DERIVATION OF EFFECTIVE THEORIES FOR THIN 3D NONLINEARLY ELASTIC RODS WITH VOIDS

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Abstract. We derive a dimension-reduction limit for a three-dimensional rod with material voids by means of Γ-convergence. Hereby, we generalize the results of the purely elastic setting [58] to a framework of free discontinuity problems. The effective one-dimensional model features a classical elastic bending-torsion energy, but also accounts for the possibility that the limiting rod can be broken apart into several pieces or folded. The latter phenomenon can occur because of the persistence of voids in the limit, or due to their collapsing into a discontinuity of the limiting deformation or its derivative. The main ingredient in the proof is a novel rigidity estimate in varying domains under vanishing curvature regularization, obtained in [33].

1. Introduction

A fundamental question in continuum mechanics is the rigorous derivation of lower dimensional theories for plates, shells, and rods in various energy scaling regimes, starting from three-dimensional models of nonlinear elasticity. Although this question has received considerable attention [8, 9], early derivations were typically based on some a priori *ansatzes*, often leading to theories which were not consistent with each other. The last decades, however, have witnessed a remarkable progress in the rigorous derivation of effective energies for thin elastic objects via variational methods, based on a fundamental cornerstone: the celebrated rigidity estimate by G. Friesecke, R.D. James, and S. Müller [41]. Ever since its appearance, this rigidity result has had numerous applications in dimension-reduction problems providing a thorough understanding of thin elastic materials. We refer the reader to the by far nonexhaustive list [1, 2, 20, 27, 35, 36, 37, 40, 41, 42, 49, 52, 53, 54, 58, 59, 60, 61, 62, 66, 67] for references.

On the contrary, beyond the purely elastic regime, when one is interested in the behavior of materials which might have defects and impurities such as *plastic slips*, *cracks*, or *stress-induced voids*, the situation is far less-well understood. The goal of this article is to advance the mathematical understanding of thin materials with voids. This corresponds to the investigation of energies that are driven by the competition between elastic and surface energies of perimeter type. Models of this form are gathered under the term *stress driven rearrangement instabilities* (SDRI), see [13, 15, 21, 28, 43, 45, 46, 50, 51, 64, 69, 70] for some mathematical and physical literature on the subject.

We start with a short overview of the literature on dimension reduction in settings beyond elasticity. Concerning plasticity, we refer the reader, e.g., to [17, 25, 26, 55, 57]. For models in brittle fracture [31], there are several results on brittle plates and shells in a linear setting [3, 5, 11, 44]. In the nonlinear framework, instead, the theory is mainly restricted to static and evolutionary models in the membrane regime [4, 10, 16]. The only result in a smaller energy regime appears to be [68] for the case of a two-dimensional thin brittle beam. In the limit of vanishing thickness, the author obtains an effective *Griffith-Euler-Bernoulli* energy defined on the midline of the possibly fractured beam, accounting also for jump discontinuities of the limiting deformation and its derivative. At the core of the arguments in [68] lies a suitable generalization of [41], namely a *quantitative piecewise geometric rigidity theorem for SBD functions* [38]. As to date this result is available only in two dimensions, the generalization of dimension-reduction results to three-dimensional fracture is still impeded. Let us however mention that analogous rigidity results in higher dimensions are available in models for nonsimple materials [32], where the elastic energy depends additionally on the second gradient of the deformation.

In the setting of material voids, a recent result [65] deals with the derivation of a plate theory in the bending energy regime. There the analysis is limited to voids with restrictive assumptions on their geometry, still allowing to resort to the classical rigidity theorem of [41]. Our goal is to derive a related result for thin rods without restriction on the void geometry. The cornerstone of our approach is a novel rigidity result in the realm of SDRI-models [33], based on a curvature regularization of the surface term. We now describe our setting in more detail.

We consider a three-dimensional thin rod with reference configuration $\Omega_h = (0, L) \times hS \subset \mathbb{R}^3$ of thickness $0 < h \ll 1$, for a cross section $S \subset \mathbb{R}^2$. For simplicity of the exposition we focus on the case $S = (-\frac{1}{2}, \frac{1}{2})^2$, but mention that adaptations to more general geometries are possible. From a variational viewpoint, models describing the formation of material voids in thin rods fall into the framework of *free discontinuity problems* [6], and typical energies take the form

$$\mathcal{F}^h_{\mathrm{el,per}}(v, E) := \int_{\Omega_h \setminus \overline{E}} W(\nabla v)\,\mathrm{d}x + \beta_h \int_{\partial E \cap \Omega_h} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^2\,. \tag{1.1}$$

Here, $E \subset \Omega_h$ represents the (sufficiently regular) void set within an elastic rod with reference configuration $\Omega_h \subset \mathbb{R}^3$, and v is the corresponding elastic deformation. The first part of (1.1) represents the nonlinear elastic energy with density W (see Section 2 for details), whereas the second one depends on a parameter $\beta_h > 0$ and on a possibly anisotropic density φ evaluated at the outer unit normal ν_E to $\partial E \cap \Omega_h$. For purely expository reasons, we will restrict ourselves to the isotropic case, i.e., $\varphi(\cdot) = |\cdot|_2$.

Regarding the energy scaling, at a heuristic level, it is well known that elastic energies of the order h^4 correspond to bending and torsion, keeping the midline unstretched, cf. [58]. At the same time, the surface area of voids completely separating the rod is of order h^2. Now, depending on the choice of β_h, different limiting models can be expected: the case $\beta_h \gg h^2$ will result in a purely elastic rod model, whereas the case $\beta_h \ll h^2$ will result in a model of purely brittle fracture. The critical regime $\beta_h \sim h^2$ is the most interesting and mathematically most challenging case, for the elastic and surface contributions are of the same order.

Therefore, we set $\beta_h := h^2$ from now on. Rescaling the energy in (1.1) by h^{-4}, the natural attempt would be to rigorously derive a corresponding effective one-dimensional theory by means of Γ-convergence [14, 23]. However, the presence of a priori unprescribed voids in the model hinders the use of the classical rigidity result of [41]. Indeed, the voids might possibly exhibit extremely complicated geometries such as densely packed thin spikes or microscopically small components with small surface measure on different length scales, see Figure 1.

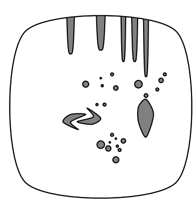

Figure 1. Densely packed thin spikes and microscopically small components leading to loss of rigidity. For simplicity, the figure illustrates and example in dimension two.

As a remedy, motivated by our work in [33], we introduce a *curvature regularization* of the form

$$\mathcal{F}^h_{\mathrm{curv}}(E) := h^2\kappa_h \int_{\partial E\cap\Omega_h} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2\,, \tag{1.2}$$

where $\boldsymbol{A}$ denotes the second fundamental form of $\partial E \cap \Omega_h$ and κ_h satisfies (2.5), which allows in particular for $\kappa_h \to 0^+$ as $h \to 0^+$ at a sufficiently slow rate. The presence of such an extra Willmore-type energy penalization allows to employ the *piecewise rigidity estimate* [33, Theorem 2.1] in the analysis. It is a *singular perturbation* for the void set E and not for the deformation v, i.e., no higher-order gradient of v is involved in the model. We refer the reader to our recent work [34], where a related discrete model is studied and an additional explanation for the presence of a microscopic analogue of the term in (1.2) is given, see [34, Subsection 2.5]. We also mention that curvature regularizations are widely used in the mathematical and physical literature of SDRI models, including the description of elastically stressed thin films or material voids, see [7, 29, 30, 47, 48, 63, 69]. In spite of possible modeling relevance, we emphasize that we include the curvature contribution in our model only for mathematical reasons as a regularization term. In particular, it does not affect the effective limiting problem.

The total energy of a pair (v, E) is then given by the sum of the two terms in (1.1) and (1.2), i.e.,

$$\mathcal{F}^h(v,E) = \mathcal{F}^h_{\mathrm{el,per}}(v,E) + \mathcal{F}^h_{\mathrm{curv}}(E)\,. \tag{1.3}$$

(We set $\beta_h = h^2$ and $\varphi(\nu) \equiv 1$ for all $\nu \in \mathbb{S}^2$.) The main result of this contribution is then Theorem 2.3, where we show that the rescaled energies $(h^{-4}\mathcal{F}^h(\cdot,\cdot))_{h>0}$ Γ-converge (in an appropriate topology) to an effective one-dimensional functional that takes the form

$$\frac{1}{2}\int_{(0,L)\setminus I} \mathcal{Q}_2(R^TR')\,\mathrm{d}x_1 + \mathcal{H}^0(\partial I\cap(0,L)) + 2\mathcal{H}^0\big((J_y\cup J_R)\setminus\partial I\big)\,. \tag{1.4}$$

Here, $I \subset (0, L)$ denotes a union of finitely many intervals in $(0, L)$ and represents the void part in the limiting one-dimensional rod. The deformation $y\colon (0, L) \to \mathbb{R}^3$ is an isometric piecewise $W^{2,2}$-regular curve that represents the deformed rod. The rotation field $R\colon (0, L) \to SO(3)$ whose first column is the velocity y' represents the *Frenet frame* with respect to y. The elastic part of the limiting energy corresponds to the one identified in the purely elastic setting [58]: it is quadratic in terms of the skew-symmetric tensor R^TR' which encodes the information for the *curvature and torsion* of the curve y. The associated quadratic form $\mathcal{Q}_2$ is defined through the quadratic form $D^2W(I)$ of linearized elasticity via a suitable minimization problem, see (2.17) for details. The second term in (1.4) accounts for the presence of voids by counting their endpoints ($\mathcal{H}^0$ stands for the counting measure in $\mathbb{R}$). The last term therein takes into account the fact that, in the limit, voids might collapse exactly into discontinuity points of the limiting y or its Frenet frame R, corresponding to cracks or kinks of the limiting rod, respectively. Accordingly, these discontinuity points should be counted twice in the energy.

Let us highlight the relation to the result in [65], where a similar model of *Blake-Zisserman type* (cf. [12, 18]) for elastic plates with voids in the Kirchhoff bending energy regime is obtained. First, in [65] plates are considered, whereas we treat the case of rods. We decided to present our approach based on the model (1.3) first for a dimension reduction from 3D-to-1D to avoid some technicalities arising in the 3D-to-2D analysis. The latter, however, can be performed as well, and is the subject of a forthcoming work, both in the Kirchhoff [41] and the von Kármán [42] regime. The fundamental difference between our work and [65] concerns the assumptions on the void set. Whereas we allow for voids with general geometry employing a mild curvature regularization, [65] is based on specific restrictive assumptions on the void geometry, namely the so-called *ψ-minimal droplet assumption*, cf. [65, Equation (6)]. This can be interpreted as an L^∞-diverging bound on the curvature of the boundary of the voids. In our setting, the curvature regularization term in (1.3) can be thought of as imposing an L^2-diverging bound on the curvature: firstly, this allows the void set to concentrate at arbitrarily small scales (independently of h) and, secondly, allows the boundary of the void set to consist of a diverging (with h) number of connected components, see Example 2.6. Our more general model comes at the expense of the necessity of more sophisticated geometric rigidity results [33] compared to [41].

1.1. **Organization of the paper and proof strategy.** The paper is organized as follows. In Section 2 we introduce our model and state the main compactness and Γ-convergence results, i.e., Theorems 2.1 and 2.3, respectively. We also include some comments on the limiting model and discuss possible boundary value problems. Section 3 contains the core of our paper by deriving a blockwise Sobolev approximation of sequences $(v_h, E_h)_{h>0}$ with

$$\sup_{h>0} h^{-4}\mathcal{F}^h(v_h, E_h) < +\infty\,.$$

We perform a careful enlargement of the voids E_h according to [33, Proposition 2.8], as well as an appropriate modification of the deformations. This is the content of Propositions 3.1 and 3.2 stated at the beginning of Section 3, where we modify the deformations v_h and their gradients ∇v_h on a small part of the rod, such that the new deformations are actually Sobolev in big blocks of the rod Ω_h with a good control on their elastic energy. Moreover, the modification is done in such a way that the jump height of the new sequence along the entire rod is suitably controlled as well as producing the correct jump points of the limiting deformation and its curvature-torsion tensor.

The main technical tools to obtain these modifications are the *piecewise rigidity estimate* [33, Theorem 2.1] and a *Korn inequality for functions with small jump set* [19], applied on long cuboids that partition Ω_h. More precisely, splitting the rod Ω_h into $\sim h^{-1}$ many long cuboids of length $\sim h$, we focus on those cuboids where the perimeter of the enlarged void is locally not large enough to produce macroscopic fracture, see (3.47)–(3.49). In these cuboids, by means of *isoperimetric arguments* and our piecewise rigidity estimate, we obtain large in volume sets in which slight modifications of v_h are approximately $W^{1,2}$-rigid in terms of the local elastic energy. As we believe that the isoperimetric inequalities may be of independent interest, we state and prove them in arbitrary space dimension, see Subsection 3.2.

Although [33, Theorem 2.1] provides an optimal estimate only in terms of the symmetrized gradient, a use of the Korn-Poincaré inequality for functions with small jump set [19] allows us to upgrade our estimate to the full gradient in all but finitely many cuboids. This leads to an optimal estimate for the difference between the rigid motions in terms of the local elastic energy, again in all but finitely many adjacent cuboids, see Proposition 3.11 and Corollary 3.12 in Subsection 3.3. In Subsection 3.4, we eventually construct the global blockwise Sobolev modifications and give the proofs of Propositions 3.1 and 3.2.

Based on these preparations, the rest of the paper is more standard and the results of the elastic case [58] can be employed directly. Section 4 is devoted to the proof of compactness (Theorem 2.1) and Section 5 to the proof of the Γ-liminf inequality of Theorem 2.3. The proof of the Γ-limsup inequality is given in Section 6 by exhibiting a recovery sequence $(v_h, E_h)_{h>0}$. Here, we use the corresponding recovery sequence from [58] for the deformations, and we construct the voids E_h with planar interfaces in order to approximate the one-dimensional limiting void sets and the jump points.

We also remark that, from a technical viewpoint, our proof strategy provides – to our view – a simplified alternative to obtain the Γ-liminf inequality compared to the methods used in [68], which were based on delicate interpolation and difference quotients estimates. The latter were dictated by the fact that, in the same fashion as the result [33], the two-dimensional piecewise rigidity estimate in SBD [38, Theorem 2.1] used in [68] provides an optimal estimate in terms of the elastic energy *only* for symmetrized gradients. Therefore, the standard difference quotients method used in [58] was not directly applicable. Our method instead leads to a blockwise Sobolev replacement with the aid of the *Korn inequality for functions with small jump set* [19]. This actually enables us to use directly the results of [58]. We emphasize that in this regard our approach is general: given any kind of geometric rigidity result delivering a sharp control for symmetrized gradients, e.g. also the result in [32], our techniques carry directly over and allow to work with blockwise Sobolev replacements.

1.2. **Notation.** We close the introduction with some basic notation. Given $U \subset \mathbb{R}^3$ open, we denote by $\mathcal{P}(U)$ the collection of subsets of finite perimeter in U. Given $E \in \mathcal{P}(U)$, for any $s \in [0,1]$ we denote by E^s the set of points with 3-dimensional density s with respect to E, and by $\partial^* E$ its essential boundary, see [6, Definition 3.60]. The family of sets of finite perimeter on a one-dimensional interval $(0,L)$ will be simply denoted by $\mathcal{P}(0,L)$. We also denote by $\mathcal{A}_{\mathrm{reg}}(U)$ the collection of all open sets $E \subset U$ such that $\partial E \cap U$ is a two-dimensional C^2-surface in $\mathbb{R}^3$. Surfaces and functions of C^2-regularity will be called C^2-regular in the following. For $E \in \mathcal{A}_{\mathrm{reg}}(U)$ we denote by $\boldsymbol{A}$ the second fundamental form of $\partial E \cap U$, i.e., $|\boldsymbol{A}| = \sqrt{\kappa_1^2 + \kappa_2^2}$, where κ_1 and κ_2 are the corresponding principal curvatures. By ν_E we indicate the outer unit normal to $\partial E \cap U$. For every $a, b \in \mathbb{R}$ we denote $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$.

For $p \in [1,\infty]$ and $d, k \in \mathbb{N}$ we denote by $L^p(U;\mathbb{R}^d)$ and $W^{k,p}(U;\mathbb{R}^d)$ the standard Lebesgue and Sobolev spaces, respectively. Partial derivatives of a function $f\colon U \to \mathbb{R}^3$ will be denoted by $(f_{,i})_{i=1,2,3}$. Given measurable sets A, B, we write χ_A for the characteristic function of A, $A \subset\subset B$ if $\overline{A} \subset B$, and $\mathrm{dist}_{\mathcal{H}}(A,B)$ for the Hausdorff distance between A and B. For $d, k \in \mathbb{N}$, we denote by $\mathcal{L}^d$ and $\mathcal{H}^k$ the d-dimensional Lebesgue measure and the k-dimensional Hausdorff measure, respectively.

We set $\mathbb{R}_+ := [0,+\infty)$. By id we denote the identity mapping on $\mathbb{R}^3$ and by $\mathrm{Id} \in \mathbb{R}^{3\times 3}$ the identity matrix. For each $F \in \mathbb{R}^{3\times 3}$ we let

$$\mathrm{sym}(F) := \frac{1}{2}\left(F + F^T\right),$$

and we also define

$$SO(3) := \{F \in \mathbb{R}^{3\times 3}\colon F^T F = \mathrm{Id},\, \det F = 1\}.$$

Moreover, we denote by $\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and $\mathbb{R}^{3\times 3}_{\mathrm{skew}}$ the space of symmetric and skew-symmetric matrices, respectively. We further write $\mathbb{S}^2 := \{\nu \in \mathbb{R}^3\colon |\nu| = 1\}$. For $\sigma > 0$, we denote by T_σ the linear transformation in $\mathbb{R}^3$ with matrix representation being given by

$$T_\sigma := \mathrm{diag}(1,\sigma,\sigma) \tag{1.5}$$

with respect to the canonical basis $\{e_1, e_2, e_3\}$.

We use standard notation for SBV-functions, cf. [6, Chapter 4] for the definition and a detailed presentation of the properties of this space. In particular, for a function $u \in SBV(U;\mathbb{R}^d)$, we write ∇u for the approximate gradient, J_u for the jump set, and $u^\pm$ for the one-sided traces on J_u. We also use the notation $[u] := u^+ - u^-$ for the corresponding jump height. We consider the space

$$SBV^2(U;\mathbb{R}^d) := \Big\{u \in SBV(U;\mathbb{R}^d)\colon \int_U |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cap U) < +\infty\Big\}.$$

In dimension one, given $a < b \in \mathbb{R}$ and $d \in \mathbb{N}$, the space $SBV^2((a,b);\mathbb{R}^d)$ coincides with the space P-$W^{1,2}((a,b);\mathbb{R}^d)$ of piecewise $W^{1,2}$-Sobolev functions, which consists of those $Y \in L^1((a,b);\mathbb{R}^d)$ for which there exists a partition

$$a =: t_0 < t_1 < \cdots < t_m < t_{m+1} := b \quad \text{such that} \quad Y \in W^{1,2}((t_{i-1},t_i);\mathbb{R}^d)\ \forall i = 1,\ldots,m+1.$$

The jump set of Y is precisely the minimal set $J_Y = \{t_1,\ldots,t_m\}$ with the above property. By taking an appropriate representative, we may then assume that Y is uniformly continuous on $\{(t_{i-1},t_i)\}_{i=1,\ldots,m+1}$ and $Y(t_i^\pm)$ are the limits of $Y(t)$ as $t \to t_i^\pm$.

Analogously, for $k \in \mathbb{N}$, we define $P\text{-}W^{k,2}((a,b);\mathbb{R}^d)$ as the space of $Y \in L^1((a,b);\mathbb{R}^d)$ for which there exists $\{a =: t_0 < t_1 < \cdots < t_{m+1} := b\}$ such that $Y \in W^{k,2}((t_{i-1},t_i);\mathbb{R}^d)$ $\forall i = 1,\dots,m+1$. For Y in this space, the minimal set $\{t_1,\dots,t_m\}$ with the above property is $\bigcup_{l=0}^{k-1} J_{Y^{(l)}}$, where $Y^{(l)}$ denotes the l-th derivate of Y.

2. The model and the main results

Model in the reference domain: We denote the reference configuration of the thin rod by

$$\Omega_h := (0,L) \times (-\tfrac{h}{2},\tfrac{h}{2})^2 \subset \mathbb{R}^3\,, \tag{2.1}$$

where $L > 0$ is a macroscopic parameter describing the length of its midline, and $0 < h \ll L$ denotes its infinitesimal thickness. For a fixed large constant $M \gg 1$, the set of *admissible pairs* of function and set is given by

$$\mathcal{A}_h := \big\{(v,E)\colon \quad E \in \mathcal{A}_{\rm reg}(\Omega_h),\ v \in W^{1,2}(\Omega_h \setminus \overline{E};\mathbb{R}^3)\,,\ v|_E \equiv \mathrm{id}\,,\ \|v\|_{L^\infty(\Omega_h)} \le M\big\}\,. \tag{2.2}$$

The third condition in (2.2) is for definiteness only. The last one is merely of technical nature to ensure compactness. At the same time, it is also justified from a physical point of view, for it corresponds to the assumption that the material under investigation is confined in a bounded region. For each pair $(v,E) \in \mathcal{A}_h$, we consider the energy

$$\mathcal{F}^h(v,E) := \int_{\Omega_h\setminus\overline{E}} W(\nabla v)\,\mathrm{d}x + h^2\mathcal{H}^2(\partial E \cap \Omega_h) + h^2\kappa_h \int_{\partial E\cap\Omega_h} |\boldsymbol{A}|^2\,\mathrm{d}\mathcal{H}^2\,. \tag{2.3}$$

Here, the first and second term correspond to the *elastic* and the *surface energy* of perimeter type, while the third term is a *curvature regularization* of Willmore-type, where $\boldsymbol{A}$ denotes the second fundamental form of $\partial E \cap \Omega_h$ and κ_h is a suitable parameter. The factor h^2 in front of the surface terms ensures that the elastic and the surface energy are of same order for our choice of the bending regime, where the elastic energy per unit volume is of order h^2. We refer to the introduction for more details.

The function $W\colon \mathbb{R}^{3\times3} \to \mathbb{R}_+$ in (2.3) represents the *stored elastic energy density*, satisfying the usual assumptions of nonlinear elasticity. Altogether, we suppose that $W \in C^0(\mathbb{R}^{3\times3};\mathbb{R}_+)$ satisfies

$$\begin{aligned}
&\text{(i) Frame indifference: } W(RF) = W(F) \text{ for all } R \in SO(3) \text{ and } F \in \mathbb{R}^{3\times3}\,,\\
&\text{(ii) Single energy-well structure: } \{W = 0\} \equiv SO(3)\,,\\
&\text{(iii) Regularity: } W \text{ is } C^2 \text{ in a neighborhood of } SO(3)\,,\\
&\text{(iv) Coercivity: There exists } c > 0 \text{ such that for all } F \in \mathbb{R}^{3\times3} \text{ it holds that}\\
&\qquad W(F) \ge c\,\mathrm{dist}^2(F,SO(3))\,.
\end{aligned} \tag{2.4}$$

Our choice of an isotropic surface energy is for simplicity only and can be generalized, as we briefly explain in Remark 2.4 below. As for the parameter $\kappa_h > 0$ in the curvature regularization, we require

$$\kappa_h h^{-52/25} \to +\infty \quad \text{ as } h \to 0\,. \tag{2.5}$$

We point out that (2.5) is a technical assumption and chosen for simplicity rather than optimality. Its role is connected to the application of suitable rigidity results [33, 19] and will become apparent along the proof, see in particular (3.40).

Rescaling of the model: As it is customary in dimension-reduction problems, we perform a change of variables to a fixed reference domain: recalling (1.5), we rescale our variables and set

$$\Omega := \Omega_1\,, \quad V := \{x \in \Omega : (x_1, hx_2, hx_3) \in E\} = T_{1/h}(E)\,. \tag{2.6}$$

We also rescale the deformations accordingly, by defining $y\colon \Omega \to \mathbb{R}^3$ via

$$y(x) := y(x_1, x_2, x_3) := v(x_1, hx_2, hx_3)\,. \tag{2.7}$$

We rescale the energy by the factor h^4 and set

$$\mathcal{E}^h(y, V) := h^{-4}\mathcal{F}^h(v, E)\,, \tag{2.8}$$

where the pair (y, V) is related to (v, E) via (2.6)–(2.7). Here, one factor h^2 corresponds to the change of volume and the other factor h^2 corresponds to the average elastic energy per unit volume, reflecting our choice of the bending energy regime.

For the corresponding rescaled gradients, we will use the notation

$$\nabla_h y(x) := \Big(\partial_1 y, \frac{1}{h}\partial_2 y, \frac{1}{h}\partial_3 y\Big)(x) = \nabla v(x_1, hx_2, hx_3)\,. \tag{2.9}$$

Therefore, by a change of variables we find

$$\mathcal{E}^h(y, V) = h^{-2}\int_{\Omega\setminus\overline{V}} W(\nabla_h y(x))\,\mathrm{d}x + \int_{\partial V\cap\Omega} \big|\big(\nu^1_V(z), h^{-1}\nu^2_V(z), h^{-1}\nu^3_V(z)\big)\big|\,\mathrm{d}\mathcal{H}^2(z) + \mathcal{E}^h_{\mathrm{curv}}(V)\,, \tag{2.10}$$

where $\nu_V(z) := \big(\nu^1_V(z), \nu^2_V(z), \nu^3_V(z)\big)$ denotes the outer unit normal to $\partial V \cap \Omega$ at the point z. (For the rescaling of the perimeter part, one can test with smooth functions and use the divergence theorem.) Here, the term $\mathcal{E}^h_{\mathrm{curv}}(V)$ denotes the curvature contribution for the rescaled set V, for which we refrain from performing the change of variables explicitly.

In view of (1.5) and (2.2), the space of rescaled admissible pairs (deformations-voids) is given by

$$\hat{\mathcal{A}}_h := \{(y, V)\colon V \in \mathcal{A}_{\mathrm{reg}}(\Omega)\,,\ y \in W^{1,2}(\Omega\setminus\overline{V};\mathbb{R}^3)\,,\ y|_V \equiv T_h(\mathrm{id})\,,\ \|y\|_{L^\infty(\Omega)} \le M\}\,. \tag{2.11}$$

Limiting model: The limiting energy will be defined on the space

$$\begin{aligned}\mathcal{A} := \big\{\big((y|\,d_2|\,d_3), I\big)\colon (y|\,d_2|\,d_3) \in SBV^2_{\mathrm{isom}}(0, L)\,,\ y|_I(x_1) \equiv x_1\,,\ (y_{,1}|\,d_2|\,d_3)|_I \equiv \mathrm{Id}\,,\\ \|y\|_{L^\infty(\Omega)} \le M\,,\ I \in \mathcal{P}(0, L)\big\}\,,\end{aligned} \tag{2.12}$$

where, recalling the definition of P-$W^{k,2}$ in Subsection 1.2, we define

$$SBV^2_{\mathrm{isom}}(0, L) := \begin{aligned}&\Big\{(y|\,d_2|\,d_3) \in \big(P\text{-}W^{2,2}\times P\text{-}W^{1,2}\times P\text{-}W^{1,2}\big)\big((0, L);\mathbb{R}^{3\times 3}\big) \text{ with}\\ &R := (y_{,1}|\,d_2|\,d_3) \in SO(3) \text{ a.e. in } (0, L)\Big\}\,.\end{aligned} \tag{2.13}$$

By a slight abuse of notation, for triplets $(\bar{y}|\,\bar{d}_2|\,\bar{d}_3)\colon \Omega \to \mathbb{R}^{3\times 3}$ we will also use the notation $(\bar{y}|\,\bar{d}_2|\,\bar{d}_3) \in SBV^2_{\mathrm{isom}}(0, L)$ if and only if

$$(\bar{y}|\,\bar{d}_2|\,\bar{d}_3)(x) = (y|\,d_2|\,d_3)(x_1) \text{ for all } x \in \Omega, \text{ for some } (y|\,d_2|\,d_3) \in SBV^2_{\mathrm{isom}}(0, L)\,. \tag{2.14}$$

In a similar fashion, we will write

$$\bar{R}(x) := (y_{,1}|\,d_2|\,d_3)(x_1) \text{ for all } x \in \Omega\,. \tag{2.15}$$

With these definitions, for each $((y|\,d_2|\,d_3), I) \in \mathcal{A}$, the limiting one-dimensional energy of Blake-Zisserman type (cf. [12, 18, 65] for analogous models in different settings) is defined as

$$\mathcal{E}^0\big((y|\,d_2|\,d_3), I\big) := \frac{1}{2}\int_{(0,L)\setminus I} \mathcal{Q}_2(R^T R_{,1})\,\mathrm{d}x_1 + \mathcal{H}^0\big(\partial^* I \cap (0, L)\big) + 2\mathcal{H}^0\big((J_y \cup J_R)\setminus \partial^* I\big)\,. \tag{2.16}$$

Here, R is defined as in (2.13), and the quadratic form $\mathcal{Q}_2\colon \mathbb{R}^{3\times 3}_{\rm skew} \to \mathbb{R}_+$ is defined through a minimization problem as

$$\mathcal{Q}_2(A) := \min_{a\in W^{1,2}\left((-\frac{1}{2},\frac{1}{2})^2;\mathbb{R}^3\right)} \int_{(-\frac{1}{2},\frac{1}{2})^2} \mathcal{Q}_3\left(A\begin{pmatrix}0\\x_2\\x_3\end{pmatrix}\Bigg|\,\alpha_{,2}\Bigg|\,\alpha_{,3}\right)\mathrm{d}x_2\,\mathrm{d}x_3 \tag{2.17}$$

for all $A\in\mathbb{R}^{3\times3}_{\rm skew}$, where, for every $G\in\mathbb{R}^{3\times3}$,

$$\mathcal{Q}_3(G) := D^2W(\mathrm{Id})[G,G] \tag{2.18}$$

is the corresponding quadratic form of linearized elasticity. Note that, as R belongs to $SO(3)$, $R^TR_{,1}$ is skew symmetric, and thus the elastic energy in (2.16) is well defined. Moreover, due to (2.4), $\mathcal{Q}_3$ vanishes on $\mathbb{R}^{3\times3}_{\rm skew}$ and is strictly positive definite on $\mathbb{R}^{3\times3}_{\rm sym}$.

As mentioned also in the introduction, the limiting one-dimensional model features the classical bending-torsion term derived in [58] and two surface terms related to the presence of voids. The first part corresponds to the energy contribution of the limiting void I, whereas the second part is associated to discontinuities or kinks of the deformation, represented by J_y and J_R, respectively. This term is due to the fact that voids may collapse to single points and hence enters the energy with a factor 2, see Figure 2.

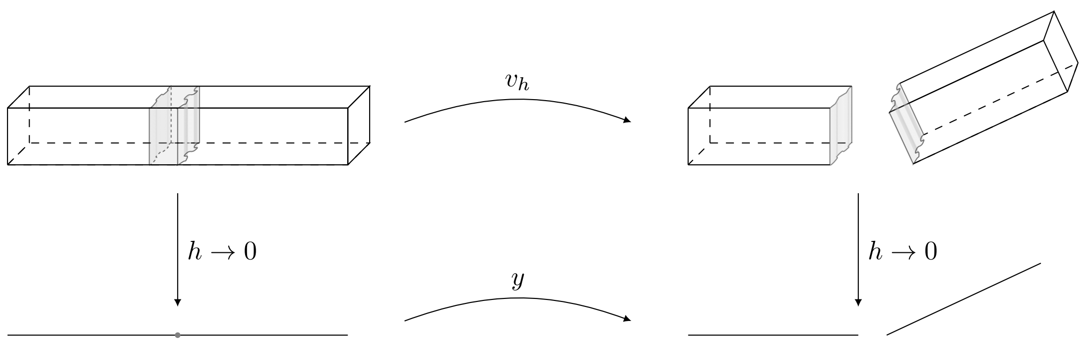

Figure 2. A collapsing void leading to a discontinuity for J_y and J_R.

Main results: Keeping in mind (2.6), (2.11), (2.12), and setting

$$V_I := I\times(-1/2,1/2)^2 \in \mathcal{P}(\Omega) \quad \text{for } I\in\mathcal{P}(0,L)\,,$$

our main results in this paper are summarized as follows.

Theorem 2.1. *(Compactness) Let $(h_j)_{j\in\mathbb{N}}\subset(0,\infty)$ with $h_j\searrow 0$ and $(y_{h_j},V_{h_j})\in\hat{\mathcal{A}}_{h_j}$ be such that*

$$\sup_{j\in\mathbb{N}}\mathcal{E}^{h_j}(y_{h_j},V_{h_j}) < +\infty\,. \tag{2.19}$$

Then, there exists $\big((y|\,d_2|\,d_3),I\big)\in\mathcal{A}$ such that up to a non-relabeled subsequence,

$$\begin{aligned} &\text{(i)} \quad \chi_{V_{h_j}} \longrightarrow \chi_{V_I} \ \ \textit{in } L^1(\Omega)\,,\\ &\text{(ii)} \quad y_{h_j} \longrightarrow \bar{y} \ \textit{in } L^1(\Omega;\mathbb{R}^3)\,,\\ &\text{(iii)} \quad \chi_{\Omega\setminus V_{h_j}}\nabla_{h_j}y_{h_j} \rightharpoonup \chi_{\Omega\setminus V_I}\bar{R} \ \ \textit{weakly in } L^2(\Omega;\mathbb{R}^{3\times3})\,, \end{aligned} \tag{2.20}$$

where $\bar{y}$ and $\bar{R}$ are meant here with the conventions made in (2.14)–(2.15).

Definition 2.2. We say that $(y_{h_j},V_{h_j}) \xrightarrow{\tau} \big((y|\,d_2|\,d_3),I\big)$ if and only if (2.20) holds.

Since (2.11) implies that $\sup_{j\in\mathbb{N}}\|y_{h_j}\|_{L^\infty(\Omega)}\le M$, the convergence in (2.20)(ii) actually holds in $L^p(\Omega;\mathbb{R}^3)$ for every $p\in[1,+\infty)$. We are now ready to state the main Γ-convergence result.

Theorem 2.3. (*Γ-convergence*) *Let $(h_j)_{j\in\mathbb{N}}\subset(0,\infty)$ with $h_j\searrow 0$. The sequence of functionals $(\mathcal{E}^{h_j})_{j\in\mathbb{N}}$ $\Gamma(\tau)$-converges to the functional $\mathcal{E}^0$, i.e., the following two inequalities hold true.*

(i) (*Γ-liminf inequality*) *Whenever $(y_{h_j},V_{h_j})\xrightarrow{\tau}\big((y|\,d_2|\,d_3),I\big)$, then*

$$\mathcal{E}^0\big((y|\,d_2|\,d_3),I\big)\le\liminf_{j\to+\infty}\mathcal{E}^{h_j}(y_{h_j},V_{h_j})\,. \tag{2.21}$$

(ii) (*Γ-limsup inequality*) *For every $\big((y|\,d_2|\,d_3),I\big)\in\mathcal{A}$ there exists a sequence $(y_{h_j},V_{h_j})_{j\in\mathbb{N}}$ with $(y_{h_j},V_{h_j})\in\hat{\mathcal{A}}_{h_j}$ for each $j\in\mathbb{N}$, such that $(y_{h_j},V_{h_j})\xrightarrow{\tau}\big((y|\,d_2|\,d_3),I\big)$, and*

$$\limsup_{j\to+\infty}\mathcal{E}^{h_j}(y_{h_j},V_{h_j})\le\mathcal{E}^0\big((y|\,d_2|\,d_3),I\big)\,. \tag{2.22}$$

Remark 2.4 (Extensions and variants)**.** (i) We could consider more general perimeter energies of the form

$$\beta_h\int_{\partial E\cap\Omega_h}\varphi(\nu_E)\,\mathrm{d}\mathcal{H}^2\,,$$

where $\lim_{h\to0}(h^{-2}\beta_h)=\beta>0$ and φ is a norm in $\mathbb{R}^3$. For simplicity of the exposition, we have chosen $\beta_h=h^2$ and φ to be the standard Euclidean norm in $\mathbb{R}^3$. The general case is completely analogous in its treatment, up to a prefactor $\varphi(e_1)$ appearing in the last two terms in (2.16).

(ii) Regarding the choice of the curvature regularization, let us mention that, in view of the results in [33, Theorem 2.1], any choice of the form

$$h^2\kappa_h\int_{\partial E\cap\Omega_h}|\boldsymbol{A}|^q\,\mathrm{d}\mathcal{H}^2$$

with $q\ge2$ would be possible, up to adjusting the condition for κ_h in (2.5) (which will then depend also on q). The choice $q\ge2$, however, is essential, see [33, Lemma 2.11 and Example 2.12]. For simplicity, we have chosen the exponent $q=2$, which corresponds to a curvature regularization of Willmore type.

(iii) Let us also remark that clamped boundary conditions and body forces can be included into the Γ-convergence statement. We refrain here from giving the details, but refer the interested reader to [68, Corollaries 2.4, 2.5] for results in this direction.

Remark 2.5 (Discussion on the limiting model)**.** The limiting model includes the Griffith-Euler-Bernoulli theory for brittle beams derived in [68], which corresponds to an energy of the form

$$\mathcal{E}_{\mathrm{GEB}}(y):=\int_0^L|\kappa_y|^2\,\mathrm{d}x_1+\mathcal{H}^0(J_y\cup J_{y'})\,, \tag{2.23}$$

where $y\in P\text{-}W^{2,2}((0,L);\mathbb{R}^2)$ is an arclength parametrization and κ_y denotes the corresponding curvature. This follows from our model for $I=\emptyset$, deformations $y=(y_1,y_2,0)$, and Frenet frames $R=(y'|\,d_2|\,d_3)$ with $d_2=(-y_2',y_1',0)$ and $d_3=(0,0,1)$. This functional is related to a one-dimensional version of the Blake-Zisserman model [12], where y is scalar and κ_y is replaced by y''. Our model (with $I=\emptyset$) and the model in (2.23) have a similar response to boundary value problems. In particular, prescribing boundary conditions $y(0)$ and $y(L)$ which are not compatible with smooth elements in $\mathcal{A}$, e.g. if $|y(0)-y(L)|>L$, we necessarily have $J_y\cup J_{y'}\ne\emptyset$. Even if the boundary conditions can be achieved by smooth elements in $\mathcal{A}$, cracks may be favorable whenever all curves connecting $y(0)$ and $y(L)$ have large curvature, e.g. if $|y(0)-y(L)|\ll L$.

Adding a volume constraint of the form $h^{-2}\mathcal{L}^3(E_h)\to0$ in our 3d model, we can easily recover (2.16) without voids, i.e., $I=\emptyset$. If we allow for voids in the limit, the interpretation of the model is a bit more delicate, as the non-smoothness could be introduced through cracks, kinks, or voids. In the extreme case, even everything could be covered by void. To avoid the latter phenomenon, one could add a volume constraint of the form $\mathcal{L}^3(E_h) \le \alpha h^2$ for $\alpha \in (0, L)$ or introduce body forces (both of which can be incorporated in the Γ-convergence result). With a body force of type

$$\int_{(0,L)\setminus I} f(x) \cdot y(x_1) \, \mathrm{d}x_1 \,,$$

cracks may be energetically favorable compared to voids. In fact, with lateral stretched boundary conditions (e.g. $y(0) = 0$, $y(L) = (L', 0, 0)$ with $L' > L$) and $f \equiv -e_3$ (corresponding to gravity force), one can check that it is convenient not to introduce void but a crack, with J_y close to 0 or L.

Example 2.6 (L^2 vs L^∞-bound on the curvature)**.** Recall (2.3) and (2.8). The following example shows that we can exhibit configurations $(v_h, E_h) \in \mathcal{A}_h$ with

$$\sup_{h>0} h^{-4} \mathcal{F}^h(v_h, E_h) < +\infty \,,$$

where E_h consists of balls which concentrate on arbitrarily small scales (independently of h), and whose number is diverging (with h). As a preparation, let $r > 0$ and observe that for $E := B_r \subset\subset \Omega_h$, where B_r is a ball of radius r, the second fundamental form of ∂E satisfies $|\mathbf{A}| = \sqrt{2} r^{-1}$, and the surface energy contribution is

$$h^{-2} \left(\mathcal{H}^2(\partial B_r) + \kappa_h \int_{\partial B_r} |\mathbf{A}|^2 \, \mathrm{d}\mathcal{H}^2 \right) = 4\pi \big(h^{-2} r^2 + 2h^{-2} \kappa_h \big) \,. \tag{2.24}$$

In the setting of [65] (cf. Remark 3.1 therein) and for void sets consisting of a disjoint union of balls compactly contained in Ω_h, an L^∞-bound of the form $|\mathbf{A}| \le Ch^{-1}$ implies a lower bound of order h for the radius of each of those balls. The energy bound $h^{-2}\mathcal{H}^2(\partial E_h \cap \Omega_h) \le C$ on the perimeter energy implies now that such voids can only consist of finitely many disjoint balls whose cardinality depends only on the a priori L^∞-bound and the energy bound.

Instead, in our setting we can construct an example of a sequence of voids with the aforementioned requirements. We perform this construction for

$$\kappa_h \to 0 \quad \text{such that} \quad h^{-2} \kappa_h \to 0 \,. \tag{2.25}$$

This rate of convergence is possible as κ_h only needs to satisfy (2.5), take e.g. $\kappa_h = h^{51/25}$. Let $N_h \in \mathbb{N}$, let $(x_{i,h})_{i=1}^{N_h} \subset \mathbb{R}^3$, $(r_{i,h})_{i=1}^{N_h} \subset (0, +\infty)$, be such that $B_{r_{i,h}}(x_{i,h}) \subset\subset \Omega_h$ for all i, with $r_{i,h} \le h N_h^{-1/2}$ for all i, $B_{r_{i,h}}(x_{i,h}) \cap B_{r_{j,h}}(x_{j,h}) = \emptyset$ for $i \ne j$. Set $E_h := \bigcup_{i=1}^{N_h} B_{r_{i,h}}(x_{i,h})$ and $v_h = \mathrm{id}$ (for definiteness only as this is not the point of this example). Then, by (2.24), we have

$$h^{-4} \mathcal{F}^h(v_h, E_h) = 4\pi \sum_{i=1}^{N_h} h^{-2} r_{i,h}^2 + 8\pi N_h h^{-2} \kappa_h \,.$$

By (2.25) and $r_{i,h} \le h N_h^{-1/2}$ for all i, we can suitably choose $N_h \to +\infty$ to obtain a sequence of void sets $(E_h)_{h>0}$ with equi-bounded surface energy, that has a diverging number of components with no (not even diverging with h) L^∞-control on the curvature.

3. Blockwise Sobolev modification of deformations

This section is devoted to two preliminary propositions which are vital in the proofs of the compactness Theorem 2.1 in Section 4 and the Γ-liminf inequality of Theorem 2.3 in Section 5. Our reasoning relies on the approximation of a sequence of deformations with equibounded energy by mappings which are blockwise Sobolev. This will allow us to use the results of [58, Theorems 2.1

and 3.1] in subsets of the domain where the modified functions are weakly differentiable. In order to control the surface contributions due to voids correctly, our arguments will also include estimates on the jump set of the blockwise Sobolev approximations. The main gain of our construction is the fact that, in contrast to the jump set of original deformations $(v_h)_{h>0}$, we are able to control the geometry of the jump set of the newly constructed sequence, namely the new jump set is contained in finitely many vertical planes. That is the reason why we call this modification *blockwise* Sobolev approximation.

In this and in the following sections, we will use the continuum subscript $h > 0$ instead of the sequential subscript notation $(h_j)_{j\in\mathbb{N}}$ for notational convenience. Before we can state the main results of this section, we need to collect some more notation. Recalling the definition of Ω_h in (2.1), for $\rho \in (0,1)$ we define the slightly smaller reference domain

$$\Omega_{h,\rho} := (\rho h, L-\rho h) \times (-\tfrac{h}{2} + \tfrac{1}{2}\rho h, \tfrac{h}{2} - \tfrac{1}{2}\rho h)^2 \,. \tag{3.1}$$

For $T \in \mathbb{N}$, $T \gg 1$, we cover the domain Ω_h with *T-cuboids*, namely

$$Q_h(i) := \big[(i-1)Th, iTh\big) \times (-\tfrac{h}{2}, \tfrac{h}{2})^2 \,,$$

for $i = 1, \ldots, N := \lfloor \frac{L}{Th} \rfloor + 1$, and let

$$\mathcal{Q}_h := \{Q_h(i) \colon i = 1, \ldots, N\} \,. \tag{3.2}$$

While we will eventually send $\rho \to 0$ in Section 4, T is fixed throughout the paper. Therefore, we refrain from including T in the notation of $\mathcal{Q}_h$. For $x \in \mathbb{R}^3$, $l \geq 0$ we introduce the *stripes*

$$S_h^l(x) := (x-l, x+l) \times (-\tfrac{h}{2}, \tfrac{h}{2})^2 \,. \tag{3.3}$$

Similarly to (3.1), for $\rho \in (0,1)$ we also introduce the smaller stripes

$$S_{h,\rho}^l(x) := (x-l+\rho l, x+l-\rho l) \times (-\tfrac{h}{2} + \tfrac{1}{2}\rho h, \tfrac{h}{2} - \tfrac{1}{2}\rho h)^2 \,. \tag{3.4}$$

For every measurable set $K \subset \mathbb{R}^3$ and $\gamma > 0$ we introduce the localized surface energy

$$\mathcal{G}^\gamma_{\mathrm{surf}}(E;K) := \mathcal{H}^2(\partial E \cap K) + \gamma \int_{\partial E \cap K} |\boldsymbol{A}|^2 \,\mathrm{d}\mathcal{H}^2 \,, \tag{3.5}$$

where for later purposes we use a general parameter γ in place of κ_h. Then, given an infinitesimal sequence $(\epsilon_h)_{h>0} \subset (0,+\infty)$, we define the total rescaled energy by

$$\mathcal{G}^h(v,E) := \frac{1}{h^2\epsilon_h}\int_{\Omega_h \setminus \overline{E}} W(\nabla v)\,\mathrm{d}x + \frac{1}{h^2}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E;\Omega_h) \tag{3.6}$$

for $(v,E) \in \mathcal{A}_h$. Note that for $\epsilon_h = h^2$ we have $\mathcal{G}^h(v,E) = h^{-4}\mathcal{F}^h(v,E)$ with $\mathcal{F}^h$ as defined in (2.3). In this section, we treat a more general scaling ϵ_h of the elastic energy in order to distinguish more clearly the scalings related to the volume of Ω_h and that of the average elastic energy per unit volume.

In the next proposition, we assume that $T \gg 1$ is chosen big enough, see (3.39) for details. We also recall the role of $M \gg 1$ in (2.2).

Proposition 3.1 (Blockwise Sobolev approximation of deformations)**.** *Let $(\epsilon_h)_{h>0} \subset (0,+\infty)$ be a sequence satisfying $\limsup_{h\to 0} \epsilon_h h^{-2} < +\infty$, and let $0 < \rho \leq \rho_0$ for some universal $\rho_0 > 0$. Then, there exists a constant $C := C(T,M) > 0$ such that for every sequence $(v_h,E_h)_{h>0}$ with $(v_h,E_h) \in \mathcal{A}_h$ and*

$$\sup_{h>0} \mathcal{G}^h(v_h,E_h) < +\infty \,, \tag{3.7}$$

there exist sequences $(w_h)_{h>0}$ *and* $(R_h)_{h>0}$ *with* $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ *and* $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times 3})$ *satisfying the following properties:*

$$\begin{aligned}
&\text{(i)} \quad \|w_h\|_{L^\infty(\Omega_{h,\rho})} \le C, \quad \|R_h\|_{L^\infty(\Omega_{h,\rho})} \le C\,,\\
&\text{(ii)} \quad J_{w_h} \cup J_{R_h} \subset \Omega_{h,\rho} \cap \bigcup_{Q_h \in \mathcal{Q}_{v_h}} \partial Q_h \quad \text{for some } \mathcal{Q}_{v_h} \subset \mathcal{Q}_h \text{ with } \#\mathcal{Q}_{v_h} \le C\,,\\
&\text{(iii)} \quad h^{-2}\mathcal{L}^3\big(\{x \in \Omega_{h,\rho} \colon w_h(x) \neq v_h(x)\}\big) \to 0, \quad h^{-2}\mathcal{L}^3\big(\Omega_{h,\rho} \cap \big\{|\nabla v_h - R_h| > \theta_h\big\}\big) \to 0\,,\\
&\text{(iv)} \quad \int_{\Omega_{h,\rho}} \mathrm{dist}^2(\nabla w_h, SO(3))\,\mathrm{d}x \le Ch^2\epsilon_h\,, \quad \int_{\Omega_{h,\rho}} |\nabla R_h|^2\,\mathrm{d}x \le C\epsilon_h\,,
\end{aligned} \tag{3.8}$$

where $(\theta_h)_{h>0} \subset (0,+\infty)$ *is a sequence with* $\theta_h \to 0$ *and* $\theta_h \epsilon_h^{-1/2} \to \infty$.

The result allows us to approximate v_h by a blockwise Sobolev function w_h by changing the mapping on an asymptotically vanishing portion of the volume, see (3.8)(iii). The important point is that the elastic energy of w_h is still of the same order, see (3.8)(iv). In the next sections we will also need some control on the second gradient of v_h, which a priori might not exist. This is achieved by a second sequence of functions $(R_h)_{h>0}$ which has bounded derivative in L^2 and suitably approximates ∇v_h, see again (3.8)(iii),(iv).

The approximation also delivers a control on the jump set, see (3.8)(ii), which corresponds to the fact that in the limit we expect functions which jump at most a finite number of times, see (2.12), (2.13). The most delicate part in the derivation of the Γ-liminf inequality for the surface energy in (2.16) is the correct factor 2 in front of $\mathcal{H}^0\big((J_y \cup J_R) \setminus \partial^* I\big)$. This will be achieved by a contradiction based fundamentally on the following lemma: suppose that along a sequence the surface energy in a set $S_h^{2l}(x)$ (see (3.3)) was less than $\sim 2h^2$, i.e., so small such that the void cannot cut through the thin rod Ω_h, see Figure 3. Then, the jump height of the sequences $(w_h)_{h>0}$, $(R_h)_{h>0}$ is small, see (3.11) below. Later in Section 5 this will allow us to exclude that the limiting functions y and R jump in the set $S_{h,\rho}^l(x) \subset S_h^{2l}(x)$.

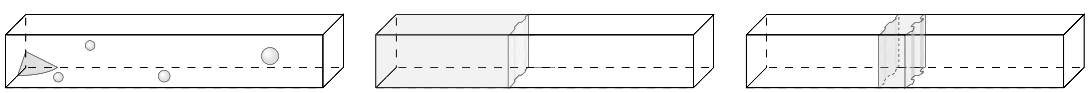

Figure 3. The void set is depicted in gray. In the situation of (3.10) or Remark 3.3(i), only case (a) can occur, whereas (b),(c) are impossible.

Recall the stripes $S_h^l(x)$ and $S_{h,\rho}^l(x)$ defined in (3.3) and (3.4), respectively.

Proposition 3.2 (Jumps of blockwise Sobolev modifications)**.** *Let* $(v_h, E_h) \in \mathcal{A}_h$ *be a sequence from Proposition 3.1 and let* $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$, $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times 3})$ *be the corresponding functions satisfying* (3.8). *Then, there exist open sets* E_h^* *with* $E_h \subset E_h^* \subset \Omega_h$, $\partial E_h^* \cap \Omega_h$ *is a union of finitely many* C^2*-regular submanifolds, that satisfy*

$$h^{-3}\mathcal{L}^3(E_h^* \setminus E_h) \to 0, \qquad \liminf_{h\to 0} h^{-2}\mathcal{H}^2(\partial E_h^* \cap \Omega_h) \le \liminf_{h\to 0} h^{-2}\mathcal{G}_{\mathrm{surf}}^{\kappa_h}(E_h;\Omega_h)\,, \tag{3.9}$$

such that the following holds for any $l \ge 6Th$, $x \in (2l, L-2l)$: *For any stripe* $S_h^{2l}(x)$ *with*

$$\begin{aligned}
&\text{(i)} \quad \frac{1}{((1-\rho)h)^2}\mathcal{H}^2\big(\partial E_h^* \cap S_h^{2l}(x)\big) < 2\,,\\
&\text{(ii)} \quad \frac{\mathcal{L}^3(E_h^* \cap S_h^{2l}(x))}{\mathcal{L}^3(S_h^{2l}(x))} \le \frac{1}{9}\,,
\end{aligned} \tag{3.10}$$

it holds that

$$\frac{1}{h^2}\int_{S^l_{h,\rho}(x)\cap J_{w_h}} \sqrt{|[w_h]|}\,\mathrm{d}\mathcal{H}^2 + \frac{1}{h^2}\int_{S^l_{h,\rho}(x)\cap J_{R_h}} \sqrt{|[R_h]|}\,\mathrm{d}\mathcal{H}^2 \to 0\,. \tag{3.11}$$

Remark 3.3. (i) One can also prove a variant of Proposition 3.2: if 2 in the right hand side of (3.10)(i) is replaced by 1, then assumption (3.10)(ii) is not needed, cf. Figure 3. (ii) In (3.11), $\sqrt{|[w_h]|}$ and $\sqrt{|[R_h]|}$ can be replaced by $|[w_h]|^{1-\beta}$ and $|[R_h]|^{1-\beta}$ for any $\beta \in (0,1)$, up to adjusting the condition for κ_h in (2.5) (which will then depend also on β). We omit details as the choice $\beta = \frac{1}{2}$ is enough for our purposes.

We refer to Remark 3.13 below for a short comment how to prove (i) and (ii).

The rest of this section is entirely devoted to the proofs of Propositions 3.1–3.2. The proofs of our main compactness and Γ-convergence results then start in Section 4. In the proofs, we will send the parameters h, ρ to zero (in this order). In order to avoid overburdening of notation, generic constants which are independent of h, ρ but may depend on the fixed parameters T, L are denoted by C. We will use a subscript notation whenever we want to highlight the dependence of a particular constant on a specific parameter.

3.1. **Rigidity results.** This subsection is devoted to recalling some rigidity results which are the basis for our proofs.

Geometric rigidity in variable domains: We first recall the result [33, Theorem 2.1]. For convenience, we will directly formulate it on the set Ω_h and its subset $\Omega_{h,\rho}$, see (2.1) and (3.1). The behavior of deformations v on (connected components of) $\Omega_h \setminus \overline{E}$ might not be rigid. We refer to [33, Example 2.6] for an explanation in that direction. A key observation in [33] is that rigidity estimates can be obtained outside of a thickened version of the voids. We start by formulating this result on the modification of the void sets.

Proposition 3.4 (Thickening of sets)**.** *Let $h, \rho > 0$, let $\gamma \in (0,1)$. Then, there exist a universal constant $C_0 > 0$, $\eta_0 = \eta_0(\rho) \in (0,1)$, and for each $\eta \in (0, \eta_0]$ the following holds:*
Given $E \in \mathcal{A}_{\mathrm{reg}}(\Omega_h)$, we can find an open set $E_{h,\eta,\gamma}$ such that $E \subset E_{h,\eta,\gamma} \subset \Omega_h$, $\partial E_{h,\eta,\gamma} \cap \Omega_h$ is a union of finitely many C^2-regular submanifolds, and

$$\begin{aligned} &\text{(i)} \quad \mathcal{L}^3(E_{h,\eta,\gamma} \setminus E) \le h\eta\gamma^{1/2}\mathcal{G}^{\gamma h^2}_{\mathrm{surf}}(E;\Omega_h), \qquad \mathrm{dist}_{\mathcal{H}}(E, E_{h,\eta,\gamma}) \le h\eta\gamma^{1/2}\,, \\ &\text{(ii)} \quad \mathcal{H}^2(\partial E_{h,\eta,\gamma} \cap \Omega_h) \le (1 + C_0\eta)\,\mathcal{G}^{\gamma h^2}_{\mathrm{surf}}(E;\Omega_h)\,. \end{aligned} \tag{3.12}$$

On the complement $\Omega_{h,\rho} \setminus \overline{E_{h,\eta,\gamma}}$ quantitative piecewise rigidity estimates hold, as the following result shows. Recall the notation $S^l_h(x)$ in (3.3).

Theorem 3.5 (Geometric rigidity in variable domains)**.** *Let $h, \rho > 0$, let $\gamma \in (0,1)$ and $l > 0$. Then, there exist a universal constant $C_0 > 0$, $\eta_0 = \eta_0(\rho) > 0$, and for each $\eta \in (0, \eta_0]$ there exists $C_\eta = C_\eta(\eta, \frac{l}{h}) > 0$ with $C_\eta \to \infty$ as $\eta \to 0$, $\frac{l}{h} \to 0$, or $\frac{l}{h} \to \infty$ such that the following holds:*
For every $E \in \mathcal{A}_{\mathrm{reg}}(\Omega_h)$, denoting by $E_{h,\eta,\gamma}$ the set of Proposition 3.4, for every $U = S^l_h(x) \subset \Omega_h$ and $\widetilde{U} = S^l_{h,\rho}(x) \subset \Omega_{h,\rho}$, for the connected components $(\widetilde{U}_j)_j$ of $\widetilde{U} \setminus \overline{E_{h,\eta,\gamma}}$ and for every $y \in W^{1,2}(\Omega_h \setminus \overline{E};\mathbb{R}^3)$ there exist corresponding rotations $(R_j)_j \subset SO(3)$ and vectors $(b_j)_j \subset \mathbb{R}^3$ such

that

$$\text{(i)}\quad \sum\nolimits_j \int_{\widetilde{U}_j} \big|\mathrm{sym}\big((R_j)^T\nabla y - \mathrm{Id}\big)\big|^2\,\mathrm{d}x \le C_0\big(1 + C_\eta\gamma^{-15/2}h^{-3}\varepsilon\big)\int_{U\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,,$$

$$\text{(ii)}\quad \sum\nolimits_j \int_{\widetilde{U}_j} \big|(R_j)^T\nabla y - \mathrm{Id}\big|^2\,\mathrm{d}x \le C_\eta\gamma^{-3}\int_{U\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,,$$

$$\text{(iii)}\quad \sum\nolimits_j \int_{\widetilde{U}_j} \frac{1}{h^2}\big|y - (R_jx + b_j)\big|^2\,\mathrm{d}x \le C_\eta\gamma^{-5}\int_{U\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x\,, \tag{3.13}$$

where for brevity $\varepsilon := \int_{U\setminus\overline{E}} \mathrm{dist}^2(\nabla y, SO(3))\,\mathrm{d}x$.

Proof of Proposition 3.4 and Theorem 3.5. The result is essentially given in [33, Theorem 2.1]. We explain here the adaptations necessary to the present version of the result, in particular the scaling in terms of the small parameter $h > 0$.

We apply [33, Theorem 2.1] for $d = 3$, $q = 2$, $\gamma \in (0,1)$ and $\varphi \equiv \|\cdot\|_2$ on the sets $h^{-1}\Omega_h \subset \mathbb{R}^3$, $\widetilde{\Omega} := h^{-1}\Omega_{h,\rho}$ and $h^{-1}E$. The constant η_0 therein depends only on $\mathrm{dist}(h^{-1}\partial\Omega_h, h^{-1}\Omega_{h,\rho})$ and can thus be chosen depending only on ρ, see (2.1) and (3.1). Then, the result first provides a set $E_{\eta,\gamma}$ with $h^{-1}E \subset E_{\eta,\gamma} \subset h^{-1}\Omega_h$ such that by [33, (2.2)] and a scaling argument the set $E_{h,\eta,\gamma} := hE_{\eta,\gamma}$ satisfies (3.12). Here, we particularly note that a change of variables implies

$$\mathcal{G}^{\gamma}_{\mathrm{surf}}(h^{-1}E; h^{-1}\Omega_h) = h^{-2}\mathcal{G}^{\gamma h^2}_{\mathrm{surf}}(E;\Omega_h)\,.$$

Then, (3.13) follows from a localized version of [33, (2.3)], see [33, Remark 2.10], first applied on the sets $h^{-1}U$, $h^{-1}\widetilde{U}$ and the function $w_h(x) = \frac{1}{h}y(hx)$ for $x \in h^{-1}(\Omega_h \setminus \overline{E})$, and then again rescaled. The factors h^{-3} and h^{-2} in (3.13)(i), (iii), respectively, ensure that all inequalities in (3.13) are scaling invariant in the sense that the constants are independent of h. The factors $\gamma^{-15/2}$, γ^{-3} and γ^{-5} follow from the choice of $d = 3$ and $q = 2$. We further observe that the constant C_η depends on η and $\mathcal{L}^3(h^{-1}U)$, see [33, Remark 2.10]. As $\mathcal{L}^3(h^{-1}U) = \frac{2l}{h}$, we indeed get that C_η depends on η and the ratio of l and h. □

In the proofs below, we will apply this rigidity result on the T-cuboids $\mathcal{Q}_h$ introduced in (3.2) or in finite unions of such cuboids. For these sets, we observe that the constant C_η depends only on η and T (as the corresponding l is approximately Th). Moreover, we will choose η and γ depending on the regime of the elastic energy ε such that $C_\eta\gamma^{-15/2}h^{-3}\varepsilon \le 1$ and $C_\eta\gamma^{-5} \le \varepsilon^{-\theta}$ for some $\theta > 0$ small. Thus, we obtain a sharp control on symmetrized gradients in terms of ε (see (3.13)(i)), while the rigidity estimate in (3.13)(ii) and the Poincaré-type estimate (3.13)(iii) yield control of order $\varepsilon^{1-\theta}$, hence being suboptimal in the exponent.

Korn and Poincaré inequalities: The issue of the suboptimal exponent can be remedied provided that the surface measure of the void set is small. This relies on delicate Korn and Poincaré inequalities in the space $GSBD^2$, see [24] for the definition of this space. We formulate the result of [19, Theorem 1.1, Theorem 1.2] in a simplified setting which does not involve functions in $GSBD^2$ but only SBV^2-functions. In the following, we say that $a\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an *infinitesimal rigid motion* if a is affine with $\mathrm{sym}(\nabla a) = 0$.

Theorem 3.6 (Korn inequality for functions with small jump set)**.** *Let $U \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, there exists a constant $c = c(U) > 0$ such that for all $u \in SBV^2(U;\mathbb{R}^3)$ there exists a set of finite perimeter $\omega \subset U$ with*

$$\mathcal{H}^2(\partial^*\omega) \le c\mathcal{H}^2(J_u)\,, \qquad \mathcal{L}^3(\omega) \le c(\mathcal{H}^2(J_u))^{3/2}\,, \tag{3.14}$$

and an infinitesimal rigid motion a such that

$$(\mathrm{diam}(U))^{-1}\|u-a\|_{L^2(U\setminus\omega)} + \|\nabla u - \nabla a\|_{L^2(U\setminus\omega)} \le c\|\mathrm{sym}(\nabla u)\|_{L^2(U)}. \tag{3.15}$$

Moreover, there exists $v \in W^{1,2}(U;\mathbb{R}^3)$ such that $v \equiv u$ on $U \setminus \omega$ and

$$\|\mathrm{sym}(\nabla v)\|_{L^2(U)} \le c\|\mathrm{sym}(\nabla u)\|_{L^2(U)}.$$

Furthermore, if $u \in L^\infty(U;\mathbb{R}^3)$ one has $\|v\|_{L^\infty(U)} \le \|u\|_{L^\infty(U)}$.

This follows from [19] (for $d = 3$ and $p = 2$) by the fact that $SBV^2 \subset GSBD^2$. Note that in [19, Theorem 1.1] $\mathcal{L}^3(\omega) \le c(\mathcal{H}^2(J_u))^{3/2}$ has not been stated explicitly, but it readily follows from $\mathcal{H}^2(\partial^*\omega) \le c\mathcal{H}^2(J_u)$ and the isoperimetric inequality. The result is indeed only relevant if $\mathcal{H}^2(J_u)$ is small since otherwise $\omega = U$ is possible and the statement is empty. In a similar fashion to the reasoning in Theorem 3.5, it is a standard matter to see that the constant in (3.14)–(3.15) is invariant under translation and rescaling of the domain.

Difference of affine maps: To estimate the difference of rigid motions, we make use of the following elementary lemma. By $B_r(x) \subset \mathbb{R}^3$ we denote the open ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$.

Lemma 3.7 (Estimate on affine maps)**.** *Let $\delta > 0$. Then there exists a constant $C > 0$ only depending on δ such for every $G \in \mathbb{R}^{3\times 3}$, $b \in \mathbb{R}^3$, $x \in \mathbb{R}^3$, and $E \subset B_r(x)$ for some $r > 0$ with $\mathcal{L}^3(E) \ge \delta r^3$ we have*

$$\|G\cdot + b\|_{L^\infty(B_r(x))} \le Cr^{-3}\mathcal{L}^3(E)^{1/2}\|G\cdot + b\|_{L^2(E)}, \quad |G| \le Cr^{-4}\mathcal{L}^3(E)^{1/2}\|G\cdot + b\|_{L^2(E)}\,.$$

Proof. For $r = 1$ and $x = 0$, the result is a special case of [39, Lemma 3.4], applied (for $d = 3$) to $\psi(t) := t^2$. In particular, in [39, (3.4)] we also use Hölder's inequality to get the control in terms of the quantity $\mathcal{L}^3(E)^{1/2}\|G\cdot + b\|_{L^2(E)}$. For general $r > 0$ and $x \in \mathbb{R}^3$, the estimates follow from a standard scaling and translation argument. □

3.2. **Isoperimetric inequalities on cuboids.** In this subsection, we present a special case of a relative isoperimetric inequality in cuboids that are long in one direction, where the isoperimetric constant is independent of the length. Such an inequality is possible for sets that have small relative perimeter as, in this case, isoperimetric sets will concentrate at one of the corners or at one of the short edges of the long cuboid. Indeed, under the small perimeter constraint, the relative boundary cannot span a cross section of the cuboid, see Figure 4. As the result may be interesting in its own right, it is formulated in arbitrary space dimension on the cuboids $S^l_\sigma(x_0) := x_0 + (-l,l)\times(-\frac{\sigma}{2},\frac{\sigma}{2})^{d-1}$, consistent with the notation (3.3). Afterwards, we will present two consequences which will be used in the sequel.

Proposition 3.8 (Relative isoperimetric inequality on cuboids)**.** *Let $l,\sigma > 0$ with $l/\sigma \ge 1$, and $x_0 \in \mathbb{R}^d$. Then, there exists a dimensional constant $C_{\mathrm{iso}} \ge 1$ independent of l and σ such that for every set of finite perimeter $P \subset S^l_\sigma(x_0)$ with*

$$\mathcal{H}^{d-1}\big(\partial^* P \cap S^l_\sigma(x_0)\big) < \sigma^{d-1}\,, \tag{3.16}$$

it holds that

$$\min\{\mathcal{L}^d(P), \mathcal{L}^d(S^l_\sigma(x_0)\setminus P)\} \le C_{\mathrm{iso}}\sigma\mathcal{H}^{d-1}(\partial^* P \cap S^l_\sigma(x_0))\,. \tag{3.17}$$

Proof. Without loss of generality, after translation and uniform rescaling, we can assume that $x_0 = 0$, $\sigma = 1$, and can without restriction reduce to showing the following assertion on the cuboid

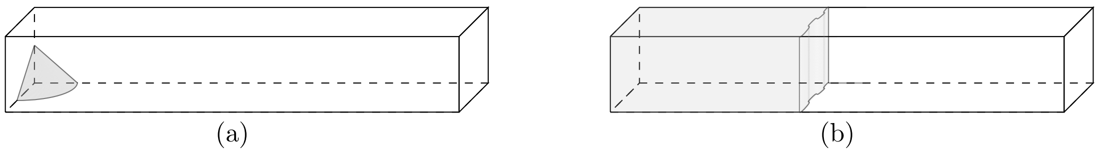

Figure 4. Void set contained in a thin rod with (a) relative perimeter less than σ^{d-1}, (b) relative perimeter bigger than σ^{d-1}.

$Q_l := S_1^l(0) = (-l,l)\times(-\frac{1}{2},\frac{1}{2})^{d-1}$. There exists a dimensional constant $c_d>0$ such that for each $l\geq 1$ and for every set of finite perimeter $P\subset Q_l$ with

$$\mathcal{L}^d(P)\leq l \quad \text{and} \quad \mathcal{H}^{d-1}(\partial^* P\cap Q_l)<1\,,$$

there holds that

$$\mathcal{L}^d(P)\leq c_d\mathcal{H}^{d-1}(\partial^* P\cap Q_l)\,. \tag{3.18}$$

Note that we can assume that $l\gg 1$ as, given $l_0>1$, we have that for all $1\leq l\leq l_0$ the statement follows directly from the classical relative isoperimetric inequality

$$\min\{\mathcal{L}^d(P),\mathcal{L}^d(Q_l\setminus P)\}\leq C_{\rm iso}(l)\mathcal{H}^{d-1}(\partial^* P\cap Q_l)\,,$$

and the fact that that $C_{\rm iso}(l)\leq C_0 l_0$, where $C_0>0$ is a dimensional constant.

We prove the assertion by induction on the dimension d, the case $d=1$ being a trivial statement. Assume now that (3.18) is true for some $d\geq 1$, and for the inductive step let us prove it in dimension $d+1$. For this purpose, let $P\subset Q_l^{d+1} := (-l,l)\times(-\frac{1}{2},\frac{1}{2})^d$ be a set of finite perimeter, with

$$\mathcal{L}^{d+1}(P)\leq l \quad \text{and} \quad \mathcal{H}^{d}(\partial^* P\cap Q_l^{d+1})<1\,. \tag{3.19}$$

For every $t\in(-\frac{1}{2},\frac{1}{2})$, let us set for notational simplicity $Q_{l,t}^d := Q_l^{d+1}\cap\{x_{d+1}=t\}$ and also $P_t := P\cap Q_{l,t}^d$. By general slicing properties of sets of finite perimeter, P_t is a subset of finite perimeter in $Q_{l,t}^d$ for $\mathcal{L}^1$-a.e. $t\in(-1/2,1/2)$. Let now $t_0\in(-1/2,1/2)$ be such that P_{t_0} is of finite perimeter and also, as a consequence of the coarea formula (cf. [56, (18.25)]) and (3.19),

$$\mathcal{H}^{d-1}(\partial^* P_{t_0}\cap Q_{l,t_0}^d)\leq\int_{-1/2}^{1/2}\mathcal{H}^{d-1}(\partial^* P_t\cap Q_{l,t}^d)\,\mathrm{d}t\leq\mathcal{H}^d(\partial^* P\cap Q_l^{d+1})<1\,. \tag{3.20}$$

By (3.19), we also find that for $\mathcal{L}^1$-a.e. $t\in(-1/2,1/2)$,

$$\begin{aligned}\big|\mathcal{L}^d(P_t)-\mathcal{L}^d(P_{t_0})\big| &\leq\mathcal{H}^d\Big(\partial^* P\cap\big((-l,l)\times(-1/2,1/2)^{d-1}\times[t_0\wedge t,t_0\vee t]\big)\Big)\\ &\leq\mathcal{H}^d(\partial^* P\cap Q_l^{d+1})<1\,.\end{aligned} \tag{3.21}$$

Note that the first inequality in (3.21) is immediate for smooth sets via a projection argument. In the general case, it can be derived by the density of smooth sets and Fubini's theorem. Therefore, we get

$$\mathcal{L}^d(P_{t_0})-1<\mathcal{L}^d(P_t)<\mathcal{L}^d(P_{t_0})+1 \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in(-1/2,1/2)\,. \tag{3.22}$$

We now claim that

$$\mathcal{L}^d(P_{t_0})\leq\mathcal{L}^d(Q_{l,t_0}^d\setminus P_{t_0})\,. \tag{3.23}$$

Indeed, if (3.23) was not true, then by (3.20) and the inductive hypothesis we would have

$$\mathcal{L}^d(Q_{l,t_0}^d\setminus P_{t_0})\leq c_d\mathcal{H}^{d-1}(\partial^* P_{t_0}\cap Q_{l,t_0}^d)<c_d \tag{3.24}$$

Then, by choosing $l\gg 1$, (3.22) together with (3.24) would imply that

$$\mathcal{L}^d(P_t)>\mathcal{L}^d(P_{t_0})-1>2l-c_d-1>l \quad \text{for } \mathcal{L}^1\text{-a.e. } t\in(-1/2,1/2)\,. \tag{3.25}$$

Thus, by (3.25) and Fubini's theorem, we would get

$$\mathcal{L}^{d+1}(P) = \int_{-1/2}^{1/2} \mathcal{L}^d(P_t)\,\mathrm{d}t > l\,,$$

contradicting the first assumption in (3.19). Therefore, indeed (3.23) holds true. By our inductive hypothesis and (3.20) this yields

$$\mathcal{L}^d(P_{t_0}) \le c_d \mathcal{H}^{d-1}(\partial^* P_{t_0} \cap Q^d_{l,t_0}) \le c_d \mathcal{H}^d(\partial^* P \cap Q^{d+1}_l)\,.$$

The last inequality together with (3.21) implies that

$$\mathcal{L}^d(P_t) \le (c_d+1)\mathcal{H}^d(\partial^* P \cap Q^{d+1}_l) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (-1/2, 1/2)\,.$$

Therefore, using Fubini's theorem again, we get

$$\mathcal{L}^{d+1}(P) = \int_{-1/2}^{1/2} \mathcal{L}^d(P_t)\,\mathrm{d}t \le (c_d+1)\mathcal{H}^d(\partial^* P \cap Q^{d+1}_l)\,,$$

finishing the induction and hence the proof. □

We proceed with two corollaries: Corollary 3.9 and Corollary 3.10 describe how a long cuboid can be partitioned by a void set. If the void set has relative perimeter less than the area of the cross section σ^{d-1}, then there is a very large dominant component and some small components whose volume can be controlled by the relative perimeter of the void set. The same is true if the void set has relative perimeter between σ^{d-1} and $2\sigma^{d-1}$ but small volume. If we drop the volume assumption, there may be two different large components - one consisting of the void set and a large complementary component. If the void set has relative perimeter bigger than $2\sigma^{d-1}$, then, even if the void set has small volume, it may separate the cuboid into two large complementary components. Some indicative cases are illustrated in Figure 3: (a) Void set with perimeter less than σ^{d-1} or small volume and perimeter less than $2\sigma^{d-1}$. (b) Void set with perimeter less than $2\sigma^{d-1}$ with large volume. (c) Void set with perimeter bigger than $2\sigma^{d-1}$ with small volume.

Corollary 3.9 (Dominant component 1)**.** *There exists $T_0 \in \mathbb{N}$ with the following property. Let $l, \sigma > 0$ with $l/\sigma \ge T_0$. Let $(P_j)_{j\ge1}$ be a Caccioppoli partition of $S^l_\sigma(x_0)$ with*

$$\mathcal{H}^{d-1}\Big(\bigcup_{j\ge1} \partial^* P_j \cap S^l_\sigma(x_0)\Big) < \sigma^{d-1} \tag{3.26}$$

and $\mathcal{L}^d(P_1) \ge \mathcal{L}^d(P_j)$ for all $j \ge 2$. Then,

$$\mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1) \le C_{\mathrm{iso}}\sigma\mathcal{H}^{d-1}\big(\partial^* P_1 \cap S^l_\sigma(x_0)\big) \quad \text{and} \quad \mathcal{L}^d(P_1) > \frac{1}{2}\mathcal{L}^d(S^l_\sigma(x_0))\,, \tag{3.27}$$

where C_{iso} is the constant in (3.17).

Proof. In view of (3.26), (3.16) holds for each P_j and Proposition 3.8 is applicable for each P_j. To prove the statement, it suffices to show that

$$\mathcal{L}^d(P_1) > \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1)\,.$$

Assume by contradiction that this was false. By $\mathcal{L}^d(P_1) \ge \mathcal{L}^d(P_j)$ for all $j \ge 2$, this would imply $\mathcal{L}^d(P_j) \le \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_j)$ for all $j \ge 1$. But then we calculate using (3.17) and (3.26),

$$\begin{aligned} 2l\sigma^{d-1} &= \mathcal{L}^d(S^l_\sigma(x_0)) = \sum_{j\ge1} \mathcal{L}^d(P_j) = \sum_{j\ge1} \min\{\mathcal{L}^d(P_j), \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_j)\} \\ &\le \sum_{j\ge1} C_{\mathrm{iso}}\sigma\mathcal{H}^{d-1}(\partial^* P_j \cap S^l_\sigma(x_0)) = 2C_{\mathrm{iso}}\sigma\mathcal{H}^{d-1}\big(\bigcup_{j\ge1} \partial^* P_j \cap S^l_\sigma(x_0)\big) < 2C_{\mathrm{iso}}\sigma^d\,, \end{aligned}$$

where we also used the local structure of Caccioppoli partitions, see [6, Theorem 4.17]. By choosing $T_0 \in \mathbb{N}$ large enough depending only on $C_{\rm iso}$ such that $l/\sigma \geq T_0 > C_{\rm iso}$, this yields a contradiction. □

Corollary 3.10 (Dominant component 2)**.** *There exists $T_0 \in \mathbb{N}$ with the following property. Let $l, \sigma > 0$ with $l/\sigma \geq T_0$. Let $E \in \mathcal{P}(S^l_\sigma(x_0))$ and let $(P_j)_{j\geq 1}$ be the connected components of $S^l_\sigma(x_0)\setminus E$ in the sense that $(P_j)_{j\geq 1} \cup \{E\}$ forms a Caccioppoli partition of $S^l_\sigma(x_0)$ with*

$$\mathcal{H}^{d-1}\big((\partial^* P_j \setminus \partial^* E) \cap S^l_\sigma(x_0)\big) = 0 \quad \text{for all } j \geq 1\,. \tag{3.28}$$

Suppose that

$$\mathcal{H}^{d-1}\Big(\bigcup_{j\geq 1} \partial^* P_j \cap S^l_\sigma(x_0)\Big) < 2\sigma^{d-1}, \quad \mathcal{L}^d(E) \leq \frac{1}{4}\mathcal{L}^d(S^l_\sigma(x_0)) \tag{3.29}$$

and $\mathcal{L}^d(P_1) \geq \mathcal{L}^d(P_j)$ for all $j \geq 2$. Then,

$$\mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1) \leq C_{\rm iso}\sigma\mathcal{H}^{d-1}\Big(\bigcup_{j\geq 1} \partial^* P_j \cap S^l_\sigma(x_0)\Big) + \mathcal{L}^d(E) \ \text{ and } \mathcal{L}^d(P_1) > \frac{1}{2}\mathcal{L}^d(S^l_\sigma(x_0))\,, \tag{3.30}$$

where $C_{\rm iso}$ is the constant in (3.17)*.*

Proof. As $\mathcal{L}^d(P_j) \leq \mathcal{L}^d(P_1)$, we first observe that

$$\mathcal{L}^d(P_j) \leq \frac{1}{2}\mathcal{L}^d(S^l_\sigma(x_0)) \quad \text{for all } j \geq 2\,. \tag{3.31}$$

The sets $(\partial^* P_j \cap S^l_\sigma(x_0))_{j\geq 1}$ are pairwise disjoint up to $\mathcal{H}^{d-1}$-negligible sets by (3.28) and the local structure of Caccioppoli partitions, see [6, Theorem 4.17]. Therefore, by (3.29) we get

$$\sum_{j\geq 1} \mathcal{H}^{d-1}\Big(\partial^* P_j \cap S^l_\sigma(x_0)\Big) = \mathcal{H}^{d-1}\Big(\bigcup_{j\geq 1} \partial^* P_j \cap S^l_\sigma(x_0)\Big) < 2\sigma^{d-1}\,. \tag{3.32}$$

This implies that at least one of the following two cases holds:

$$\text{(a)}\ \ \mathcal{H}^{d-1}\Big(\partial^* P_1 \cap S^l_\sigma(x_0)\Big) < \sigma^{d-1}, \qquad \text{(b)}\ \ \sum_{j\geq 2}\mathcal{H}^{d-1}\Big(\partial^* P_j \cap S^l_\sigma(x_0)\Big) < \sigma^{d-1}\,.$$

We first assume that (a) holds. An application of Proposition 3.8 yields

$$\min\{\mathcal{L}^d(P_1), \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1)\} \leq C_{\rm iso}\sigma\mathcal{H}^{d-1}(\partial^* P_1 \cap S^l_\sigma(x_0)) \leq C_{\rm iso}\sigma^d\,. \tag{3.33}$$

Then, in the case $\mathcal{L}^d(P_1) \geq \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1)$, we find

$$\mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1) \leq C_{\rm iso}\sigma\mathcal{H}^{d-1}(\partial^* P_1 \cap S^l_\sigma(x_0)) \leq C_{\rm iso}\sigma^d\,.$$

This shows the first part of (3.30). The second part follows by choosing $T_0 \in \mathbb{N}$ large enough depending on $C_{\rm iso}$ noting that $\mathcal{L}^d(S^l_\sigma(x_0)) = 2l\sigma^{d-1} \geq 2T_0\sigma^d$.

We show that the case $\mathcal{L}^d(P_1) < \mathcal{L}^d(S^l_\sigma(x_0) \setminus P_1)$ leads to a contradiction. Indeed, if that was the case, by (3.33) we would have

$$\mathcal{L}^d(P_j) \leq \mathcal{L}^d(P_1) \leq C_{\rm iso}\sigma^d \quad \text{for all } j \geq 2\,. \tag{3.34}$$

From this we derive that

$$\mathcal{L}^d(P_j) \leq C_{\rm iso}\sigma\mathcal{H}^{d-1}(\partial^* P_j \cap S^l_\sigma(x_0)) \quad \text{for all } j \geq 1\,. \tag{3.35}$$

Indeed, for $j = 1$ this is a consequence of (3.33). For $j \geq 2$ instead, if $\mathcal{H}^{d-1}(\partial^* P_j \cap S^l_\sigma(x_0)) \geq \sigma^{d-1}$, this follows from (3.34). If $\mathcal{H}^{d-1}(\partial^* P_j \cap S^l_\sigma(x_0)) < \sigma^{d-1}$, (3.16) holds and the estimate follows from an application of Proposition 3.8 and (3.31).

Now, by (3.32) and (3.35) we obtain

$$\mathcal{L}^d(S^l_\sigma(x_0)\setminus \overline{E}) = \sum_{j\geq 1}\mathcal{L}^d(P_j) \leq C_{\rm iso}\sigma\sum_{j\geq 1}\mathcal{H}^{d-1}(\partial^* P_j\cap S^l_\sigma(x_0)) \leq 2C_d\sigma^d\,.$$

Using that $\mathcal{L}^d(S^l_\sigma(x_0)) \geq 2T_0\sigma^d$, by choosing $T_0\in\mathbb{N}$ large enough, this would imply

$$\mathcal{L}^d(S^l_\sigma(x_0)\setminus \overline{E}) < \frac{3}{4}\mathcal{L}^d(S^l_\sigma(x_0))\,.$$

This however contradicts the fact that $\mathcal{L}^d(E)\leq \frac{1}{4}\mathcal{L}^d(S^l_\sigma(x_0))$, see (3.29).

We are left with case (b). Here, we can again apply Proposition 3.8 on each P_j, $j\geq 2$, to find

$$\mathcal{L}^d(P_j) = \min\{\mathcal{L}^d(P_j),\mathcal{L}^d(S^l_\sigma(x_0)\setminus P_j)\} \leq C_{\rm iso}\sigma\mathcal{H}^{d-1}(\partial^* P_j\cap S^l_\sigma(x_0))\,,$$

where the first identity follows from (3.31). Now, by using (3.32) we estimate

$$\begin{aligned}\mathcal{L}^d(S^l_\sigma(x_0)\setminus P_1) &\leq \mathcal{L}^d(E) + \sum_{j\geq 2}\mathcal{L}^d(P_j) \leq \mathcal{L}^d(E) + \sum_{j\geq 2} C_{\rm iso}\sigma\mathcal{H}^{d-1}\Big(\partial^* P_j\cap S^l_\sigma(x_0)\Big)\\ &\leq C_{\rm iso}\sigma\mathcal{H}^{d-1}\Big(\bigcup_{j\geq 1}\partial^* P_j\cap S^l_\sigma(x_0)\Big) + \mathcal{L}^d(E)\,.\end{aligned}$$

This shows the first part of (3.30). The second part again follows for some $T_0\in\mathbb{N}$ large enough, using that $\mathcal{L}^d(E)\leq\frac{1}{4}\mathcal{L}^d(S^l_\sigma(x_0))$. □

3.3. **Local estimates and Sobolev extension on cuboids.** In the following, we set up the necessary notation and definitions for the remainder of Section 3. We introduce the thickened void set and partition our reference domain $\Omega_{h,\rho}$ into cuboids, where we partition with respect to the surface area of the boundary of the thickened void. We let $(v_h,E_h)_{h>0}$ be a sequence of admissible deformations and void sets in the thin rod Ω_h, where for convenience we use a continuum index h in the notation for the sequences. Recalling (3.6), we suppose that

$$\sup_{h>0}\mathcal{G}^h(v_h,E_h) < +\infty\,. \tag{3.36}$$

We fix $0<\rho\leq\rho_0 := 1-(19/20)^{1/3}$ as in Proposition 3.1. Recall the sequence $(\kappa_h)_{h>0}$ as in (2.5). For technical reasons, we need to assume that $(\kappa_h)_{h>0}$ converges to zero sufficiently fast. Therefore, we introduce

$$\bar\kappa_h := \min\{\kappa_h, h^2\}\,, \tag{3.37}$$

and observe that

$$\mathcal{G}^{\bar\kappa_h}_{\rm surf}(E_h;\Omega_h) \leq \mathcal{G}^{\kappa_h}_{\rm surf}(E_h;\Omega_h)\,. \tag{3.38}$$

Recall T as introduced before (3.2). From now on, we will tacitly assume that T is chosen sufficiently large such that Corollaries 3.9–3.10 are applicable. After possibly increasing T, we can assume that

$$T\geq 80C_{\rm iso}\,, \tag{3.39}$$

where $C_{\rm iso}\geq 1$ is the constant in Proposition 3.8. Let $\eta_0=\eta_0(\rho)\in(0,1)$ be the constant in Proposition 3.4. In view of (2.5) and (3.37), we can choose a sequence $(\eta_h)_{h>0}\subset(0,\eta_0)$ converging to zero sufficiently slow such that the constant C_{η_h} in (3.13), applying Theorem 3.5 for ρ, $l=3Th$, $\eta=\eta_h$, and $\gamma=\bar\kappa_h/h^2$, satisfies

$$\limsup_{h\to 0}\, C_{\eta_h}\Big(\frac{h^2}{\bar\kappa_h}\Big)^5 h^{2/5} < +\infty\,. \tag{3.40}$$

Then, by Proposition 3.4 applied for ρ, $\eta = \eta_h$, and $\gamma = \bar{\kappa}_h/h^2$, for all $h > 0$ we can find open sets E_h^* with $E_h \subset E_h^* \subset \Omega_h$ such that $\partial E_h^* \cap \Omega_h$ is a union of finitely many C^2-regular submanifolds and

$$\begin{aligned} &\text{(i)} \quad h^{-3}\mathcal{L}^3(E_h^* \setminus E_h) \to 0, \qquad h^{-1}\operatorname{dist}_{\mathcal{H}}(E_h^*, E_h) \to 0 \quad \text{as } h \to 0\,, \\ &\text{(ii)} \quad \liminf_{h\to 0} h^{-2}\mathcal{H}^2(\partial E_h^* \cap \Omega_h) \le \liminf_{h\to 0}\, h^{-2}\mathcal{G}^{\bar{\kappa}_h}_{\rm surf}(E_h;\Omega_h) \le \liminf_{h\to 0}\, h^{-2}\mathcal{G}^{\kappa_h}_{\rm surf}(E_h;\Omega_h)\,. \end{aligned} \tag{3.41}$$

Here, we used (3.12)(i), (ii), $\eta_h \to 0$, (3.37), and that $h^{-2}\mathcal{G}^{\gamma h^2}_{\rm surf}(E_h;\Omega_h) = h^{-2}\mathcal{G}^{\bar{\kappa}_h}_{\rm surf}(E_h;\Omega_h)$ is uniformly bounded by (3.6), (3.36), and (3.38). This is the sequence of sets in Proposition 3.2 and we note that (3.41) implies (3.9). In the rigidity estimate (3.13), the behavior of the deformation inside E_h^* cannot be controlled. Thus, in a similar fashion to (2.2), for definiteness we can assume that the deformation is the identity inside E_h^*, i.e., we introduce the modification $v_h^*\colon \Omega_h \to \mathbb{R}^3$ by

$$v_h^*(x) := \begin{cases} v_h(x) & \text{if } x \in \Omega_h \setminus E_h^*, \\ \mathrm{id} & \text{if } x \in E_h^*\,. \end{cases} \tag{3.42}$$

Note that by (3.41) we get

$$h^{-3}\mathcal{L}^3(\{v_h \neq v_h^*\}) \le h^{-3}\mathcal{L}^3(E_h^* \setminus E_h) \to 0 \quad \text{as } h \to 0\,. \tag{3.43}$$

Recall the definition of the T-cuboids in the family $\mathcal{Q}_h$ in (3.2). For $i = 2, \dots, N-1$, we also introduce the $3T$-cuboids by

$$Q_h^3(i) := Q_h(i-1) \cup Q_h(i) \cup Q_h(i+1)\,.$$

Our idea is to apply Theorem 3.5 for $U := Q_h^3(i)$. To this end, we also need the slightly smaller cuboids, defined by

$$Q^3_{h,\rho}(i) := x_i + (1-\rho)\big(Q_h^3(i) - x_i\big) \subset \Omega_{h,\rho}\,, \tag{3.44}$$

where $x_i = ((i-1/2)Th, 0, 0)$ denotes the center of the cuboid $Q_h(i)$. As we suppose that $0 < \rho \le 1 - (19/20)^{1/3}$, (3.44) implies that

$$\mathcal{L}^3(Q^3_{h,\rho}(i)) \ge \frac{19}{20}\mathcal{L}^3(Q^3_h(i))\,. \tag{3.45}$$

We also introduce the (small) parameter

$$\alpha = \Big(\frac{T}{10c_T}\Big)^{2/3}, \tag{3.46}$$

where $c_T := c(T) > 0$ denotes the constant of Theorem 3.6 applied on the cuboid $(0, 3T) \times (-\frac{1}{2}, \frac{1}{2})^2$. We will distinguish three classes of cuboids: first, we consider the family of indices associated to *good cuboids*, defined by

$$I^h_{\rm g} := \Big\{ i = 2, \dots, N-1\colon\ \mathcal{H}^2(\partial E_h^* \cap Q^3_{h,\rho}(i)) \le \alpha h^2 \Big\}\,. \tag{3.47}$$

This will be the family of cuboids for which Theorem 3.6 can be applied without introducing a too large exceptional set, cf. (3.14). Next, we collect the family of *bad cuboids* in the index set

$$\begin{aligned} I^h_{\rm b} :=& \Big\{ i \in \{2,\dots,N-1\} \setminus I^h_{\rm g}\colon \mathcal{H}^2\big(\partial E_h^* \cap Q^3_{h,\rho}(i)\big) < (1-\rho)^2 h^2 \Big\} \\ &\cup \Big\{ i \in \{2,\dots,N-1\} \setminus I^h_{\rm g}\colon \mathcal{H}^2\big(\partial E_h^* \cap Q^3_{h,\rho}(i)\big) < 2(1-\rho)^2 h^2\,,\ \mathcal{L}^3(E_h^* \cap Q^3_{h,\rho}(i)) \le 2C_{\rm iso} h^3 \Big\}\,. \end{aligned} \tag{3.48}$$

For the cuboids in $I^h_{\rm b}$, it might not be possible to apply Theorem 3.6, but due to the relative isoperimetric inequality, see Corollaries 3.9–3.10, we can still find a dominant component which will allow us to compare rigid motions on adjacent cuboids via Lemma 3.7.

On the remaining set of indices

$$I_{\rm u}^h := \{i = 1, \ldots, N\colon i \notin I_{\rm g}^h \cup I_{\rm b}^h\}\,, \tag{3.49}$$

corresponding to the so called *ugly cuboids*, where the thickened void may cut through the rod and thus the behavior of v_h^* cannot be controlled.

Note that for each $i = 2, \ldots, N-1$, we have

$$\#\big\{j \in \{2, \ldots, N-1\}\colon Q^3_{h,\rho}(i) \cap Q^3_{h,\rho}(j) \neq \emptyset\big\} \leq 5\,. \tag{3.50}$$

By (3.47)–(3.50), (3.41)(ii), and (3.36), for $h > 0$ small enough, we obtain

$$\alpha\#\big(I_{\rm b}^h \cup I_{\rm u}^h\big) \leq h^{-2} \sum_{i \in I_{\rm b}^h \cup I_{\rm u}^h} \mathcal{H}^2(\partial E_h^* \cap Q^3_{h,\rho}(i)) \leq Ch^{-2}\mathcal{H}^2(\partial E_h^* \cap \Omega_h) \leq Ch^{-2}\mathcal{G}^{\kappa_h}_{\rm surf}(E_h; \Omega_h) \leq C\,.$$

Thus, we deduce that

$$\#\big(I_{\rm b}^h \cup I_{\rm u}^h\big) = \#\big(\{1, \ldots, N\} \setminus I_{\rm g}^h\big) \leq C \tag{3.51}$$

for $C = C(\alpha) > 0$, i.e., there are only a bounded number of indices in $I_{\rm b}^h \cup I_{\rm u}^h$ independently of h.

We now formulate a local rigidity estimate on cuboids. As a final preparation, we introduce the localized elastic energy by

$$\varepsilon_{i,h} := \int_{Q^3_h(i) \setminus \overline{E_h}} {\rm dist}^2(\nabla v_h, SO(3))\,{\rm d}x\,, \tag{3.52}$$

and use (2.4)(iv), (3.6), (3.36), and (3.50) to find

$$\sum_{i=2}^{N-1} \varepsilon_{i,h} \leq C \int_{\Omega_h \setminus \overline{E_h}} {\rm dist}^2(\nabla v_h, SO(3))\,{\rm d}x \leq Ch^2\epsilon_h\,. \tag{3.53}$$

Proposition 3.11 (Local rigidity estimate and Sobolev approximation)**.** *Let $0 < \rho \leq \rho_0$. There exists a constant $C = C(T) > 0$ independent of h such that for all $h > 0$ and for every $i \in I_{\rm g}^h \cup I_{\rm b}^h$ there exists a set of finite perimeter $D^3_{i,h} \subset Q^3_{h,\rho}(i)$ satisfying*

$$\mathcal{L}^3\big(Q^3_{h,\rho}(i) \setminus D^3_{i,h}\big) \leq Ch\mathcal{H}^2\big(\partial E_h^* \cap Q^3_h(i)\big), \quad \mathcal{L}^3(Q^3_h(i) \setminus D^3_{i,h}) \leq \frac{1}{5}\mathcal{L}^3(Q^3_h(i))\,, \tag{3.54}$$

and a corresponding rigid motion $r_{i,h}(x) := R_{i,h}x + b_{i,h}$, where $R_{i,h} \in SO(3)$ and $b_{i,h} \in \mathbb{R}^3$ with $|b_{i,h}| \leq CM$ (see (2.2) *for the definition of M) such that*

$$h^{-2} \int_{D^3_{i,h}} \big|v_h^*(x) - r_{i,h}(x)\big|^2\,{\rm d}x + \int_{D^3_{i,h}} \big|\nabla v_h^*(x) - R_{i,h}\big|^2\,{\rm d}x \leq C\varepsilon_{i,h}^{9/10}\,, \tag{3.55}$$

where v_h^* is defined in (3.42)*.*

Moreover, for $i \in I_{\rm g}^h$ there exists a Sobolev map $z_{i,h} \in W^{1,2}(Q^3_{h,\rho}(i); \mathbb{R}^3)$ such that

$$\begin{aligned}
&\text{(i)} \quad z_{i,h} \equiv v_h^* \quad \textit{on} \quad D^3_{i,h}\,,\\
&\text{(ii)} \quad h^{-2} \int_{Q^3_{h,\rho}(i)} \big|z_{i,h}(x) - r_{i,h}(x)\big|^2\,{\rm d}x + \int_{Q^3_{h,\rho}(i)} \big|\nabla z_{i,h}(x) - R_{i,h}\big|^2\,{\rm d}x \leq C\varepsilon_{i,h}\,,\\
&\text{(iii)} \quad \|z_{i,h}\|_{L^\infty(Q^3_{h,\rho}(i))} \leq CM\,.
\end{aligned} \tag{3.56}$$

In the following, we refer to $D^3_{i,h}$ as the *dominant component* since $\mathcal{L}^3(Q^3_h(i) \setminus D^3_{i,h})$ is small, see (3.54). Accordingly, $r_{i,h}$ denotes the *dominant rigid motion* which approximates v_h^* in $Q^3_{h,\rho}(i)$. Note that $D^3_{i,h} \subset E_h^*$ is also possible which means that the void has a large volume inside $Q^3_{h,\rho}(i)$.

Observe that the estimate (3.55) is actually better for $i \in I_{\rm g}^h$ as $\varepsilon_{i,h}^{9/10}$ can be replaced by $\varepsilon_{i,h}$. This follows directly from (3.56). This improvement is possible due to the application of a Korn-Poincaré inequality in case of void sets with small surface measure, see Theorem 3.6. We also note that the choice of the exponent 9/10 is for definiteness only and can be enhanced to any exponent smaller than 1, provided the sequence $(\kappa_h)_{h>0}$ in (2.5) is chosen appropriately. Before starting with the proof, let us recall that we use the notation $C > 0$ for generic constants which are independent of h, ρ but may depend on the fixed parameters T, L.

Proof of Proposition 3.11. We use Theorem 3.5 for $\rho > 0$, $l = 3Th$, $\gamma := \bar\kappa_h/h^2$ with $\bar\kappa_h$ from (3.37), and the sequence $\eta_h \to 0$ such that (3.40) holds. We apply the rigidity result to v_h^* in the cuboid $U := Q_h^3(i)$ for $i \in I_{\rm g}^h \cup I_{\rm b}^h$ and the compactly contained cuboid $\widetilde U := Q^3_{h,\rho}(i)$. We denote by

$$\mathcal{P}_{i,h} := \Big\{(P^j_{i,h})_j \text{ the connected components of } Q^3_{h,\rho}(i)\setminus \overline{E_h^*}\Big\} \cup \{E_h^*\}\,,$$

where the enumeration is such that $\mathcal{L}^3(P^1_{i,h})$ is always maximal.

Recall the definitions in (3.47)–(3.48). In the case $i \in I_{\rm g}^h$ or in the case that $i \in I_{\rm b}^h$ with $\mathcal{H}^2\big(\partial E_h^* \cap Q^3_{h,\rho}(i)\big) < (1-\rho)^2 h^2$ we can apply Corollary 3.9 on $Q^3_{h,\rho}(i)$ to obtain a dominant component. If $i \in I_{\rm b}^h$ with $\mathcal{H}^2\big(\partial E_h^* \cap Q^3_{h,\rho}(i)\big) \geq (1-\rho)^2 h^2$ instead, we can apply Corollary 3.10 on $Q^3_{h,\rho}(i)$, where we note that the volume condition in (3.29) is indeed satisfied by the definition of $I_{\rm b}^h$, (3.45), and the fact that $T \geq 80C_{\rm iso}$, see (3.39).

In both cases, using that $\bigcup_{j\geq 1} \partial P^j_{i,h} \cap Q^3_{h,\rho}(i) = \partial E_h^* \cap Q^3_{h,\rho}(i)$, we get a dominant component $P^1_{i,h} \subset Q^3_{h,\rho}(i)$ which by (3.27) or (3.30), respectively, and (3.48) satisfy

$$\mathcal{L}^3(Q^3_{h,\rho}(i)\setminus P^1_{i,h}) \leq C_{\rm iso} h \mathcal{H}^2(\partial E_h^* \cap Q_h^3(i)) \leq C_{\rm iso} h^3 \tag{3.57}$$

or

$$\mathcal{L}^3(Q^3_{h,\rho}(i)\setminus P^1_{i,h}) \leq C_{\rm iso} h \mathcal{H}^2(\partial E_h^* \cap Q_h^3(i)) + \mathcal{L}^3(E_h^* \cap Q^3_{h,\rho}(i)) \leq 2C_{\rm iso}h^3 + 2C_{\rm iso}h^3 = 4C_{\rm iso}h^3\,. \tag{3.58}$$

Therefore, in both cases, we get by (3.39) and (3.45) that

$$\mathcal{L}^3(Q^3_h(i)\setminus P^1_{i,h}) \leq 4C_{\rm iso}h^3 + \mathcal{L}^3(Q^3_h(i)\setminus Q^3_{h,\rho}(i)) \leq \frac{1}{20}Th^3 + \frac{1}{20}\mathcal{L}^3(Q^3_h(i)) \leq \frac{1}{10}\mathcal{L}^3(Q^3_h(i))\,, \tag{3.59}$$

and moreover

$$\mathcal{L}^3\big(Q^3_{h,\rho}(i)\setminus P^1_{i,h}\big) \leq Ch\mathcal{H}^2(\partial E_h^* \cap Q_h^3(i))\,. \tag{3.60}$$

Indeed, in the first case this directly follows from (3.57). In the second case, it follows from (3.58) and the fact $\mathcal{H}^2\big(\partial E_h^* \cap Q^3_{h,\rho}(i)\big) \geq (1-\rho)^2h^2 \geq \frac14 h^2$ (as $0 < \rho \leq \frac12$), where the absolute constant $C > 0$ needs to be chosen sufficiently large.

We now distinguish the cases

$$\text{(a)}\;\; P^1_{i,h} = E_h^*, \quad \text{(b)}\;\; P^1_{i,h}\cap E_h^* = \emptyset,\; i \in I_{\rm b}^h, \quad \text{(c)}\;\; P^1_{i,h}\cap E_h^* = \emptyset,\; i \in I_{\rm g}^h\,.$$

Case (a) : If $P^1_{i,h} = E_h^*$, we define $D^3_{i,h} := P^1_{i,h}$, $R_{i,h} := \mathrm{Id}$, $b_{i,h} := 0$, and $z_{i,h} \in W^{1,2}(Q^3_{h,\rho}(i);\mathbb{R}^3)$ by $z_{i,h} := \mathrm{id}$. Then, (3.54) holds by (3.59)–(3.60) and (3.55)–(3.56) are trivially satisfied (recall (3.42)). Note that in this case we can define a Sobolev modification $z_{i,h}$ also if $i \in I_{\rm b}^h$.

Preparations for (b) *and* (c) : We proceed with preparations for (b) and (c). Suppose that $P^1_{i,h} \cap E_h^* = \emptyset$. Then, (3.13) in Theorem 3.5 provides a rotation $R^1_{i,h} \in SO(3)$ and $b^1_{i,h} \in \mathbb{R}^3$ such that

(i) $\displaystyle\int_{P^1_{i,h}} \big|\mathrm{sym}\big((R^1_{i,h})^T\nabla v_h^* - \mathrm{Id}\big)\big|^2\,\mathrm{d}x \leq C\big(1 + C_{\eta_h}(h^{-2}\bar\kappa_h)^{-15/2}h^{-3}\varepsilon_{i,h}\big)\varepsilon_{i,h}\,,$

(ii) $\displaystyle h^{-2}\int_{P^1_{i,h}} |v_h^* - (R^1_{i,h}x + b^1_{i,h})|^2\,\mathrm{d}x + \int_{P^1_{i,h}} \big|(R^1_{i,h})^T\nabla v_h^* - \mathrm{Id}\big|^2\,\mathrm{d}x \leq C_{\eta_h}(h^{-2}\bar\kappa_h)^{-5}\varepsilon_{i,h}\,,$

where we use the notation in (3.52) and recall that we set $\gamma = \bar\kappa_h/h^2 \in (0,1]$. By the choice of $(\eta_h)_{h>0}$ before (3.40), $\limsup_{h\to 0}\epsilon_h h^{-2} < +\infty$, and (3.53) we obtain

$$\begin{aligned} &\text{(i)} \quad \int_{P^1_{i,h}} \big|\mathrm{sym}\big((R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big)\big|^2\,\mathrm{d}x \le C_0\varepsilon_{i,h}\,,\\ &\text{(ii)} \quad \Big(h^{-2}\int_{P^1_{i,h}} |v^*_h - (R^1_{i,h}x + b^1_{i,h})|^2\,\mathrm{d}x + \int_{P^1_{i,h}} \big|(R^1_{i,h})^T\nabla v^*_h - \mathrm{Id}\big|^2\,\mathrm{d}x\Big) \le C_0 h^{-2/5}\varepsilon_{i,h}\,, \end{aligned} \tag{3.61}$$

for a universal constant $C_0 > 0$. We now show that

$$|b^1_{i,h}| \le CM\,, \tag{3.62}$$

for a constant $C > 0$ that depends on $L, T > 0$, but is independent of $h > 0$. As $\|v_h\|_{L^\infty(\Omega_h)} \le M$ for some $M \ge 1$, the triangle inequality implies

$$\mathcal{L}^3(P^1_{i,h})|b^1_{i,h}|^2 \le C\int_{P^1_{i,h}} |v^*_h(x) - (R^1_{i,h}x + b^1_{i,h})|^2\,\mathrm{d}x + C\mathcal{L}^3(P^1_{i,h})\big(\|v_h\|^2_{L^\infty(\Omega_h)} + (\mathrm{diam}(\Omega_h))^2\big)\,. \tag{3.63}$$

Thus, by (3.53), (3.59), and (3.61)(ii) we get

$$|b^1_{i,h}|^2 \le Ch^{-3}h^2h^{-2/5}\varepsilon_{i,h} + C(M^2 + C) \le C(M^2 + C)\,, \tag{3.64}$$

and thus $|b^1_{i,h}| \le CM$. After these preparations, we continue with the cases (b) and (c).

Case (b) : We first suppose that $i \in I^h_{\mathrm{b}}$. We set $D^3_{i,h} := P^1_{i,h}$. Then, (3.54) follows from (3.59)–(3.60) and (3.55) follows from (3.61)(ii) by setting $R_{i,h} := R^1_{i,h}$ and $b_{i,h} := b^1_{i,h}$, where we use $\varepsilon^{1/10}_{i,h} \le C(h^4)^{1/10}$ by (3.53) and $\limsup_{h\to 0}\epsilon_h h^{-2} < +\infty$. Observe that $|b_{i,h}| \le CM$ by (3.62).

Case (c) : Let us now assume that $i \in I^h_{\mathrm{g}}$. We will use Theorem 3.6 to obtain a Sobolev function which satisfies (3.56). First, let us introduce the function $u_{i,h} \in SBV^2(Q^3_{h,\rho}(i);\mathbb{R}^3)$ by

$$u_{i,h}(x) := \chi_{P^1_{i,h}}(x)\big[(R^1_{i,h})^T v^*_h(x) - x - (R^1_{i,h})^T b^1_{i,h}\big]\,, \tag{3.65}$$

and note that by its definition $J_{u_{i,h}} \subset \partial E^*_h \cap Q^3_{h,\rho}(i)$. Now, (3.53), (3.61), (3.65), as well as $\limsup_{h\to 0}\epsilon_h h^{-2} < +\infty$ imply the bounds

$$\begin{aligned} &\text{(i)} \quad \int_{Q^3_{h,\rho}(i)} |\mathrm{sym}\big(\nabla u_{i,h}\big)|^2\,\mathrm{d}x \le C\varepsilon_{i,h}\,,\\ &\text{(ii)} \quad h^{-2}\int_{Q^3_{h,\rho}(i)} |u_{i,h}|^2\,\mathrm{d}x + \int_{Q^3_{h,\rho}(i)} |\nabla u_{i,h}|^2\,\mathrm{d}x \le C\varepsilon^{9/10}_{i,h}\,. \end{aligned} \tag{3.66}$$

By the scaling invariance of Theorem 3.6 we note that the constant therein is given by c_T appearing in (3.46). Theorem 3.6 for the map $u_{i,h}$ and the definition of I^h_{g} in (3.47) provide a set of finite perimeter $\omega_{i,h} \subset Q^3_{h,\rho}(i)$ satisfying

$$\mathcal{L}^3(\omega_{i,h}) \le c_T\big(\mathcal{H}^2(J_{u_{i,h}})\big)^{3/2} \le c_T\big(\mathcal{H}^2(\partial E^*_h \cap Q^3_{h,\rho}(i))\big)^{3/2} \le c_T\alpha^{1/2}h\mathcal{H}^2(\partial E^*_h \cap Q^3_{h,\rho}(i)) \le c_T\alpha^{3/2}h^3 \tag{3.67}$$

and a Sobolev map $\zeta_{i,h} \in W^{1,2}(Q^3_{h,\rho}(i);\mathbb{R}^3)$ such that

$$\begin{aligned} &\text{(i)} \quad \zeta_{i,h} \equiv u_{i,h} \quad \text{on} \quad Q^3_{h,\rho}(i)\setminus\omega_{i,h}\,,\\ &\text{(ii)} \quad \|\mathrm{sym}(\nabla\zeta_{i,h})\|_{L^2(Q^3_{h,\rho}(i))} \le c_T\|\mathrm{sym}(\nabla u_{i,h})\|_{L^2(Q^3_{h,\rho}(i))}\,,\\ &\text{(iii)} \quad \|\zeta_{i,h}\|_\infty \le \|u_{i,h}\|_\infty \le CM\,, \end{aligned} \tag{3.68}$$

where the last estimate in (iii) follows from $\|v_h^*\|_\infty \le M$, (3.62), and the definition of $u_{i,h}$ in (3.65). In view of (3.46) and (3.67), we get

$$\mathcal{L}^3(\omega_{i,h}) \le \frac{1}{10}Th^3 \le \frac{1}{10}\mathcal{L}^3(Q_h^3(i)) .$$

We define the dominant component

$$D_{i,h}^3 := P_{i,h}^1 \setminus \omega_{i,h} \tag{3.69}$$

and observe by (3.59)–(3.60) and (3.67) that (3.54) holds.

By the classical Korn's inequality in $W^{1,2}$ we find $A_{i,h} \in \mathbb{R}^{3\times 3}_{\rm skew}$ such that

$$\int_{Q_{h,\rho}^3(i)} |\nabla \zeta_{i,h} - A_{i,h}|^2 \,\mathrm{d}x \le C_T \int_{Q_{h,\rho}^3(i)} |\mathrm{sym}(\nabla u_{i,h})|^2 \,\mathrm{d}x \le CC_T\varepsilon_{i,h} , \tag{3.70}$$

where we used (3.68)(ii) and the last step follows from (3.66)(i). Therefore, setting

$$z_{i,h} := R_{i,h}^1\zeta_{i,h} + R_{i,h}^1\mathrm{id} + b_{i,h}^1 \in W^{1,2}(Q_{h,\rho}^3(i);\mathbb{R}^3)$$

we observe by (3.65), (3.68)(i), and (3.69) that $z_{i,h} \equiv v_h^*$ on $D_{i,h}^3$. This yields (3.56)(i). Moreover, (3.56)(iii) follows from (3.68)(iii) and (3.62).

We proceed to show (3.56)(ii). We start with the observation that (3.70) implies

$$\int_{Q_{h,\rho}^3(i)} |\nabla z_{i,h} - R_{i,h}^1(\mathrm{Id} + A_{i,h})|^2 \,\mathrm{d}x \le C\varepsilon_{i,h} . \tag{3.71}$$

Now, we need to replace $R_{i,h}^1(\mathrm{Id} + A_{i,h})$ suitably by a rotation. We claim that there exists $R_{i,h} \in SO(3)$ such that

$$\mathcal{L}^3(Q_{h,\rho}^3(i))|R_{i,h}^1(\mathrm{Id} + A_{i,h}) - R_{i,h}|^2 \le C\varepsilon_{i,h} . \tag{3.72}$$

In order to show (3.72), we argue as follows. By (3.54) together with (3.68)(i), (3.66)(ii), and (3.70) we get

$$\begin{aligned}\tfrac{4}{5}\mathcal{L}^3(Q_h(i))|A_{i,h}|^2 \le \mathcal{L}^3(D_{i,h}^3)|A_{i,h}|^2 &= \int_{D_{i,h}^3} \big|\nabla u_{i,h} + A_{i,h} - \nabla\zeta_{i,h}\big|^2 \,\mathrm{d}x \\ &\le 2\Big(\int_{D_{i,h}^3} |\nabla u_{i,h}|^2 \,\mathrm{d}x + \int_{D_{i,h}^3} |\nabla \zeta_{i,h} - A_{i,h}|^2 \,\mathrm{d}x\Big) \le C(\varepsilon_{i,h} + \varepsilon_{i,h}^{9/10}) .\end{aligned}$$

By using the fact that $\varepsilon_{i,h} \le Ch^2\epsilon_h \le Ch^4$, see (3.53), and $\mathcal{L}^3(Q_h(i)) = Th^3$, we obtain an estimate on $A_{i,h}$, namely

$$|A_{i,h}|^2 \le Ch^{-3}\varepsilon_{i,h}^{9/10} \le Ch^{-7/5}\varepsilon_{i,h}^{1/2} .$$

Therefore, the Taylor expansion (see [41, Equation (33)])

$$\mathrm{dist}(G, SO(3)) = |\mathrm{sym}(G) - \mathrm{Id}| + \mathrm{O}(|G - \mathrm{Id}|^2)$$

allows us to estimate

$$\mathrm{dist}^2\big((\mathrm{Id} + A_{i,h}), SO(3)\big) \le C|A_{i,h}|^4 \le Ch^{-14/5}\varepsilon_{i,h} ,$$

i.e., there exists indeed $R_{i,h} \in SO(3)$ for which

$$|R_{i,h}^1(\mathrm{Id} + A_{i,h}) - R_{i,h}|^2 \le Ch^{1/5}h^{-3}\varepsilon_{i,h} \le Ch^{-3}\varepsilon_{i,h} \le C(\mathcal{L}^3(Q_{h,\rho}^3(i)))^{-1}\varepsilon_{i,h} .$$

This proves (3.72). Hence, in view of (3.71) and(3.72) we get

$$\int_{Q_{h,\rho}^3(i)} |\nabla z_{i,h} - R_{i,h}|^2 \,\mathrm{d}x \le C\varepsilon_{i,h} ,$$

which yields the second part of (3.56)(ii). Finally, the Poincaré inequality on $W^{1,2}(Q^3_{h,\rho}(i);\mathbb{R}^3)$ also implies that there exists a vector $b_{i,h} \in \mathbb{R}^3$ such that the rigid motion $r_{i,h}(x) := R_{i,h}x + b_{i,h}$ satisfies

$$h^{-2} \int_{Q^3_{h,\rho}(i)} |z_{i,h}(x) - r_{i,h}(x)|^2 \,\mathrm{d}x \le C\varepsilon_{i,h}\,.$$

This concludes the proof of (3.56)(ii). Eventually, in the case $i \in I^h_{\mathrm{g}}$, we note that estimate (3.55) is an immediate consequence of (3.56). Therefore, by repeating exactly the argument in (3.63)–(3.64) with $b_{i,h}$ in place of $b^1_{i,h}$ we also get that $|b_{i,h}| \le CM$. This concludes the proof. □

As a consequence, we can estimate the difference of two dominant rigid motions on adjacent cuboids.

Corollary 3.12 (Difference of rigid motions)**.** *Suppose $i, i+1 \in I^h_{\mathrm{g}} \cup I^h_{\mathrm{b}}$. The rigid motions $r_{i,h}$, $r_{i+1,h}$ given in Proposition 3.11 satisfy*

$$h^{-2}\|r_{i,h} - r_{i+1,h}\|^2_{L^\infty(Q^3_h(i)\cup Q^3_h(i+1))} + \big|R_{i,h} - R_{i+1,h}\big|^2 \le Ch^{-3}(\varepsilon^{9/10}_{i,h} + \varepsilon^{9/10}_{i+1,h})\,. \tag{3.73}$$

If $i, i+1 \in I^h_{\mathrm{g}}$, the better estimate

$$h^{-2}\|r_{i,h} - r_{i+1,h}\|^2_{L^\infty(Q^3_h(i)\cup Q^3_h(i+1))} + \big|R_{i,h} - R_{i+1,h}\big|^2 \le Ch^{-3}(\varepsilon_{i,h} + \varepsilon_{i+1,h}) \tag{3.74}$$

holds.

Proof. By (3.55) and the triangle inequality we have

$$\begin{aligned}\int_{D^3_{i,h}\cap D^3_{i+1,h}} \big|r_{i,h} - r_{i+1,h}\big|^2 \,\mathrm{d}x &\le 2\int_{D^3_{i,h}} \big|v^*_h(x) - r_{i,h}(x)\big|^2 \,\mathrm{d}x + 2\int_{D^3_{i+1,h}} \big|v^*_h(x) - r_{i+1,h}(x)\big|^2 \,\mathrm{d}x \\ &\le Ch^2(\varepsilon^{9/10}_{i,h} + \varepsilon^{9/10}_{i+1,h})\,. \end{aligned} \tag{3.75}$$

Note that $\mathcal{L}^3(Q^3_h(i) \cap Q^3_h(i+1)) = 2Th^3$ and $\mathcal{L}^3(Q^3_h(j) \setminus D^3_{j,h}) \le \frac{3}{5}Th^3$ by (3.54) for $j = i, i+1$. This yields

$$\mathcal{L}^3(D^3_{i,h} \cap D^3_{i+1,h}) \ge \mathcal{L}^3(Q^3_h(i) \cap Q^3_h(i+1)) - \mathcal{L}^3(Q^3_h(i) \setminus D^3_{i,h}) - \mathcal{L}^3(Q^3_h(i+1) \setminus D^3_{i+1,h}) \ge \frac{4}{5}Th^3\,.$$

Moreover, we observe that $Q^3_h(i) \cup Q^3_h(i+1)$ is contained in a ball of radius $r = cTh$ for a universal constant $c > 0$. This along with (3.75) and Lemma 3.7 shows (3.73). Estimate (3.74) follows in the same fashion noting that with (3.56)(ii) in place of (3.55) the exponents 9/10 in (3.75) can be replaced by 1. □

3.4. **Construction of blockwise Sobolev modifications and proofs of the propositions.** This subsection is devoted to the construction of w_h and R_h, as well as to the proofs of Proposition 3.1 and Proposition 3.2.

We start with the construction of $(w_h)_{h>0}$ and $(R_h)_{h>0}$. To this end, let $\psi^h \in C^\infty(\mathbb{R}^3)$ be a cut-off function satisfying $\psi^h(x) = \psi^h(x_1, 0, 0)$ for $x \in \mathbb{R}^3$, $0 \le \psi^h \le 1$, $\psi^h \equiv 1$ on $\{x_1 \le -h\}$, and $\psi^h \equiv 0$ on $\{x_1 \ge h\}$ such that

$$\|\nabla\psi^h\|_\infty \le Ch^{-1}\,. \tag{3.76}$$

Recalling (3.2), for each $i = 1, \dots, N-1$, we set $\psi^h_{i,i+1}(x) = \psi^h(x - iThe_1)$. For $i = 1, \dots, N-1$, we also define the sets

$$\Psi^h_{i,i+1} := \begin{cases} \big((iTh - h, iTh + h) \times \mathbb{R}^2\big) \cap \Omega_{h,\rho} & \text{if } i, i+1 \in I^h_{\mathrm{g}}, \\ \emptyset & \text{else}, \end{cases}$$

i.e., $\{\psi^h_{i,i+1} \in (0,1)\} \cap \Omega_{h,\rho} \subset \Psi^h_{i,i+1}$, provided that $i, i+1 \in I^h_{\rm g}$. Note that the sets $(\Psi^h_{i,i+1})_i$ are pairwise disjoint by (3.39). Moreover, since $0 < \rho \le 1 - (19/20)^{1/3} \le 0.017$, we get that

$$\big(Q_h(i) \cup \Psi^h_{i-1,i} \cup \Psi^h_{i,i+1}\big) \cap \Omega_{h,\rho} \subset Q^3_{h,\rho}(i) \quad \text{for all } i = 2, \dots, N-1, \tag{3.77}$$

cf. (3.44). To see this, by (3.39), it suffices to note that $(T - \frac{3}{2}T\rho - 1)h \ge C_{\rm iso}(79 - 120\rho)h > 0$.

We now define the sequences $(w_h)_{h>0}$ and $(R_h)_{h>0}$. First, we construct $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$ as follows. We set

$$w_h := {\rm id} \qquad \text{on } \ Q_h(i) \cap \Omega_{h,\rho} \ \text{ for all } i \in I^h_{\rm u}\,, \tag{3.78}$$

and

$$w_h := r_{i,h} \quad \text{ on } Q_h(i) \cap \Omega_{h,\rho} \ \text{ for all } i \in I^h_{\rm b}\,, \tag{3.79}$$

where $r_{i,h}$ denotes the rigid motion given in (3.55). Eventually, recalling the definition of the Sobolev maps $z_{i,h} \in W^{1,2}(Q^3_{h,\rho}(i);\mathbb{R}^3)$ in Proposition 3.11, given $i \in I^h_{\rm g} \subset \{2, \dots, N-1\}$, and $x \in Q_h(i) \cap \Omega_{h,\rho}$, we define

$$w_h(x) := \begin{cases} z_{i,h}(x) & \text{if } x \in Q_h(i) \setminus (\Psi^h_{i-1,i} \cup \Psi^h_{i,i+1})\,, \\ \psi^h_{i-1,i}(x) z_{i-1,h}(x) + (1 - \psi^h_{i-1,i}(x)) z_{i,h}(x) & \text{if } x \in Q_h(i) \cap \Psi^h_{i-1,i}\,, \\ \psi^h_{i,i+1}(x) z_{i,h}(x) + (1 - \psi^h_{i,i+1}(x)) z_{i+1,h}(x) & \text{if } x \in Q_h(i) \cap \Psi^h_{i,i+1}\,, \end{cases} \tag{3.80}$$

where the second and third part of the definition might be empty if $\Psi^h_{i-1,i} = \emptyset$ or $\Psi^h_{i,i+1} = \emptyset$, respectively. Note that this is well defined by (3.77), and the fact that $z_{i-1,h}$ or $z_{i+1,h}$ exist if $\Psi^h_{i-1,i} \neq \emptyset$ or $\Psi^h_{i,i+1} \neq \emptyset$, respectively.

In the absence of information on the second derivatives of $(v_h)_{h>0}$, we construct another sequence of functions $(R_h)_{h>0}$ with $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times 3})$ which approximate ∇v_h and whose derivative can be controlled. We define

$$R_h := \nabla w_h \quad \text{ on } Q_h(i) \cap \Omega_{h,\rho} \ \text{ for all } i \in I^h_{\rm b} \cup I^h_{\rm u}\,, \tag{3.81}$$

and for $x \in Q_h(i) \cap \Omega_{h,\rho}$, $i \in I^h_{\rm g}$, we let

$$R_h(x) := \begin{cases} R_{i,h} & \text{if } x \in Q_h(i) \setminus (\Psi^h_{i-1,i} \cup \Psi^h_{i,i+1})\,, \\ \psi^h_{i-1,i}(x) R_{i-1,h} + (1 - \psi^h_{i-1,i}(x)) R_{i,h} & \text{if } x \in Q_h(i) \cap \Psi^h_{i-1,i}\,, \\ \psi^h_{i,i+1}(x) R_{i,h} + (1 - \psi^h_{i,i+1}(x)) R_{i+1,h} & \text{if } x \in Q_h(i) \cap \Psi^h_{i,i+1}\,, \end{cases} \tag{3.82}$$

where $R_{i,h}$ are given by Proposition 3.11.

Note that the construction implies that indeed $w_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^3)$, $R_h \in SBV^2(\Omega_{h,\rho};\mathbb{R}^{3\times 3})$, and the jump sets satisfy

$$J_{w_h} \cup J_{R_h} \subset \Omega_{h,\rho} \cap \bigcup_{i \in I^h_{\rm b} \cup I^h_{\rm u}} \partial Q_h(i)\,. \tag{3.83}$$

We are now ready to give the proofs of the propositions.

Proof of Proposition 3.1. First, (3.8)(i) follows from the construction (3.78)–(3.82), the uniform control in (3.56)(iii), the bound $|b_{i,h}| \le CM$ for $i \in I^h_{\rm b}$, and the fact that $SO(3) \subset \mathbb{R}^{3\times 3}$ is compact. To see (3.8)(ii), we use (3.83) and the fact that $\#(I^h_{\rm b} \cup I^h_{\rm u}) \le C$, see (3.51).

We proceed to show (3.8)(iii),(iv) for w_h and defer the proof for R_h to the end. Regarding (3.8)(iii), we note that by the definition of w_h and (3.56)(i),

$$\{w_h \neq v_h\} \subset B_h := \bigcup_{i \in I^h_{\rm b} \cup I^h_{\rm u}} Q^3_h(i) \cup \bigcup_{i \in I^h_{\rm g}} \big(Q^3_{h,\rho}(i) \setminus D^3_{i,h}\big) \cup \{v_h \neq v^*_h\}\,.$$

Since $\#(I^h_{\rm b}\cup I^h_{\rm u})\le C$, using also (3.36), (3.41)(ii), and (3.54), we find

$$\begin{aligned}\mathcal{L}^3(\{w_h\neq v_h\})\le \mathcal{L}^3(B_h)&\le CTh^3+Ch\sum_{i\in I^h_{\rm g}}\mathcal{H}^2(\partial E^*_h\cap Q^3_h(i))+\mathcal{L}^3(\{v_h\neq v^*_h\})\\ &\le CTh^3+Ch\mathcal{H}^2(\partial E^*_h\cap\Omega_h)+\mathcal{L}^3(\{v_h\neq v^*_h\})\le Ch^3+\mathcal{L}^3(\{v_h\neq v^*_h\}),\end{aligned}\tag{3.84}$$

where we also used the fact that each cuboid $Q^3_h(i)$ overlaps only with neighboring ones, cf. (3.50). This along with (3.43) shows (3.8)(iii) for w_h.

Eventually, we show (3.8)(iv) for w_h. First, by (3.78)–(3.79) we observe that

$$\int_{\Omega_{h,\rho}}{\rm dist}^2(\nabla w_h,SO(3))\,{\rm d}x=\sum_{i\in I^h_{\rm g}}\int_{Q_h(i)\cap\Omega_{h,\rho}}{\rm dist}^2(\nabla w_h,SO(3))\,{\rm d}x\,.\tag{3.85}$$

Moreover, by (3.80) and (3.56)(ii) we compute, for $i\in I^h_{\rm g}$,

$$\begin{aligned}\int_{(Q_h(i)\cap\Omega_{h,\rho})\setminus(\Psi^h_{i-1,i}\cup\Psi^h_{i,i+1})}{\rm dist}^2(\nabla w_h,SO(3))\,{\rm d}x&\le\int_{Q^3_{h,\rho}(i)}{\rm dist}^2(\nabla z_{i,h},SO(3))\,{\rm d}x\\ &\le\int_{Q^3_{h,\rho}(i)}|\nabla z_{i,h}-R_{i,h}|^2\,{\rm d}x\le C\varepsilon_{i,h}\,.\end{aligned}$$

This along with (3.53) shows

$$\sum_{i\in I^h_{\rm g}}\int_{(Q_h(i)\cap\Omega_{h,\rho})\setminus(\Psi^h_{i-1,i}\cup\Psi^h_{i,i+1})}{\rm dist}^2(\nabla w_h,SO(3))\,{\rm d}x\le C\sum_{i\in I^h_{\rm g}}\varepsilon_{i,h}\le Ch^2\epsilon_h\,.\tag{3.86}$$

For all $\Psi^h_{i,i+1}\neq\emptyset$, i.e., $i,i+1\in I^h_{\rm g}$, we estimate using (3.56)(ii), (3.77), and (3.80)

$$\begin{aligned}\int_{\Psi^h_{i,i+1}}{\rm dist}^2(\nabla w_h,SO(3))&=\int_{\Psi^h_{i,i+1}}{\rm dist}^2\big(\nabla\big(z_{i+1,h}+\psi^h_{i,i+1}(z_{i,h}-z_{i+1,h})\big),SO(3)\big)\\ &\le C\int_{Q^3_{h,\rho}(i+1)}{\rm dist}^2(\nabla z_{i+1,h},SO(3))\\ &\quad+C\int_{\Psi^h_{i,i+1}}(|\nabla\psi^h_{i,i+1}|^2|z_{i,h}-z_{i+1,h}|^2+|\psi^h_{i,i+1}|^2|\nabla z_{i,h}-\nabla z_{i+1,h}|^2)\\ &\le C\varepsilon_{i+1,h}+C\int_{\Psi^h_{i,i+1}}\big(h^{-2}|z_{i,h}-z_{i+1,h}|^2+|\nabla z_{i,h}-\nabla z_{i+1,h}|^2\big)\,,\end{aligned}\tag{3.87}$$

where in the last step we used that $0\le\psi^h_{i,i+1}\le 1$ and $\|\nabla\psi^h_{i,i+1}\|_\infty\le Ch^{-1}$, see (3.76). Since $\Psi^h_{i,i+1}\subset Q^3_{h,\rho}(i),Q^3_{h,\rho}(i+1)$ by (3.77), we compute by (3.56)(ii), (3.74), and the triangle inequality

$$\begin{aligned}\int_{\Psi^h_{i,i+1}}|z_{i,h}-z_{i+1,h}|^2\,{\rm d}x&\le C\int_{Q^3_{h,\rho}(i)}|z_{i,h}-r_{i,h}|^2\,{\rm d}x+C\int_{Q^3_{h,\rho}(i+1)}|z_{i+1,h}-r_{i+1,h}|^2\,{\rm d}x\\ &\quad+C\int_{Q^3_{h,\rho}(i)\cup Q^3_{h,\rho}(i+1)}|r_{i,h}-r_{i+1,h}|^2\,{\rm d}x\le Ch^2(\varepsilon_{i,h}+\varepsilon_{i+1,h})\,,\end{aligned}$$

where we also used that $\mathcal{L}^3(Q^3_{h,\rho}(i)\cup Q^3_{h,\rho}(i+1))\le CTh^3$. In a similar fashion, (3.56)(ii) and (3.74) also imply

$$\begin{aligned}\int_{\Psi^h_{i,i+1}}|\nabla z_{i,h}-\nabla z_{i+1,h}|^2\,{\rm d}x&\le C\int_{Q^3_{h,\rho}(i)}|\nabla z_{i,h}-R_{i,h}|^2\,{\rm d}x+C\int_{Q^3_{h,\rho}(i+1)}|\nabla z_{i+1,h}-R_{i+1,h}|^2\,{\rm d}x\\ &\quad+Ch^3|R_{i,h}-R_{i+1,h}|^2\le C(\varepsilon_{i,h}+\varepsilon_{i+1,h})\,.\end{aligned}$$

The last two estimates along with (3.87) and (3.53) show

$$\sum_{i,i+1\in I^h_{\rm g}} \int_{\Psi^h_{i,i+1}} \mathrm{dist}^2(\nabla w_h, SO(3))\,\mathrm{d}x \le C\sum_{i=2}^{N-1} \varepsilon_{i,h} \le Ch^2\epsilon_h\,.$$

This together with (3.85)–(3.86) concludes the proof of the first inequality in (3.8)(iv).

We now continue with the proof of (3.8)(iii) for R_h. By (3.84) the set B_h satisfies $h^{-2}\mathcal{L}^3(B_h)\to 0$ as $h\to 0$. To obtain an estimate on the complement $\Omega_{h,\rho}\setminus B_h$, we recall the definition of w_h and R_h in (3.80) and (3.82), respectively. In particular, as $z_{i,h}=z_{i+1,h}$ on $\Psi^h_{i,i+1}\setminus B_h$, see (3.56)(ii), we have $\nabla v_h=\nabla w_h=\nabla z_{i,h}=\nabla z_{i+1,h}=\psi^h_{i,i+1}\nabla z_{i,h}+(1-\psi^h_{i,i+1})\nabla z_{i+1,h}$ on $\Psi^h_{i,i+1}\setminus B_h$. Therefore, one can check that

$$\int_{\Omega_{h,\rho}\setminus B_h} |\nabla v_h - R_h|^2\,\mathrm{d}x = \int_{\Omega_{h,\rho}\setminus B_h} |\nabla w_h - R_h|^2\,\mathrm{d}x \le C\sum_{i\in I^h_{\rm g}}\int_{Q^3_{h,\rho}(i)} |\nabla z_{i,h}-R_{i,h}|^2\,\mathrm{d}x \le Ch^2\epsilon_h\,,$$

where the last step follows from (3.56)(ii) and (3.53). Let $(\theta_h)_{h>0}\subset(0,+\infty)$ be an infinitesimal sequence such that $\theta_h\epsilon_h^{-1/2}\to\infty$. Then, by using $h^{-2}\mathcal{L}^3(B_h)\to 0$ we compute

$$\begin{aligned} h^{-2}\mathcal{L}^3\big(\Omega_{h,\rho}\cap\big\{|\nabla v_h-R_h|>\theta_h\big\}\big) &\le h^{-2}\mathcal{L}^3\big((\Omega_{h,\rho}\setminus B_h)\cap\big\{|\nabla v_h-R_h|>\theta_h\big\}\big)+h^{-2}\mathcal{L}^3(B_h)\\ &\le h^{-2}\theta_h^{-2}\int_{\Omega_{h,\rho}\setminus B_h}|\nabla v_h-R_h|^2\,\mathrm{d}x + h^{-2}\mathcal{L}^3(B_h)\\ &\le C\epsilon_h\theta_h^{-2}+h^{-2}\mathcal{L}^3(B_h)\to 0\,. \end{aligned}$$

This shows (3.8)(iii) for R_h.

We finally show the second estimate in (3.8)(iv). We observe $\nabla R_h=0$ on $\Omega_{h,\rho}\setminus\bigcup_i\Psi^h_{i,i+1}$ (recall (3.81), (3.82)). For all $\Psi^h_{i,i+1}\neq\emptyset$, i.e., $i,i+1\in I^h_{\rm g}$, by (3.74) and the fact that $\mathcal{L}^3(\Psi^h_{i,i+1})\le 2h^3$, we compute

$$\begin{aligned} \int_{\Psi^h_{i,i+1}} |\nabla R_h|^2\,\mathrm{d}x &= \int_{\Psi^h_{i,i+1}} |\nabla\psi^h_{i,i+1}|^2|R_{i,h}-R_{i+1,h}|^2\,\mathrm{d}x\\ &\le Ch^{-2}\int_{\Psi^h_{i,i+1}}|R_{i,h}-R_{i+1,h}|^2\,\mathrm{d}x \le Ch^{-2}(\varepsilon_{i,h}+\varepsilon_{i+1,h})\,, \end{aligned}$$

where we again used that $\|\nabla\psi^h_{i,i+1}\|\le Ch^{-1}$. Summing over all $i\in I^h_{\rm g}$ and using (3.53) we conclude

$$\sum_{i\in I^h_{\rm g}}\int_{\Psi^h_{i,i+1}}|\nabla R_h|^2\,\mathrm{d}x\le C\epsilon_h\,.$$

This concludes the proof of the second estimate in (3.8)(iv). □

We close this section with the proof of Proposition 3.2.

Proof of Proposition 3.2. Fix the stripe $S^{2l}_h(x)$ as in the statement. We start the proof with the following observation: Assumption (3.10) implies that

$$i\in I^h_{\rm g}\cup I^h_{\rm b}\quad\text{for each}\quad i\in I^h:=\{i\colon Q^3_h(i)\cap S^l_{h,\rho}(x)\neq\emptyset\}\,. \tag{3.88}$$

Indeed, since $l\ge 6Th$, we first get that $Q^3_h(i)\subset S^{3l/2}_h(x)$ for all $i\in I^h$, see (3.3)–(3.4). Then, in view of (3.10)(i), we find $\mathcal{H}^2\big(\partial E^*_h\cap Q^3_{h,\rho}(i)\big)<2(1-\rho)^2h^2$ for all $i\in I^h$. Thus, recalling (3.47)–(3.48), to show that $i\in I^h_{\rm g}\cup I^h_{\rm b}$ it suffices to check that

$$\mathcal{L}^3\big(E^*_h\cap S^{3l/2}_h(x)\big)\le 2C_{\rm iso}h^3\,. \tag{3.89}$$

Thus, let us check (3.89). Again using (3.10)(i) we can find a partition $S_h^{2l}(x) = U_- \cup U_+$ (up to a set of negligible $\mathcal{L}^3$- measure) with disjoint open cuboids U_-, U_+ such that

$$\mathcal{H}^2\big(\partial E_h^* \cap U_\pm\big) < h^2 \,. \tag{3.90}$$

($\pm$ is a shorthand for $+$ or $-$.) If $U_\pm \cap S_h^{3l/2}(x) = \emptyset$, the set is irrelevant for showing (3.89). Thus, we suppose that $U_\pm \cap S_h^{3l/2}(x) \neq \emptyset$. Then, Proposition 3.8 implies

$$\min\{\mathcal{L}^3(E_h^* \cap U_\pm), \mathcal{L}^3(U_\pm \setminus E_h^*)\} \le C_{\rm iso} h \mathcal{H}^2(\partial E_h^* \cap U_\pm)\,.$$

(Note that the proposition is applicable as $U_\pm$ contains at least one T-cuboid.) By (3.90) we get

$$\min\{\mathcal{L}^3(E_h^* \cap U_\pm), \mathcal{L}^3(U_\pm \setminus E_h^*)\} \le C_{\rm iso} h^3\,. \tag{3.91}$$

For $U_\pm \cap S_h^{3l/2}(x) \neq \emptyset$ we have $\mathcal{L}^3(U_\pm) \ge \frac{1}{8}\mathcal{L}^3(S_h^{2l}(x))$. Then, necessarily, $\mathcal{L}^3(E_h^* \cap U_\pm) \le \mathcal{L}^3(U_\pm \setminus E_h^*)$, since otherwise by (3.10)(ii)

$$\frac{1}{8}\mathcal{L}^3(S_h^{2l}(x)) \le \mathcal{L}^3(U_\pm) \le \mathcal{L}^3(U_\pm \setminus E_h^*) + \mathcal{L}^3(U_\pm \cap E_h^*) \le C_{\rm iso} h^3 + \frac{1}{9}\mathcal{L}^3(S_h^{2l}(x))\,.$$

This yields a contradiction, since

$$\mathcal{L}^3(S_h^{2l}(x)) = 4lh^2 \ge 24Th^3 \ge 1920 C_{\rm iso} h^3\,,$$

see (3.39). Thus, using (3.91) we conclude

$$\mathcal{L}^3\big(E_h^* \cap S_h^{3l/2}(x)\big) \le \mathcal{L}^3\big(E_h^* \cap U_- \cap S_h^{3l/2}(x)\big) + \mathcal{L}^3\big(E_h^* \cap U_+ \cap S_h^{3l/2}(x)\big) \le 2C_{\rm iso} h^3.$$

This shows (3.89), and thus (3.88) holds.

We are now ready to verify (3.11). In view of (3.88), (3.83) yields that

$$J_{w_h} \cap S_{h,\rho}^l(x) \subset \bigcup_{i \in I_{\rm b}^h} \partial Q_h(i) \cap \Omega_{h,\rho}\,.$$

Note that in each cuboid $Q_h(i)$, $i \in I_{\rm b}^h$, the trace $\mathrm{tr}(w_h)$ on $\partial Q_h(i) \cap \Omega_{h,\rho}$ coincides with $r_{i,h}$ (recall (3.79)). If the neighboring cuboid is good, i.e., $i-1 \in I_{\rm g}^h$ or $i+1 \in I_{\rm g}^h$, by (3.56)(ii), (3.80), and the trace estimate on $Q_h(j)$ (with its scaling), for $j = i-1, i+1$, the trace $\mathrm{tr}(w_h)$ satisfies

$$\int_{\partial Q_h(i) \cap \partial Q_h(j) \cap \Omega_{h,\rho}} |\mathrm{tr}(w_h) - r_{j,h}|^2 \,\mathrm{d}\mathcal{H}^2 \le Ch \int_{Q_h(j)} \big(h^{-2}|w_h - r_{j,h}|^2 + |\nabla w_h - R_{j,h}|^2\big)\,\mathrm{d}x \le Ch\varepsilon_{j,h}\,.$$

Thus, by a discrete Hölder's inequality we find for each $i \in I_{\rm b}^h$ that

$$\gamma_i := \sum_{j=i-1,i+1} \int_{\partial Q_h(i) \cap \partial Q_h(j) \cap \Omega_{h,\rho}} |\mathrm{tr}(w_h) - r_{j,h}|^{1/2} \,\mathrm{d}\mathcal{H}^2 \le C(h^2)^{3/4} h^{1/4}\big((\varepsilon_{i-1,h})^{1/4} + (\varepsilon_{i+1,h})^{1/4}\big)\,.$$

Now, by (3.73) we compute

$$\begin{aligned}
\int_{J_{w_h} \cap S_{h,\rho}^l(x)} |[w_h]|^{1/2} \,\mathrm{d}\mathcal{H}^2 &\le \sum_{i \in I_{\rm b}^h} \int_{\partial Q_h(i) \cap \Omega_{h,\rho}} |(w_h)_+ - (w_h)_-|^{1/2} \,\mathrm{d}\mathcal{H}^2 \\
&\le C \sum_{i \in I_{\rm b}^h} \big(h^2 \|r_{i-1,h} - r_{i,h}\|^{1/2}_{L^\infty(Q_h^3(i))} + h^2 \|r_{i,h} - r_{i+1,h}\|^{1/2}_{L^\infty(Q_h^3(i))} + \gamma_i\big) \\
&\le C \sum_{i \in I_{\rm b}^h} \Big(h^{7/4}\big(\varepsilon_{i-1,h}^{9/40} + \varepsilon_{i,h}^{9/40} + \varepsilon_{i+1,h}^{9/40}\big) + h^{7/4}\big((\varepsilon_{i-1,h})^{1/4} + (\varepsilon_{i+1,h})^{1/4}\big)\Big)\,.
\end{aligned}$$

A discrete Hölder's inequality along with $\#I_{\rm b}^h \le C$ (recall (3.51)) and (3.53) then yields

$$\int_{J_{w_h}\cap S^l_{h,\rho}(x)} |[w_h]|^{1/2}\,\mathrm{d}\mathcal{H}^2 \le Ch^{7/4}\Big(\sum_{i=2}^{N-1}\varepsilon_{i,h}\Big)^{9/40} \le Ch^{11/5}\epsilon_h^{9/40}\,.$$

This shows the first part of (3.11). For the second part, we compute in a similar fashion, again using (3.53) and (3.73), and the construction in (3.81)–(3.82)

$$\begin{aligned}\int_{J_{R_h}\cap S^l_{h,\rho}(x)} |[R_h]|^{1/2}\,\mathrm{d}\mathcal{H}^2 &\le \sum_{i\in I_{\rm b}^h}\int_{\partial Q_h(i)\cap\Omega_{h,\rho}} |(R_h)_+ - (R_h)_-|^{1/2}\,\mathrm{d}\mathcal{H}^2\\ &\le Ch^2\sum_{i\in I_{\rm b}^h}\big(|R_{i-1,h}-R_{i,h}|^{1/2}+|R_{i,h}-R_{i+1,h}|^{1/2}\big)\\ &\le Ch^{2-3/4}\sum_{i\in I_{\rm b}^h}\big(\varepsilon_{i-1,h}^{9/40}+\varepsilon_{i,h}^{9/40}+\varepsilon_{i+1,h}^{9/40}\big)\le Ch^{17/10}\epsilon_h^{9/40}\,.\end{aligned}$$

This along with $\limsup_{h\to 0}\epsilon_h h^{-2} < +\infty$ concludes the proof. □

Remark 3.13 (Variant of Proposition 3.2)**.** Let us briefly comment on Remark 3.3. The proof of (i) basically follows from the previous proof by noting that the assumption (3.10)(i) with 1 in place of 2 (on the right hand side) excludes the presence of ugly cuboids. For (ii), we also follow the estimates above and observe that, in the worst case $\epsilon_h \sim h^2$, the integral over jump heights $|[w_h]|^{1-\beta}$ and $|[R_h]|^{1-\beta}$ can be estimated by $h^2h^{13(1-\beta)/10}$ and $h^2h^{3(1-\beta)/10}$, respectively.

4. Compactness

This section is devoted to the proof of Theorem 2.1. We again use the continuum subscript $h > 0$ instead of the sequential subscript notation $(h_j)_{j\in\mathbb{N}}$ for convenience. We first recall the relevant result from the Sobolev setting.

Lemma 4.1 (Compactness in the Sobolev setting)**.** *Let $\Omega_{\ell_1,\ell_2} := (0,\ell_1)\times(-\ell_2,\ell_2)^2$ for $\ell_1,\ell_2>0$, and let $(\widetilde{w}_h)_{h>0}$ be a bounded sequence in $W^{1,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3)$ such that*

$$\limsup_{h\to0}\frac{1}{h^2}\int_{\Omega_{\ell_1,\ell_2}}\mathrm{dist}^2(\nabla_h\widetilde{w}_h,SO(3))\,\mathrm{d}x + \|\widetilde{w}_h\|_{W^{1,2}(\Omega_{\ell_1,\ell_2})} \le C_0 < +\infty\,.$$

Then, there exist $\bar y\in W^{2,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3)$ and $\bar d_2,\bar d_3\in W^{1,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3)$, all independent of (x_2,x_3), and a subsequence (not relabeled) such that

$$\widetilde{w}_h \rightharpoonup \bar y \quad \text{weakly in } W^{1,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3),\qquad \nabla_h\widetilde{w}_h\to\big(\bar y_{,1}\big|\,\bar d_2\big|\,\bar d_3\big) \text{ strongly in } L^2(\Omega_{\ell_1,\ell_2};\mathbb{R}^{3\times3})\,. \tag{4.1}$$

Moreover, $(\bar y_{,1}|\,\bar d_2|\,\bar d_3)\in SO(3)$ a.e. in Ω_{ℓ_1,ℓ_2}, and

$$\|\bar y\|_{W^{2,2}(\Omega_{\ell_1,\ell_2})} + \|\bar d_2\|_{W^{1,2}(\Omega_{\ell_1,\ell_2})} + \|\bar d_3\|_{W^{1,2}(\Omega_{\ell_1,\ell_2})} \le C \tag{4.2}$$

for a constant $C>0$ only depending on C_0.

For the proof we refer to [58, Theorem 2.1]. The weak convergence to $\bar y$ has not been mentioned in the original statement, but follows directly from weak compactness. A simple argument shows that $\bar y$ is indeed independent of (x_2,x_3). In fact, (4.1) and (2.9) yield $\bar y_{,i}\equiv 0$ for $i=2,3$. Property (4.2) has not been stated explicitly in [58, Theorem 2.1], but is a consequence of the proof of [58, Theorem 2.1] (cf. the last estimate therein) and the equivalent characterization of Sobolev spaces via finite differences, see e.g. [22, Theorem 1.36]. We now proceed with the proof of our compactness result.

Proof of Theorem 2.1. By the energy bound (2.19) and (2.10) we have that

$$h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h(x))\,\mathrm{d}x+\int_{\partial V_h\cap\Omega}\big|\big(\nu^1_{V_h}(z),h^{-1}\nu^2_{V_h}(z),h^{-1}\nu^3_{V_h}(z)\big)\big|\,\mathrm{d}\mathcal{H}^2(z)\le C\,, \tag{4.3}$$

where $\nu_{V_h}(z):=(\nu^1_{V_h}(z),\nu^2_{V_h}(z),\nu^3_{V_h}(z))$ denotes the outward pointing unit normal to $\partial V_h\cap\Omega$ at the point z. Note that (4.3) implies

$$\sup_{h>0}\big(\mathcal{L}^3(V_h)+\mathcal{H}^2(\partial V_h\cap\Omega)\big)\le C\,.$$

Therefore, by a compactness result for sets of finite perimeter (see [6, Theorem 3.39]), there exists $V\in\mathcal{P}(\Omega)$ such that, up to a non-relabeled subsequence, we have

$$\chi_{V_h}\to\chi_V\quad\text{in } L^1(\Omega)\,.$$

By Reshetnyak's lower semicontinuity theorem (cf. [6, Theorem 2.38]) applied to the lower semicontinuous, positively 1-homogeneous, convex function $\phi:\mathbb{S}^2\to[0,+\infty)$ with $\phi(\nu):=|(0,\nu^2,\nu^3)|$, we get, using again (4.3), that

$$\int_{\partial^*V\cap\Omega}|(0,\nu^2_V,\nu^3_V)|\,\mathrm{d}\mathcal{H}^2\le\liminf_{h\to0}\int_{\partial V_h\cap\Omega}|(0,\nu^2_{V_h},\nu^3_{V_h})|\,\mathrm{d}\mathcal{H}^2\le C\liminf_{h\to0}h=0\,,$$

where ν_V denotes the measure-theoretic outer unit normal to ∂^*V. This implies $\nu^2_V(x)=\nu^3_V(x)=0$ for $\mathcal{H}^2$-a.e. $x\in\partial^*V\cap\Omega$. We denote by $I:=\{x_1\in(0,L)\colon\mathcal{H}^2((\{x_1\}\times\mathbb{R}^2)\cap V)>0\}$ the measure-theoretic projection of V onto the x_1-axis. The previous argument shows that indeed $V=V_I:=I\times(-1/2,1/2)^2$ for some $I\in\mathcal{P}(0,L)$, up to a set of negligible $\mathcal{L}^3$-measure. This proves (2.20)(i). From now on (and also in the next sections) we will without loss of generality consider the representative in the $\mathcal{L}^1$-equivalence class of I which consists of a finite union of open subintervals of $(0,L)$, without mentioning it further.

We now proceed with the compactness for the deformations. Denote by $(v_h,E_h)_{h>0}$ the sequence related to $(y_h,V_h)_{h>0}$ via (2.6)–(2.7). We first derive a compactness result for the blockwise Sobolev modifications constructed in Proposition 3.1 and then we will show (2.20)(ii),(iii) for the sequence $(y_h)_{h>0}$ afterwards. To this end, we fix $\rho>0$ sufficiently small. We apply Proposition 3.1 on $(v_h,E_h)_{h>0}$ and $\epsilon_h:=h^2$ to find a sequence $(w_h)_{h>0}$. (Observe that (2.19) implies (3.7).) Then, we consider the sequence $(\widetilde{w}_h)_{h>0}\subset SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ defined by

$$\widetilde{w}_h(x):=\widetilde{w}_h(x_1,x_2,x_3):=w_h(x_1,hx_2,hx_3)\,. \tag{4.4}$$

After passing to a subsequence, we may assume that there exists $n\in2\mathbb{N}$ such that the sets $\mathcal{Q}_{v_h}$ in (3.8)(ii) satisfy $\#\mathcal{Q}_{v_h}=n/2$ for all $h>0$. Thus, by (3.8)(ii) we get $(x^h_i)^n_{i=1}\subset(0,L)$ such that

$$J_{w_h}\subset\Omega_{h,\rho}\cap\bigcup_{i=1}^n\big(\{x^h_i\}\times\mathbb{R}^2\big)\,.$$

Up to a further subsequence, we can suppose that for each $i=1,\dots,n$ we have $x^h_i\to x_i$ as $h\to0$ for suitable $x_i\in[0,L]$. Thus, fixing an arbitrary $\delta>0$ and defining the set

$$\Omega^\delta_\rho:=\Omega_{1,\rho}\setminus\bigcup_{i=1}^n\Big([x_i-\delta,x_i+\delta]\times\mathbb{R}^2\Big)\,, \tag{4.5}$$

we find that

$$\widetilde{w}_h|_{\Omega^\delta_\rho}\in W^{1,2}(\Omega^\delta_\rho;\mathbb{R}^3)\quad\text{for all } h>0 \text{ small enough.}$$

A change of variables together with (3.8)(iv) (for $\epsilon_h:=h^2$) imply that

$$h^{-2}\int_{\Omega^\delta_\rho}\mathrm{dist}^2(\nabla_h\widetilde{w}_h,SO(3))\,\mathrm{d}x\le h^{-4}\int_{\Omega_{h,\rho}}\mathrm{dist}^2(\nabla w_h,SO(3))\,\mathrm{d}x\le C\,, \tag{4.6}$$

for a constant $C > 0$ independent of h, δ, and ρ. This along with (3.8)(i) and (4.4) shows that the sequence $(\widetilde{w}_h)_{h>0}$ is equibounded in $W^{1,2}(\Omega^\delta_\rho;\mathbb{R}^3)$, i.e.,

$$\|\widetilde{w}_h\|_{W^{1,2}(\Omega^\delta_\rho)} \le C\,. \tag{4.7}$$

Therefore, by Lemma 4.1 applied to the sequence $(\widetilde{w}_h)_{h>0}$ on the connected components of the fixed domain Ω^δ_ρ, we obtain a map $y^\delta_\rho \in W^{2,2}(\Omega^\delta_\rho;\mathbb{R}^3)$, and $(d_2)^\delta_\rho, (d_3)^\delta_\rho \in W^{1,2}(\Omega^\delta_\rho;\mathbb{R}^3)$, all independent of the (x_2, x_3)-coordinates, such that

$$\widetilde{w}_h \rightharpoonup y^\delta_\rho \ \text{ weakly in } W^{1,2}(\Omega^\delta_\rho;\mathbb{R}^3) \quad \text{and} \quad \nabla_h \widetilde{w}_h \to R^\delta_\rho \ \text{ strongly in } L^2(\Omega^\delta_\rho;\mathbb{R}^{3\times 3})\,, \tag{4.8}$$

where

$$R^\delta_\rho := \big(y^\delta_{\rho,1}\big|\,(d_2)^\delta_\rho\big|\,(d_3)^\delta_\rho\big) \in SO(3) \text{ a.e. in } \Omega^\delta_\rho\,. \tag{4.9}$$

Moreover, by (4.2), (4.6), and (4.7) we have

$$\|y^\delta_\rho\|_{W^{2,2}(\Omega^\delta_\rho)} + \|(d_2)^\delta_\rho\|_{W^{1,2}(\Omega^\delta_\rho)} + \|(d_3)^\delta_\rho\|_{W^{1,2}(\Omega^\delta_\rho)} \le C \tag{4.10}$$

for a constant $C > 0$ independent of ρ and δ.

We now replace $\widetilde{w}_h$ by y_h in (4.8). By (2.7), (4.4), a scaling argument, and (3.8)(iii) we have that

$$\mathcal{L}^3\big(\{x \in \Omega^\delta_\rho \colon y_h(x) \neq \widetilde{w}_h(x)\}\big) \le h^{-2}\mathcal{L}^3\big(\{x \in \Omega_{h,\rho} \colon v_h(x) \neq w_h(x)\}\big) \to 0 \tag{4.11}$$

as $h \to 0$. Thus, from (4.8) and (4.11) we deduce that

$$y_h \to y^\delta_\rho \quad \text{in measure on } \Omega^\delta_\rho \quad \text{and} \quad \nabla_h y_h \to R^\delta_\rho \quad \text{in measure on } \Omega^\delta_\rho\,. \tag{4.12}$$

Now, by (4.9), (4.10), (4.12), and a monotonicity argument for $\rho \to 0$ and $\delta \to 0$ we find $(y|\,d_2|\,d_3) \in \big(P\text{-}W^{2,2} \times P\text{-}W^{1,2} \times P\text{-}W^{1,2}\big)((0, L);\mathbb{R}^{3\times 3}\big)$, such that the corresponding functions $\bar{y}, \bar{d}_2, \bar{d}_3 \colon \Omega \to \mathbb{R}^3$ defined in (2.14) satisfy

$$\bar{y} = y^\delta_\rho \quad \text{on } \Omega^\delta_\rho, \qquad \bar{R} := (\bar{y}_{,1}|\,\bar{d}_2|\,\bar{d}_3) = R^\delta_\rho \quad \text{on } \Omega^\delta_\rho\,, \tag{4.13}$$

and

$$y_h \to \bar{y} \quad \text{in measure on } \Omega \quad \text{and} \quad \nabla_h y_h \to \bar{R} \quad \text{in measure on } \Omega\,. \tag{4.14}$$

Property (4.9) also implies that $(\bar{y}_{,1}\big|\,\bar{d}_2\big|\,\bar{d}_3) \in SO(3)$ a.e. in Ω, i.e., $(y|\,d_2|\,d_3) \in SBV^2_{\mathrm{isom}}(0, L)$, see (2.13) and the convention introduced right after it. The measure convergence $y_h \to \bar{y}$ in (4.14) together with $\|y_h\|_{L^\infty(\Omega)} \le M$ shows (2.20)(ii). By (2.4)(iv), (4.3), and the fact that $\nabla_h y_h = \mathrm{Id}$ on V_h we have that

$$\sup_{h>0} \int_\Omega |\nabla_h y_h|^2 \,\mathrm{d}x < +\infty\,.$$

A compactness argument and (4.14) show that $\nabla_h y_h \rightharpoonup \bar{R}$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Recalling that $\chi_{V_h} \to \chi_{V_I}$ in $L^1(\Omega)$ by (2.20)(i), the proof of (2.20)(iii) is concluded.

It finally remains to show that $((y|\,d_2|\,d_3), I) \in \mathcal{A}$. In fact, we have $\|y\|_{L^\infty(\Omega)} \le M$ by $\|y_h\|_{L^\infty(\Omega)} \le M$ for all $h > 0$ and (4.14). Moreover, the second part of (4.14), (2.20)(i), and the fact that $y = T_h(\mathrm{id})$, $\nabla_h y_h = \mathrm{Id}$ on V_h (see (2.11)) show that $\bar{R} \equiv \mathrm{Id}$ on V_I and thus $y(x_1) \equiv x_1$ and $(y_{,1}|\,d_2|\,d_3) \equiv \mathrm{Id}$ on I. As above we have already seen that $(y|\,d_2|\,d_3) \in SBV^2_{\mathrm{isom}}(0, L)$, the proof is concluded. □

5. The Γ-liminf inequality

This section is devoted to the proof of the lower bound (2.21) of Theorem 2.3, which is split into proving the lower bound for the bulk and the surface part of the energy separately.

Recalling Definition 2.2, we consider a sequence $(y_h, V_h)_{h>0}$ and $((y|\, d_2|\, d_3), I) \in \mathcal{A}$ such that $(y_h, V_h) \xrightarrow{\tau} ((y|\, d_2|\, d_3), I)$, i.e., (2.20)(i)–(iii) hold true.

Regarding the lower bound for the elastic energy, we will use the following result from the purely elastic case. For its formulation, recall the definition of the elastic part of the limiting energy, as introduced in (2.16)–(2.18), and the convention after (2.13).

Lemma 5.1 (Lower bound in the Sobolev setting)**.** *Let $\Omega_{\ell_1,\ell_2} := (0,\ell_1) \times (-\ell_2,\ell_2)^2$ for $\ell_1, \ell_2 > 0$, and let $(\widetilde{w}_h)_{h>0}$ be a sequence in $W^{1,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3)$ such that*

$$\widetilde{w}_h \rightharpoonup \bar{y} \quad \text{weakly in } W^{1,2}(\Omega_{\ell_1,\ell_2};\mathbb{R}^3), \quad \nabla_h \widetilde{w}_h \to \bar{R} = \big(\bar{y}_{,1}\big|\, \bar{d}_2\big|\, \bar{d}_3\big) \text{ strongly in } L^2(\Omega_{\ell_1,\ell_2};\mathbb{R}^{3\times 3})\,, \tag{5.1}$$

where $\bar{y}$, $\bar{d}_2$, and $\bar{d}_3$ are independent of (x_2,x_3). Then, there exists a sequence of piecewise constant functions $\mathcal{R}_h \colon \Omega_{\ell_1,\ell_2} \to SO(3)$ and a limiting function $G \in L^2(\Omega_{\ell_1,\ell_2};\mathbb{R}^{3\times 3})$ such that

$$\begin{aligned} &\text{(i)} \quad G_h := \frac{\mathcal{R}_h^T \nabla_h \widetilde{w}_h - \mathrm{Id}}{h} \rightharpoonup G \quad \text{weakly in } L^2(\Omega_{\ell_1,\ell_2};\mathbb{R}^{3\times 3})\,, \\ &\text{(ii)} \quad \liminf_{h\to 0} \frac{1}{h^2} \int_{\Omega_{\ell_1,\ell_2}} W(\nabla_h \widetilde{w}_h)\,\mathrm{d}x \geq \frac{1}{2} \int_{\Omega_{\ell_1,\ell_2}} \mathcal{Q}_3(G)\,\mathrm{d}x\,, \\ &\text{(iii)} \quad \frac{1}{2} \int_{\Omega_{\ell_1,\ell_2}} \mathcal{Q}_3(G)\,\mathrm{d}x \geq \frac{1}{2}(2\ell_2)^4 \int_{(0,\ell_1)} \mathcal{Q}_2(R^T R_{,1})\,\mathrm{d}x_1\,, \end{aligned} \tag{5.2}$$

where R is defined via (2.13)–(2.15).

The proof can be found in [58, Theorem 3.1(i)]. In particular, we refer to [58, (3.4)–(3.6), (3.16), and Remark 3.2]. The result is stated there only for cross sections with area 1, corresponding to $\ell_2 = \frac{1}{2}$. However, a standard scaling argument shows that

$$(2\ell_2)^4 \mathcal{Q}_2(A) = \mathcal{Q}_2^{\ell_2}(A) := \min_{a \in W^{1,2}((-\ell_2,\ell_2)^2;\mathbb{R}^3)} \int_{(-\ell_2,\ell_2)^2} \mathcal{Q}_3\left(A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \Bigg|\, \alpha_{,2} \Bigg|\, \alpha_{,3} \right) \mathrm{d}x_2\,\mathrm{d}x_3\,,$$

where $\mathcal{Q}_2(A)$ is given by (2.17). This implies (5.2)(iii) in the present form.

The issue in our framework is that Lemma 5.1 cannot be applied directly since the sequence $(y_h)_{h>0}$ is only in $W^{1,2}(\Omega \setminus \overline{E}_h;\mathbb{R}^3)$ and the geometry of $(E_h)_{h>0}$ cannot be controlled a priori. Therefore, as in the proof of Theorem 2.1, we will use the modification $(w_h)_{h>0}$ constructed in Proposition 3.1. The advantage here is that, due to (3.8)(ii), the geometry of the jump set of $(w_h)_{h>0}$ is well controlled in the sense that it is contained in the vertical faces of finitely many cuboids. Therefore, far from these cuboids, we can reduce to the Sobolev setting.

Lemma 5.2. *Suppose that $(y_h, V_h) \xrightarrow{\tau} ((y|\, d_2|\, d_3), I)$ for some $((y|\, d_2|\, d_3), I) \in \mathcal{A}$. Then,*

$$\liminf_{h\to 0} \Big(h^{-2} \int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x \Big) \geq \frac{1}{2} \int_{(0,L)\setminus I} \mathcal{Q}_2(R^T R_{,1})\,\mathrm{d}x_1\,. \tag{5.3}$$

Proof. We apply Proposition 3.1 for $\rho > 0$ small and $\epsilon_h := h^2$ and the sequence $(v_h, E_h)_{h>0}$ related to $(y_h, V_h)_{h>0}$ via (2.6)–(2.7). Here, we note that it is not restrictive to assume that $(\mathcal{E}^h(y_h, V_h))_{h>0}$ is bounded, and thus (3.7) holds. We denote the resulting sequence by $(w_h)_{h>0}$, and as in the proof of the compactness result, we consider the sequence $(\widetilde{w}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ defined by

$$\widetilde{w}_h(x) := \widetilde{w}_h(x_1,x_2,x_3) := w_h(x_1, hx_2, hx_3)\,. \tag{5.4}$$

Similarly to the reasoning in the proof of Theorem 2.1, see (4.5), (4.8), and (4.13), we can define a set Ω_ρ^δ for $\rho, \delta > 0$ with $\mathcal{L}^3(\Omega \setminus \Omega_\rho^\delta) \to 0$ as $\rho, \delta \to 0$ such that

$$\widetilde{w}_h|_{\Omega_\rho^\delta} \in W^{1,2}(\Omega_\rho^\delta; \mathbb{R}^3) \quad \text{for all } h > 0 \text{ small enough},$$

and

$$\widetilde{w}_h \rightharpoonup \bar{y} \ \text{ weakly in } W^{1,2}(\Omega_\rho^\delta; \mathbb{R}^3) \quad \text{and} \quad \nabla_h \widetilde{w}_h \to \bar{R} \ \text{ strongly in } L^2(\Omega_\rho^\delta; \mathbb{R}^{3\times 3}).$$

This means that (5.1) is satisfied and we can thus apply Lemma 5.1 on each connected component of Ω_ρ^δ to find corresponding G_h and G such that (5.2) holds on the set Ω_ρ^δ. The main part of the proof will consist in confirming that (5.2)(ii) also holds with y_h in place of $\widetilde{w}_h$. Then, the liminf inequality follows from (5.2)(iii).

To show (5.2)(ii) for y_h in place of $\widetilde{w}_h$, we will perform a by now classical linearization argument which we sketch here for convenience: we consider a sequence of positive numbers $(\lambda_h)_{h>0} \subset (0, \infty)$ with

$$\lambda_h \to \infty, \ h\lambda_h \to 0 \ \text{ as } h \to 0, \tag{5.5}$$

and define

$$\Theta_h := \{x \in \Omega_\rho^\delta \colon \widetilde{w}_h(x) = y_h(x)\} \cap \{x \in \Omega_\rho^\delta \colon |G_h(x)| \le \lambda_h\}. \tag{5.6}$$

Note that $\mathcal{L}^3(\{\widetilde{w}_h \neq y_h\}) \to 0$ by (3.8)(iii) and a scaling argument. This together with the fact that $\|G_h\|_{L^2(\Omega_\rho^\delta)} \le C$, see (5.2)(i), $\lambda_h \to +\infty$, and Chebyshev's inequality implies that

$$\mathcal{L}^3(\Omega_\rho^\delta \setminus \Theta_h) \to 0 \quad \text{as } h \to 0. \tag{5.7}$$

This yields $\chi_{\Theta_h} \to 1$ boundedly in measure in Ω_ρ^δ as $h \to 0$. By (2.11), $W(\mathrm{Id}) = 0$, $W \ge 0$, and by the definition of Θ_h we get

$$\begin{aligned}\liminf_{h\to 0} \Big(h^{-2} \int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) &= \liminf_{h\to 0} \Big(h^{-2} \int_{\Omega} W(\nabla_h y_h)\,\mathrm{d}x\Big) \\ &\ge \liminf_{h\to 0} \Big(h^{-2} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} W(\nabla_h \widetilde{w}_h)\,\mathrm{d}x\Big).\end{aligned}$$

By the regularity and the structural hypotheses on W (recall (2.4)) we get

$$W(\mathrm{Id} + F) = \tfrac{1}{2}\mathcal{Q}_3(\mathrm{sym}(F)) + \Phi(F),$$

where $\Phi\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is a function satisfying

$$\sup\big\{\tfrac{|\Phi(F)|}{|F|^2} \colon |F| \le \sigma\big\} \to 0 \quad \text{ as } \sigma \to 0. \tag{5.8}$$

Then, together with the definition of G_h in (5.2)(i) this gives

$$\begin{aligned}\liminf_{h\to 0} \Big(h^{-2} \int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x\Big) &\ge \liminf_{h\to 0} \Big(h^{-2} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} W(\mathrm{Id} + hG_h)\,\mathrm{d}x\Big) \\ &\ge \liminf_{h\to 0} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} \Big(\tfrac{1}{2}\mathcal{Q}_3(\mathrm{sym}(G_h)) + h^{-2}\Phi(hG_h)\Big)\,\mathrm{d}x \\ &= \liminf_{h\to 0} \frac{1}{2} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} \mathcal{Q}_3(\mathrm{sym}(G_h))\,\mathrm{d}x.\end{aligned} \tag{5.9}$$

Here, in the last step we used that

$$\limsup_{h\to 0} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} h^{-2} |\Phi(hG_h)|\,\mathrm{d}x \le \limsup_{h\to 0} \bigg(\sup\big\{\tfrac{|\Phi(hG_h)|}{|hG_h|^2} \colon |hG_h| \le h\lambda_h\big\} \int_{\Omega_\rho^\delta} \chi_{\Theta_h} |G_h|^2\,\mathrm{d}x\bigg) = 0,$$

which follows from the fact that $(G_h)_h$ is bounded in $L^2(\Omega^\delta_\rho;\mathbb{R}^{3\times3})$, (5.6), (5.8), and $h\lambda_h \to 0$ (see (5.5)). Hence, (5.2)(i),(iii), the fact that $\chi_{\Theta_h} \to 1$ boundedly in measure in Ω^δ_ρ, see (5.7), and the convexity of $\mathcal{Q}_3$ imply that

$$\liminf_{h\to0} \frac{1}{2}\int_{\Omega^\delta_\rho} \chi_{\Theta_h}\,\mathcal{Q}_3(\mathrm{sym}(G_h))\,\mathrm{d}x \ge \frac{1}{2}\int_{\Omega^\delta_\rho} \mathcal{Q}_3(G)\,\mathrm{d}x \ge \frac{1}{2}(1-\rho)^4\int_{\pi_1(\Omega^\delta_\rho)} \mathcal{Q}_2\big(R^T R_{,1}\big)\,\mathrm{d}x_1\,, \quad (5.10)$$

where π_1 is the projection onto the x_1-axis, and Ω^δ_ρ is defined in (3.1) and (4.5). As $\mathcal{L}^3(\Omega\setminus\Omega^\delta_\rho)\to0$ for $\rho,\delta\to0$, we also get that $\mathcal{L}^1((0,L)\setminus\pi_1(\Omega^\delta_\rho))\to0$ as $\rho,\delta\to0$. Thus, (5.9), (5.10), and monotone convergence yield the lower bound (5.3). Note that the last integral can also be taken on $(0,L)\setminus I$ only since $\mathcal{Q}_2(0)=0$ and $R=\mathrm{Id}$ on I, see (2.12), (2.13) and (2.17). This concludes the proof. □

We now proceed with the lower bound for the surface part of the energy, namely

$$\mathcal{E}^h_{\mathrm{surf}}(V_h) := \mathcal{E}^h(y_h,V_h) - h^{-2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x = h^{-2}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E_h;\Omega_h)\,, \quad (5.11)$$

where we refer to the definitions in (2.8) and (3.5). Our approach deviates significantly from the proof of lower bounds in relaxation results for energies defined on pairs of functions and sets, cf. [15] or [21]. This is mainly due to the fact that the nonlinear geometric rigidity result allows us to control the elastic energy only in a large part of $\Omega\setminus\overline{V_h}$. Our argument to derive the lower bound for the surface energy term related to collapsing voids correctly hinges on Proposition 3.2 along with an argument by contradiction. We again suppose that I is the representative consisting of a finite union of open intervals.

Lemma 5.3. *Suppose that $(y_h,V_h)\xrightarrow{\tau}((y|\,d_2|\,d_3),I)$ for some $((y|\,d_2|\,d_3),I)\in\mathcal{A}$. Then,*

$$\liminf_{h\to0}\mathcal{E}^h_{\mathrm{surf}}(V_h) \ge \mathcal{H}^0(\partial I\cap(0,L)) + 2\mathcal{H}^0\big((J_y\cup J_R)\setminus\partial I\big)\,. \quad (5.12)$$

Proof. Let $(E_h)_{h>0}$ be the void sets associated to $(V_h)_{h>0}$ according to (2.6). By $(E^*_h)_{h>0}$ we denote the open sets given by Proposition 3.2 satisfying $E_h\subset E^*_h\subset\Omega_h$ and (3.9). We also introduce the rescaled sets

$$V^*_h := T_{1/h}(E^*_h)\,, \quad (5.13)$$

and note by (3.9), a scaling argument, and (2.20)(i) that

$$\chi_{V^*_h}\longrightarrow\chi_{V_I}\quad\text{in } L^1(\Omega)\,. \quad (5.14)$$

By (3.9) and (5.11) we have

$$\liminf_{h\to0}\mathcal{E}^h_{\mathrm{surf}}(V_h) = \liminf_{h\to0} h^{-2}\mathcal{G}^{\kappa_h}_{\mathrm{surf}}(E_h;\Omega_h) \ge \liminf_{h\to0} h^{-2}\mathcal{H}^2(\partial E^*_h\cap\Omega_h)\,. \quad (5.15)$$

Since $((y|\,d_2|\,d_3),I)\in\mathcal{A}$, there exist finitely many $(x_j)_{j=1}^n\subset(0,L)$ such that

$$\{x_1,\dots,x_n\} = (\partial I\cap(0,L))\cup J_y\cup J_R\,.$$

We choose $\delta>0$ sufficiently small such that the sets $S^{2\delta}_h(x_j)$, $j=1,\dots,n$, are pairwise disjoint and contained in Ω_h, cf. (3.3). Our goal is to prove

$$\begin{aligned}
&\text{(i)}\quad \liminf_{h\to0} h^{-2}\mathcal{H}^2\big(\partial E^*_h\cap S^{2\delta}_h(x_j)\big)\ge1 \quad \text{if } x_j\in\partial I\cap(0,L)\,,\\
&\text{(ii)}\quad \liminf_{h\to0} h^{-2}\mathcal{H}^2\big(\partial E^*_h\cap S^{2\delta}_h(x_j)\big)\ge2 \quad \text{if } x_j\in J_y\setminus\partial I\,,\\
&\text{(iii)}\quad \liminf_{h\to0} h^{-2}\mathcal{H}^2\big(\partial E^*_h\cap S^{2\delta}_h(x_j)\big)\ge2 \quad \text{if } x_j\in J_R\setminus\partial I\,.
\end{aligned} \quad (5.16)$$

Once (5.16) is shown, we can conclude as follows. By (5.15) and the fact that the sets $S_h^{2\delta}(x_j) \subset \Omega_h$, $j = 1, \ldots, n$, are pairwise disjoint, we get

$$\liminf_{h\to 0} \mathcal{E}^h_{\rm surf}(V_h) \geq \liminf_{h\to 0} \sum_{j=1}^n h^{-2}\mathcal{H}^2(\partial E_h^* \cap S_h^{2\delta}(x_j)) \geq \mathcal{H}^0(\partial I \cap (0,L)) + 2\mathcal{H}^0\big((J_y \cup J_R)\setminus \partial I\big)\,.$$

This shows (5.12).

We now proceed with the proof of the properties stated in (5.16). We start with (i). By a change of variables and the definition in (5.13) we find

$$\liminf_{h\to 0} h^{-2}\mathcal{H}^2\big(\partial E_h^* \cap S_h^{2\delta}(x_j)\big) = \liminf_{h\to 0} \int_{\partial V_h^* \cap S_1^{2\delta}(x_j)} |(\nu^1_{V_h^*}, h^{-1}\nu^2_{V_h^*}, h^{-1}\nu^3_{V_h^*})| \,\mathrm{d}\mathcal{H}^2$$

where we use the notation $\nu_{V_h^*} = (\nu^1_{V_h^*}, \nu^2_{V_h^*}, \nu^3_{V_h^*}) \in \mathbb{S}^2$ for the outer unit normal to V_h^*. By (5.14) and the lower semicontinuity of the perimeter (cf. [6, Proposition 3.38]), we get

$$\liminf_{h\to 0} h^{-2}\mathcal{H}^2\big(\partial E_h^* \cap S_h^{2\delta}(x_j)\big) \geq \liminf_{h\to 0} \mathcal{H}^2\big(\partial V_h^* \cap S_1^{2\delta}(x_j)\big) \geq \mathcal{H}^2\big(\partial V_I \cap S_1^{\delta}(x_j)\big)\,.$$

The fact that $\{x_j\} \times (-\frac{1}{2}, \frac{1}{2}) \subset \partial V_I$ yields (i).

We proceed to show (ii). Suppose the statement was wrong, i.e., there exists $0 < \mu < 1$ and a subsequence (not relabeled) such that $2\mu \geq h^{-2}\mathcal{H}^2(\partial E_h^* \cap S_h^{2\delta}(x_j))$ for all $h > 0$. Choose $\rho > 0$ small enough such that $\frac{\mu}{(1-\rho)^2} < 1$. Then, we get

$$\mathcal{H}^2\big(\partial E_h^* \cap S_h^{2\delta}(x_j)\big) \leq 2\mu h^2 < 2((1-\rho)h)^2\,.$$

This implies that (3.10)(i) (for $l = \delta$) holds. The fact that the sets $S_1^{2\delta}(x_j)$, $j = 1, \ldots, n$, are pairwise disjoint implies that $S_1^{2\delta}(x_j) \cap V_I = \emptyset$. Thus, by (5.14) we get $\lim_{h\to\infty} \mathcal{L}^3(V_h^* \cap S_1^{2\delta}(x_j)) = 0$. By the definition of V_h^* and a change of variables we find

$$\frac{\mathcal{L}^3(E_h^* \cap S_h^{2\delta}(x_j))}{\mathcal{L}^3(S_h^{2\delta}(x_j))} \leq \frac{1}{9}$$

for $h > 0$ sufficiently small, i.e., (3.10)(ii) is satisfied. Let $(w_h)_{h>0}$ be the sequence from Proposition 3.1 and let again $(\widetilde{w}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^3)$ be the rescaled sequence, see (5.4). Thus, by a change of variables and by (3.11), we find that

$$\begin{aligned}\int_{S^\delta_{1,\rho}(x_j)\cap J_{\widetilde{w}_h}} \sqrt{|[\widetilde{w}_h]|}\,\mathrm{d}\mathcal{H}^2 &\leq \int_{S^\delta_{1,\rho}(x_j)\cap J_{\widetilde{w}_h}} \sqrt{|[\widetilde{w}_h]|}\ |(\nu^1_{\widetilde{w}_h}, h^{-1}\nu^2_{\widetilde{w}_h}, h^{-1}\nu^3_{\widetilde{w}_h})|\,\mathrm{d}\mathcal{H}^2\\ &= \frac{1}{h^2}\int_{S^\delta_{h,\rho}(x_j)\cap J_{w_h}} \sqrt{|[w_h]|}\,\mathrm{d}\mathcal{H}^2 \to 0\,.\end{aligned}$$

By (3.8)(ii)–(iv), (2.7), (2.20)(ii), and(5.4) we get $\widetilde{w}_h \to \bar{y}$ in $L^1(\Omega_{1,\rho};\mathbb{R}^3)$ and

$$\sup_{h>0}\Big(\int_{\Omega_{1,\rho}} |\nabla \widetilde{w}_h|^2 \mathrm{d}x + \mathcal{H}^2(J_{\widetilde{w}_h})\Big) \leq C\,,$$

for a constant $C > 0$ independent of $h > 0$. By Ambrosio's lower semicontinuity theorem in SBV (cf. [6, Theorem 4.7]) and the fact that $\widetilde{w}_h \to \bar{y}$ in $L^1(\Omega;\mathbb{R}^3)$ we get

$$\int_{S^\delta_{1,\rho}(x_j)\cap J_{\bar{y}}} \sqrt{|[\bar{y}]|}\,\mathrm{d}\mathcal{H}^2 \leq \liminf_{h\to 0} \int_{S^\delta_{1,\rho}(x_j)\cap J_{\widetilde{w}_h}} \sqrt{|[\widetilde{w}_h]|}\,\mathrm{d}\mathcal{H}^2 = 0\,.$$

This shows that $J_{\bar{y}}$ does not jump on $(\{x_j\} \times \mathbb{R}^2) \cap \Omega_{1,\rho}$ which contradicts the fact that $x_j \in J_y$.

For (iii) we proceed in a similar fashion and first get that (3.10) is satisfied. We let $(R_h)_{h>0}$ be the sequence in Proposition 3.2 and introduce the rescaled sequence $(\widetilde{R}_h)_{h>0} \subset SBV^2(\Omega_{1,\rho};\mathbb{R}^{3\times 3})$ by

$$\widetilde{R}_h(x) := \widetilde{R}_h(x_1,x_2,x_3) := R_h(x_1,hx_2,hx_3)\,.$$

By a change of variables, the properties (3.8)(ii)–(iv), as well as (2.7) and (2.9) we get

$$\sup_{h>0}\Big(\int_{\Omega_{1,\rho}} |\nabla \widetilde{R}_h|^2 \mathrm{d}x + \mathcal{H}^2(J_{\widetilde{R}_h})\Big) \le C, \qquad |\nabla_h y_h - \widetilde{R}_h| \to 0 \text{ in measure on } \Omega_{1,\rho}\,,$$

for a constant $C>0$ again independent of $h>0$. Then, again by Ambrosio's lower semicontinuity theorem, (3.11), (2.20)(iii), and the fact that $S^\delta_{1,\rho}(x_j)\cap V_I = \emptyset$, we derive

$$\int_{S^\delta_{1,\rho}(x_j)\cap J_{\bar R}} \sqrt{|[\bar R]|}\,\mathrm{d}\mathcal{H}^2 \le \liminf_{h\to 0}\int_{S^\delta_{1,\rho}(x_j)\cap J_{\widetilde{R}_h}} \sqrt{|[\widetilde{R}_h]|}\,\mathrm{d}\mathcal{H}^2 = 0\,.$$

As in (ii), this yields a contradiction, and the proof of (iii) is concluded. □

6. The Γ-limsup inequality

In this last section, we construct recovery sequences for admissible limits $((y|\,d_2|\,d_3), I) \in \mathcal{A}$. We start by recalling the relevant result for elastic rods, using again the convention in (2.14)–(2.15).

Lemma 6.1 (Recovery sequences in the Sobolev setting)**.** *Let $\Omega_\ell := (0,\ell)\times(-\frac12,\frac12)^2$ for $\ell>0$. Let $((y|\,d_2|\,d_3), I)\in\mathcal{A}$ be such that $\bar y|_{\Omega_\ell} \in W^{2,2}(\Omega_\ell;\mathbb{R}^3)$ and $\bar d|_{\Omega_\ell}, \bar d_3|_{\Omega_\ell} \in W^{1,2}(\Omega_\ell;\mathbb{R}^3)$. Then, there exists a sequence $(y_h)_{h>0}\subset W^{1,2}(\Omega_\ell;\mathbb{R}^3)$ such that*

$$y_h \to \bar y \ \text{ strongly in } W^{1,2}(\Omega_\ell;\mathbb{R}^3), \qquad \nabla_h y_h \to (\bar y_{,1}|\,\bar d_2|\,\bar d_3) \text{ strongly in } L^2(\Omega_\ell;\mathbb{R}^{3\times3})\,, \tag{6.1}$$

and we have

$$\lim_{h\to0}\frac{1}{h^2}\int_{\Omega_\ell} W(\nabla_h y_h)\,\mathrm{d}x = \frac12\int_{(0,\ell)} \mathcal{Q}_2(R^TR_{,1})\,\mathrm{d}x_1\,, \tag{6.2}$$

where $R := (y_{,1}|\,d_2|\,d_3)$. Moreover, if $\bar y\in L^\infty(\Omega_\ell;\mathbb{R}^3)$, it holds that $\limsup_{h\to\infty}\|y_h\|_\infty \le \|\bar y\|_\infty$.

For the proof we refer to [58, Theorem 3.1(ii)]. We now proceed with the construction of recovery sequences.

Proof of Theorem 2.3(ii). Consider an admissible limit $((y|\,d_2|\,d_3), I)\in\mathcal{A}$, see (2.12)–(2.13). We will combine ideas from [58, Section 3] and [68, Subsection 5.4]. We first treat the case $\|y\|_\infty < M$ and address the changes for $\|y\|_\infty = M$ at the end of the proof. By choosing a suitable representative, we can assume that I is the union of finitely many open subintervals of $(0,L)$. We also denote

$$(J_y\cup J_R)\setminus \partial I := \{t_1,\dots,t_m\}\subset (0,L)\,, \tag{6.3}$$

where $0<t_1<\dots<t_m<L$. Denote by $(J_i)_i^n$ the connected components of $(0,L)\setminus(I\cup\{t_1,\dots,t_m\})$.

We apply Lemma 6.1 on each connected component J_i to find recovery sequences $y_h^i\in W^{1,2}(\widetilde{J}_i;\mathbb{R}^3)$, where $\widetilde{J}_i := J_i\times(-\frac12,\frac12)^2$ such that (6.1)–(6.2) are satisfied for the respective functions on the respective sets. For $h>0$ sufficiently small, consider the sets $(V_h)_{h>0}\subset\mathcal{A}_{\mathrm{reg}}(\Omega)$ defined by

$$V_h := \Big(I\cup\bigcup_{i=1}^m (t_i-h,t_i+h)\Big)\times(-\tfrac12,\tfrac12)^2\,. \tag{6.4}$$

Recalling (1.5), we introduce the deformations $(y_h)_{h>0}\subset SBV^2(\Omega;\mathbb{R}^3)$ defined by

$$y_h(x) := \begin{cases} y_h^i(x) & \text{if } x\in\widetilde{J}_i\setminus V_h,\\ T_h(\mathrm{id}) & \text{if } x\in V_h\,.\end{cases} \tag{6.5}$$

Since $\|y\|_\infty < M$, Lemma 6.1 also implies that $\|y_h\|_\infty \le M$ for $h > 0$ sufficiently small. This shows that $(y_h, V_h)_{h>0} \subset \hat{\mathcal{A}}_h$, cf. (2.11). Clearly, in view of (6.4), we have $\chi_{V_h} \to \chi_{V_I}$ in $L^1(\Omega)$. Moreover, (6.1) and (6.5) show that $(y_h)_{h>0}$ also satisfies (2.20)(ii),(iii). Thus, by the definition of τ-convergence in Definition 2.2 we have $(y_h, V_h) \xrightarrow{\tau} ((y|\,d_2|\,d_3), I)$ as $h \to 0$.

Regarding the elastic part of the energy, from (6.2) and (6.5) we directly infer

$$\limsup_{h\to 0} \frac{1}{h^2}\int_{\Omega\setminus\overline{V_h}} W(\nabla_h y_h)\,\mathrm{d}x \le \lim_{h\to 0}\frac{1}{h^2}\sum_{i=1}^n \int_{\widetilde{J}_i} W(\nabla_h y_h)\,\mathrm{d}x$$
$$= \frac{1}{2}\sum_{i=1}^n \int_{J_i} \mathcal{Q}_2(R^T R_{,1})\,\mathrm{d}x_1 = \frac{1}{2}\int_{(0,L)\setminus I} \mathcal{Q}_2(R^T R_{,1})\,\mathrm{d}x_1\,. \tag{6.6}$$

We now address the surface part of the energy introduced in (5.11). First, we set $E_h := T_h(V_h)$, where $(V_h)_{h>0}$ are defined in (6.4). By (5.11), (3.5), and the fact that $\partial E_h \cap \Omega_h$ consists of planar interfaces with unit normal $\pm e_1$, we have that

$$\begin{aligned}\lim_{h\to 0}\mathcal{E}^h_{\rm surf}(V_h) &= \lim_{h\to 0} h^{-2}\Big(\mathcal{H}^2(\partial E_h\cap\Omega_h) + \kappa_h\int_{\partial E_h\cap\Omega_h} |\boldsymbol{A}_h|^2\,\mathrm{d}\mathcal{H}^2\Big)\\ &= \lim_{h\to 0} h^{-2}\mathcal{H}^2\Big(\partial\Big(\Big(I\cup\bigcup_{i=1}^m (t_i-h,t_i+h)\Big)\times(-\tfrac{h}{2},\tfrac{h}{2})^2\Big)\cap\Omega_h\Big)\\ &= \mathcal{H}^0(\partial I\cap(0,L)) + 2m = \mathcal{H}^0(\partial I\cap(0,L)) + 2\mathcal{H}^0((J_y\cup J_R)\setminus\partial I)\,,\end{aligned} \tag{6.7}$$

where the last step follows from (6.3). Now, (6.6) and (6.7) show (2.22) in the case $\|y\|_{L^\infty} < M$.

We conclude the proof by addressing the case $\|y\|_\infty = M$. In this case, we extend y, d_2, d_3 on $(L, L+1)$ such that $y_{,1}, d_2, d_3$ are constant on $[L, L+1)$ and $(y|\,d_2|\,d_3) \in SBV^2_{\rm isom}(0, L+1)$. For $0 < \sigma < 1$, we consider the functions $y^\sigma(x_1) := \sigma y(x_1/\sigma)$, $d_2^\sigma(x_1) := d_2(x_1/\sigma)$, and $d_3^\sigma(x_1) := d_3(x_1/\sigma)$ on $(0, \sigma(L+1))$. Now, $\|y^\sigma\|_\infty < M$ and we can construct a recovery sequence as above. Moreover, one can check that $\lim_{\sigma\to 0}\mathcal{E}^0((y^\sigma|\,d_2^\sigma|\,d_3^\sigma), \sigma I) = \mathcal{E}^0((y|\,d_2|\,d_3), I)$. Thus, the conclusion follows by a standard diagonal sequence argument in the theory of Γ-convergence. □

Acknowledgements

This work was supported by the DFG project FR 4083/3-1 and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044 -390685587, Mathematics Münster: Dynamics–Geometry–Structure. The research of LK was supported by the DFG through the Emmy Noether Programme (project number 509436910). The authors would like to thank the anonymous referees for their valuable comments and suggestions for the final version of this manuscript.

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