Prescribing scalar curvatures: on the negative Yamabe case

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Abstract

The problem of prescribing conformally the scalar curvature on a closed Riemannian manifold of negative Yamabe invariant is always solvable, when the function K to be prescribed is strictly negative, while sufficient and necessary conditions are known for $K \leq 0$. For sign changing K Rauzy [21] showed solvability, provided K is not too positive. We revisit this problem in a different variational context, thereby recovering and quantifying the principal existence result of Rauzy and show under additional assumptions, that for a sign changing K solutions to the conformally prescribed scalar curvature problem, while existing, are not unique.

Key Words: conformal geometry, scalar curvature, calculus of variations, nonlinear analysis
MSC : 35A15, 35J60, 53C21

Contents

1 Introduction

Let $M = (M^n, g_0)$ be a closed Riemannian manifold with $n \geq 3$. We consider the classical conformally prescribed scalar curvature problem, i.e., given a smooth function K on M, we ask for the existence of a conformal metric g to g_0, whose scalar curvature is K.

If we denote by $g = g_u = u^{\frac{4}{n-2}} g_0$ with $u > 0$ a conformal metric to g_0, this problem is equivalent to finding a positive solution $u > 0$ of the equation

$$L_{g_0} u = -c_n \Delta_{g_0} u + R_{g_0} u = K u^{\frac{n+2}{n-2}}, \; c_n = \frac{4(n-1)}{n-2}. \tag{1}$$

Here R_{g_0} denotes the scalar curvature with respect to the metric g_0. In particular, when K is constant, (1) reduces to the Yamabe problem, which has been, as is well known, completely solved by the works of Yamabe, Trudinger, Aubin and Schoen.

The prescribed scalar curvature problem for non constant K has been widely studied as well, especially in case of a positive Yamabe invariant, in particular on the standard sphere S^n, see for instance [15] and the references therein. Here we are interested in the case of a negative Yamabe invariant, i.e. when

$$Y(M) = \inf_{\substack{u \in H^1(M) \\ u > 0}} \frac{\int_M L_{g_0} u u d\mu_{g_0}}{(\int_M u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} < 0,$$

and we refer to [2] for a comprehensive introduction. By the resolution of the Yamabe problem [12] we may assume $R_{g_0} \equiv -1$, whence

$$L_{g_0} = -c_n \Delta_{g_0} - 1$$

and the constant functions become the first and a negative eigenspace of L_{g_0}.

If $K < 0$, then (1) is always solvable, and still, if $K \leq 0$, solutions are unique [12]. Moreover a necessary condition for solvability of (1) is

$$\nu_1(\Omega_K) = \nu_1(L_{g_0}, \Omega_K) > 0, \tag{2}$$

as was first proved by Rauzy [21], where $\Omega_K = \{K \geq 0\}$,

$$\nu_1(L_{g_0}, \Omega_K) = \sup_{\Omega_K \subset \Omega \text{ smooth}} \nu_1(L_{g_0}, \Omega)$$

is the first Dirichlet eigenvalue and here for a smooth subset $\Omega \subset M$

$$\nu_1(L_{g_0}, \Omega) = \inf_{\mathcal{A}} \frac{\int_\Omega L_{g_0} u u d\mu_{g_0}}{\int_\Omega u^2 d\mu_{g_0}}, \; \mathcal{A} = \{u \in C_0^\infty(\Omega) \; : \; u > 0 \;\text{ in }\; \Omega \;\}.$$

And actually, if $K \leq 0$, the necessary condition $\nu_1(\Omega_K) > 0$ is also sufficient to guarantee solvability, see [19, 21, 24, 25]. Furthermore an additional necessary condition, which is automatically satisfied for $0 \not\equiv K \leq 0$, is the positivity of the unique solution $\bar{w} > 0$ of

$$\mathcal{L}_{g_0}\bar{w} = -(n-1)\Delta_{g_0}\bar{w} + \bar{w} = -K, \tag{3}$$

see [12], and in particular necessarily $\int K d\mu_{g_0} < 0$. Indeed $\bar{u} = u^{-\frac{4}{n-2}}$ for a solution $u > 0$ of (1) is a subsolution to (3) and thus the maximum principle tells us, that necessarily $\bar{u} \leq \bar{w}$. Finally, when K changes sign, Rauzy [21] used a subcritical approximation argument to obtain solvability under a smallness assumption on $\sup K$, which later on Aubin&Bismuth [3] quantified, see Remark 1.1 below and for $n = 2$ the analogous work [7] by Bismuth.

In view of these results we will study for sign changing K the conformally prescribed scalar curvature problem in a variational setting *different* from the one used by Rauzy [21] and assume throughout the necessary conditions

(i) $\nu_1(\Omega_K) > 0$

(ii) $\int_M K d\mu_{g_0} < 0$.

Let us first introduce some notations. Let

$$X = \{u > 0\} \cap \{r < 0\} \cap \{k < 0\} \cap \{\|u\|_{L^{\frac{2n}{n-2}}} = 1\} \subset C^\infty(M),$$

where

$$r = r_{g_u} = \int_M R_g d\mu_g = \int_M L_{g_0} u u d\mu_{g_0} \text{ and } k = k_{g_u} = \int_M K u^{\frac{2n}{n-2}} d\mu_{g_0}. \tag{4}$$

Clearly $X \neq \emptyset$, as some constant function is in X, and we consider

$$J = \frac{-k}{(-r)^{\frac{n}{n-2}}} > 0$$

as a scaling invariant functional on X with derivative

$$\partial J(u) = \frac{2^*}{(-r)^{\frac{n}{n-2}}} \left(\frac{-k}{-r} L_{g_0} u - K u^{\frac{n+2}{n-2}} \right),\ 2^* = \frac{2n}{n-2}$$

and a Yamabe type flow

$$\partial_t u = -(\frac{-k}{-r} R - K) u = -u^{-\frac{4}{n-2}} (\frac{-k}{-r} L_{g_0} u - K u^{\frac{n+2}{n-2}}). \tag{5}$$

In this way J becomes a variational functional and X a variational space, as any solution to $\partial J = 0$ on X is a solution to (1). Note, that the choice of X is somewhat natural, since for $K \leq 0$ any normalized solution to the conformally prescribed scalar curvature problem must be in X. On the other hand J is not necessarily bounded from below and, flowing along (5), while the factors $-k, -r > 0$ are readily uniformly bounded away from infinity on X, generally

$$0 < -k \longrightarrow 0 \text{ or } 0 < -r \longrightarrow 0 \tag{6}$$

may happen and it is natural to ask, how to prevent this scenario.

We will show first, that under conditions similar to, but generally weaker than those of Aubin&Bismuth [3] a certain integral condition holds, to which we refer as a global A-B-inequality, namely

$$\|u\|_{H^1}^2 \leq Ar + B|k|^{\frac{n-2}{n}} \text{ for all } u \in H^1(M),$$

see Proposition 3.1, and second, that an A-B-inequality holding on X guarantees $\inf_X J > 0$ and (6) does not occur, whence J becomes an energy and the flow generated by (5) can be used for variational arguments. In particular, by choosing appropriate initial data, we find a solution of (1), which is a global minimizer of the functional J on X. Note, that a minimizing property or saddle point structure of the solution obtained via approximation and variational means by Rauzy or via perturbation arguments by Aubin&Bismuth is unknown, while here we argue on the critical equation directly and do not rely on the method of sub- and supersolutions for instance and in contrast to e.g. [3], [12].

Theorem 1. *If an A-B-inequality holds on some sublevel*

$$J^L = \{J \leq L\} \neq \emptyset,$$

then J admits a global minimizer on X, which is a solution of equation (1).

For the proof see Section 3.2. Combined with Proposition 3.1, Theorem 1 quantifies the smallness assumption in Rauzy [21].

Remark 1.1. *Let us review the Aubin&Bismuth result in the corresponding notations of Remark 6.13 in [2] and Theorem 6 in [3], namely solvability of*

$$L_{g_0} u = f u^{\frac{n+2}{n-2}},\ u > 0 \ \textit{ for } \ f \in \mathcal{F}_{\alpha,K} = \{f \in C^\alpha(M) \ : \ \{f \geq 0\} = K\},$$

where $0 < \alpha < 1$ and $K \subset M$ are fixed.

a) In Remark 6.13 in [2] solvability is claimed, provided, that for some smooth

$$\Omega \supset K \ \textit{ with } \ \lambda(\Omega) > -\tilde{R} \tag{7}$$

there holds

$$\sup f \leq C(K) \inf_{M \setminus \Omega} (-f),$$

where $C(K)$ supposedly depends on $K = \{f \geq 0\}$ only. But this is not substantiated by the reference to [3], see Theorem 6 there, whose statement requires specific choices of neighbourhoods $\Omega \supset \theta \supset K$, see b) below, and thus Ω as in (7) *is not arbitrary.*

b) In Theorem 6 in [3] solvability is claimed, provided

$$\sup K \leq C(K) \inf_{M \setminus \theta} (-f), \tag{8}$$

where according to the Definition before Theorem 6

$$\Omega \supset\supset \theta \supset\supset K \tag{9}$$

are smooth neighbourhoods of K, which in particular satisfy

$$\frac{1}{2}\lambda(K) < \lambda(\theta) < \lambda(K) \ \ \text{and} \ \ \frac{1}{2}\lambda(\theta) < \lambda(\Omega) < \lambda(\theta)$$

for the first Dirichlet eigenvalues λ of these sets, whence specific choices for θ and Ω are required in contrast to Proposition 3.1 below.

c) *Moreover $C(K)$ in (8) is supposed to depend on $K = \{f \geq 0\}$ only, which is surprising, since the smallness constant in our Proposition 3.1 depends on a distance corresponding to $d(\partial\theta, \partial\Omega)$, cf. (9). And in fact the upper bound of ψ in Proposition 1 of [3], claimed to be*

$$\sup \psi \leq C(K),$$

tends to infinity, when $d(\partial\Omega, \partial\theta) \longrightarrow 0$. To be precise, if $d(\partial\Omega, \partial\theta) \longrightarrow 0$, then $\inf \varphi \longrightarrow 0$ from (15) and the definition of φ two lines below, while

$$u^+ = \xi\varphi \ \ \text{for a constant} \ \ \xi > 0 \ \ \text{and} \ \ u^- = |R|^{\frac{n-2}{4}}$$

shall act as a super- and subsolution respectively. To apply the method of sub- and supersolutions we then require

$$u^- \leq u^+ \ \ \text{pointwise and thus} \ \ \xi \longrightarrow \infty,$$

as $\sup \varphi \not\longrightarrow 0$. So a uniform upper bound for ψ is not feasible.

Either way, while Aubin&Bismuth in [3] qualify via (8) the smallness condition of Rauzy in [21], Proposition 3.1 quantifies the smallness constant $C(K) > 0$ in (8). Concerning non existence and recalling (2) and (3) we have

Lemma 1.2. *There holds*

$$\nu_1(\Omega_K) > 0 \ \begin{array}{c}\not\Longleftarrow \\ \not\Longrightarrow\end{array} \ \bar{w} > 0 \ \Longrightarrow \ \int_M K d\mu_{g_0} < 0.$$

Thus (i) or (ii) above are alone not sufficient to guarantee solvability. To see Lemma 1.2, whose demonstration we believe to be instructive, first note, that the last implication is immediate from testing (3) against a constant. Secondly, to see $\bar{w} > 0 \ \not\Longrightarrow \ \nu_1(\Omega_K) > 0$, note, that $\bar{w} > 0$ for $0 \not\equiv K \leq 0$ by the maximum principle, while we may choose $\Omega_K = M \setminus B_\epsilon(x_0)$. Then for suitable $0 \leq \eta_{\epsilon,x_0} \leq 1$ with $\eta_{\epsilon,x_0} \equiv 0$ on $B_\epsilon(x_0)$ we have

$$\int_M L_{g_0}\eta_{\epsilon,x_0}\eta_{\epsilon,x_0} < 0,$$

whence $\nu_1(\Omega_K) < 0$. Finally we may even construct $K \in C^\infty(M)$, for which (i) and (ii) hold, but $\bar{w} \not> 0$. Indeed $\mathcal{L}_{g_0} > 0$ has a Green's function $G_{\mathcal{L}_{g_0}}$ satisfying

$$\inf_{(M\times M)\setminus\{diag(M)\}} G_{\mathcal{L}_{g_0}} > 0 \ \text{ with a principal term } G_{\mathcal{L}_{g_0}}(x,y) \simeq d_{g_0}^{2-n}(x,y)$$

and we consider for $\min\{2, n/2\} < p < n$, $0 < \epsilon \ll 1$ and $\epsilon\lambda^{n-p} \gg 1$

$$K = -\epsilon + \eta_{\epsilon,x_0}\Big(\frac{\lambda}{1+\lambda^2 d_{g_0}^2(x_0,x)}\Big)^p$$

with a suitable cut-off function η_{ϵ,x_0}, $\eta_{\epsilon,x_0} \equiv 1$ on $B_\epsilon(x_0)$. We then easily check

$$\nu_1(\Omega_K) > 0 \ \text{ and } \ \int_M K d\mu_{g_0} < 0,$$

while

$$\bar{w}(x_0) = -G_{\mathcal{L}_{g_0}}(x_0,\cdot) * K \simeq -\lambda^{p-2} < 0.$$

In particular the last argument shows, that some kind of smallness assumption to guarantee solvability is natural.

On the other hand, once solvability of (1) is given, we naturally ask for uniqueness, which, as we recall, holds true in case $K \leq 0$. Surprisingly, as our second theorem shows, we may *either* lose existence *or* uniqueness, when passing from $K < 0$ strictly negative to K sign changing.

In the latter case, when we lose uniqueness in that passage, but not existence, we find at least two solutions, one inducing a totally *negative* scalar curvature $r = k < 0$, see (4), while more surprisingly the other solution induces a totally *positive* one, see Theorem 2 below, although, fixing some suitable $0 \not\equiv K_- \leq 0$, we may choose $0 \not\equiv K_+ \geq 0$ such, that for $K = K_+ + K_-$ the positive maximum of K is as small as we wish.

For the sake of simple statements we say, that $Cond_n$ holds at $a \in M$, if

$$(Cond_n) \quad \begin{cases} 3 \leq n \leq 5 \ \text{ and no restrictions on } a \in M \text{ are required} \\ \quad\ n \geq 6 \ \text{ and } M \text{ is locally conformally flat around } a \in M. \end{cases}$$

In particular $Cond_n$ is satisfied for all $a \in M$, if $3 \leq n \leq 5$.

Theorem 2. *Suppose, that L_{g_0} is invertible, and consider a sign changing function $K \in C^\infty(M)$, for which a global A-B-inequality holds. Then, if $Cond_n$ is satisfied at some $a \in \{K = \max K\}$ and*

$$\nabla^l K(a) = 0 \ \text{ for all } \ \frac{n-2}{2} \geq l \in \mathbb{N},$$

there exists $C = C(A,B)$ such, that the conformally prescribed scalar curvature problem admits at least two solutions u_0, u_1, provided $\sup K \leq C$, in which case

$$r_{u_0}, k_{u_0} < 0 \ \text{ and } \ r_{u_1}, k_{u_1} > 0.$$

For the proof see Section 4.3. Some remarks on Theorem 2 are in order.

Remark 1.3. *(i) The function u_0 is the minimizer from Theorem 1, while the second solution u_1 is a minimizer of $I = \frac{r}{k^{\frac{n-2}{n}}}$ on the natural domain*

$$Y = \{r > 0\} \cap \{k > 0\} \cap \{u > 0\} \cap \{\| \cdot \|_{L^{\frac{2n}{n-2}}} = 1\} \subset C^\infty(M).$$

Clearly $Y = \emptyset$ for $K \leq 0$ and, if an A-B-inequality holds, then $\inf_Y I > 0$.

(ii) Let $J_\epsilon = J_{K_\epsilon}$, where $K_\epsilon = K_0 + \epsilon K_1$, $K_0 \leq 0$ and $\sup K_1 > 0$. Suppose, that an A-B-inequality holds for K_{ϵ_0}, whence for all $0 \leq \epsilon \leq \epsilon_0$ in particular the same A-B-inequality holds for K_ϵ, and thus we can minimize J_ϵ from Theorem 1. Then, at least if Theorem 2 is applicable, we have

1) a unique solution u_0 for $\epsilon = 0$, namely the minimizer of J_0

2) at least two solutions

$$u_{0,\epsilon} \neq u_{1,\epsilon},$$

namely the minimizers of J_ϵ and I_ϵ respectively.

Thus $u_{1,\epsilon}$ must cease to exist in the passage $0 < \epsilon \longrightarrow 0$. Indeed

$$0 < k_{u_{1,\epsilon}} = \int K_\epsilon u_{1,\epsilon}^{\frac{2n}{n-2}} d\mu_{g_0} \leq \epsilon \int K_1 u_{1,\epsilon}^{\frac{2n}{n-2}} d\mu_{g_0} \leq \epsilon \max K_1 \xrightarrow{\epsilon \to 0} 0,$$

while from the validity of an A-B-inequality $r_{u_{1,\epsilon}} \neq o_\epsilon(1)$. Hence in fact

$$\inf I_\epsilon = \frac{r_{u_{1,\epsilon}}}{k_{u_{1,\epsilon}}^{\frac{n-2}{n}}} \xrightarrow{\epsilon \to 0} \infty.$$

(iii) The dimensional dependence of Theorem 2 is reminiscent of the distinction in Theorem 2.1 and Theorem 2.3 in [8]. The differences are explained in terms of, for which dimensions vanishing of $\nabla^l K$is assumed, and, if vanishing is assumed, to which degree l, played against the different scales of deconcentration, i.e. the principal quantities, that prevent blow-up. In case of Escobar&Schoen this is the positive mass H, when considering a single bubbling blow-up of type $u = \varphi_{a,\lambda}$, for us it is the solution u_0 for a mixed type blow up $u = u_0 + \varphi_{a,\lambda}$. The situation is as follows

	Escobar&Schoen	*Mayer&Zhu*
no vanishing assumption	$n = 3$	$3 \leq n \leq 5$
vanishing assumption	$n \geq 4$	$n \geq 6$
vanishing degree	$l \leq n - 2$	$l \leq {}^{(n-2)}/_2$
deconcentration term	$^{H(a)}/_{\lambda^{n-2}}$	$^{u_0(a)}/_{\lambda^{\frac{n-2}{2}}}$

where $a \in \{K = \max K\}$ and in particular $\nabla K(a) = 0$. On the other hand a derivative $\nabla^l K(a)$ contributes in the corresponding energy expansion at most a quantity of order $O(^1/_{\lambda^l} + {}^1/_{\lambda^n})$. Then, as is easy to see from the table above, the deconcentration terms are in fact dominant. We refer to [1, 5, 16] for related studies on the mixed type blow-up case.

(iv) *For Theorem 2 to be meaningful, invertibility of L_{g_0} and local flatness at least* somewhere *on M must be compatible. Indeed, if L_{g_0} is not invertible, let us consider the finite dimensional space $ker(L_{g_0})$, for each eigenfunction $e_i \in ker L_{g_0}$ its nodal set $N_i = \{e_i = 0\}$ and an open set $O_i \subset M$ with $N_i \cap O_i \neq \emptyset$ such, that $O_i \cap O_j = \emptyset$ for $i \neq j$. Following and slightly modifying the arguments in [10], we then can perturb g_0 to gain invertibility, while the perturbation leaves g_0 on $M \setminus \cup_i O_i$ unchanged. The required localization of the perturbation in [10] is based on adding for each $i = 1, \ldots, dim(ker(L_{g_0}))$ a suitable cut-off function $\eta = \eta_i$, living on O_i, to the definition of $h = h_i$ in line 5, page 799 in [10], where $\psi = e_i$, i.e.*

$$h = \eta \left(c_n \psi (2\psi \mathring{\nabla}^2 \psi - \psi^2 \mathring{Ric}) + (2c_n - 1)(d\psi \otimes d\psi)^o \right).$$

We leave it to the reader to verify, that the arguments in [10] proceed with only minor modifications. In this way, if $M = (M^n, g_0)$ is of negative Yamabe invariant and locally conformally flat on some $A \subset M \setminus \cup_i O_i$, then we may slightly change the metric g_0 to gain invertibility of L_{g_0}, while local conformal flatness on A is unchanged and, as the modification is only slight, the Yamabe invariant remains negative.

Concerning Theorem 2 we also mention Rauzy [22] for complementary and under much stronger assumptions [20] for similar results. Note, that extensions of Theorem 2 are possible under suitable flatness assumptions played against a non vanishing Weyl tensor at a maximum point of K, thereby recovering the results of [22]. On the other hand for generic functions K in higher dimensions the relevant arguments for direct minimization of J or I cannot be applied, as we will discuss in [18].

Finally we wish to thank Prof. Daniele Bartolucci for bringing this problem to our attention during our time at the University of Rome "Tor Vergata".

2 Preliminaries

We start with providing the fundamental properties of the flow generated by (5), whose short time existence is standard, cf. [9], provided $k_{u_0}, r_{u_0} < 0$ for an initial data, which we assume.

Proposition 2.1. *For a positive flow line*

$$u : [0,T) \times C^\infty(M, \mathbb{R}_+) \longrightarrow C^\infty(M, \mathbb{R}_+) : (t, u_0) \longrightarrow u \;\; \text{with} \;\; k_{u_0}, r_{u_0} < 0$$

generated by (5) *and satisfying* $k, r < 0$ *on* $[0,T)$ *there holds for all* $0 \leq t < T$

(i) *conservation of the volume, i.e.* $\partial_t \mu_{g_u} = \partial_t \int_M u^{\frac{2n}{n-2}} d\mu_{g_0} = 0.$

(ii) *non growth of J, precisely*

$$\partial_t J(u) = -\frac{2^*}{(-r)^{\frac{n}{n-2}}} |\delta J|^2(u) \leq -\frac{(-r)^{\frac{n}{n-2}}}{2^* S^2 \|u\|_{L^{2^*}}^{\frac{4}{n-2}}} |\partial J|^2(u)$$

where $|\delta J|^2(u) = \int_M |\frac{-k}{-r}R - K|^2 u^{\frac{2n}{n-2}} d\mu_{g_0}$ *and* S *is the Sobolev constant.*

(iii) preservation of positivity, precisely

$$\min\{1/C, u_{min}(0)\} \leq u(t) \leq u_{max}(0)e^{Ct},$$

where $C = C(\sup_{[0,t]}(\frac{-k}{-r} + \frac{-r}{-k}))$.

Proof. Property (i) is easy to check by direct computation, as is

$$\partial_t J(u) \quad = \quad -\frac{2^*}{(-r)^{\frac{n}{n-2}}} \int_M |\frac{-k}{-r}R - K|^2 u^{\frac{2n}{n-2}} d\mu_{g_0}.$$

Moreover

$$\begin{aligned}|\partial J|(u) &= \sup_{\|\varphi\|_{H^1(M)} \leq 1} \int_M \partial J(u) \cdot \varphi d\mu_{g_0} \\ &= \frac{2^*}{(-r)^{\frac{n}{n-2}}} \sup_{\|\varphi\|_{H^1(M)} \leq 1} \int_M (\frac{-k}{-r}R - K)u\varphi u^{\frac{4}{n-2}} d\mu_{g_0}.\end{aligned}$$

Denote $dw = u^{\frac{4}{n-2}} d\mu_{g_0}$ with corresponding L^2_w-norm

$$\|\varphi\|^2_{L^2_w} = \int_M \varphi^2 dw = \int_M \varphi^2 u^{\frac{4}{n-2}} d\mu_{g_0}.$$

Then for any $\varphi \in H^1(M)$ by Hölder's inequality we have

$$\|\varphi\|^2_{L^2_w} \leq (\int_M \varphi^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}} (\int_M u^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{2}{n}} \leq S^2 \|u\|^{\frac{4}{n-2}}_{L^{2^*}} \|\varphi\|^2_{H^1(M)},$$

whence by L^2-duality

$$\begin{aligned}|\partial J|(u) &\leq \frac{2^*}{(-r)^{\frac{n}{n-2}}} \sup_{\|\varphi\|_{L^2_w(M)} \leq S\|u\|^{\frac{2}{n-2}}_{L^{2^*}}} \int_M (\frac{-k}{-r}R - K)u\varphi u^{\frac{4}{n-2}} d\mu_{g_0} \\ &\leq \frac{2^* S\|u\|^{\frac{2}{n-2}}_{L^{2^*}}}{(-r)^{\frac{n}{n-2}}} \int_M (\frac{-k}{-r}R - K)u \cdot \frac{(\frac{-k}{-r}R - K)u}{\|(\frac{-k}{-r}R - K)u\|_{L^2_w}} dw \\ &= \frac{2^* S\|u\|^{\frac{2}{n-2}}_{L^{2^*}}}{(-r)^{\frac{n}{n-2}}} \|(\frac{-k}{-r}R - K)u\|_{L^2_w},\end{aligned}$$

where S is the Sobolev constant and (ii) is immediate. Recalling finally (5), the lower bound in (iii) follows from the maximum principle, while the upper one is due to Gronwall's lemma. □

Note, that we will ensure from an A-B-inequality, that a priori $|k|$ and $|r|$ are uniformly bounded away from zero and infinity, at least on energy sublevels, and then long time existence follows from the next two lemmata, since we already know, that $\inf u \longrightarrow 0$ is impossible, cf. [4, 16].

Lemma 2.2. *For any* $1 \le p \le \frac{n^2}{2(n-2)}$ *we have*

$$\int_M |\frac{-k}{-r}R - K|^p d\mu_g \le e^{\omega e^{\omega T}},$$

where $\omega \ge 1$ *is bounded against* $\sup_{0\le t\le T} \frac{1+|r|}{|k|}$.

Proof. Using (5) by direct calculation, we have

$$\begin{aligned}
&\partial_t \int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
= & -\frac{4(p-1)}{p} c_n(\frac{-k}{-r}) \int_M |\nabla_g |\frac{-k}{-r}R - K|^{\frac{p}{2}}|^2 d\mu_g \\
& + \frac{4p-2n}{n-2} \int_M |\frac{-k}{-r}R - K|^p (\frac{-k}{-r}R - K) d\mu_g + \frac{4p}{n-2}\int_M |\frac{-k}{-r}R - K|^p K d\mu_g \\
& - \frac{2p}{-k}\int_M |\frac{-k}{-r}R - K|^2 d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
& - \frac{2p}{-k}\int_M |\frac{-k}{-r}R - K|^2 d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^{p-2} (\frac{-k}{-r}R - K) K d\mu_g \\
& + \frac{4p}{n-2}\cdot\frac{1}{-k}\int_M (\frac{-k}{-r}R - K) K d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
& + \frac{4p}{n-2}\cdot\frac{1}{-k}\int_M (\frac{-k}{-r}R - K) K d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^{p-2} (\frac{-k}{-r}R - K) K d\mu_g.
\end{aligned}$$

Applying the Sobolev and Hölder inequalities, we then estimated

$$\begin{aligned}
&\partial_t \int_M |\frac{-k}{-r}R - K|^p d\mu_g + \omega(\int_M |\frac{-k}{-r}R - K|^{\frac{pn}{n-2}} d\mu_g)^{\frac{n-2}{n}} \\
\le & \frac{4p-2n}{n-2}\int_M |\frac{-k}{-r}R - K|^{p+1} d\mu_g + C\int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
& - \frac{2p}{-k}\int_M |\frac{-k}{-r}R - K|^2 d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
& - \frac{2p}{-k}\int_M |\frac{-k}{-r}R - K|^2 d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^{p-2} (\frac{-k}{-r}R - K) K d\mu_g \\
& + \frac{4p}{n-2}\cdot\frac{1}{-k}\int_M (\frac{-k}{-r}R - K) K d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^p d\mu_g \\
& + \frac{4p}{n-2}\cdot\frac{1}{-k}\int_M (\frac{-k}{-r}R - K) K d\mu_g \cdot \int_M |\frac{-k}{-r}R - K|^{p-2} (\frac{-k}{-r}R - K) K d\mu_g,
\end{aligned}$$

whence

$$\begin{aligned}
\partial_t \int_M |\frac{-k}{-r}R - K|^p d\mu_g &+ \omega(\int_M |\frac{-k}{-r}R - K|^{\frac{pn}{n-2}} d\mu_g)^{\frac{n-2}{n}} \\
&\le \frac{4p-2n}{n-2}\int_M |\frac{-k}{-r}R - K|^{p+1} d\mu_g + \omega\int_M |\frac{-k}{-r}R - K|^p d\mu_g \qquad (10)\\
&\quad + \omega\int_M |\frac{-k}{-r}R - K|^2 d\mu_g + \omega.
\end{aligned}$$

Let $p = \frac{n}{2}$. We then estimate in case $n \geq 4$

$$\partial_t \int_M |\frac{-k}{-r}R - K|^{\frac{n}{2}} d\mu_g \leq \omega \int_M |\frac{-k}{-r}R - K|^{\frac{n}{2}} d\mu_g + \omega,$$

whence we obtain the logarithmic type estimate

$$\int_M |\frac{-k}{-r}R - K|^{\frac{n}{2}} d\mu_g \leq (\int_M |\frac{-k}{-r}R - K|^{\frac{n}{2}} d\mu_g|_{t=0} + \omega)e^{\omega t},$$

while for $n = 3$ we have

$$\partial_t \int_M |\frac{-k}{-r}R - K|^{\frac{3}{2}} d\mu_g \leq \omega \int_M |\frac{-k}{-r}R - K|^2 d\mu_g + \omega,$$

whence

$$\int_M |\frac{-k}{-r}R - K|^{\frac{3}{2}} d\mu_g - \int_M |\frac{-k}{-r}R - K|^{\frac{3}{2}} d\mu_g|_{t=0}$$
$$\leq \omega \int_0^t \int_M |\frac{-k}{-r}R - K|^2 d\mu_g dt + \omega t \leq \omega J(u_0) + \omega t$$

and therefore

$$\int_M |\frac{-k}{-r}R - K|^{\frac{3}{2}} d\mu_g \leq \omega t + \omega.$$

In conclusion for any $n \geq 3$ we get

$$\int_M |\frac{-k}{-r}R - K|^{\frac{n}{2}} d\mu_g \leq \omega e^{\omega t}.$$

Letting $p = \frac{n}{2}$ in (10) and integrating from 0 to t, we therefore have

$$\int_0^t (\int_M |\frac{-k}{-r}R - K|^{\frac{n^2}{2(n-2)}} d\mu_g)^{\frac{n-2}{n}} dt \leq \omega e^{\omega t} + \omega.$$

Returning to (10) and applying the Hölder's and Young's inequality to the term

$$\int_M |\frac{-k}{-r}R - K|^{p+1} d\mu_g,$$

we obtain

$$\partial_t \int_M |\frac{-k}{-r}R \ - K|^p d\mu_g \leq \omega(\int_M |\frac{-k}{-r}R - K|^p d\mu_g)^{\frac{2p-n+2}{2p-n}}$$
$$+ \omega \int_M |\frac{-k}{-r}R - K|^p d\mu_g + \omega \int_M |\frac{-k}{-r}R - K|^2 d\mu_g + \omega.$$

Taking $p = \frac{n^2}{2(n-2)}$ and setting $y = \int_M |\frac{-k}{-r}R - K|^{\frac{n^2}{2(n-2)}} d\mu_g$ we then have

$$\partial_t y \leq \omega y^{1+\frac{n-2}{n}} + \omega y + \omega$$

and therefore

$$\log \frac{y+\omega}{(y+\omega)|_{t=0}} \leq \omega \int_0^t y^{\frac{n-2}{n}} dt + \omega t,$$

which implies

$$\int_M |\frac{-k}{-r} R - K|^{\frac{n^2}{2(n-2)}} d\mu_g \leq \omega e^{\omega e^{\omega t}}.$$

The assertion follows. □

Lemma 2.3. *A flow line, for which* $\frac{1+|r|}{|k|}$ *is upper bounded, exists for all time.*

Proof. By Lemma 2.2 we have

$$\int_M |R|^p d\mu_g \leq e^{\omega e^{\omega T}}$$

for any $1 \leq p \leq \frac{n^2}{2(n-2)}$. Thus, denoting by $\tilde{\omega} \geq 1$ any quantity bounded against

$$\sup_{0 \leq t \leq T} (\|u\|_{L^\infty} + \|1/u\|_{L^\infty} + \frac{1+|r|}{|k|}),$$

we deduce

$$\int_M |\Delta_{g_0} u|^p d\mu_g \leq e^{\tilde{\omega} e^{\tilde{\omega} T}},$$

since u is time-dependently bounded. Then Morrey's inequality shows

$$|u(t,x_1) - u(t,x_2))| \leq C(\tilde{\omega}, T) d(x_1,x_2)^\alpha$$

for $x_1, x_2 \in M$, $t \in [0,T)$ and

$$\alpha = 2 - \frac{n}{p}, \ \frac{n}{2} < p < \min\{\frac{n^2}{2(n-2)}, n\}.$$

Moreover from Lemma 2.2

$$\int_M |\partial_t u|^p d\mu_{g_0} \leq C(\tilde{\omega}, T).$$

We then obtain for $0 < t_1 - t_2 < 1$

$$\begin{aligned}
|u(t_1,x) - u(t_2,x)| &= \frac{1}{|B_{\sqrt{t_1-t_2}}(x)|} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1,x) - u(t_2,x)| d\mu_{g_0}(y) \\
&= \frac{1}{|B_{\sqrt{t_1-t_2}}(x)|} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1,x) - u(t_1,y)| d\mu_{g_0}(y) \\
&\quad + \frac{1}{|B_{\sqrt{t_1-t_2}}(x)|} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_1,y) - u(t_2,y)| d\mu_{g_0}(y) \\
&\quad + \frac{1}{|B_{\sqrt{t_1-t_2}}(x)|} \int_{B_{\sqrt{t_1-t_2}}(x)} |u(t_2,y) - u(t_2,x)| d\mu_{g_0}(y) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The first term on the right hand of the above equality yields to

$$I_1 \leq C(\tilde{\omega},T)|t_1-t_2|^{-\frac{2}{n}} \int_{B_{\sqrt{t_1-t_2}}(x)} |x-y|^{\alpha} d\mu_{g_0}(y) \leq C(\tilde{\omega},T)|t_1-t_2|^{\frac{\alpha}{2}}$$

and similarly $I_3 \leq C(\tilde{\omega},T)|t_1-t_2|^{\frac{\alpha}{2}}$. We finally estimate

$$\begin{aligned} I_2 &\leq C|t_1-t_2|^{-\frac{n}{2}} \sup_{t_2\leq t\leq t_1} \int_{B_{\sqrt{t_1-t_2}}(x)} |\frac{\partial u}{\partial t}||t_1-t_2| d\mu_{g_0}(y) \\ &\leq C|t_1-t_2|^{-\frac{n}{2}+1} \sup_{t_2\leq t\leq t_1} (\int_{B_{\sqrt{t_1-t_2}}(x)} |\frac{\partial u}{\partial t}|^p d\mu_{g_0})^{\frac{1}{p}} (\int_{B_{\sqrt{t_1-t_2}}(x)} \mathbb{1})^{1-\frac{1}{p}} \\ &\leq C(\tilde{\omega},T)|t_1-t_2|^{1-\frac{n}{2p}} = C(\tilde{\omega},T)|t_1-t_2|^{\frac{\alpha}{2}}, \end{aligned}$$

and long time existence follows immediately. □

3 Existence

We essentially show Theorem 1, whose proof is found at this section's end.

3.1 The A-B-inequality

Recalling (2), we start with proving, that the A-B-conditions (i) and (ii) below imply some A-B-inequality (11).

Proposition 3.1. *There exists $\epsilon > 0$ such, that for any $K \in C^{\infty}(M)$, if*

$$\{K \geq 0\} = \Omega_K \subset\subset \Omega \subset\subset D$$

with smooth $\Omega, D \subset M$ and

(i) $\sup_M K < \epsilon[\, dist^{2\frac{n-1}{n-2}}(\partial\Omega,\partial D)(\frac{\nu_1(D)}{\nu_1(D)+1})^{\frac{n}{n-2}}\,] \inf_{M\setminus\Omega}(-K)$

(ii) $\nu_1(D) = \nu_1(L_{g_0},D) > 0,$

then for some constants $A, B > 0$ there holds

$$\|u\|^2_{H^1} \leq Ar + B|k|^{\frac{n-2}{n}} \quad \textit{for all} \quad u \in H^1(M). \tag{11}$$

We say, that an A-B-inequality holds globally, if (11) is satisfied, and likewise holding on X means, that (11) holds on X instead of $H^1(M)$.

Proof. Recalling (4) and by rescaling we have to show

$$\int_M L_{g_0} uu d\mu_{g_0} + B|\int_M K u^{\frac{2n}{n-2}} d\mu_{g_0}|^{\frac{n-2}{n}} \geq \epsilon_0 \;\text{ on }\; \{\|\cdot\|_{H^1} = 1\}.$$

Let $D \subset M$ satisfying

$$\Omega_K \subset\subset \Omega \subset\subset D \;\text{ and }\; \nu_1(D) = \nu_1(L_{g_0},D) > 0$$

and choose a cut off function $\eta \in C_0^\infty(D)$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ in Ω and $|\nabla\eta| \le C/d$, where $d = \mathrm{dist}(\partial\Omega, \partial D)$. We then decompose

$$u = \eta u + (1-\eta)u, \ \eta u \in H_0^1(D)$$

and observe, that from $\nu_1(D) > 0$

$$c_n \int_M |\nabla(\eta u)|^2 d\mu_{g_0} - \int_M |\eta u|^2 d\mu_{g_0} = \langle L_{g_0}(\eta u), \eta u\rangle \ge \nu_1(D) \int_M |\eta u|^2 d\mu_{g_0},$$

whence $c_n \int_M |\nabla(\eta u)|^2 d\mu_{g_0} \ge (\nu_1(D)+1) \int_M |\eta u|^2 d\mu_{g_0}$ and therefore

$$\langle L_{g_0}(\eta u), \eta u\rangle \ge \frac{c_n \nu_1(D)}{\nu_1(D)+1} \int |\nabla(\eta u)|^2 d\mu_{g_0}.$$

As a consequence

$$\begin{aligned}
\langle L_{g_0} u, u\rangle = & \int_M L_{g_0}(\eta u)(\eta u) d\mu_{g_0} + 2\int_M L_{g_0}(\eta u)\big((1-\eta)u\big) d\mu_{g_0} \\
& + \int_M L_{g_0}\big((1-\eta)u\big)\big((1-\eta)u\big) d\mu_{g_0} \\
\ge & \frac{c_n \nu_1(D)}{\nu_1(D)+1} \int_M |\nabla(\eta u)|^2 d\mu_{g_0} + c_n \int_M (1-\eta^2)|\nabla u|^2 d\mu_{g_0} \qquad (12) \\
& - 2c_n \int_M u \nabla\eta \cdot \nabla(\eta u) d\mu_{g_0} + c_n \int_M |\nabla\eta|^2 u^2 d\mu_{g_0} \\
& - \int_M (1-\eta^2) u^2 d\mu_{g_0}.
\end{aligned}$$

From Hölder's inequality we then have

$$\begin{aligned}
\Big|\int_M u\nabla\eta \cdot \nabla(\eta u) d\mu_{g_0}\Big| &\le \frac{C}{d}\Big(\int_{D\setminus\Omega} u^2 d\mu_{g_0}\Big)^{\frac{1}{2}} \Big(\int_{D\setminus\Omega} |\nabla(\eta u)|^2 d\mu_{g_0}\Big)^{\frac{1}{2}} \\
&\le \frac{C}{d}\Big(\int_{D\setminus\Omega} u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{2n}} \Big(\int_{D\setminus\Omega} d\mu_{g_0}\Big)^{\frac{1}{n}} \Big(\int_{D\setminus\Omega} |\nabla(\eta u)|^2 d\mu_{g_0}\Big)^{\frac{1}{2}} \\
&\le \frac{C|D\setminus\Omega|^{\frac{1}{n}}}{d}\Big(\int_{D\setminus\Omega} u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{2n}} \Big(\int_{D\setminus\Omega} |\nabla(\eta u)|^2 d\mu_{g_0}\Big)^{\frac{1}{2}}
\end{aligned}$$

and with $C_3 > 0$

$$\begin{aligned}
\Big|\int_M (1-\eta^2) u^2 d\mu_{g_0}\Big| &\le \int_{M\setminus\Omega} u^2 d\mu_{g_0} \\
&\le \Big(\int_{M\setminus\Omega} u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{n}} \Big(\int_{M\setminus\Omega} d\mu_{g_0}\Big)^{\frac{2}{n}} \le C_3 \Big(\int_{M\setminus\Omega} u^{\frac{2n}{n-2}} d\mu_{g_0}\Big)^{\frac{n-2}{n}}.
\end{aligned}$$

Plugging these into (12), whose second last summand is positive anyway, we obtain from Young's inequality with a constant $C_2 > 0$ and for any $0 < a < 1$

$$
\begin{aligned}
\langle L_{g_0}u,u\rangle \geq &\frac{c_n\nu_1(D)}{\nu_1(D)+1}\int_M |\nabla(\eta u)|^2 d\mu_{g_0} \\
&- \frac{2Cc_n|D\setminus\Omega|^{\frac{1}{n}}}{d}\Big(\int_{D\setminus\Omega} u^{\frac{2n}{n-2}}d\mu_{g_0}\Big)^{\frac{n-2}{2n}}\Big(\int_{D\setminus\Omega}|\nabla(\eta u)|^2 d\mu_{g_0}\Big)^{\frac{1}{2}} \\
&- C_3\Big(\int_{M\setminus\Omega} u^{\frac{2n}{n-2}}d\mu_{g_0}\Big)^{\frac{n-2}{n}} \\
\geq &\frac{c_n(1-a)\nu_1(D)}{\nu_1(D)+1}\int_M |\nabla(\eta u)|^2 d\mu_{g_0} \\
&- \Big(C_2\frac{\nu_1(D)+1}{\nu_1(D)}\frac{|D\setminus\Omega|^{\frac{2}{n}}}{ad^2}+C_3\Big)\Big(\int_{M\setminus\Omega} u^{\frac{2n}{n-2}}d\mu_{g_0}\Big)^{\frac{n-2}{n}}.
\end{aligned}
\tag{13}
$$

On the other hand, recalling $\{K \geq 0\} = \Omega_K \subset\subset \Omega$,

$$
\begin{aligned}
|\int_M Ku^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} \geq &\,|\int_{\Omega_K^c} Ku^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} - |\int_{\Omega_K} Ku^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} \\
\geq &\inf_{M\setminus\Omega}|K|^{\frac{n-2}{n}}|\int_{M\setminus\Omega} u^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} - \sup_M K^{\frac{n-2}{n}}|\int_\Omega u^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}}.
\end{aligned}
\tag{14}
$$

Thus combining (13) with (14) via

$$
\begin{aligned}
\langle L_{g_0}u,u\rangle + B|\int_M Ku^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} \geq &\frac{c_n(1-a)\nu_1(D)}{\nu_1(D)+1}\int_M|\nabla(\eta u)|^2 d\mu_{g_0} \\
&+ (B\inf_{M\setminus\Omega}|K|^{\frac{n-2}{n}} - C_2\frac{\nu_1(D)+1}{\nu_1(D)}\frac{|D\setminus\Omega|^{\frac{2}{n}}}{ad^2} - C_3)\|u\|^2_{L^{\frac{2n}{n-2}}(M\setminus\Omega)} \\
&- B\sup_M K^{\frac{n-2}{n}}\|u\|^2_{L^{\frac{2n}{n-2}}(\Omega)}.
\end{aligned}
$$

Choosing $a = \frac{1}{2}$, denoting by S the Sobolev constant, supposing

1) $B^{\frac{n}{n-2}} > 2^{\frac{n}{n-2}}(2C_2\frac{\nu_1(D)+1}{\nu_1(D)}\frac{|D\setminus\Omega|^{\frac{2}{n}}}{d^2}+C_3)^{\frac{n}{n-2}}/\inf_{M\setminus\Omega}|K|$

2) $\sup_M K < C_1/B^{\frac{n}{n-2}}$ with $C_1^{\frac{n-2}{n}}S^2 < \frac{c_n\nu_1(D)}{4(\nu_1(D)+1)}$,

and recalling $\Omega \subset\subset D$, we then find from the Sobolev embedding on $H^1(M)$

$$
\begin{aligned}
\langle L_{g_0}u,u\rangle + B|\int_M Ku^{\frac{2n}{n-2}}d\mu_{g_0}|^{\frac{n-2}{n}} \geq &\frac{c_n\nu_1(D)}{4(\nu_1(D)+1)}\int_M|\nabla(\eta u)|^2 d\mu_{g_0} \\
&+ (C_2\frac{\nu_1(D)+1}{\nu_1(D)}\frac{|D\setminus\Omega|^{\frac{2}{n}}}{2d^2}+C_3)\|u\|^2_{L^{\frac{2n}{n-2}}(M\setminus\Omega)} \\
\geq &\, C(\Omega,D)\|u\|^2_{L^{\frac{2n}{n-2}}}.
\end{aligned}
\tag{15}
$$

Adding some $\varepsilon\langle L_{g_0}u,u\rangle$ with $\varepsilon > 0$ sufficiently small to both sides of (15), estimate (11) readily follows under 1) and 2) above, which both hold true, if

$$\sup_M K < \frac{\epsilon \inf_{M\setminus\Omega}(-K)}{(\frac{\nu_1(D)+1}{\nu_1(D)}\frac{|D\setminus\Omega|^{\frac{2}{n}}}{d^2}+1)^{\frac{n}{n-2}}}$$

for some universal $\epsilon > 0$. Recalling $d = dist(\partial\Omega, \partial D)$, the assertion follows. □

Remark 3.2. *In view of the non existence results, discussed in Section 1, condition (i) in Proposition 3.1 displays the correct behaviour in that* $\sup(K,0) \longrightarrow 0$, *as* $\inf(K,0) \longrightarrow 0$ *or* $\nu_1(\Omega_K) \longrightarrow 0$. *Also note, that* $\nu_1(\Omega_K) \longrightarrow \infty$ *does not really relax the smallness assumption.*

Lemma 3.3. *If an A-B-inequality holds on X, then*

(i) $\inf_X(-k), \inf_X J > 0$ *and* $\sup_X(-k), \sup_X(-r) < \infty$

(ii) $\inf_X(-r) > 0$ *on any sublevel* $\{J \leq L\}$.

In particular $\|\cdot\|_{H^1}$ *is uniformly bounded on X.*

Proof. Since $\|\cdot\|_{L^{\frac{2n}{n-2}}} = 1$ on X, we have

1) $-k \leq \|K\|_{L^\infty}\int u^{\frac{2n}{n-2}}d\mu_{g_0} = \|K\|_{L^\infty}$

2) $0 < -r = -\int(c_n|\nabla u|^2 - u^2)d\mu_{g_0} \leq \int u^2 d\mu_{g_0} \leq C_0\|u\|^2_{L^{\frac{2n}{n-2}}} = C_0$

and may from the Sobolev embedding assume, that $1 \leq Ar + B|k|^{\frac{n-2}{n}}$. Then $k \longrightarrow 0 \Longrightarrow 2r > 1/A$, while $r < 0$ on X by definition. Thus

$$-k \geq c_0 > 0 \ \text{ on } \ X$$

and therefore $J(u) = \frac{-k}{(-r)^{\frac{n}{n-2}}} \geq \frac{c_0}{C_0^{\frac{n}{n-2}}}$. We conclude

$$L \geq J(u) = \frac{-k}{(-r)^{\frac{n}{n-2}}} \Longrightarrow (-r)^{\frac{n}{n-2}} \geq \frac{-k}{L} \geq \frac{c_0}{L}.$$

Finally $\|u\|^2_{H^1} \leq C_1 r + C_2\|u\|^2_{L^{\frac{2n}{n-2}}}$ by Hölder's inequality. □

3.2 Solvability by Minimization

We show Theorem 1 by direct minimization, using the Yamabe type flow (5) in order to pass from a minimizing sequence to a minimizing sequence of solutions.

Proof of Theorem 1. Choose a minimizing sequence of initial data $(u_0^i) \subset X$, $J(u_0^i) \longrightarrow \inf_X J$, and for $i \in \mathbb{N}$ fixed the flow line generated by (5)

$$u^i = u^i(t,\cdot),\ u(0,\cdot) = u_0^i.$$

Combining Lemmata 3.3, 2.3 and Proposition 2.1 we find

$$\exists\, 0 < t^i_j \longrightarrow \infty \ : \ |\partial J|(u^i_{t_j}) \longrightarrow 0,\ u^i_{t_j} = u(t^i_j, \cdot).$$

Moreover, passing to a subsequence, we may assume, that

$$r_{u^i_{t_j}} \longrightarrow r^i_\infty < 0 \ \text{ and } \ k_{u^i_{t_j}} \longrightarrow k^i_\infty < 0,$$

and, since $\sup_{u\in X} \|u\|_{H^1} < C$ on X, as $r < 0$ and $\|\cdot\|_{L^{\frac{2n}{n-2}}} = 1$ thereon, that

$$u^i_{t_j} \rightharpoonup u^i_\infty \ \text{ weakly in } \ H^1(M) \ \text{ and } \ \sup_{i,j} \|u^i_{t_j}\|_{H^1} < C. \tag{16}$$

In particular $u^i_\infty > 0$ from (iii) in Proposition 2.1 and is a weak solution of

$$\frac{-k^i_\infty}{-r^i_\infty} L_{g_0} u^i_\infty = K(u^i_\infty)^{\frac{n+2}{n-2}},$$

which implies

$$\frac{r_{u^i_\infty}}{k_{u^i_\infty}} = \frac{r^i_\infty}{k^i_\infty}. \tag{17}$$

As a consequence of (17) and the weak lower semicontinuity of the norm $\|\cdot\|_{H^1}$.

$$\begin{aligned} J(u^i_0) \geq & \lim_{j\to\infty} J(u^i_{t_j}) = \lim_{j\to\infty} \frac{-k_{u^i_{t_j}}}{(-r_{u^i_{t_j}})^{\frac{n}{n-2}}} = \lim_{j\to\infty} \big(\frac{-k_{u^i_{t_j}}}{-r_{u^i_{t_j}}} (-r_{u^i_{t_j}})^{-\frac{2}{n-2}}\big) \\ \geq & \frac{-k^i_\infty}{-r^i_\infty} \liminf_{j\to\infty} (-r_{u^i_{t_j}})^{-\frac{2}{n-2}} \geq \frac{-k^i_\infty}{-r^i_\infty} (-r_{u^i_\infty})^{-\frac{2}{n-2}} = \frac{-k_{u^i_\infty}}{(-r_{u^i_\infty})^{\frac{n}{n-2}}} \\ = & J(u^i_\infty), \end{aligned} \tag{18}$$

and we deduce $J(u^i_\infty) \longrightarrow \inf_X J$, i.e. we have found a minimizing sequence of solutions. Finally, passing again to a subsequence, if necessary, from (16) there exist $0 \leq u^\infty_\infty \in H^1(M)$ as a weak limit of $u^i_\infty > 0$ satisfying

$$\frac{-k_\infty}{-r_\infty} L_{g_0} u^\infty_\infty = K(u^\infty_\infty)^{\frac{n+2}{n-2}}.$$

Then by standard regularity $0 \leq u^\infty_\infty \in C^\infty(M)$ and, writing $\partial J(u^\infty_\infty) = 0$ as

$$-c_n \Delta_{g_0} u^\infty_\infty - (1 + \frac{-r_\infty}{-k_\infty} K(u^\infty_\infty)^{\frac{4}{n-2}}) u_\infty = 0,$$

then $u^\infty_\infty > 0$ follows from the weak Harnack inequality. Finally, repeating the argument of (18), we find $J(u^\infty_\infty) = \inf_X J$ and the proof is complete. □

4 Non Uniqueness

In this section we study the occurrence of critical points at infinity, cf. [17], associated to variational formulations of $R = K$ and rule them out, as needed, cf. Remark 4.12. We shall throughout this section assume the validity of some global A-B-inequality (11), which ensures, that

$$\inf_X J = \min_X J > 0,$$

i.e. a lower, positive bound and the existence of a minimizer $u_0 \in X$ of J, see Theorem 1. We also assume invertibility of L_{g_0}, a generic property [10] implying the existence of a Green's function G_{g_0}, see Theorems 2.2 and 3.7 in [23], which due to non positivity of L_{g_0} is necessarily sign changing. Recall, that on X we reduce J by flowing along

$$\partial_t u = -(\frac{k}{r}R - K)u = \frac{k}{r}u^{-\frac{4}{n-2}}(c_n \Delta_{g_0} u + u + \frac{r}{k}Ku^{\frac{n+2}{n-2}})$$

and that along every flow line we have

$$c < -k, -r < C,$$

see Lemma 3.3, while as a consequence of the parabolic maximum principle every flow line starting strictly positive remains strictly positive, and thus a zero weak limit can never occur along (5), see Proposition 2.1, *in contrast to what happens in case of positive Yamabe invariants.*

On the other hand, if K is sign changing, then

$$\emptyset \neq Y = \{r > 0\} \cap \{k > 0\} \cap \{\|u\|_{L^{\frac{2n}{n-2}}} = 1\} \cap \{u > 0\} \subset C^{\infty}(M),$$

as is easily seen from taking some $0 \leq u_0 \in C^{\infty}$ with

$$supp(u_0) \subset B_{\varepsilon}(a_0) \subset \{K > 0\}.$$

In fact $k_{u_0} > 0$ and also $r_{u_0} > 0$ for $0 < \varepsilon \ll 1$ sufficiently small, as the first Dirichlet eigenvalue of L_{g_0} on $B_{\varepsilon}(a_0)$ is positive, whence $u = \alpha(u_0 + \delta) \in Y$ for suitable $\alpha > 0$ and $0 < \delta \ll 1$. Moreover we may define analogously to J on X the scaling invariant functional

$$I = \frac{r}{k^{\frac{n-2}{n}}} \text{ on } Y$$

with naturally associated Yamabe type flow

$$\partial_t u = -(R - \frac{r}{k}K)u, \tag{19}$$

for which the same arguments as for (5) in Section 2 apply, and so the flow generated by (19) exists for all times, preserves Y and is a pseudo gradient flow for I, provided $c < k, r < C$ remain uniformly bounded away from zero and infinity along each flow line, which by the following lemma holds true.

Lemma 4.1. *If an A-B-inequality holds on $H^1(M)$, then*

(i) $\sup_Y k < \infty$ *and* $\inf_Y r, \inf_Y I > 0$

(ii) $\sup_Y r < \infty$ *and* $\inf_Y k > 0$ *on any sublevel* $\{I \leq L\}$.

In particular $\|\cdot\|_{H^1}$ *is uniformly bounded on any sublevel.*

Proof. By $\|\cdot\|_{L^{\frac{2n}{n-2}}} = 1$ on Y, clearly $k \leq \max K$, and we may assume

$$1 \leq Ar + B|k|^{\frac{n-2}{n}} \text{ on } H^1 \cap \{\|\cdot\|_{L^{\frac{2n}{n-2}}} = 1\}. \tag{20}$$

Moreover suppose, that $\inf_Y r = 0$. Then there exists $(u_m) \subset Y$ with $r_{u_m} \longrightarrow 0$ and hence for any $\epsilon > 0$ some $u_0 \in Y$ with $r_{u_0} \leq \epsilon$. Then consider for $\tau \in [0,1]$

$$u_\tau = \frac{(1-\tau)u_0 + \tau\mathbb{1}}{\|(1-\tau)u_0 + \tau\mathbb{1}\|_{L^{\frac{2n}{n-2}}}} \in H^1 \cap \{\|\cdot\|_{L^{\frac{2n}{n-2}}} = 1\},$$

for which we readily have $r_{u_\tau} \leq \epsilon$, while from $k_0 = k_{u_0} > 0$ and $k_1 = k_{\mathbb{1}} < 0$ there exists some $0 < t_0 < 1$ with $k_{t_0} = 0$. Then for some $0 < t_1 < t_0$ we have $r_{u_{t_1}} \leq \epsilon$ and $0 < k_{u_{t_1}} \leq \epsilon$, contradicting (20) for $0 < \epsilon \ll 1$. Thus

$$\sup_Y k < \infty, \inf_Y r > 0 \text{ and in particular } I = \frac{r}{k^{\frac{n-2}{n}}} \geq C,$$

whence (i) is shown. To see (ii), for $u \in \{I \leq L\}$ from (i) we have

(i) $L \geq I = \frac{r}{k^{\frac{n-2}{n}}} \Longrightarrow r \leq Lk^{\frac{n-2}{n}} \leq CL$

(ii) $L \geq I = \frac{r}{k^{\frac{n-2}{n}}} \Longrightarrow k^{\frac{n-2}{n}} \geq {r}/{L} \geq {C}/{L}$.

Finally $\|u\|^2_{H^1} \leq C_1 r + C_2\|u\|^2_{L^{\frac{2n}{n-2}}}$ and the last assertion follows. ☐

For the same reasons as for J a flow line u for I along (19) with a strictly positive initial data u_0 will remain strictly positive and hence also for I a zero weak limit will never occur. Be it as it may, clearly I is a valid variational energy just like J, but in contrast to J we cannot pass to weak limits $u_{t_k} \rightharpoonup u_\infty$ within Y, since in that passage $r = \int L_{g_0} u u d\mu_{g_0} > 0$ as a just *lower* semicontinuous function may become non positive, i.e. $r_{u_\infty} \leq 0$. On the other hand, if we can exclude blow-up for I, then we will find a minimizer for I, yielding a second solution to the problem besides the minimizer for J. This will show Theorem 2.

4.1 Bubbles and Estimates

Let us recall the construction of *conformal normal coordinates* from [13]. Given $a \in M$, one chooses a special conformal metric

$$g_a = u_a^{\frac{4}{n-2}} g_0 \text{ with } u_a = 1 + O(d^2_{g_0}(a,\cdot)),$$

whose volume element in g_a-geodesic normal coordinates coincides with the Euclidean one, see also [11]. In particular

$$(exp_a^{g_0})^- \circ \exp_a^{g_a}(x) = x + O(|x|^3)$$

for the exponential maps centered at a. We then denote by r_a the geodesic distance from a with respect to the metric g_a just introduced. With these choices the expression of the Green's function G_{g_a} for the conformal Laplacian L_{g_a} with pole at $a \in M$, denoted by $G_a = G_{g_a}(a, \cdot)$, simplifies to

$$G_a = \frac{1}{4n(n-1)\omega_n}(r_a^{2-n} + H_a), \; r_a = d_{g_a}(a, \cdot), \; H_a = H_{r,a} + H_{s,a}, \tag{21}$$

where $\omega_n = |S^{n-1}|$, cf. Section 6 in [13]. Here $H_{r,a} \in C^{2,\alpha}$, while

$$H_{s,a} = O\begin{pmatrix} 0 & \text{for } n=3 \\ r_a^2 \ln r_a & \text{for } n=4 \\ r_a & \text{for } n=5 \\ \ln r_a & \text{for } n=6 \\ r_a^{6-n} & \text{for } n \geq 7 \end{pmatrix} \tag{22}$$

and $H_{s,a} \equiv 0$, if g_a is flat around a. In fact the argument in [13] of a successive polynomial killing of polynomial deficits, created by the local geometry, is purely local and thus applies here as well. Proceeding hence as in [13] we reach

$$L_{g_a}\tilde{G}_a = \delta_a + \tilde{f}_a \; \text{ with } \; \tilde{G}_a \; \text{ as in (21) and } \; \tilde{f}_a \in C^{0,\alpha}$$

and, solving $L_{g_a}\tilde{F}_a = -\tilde{f}_a$ by assumed invertibility of L_{g_a}, we find $\tilde{F}_a \in C^{2,\alpha}$ from Schauder estimates and then $G_a = \tilde{G}_a + \tilde{F}_a$. On $\{G_a > 0\}$ let for $\lambda > 0$

$$\theta_{a,\lambda} = u_a\Big(\frac{\lambda}{1 + \lambda^2\gamma_n G_a^{\frac{2}{2-n}}}\Big)^{\frac{n-2}{2}}, \; \gamma_n = (4n(n-1)\omega_n)^{\frac{2}{2-n}},$$

see [14] or [16]. Extend $\theta_{a,\lambda} = 0$ on $\{G_a \leq 0\}$ and with a smooth cut-off function

$$\eta_a = \eta(d_{g_a}(a, \cdot)) = \begin{cases} 1 & \text{on} \quad B_\epsilon(a) \quad = \quad B_\epsilon^{d_{g_a}}(a) \\ 0 & \text{on} \quad B_{2\epsilon}(a)^c \quad = \quad M \setminus B_{2\epsilon}^{d_{g_a}}(a)\}, \end{cases}$$

where $0 < \epsilon \ll 1$ is independent of $a \in M$ and such, that on $B_{4\epsilon}^{d_{g_a}}(a)$ the conformal normal coordinates from g_a are well defined and $G_{g_a} > 0$, define

$$\varphi_{a,\lambda} = \eta_a \theta_{a,\lambda}. \tag{23}$$

Note, that $\gamma_n G_a^{\frac{2}{2-n}}(x) = d_{g_a}^2(a,x) + o(d_{g_a}^2(a,x))$, as $x \longrightarrow a$.

Lemma 4.2. *If $Cond_n$ holds at $a \in M$, then*

$$L_{g_0}\varphi_{a,\lambda} = 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} + O\Big(\frac{\chi_{B_{2\epsilon}(a)\setminus B_\epsilon(a)}}{\lambda^{\frac{n-2}{2}}}\Big) + o_{\frac{1}{\lambda}}\Big(\frac{1}{\lambda^{\frac{n-2}{2}}}\Big) \; \text{ in } \; W^{-1,2}(M).$$

The expansion above persists upon taking the $\lambda\partial_\lambda$ and $\frac{\nabla_a}{\lambda}$ derivatives.

Proof. As in Lemma 3.3 in [16] and denoting $r_a = d_{g_a}(a,\cdot)$ we find

$$\begin{aligned}L_{g_0}\theta_{a,\lambda} =& 4n(n-1)\theta_{a,\lambda}^{\frac{n+2}{n-2}} + \frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\theta_{a,\lambda}^{\frac{n}{n-2}} \\ & - 2nc_n(1+o_{r_a}(1))r_a^{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)\theta_{a,\lambda}^{\frac{n+2}{n-2}}\end{aligned}$$

pointwise and thus

$$\begin{aligned}L_{g_0}\varphi_{a,\lambda} =& \ 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}} \\ & - c_n\Delta_{g_0}\eta_a\theta_{a,\lambda} - 2c_n\langle\nabla\eta_a,\nabla\theta_{a,\lambda}\rangle_{g_0} + 4n(n-1)(\eta_a - \eta_a^{\frac{n+2}{n-2}})\theta_{a,\lambda}^{\frac{n+2}{n-2}} \\ & - 2nc_n(1+o_{r_a}(1))r_a^{n-2}((n-1)H_a + r_a\partial_{r_a}H_a)\eta_a\theta_{a,\lambda}^{\frac{n+2}{n-2}} + \frac{u_a^{\frac{2}{n-2}}R_{g_a}}{\lambda}\eta_a\theta_{a,\lambda}^{\frac{n}{n-2}}.\end{aligned}$$

Readily the terms of the second line above can be subsumed under some

$$O(\lambda^{\frac{2-n}{2}})\chi_{B_{2\epsilon}(a)\setminus B_\epsilon(a)},$$

while those of the third are of order $o(\lambda^{\frac{2-n}{2}})$ in $W^{-1,2}(M)$, as follows from

(i) $R_{g_a} = O(r_a^2)$ and (22) in case $3 \leq n \leq 5$

(ii) $H_{s,a} = R_{g_a} = 0$ close to a in case of local flatness around a for $n \geq 6$

and simple integral estimates. ☐

Since we target $\lambda^{\frac{2-n}{2}}$ as the level of precision and from Lemma 4.2

$$\int |L_{g_0}\varphi_{a,\lambda} - 4n(n-1)\varphi_{a,\lambda}^{\frac{n+2}{n-2}}|\theta_{a,\lambda}d\mu_{g_0} = o_{\frac{1}{\lambda}}(\frac{1}{\lambda^{\frac{n-2}{2}}}),$$

the error terms in Lemma 4.2 will be of no concern.

Notation. For points $a_i \in M$ we will denote by $K_i, \nabla K_i$ and ΔK_i for instance

$$K(a_i),\ \nabla K(a_i) = \nabla_{g_0}K(a_i) \ \text{ and } \ \Delta K(a_i) = \Delta_{g_0}K(a_i).$$

For $k,l = 1,2,3$ and $\lambda_i > 0$, $a_i \in M$, $i = 1,\ldots,q$ let

(i) $\varphi_i = \varphi_{a_i,\lambda_i}$ and $(d_{1,i}, d_{2,i}, d_{3,i}) = (1, -\lambda_i\partial_{\lambda_i}, \frac{1}{\lambda_i}\nabla_{a_i})$;

(ii) $\phi_{1,i} = \varphi_i,\ \phi_{2,i} = -\lambda_i\partial_{\lambda_i}\varphi_i,\ \phi_{3,i} = \frac{1}{\lambda_i}\nabla_{a_i}\varphi_i$, so $\phi_{k,i} = d_{k,i}\varphi_i$.

Note, that with the above definitions $\phi_{k,i}$ is uniformly bounded in $H^1(M)$.

Lemma 4.3. *Let $k,l = 1,2,3$ and $i,j = 1,\ldots,q$. Then for*

$$\varepsilon_{i,j} = \eta(d_{g_0}(a_i,a_j))(\frac{\lambda_j}{\lambda_i} + \frac{\lambda_i}{\lambda_j} + \lambda_i\lambda_j\gamma_n G_{g_0}^{\frac{2}{2-n}}(a_i,a_j))^{\frac{2-n}{2}}$$

with a suitable cut-off function

$$\eta = \begin{cases} 1 & on \quad r < 4\epsilon \\ 0 & on \quad r \geq 6\epsilon \end{cases}$$

and $\epsilon > 0$ sufficiently small there holds

(i) $|\phi_{k,i}|, |\lambda_i \partial_{\lambda_i} \phi_{k,i}|, |\frac{1}{\lambda_i} \nabla_{a_i} \phi_{k,i}| \leq C\varphi_i + o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})$

(ii) $\int \varphi_i^{\frac{4}{n-2}} \phi_{k,i} \phi_{k,i} d\mu_{g_0} = c_k \cdot id + O(\frac{1}{\lambda_i^2}) + o(\frac{1}{\lambda_i^{\frac{n-2}{2}}}),\ c_k > 0$

(iii) for $i \neq j$ up to some $O(\sum_{i\neq j}(\frac{1}{\lambda_i^4} + \varepsilon_{i,j}^{\frac{n+2}{n}})) + o(\sum_{i\neq j} \frac{1}{\lambda_i^{\frac{n-2}{2}}})$

$$\int \varphi_i^{\frac{n+2}{n-2}} \phi_{k,j} d\mu_{g_0} = b_k d_{k,i} \varepsilon_{i,j} = \int \varphi_i d_{k,j} \varphi_j^{\frac{n+2}{n-2}} d\mu_{g_0}$$

(iv) $\int \varphi_i^{\frac{4}{n-2}} \phi_{k,i} \phi_{l,i} d\mu_{g_0} = O(\frac{1}{\lambda_i^2}) + o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})$ *for $k \neq l$ and for $k = 2, 3$*

$$\int \varphi_i^{\frac{n+2}{n-2}} \phi_{k,i} d\mu_{g_0} = O(\frac{1}{\lambda_i^4}) + o(\frac{1}{\lambda_i^{\frac{n-2}{2}}})$$

(v) $\int \varphi_i^{\alpha} \varphi_j^{\beta} d\mu_{g_0} = O(\varepsilon_{i,j}^{\beta})$ *for $i \neq j,\ \alpha + \beta = \frac{2n}{n-2},\ \alpha > \frac{n}{n-2} > \beta \geq 1$*

(vi) $\int \varphi_i^{\frac{n}{n-2}} \varphi_j^{\frac{n}{n-2}} d\mu_{g_0} = O(\varepsilon_{i,j}^{\frac{n}{n-2}} \ln \varepsilon_{i,j}),\ i \neq j$

(vii) $(1, \lambda_i \partial_{\lambda_i}, \frac{1}{\lambda_i} \nabla_{a_i}) \varepsilon_{i,j} = O(\varepsilon_{i,j}) + o(\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \frac{1}{\lambda_j^{\frac{n-2}{2}}}),\ i \neq j,$

with constants $b_k = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}}$ *for $k = 1, 2, 3$ and*

$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n},\ c_2 = \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{(r^2-1)^2 dx}{(1+r^2)^{n+2}},\ c_3 = \frac{(n-2)^2}{n} \int_{\mathbb{R}^n} \frac{r^2 dx}{(1+r^2)^{n+2}}.$$

Proof. First note, that $\varepsilon_{i,j}$ is well defined, as $G_{g_0}(a_i, a_j) > 0$, if a_i, a_j are close. Secondly the estimates above are just those of Lemma 3.5[1] in [16] up to some

$$o(\lambda_i^{-\frac{n-2}{2}}) \ \text{ or } \ o(\lambda_i^{-\frac{n-2}{2}} + \lambda_j^{-\frac{n-2}{2}})$$

respectively, depending on whether we calculate an inter- or selfactions of bubbles. These errors account for the cut-off functions in (23). Concerning

[1]See also Lemma 3.5 and its proof in the more detailed version of [16] found at `http://geb.uni-giessen.de/geb/volltexte/2015/11691/`

(i) interactions of two bubbles φ_i and φ_j, $i \neq j$, as in e.g. (iii), these

   (1) are zero, if a_i, a_j are far, e.g. $d_{g_0}(a_i, a_j) > 4\epsilon$
   (2) are of order $o(1/\lambda_i^{\frac{n-2}{2}} + 1/\lambda_j^{\frac{n-2}{2}})$, in case $d_{g_0}(a_i, a_j) \geq \frac{\epsilon}{4}$
   (3) or else expand as in Lemma 3.5 in [16].

(ii) the various selfactions of one bubble, e.g. (ii), in view of Lemma 4.2 and

$$\varphi_{a_i,\lambda_i} = O(\lambda_i^{-\frac{n-2}{2}}) \text{ on } B_\epsilon(a_i)^c$$

the estimates are up to some $o(\lambda_i^{-\frac{n-2}{2}})$ the same as in Lemma 3.5 in [16].

With these hints in mind we leave the details to the reader. □

4.2 Blow-up Description

The general characterization of Palais-Smale sequences hold for I and J alike.

Proposition 4.4. *Let $K \in C^\infty(M)$ satisfy some A-B-inequality on $H^1(M)$ and*

$$(u_m) \subset X \quad or \quad (u_m) \subset Y$$

be a sequence, for which

$$J(u_m) = \frac{-k_{u_m}}{(-r_{u_m})^{\frac{n}{n-2}}} \longrightarrow c > 0 \ \text{ and } \ \partial J(u_m) \longrightarrow 0 \ \text{ in } \ W^{-1,2}(M).$$

or

$$I(u_m) = \frac{r_{u_m}}{(k_{u_m})^{\frac{n-2}{n}}} \longrightarrow c > 0 \ \text{ and } \ \partial I(u_m) \longrightarrow 0 \ \text{ in } \ W^{-1,2}(M)$$

respectively. Then up to a subsequence

$$r_{u_m} = r_m \longrightarrow r_\infty < 0, \ k_{u_m} = k_m \longrightarrow k_\infty < 0$$

or

$$r_{u_m} = r_m \longrightarrow r_\infty > 0, \ k_{u_m} = k_m \longrightarrow k_\infty > 0$$

respectively and in either case there exist a solution

$$0 \leq u_\infty \in C^\infty(M) \ \text{ to } \ L_{g_0} u_\infty = K u_\infty^{\frac{n+2}{n-2}} \ \text{ with either } \ u_\infty \equiv 0 \ \text{ or } \ u_\infty > 0,$$

a sequence $\mathbb{R}_+ \supset (\alpha_m) \longrightarrow \alpha_\infty$, $q \in \mathbb{N}_0$ and for $i = 1, \ldots, q$ sequences

$$M \supset (a_{i,m}) \xrightarrow{m \to \infty} a_{i_\infty} \ \text{ and } \ \mathbb{R}_+ \supset \lambda_{i,m} \xrightarrow{m \to \infty} \infty$$

such, that $u_m = \alpha_m u_\infty + \sum_{i=1}^q \alpha_{i,m} \varphi_{a_{i,m},\lambda_{i,m}} + v_m$ with

$$\|v_m\| \longrightarrow 0, \ \frac{r_\infty K(a_{i_\infty}) \alpha_{i_\infty}^{\frac{4}{n-2}}}{4n(n-1)k_\infty} = 1 \ \text{ and, if } \ u_\infty \not\equiv 0, \ \text{ then } \ \frac{r_\infty \alpha_\infty^{\frac{4}{n-2}}}{k_\infty} = 1.$$

Moreover, if $q > 1$, then $(\varepsilon_{i,j})_m \xrightarrow{m \to \infty} 0$ for each pair $1 \leq i < j \leq q$.

Proof. This follows as in [14], which does not rely on the positivity of L_{g_0}, but on being able to pass $r_m \longrightarrow r_\infty \neq 0$ and $k \longrightarrow k_\infty \neq 0$, which we may assume here thanks to Lemmata 3.3 and 4.1. □

Note, that $L_{g_0}u_\infty = Ku_\infty^{\frac{n+2}{n-2}}$ implies $r_{u_\infty} = k_{u_\infty}$, whence $r_{u_\infty} = 0$ or $k_{u_\infty} = 0$ are impossible from the presumed validity of an A-B-inequality. Moreover note, that necessarily $K(a_{i_\infty}) > 0$, that is, if blow-up happens, then on $\{K > 0\}$.

In order to have a unique representation of a blow-up scenario, we perform a standard reduction procedure, see [16], as follows.

Definition 4.5. *For $\varepsilon > 0$, $q \in \mathbb{N}$ and $u \in H^1(M)$ let*

(i) $A_u(u_\infty, q, \varepsilon) = \{(\alpha, \alpha_i, \lambda_i, a_i) \in \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q_+ \times M^q \;:\; \underset{i\neq j}{\forall}\; \lambda_i^{-1}, \lambda_j^{-1}, \varepsilon_{i,j},$

$$|1 - \frac{r\alpha^{\frac{4}{n-2}}}{k}|, |1 - \frac{r\alpha_i^{\frac{4}{n-2}}K(a_i)}{4n(n-1)k}|, \|u - \alpha u_\infty - \alpha^i \varphi_{a_i,\lambda_i}\| \leq \varepsilon\}$$

(ii) $V(u_\infty, q, \varepsilon) = \{u \in W^{1,2}(M) \mid A_u(u_\infty, q, \varepsilon) \neq \emptyset\}$.

Here $\|\cdot\| = \|\cdot\|_{D_{g_0}}$ denotes the Sobolev norm induced by $D_{g_0} = L_{g_0} + 2$.

As shown in the Appendix, given $u \in V(u_\infty, q, \varepsilon)$ and fixed $(\bar{a}_i, \bar{\lambda}_i)$, by invertibility of the linear operator L_{g_0} we may *uniquely* write

$$u = \bar{\alpha} u_\infty + \bar{\alpha}^i \varphi_{\bar{a}_i, \bar{\lambda}_i} + \bar{v} \;\text{ with }\; \bar{v} \perp_{L_{g_0}} \mathit{span}\{u_\infty, \varphi_{\bar{a}_i,\bar{\lambda}_i}\} \tag{24}$$

and then minimize along natural projection Π induced by (24), as follows.

Lemma 4.6. *For every $\varepsilon_0 > 0$ there exists $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$ such, that for any*

$$u \in V(u_\infty, q, \varepsilon_2)$$

the problem

$$\inf_{(\tilde{a}_i,\tilde{\lambda}_i) \in \Pi_{(a_i,\lambda_i)} A_u(u_\infty, q, 2\varepsilon_1)} \int u^{\frac{4}{n-2}} |u - \tilde{\alpha} u_\infty - \tilde{\alpha}^i \varphi_{a_i,\lambda_i}|^2 d\mu_{g_0}$$

admits a unique minimizer (a_i, λ_i) with $(\alpha, \alpha_i, a_i, \lambda_i) \in A_u(u_\infty, q, \varepsilon_1)$. Setting

$$\varphi_i = \varphi_{a_i,\lambda_i},\; v = u - \alpha u_\infty - \alpha^i \varphi_i,$$

we have in addition to $v \perp_{L_{g_0}} \mathit{span}\{u_\infty, \varphi_{a_i,\lambda_i}\}$ from (24), that

(i) the quantities $\langle \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i}, v \rangle_{L_{g_0}}$ and $\int u^{\frac{4}{n-2}} \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i} v \, d\mu_{g_0}$ are of order

$$O(\sum_i \frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j} \varepsilon_{i,j}^2 + \|v\|^2) + O(\sum_i |\langle \varphi_{a_i,\lambda_i}, \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i} \rangle_{L_{g_0}}|^2)$$

$$+ O(\sum_i \|\lambda_i \partial_{\lambda_i} L_{g_0} \varphi_{a_i,\lambda_i} - 4n(n-1)\lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i}^{\frac{n+2}{n-2}}\|^2_{L^2_{g_0}})$$

and, if $Cond_n$ holds at all a_i, of order $o_{\varepsilon_1}(\sum_i \frac{1}{\lambda_i^{\frac{n-2}{2}}}) + O(\sum_{i\neq j} \varepsilon_{i,j}^2 + \|v\|^2)$

(ii) the quantities $\langle \frac{\nabla_{a_i}}{\lambda_i}\varphi_{a_i,\lambda_i}, v\rangle_{L_{g_0}}$ *and* $\int u^{\frac{4}{n-2}}\frac{\nabla_{a_i}}{\lambda_i}\varphi_{a_i,\lambda_i} v d\mu_{g_0}$ *are of order*

$$O(\sum_i \frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j}\varepsilon_{i,j}^2 + \|v\|^2) + O(\sum_i |\langle \varphi_{a_i,\lambda_i}, \frac{\nabla_{a_i}}{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}|^2)$$
$$+ O(\sum_i \|\lambda_i \partial_{\lambda_i} L_{g_0}\varphi_{a_i,\lambda_i} - 4n(n-1)\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}^{\frac{n+2}{n-2}}\|^2_{L^2_{g_0}})$$

and, if $Cond_n$ *holds at all* a_i*, of order* $o_{\varepsilon_1}(\sum_i \frac{1}{\lambda_i^{\frac{n-2}{2}}}) + O(\sum_{i\neq j}\varepsilon_{i,j}^2 + \|v\|^2)$.

The proof, postponed to the Appendix, is technically analogous to the case of a positive Yamabe invariant, cf. Appendix A in [6] and Proposition 3.10 in [16], where thanks to $L_{g_0} > 0$ we may minimize over all variables

$$\inf_{(\alpha,\alpha_i,a_i,\lambda_i)\in A_u(u_\infty,q,2\varepsilon_1)} \|u - \alpha u_\infty - \alpha^i\varphi_{a_i,\lambda_i}\|^2_{L_{g_0}},$$

which is not feasible in our context. Here the *linear* variables (α, α_i) are chosen for the sake of the L_{g_0}-orthogonalities in (24), since for proper estimates and expansions, which we will require in [18], at least $\langle u_\infty, v\rangle_{L_{g_0}} = 0$ is indispensable. Fortunately Lemma 4.6 still provides sufficient *almost*-orthogonalities in (λ_i, a_i).

Remark 4.7. *With these notions at hand the lack of zero weak limit blow-ups along flow lines and in particular of, as they are referred to, pure critical points at infinity becomes* heuristically *clear. Indeed consider* $u = \alpha\varphi + v \in Y$ *and*

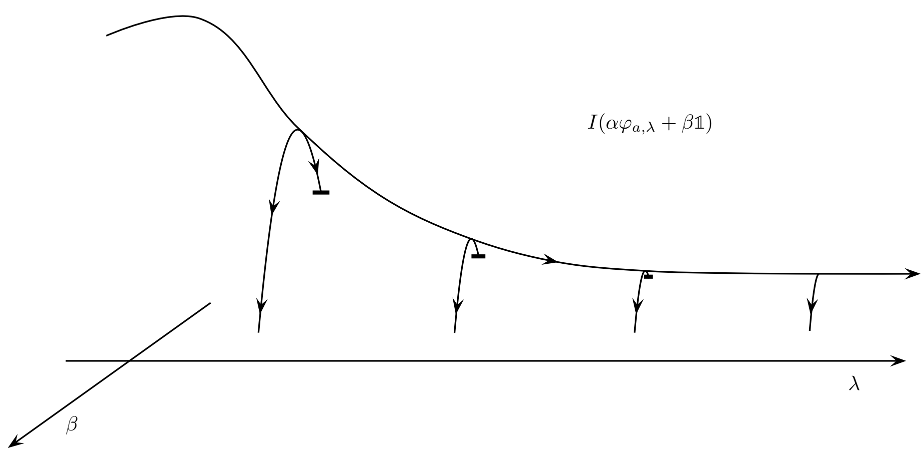

Figure 1

compute from Proposition 4.4 for instance and up to some $o(\|v\|^2)$

$$
\begin{aligned}
I(u) = \frac{r}{k^{\frac{n-2}{n}}} = & \frac{r_\varphi}{k_\varphi^{\frac{n-2}{n}}} + o_{\frac{1}{\lambda}}(\|v\|) \\
& + \frac{1}{k^{\frac{n-2}{n}}}(\int L_{g_0} vv d\mu_{g_0} - \frac{n+2}{n-2}\frac{r\alpha^{\frac{4}{n-2}}}{k}\int K\varphi^{\frac{4}{n-2}}v^2 d\mu_{g_0}) \\
= & \frac{r_\varphi}{k_\varphi^{\frac{n-2}{n}}} + o_{\frac{1}{\lambda}}(\|v\|) + \frac{c_n}{k^{\frac{n-2}{n}}}(\int \frac{L_{g_0}vv}{c_n} d\mu_{g_0} - n(n+2)\int \varphi^{\frac{4}{n-2}}v^2 d\mu_{g_0}).
\end{aligned}
$$

Thus for $\lambda \gg 1$ the constant functions $\pm\mathbb{1}$ become principally negative directions, see Figure 1, i.e. directions along which we may decrease energy, in contrast to the positive Yamabe case, where in a suitable Yamabe metric $L_{g_0} = -c_n\Delta_{g_0} + 1$ instead of, as here, $L_{g_0} = -c_n\Delta_{g_0} - 1$. That is, in the positive Yamabe case it is generally opportune to decrease v, while here it is not and we furthermore cannot reasonably increase v along $-\mathbb{1}$, since then $u > 0$ would lose positivity. Conversely $+\mathbb{1}$ becomes a preferred direction to decrease energy. On the other hand, if $u = \alpha_0 u_\infty + \alpha\varphi_{a,\lambda} + v$ with $u_\infty > 0$ these issues clearly do not occur.

4.3 Compactness and Existence

As discussed above, the case $u_\infty = 0$ does not occur, when flowing, while non zero weak limit blow-ups may principally occur for I or J. The latter case is ruled out under a smallness assumption on K.

Lemma 4.8. *If an A-B-inequality holds on $H^1(M)$, then the flow generated by*

$$\partial_t u = -(\frac{k}{r}R - K)u$$

is compact on X, provided $0 \le \sup_M K$ sufficiently small.

Proof. If blow-up occurs for J along a flow line for a sequence in time, i.e. for some $u_m = u_{t_m} \in X$, then from Proposition 4.4 and Lemma 4.3

$$
\begin{aligned}
0 > r_{u_m} = & \int L_{g_0} u_m u_m d\mu_{g_0} = \alpha_\infty^2 r_{u_\infty} + c_0 \sum_i \alpha_i^2 + o(1) \\
& = (\frac{k_\infty}{r_\infty})^{\frac{n-2}{2}}(r_{u_\infty} + \sum_i \frac{c_0(4n(n-1))^{\frac{n-2}{2}}}{K^{\frac{n-2}{2}}(a_i)}) + o(1).
\end{aligned}
\tag{25}
$$

On the other hand $r_{u_\infty} < 0$ by lower semicontinuity, whence for suitable $\alpha > 0$

$$\alpha u_\infty \in X \text{ and } 0 < \inf_X J \le J(\alpha u_\infty) = \frac{-k_{u_\infty}}{(-r_{u_\infty})^{\frac{n}{n-2}}} = (-r_{u_\infty})^{-\frac{2}{n}}.$$

Thus $r_{u_\infty} > -(\inf_X J)^{-\frac{n}{2}}$ in contradiction to (25) for $0 < \sup_M K \ll 1$. $\square$

Remark 4.9. *The required smallness of* $\sup_M K > 0$ *in Lemma 4.8 is determined by* $\inf_X J$. *On the other hand, since we assume the validity of an A-B-inequality, this infimum is lower bounded as follows. From*

$$1 = \|u\|_{L^{\frac{2n}{n-2}}} \leq Ar + B|k|^{\frac{n-2}{n}} \leq B|k|^{\frac{n-2}{n}}$$

we find

$$J = \frac{-k}{(-r)^{\frac{n}{n-2}}} \geq (-Br)^{-\frac{n}{n-2}}, \text{ while } r \geq -\|u\|^2_{L^2} \geq -c\|u\|^{\frac{n-2}{n}}_{L^{\frac{2n}{n-2}}} = -c$$

and so $\inf_X J \geq \gamma B^{-\frac{n}{n-2}}$, $\gamma = \gamma(M)$.

Similarly from Proposition 4.4 we have the following rough energy estimate.

Proposition 4.10. *Suppose, that an A-B-inequality holds on* $H^1(M)$. *Then for* $0 < \sup_M K$ *sufficiently small, if a flow line for* I *along a sequence in time blows up as* $u = \alpha u_\infty + \alpha^i \varphi_i + v$, *there holds*

$$I(u) \longrightarrow (E(u_\infty) + c_0 \sum (\frac{4n(n-1)}{K(a_{i_\infty})})^{\frac{n-2}{2}})^{\frac{2}{n}},$$

where

(i) $E(u_\infty) = -J^{-\frac{n-2}{2}}(u_\infty/\|u_\infty\|_{L^{\frac{2n}{n-2}}})$, *if* $u_\infty/\|u_\infty\|_{L^{\frac{2n}{n-2}}} \in X$

(ii) $E(u_\infty) = I^{\frac{n}{2}}(u_\infty/\|u_\infty\|_{L^{\frac{2n}{n-2}}})$, *if* $u_\infty/\|u_\infty\|_{L^{\frac{2n}{n-2}}} \in Y$.

Proof. From Proposition 4.4 we find

$$I(u) = \frac{\alpha^2 \int L_{g_0} u_\infty u_\infty d\mu_{g_0} + c_0 \sum \alpha_i^2}{(\alpha^{\frac{2n}{n-2}} \int K u_\infty^{\frac{2n}{n-2}} d\mu_{g_0} + c_1 K(a_i) \alpha_i^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} + o(1).$$

where

$$c_1 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^n} \text{ and } c_0 = 4n(n-1)c_1, \tag{26}$$

cf. Lemmata 4.2, 4.3. As $L_{g_0} u_\infty = K u_\infty^{\frac{n+2}{n-2}}$, see Proposition 4.4, we may write

$$I(u) = \frac{\alpha^2 k_{u_\infty} + c_0 \sum \alpha_i^2}{(\alpha^{\frac{2n}{n-2}} k_{u_\infty} + c_1 \sum K(a_i) \alpha_i^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} + o(1)$$

and obtain, again from Proposition 4.4,

$$\begin{aligned} I(u) &= \frac{(\frac{k_\infty}{r_\infty})^{\frac{n-2}{2}} k_{u_\infty} + c_0 \sum (\frac{4n(n-1)k_\infty}{r_\infty K(a_{i_\infty})})^{\frac{n-2}{2}}}{((\frac{k_\infty}{r_\infty})^{\frac{n}{2}} k_{u_\infty} + c_1 \sum K(a_{i_\infty})(\frac{4n(n-1)k_\infty}{r_\infty K(a_{i_\infty})})^{\frac{n}{2}})^{\frac{n-2}{n}}} + o(1) \\ &= \frac{k_{u_\infty} + c_0 \sum (\frac{4n(n-1)}{K(a_{i_\infty})})^{\frac{n-2}{2}}}{(k_{u_\infty} + c_1 \sum K(a_{i_\infty})(\frac{4n(n-1)}{K(a_{i_\infty})})^{\frac{n}{2}})^{\frac{n-2}{n}}} + o(1), \end{aligned}$$

which due to $c_0 = 4n(n-1)c_1$ simplifies to

$$I(u) = (k_{u_\infty} + c_0 \sum (\frac{4n(n-1)}{K(a_{i_\infty})})^{\frac{n-2}{2}})^{\frac{2}{n}} + o(1).$$

Note, that $r_{u_\infty} = k_{u_\infty}$ and thus $u_\infty / \|u_\infty\|_{L^{\frac{2n}{n-2}}}$ is either in X or Y, whence

(i) $J(u_\infty / \|u_\infty\|_{L^{\frac{2n}{n-2}}}) = (-k_{u_\infty})^{-\frac{2}{n-2}}$, if $k_{u_\infty} < 0$, i.e. $u_\infty / \|u_\infty\|_{L^{\frac{2n}{n-2}}} \in X$

(ii) $I(u_\infty / \|u_\infty\|_{L^{\frac{2n}{n-2}}}) = k_{u_\infty}^{\frac{2}{n}}$, if $k_{u_\infty} > 0$, i.e. $u_\infty / \|u_\infty\|_{L^{\frac{2n}{n-2}}} \in Y$.

The proof is complete. □

Note, that from Proposition 4.10 evidently the least possible blow-up energy I_∞ for I occurs in the single bubbling case $u = \alpha_\infty u_\infty + \alpha \varphi_{a,\lambda} + v$, and, when $K(a) = \max K > 0$. On the other hand, in order to show the existence of a second solution besides a global minimizer

$$u_0 \in X \quad \text{with} \quad \inf_X J = \min_X J = J(u_0),$$

which by Theorem 1 exists, we may argue by contradiction and assume, that u_0 is the only solution to

$$L_{g_0} u = \frac{1}{\beta} K u^{\frac{n+2}{n-2}}, \beta > 0.$$

Then in return by Proposition 4.4 the only possible simple blow-up scenario for I is a mixed bubbling of type $u = \alpha u_0 + \alpha_1 \varphi_{a_1,\lambda_1} + v$, where $u_\infty = u_0$.

Proposition 4.11. *Under the assumptions of Theorem 2 for suitable*

$$a \in \{K = \max K\},$$

choices $\alpha_0, \alpha_1 > 0$ *and* $u = \alpha_0 u_0 + \alpha_1 \varphi_{a,\lambda}$ *there holds*

$$I(u / \|u\|_{L^{\frac{2n}{n-2}}}) < I_\infty,$$

where I_∞ *denotes the least possible blow-up energy of* I.

In particular Theorem 2 follows, since by Proposition 4.11 we may consider a minimizing sequence $(u_k) \subset \{I < I_\infty\}$, which by virtue of Proposition 4.4 then leads to a minimizing sequence $(w_k) \subset \{I < I_\infty\}$ of solutions $\partial I(w_k) = 0$, which again by Proposition 4.4 converges to a minimizer of I.

Proof of Proposition 4.11. We consider

$$u = \gamma(\alpha_0 u_0 + \alpha_1 \varphi_{a_1,\lambda_1}) \in Y$$

for a suitable choice of $\gamma > 0$ and have

$$I(u) = \frac{\int L_{g_0}(\alpha_0 u_0 + \alpha_1 \varphi_{a_1,\lambda_1})(\alpha_0 u_0 + \alpha_1 \varphi_{a_1,\lambda_1}) d\mu_{g_0}}{(\int K(\alpha_0 u_0 + \alpha_1 \varphi_{a_1,\lambda_1})^{\frac{2n}{n-2}} d\mu_{g_0})^{\frac{n-2}{n}}} = \frac{N}{D}.$$

We first consider the case $3 \leq n \leq 5$. From Lemma 4.2 we then find

$$N = \alpha_0^2 r_{u_0} + 2\alpha_0\alpha_1 \int L_{g_0} u_0 \varphi_{a_1,\lambda_1} + \alpha_1^2 \int L_{g_0} \varphi_{a_1,\lambda_1} \varphi_{a_1,\lambda_1} d\mu_{g_0}$$
$$= \alpha_0^2 r_{u_0} + 2\alpha_0\alpha_1 \int L_{g_0} u_0 \varphi_{a_1,\lambda_1} + 4n(n-1)\alpha_1^2 \int \varphi_{a_1,\lambda_1}^{\frac{2n}{n-2}} d\mu_{g_0} + o(\frac{1}{\lambda_1^{\frac{n-2}{2}}})$$

and thus from (21)-(23) by expansion

$$N = \alpha_0^2 r_{u_0} + 2\alpha_0\alpha_1 \int L_{g_0} u_0 \varphi_{a_1,\lambda_1} + c_0\alpha_1^2 + o(\frac{1}{\lambda_1^{\frac{n-2}{2}}}),$$

where $c_0 = 4n(n-1)c_1$, cf. (26). On the other hand

$$L_{g_0} u_0 = \frac{1}{\beta} K u_0^{\frac{n+2}{n-2}} \quad \text{for some} \quad \beta > 0$$

and again we derive by expansion up to some $o(\frac{1}{\lambda_1^{\frac{n-2}{2}}})$

$$\begin{aligned} D^{\frac{n}{n-2}} = & \int K(\alpha_0^{\frac{2n}{n-2}} u_0^{\frac{2n}{n-2}} + \alpha_1^{\frac{2n}{n-2}} \varphi_{a_1,\lambda_1}^{\frac{2n}{n-2}} \\ & + \frac{2n\alpha_1\alpha_0^{\frac{n+2}{n-2}}}{n-2} u_0^{\frac{n+2}{n-2}} \varphi_{a_1,\lambda_1} + \frac{2n\alpha_0\alpha_1^{\frac{n+2}{n-2}}}{n-2} \varphi_{a_1,\lambda_1}^{\frac{n+2}{n-2}} u_0) d\mu_{g_0} \\ = & \beta\alpha_0^{\frac{2n}{n-2}} r_{u_0} + c_1\alpha_1^{\frac{2n}{n-2}} (K(a_1) + O(\frac{1}{\lambda_1^2})) + \frac{2nc_4}{n-2}\alpha_0\alpha_1^{\frac{n+2}{n-2}} \frac{u_0(a_1)}{\lambda_1^{\frac{n-2}{2}}} \\ & + \frac{2n}{n-2}\beta\alpha_1\alpha_0^{\frac{n+2}{n-2}} \int L_{g_0} u_0 \varphi_{a_1,\lambda_1} d\mu_{g_0}, \end{aligned} \tag{27}$$

where $c_4 = \int_{\mathbb{R}^n} \frac{dx}{(1+r^2)^{\frac{n+2}{2}}}$. Moreover

$$\frac{1}{\lambda_1^2} = o(\frac{1}{\lambda_1^{\frac{n-2}{2}}}), \quad \text{as} \quad 3 \leq n \leq 5,$$

and, since $\int L_{g_0} u_0 \varphi_{a_1,\lambda_1} d\mu_{g_0} = O(\frac{1}{\lambda_1^{\frac{n-2}{2}}})$, we obtain up to some $o(\frac{1}{\lambda_1^{\frac{n-2}{2}}})$

$$\begin{aligned} I(u) = & \frac{\alpha_0^2 r_{u_0} + c_0\alpha_1^2}{(\beta\alpha_0^{\frac{2n}{n-2}} r_{u_0} + c_1 K(a_1)\alpha_1^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} \\ & - \frac{2c_4(\alpha_0^2 r_{u_0} + c_0\alpha_1^2)\alpha_0\alpha_1^{\frac{n+2}{n-2}}}{(\beta\alpha_0^{\frac{2n}{n-2}} r_{u_0} + c_1 K(a_1)\alpha_1^{\frac{2n}{n-2}})^{\frac{n-2}{n}+1}} \frac{u_0(a_1)}{\lambda_1^{\frac{n-2}{2}}} \\ & + \frac{2\alpha_0\alpha_1 \int L_{g_0} u_0 \varphi_{a_1,\lambda_1} d\mu_{g_0}}{(\beta\alpha_0^{\frac{2n}{n-2}} r_{u_0} + c_1 K(a_1)\alpha_1^{\frac{2n}{n-2}})^{\frac{n-2}{n}}} [1 - \frac{(\alpha_0^2 r_{u_0} + c_0\alpha_1^2)\alpha_0^{\frac{4}{n-2}}\beta}{\beta\alpha_0^{\frac{2n}{n-2}} r_{u_0} + c_1 K(a_1)\alpha_1^{\frac{2n}{n-2}}}]. \end{aligned}$$

Choosing e.g.

$$\alpha_0^{\frac{4}{n-2}} = 1/\beta \;\text{ and }\; \alpha_1^{\frac{4}{n-2}} = \frac{c_0}{c_1 K(a_1)} = \frac{4n(n-1)}{K(a_1)},$$

the latter summand vanishes and, as $u_0 > 0$, we obtain

$$I(u) = (\beta^{-\frac{n-2}{2}} r_{u_0} + c_0(\frac{4n(n-1)}{K(a_i)})^{\frac{n-2}{2}})^{\frac{2}{n}} - O^+(\frac{1}{\lambda_1^{\frac{n-2}{2}}}).$$

Recalling $L_{g_0} u_0 = \frac{1}{\beta} K u_0^{\frac{n+2}{n-2}}$ and thus

$$J(\frac{u_0}{\|u_0\|_{L^{\frac{2n}{n-2}}}}) = \frac{-k_{u_0}}{(-r_{u_0})^{\frac{n}{n-2}}} = \frac{-\beta r_{u_0}}{(-r_{u_0})^{\frac{n}{n-2}}} = \frac{\beta}{(-r_{u_0})^{\frac{2}{n-2}}} = (-\beta^{-\frac{n-2}{2}} r_{u_0})^{-\frac{2}{n-2}},$$

we finally derive

$$I(u) = (-J^{-\frac{n-2}{2}}(u_0) + c_0(\frac{4n(n-1)}{K(a_1)}))^{\frac{2}{n}} - O^+(\frac{1}{\lambda_1^{\frac{n-2}{2}}}).$$

Recalling Proposition 4.10 (i), the assertion follows for $\lambda \gg 1$ in case $3 \leq n \leq 5$. The case $n \geq 6$ with local flatness around $a \in \{K = \max K\}$ and

$$\nabla^l K(a) = 0 \;\text{ for all }\; \frac{n-2}{2} \geq l \in \mathbb{N}$$

then follows as when $3 \leq n \leq 5$ upon replacing in (27)

$$O(\frac{1}{\lambda^2}) \;\text{ by }\; O(\frac{1}{\lambda^{l+1}}),$$

noticing $\frac{1}{\lambda^{l+1}} = o_{\frac{1}{\lambda}}(\frac{1}{\lambda^{\frac{n-2}{2}}})$ and recalling $H_{s,a} \equiv 0$, when expanding. □

Remark 4.12. *The argument of the proof reflects the question of existence or non existence of mixed type critical points at infinity [5], depending on the usual non degeneracy and flatness conditions. Although here pure type critical points at infinity do not exist, mixed type ones will generally occur, provided of course, that classical solutions exist in the first place.*

5 Appendix

First note, that we may write any $u = \tilde{\alpha} u_\infty + \tilde{\alpha}^i \varphi_{a_i,\lambda_i} + \tilde{v} \in V(u_\infty, q, \varepsilon_2)$ as

$$u = \alpha u_\infty + \alpha^i \varphi_{a_i,\lambda_i} + v \;\text{ with }\; v \perp_{L_{g_0}} span\{u_\infty, \varphi_{a_i,\lambda_i}\}$$

by solving in the (α, α_i)-variables the linear system

$$\begin{cases} \langle u, u_\infty \rangle_{L_{g_0}} & = & \alpha \langle u_\infty, u_\infty \rangle_{L_{g_0}} & + & \alpha^i \langle \varphi_{a_i,\lambda_i}, u_\infty \rangle_{L_{g_0}} \\ \langle u, \varphi_{a_1,\lambda_1} \rangle_{L_{g_0}} & = & \alpha \langle u_\infty, \varphi_{a_1,\lambda_1} \rangle_{L_{g_0}} & + & \alpha^i \langle \varphi_{a_i,\lambda_i}, \varphi_{a_1,\lambda_1} \rangle_{L_{g_0}} \\ \vdots & & \vdots & & \vdots \\ \langle u, \varphi_{a_q,\lambda_q} \rangle_{L_{g_0}} & = & \alpha \langle u_\infty, \varphi_{a_q,\lambda_q} \rangle_{L_{g_0}} & + & \alpha^i \langle \varphi_{a_i,\lambda_i}, \varphi_{a_q,\lambda_q} \rangle_{L_{g_0}} \end{cases}$$

which due to $|\langle u_\infty, u_\infty\rangle_{L_{g_0}}|, \langle \varphi_{a_i,\lambda_i}, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \geq c > 0$ and

$$\langle \varphi_{a_i,\lambda_i}, u_\infty\rangle_{L_{g_0}}, \langle L_{g_0}\varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} = o_{\varepsilon_2}(1) \text{ for } j \neq i$$

clearly admits a unique solution (α, α_i), whose dependence on (a_i, λ_i) we clarify as follows. From the above let us write some *fixed* $u \in V(u_\infty, q, \varepsilon_2)$ as

$$u = \alpha u_\infty + \alpha^i \varphi_{a_i,\lambda_i} + v,\ v \perp_{L_{g_0}} span\{u_\infty, \varphi_{a_i,\lambda_i}\}.$$

Varying in (a_i, λ_i) this representation of u, due to $\langle v, u_\infty\rangle_{L_{g_0}} = 0$ we have

$$\begin{aligned} 0 = \lambda_i\partial_{\lambda_i}\langle u, u_\infty\rangle_{L_{g_0}} &= \lambda_i\partial_{\lambda_i}\langle \alpha u_\infty + \alpha^j\varphi_{a_j,\lambda_j}, u_\infty\rangle_{L_{g_0}} \\ &= \lambda_i\partial_{\lambda_i}\alpha\langle u_\infty, u_\infty\rangle_{L_{g_0}} \\ &\quad + \lambda_i\partial_{\lambda_i}\alpha^j\langle \varphi_{a_j,\lambda_j}, u_\infty\rangle_{L_{g_0}} + \alpha_i\langle \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}, u_\infty\rangle_{L_{g_0}}, \end{aligned} \tag{28}$$

whence

$$\lambda_i\partial_{\lambda_i}\alpha\langle u_\infty, u_\infty\rangle_{L_{g_0}} = o_{\varepsilon_2}(\sum_{i,j}|\lambda_i\partial_{\lambda_i}\alpha_j|) + O(\frac{1}{\lambda_i^{\frac{n-2}{2}}}). \tag{29}$$

Likewise for $i \neq j$ and due to $\langle v, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} = 0$

$$\begin{aligned} 0 = \lambda_i\partial_{\lambda_i}\langle u, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} &= \lambda_i\partial_{\lambda_i}\langle \alpha u_\infty + \alpha^k\varphi_{a_k,\lambda_k}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} \\ &= \lambda_i\partial_{\lambda_i}\alpha_j\langle \varphi_{a_j,\lambda_j}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} + \alpha_i\langle \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} \\ &\quad + \lambda_i\partial_{\lambda_i}\alpha\langle u_\infty, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} + \sum_{j\neq k=1}^{q}\lambda_i\partial_{\lambda_i}\alpha_k\langle \varphi_{a_k,\lambda_k}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}}, \end{aligned} \tag{30}$$

whence

$$\lambda_i\partial_{\lambda_i}\alpha_j\langle \varphi_{a_j,\lambda_j}, \varphi_{a_j,\lambda_j}\rangle_{L_{g_0}} = o_{\varepsilon_2}(|\lambda_i\partial_{\lambda_i}\alpha| + \sum_{j\neq k=1}^{q}|\lambda_i\partial_{\lambda_i}\alpha_k|) + O(\varepsilon_{i,j}), \tag{31}$$

cf. Lemma 4.3. Similarly we compute on one hand

$$\begin{aligned} \lambda_i\partial_{\lambda_i}\langle u, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} &= \langle u, \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\ &= \langle \alpha u_\infty + \alpha^j\varphi_{a_j,\lambda_j} + v, \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}, \end{aligned}$$

while on the other due $\langle v, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} = 0$

$$\begin{aligned} \lambda_i\partial_{\lambda_i}\langle u, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} = &\ \lambda_i\partial_{\lambda_i}\langle \alpha u_\infty + \alpha^j\varphi_{a_j,\lambda_j}, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\ &= \langle \alpha u_\infty + \alpha^j\varphi_{a_j,\lambda_j}, \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\ &\quad + \lambda_i\partial_{\lambda_i}\alpha_i\langle \varphi_{a_i,\lambda_i}, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} + \alpha_i\langle \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\ &\quad + \lambda_i\partial_{\lambda_i}\alpha\langle u_\infty, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} + \sum_{i\neq j=1}^{q}\lambda_i\partial_{\lambda_i}\alpha_j\langle \varphi_{a_j,\lambda_j}, \varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}, \end{aligned}$$

and so we conclude by subtracting

$$
\begin{aligned}
&\lambda_i\partial_{\lambda_i}\alpha_i\langle\varphi_{a_i,\lambda_i},\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\
&\quad= -\lambda_i\partial_{\lambda_i}\alpha\langle u_\infty,\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} - \sum_{i\neq j=1}^{q}\lambda_i\partial_{\lambda_i}\alpha_j\langle\varphi_{a_j,\lambda_j},\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} \\
&\qquad -\alpha_i\langle\varphi_{a_i,\lambda_i},\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} + \langle v,\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}},
\end{aligned} \tag{32}
$$

which we estimate as

$$
\begin{aligned}
\lambda_i\partial_{\lambda_i}\alpha_i\langle\varphi_{a_i,\lambda_i},\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} = {} & o_{\varepsilon_2}(|\lambda_i\partial_{\lambda_i}\alpha| + \sum_{i\neq j=1}^{q}|\lambda_i\partial_{\lambda_i}\alpha_j|) \\
& + \langle v,\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}} - \alpha_i\langle\varphi_{a_i,\lambda_i},\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}.
\end{aligned} \tag{33}
$$

Since

$$\langle u_\infty,u_\infty\rangle_{L_{g_0}} = r_{u_\infty}\neq 0 \ \text{ and } \ \langle\varphi_{a_i,\lambda_i},\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}\neq o(1),$$

we then find from (29), (31) and (33)

$$
\begin{aligned}
\sum_i(|\lambda_i\partial_{\lambda_i}\alpha| + \sum_j|\lambda_i\partial_{\lambda_i}\alpha_j|) = {} & O(\sum_i\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j}\varepsilon_{i,j} + \|v\|) \\
& + O(\sum_i|\langle\varphi_{a_i,\lambda_i},\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}|) \\
= {} & o_{\varepsilon_2}(1)
\end{aligned} \tag{34}
$$

and therefore, if $Cond_n$ holds at all a_i, $i=1,\ldots,q$, then

$$\sum_i(|\lambda_i\partial_{\lambda_i}\alpha| + \sum_j|\lambda_i\partial_{\lambda_i}\alpha_j|) = O(\sum_i\frac{1}{\lambda_i^{\frac{n-2}{2}}} + \sum_{i\neq j}\varepsilon_{i,j} + \|v\|), \tag{35}$$

as follows from Lemma 4.2, see Lemma 4.3 (iv). Finally and by the same reasoning (34) and (35) hold for $\lambda_i\partial_{\lambda_i}$ replaced by ∇_{a_i}/λ_i.

With this in mind the proof of Lemma 4.6 decomposes into three steps, first showing that the infimum is attained in the interior, secondly that the resulting minimum is unique and finally justifying the estimates yielding sufficient *almost*-orthogonalities for the v-part.

Proof of Lemma 4.6. Consider some fixed

$$u = \hat{\alpha}u_\infty + \hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i} + \hat{v}\in V(u_\infty,q,\varepsilon_2).$$

Then at some $(\alpha,\alpha_i,\lambda_i,a_i)\in A_u(u_\infty,q,2\varepsilon_1)$

$$\inf_{(\tilde{a}_i,\tilde{\lambda}_i)\in\Pi_{(a_i,\lambda_i)}A_u(u_\infty,q,2\varepsilon_1)}\int u^{\frac{4}{n-2}}|u-\tilde{\alpha}u_\infty-\tilde{\alpha}^i\varphi_{\tilde{a}_i,\tilde{\lambda}_i}|^2 d\mu_{g_0}$$

is attained and, since $\|\hat{v}\| \leq \varepsilon_2$, there holds

$$\begin{aligned} o_{\varepsilon_2}(1) = & \int u^{\frac{4}{n-2}} |u - \alpha u_\infty - \alpha^i \varphi_{a_i,\lambda_i}|^2 d\mu_{g_0} \\ = & \, o_{\varepsilon_2}(1) + \int |\hat{\alpha} u_\infty + \hat{\alpha}^i \varphi_{\hat{a}_i,\hat{\lambda_i}}|^{\frac{4}{n-2}} \\ & \qquad (\hat{\alpha} u_\infty - \alpha u_\infty + \hat{\alpha}^i \varphi_{\hat{a}_i,\hat{\lambda}_i} - \alpha^j \varphi_{a_j,\lambda_j})^2 d\mu_{g_0}, \end{aligned}$$

whence after possibly relabelling the indices $j = 1, \ldots, q$ for

$$A = |\alpha - \hat{\alpha}|, \; A_i = |\alpha_i - \hat{\alpha}_i|, \; L_i = |\frac{\hat{\lambda}_i}{\lambda_i} - 1|, \; D_i = \sqrt{\lambda_i \hat{\lambda}_i d^2_{g_0}(a_i, \hat{a}_i)}$$

we necessarily have

$$A + \sum_i (A_i + L_i + D_i) = o_{\varepsilon_2}(1). \tag{36}$$

In particular, since

$$v = u - \alpha u_\infty - \alpha^i \varphi_{a_i,\lambda_i} = \hat{\alpha} u_\infty - \alpha u_\infty + \hat{\alpha}^i \varphi_{\hat{a}_i,\hat{\lambda}_i} - \alpha^i \varphi_{a_i,\lambda_i} + \hat{v}, \tag{37}$$

we find $\|v\| = o_{\varepsilon_2}(1)$ and thus for ε_2 sufficiently small with $o_{\varepsilon_2}(1) < \varepsilon_1$, that

$$(\alpha, \alpha_i, \lambda_i, a_i) \in A_u(u_\infty, q, o_{\varepsilon_2}(1)) \subset A_u(u_\infty, q, \varepsilon_1) \subset A_u(u_\infty, q, 2\varepsilon_1),$$

i.e. the infimum is attained as an interior minimum. To show uniqueness as a critical point of an interior minimizer, assume there are two, say

$$u = \hat{\alpha} u_\infty + \hat{\alpha}^i \varphi_{\hat{a}_i,\hat{\lambda_i}} + \hat{v} = \alpha u_\infty + \alpha^i \varphi_{a_i,\lambda_i} + v \in V(u_\infty, q, \varepsilon_1). \tag{38}$$

Then (36) holds, by construction we have

$$\hat{v} \perp_{L_{g_0}} span\{u_\infty, \varphi_{\hat{a}_i,\hat{\lambda}_i}\}, \; v \perp_{L_{g_0}} span\{u_\infty, \varphi_{a_i,\lambda_i}\}$$

and due to minimality, taking the derivatives in $(\hat{\lambda}_i, \hat{a}_i)$ or (λ_i, a_i),

$$\begin{aligned} 0 = & \int u^{\frac{4}{n-2}} \hat{v} \hat{\lambda}_i \partial_{\hat{\lambda}_i} (\hat{\alpha} u_\infty + \hat{\alpha}^j \varphi_{\hat{a}_j,\hat{\lambda}_j}) d\mu_{g_0} \\ = & \int u^{\frac{4}{n-2}} v \lambda_i \partial_{\lambda_i} (\alpha u_\infty + \alpha^j \varphi_{a_i,\lambda_i}) d\mu_{g_0} \end{aligned} \tag{39}$$

and

$$\begin{aligned} 0 = & \int u^{\frac{4}{n-2}} \hat{v} \frac{\nabla_{\hat{a}_i}}{\hat{\lambda}_i} (\hat{\alpha} u_\infty + \hat{\alpha}^j \varphi_{\hat{a}_j,\hat{\lambda}_j}) d\mu_{g_0} \\ = & \int u^{\frac{4}{n-2}} v \frac{\nabla_{a_i}}{\lambda_i} (\alpha u_\infty + \alpha^j \varphi_{a_j,\lambda_j}) d\mu_{g_0}. \end{aligned} \tag{40}$$

Uniqueness then follows from

$$A+\sum_i(A_i+L_i+D_i)=o_{\varepsilon_1}(A+\sum_i(A_i+L_i+D_i)). \tag{41}$$

To show (41), we start with using $\langle\hat{v},u_\infty\rangle_{L_{g_0}}=0=\langle v,u_\infty\rangle_{L_{g_0}}=0$ to get

$$\begin{aligned}
0=&\int L_{g_0}u_\infty\hat{v}d\mu_{g_0}=\int L_{g_0}u_\infty(u-\hat{\alpha}u_\infty-\hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i})d\mu_{g_0}\\
=&\int L_{g_0}u_\infty(\alpha u_\infty+\alpha^i\varphi_{a_i,\lambda_i}-\hat{\alpha}u_\infty-\hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i})d\mu_{g_0}\\
=&(\alpha-\hat{\alpha})\int L_{g_0}u_\infty u_\infty d\mu_{g_0}+o_{\varepsilon_1}(\sum_i(A_i+L_i+D_i)),
\end{aligned}\tag{42}$$

and likewise from $\langle\hat{v},\varphi_{\hat{a}_i,\hat{\lambda}_i}\rangle_{L_{g_0}}=0=\langle v,\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}=0$ we obtain

$$\begin{aligned}
0=&\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}\hat{v}d\mu_{g_0}=\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}(u-\hat{\alpha}u_\infty-\hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i})d\mu_{g_0}\\
=&\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}(\alpha u_\infty+\alpha^i\varphi_{a_i,\lambda_i}+v-\hat{\alpha}u_\infty-\hat{\alpha}^i\varphi_{\hat{a}_i,\hat{\lambda}_i})d\mu_{g_0}\\
=&(\alpha_j-\hat{\alpha}_j)\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}j}\varphi_{\hat{a}_j,\hat{\lambda}_j}d\mu_{g_0}\\
&+\alpha_j\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}(\varphi_{a_j,\lambda_j}-\varphi_{\hat{a}_j,\hat{\lambda}_j})d\mu_{g_0}\\
&+o_{\varepsilon_1}(A+\sum_{j\neq i=1}^{q}A_i+\sum_{i=1}^{q}(L_i+D_i))\\
=&(\alpha_j-\hat{\alpha}_j)\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}\varphi_{\hat{a}_j,\hat{\lambda}_j}d\mu_{g_0}+o_{\varepsilon_1}(A+\sum_i(A_i+L_i+D_i)),
\end{aligned}\tag{43}$$

where we made use of Lemma 4.3 and to treat the term

$$\int L_{g_0}\varphi_{\hat{a}_j,\hat{\lambda}_j}(\varphi_{a_j,\lambda_j}-\varphi_{\hat{a}_j,\hat{\lambda}_j})d\mu_{g_0}$$

also of (36) and

$$\varphi_{\hat{a}_i,\hat{\lambda}_i}-\varphi_{a_i,\lambda_i}=\begin{pmatrix}\frac{1}{\lambda_i}\nabla_{a_i}\\ \lambda_i\partial_{\lambda_i}\end{pmatrix}\varphi_{a_i,\lambda_i}\begin{pmatrix}\lambda_i(\hat{a}_i-a_i)\\ \frac{\hat{\lambda}_i}{\lambda_i}-1\end{pmatrix}+o_{L_i+D_i}(L_i+D_i) \tag{44}$$

as well as

$$\lambda\partial_\lambda\int_{\mathbb{R}^n}\delta_{a,\lambda}^{\frac{2n}{n-2}}=\frac{1}{\lambda}\nabla_a\int_{\mathbb{R}^n}\delta_{a,\lambda}^{\frac{2n}{n-2}}=0\quad\text{for}\quad\delta_{a,\lambda}=(\frac{\lambda}{1+\lambda^2|\cdot-a|^2})^{\frac{n-2}{2}}.$$

Combining (42) and (43) we conclude

$$A+\sum_i A_i=o_{\varepsilon_1}(A+\sum_i(A_i+L_i+D_i)). \tag{45}$$

We proceed by employing (39). First note, that due to (37)

$$\|v-\hat{v}\| = O(A+\sum_i (A_i+L_i+D_i)). \tag{46}$$

Secondly (28), (30), (32) and (34) hold for both representations of u in (38), whence by subtracting the corresponding versions of (28), (30) and (32) we find

$$\begin{aligned}\sum_i (|\lambda_i\partial_{\lambda_i}\alpha-\hat{\lambda}_i\partial_{\hat{\lambda}_i}\hat{\alpha}| + \sum_j |\lambda_i\partial_{\lambda_i}\alpha_j-\hat{\lambda}_i\partial_{\hat{\lambda}_i}\hat{\alpha}_j|) \\ = O(A+\sum_i (A_i+L_i+D_i)).\end{aligned} \tag{47}$$

Thus, when subtracting in (39), the estimates (34), (46) and (47) yield

$$\int u^{\frac{4}{n-2}}(\alpha_i\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}v-\hat{\alpha}_i\hat{\lambda}_i\partial_{\hat{\lambda}_i}\varphi_{\hat{a}_i,\hat{\lambda}_i}\hat{v})d\mu_{g_0} = o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i)),$$

whence, as $|\alpha_i-\hat{\alpha}_i| = A_i$ and $\|v\|, \|\hat{v}\| = o_{\varepsilon_1}(1)$, and due to

$$\|\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}-\hat{\lambda}_i\partial_{\hat{\lambda}_i}\varphi_{\hat{a}_i,\hat{\lambda}_i}\| = O(L_i+D_i)$$

we obtain

$$\int u^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}(v-\hat{v})d\mu_{g_0} = o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i)).$$

Recalling (38) and estimating as above, we then get from (44)

$$\begin{aligned}o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i)) &= \int u^{\frac{4}{n-2}}\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}(\varphi_{\hat{a}_i,\hat{\lambda}_i}-\varphi_{a_i,\lambda_i})d\mu_{g_0} \\ &= (\frac{\hat{\lambda}_i}{\lambda_i}-1)\int u^{\frac{4}{n-2}}|\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}|^2 d\mu_{g_0} + o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i)).\end{aligned}$$

By by simple expansions of $u^{\frac{4}{n-2}}$ for $u=\alpha u_\infty+\alpha^i\varphi_{a_i,\lambda_i}+v$ we thus obtain

$$L_i = o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i))$$

and by analogous arguments, employing (40) instead of (39), also

$$D_i = o_{\varepsilon_1}(A+\sum_i (A_i+L_i+D_i)).$$

Combining these estimates with (45) establishes (41) and therefore the desired uniqueness of an interior minimizer as a critical point. We finally turn to proving the *almost*-orthogonalities (i) and (ii). From (34) and (39) we find

$$\begin{aligned}&\int u^{\frac{4}{n-2}}v\lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}d\mu_{g_0} \\ &\quad = O(\sum_i \frac{1}{\lambda_i^{n-2}} + \sum_{i\neq j}\varepsilon_{i,j}^2 + \|v\|^2) + O(\sum_i |\langle\varphi_{a_i,\lambda_i}, \lambda_i\partial_{\lambda_i}\varphi_{a_i,\lambda_i}\rangle_{L_{g_0}}|^2)\end{aligned}$$

and by simple expansions of $u^{\frac{4}{n-2}}$ for $u = \alpha u_\infty + \alpha^i \varphi_{a_i,\lambda_i} + v$, that also

$$\int \varphi_{a_i,\lambda_i}^{\frac{4}{n-2}} \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i} v d\mu_{g_0} = O(\sum_i \frac{1}{\lambda_i^{n-2}} + \sum_{i \neq j} \varepsilon_{i,j}^2 + \|v\|^2) + O(\sum_i |\langle \varphi_{a_i,\lambda_i}, \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i} \rangle_{L_{g_0}}|^2).$$

Now assertion (i) of Lemma 4.2 follows from writing

$$\langle \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i}, v \rangle_{L_{g_0}} = 4n(n-1) \int \varphi_{a_i,\lambda_i}^{\frac{4}{n-2}} \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i} v d\mu_{g_0} + \int (\lambda_i \partial_{\lambda_i} L_{g_0} \varphi_{a_i,\lambda_i} - 4n(n-1) \lambda_i \partial_{\lambda_i} \varphi_{a_i,\lambda_i}^{\frac{n+2}{n-2}}) v d\mu_{g_0},$$

while assertion (ii) follows analogously, relying on (40) instead of (39). □

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