \in SBV(\Omega)$ for every $\phi : \mathbf{R} \to \mathbf{R}$ uniformly Lipschitz continuous with $\phi(0) = 0$.

Denote by $W^{1,p}(\Omega)$ $(p \geq 1)$ the Sobolev space of functions $u \in L^p(\Omega)$ such that $Du \in L^p(\Omega; \mathbf{R}^n)$; then we remark that, for $u \in SBV(\Omega)$,

$$u \in W^{1,p}(\Omega) \quad \text{iff} \quad \mathcal{H}_{n-1}(S_u \cap \Omega) = 0 \quad \text{and} \quad \int_\Omega (|\nabla u|^p + |u|^p) dx < +\infty$$

(see e.g. [18], 4.5.9(30)).

LEMMA 2.3. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with $\mathcal{H}_{n-1}(\partial\Omega) < +\infty$. Let $w \in W^{1,1}(\mathbf{R}^n)$. Let $K \subset \overline{\Omega}$ be a closed set with $\mathcal{H}_{n-1}(K) < +\infty$ and let $u \in C^1(\Omega \setminus K) \cap C^0(\overline{\Omega} \setminus K) \cap L^\infty(\Omega \setminus K)$ with $\int_{\Omega\setminus K} |\nabla u| dx < +\infty$.*
Set

$$u'(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \setminus K, \\ w(x) & \text{if } x \in \mathbf{R}^n \setminus \overline{\Omega}, \end{cases}$$

then

(i) $u' \in SBV(\mathbf{R}^n)$,

(ii) $\mathcal{H}_{n-1}(S_{u'} \setminus (K \cup \{x \in \partial\Omega \setminus K; u(x) \neq \tilde{w}(x)\})) = 0$.

Proof. We have $u' \in SBV(\Omega)$ and $S_{u'} \cap \Omega \subseteq K$ by Lemma 2.3 in [14]. As in [4] (see Theorem 3), we have $u' \in BV(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} |Du'| = \int_\Omega |Du| + \int_{\mathbf{R}^n\setminus\overline{\Omega}} |\nabla u| dx + \int_{\partial\Omega} |\mathrm{tr}^-(x,u,\nu) - \tilde{w}| d\mathcal{H}_{n-1},$$

where ν is the outward unit normal to Ω.
Therefore $u' \in SBV(\mathbf{R}^n)$ and, since $\mathcal{H}_{n-1}(S_{u'} \cap (\mathbf{R}^n \setminus \overline{\Omega})) = 0$ and $\mathrm{tr}^-(x,u,\nu) = u(x)$ for $\mathcal{H}_{n-1}$-almost all $x \in \partial\Omega \setminus K$, we infer also (ii). **q.e.d.**

For further results on the functions in $SBV(\Omega)$ we refer to [13], [2], [3].
In this paper we use the following semicontinuity theorem in $SBV(\Omega)$, that is an obvious consequence of a result by L. Ambrosio (see Theorems 2.1 and 3.4 of [3]) and of Remark 2.2.

THEOREM 2.4. *Let $p > 1$. Let $u_h \in SBV(\Omega)$ be such that*

$$u_h \to u \quad \text{in } L^1_{\mathrm{loc}}(\Omega),$$

$$\sup_{h\in\mathbf{N}} \left\{ \int_\Omega |\nabla u_h|^p dx + \mathcal{H}_{n-1}(S_{u_h} \cap \Omega) \right\} < +\infty,$$

$$\sup_{h\in\mathbf{N}} \left\{ \int_\Omega |Du_h| + \int_\Omega |u_h| dx \right\} < +\infty.$$

Then

(i) $u \in SBV(\Omega)$,

(ii) $\mathcal{H}_{n-1}(S_u \cap \Omega) \leq \liminf_h \mathcal{H}_{n-1}(S_{u_h} \cap \Omega)$,

(iii) $\int_\Omega |\nabla u|^p dx \leq \liminf_h \int_\Omega |\nabla u_h|^p dx.$

We remark that the previous Theorem 2.4 is not true for $p = 1$ because, in this case, it is possible to approximate every $u \in BV(\Omega)$ by a sequence of smooth functions (which are also functions of class $SBV(\Omega)$).
In [14], section 3, a Poincaré-Wirtinger type inequality for functions of the class SBV in a ball and two consequences have been proved. Here, for completeness and the reader's convenience, we give the statements.

Let B be a ball in $\mathbf{R}^n$, $n \geq 2$; for every measurable function $u : B \to \mathbf{R}$, we consider the non decreasing rearrangement of u

$$u_*(s, B) = \inf\{\ t \in \mathbf{R};\ |\{u < t\} \cap B| \geq s\} \qquad \text{for } 0 \leq s \leq |B|,$$

and we set

$$\mathrm{med}(u, B) = u_* \left(\frac{1}{2}|B|, B\right);$$

moreover for every $u \in SBV(B)$ such that $\quad (2\gamma_n \mathcal{H}_{n-1}(S_u \cap B))^{\frac{n}{n-1}} < \frac{1}{2}|B| \quad$ we set

$$\tau'(u, B) = u_* \left((2\gamma_n \mathcal{H}_{n-1}(S_u \cap B))^{\frac{n}{n-1}}, B\right),$$

$$\tau''(u, B) = u_* \left(|B| - (2\gamma_n \mathcal{H}_{n-1}(S_u \cap B))^{\frac{n}{n-1}}, B\right),$$

where γ_n is the isoperimetric constant relative to the balls of $\mathbf{R}^n$.

THEOREM 2.5 *Let $B \subset \mathbf{R}^n$ be a ball, $n \geq 2$, $1 \leq p < n$ and $p^* = \frac{np}{n-p}$. Let $u \in SBV(B)$, $\mathcal{H}_{n-1}(S_u \cap B) < \frac{1}{2\gamma_n} \left(\frac{1}{2}|B|\right)^{\frac{n-1}{n}}$, and*

$$\overline{u} = (u \wedge \tau''(u, B)) \vee \tau'(u, B).$$

Then

$$\|\overline{u} - \mathrm{med}(u, B)\|_{L^{p^*}(B)} \leq \frac{2\gamma_n p(n-1)}{n-p} \|\nabla u\|_{L^p(B)}.$$

THEOREM 2.6. *Let $B \subset \mathbf{R}^n$ be a ball, $u_h \in SBV(B)$, $p > 1$, and let*

$$\sup_{h \in \mathbf{N}} \int_B |\nabla u_h|^p dx < +\infty,$$

$$\lim_h \mathcal{H}_{n-1}(S_{u_h} \cap B) = 0.$$

Then there exist a subsequence (u_{h_i}) and a function $u_\infty \in W^{1,p}(B)$ such that

$$\lim_i [\overline{u}_{h_i} - \mathrm{med}(u_{h_i}, B)] = u_\infty \qquad \text{in } L^r(B)$$

for every $1 \leq r < \frac{np}{n-p}$ if $1 < p < n$, and for every $r \geq 1$ if $p \geq n$; moreover

$$\lim_i [u_{h_i} - \mathrm{med}(u_{h_i}, B)] = u_\infty \qquad \text{a.e. on } B.$$

THEOREM 2.7 *Let $n \in \mathbf{N}$, $n \geq 2$, $p > 1$; let $u \in SBV(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$. If*

$$\lim_{\rho \to 0} \rho^{1-n} \left[\int_{B_\rho(x)} |\nabla u|^p dy + \mathcal{H}_{n-1}(S_u \cap B_\rho(x))\right] = 0,$$

then $x \notin S_u$ and $\tilde{u}(x) \in \mathbf{R}$.

3. A limit theorem and some estimates at the boundary.

Given Ω and w as in Theorem 1.1, in this section we will study some properties of a function $u \in SBV(\mathbf{R}^n)$ solution of the following minimum problem (see Lemma 4.1 for the existence of u)

$$\min\left\{\int_\Omega |\nabla v|^2 dx + \lambda\mathcal{H}_{n-1}(S_v);\ v \in SBV(\mathbf{R}^n),\ v = w_e \text{ in } \mathbf{R}^n \setminus \overline{\Omega}\right\}, \tag{3.1}$$

where $w_e \in W^{1,1}(\mathbf{R}^n)$ is an extension of the boundary datum w such that $\tilde{w}_e = w$ $\mathcal{H}_{n-1}$-a.e. on $\partial\Omega$.

In particular we prove that, setting

$$\Omega_0 = \left\{x \in \overline{\Omega} \setminus M;\quad \lim_{\rho\to 0} \rho^{1-n}\mathcal{H}_{n-1}(S_u \cap \overline{B_\rho(x)}) = 0\right\},$$

then $\overline{\Omega} \setminus \Omega_0$ is closed and $\mathcal{H}_{n-1}\left((\overline{\Omega} \setminus \Omega_0) \triangle S_u\right) = 0$, where $A \triangle B$ denotes the symmetric difference of the sets A and B.

Such partial regularity result (see Theorem 3.12 and Remark 3.13) shall allow us to prove Theorem 1.1 and Proposition 1.2 in the next section, by showing that the pair $(\overline{\Omega} \setminus \Omega_0, u)$ is a solution of the minimum problem considered in this paper.

DEFINITION 3.1. Let $u \in SBV(\mathbf{R}^n)$ and $0 < c < +\infty$. Let $K \subset \mathbf{R}^n$ be closed. We set

$$\mathcal{F}(u, c, K) = \int_K |\nabla u|^2 dx + c\mathcal{H}_{n-1}(S_u \cap K),$$

$$\Phi(u, c, K) = \inf\ \{\ \mathcal{F}(v, c, K)\ ;\ v \in SBV(\mathbf{R}^n),\quad v = u \text{ in } \mathbf{R}^n \setminus K\ \};$$

moreover, if $\Phi(u, c, K) < +\infty$, we set

$$\Psi(u, c, K) = \mathcal{F}(u, c, K) - \Phi(u, c, K).$$

We first state some technical lemmas.

LEMMA 3.2. *Let $u \in SBV(B_r)$. For every $0 < c < +\infty$ the functions*

$\rho \to \mathcal{F}(u, c, \overline{B_\rho})$

and

$\rho \to \Psi(u, c, \overline{B_\rho})$

are non-decreasing in $(0, r)$.

LEMMA 3.3. *Let $u \in SBV(B_r(x_0))$, $\rho < r$. Set $u_\rho(x) = \rho^{-1/2}\ u(x_0 + \rho x)$ for every $x \in B_{r/\rho}$, then*

$u_\rho \in SBV(B_{r/\rho})$,

$\mathcal{F}(u_\rho, c, \overline{B_1}) = \rho^{1-n}\mathcal{F}(u, c, \overline{B_\rho(x_0)})$

and

$$\Phi(u_\rho, c, \overline{B_1}) = \rho^{1-n}\Phi(u, c, \overline{B_\rho(x_0)}).$$

LEMMA 3.4 *Let $u \in SBV(\mathbf{R}^n)$ and $0 < c < +\infty$. If $\Psi(u, c, K) = 0$ for some closed set $K \subset \mathbf{R}^n$, then*

$$\mathcal{F}(u, c, \overline{B_\rho(x)}) \leq cn\omega_n\rho^{n-1}$$

for every $\overline{B_\rho(x)} \subset K$.

Proof. Because of the minimality of u we have

$$\mathcal{F}(u, c, \overline{B_\rho(x)}) \leq \mathcal{F}(u\chi_{\mathbf{R}^n\setminus\overline{B_\rho(x)}}, c, \overline{B_\rho(x)}) \leq cn\omega_n\rho^{n-1}.$$ **q.e.d.**

The proofs of the following Lemma 3.5 and Lemma 3.6 are similar to the ones exhibited in [14] for Lemma 4.6 and Lemma 4.7 respectively.

LEMMA 3.5 *Let $u,\ v \in SBV(B_r)$, $0 < c < +\infty$ and $0 < \rho < r$. Suppose*

$$\mathcal{H}_{n-1}(S_u \cap \partial B_\rho) = \mathcal{H}_{n-1}(S_v \cap \partial B_\rho) = 0.$$

Set

$$w(x) = \begin{cases} u(x) & \text{if } x \in \overline{B_\rho}, \\ v(x) & \text{if } x \in B_r \setminus \overline{B_\rho}, \end{cases}$$

then

$$\mathcal{F}(w, c, \overline{B_\rho}) \leq \mathcal{F}(u, c, \overline{B_\rho}) + c\mathcal{H}_{n-1}\left(\{\tilde{u} \neq \tilde{v}\} \cap \partial B_\rho\right).$$

LEMMA 3.6. *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \mathbf{R}^n$ two open sets. Let $\gamma \in C_0^1(\Omega_2)$ with $|\gamma| \leq 1,\ \gamma \equiv 1$ in a neighborhood of $\overline{\Omega_1}$, $|\nabla\gamma| \leq L$. Let $u,\ v \in SBV(\mathbf{R}^n)$ and $w = \gamma u + (1-\gamma)v$. For every $0 < c < +\infty$ and for every $0 < \lambda < 1$ it is true that*

$$\mathcal{F}(w, c, \overline{\Omega_2}) \leq \frac{1}{1-\lambda}\left[\mathcal{F}(u, c, \overline{\Omega_2}) + \mathcal{F}(v, c, \overline{\Omega_2} \setminus \Omega_1)\right] + \frac{L^2}{\lambda}\int_{\Omega_2\setminus\Omega_1} |u - v|^2 dy.$$

To treat *blowing-up* at a boundary point of Ω for a function $u \in SBV(\mathbf{R}^n)$ minimizing the functional $\mathcal{F}$ in Ω with given Dirichlet boundary datum, we introduce the following notation.

For any C^1 function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ with $\varphi(0) = 0 = |\nabla\varphi(0)|$, $Lip\,\varphi \leq 1$, let

$$\Omega_\varphi = \{x \in B_1;\ x_n > \varphi(x')\}$$

where $x' = (x_1, \ldots, x_{n-1})$.

We are now in a position to prove the following limit theorem.

THEOREM 3.7. *Let $\varphi_h : \mathbf{R}^{n-1} \to \mathbf{R}$ be a sequence of C^1 functions such that $\varphi_h(0) = 0 = |\nabla\varphi_h(0)|$, $Lip\,\varphi_h \le 1$, $\lim_h \|\nabla\varphi_h\|_{L^\infty} = 0$. Let $w_h : B_1 \to \mathbf{R}$ be a sequence of C^1 functions such that $\lim_h (\|w_h\|_{L^\infty} + \|\nabla w_h\|_{L^\infty}) = 0$. Let $c_h \in \mathbf{R}$ and $u_h \in SBV(\mathbf{R}^n)$ such that $u_h = w_h$ in $B_1 \setminus \overline{\Omega}_{\varphi_h}$ for every $h \in \mathbf{N}$; let $u_\infty \in W^{1,2}(B_1)$. Assume that*

(1) $\lim_h \mathcal{F}(u_h, c_h, \overline{B_\rho}) = \alpha(\rho) < +\infty$ *for almost all* $\rho < 1$,

(2) $\lim_h \Psi(u_h, c_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) = 0$ *for almost all* $\rho < 1$,

(3) $\lim_h c_h = +\infty$,

(4) $\lim_h u_h = u_\infty$ *a.e. on* B_1.

Then

the function $u_\infty \in C^0(B_1)$, u_∞ is harmonic in $\{x_n > 0\} \cap B_1$, $u_\infty \equiv 0$ in $\{x_n \le 0\} \cap B_1$ and $\alpha(\rho) = \int_{B_\rho} |\nabla u_\infty|^2 dy$ for almost all $\rho < 1$.

Proof. By the hypothesis on the functions φ_h, we infer that, for every $\delta > 0$ and for every h large enough, $-\delta < \varphi_h(x') < \delta$.
By the hypothesis on the functions w_h and by the assumption $u_h = w_h$ in $B_1 \setminus \overline{\Omega}_{\varphi_h}$, we infer that $\lim_h u_h = 0$ in $\{x_n < -\delta\} \cap B_1$. Therefore, by (4), $u_\infty \equiv 0$ in $\{x_n < 0\} \cap B_1$.
On the other hand, for every $\delta > 0$ and for every ball $B_r(x)$ such that $\overline{B_r(x)} \subset \{x_n > \delta\} \cap B_1$, we have by Lemma 3.2 and by (1) and (2) respectively,

$$\sup_h \mathcal{F}(u_h, c_h, \overline{B_r(x)}) < +\infty$$

and

$$\lim_h \Psi(u_h, c_h, \overline{B_r(x)}) = 0.$$

By using Theorem 4.8 of [14] and by the arbitrariness of δ , we infer that u_∞ is harmonic in $\{x_n > 0\} \cap B_1$. By well-known results (see e.g. [21], chap. II, Appendices) it follows that $u_\infty \in C^0(B_1)$, so $u_\infty \equiv 0$ in $\{x_n \le 0\} \cap B_1$.
By the hypothesis (1) and by the semicontinuity theorem 2.4 we have, for any $c > 0$,

$$\int_{B_\rho} |\nabla u_\infty|^2 dy = \mathcal{F}(u_\infty, c, \overline{B_\rho}) \le \liminf_h \mathcal{F}(u_h, c, \overline{B_\rho}) \le \lim_h \mathcal{F}(u_h, c_h, \overline{B_\rho}) = \alpha(\rho)$$

for almost all $\rho < 1$.
The proof will be completed by proving the following inequality

$$\alpha(\rho) \le \int_{B_\rho} |\nabla u_\infty|^2 dy \quad \text{for almost all} \quad \rho < 1. \tag{3.2}$$

We may suppose $\|w_h\|_{L^\infty} < 1/h$. Set, with the notations of Theorem 2.5,

$$\hat{u}_h = u_h \wedge (\tau''(u_h, B_1) \vee 1/h) \vee (\tau'(u_h, B_1) \wedge (-1/h)),$$

we have $\hat{u}_h = w_h$ in $B_1 \setminus \overline{\Omega}_{\varphi_h}$ and, by Theorem 2.6,

$$\lim_h \hat{u}_h = u_\infty \quad \text{in} \quad L^2(B_1). \tag{3.3}$$

Moreover, as in Theorem 4.8 of [14], there exists a subsequence of $(\hat{u}_h)$ (for brevity's sake still denoted by $(\hat{u}_h)$) such that

$$\lim_h c_h \mathcal{H}_{n-1}(\{\tilde{\hat{u}}_h \neq \tilde{u}_h\} \cap \partial B_\rho) = 0 \tag{3.4}$$

for almost all $\rho < 1$.
Set now

$$w'_h(x) = \begin{cases} w_h(x) & \text{if } x \in B_1 \setminus \overline{\Omega}_{\varphi_h}, \\ w_h(x', \varphi_h(x')) + u_\infty(x', x_n - \varphi_h(x')) & \text{if } x \in \overline{\Omega}_{\varphi_h}, \end{cases}$$

we notice that the functions w'_h are Lipschitz continuous in B_ρ uniformly with respect to h and $\lim_h w'_h = u_\infty$ in $L^\infty(B_\rho)$ for every $\rho < 1$.
Finally we may prove (3.2).
Since the function $\rho \to \alpha(\rho)$ is non-decreasing, it is also a continuous function for almost all $\rho < 1$. Let $\rho < 1$ be such that $\alpha(\cdot)$ is continuous in ρ and the hypothesis (1) and the condition (3.4) are fulfilled. Let $L > 0$ be such that $\sup_{B_\rho} |\nabla w'_h| \leq L$ for every $h \in \mathbf{N}$.
Fixed $\epsilon > 0$, let $0 < \rho_1 < \rho$ be such that $\alpha(\rho) - \alpha(\rho_1) < \epsilon$ and $L^2 |B_\rho \setminus B_{\rho_1}| < \epsilon$; moreover let ρ_2 be such that $\rho_1 < \rho_2 < \rho$ and the hypothesis (1) is fulfilled.
For $0 < r < 1$ and $-r < \delta < r$ we set $B_{r,\delta} = B_r \cap \{x_n > \delta\}$. Let $0 < \rho_3 < \rho_4$ and $0 < \delta_2 < \delta_1$ be such that

$$B_{\rho_1,\delta_1} \subset\subset B_{\rho_2,\delta_2} \subset\subset B_{\rho_3} \subset\subset B_{\rho_4} \subset\subset B_\rho$$

and $L^2 |B_{\rho,-\delta_2} \setminus B_{\rho_1,\delta_1}| < \epsilon$.
Let γ_1 and γ_2 be two C^1 functions such that

$$|\gamma_1| \leq 1,\ \gamma_1 \equiv 1 \text{ in a neighborhood of } B_{\rho_1,\delta_1},\quad \text{spt}\gamma_1 \subset B_{\rho_2,\delta_2},\ |\nabla\gamma_1|^2 \leq \tfrac{2}{(\rho_2-\rho_1)^2} + \tfrac{2}{(\delta_2-\delta_1)^2},$$

and

$$|\gamma_2| \leq 1,\ \gamma_2 \equiv 1 \text{ in a neighborhood of } B_{\rho_3,-\delta_2},\quad \text{spt}\gamma_2 \subset B_{\rho_4},\ |\nabla\gamma_2|^2 \leq \tfrac{2}{(\rho_4-\rho_3)^2}.$$

Now we define the following three sequences of functions in $SBV(B_1)$

$$\zeta_h = \gamma_1 u_\infty + (1-\gamma_1) w'_h$$

$$\xi_h = \gamma_2 \zeta_h + (1-\gamma_2)\hat{u}_h$$

$$z_h = \begin{cases} \xi_h & \text{in } \overline{B_\rho}, \\ u_h & \text{in } B_1 \setminus \overline{B_\rho}. \end{cases}$$

We notice that $z_h = u_h$ in $B_1 \setminus \overline{B_\rho}$ for every $h \in \mathbf{N}$, and $z_h = w_h$ in $B_1 \setminus \overline{\Omega}_{\varphi_h}$ for h large enough in order to have $\|\varphi_h\|_{L^\infty} < \delta_2$. Then, setting $\epsilon_h = \Psi(u_h, c_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho})$, by Lemma 3.5 and 3.6, for every $0 < \lambda < 1$ we have

$$\mathcal{F}(u_h, c_h, \overline{B_\rho}) - \epsilon_h \leq \mathcal{F}(z_h, c_h, \overline{B_\rho}) \leq \mathcal{F}(\xi_h, c_h, \overline{B_\rho}) + c_h \mathcal{H}_{n-1}\left(\{\tilde{\hat{u}}_h \neq \tilde{u}_h\} \cap \partial B_\rho\right) \leq$$

$$\leq \frac{1}{1-\lambda}\left[\mathcal{F}(\zeta_h, c_h, \overline{B_\rho}) + \mathcal{F}(\hat{u}_h, c_h, \overline{B_\rho} \setminus B_{\rho_3,-\delta_2})\right] +$$

$$+\frac{2}{\lambda(\rho_4-\rho_3)^2}\int_{B_\rho\setminus B_{\rho_3,-\delta_2}}|\hat{u}_h-\zeta_h|^2dy+c_h\mathcal{H}_{n-1}(\{\tilde{\hat{u}}_h\neq\tilde{u}_h\}\cap\partial B_\rho).$$

By using again Lemma 3.6 we have

$$\mathcal{F}(u_h,c_h,\overline{B_\rho})-\epsilon_h\leq\frac{1}{1-\lambda}\left[\frac{1}{1-\lambda}\left(\int_{B_\rho}|\nabla u_\infty|^2dy+\int_{B_\rho\setminus B_{\rho_1,\delta_1}}|\nabla w_h'|^2dy\right)+\right.$$

$$\left.+\frac{1}{\lambda}\left(\frac{2}{(\rho_2-\rho_1)^2}+\frac{2}{(\delta_2-\delta_1)^2}\right)\int_{B_\rho\setminus B_{\rho_1,\delta_1}}|w_h'-u_\infty|^2dy+\mathcal{F}(\hat{u}_h,c_h,\overline{B_\rho}\setminus B_{\rho_3,-\delta_2})\right]+$$

$$+\frac{2}{\lambda(\rho_4-\rho_3)^2}\int_{B_\rho\setminus B_{\rho_3,-\delta_2}}|\hat{u}_h-\zeta_h|^2dy+c_h\mathcal{H}_{n-1}(\{\tilde{\hat{u}}_h\neq\tilde{u}_h\}\cap\partial B_\rho)\ ;$$

then, letting $h\to+\infty$ and taking into account (3.3), (3.4) and the hypotheses (1), (2), we obtain

$$\alpha(\rho)\leq\frac{1}{1-\lambda}\left[\frac{1}{1-\lambda}\left(\int_{B_\rho}|\nabla u_\infty|^2dy+\epsilon\right)+\limsup_h\mathcal{F}(\hat{u}_h,c_h,\overline{B_\rho}\setminus B_{\rho_3,-\delta_2})\right]\leq$$

$$\leq\left(\frac{1}{1-\lambda}\right)^2\left(\int_{B_\rho}|\nabla u_\infty|^2dy+\epsilon\right)+\frac{1}{1-\lambda}\limsup_h\mathcal{F}(u_h,c_h,\overline{B_\rho}\setminus B_{\rho_2})\leq$$

$$\leq\left(\frac{1}{1-\lambda}\right)^2\left(\int_{B_\rho}|\nabla u_\infty|^2dy+\epsilon\right)+\frac{1}{1-\lambda}\left(\alpha(\rho)-\alpha(\rho_2)\right)\leq$$

$$\leq\left(\frac{1}{1-\lambda}\right)^2\left(\int_{B_\rho}|\nabla u_\infty|^2dy+\epsilon\right)+\frac{\epsilon}{1-\lambda}.$$

For the arbitrariness of ϵ and λ the assertion follows. **q.e.d.**

COROLLARY 3.8. *Let (φ_h) and (w_h) be as in Theorem 3.7. Let $\lambda_h\in\mathbf{R}$ with $0<c\leq\lambda_h<+\infty$ for every $h\in\mathbf{N}$; let $u_h\in SBV(\mathbf{R}^n)$ such that $u_h=w_h$ in $B_1\setminus\overline{\Omega}_{\varphi_h}$, and let $u_\infty\in W^{1,2}(B_1)$. Assume that*

(1) $\lim_h\mathcal{F}(u_h,\lambda_h,\overline{B_\rho})=\alpha(\rho)<+\infty$ *for almost all* $\rho<1$,

(2) $\lim_h\Psi(u_h,\lambda_h,\overline{\Omega}_{\varphi_h}\cap\overline{B_\rho})=0$ *for almost all* $\rho<1$,

(3) $\lim_h\mathcal{H}_{n-1}(S_{u_h}\cap\overline{B_1})=0$,

(4) $\lim_h u_h=u_\infty$ a.e. *on* B_1.

Then the same thesis of Theorem 3.7 is true.

Proof. If $\limsup_h\lambda_h=+\infty$ the assertion follows by Theorem 3.7. If $\limsup_h\lambda_h<+\infty$, setting $c_h=\lambda_h\vee\left(\mathcal{H}_{n-1}(S_{u_h}\cap\overline{B_1})+\frac{1}{h}\right)^{-1/2}$, we have $\lim_h c_h=+\infty$ and

$$\lim_h \mathcal{F}(u_h, c_h, \overline{B_\rho}) = \alpha(\rho) < +\infty \quad \text{for almost all } \rho < 1.$$

Since

$$\mathcal{F}(u_h, c_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) = \mathcal{F}(u_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) + (c_h - \lambda_h)\mathcal{H}_{n-1}(S_{u_h} \cap \overline{B_\rho}) =$$

$$= \Phi(u_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) + \Psi(u_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) + (c_h - \lambda_h)\mathcal{H}_{n-1}(S_{u_h} \cap \overline{B_\rho}) \leq$$

$$\leq \Phi(u_h, c_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) + \Psi(u_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) + (c_h - \lambda_h)\mathcal{H}_{n-1}(S_{u_h} \cap \overline{B_\rho}),$$

we have also

$$\lim_h \Psi(u_h, c_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_\rho}) = 0 \quad \text{for almost all } \rho < 1.$$

Then the assertion follows again by Theorem 3.7. **q.e.d.**

From Corollary 3.8 we infer the following decay estimates near a boundary point.

LEMMA 3.9. *For every $n \in \mathbf{N}$, $n \geq 2$, and every $0 < c < +\infty$, $0 < \alpha < 1$, $0 < \beta < 1$ and $L > 0$ there exist $\epsilon = \epsilon(n, c, \alpha, \beta, L)$ and $\vartheta = \vartheta(n, c, \alpha, \beta, L)$ such that:*

for every $\varphi \in C^1(\mathbf{R}^{n-1})$ with $\varphi(0) = 0 = |\nabla\varphi(0)|$, $Lip\,\varphi \leq 1$ and for every $w \in C^1(B_2)$, with $Lip\,w < L$, if $u \in SBV(B_2)$, $u = w$ in $B_1 \setminus \overline{\Omega}_\varphi$, $\Psi(u, c, \overline{\Omega}_\varphi \cap \overline{B_1}) = 0$ and

$$\mathcal{H}_{n-1}(S_u \cap \overline{B_1}) \leq \epsilon,$$

then

$$\mathcal{F}(u, c, \overline{B_\alpha}) \leq \alpha^{n-\beta} \max\left\{\mathcal{F}(u, c, \overline{B_1}), \vartheta\left[(Lip\,\varphi)^2 + (Lip\,w)^2\right]\right\}.$$

Proof. Suppose the lemma is not true. Then there exist $n \geq 2$, $c > 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $L > 0$, two sequences (ϵ_h), (ϑ_h) such that $\lim_h \epsilon_h = 0$, $\lim_h \vartheta_h = +\infty$, a sequence $\varphi_h \in C^1(\mathbf{R}^{n-1})$ with $\varphi_h(0) = 0 = |\nabla\varphi_h(0)|$, $Lip\,\varphi_h \leq 1$, a sequence $w_h \in C^1(B_2)$ with $Lip\,w_h < L$, a sequence $u_h \in SBV(B_2)$ with $u_h = w_h$ in $B_1 \setminus \overline{\Omega}_{\varphi_h}$, $\Psi(u_h, c, \overline{\Omega}_{\varphi_h} \cap \overline{B_1}) = 0$ and

$$\mathcal{H}_{n-1}(S_{u_h} \cap \overline{B_1}) = \epsilon_h\ ,$$

$$\mathcal{F}(u_h, c, \overline{B_\alpha}) > \alpha^{n-\beta}\vartheta_h\left[(Lip\,\varphi_h)^2 + (Lip\,w_h)^2\right], \tag{3.5}$$

$$\mathcal{F}(u_h, c, \overline{B_\alpha}) > \alpha^{n-\beta}\mathcal{F}(u_h, c, \overline{B_1}). \tag{3.6}$$

Set $\lambda_h = \frac{c}{\mathcal{F}(u_h, c, \overline{B_1})}$ and $v_h = \left(\frac{\lambda_h}{c}\right)^{\frac{1}{2}} u_h$, we have

$$\mathcal{F}(v_h, \lambda_h, \overline{B_1}) = 1,$$

$$\Psi(v_h, \lambda_h, \overline{\Omega}_{\varphi_h} \cap \overline{B_1}) = 0,$$

$$v_h = \left(\frac{\lambda_h}{c}\right)^{\frac{1}{2}} w_h \quad \text{in} \quad B_1 \setminus \overline{\Omega}_{\varphi_h}.$$

Since by (3.5)

$$\max\left\{(Lip\,\varphi_h)^2, (Lip\,w_h)^2\right\} < \frac{c}{\lambda_h \alpha^{n-\beta} \vartheta_h},$$

and since (as in Lemma 3.4) $\inf_h \lambda_h > 0$, we have

$$\lim_h Lip\left[\left(\frac{\lambda_h}{c}\right)^{\frac{1}{2}} w_h\right] = 0 \quad \text{and} \quad \lim_h Lip\,\varphi_h = 0. \tag{3.7}$$

Moreover, since $\lim_h \epsilon_h = 0$, by Theorem 2.6 there exist a subsequence, still denoted by (v_h), and a function $v_\infty \in W^{1,2}(B_1)$ such that

$$\lim_h \left[v_h - \text{med}(v_h, B_1)\right] = v_\infty \qquad \textit{a.e. on } B_1.$$

By (3.7) the sequence

$$\left(\left(\frac{\lambda_h}{c}\right)^{\frac{1}{2}} w_h - \text{med}(v_h, B_1)\right)$$

uniformly converges to zero. Then, by Corollary 3.8, $v_\infty \in C^0(B_1)$, v_∞ is harmonic in $\{x_n > 0\} \cap B_1$ and $v_\infty \equiv 0$ in $\{x_n \leq 0\} \cap B_1$. By Schwarz reflection principle there exists a function V harmonic in B_1 defined by

$$V(x) = \begin{cases} v_\infty(x) & \text{if } x_n \geq 0, \\ -v_\infty(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

for which we have

$$\int_{B_\alpha} |\nabla v_\infty|^2 dy = \frac{1}{2} \int_{B_\alpha} |\nabla V|^2 dy \leq \frac{\alpha^n}{2} \int_{B_1} |\nabla V|^2 dy = \alpha^n \int_{B_1} |\nabla v_\infty|^2 dy.$$

Therefore, still by Corollary 3.8, we obtain

$$\limsup_h \mathcal{F}(v_h, \lambda_h, \overline{B_\alpha}) \leq \alpha^n \int_{B_1} |\nabla v_\infty|^2 dy \leq \alpha^n,$$

whereas by (3.6) we have

$$\mathcal{F}(v_h, \lambda_h, \overline{B_\alpha}) > \alpha^{n-\beta}.$$

So we obtain a contradiction. **q.e.d.**

LEMMA 3.10. *Let $n \in \mathbf{N}$, $n \geq 2$, let $0 < c < +\infty$, $0 < \alpha < 1$ and $0 < \beta < 1$. Let $\varphi \in C^1(\mathbf{R}^{n-1})$ with $\varphi(0) = 0 = |\nabla\varphi(0)|$, $Lip\,\varphi \leq 1$, let $w \in C^1(\overline{B_2})$ and $L = \max_{x \in \overline{B_1}} |\nabla w(x)|$. There exist $\epsilon' > 0$, $0 < r < 1$ such that if $u \in SBV(B_2)$, $u = w$ in $B_r \setminus \overline{\Omega}_\varphi$, $\Psi(u, c, \overline{\Omega}_\varphi \cap \overline{B_r}) = 0$ and if $\mathcal{H}_{n-1}(S_u \cap \overline{B_\rho}) \leq \epsilon' \rho^{n-1}$ for some $0 < \rho \leq r$, then*

$$\lim_{t \to 0} t^{1-n} \mathcal{F}(u, c, \overline{B_t}) = 0 \ .$$

Proof. Let ϵ, ϑ be as in lemma 3.9; let $\alpha' \in (0, 1)$ be such that $(\alpha')^{1-\beta} cn\omega_n < \epsilon$ and let ϵ' and ϑ' be the constants depending on n, c, α', β, L given by lemma 3.9.

Set $u_\rho(x) = \rho^{-1/2} u(\rho x)$. We have $u_\rho \in SBV(B_{2/\rho})$ and

$$\Psi(u_\rho, c, \overline{\Omega}_{\varphi_\rho} \cap \overline{B_1}) = 0 \ ,$$

where $\varphi_\rho(x') = \rho^{-1}\varphi(\rho x')$ for every $x' \in \mathbf{R}^{n-1}$; moreover $u_\rho = w_\rho$ in $B_1 \setminus \overline{\Omega}_{\varphi_\rho}$, where $w_\rho(x) = \rho^{-1/2} w(\rho x)$. Let $r > 0$ be such that

$$(\vartheta \vee \vartheta') \left[\left(Lip_{B_r} \varphi \right)^2 + rL^2 \right] < \epsilon \ .$$

Assume that $\mathcal{H}_{n-1}(S_u \cap \overline{B_\rho}) \leq \epsilon' \rho^{n-1}$ for some $0 < \rho \leq r$. Then by lemma 3.3 and lemma 3.9 we have

$$\mathcal{F}(u_\rho, c, \overline{B'_\alpha}) \leq (\alpha')^{n-\beta} \left(\mathcal{F}(u_\rho, c, \overline{B_1}) \vee \epsilon \right)$$

hence by lemma 3.4

$$\mathcal{F}(u, c, \overline{B}_{\alpha'\rho}) \leq (\alpha')^{n-\beta} \left(\mathcal{F}(u, c, \overline{B_\rho}) \vee \epsilon\rho^{n-1} \right) \leq (\alpha')^{n-\beta} (cn\omega_n \rho^{n-1} \vee \epsilon\rho^{n-1}) \leq \epsilon(\alpha'\rho)^{n-1} \ .$$

Set $\rho' = \alpha'\rho$. Since we have

$$\mathcal{H}_{n-1}(S_u \cap \overline{B}_{\rho'} \leq \epsilon(\rho')^{n-1}$$

then, by lemma 3.9, we obtain

$$\mathcal{F}(u, c, \overline{B}_{\alpha\rho'}) \leq \alpha^{n-\beta} \max\left\{ \mathcal{F}(u, c, \overline{B}_{\rho'}), \epsilon(\rho')^{n-1} \right\} \leq \alpha^{1-\beta} \epsilon(\alpha\rho')^{n-1}.$$

By induction we obtain for every $h \in \mathbf{N}$

$$\mathcal{F}(u, c, \overline{B}_{\alpha^h\rho'}) \leq \alpha^{h(1-\beta)} \epsilon(\alpha^h\rho')^{n-1}. \tag{3.8}$$

Now let $t < \rho'$ and let $\alpha^h\rho' \leq t < \alpha^{h-1}\rho'$; then by (3.8) we have

$$t^{1-n} \mathcal{F}(u, c, \overline{B}_t) \leq (\alpha^h\rho')^{1-n} \mathcal{F}(u, c, \overline{B}_{\alpha^{h-1}\rho'}) \leq \alpha^{1-n} \epsilon \alpha^{(h-1)(1-\beta)},$$

hence the assertion follows. **q.e.d.**

REMARK 3.11. Let $\Omega \in \mathbf{R}^n$ be a bounded domain with $\mathcal{H}_{n-1}(\partial\Omega) < +\infty$ and let $u \in SBV(\mathbf{R}^n)$ be a solution of the minimum problem (3.1). Let $w'_e \in W^{1,1}(\mathbf{R}^n)$ such that $\tilde{w'_e} = \tilde{w}_e$ $\mathcal{H}_{n-1}$-a.e. on $\partial\Omega$. Then the function

$$u'(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega}, \\ \\ w'_e(x) & \text{if } x \in \mathbf{R}^n \setminus \overline{\Omega} \end{cases}$$

is a minimizer for the functional

$$\int_\Omega |\nabla v|^2 dx + \lambda \mathcal{H}_{n-1}(S_v)$$

among the functions $v \in SBV(\mathbf{R}^n)$, $v = w'_e$ in $\mathbf{R}^n \setminus \overline{\Omega}$.

THEOREM 3.12. (Partial regularity) *Let $n \in \mathbf{N}$, $n \geq 2$, let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $0 < \lambda < +\infty$. Assume that a closed set M exists such that $\mathcal{H}_{n-1}(M) = 0$ and $\partial\Omega \setminus M$ is a C^1 surface; let $w \in C^1(\partial\Omega \setminus M)$ and let $w_e \in W^{1,1}(\mathbf{R}^n)$ be such that $\tilde{w}_e = w$ $\mathcal{H}_{n-1}$-a.e. on $\partial\Omega$. Assume that $u \in SBV(\mathbf{R}^n)$ satisfies the conditions $\Psi(u, \lambda, \overline{\Omega}) = 0$, $u = w_e$ in $\mathbf{R}^n \setminus \overline{\Omega}$. Set*

$$\Omega_0 = \left\{ x \in \overline{\Omega} \setminus M \ ; \ \lim_{\rho \to 0} \rho^{1-n} \mathcal{F}(u, \lambda, \overline{\Omega} \cap \overline{B_\rho(x)}) = 0 \ \right\},$$

then

(i) Ω_0 *is relatively open in* $\overline{\Omega}$,

(ii) $\mathcal{H}_{n-1}\left((\overline{\Omega} \setminus \Omega_0) \triangle S_u\right) = 0$.

Proof. By Theorem 4.12 in [14] $\Omega_0 \cap \Omega$ is an open set. Let now $x \in \Omega_0 \cap \partial\Omega$; by virtue of the hypotheses, there exists $B_r(x)$ such that $B_r(x) \cap M = \emptyset$ and $\partial\Omega \cap B_r(x)$ is a C^1 surface. By Remark 3.11 we may assume $w_e \in C^1(B_r(x))$, hence $\lim_{\rho \to 0} \rho^{1-n} \mathcal{F}(u, \lambda, \overline{B_\rho(x)}) = 0$. Provided that r has been selected small enough, by Lemma 3.10 we have

$$\partial\Omega \cap B_{r/2}(x) \subset \Omega_0.$$

On the other hand, provided that $r' < r/2$ be small enough in order to apply Lemma 4.11 of [14], we have also that

$$\Omega \cap B_{r'}(x) \subset \Omega_0.$$

Thus (i) is proved. By Theorem 2.7 we have $S_u \cap \overline{\Omega} \subset \overline{\Omega} \setminus \Omega_0$. Finally, by a covering argument (see e.g. Lemma 2.6 in [14]), we have $\mathcal{H}_{n-1}\left((\overline{\Omega} \setminus \Omega_0) \setminus S_u\right) = 0$, so also (ii) is proved. **q.e.d.**

REMARK 3.13. If u is a solution of the minimum problem (3.1) and if for $x \in \partial\Omega \setminus M$ we have

$$\lim_{\rho \to 0} \rho^{1-n} \mathcal{H}_{n-1}(S_u \cap \overline{B_\rho(x)}) = 0,$$

then, by Lemma 3.10, we have also

$$\lim_{\rho \to 0} \rho^{1-n} \mathcal{F}(u, \lambda, \overline{\Omega} \cap \overline{B_\rho(x)}) = 0.$$

We remark that such a result also is true for $x \in \Omega$. Indeed, to this aim, it is enough to prove a corollary of Theorem 4.8 of [14], similar to Corollary 3.8 of this paper, so that we may assume in the hypotheses of Lemma 4.9 of [14] the weaker condition

$$\mathcal{H}_{n-1}(S_u \cap \overline{B_\rho(x)}) \leq \epsilon \rho^{n-1}$$

instead of

$$\mathcal{F}(u, \lambda, \overline{\Omega} \cap \overline{B_\rho(x)}) \leq \epsilon \rho^{n-1}.$$

Therefore we conclude that the set Ω_0 defined in Theorem 3.12 is equal to the set

$$\left\{ x \in \overline{\Omega} \setminus M; \quad \lim_{\rho \to 0} \rho^{1-n} \mathcal{H}_{n-1}(S_u \cap \overline{B_\rho(x)}) = 0 \right\}.$$

4. Proof of the existence theorem.

We begin this section by proving the existence of a solution for the minimum problem (3.1) in $SBV(\mathbf{R}^n)$.

LEMMA 4.1. *Under the hypotheses of Theorem 1.1, there exists* $w_e \in W^{1,1}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ *such that* $\tilde{w}_e = w$ $\mathcal{H}_{n-1}$*-a.e. on* $\partial\Omega$ *and* $\|w_e\|_{L^\infty} = \|w\|_{L^\infty}$ *; moreover there exists*

$$\min\left\{\int_\Omega |\nabla v|^2 dx + \lambda\mathcal{H}_{n-1}(S_v);\ v \in SBV(\mathbf{R}^n),\ v = w_e \ \text{ in } \ \mathbf{R}^n \setminus \overline{\Omega}\right\}$$

and it is smaller than, or equal to,

$$\inf\left\{\int_{\Omega\setminus K} |\nabla v|^2 dx + \lambda\mathcal{H}_{n-1}(K)\right\}$$

where the infimum is taken over all the closed sets $K \subset \overline{\Omega}$ *and the functions* $v \in C^1(\Omega\setminus K)\cap C^0(\overline{\Omega}\setminus (M \cup K))$ *with* $v = w$ on $\partial\Omega \setminus (M \cup K)$.

Proof. The existence of the function w_e follows by Theorem 9 in [4]. We remark that, setting $v_0 = w_e \cdot \chi_{\mathbf{R}^n\setminus\overline{\Omega}}$, we have $v_0 \in SBV(\mathbf{R}^n)$ and

$$\mathcal{F}(v_0, \lambda, \overline{\Omega}) \leq \lambda\mathcal{H}_{n-1}(\partial\Omega) < +\infty.$$

Let $(v_h) \subset SBV(\mathbf{R}^n)$ be a minimizing sequence for $\mathcal{F}(\cdot, \lambda, \overline{\Omega})$, with $v_h = w_e$ in $\mathbf{R}^n \setminus \overline{\Omega}$ for every $h \in \mathbf{N}$; by Remark 2.2 we may suppose $\|v_h\|_{L^\infty} \leq \|w_e\|_{L^\infty}$. By (2.2) the sequence (v_h) is uniformly bounded in $BV(\mathbf{R}^n)$, hence, by the compactness theorem in $BV(\mathbf{R}^n)$ (see e.g. [19], Theorem 1.19), there exist a subsequence, still denoted by (v_h), and a function $u \in BV(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ such that $v_h \to u$ in $L^1(\mathbf{R}^n)$. By Theorem 2.4 $u \in SBV(\mathbf{R}^n)$ and

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_h \int_\Omega |\nabla v_h|^2 dx;$$

moreover, by a covering argument and by Remark 3.11, we may assume that $w_e \in W^{1,2}$ near any point of $\partial\Omega \setminus M$, hence we may use again Theorem 2.4 to obtain

$$\mathcal{H}_{n-1}(S_u) \leq \liminf_h \mathcal{H}_{n-1}(S_{v_h}).$$

Then we have

$$\int_\Omega |\nabla u|^2 dx + \lambda\mathcal{H}_{n-1}(S_u) \leq \lim_h \left[\int_\Omega |\nabla v_h|^2 dx + \lambda\mathcal{H}_{n-1}(S_{v_h})\right],$$

therefore u is a minimizer for $\mathcal{F}(\cdot, \lambda, \overline{\Omega})$ over all the functions $v \in SBV(\mathbf{R}^n)$ having prescribed value w_e in $\mathbf{R}^n \setminus \overline{\Omega}$.

Now let $K \subset \overline{\Omega}$ be a closed set, let $v \in C^1(\Omega\setminus K)\cap C^0(\overline{\Omega}\setminus(M\cup K))$ with $v = w$ on $\partial\Omega\setminus(M\cup K)$ and $\int_{\Omega\setminus K} |\nabla v|^2 dx + \lambda\mathcal{H}_{n-1}(K) < +\infty$. If $\phi : \mathbf{R} \to \mathbf{R}$ is a $C^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ function with $0 \leq \phi' \leq 1$ and

$\phi(t) = t$ for $|t| \leq \|w\|_{L^\infty}$, then we may apply Lemma 2.3 to the function $\phi(v)$ obtaining a function $v' \in SBV(\mathbf{R}^n)$ such that

$$\mathcal{F}(v', \lambda, \overline{\Omega}) \leq \int_{\Omega \setminus K} |\nabla v|^2 dx + \lambda \mathcal{H}_{n-1}(K),$$

so the assertion follows. **q.e.d.**

Proof of Theorem 1.1. By Lemma 4.1 there exist $w_e \in W^{1,1}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ such that $\tilde{w}_e = w$ $\mathcal{H}_{n-1}$-a.e. on $\partial\Omega$ and a function $u \in SBV(\mathbf{R}^n)$ which is a minimizer for $\mathcal{F}(\cdot, \lambda, \overline{\Omega})$ over all the functions $v \in SBV(\mathbf{R}^n)$ having prescribed value w_e in $\mathbf{R}^n \setminus \overline{\Omega}$. With the same notation as in Theorem 3.12, setting $K = \overline{\Omega} \setminus \Omega_0$, we have that K is closed and

$$\mathcal{H}_{n-1}(K \triangle S_u) = 0. \tag{4.1}$$

Let $\overline{B_r(x)} \subset \Omega \setminus K$; by (4.1) we have that $u \in W^{1,2}(B_r(x))$ and

$$\int_{B_r(x)} |\nabla u|^2 dy \leq \int_{B_r(x)} |\nabla v|^2 dy$$

for every $v \in u + W_0^{1,2}(B_r(x))$. Thus u is harmonic in $\Omega \setminus K$. Moreover for every $\xi \in \partial\Omega \setminus (M \cup K)$ there exists $r > 0$ such that $B_r(\xi) \cap (M \cup K) = \emptyset$, $\partial\Omega \cap B_r(\xi)$ is a C^1 surface and $w \in C^1(\partial\Omega \cap B_r(\xi))$. By well-known results on elliptic Dirichlet problems (see e.g. [21], Chap. II, Appendices) it follows that $u \in C^0(\overline{\Omega} \cap B_r(\xi))$ and $u = w$ on $\partial\Omega \cap B_r(\xi)$. Therefore $u \in C^\omega(\Omega \setminus K) \cap C^0(\overline{\Omega} \setminus (M \cup K))$ and we have

$$\mathcal{G}(K, u) = \int_{\Omega \setminus K} |\nabla u|^2 dx + \lambda \mathcal{H}_{n-1}(K) = \mathcal{F}(u, \lambda, \overline{\Omega}).$$

By Lemma 4.1 we conclude that the pair (K, u) gives a solution of the minimum problem considered in Theorem 1.1. **q.e.d.**

REMARK 4.2. We notice that the minimizing pair (K, u) whose existence has been proved in Theorem 1.1 satisfies also the conditions $\|u\|_{L^\infty} \leq \|w\|_{L^\infty}$ and $u \in C^\omega(\Omega \setminus K)$.

REMARK 4.3 By the proof of Theorem 1.1 we conclude that

(a) *if $u \in SBV(\mathbf{R}^n)$ is a minimizer for $\mathcal{F}(\cdot, \lambda, \overline{\Omega})$ over all the functions $v \in SBV(\mathbf{R}^n)$ with $v = w_e$ in $\mathbf{R}^n \setminus \overline{\Omega}$, then the pair $(\overline{\Omega} \setminus \Omega_0, u)$ gives the minimum that one is looking for in Theorem 1.1 and*

$$\mathcal{G}(\overline{\Omega} \setminus \Omega_0, u) = \mathcal{F}(u, \lambda, \overline{\Omega});$$

viceversa,

(b) *if (K, u) is a minimizing pair given by Theorem 1.1 with $u \in L^\infty$, setting*

$$u'(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \setminus K, \\ w_e(x) & \text{if } x \in \mathbf{R}^n \setminus \overline{\Omega}, \end{cases}$$

then, by Lemma 2.3, $u' \in SBV(\mathbf{R}^n)$ and, by (a), $\mathcal{H}_{n-1}(S_{u'} \triangle K) = 0$, $\Psi(u', \lambda, \overline{\Omega}) = 0$ and $\mathcal{F}(u', \lambda, \overline{\Omega}) = \mathcal{G}(K, u)$.

Proof of Proposition 1.2. Let (K, u) be a minimizing pair given by Theorem 1.1 and let $u' \in SBV(\mathbf{R}^n)$ defined as in (b) of the previous Remark 4.3. The assertion (i) immediately follows by (b) of Remark 4.3 since $S_{u'}$ is $(\mathcal{H}_{n-1}, n-1)$-rectifiable.
In order to prove (ii), it is enough to choose

$$K' = \overline{\Omega} \setminus \{x \in \overline{\Omega} \setminus M; \quad \lim_{\rho \to 0} \rho^{1-n} \mathcal{H}_{n-1}(S_{u'} \cap \overline{B_\rho(x)}) = 0\}.$$

Indeed, by Theorem 3.12 and Remark 3.13, we have another minimizing pair (K', u') which has the required properties. Obviously the pair (K', u') is uniquely determined by the properties (ii).
q.e.d.

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