UNIFORM SOBOLEV, INTERPOLATION AND GEOMETRIC CALDERÓN–ZYGMUND INEQUALITIES FOR GRAPH HYPERSURFACES

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ABSTRACT. In this note, our aim is to show that families of smooth hypersurfaces of $\mathbb{R}^{n+1}$ which are all C^1–close enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for mathematical analysis, like Sobolev, Gagliardo–Nirenberg and "geometric" Calderón–Zygmund inequalities. This technical result is quite useful, in particular, in the study of the geometric flows of hypersurfaces.

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1. INTRODUCTION AND PRELIMINARIES

In this note, our aim is to show that families of smooth hypersurfaces of $\mathbb{R}^{n+1}$ which are all C^1–close enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for mathematical analysis, like Sobolev, Gagliardo–Nirenberg, "geometric" Calderón–Zygmund, trace and extension inequalities. This technical result is quite useful, in particular, in the analysis of the geometric flows of hypersurfaces, when one studies the behavior of the hypersurfaces close (in some norm, for instance in C^1–norm) to critical ones (possibly "stable") or the asymptotic limits of flows existing for all times (see for instance [2,3,10,13], where such controls on the constants are necessary).

We start by setting up some notation and recall some basic facts about hypersurfaces in Euclidean spaces that we need in the sequel, possible references are [1,6,15].

We will consider smooth, n–dimensional, compact hypersurfaces M, embedded in $\mathbb{R}^{n+1}$, getting a Riemannian metric g by pull–back of the standard scalar product $\langle \cdot \,|\, \cdot \rangle$ of $\mathbb{R}^{n+1}$ via the embedding map $\varphi : M \to \mathbb{R}^{n+1}$, hence, turning it into a Riemannian manifold (M, g). Then, we use ∇ for the associated Levi–Civita covariant derivative and μ for the canonical

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measure induced by the metric g, which actually coincides with the n–dimensional Hausdorff measure $\mathcal{H}^n$ of $\mathbb{R}^{n+1}$ restricted to M. Then, the components of g in a local chart are

$$g_{ij} = \left\langle \frac{\partial \varphi}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

and the "canonical" measure μ, induced on M by the metric g is then locally described by $\mu = \sqrt{\det g_{ij}}\, \mathscr{L}^n$, where $\mathscr{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.
The inner product on M, extended to tensors, is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l}$$

where g_{ij} is the matrix of the coefficients of the metric tensor in the local coordinates and g^{ij} is its inverse. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T, T)}\,.$$

The induced Levi–Civita covariant derivative on (M, g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k\,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji} \omega_k\,,$$

where Γ^i_{jk} are the Christoffel symbols of the connection ∇, expressed by the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \Big(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \Big)\,. \tag{1.1}$$

With $\nabla^m T$ we will mean the m–th iterated covariant derivative of a tensor T.

Being M embedded, we can assume it is a subset of $\mathbb{R}^{n+1}$ (hence the embedding map is the identity) and we denote with $\nu : M \to \mathbb{R}^n$ its global unit normal vector field, *pointing outward*. It is indeed well known (theorem of Jordan–Brouwer, see [6, Proposition 12.2], for instance) that any compact, embedded M "divides" $\mathbb{R}^{n+1}$ in two connected components, one of them bounded (called "the interior"), both having M as its smooth boundary, hence the hypersurface is orientable and such field ν exists.

Then, we define the *second fundamental form* B which is a symmetric bilinear form given, in a local charts, by its components

$$h_{ij} = -\left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \,\Big|\, \nu \right\rangle$$

and whose trace is the *mean curvature* $\mathrm{H} = g^{ij} h_{ij}$ of the hypersurface (with these choices, the standard sphere of $\mathbb{R}^n$ has positive mean curvature).

Remark 1.1. If the hypersurface M is locally the graph of a function $f : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^n$, that is, $M = \{(x, f(x)) \,:\, x \in U\}$, then we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}\,, \qquad \nu = -\frac{(\nabla f, -1)}{\sqrt{1 + |\nabla f|^2}}\,,$$

$$h_{ij} = \frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}}\,, \tag{1.2}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess}\, f(\nabla f, \nabla f)}{\big(\sqrt{1 + |\nabla f|^2}\big)^3} = \mathrm{div}\Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \Big) \tag{1.3}$$

where $\mathrm{Hess}\, f$ is the (standard) Hessian of the function f.

Then, the following *Gauss–Weingarten relations* hold,

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = \Gamma^k_{ij}\frac{\partial\varphi}{\partial x_k} - h_{ij}\nu \qquad\qquad \frac{\partial\nu}{\partial x_j} = h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\,, \tag{1.4}$$

which easily imply

$$\nabla^2\varphi = -\mathrm{B}\nu \qquad \text{and} \qquad \Delta\varphi = -\mathrm{H}\nu\,. \tag{1.5}$$

The symmetry properties of the covariant derivative of B are given by the following Codazzi equations,

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}$$

which imply the following *Simons' identity* (see [23]),

$$\Delta h_{ij} = \nabla_i\nabla_j\mathrm{H} + \mathrm{H}\, h_{il}g^{ls}h_{sj} - |\mathrm{B}|^2 h_{ij}\,. \tag{1.6}$$

Finally, the Riemann tensor can be expressed as (*Gauss equations*),

$$\mathrm{R}_{ijkl} \;=\; h_{ik}h_{jl} - h_{il}h_{jk}\,. \tag{1.7}$$

If now we choose a fixed smooth, compact, embedded hypersurface of $\mathbb{R}^{n+1}$, it is well known (by its compactness and smoothness) that, for $\varepsilon > 0$ small enough, M_0 has a *tubular neighborhood*

$$N_\varepsilon = \big\{x \in \mathbb{R}^{n+1} \,:\, d(x, M_0) < \varepsilon\big\}$$

(where d is the Euclidean distance on $\mathbb{R}^{n+1}$) such that the *orthogonal projection map* $\pi : N_\varepsilon \to M_0$ giving the (unique) closest point on M_0, is well defined and smooth. Then, if E is "the interior" of M_0, the *signed distance function* $d_E : N_\varepsilon \to \mathbb{R}$ from M_0

$$d_E(x) = \begin{cases} d(x, M_0) & \text{if } x \notin E\\ -d(x, M_0) & \text{if } x \in E\end{cases}$$

is well defined and smooth in N_ε and $\nu(x) = \nabla d_E(x)$, for every $x \in M_0$. Moreover, for every $x \in N_\varepsilon$, the projection map π is given explicitly by

$$\pi_E(x) = x - \nabla d^2_E(x)/2 = x - d_E(x)\nabla d_E(x)$$

(indeed, actually $\nu(x) = \nabla d_E(x)$ for every $x \in M_0$).
This implies that, every smooth hypersurface M which is C^1–close enough to M_0, can be written (possibly after reparametrization) as

$$M = \big\{x + \psi(x)\nu(x) \,:\, x \in M_0\big\}, \tag{1.8}$$

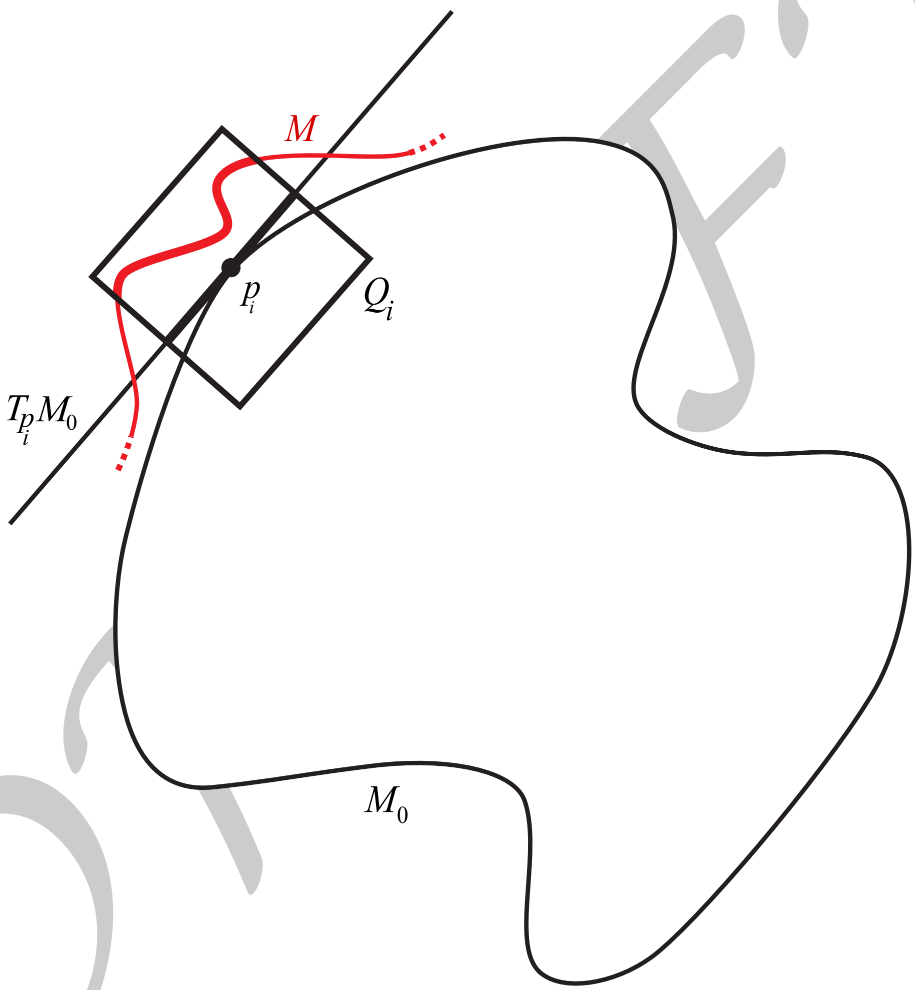
for a smooth function $\psi : M_0 \to \mathbb{R}$ with $\|\psi\|_{C^1(M_0)} < \varepsilon$. Indeed, if $\varphi_0 : \widetilde{M} \to \mathbb{R}^{n+1}$ and $\varphi : \widetilde{M} \to \mathbb{R}^{n+1}$ are two smooth immersions (φ_0 embedding) of a differentiable manifold $\widetilde{M}$, describing respectively M_0 and M, close in C^1, then the map $\pi \circ \varphi \circ \varphi_0^{-1} : M_0 \to M_0$ is a diffeomorphism, which implies that $\pi|_M : M \to M_0$ is also a diffeomorphism. Then, the map ψ above in expression (1.8), is uniquely given by $\psi(x) = d_E(\pi|_M^{-1}(x))$, which has small C^1–norm, as $\pi|_M$ gets C^1–closer and closer to the identity, as φ is C^1–close to φ_0.

Hence, from now on, we will consider families of hypersurfaces (clearly all containing M_0)

$$\mathfrak{C}^1_\delta(M_0) = \Big\{M = \big\{x + \psi(x)\nu(x) \,:\, x \in M_0\big\} \text{ for a smooth } \psi : M_0 \to \mathbb{R} \text{ with } \|\psi\|_{C^1(M_0)} < \delta\Big\}$$

where $\delta \in (0, \varepsilon)$. We are going to see that the constants in Sobolev, Gagliardo–Nirenberg, some geometric Calderón–Zygmund inequalities, trace and extension inequalities are uniformly bounded, depending only on M_0 and δ.

Before starting discussing that, we introduce another technical construction. We notice that, possibly choosing a smaller $\varepsilon > 0$, the tubular neighborhood N_ε of M_0 defined above, can be covered by a finite number of open hypercubes $Q_1, \dots, Q_k \subseteq \mathbb{R}^{n+1}$ respectively centered at some points $p_1, \dots, p_k \in M_0$, such that, for every $i \in \{1, \dots, k\}$ and every $M \in \mathfrak{C}^1_\delta(M_0)$, the "pieces" of hypersurfaces $M \cap Q_i$ can be written as graphs on the tangent hyperplanes to M_0 at the points $p_i \in M_0$, as in the following figure.



Then, we let $\rho_i : \mathbb{R}^{n+1} \to [0, 1]$ a smooth partition of unity (with compact support) for N_ε, associated to the open covering Q_i, hence, if $M \in \mathfrak{C}^1_\delta(M_0)$ and $u : M \to \mathbb{R}$, there holds

$$u(y) = \sum_{i=1}^{k} u(y)\rho_i(y) = \sum_{i=1}^{k} u_i(y)$$

with the compact support of $u_i : M \to \mathbb{R}$ contained in the piece $M \cap Q_i$ of the hypersurface M, which is described as the graph of a smooth function $\theta_i : T_{p_i}M_0 \to \mathbb{R}$, that is, $M \cap Q_i$ is the image of the map $x \mapsto \theta_i(x)\nu(p_i)$ on $T_{p_i}M_0 \cap Q_i$. It is then easy to see that $\|\theta_i\|_{C^1(T_{p_i}M_0)} \leq 2\delta$, for every $i \in \{1, \dots, k\}$.

We notice and underline that the family (and the number) of the hypercubes Q_i, as well as the width $\varepsilon > 0$ of the tubular neighborhood N_ε that we considered for this construction, only depend on M_0, precisely on its local and global geometry (in particular, on its second fundamental form B_0 – see [9] for more details).

We highlight to the reader that in the following, we will often denote with C a constant which may vary from a line to another.

2. Sobolev, Poincaré and Gagliardo–Nirenberg interpolation inequalities

We start discussing the Sobolev constants $C_S(p,n)$ of any compact hypersurface M, for every $p \in [1,n)$, entering in the following inequalities (which are known to hold, see [5, Chapter 2], for instance),

$$\|u\|_{L^{p^*}} = \Big(\int_M |u|^{p^*}\,d\mu\Big)^{1/p^*} \leq C_S(p,n)\Big(\int_M |\nabla u|^p + |u|^p\,d\mu\Big)^{1/p} = C_S(p,n)\|u\|_{W^{1,p}}$$

for every C^1–function $u : M \to \mathbb{R}$ (or $u \in W^{1,p}(M)$), where $p^* = \frac{np}{n-p}$ is the *Sobolev conjugate exponent* of p. It is well known that a bound on $C_S(1,n)$ implies a bound on $C_S(p,n)$, for every $p \in [1,n)$ (see [5, Chapter 2, Section 5], for instance), hence we concentrate on the case $p = 1$, where $1^* = \frac{n}{n-1}$.

We first want to argue localizing things by means of the construction of the previous section. We then have a finite family of hypercubes Q_i centered at $p_i \in M_0$, the partition of unity ρ_i and functions $\theta_i : T_{p_i}M_0 \to \mathbb{R}$, describing the pieces $M \cap Q_i$ of any smooth hypersurface in $\mathfrak{C}^1_\delta(M_0)$, with $\|\theta_i\|_{C^1(T_{p_i}M_0)} \leq 2\delta$, for every $i \in \{1, \dots, k\}$ and $\delta > 0$. Hence, we can write

$$\Big(\int_M |u|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} = \Big(\int_M \Big|\sum_{i=1}^k u\rho_i\Big|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leq \sum_{i=1}^k \Big(\int_{M\cap Q_i} |u\rho_i|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}}$$

as the compact support of $u\rho_i$ is contained in $M \cap Q_i$.
Then, for every C^1 function $v : M \to \mathbb{R}$, with compact support in $M \cap Q_i$, we have, setting $\nu_i = \nu(p_i)$,

$$\begin{aligned}\Big(\int_{M\cap Q_i} |v(y)|^{\frac{n}{n-1}}\,d\mu(y)\Big)^{\frac{n-1}{n}} &= \Big(\int_{T_{p_i}M_0\cap Q_i} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\sqrt{1+|\nabla\theta_i(x)|^2}\,dx\Big)^{\frac{n-1}{n}}\\ &\leq (1+2\delta)^{\frac{n-1}{n}}\Big(\int_{T_{p_i}M_0} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\,dx\Big)^{\frac{n-1}{n}}\end{aligned}$$

and applying the Sobolev inequality inequality in $\mathbb{R}^n \approx T_{p_i}M_0$, for C^1 functions with compact support, we have

$$\begin{aligned}\Big(\int_{T_{p_i}M_0} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\,dx\Big)^{\frac{n-1}{n}} &\leq C\int_{T_{p_i}M_0} \big|\nabla v(x+\theta_i(x)\nu_i)\circ\big(\mathrm{Id}+\nabla\theta_i(x)\otimes\nu_i\big)\big|\,dx\\ &\leq C\int_{T_{p_i}M_0} |\nabla v(x+\theta_i(x)\nu_i)|\,\big|\mathrm{Id}+\nabla\theta_i(x)\otimes\nu_i\big|\,dx\\ &\leq C\int_{T_{p_i}M_0} |\nabla v(x+\theta_i(x)\nu_i)|\,\big(1+|\nabla\theta_i(x)|\big)\,dx\\ &\leq C(1+2\delta)\int_{T_{p_i}M_0} |\nabla v(x+\theta_i(x)\nu_i)|\,dx\\ &\leq C\frac{1+2\delta}{1-2\delta}\int_{T_{p_i}M_0} |\nabla v(x+\theta_i(x)\nu_i)|\,\big|\mathrm{Id}+\nabla\theta_i(x)\otimes\nu_i\big|\,dx\\ &= C\frac{1+2\delta}{1-2\delta}\int_M |\nabla v(y)|\,d\mu(y)\,. \end{aligned} \tag{2.1}$$

Hence,

$$\Big(\int_{M\cap Q_i} |v|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leq C\frac{(1+2\delta)^{\frac{2n-1}{n}}}{1-2\delta}\int_M |\nabla v|\,d\mu\,.$$

and setting $v_i = u\rho_i$, after summing on $i\in\{1,\dots,k\}$, we conclude

$$\begin{aligned}\Big(\int_M |u|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} &\leq \sum_{i=1}^k\Big(\int_{M\cap Q_i}|v_i|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}}\\ &\leq C(\delta)\sum_{i=1}^k\int_M |\nabla v_i|\,d\mu\\ &= C(\delta)\sum_{i=1}^k\int_M |\nabla u|\rho_i + |u|\,|\nabla\rho_i|\,d\mu\\ &\leq C(\delta)\int_M |\nabla u|\,d\mu + C(\delta)C'(M_0)\int_M |u|\,d\mu\,, \end{aligned} \tag{2.2}$$

for a constant $C'(M_0)$ such that $|\nabla\rho_i| \leq C'(M_0)$, for every $i\in\{1,\dots,k\}$. This clearly gives a uniform bound on $C_S(1,n)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, depending only on M_0 (in particular, on its second fundamental form B_0, as we said in the previous section) and $\delta>0$.

Let now see an alternate line, based on the graph representation of the hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$ over M_0.
For every C^1 function $u: M\to\mathbb{R}$, we have

$$\Big(\int_M |u(y)|^{\frac{n}{n-1}}\,d\mu(y)\Big)^{\frac{n-1}{n}} = \Big(\int_{M_0} |u(x+\psi(x)\nu(x))|^{\frac{n}{n-1}}\,J\Psi(x)\,d\mu_0(x)\Big)^{\frac{n-1}{n}}$$

where $J\Psi$ is the (*tangential*) Jacobian of the map $\Psi : M_0 \to M$ given by $\Psi(x) = x + \psi(x)\nu(x)$ (by the *area formula*, see [22, Chapter 2, Section 8]), which is given by

$$J\Psi(x) = \sqrt{\det(d\Psi_x^T \circ d\Psi_x)}$$

and it is an easy check that, at every point $x \in M_0$, there holds

$$\frac{1}{C(\mathrm{B}_0, \delta)} \leq J\Psi \leq C(\mathrm{B}_0, \delta)\,,$$

for some constant $C(\mathrm{B}_0, \delta) > 0$, where B_0 is the second fundamental form of M_0. Moreover, $C(\mathrm{B}_0, \delta)$ goes to 1 as $\delta \to 0$. Notice that the fact that B_0 appears here can be seen from the expression of $d\Psi$, that is

$$d\Psi(x) = \mathrm{Id} + \nabla\psi(x) \otimes \nu + \psi(x)d\nu(x) = \mathrm{Id} + \nabla\psi(x) \otimes \nu + \psi(x)\mathrm{B}_0(x)\,,$$

by the Gauss–Weingarten relations (1.4).
Then, by applying the Sobolev inequality holding for M_0, we have

$$\begin{aligned}\Big(\int_{M_0} |u(x + \psi(x)\nu(x))|^{\frac{n}{n-1}}\, d\mu_0(x)\Big)^{\frac{n-1}{n}} &\leq C(M_0, \mathrm{B}_0, \delta) \int_{M_0} \big|\nabla[u(x + \psi(x)\nu(x))]\big|\, d\mu_0(x)\\ &= C(M_0, \mathrm{B}_0, \delta) \int_M |\nabla u(y)|\, J\Psi^{-1}(y)\, d\mu(y)\\ &\leq C(M_0, \mathrm{B}_0, \delta) \int_M |\nabla u(y)|\, d\mu(y)\,.\end{aligned}$$

Hence,

$$\Big(\int_M |u|^{\frac{n}{n-1}}\, d\mu\Big)^{\frac{n-1}{n}} \leq C(M_0, \mathrm{B}_0, \delta) \int_M |\nabla u|\, d\mu\,.$$

As before, this means that the constant $C(M_0, \mathrm{B}_0, \delta)$ is a uniform bound on $C_S(1, n)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, moreover, since $C(M_0, \mathrm{B}_0, \delta) \to 1$, as $\delta \to 0$, it also shows the continuous dependence of $C_S(1, n)$ under the C^1–convergence of the hypersurfaces.

Theorem 2.1. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$. Then, there exist uniform bounds, depending only on M_0 and δ (more precisely, on the "C^1– structure" of the immersion of M_0 in $\mathbb{R}^{n+1}$, its dimension and its second fundamental form), for all the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ on:*

- (i) *the volume of M,*
- (ii) *the Sobolev constants for $p \in [1, n)$ of the embeddings $W^{1,p}(M) \hookrightarrow L^{p^*}(M)$,*
- (iii) *the Sobolev constants for $p \in (n, \infty]$ of the embeddings $W^{1,p}(M) \hookrightarrow C^{0,1-n/p}(M)$,*
- (iv) *the constants in the Poincaré–Wirtinger inequalities on M, for $p \in [1, +\infty]$,*
- (v) *the Sobolev constant of the embedding $W^{1,n}(M) \hookrightarrow BMO(M)$,*
- (vi) *all the constants in the embeddings of the fractional Sobolev spaces $W^{s,p}(M)$,*
- (vii) *all the constants in the Gagliardo–Nirenberg interpolation inequalities on M.*

Moreover, all these bounds go to the corresponding constants for M_0, as $\delta \to 0$.

Proof.
(i) This is trivial due to the C^1–closedness of M to M_0.
(ii) As explained at the beginning of the section, we can estimate the constant in the Sobolev inequality for $p \in [1, n)$, by means of $C_S(1, n)$, which is uniformly bounded for all the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, by the above discussion.

(iii) If $p > n$, we show that there exists a uniform constant $C = C(p, n, M_0, \mathrm{B}_0, \delta)$ such that

$$\|u\|_{C^{0,\gamma}(M)} \leq C\|u\|_{W^{1,p}(M)} \tag{2.3}$$

with $\gamma = 1 - n/p$ and $\|u\|_{C^{0,\gamma}} = \sup_M |u| + \sup_{x\neq y} \frac{u(x)-u(y)}{|x-y|^\gamma}$.
As before, given the partition of unity ρ_i and the cubes Q_i, the following holds

$$\sup_{y\in M} |u(y)| \leq \sup_{y\in M\cap Q_i} \sum_{i=1}^k |u\rho_i|.$$

Then, for every C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, setting $\nu_i = \nu(p_i)$, we have

$$\sup_{y\in M\cap Q_i} |v| = \sup_{x\in T_{p_i}M_0} |v(x + \theta_i(x)\nu_i)| \leq C\|\nabla v\|_{L^p(T_{p_i}M_0)} \leq C\|\nabla v\|_{L^p(M)}$$

with $C = C(p, n, M_0, \mathrm{B}_0, \delta)$, where the inequalities follows by applying the Sobolev inequality for $p > n$ in $T_{p_i}M_0 \approx \mathbb{R}^n$, then arguing as in obtaining estimate (2.1). Then, setting $v_i = u\rho_i$ and estimating as in getting inequality (2.2), we conclude

$$\begin{aligned}
\sup_M |u| &\leq C(\delta) \sum_{i=1}^k \Big(\int_M |\nabla v_i|^p \, d\mu\Big)^{1/p} \\
&\leq C(\delta) \sum_{i=1}^k \Big(\int_M |\nabla u|^p \rho_i^p + |u|^p \, |\nabla \rho_i|^p \, d\mu\Big)^{1/p} \\
&\leq C(\delta)\Big(\int_M |\nabla u|^p \, d\mu\Big)^{1/p} + C(\delta)C'(M_0)\Big(\int_M |u|^p \, d\mu\Big)^{1/p}.
\end{aligned} \tag{2.4}$$

Regarding the seminorm $[u]_{C^{0,\gamma}} = \sup_{x\neq y} \frac{u(x)-u(y)}{|x-y|^\gamma}$, given two points $y, y^* \in M$, we have

$$|u(y^*) - u(y)| = \Big|\sum_{i=1}^k v_i(y^*) - v_i(y)\Big| \leq \sum_{i=1}^k |v_i(y^*) - v_i(y)|\,, \tag{2.5}$$

hence, given a C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, we have

$$\begin{aligned}
|v(y^*) - v(y)| &= |v(x^* + \theta_i(x^*)\nu_i(x^*)) - v(x + \theta_i(x)\nu_i(x))| \\
&\leq C(n, p, M_0) \, |x^* + \theta_i(x^*)\nu_i(x^*) - (x + \theta_i(x)\nu_i(x))|^\gamma \, \|\nabla v\|_{L^p(T_{p_i}M_0)} \\
&\leq C(n, p, M_0) \, |y^* - y|^\gamma \, \|\nabla v\|_{L^p(M)}
\end{aligned} \tag{2.6}$$

where the first inequality follows by arguing as in Theorem 5.6.4 in [14]). Then, collecting inequalities (2.5) and (2.6), we conclude

$$|u(y^*) - u(y)| \leq \sum |v_i(y^*) - v_i(y)| \leq C(p, n, \delta, M_0)|y^* - y|^\gamma \|\nabla u\|_{W^{1,p}(M)}$$

which together with inequality (2.4) give the desired estimate (2.3).

(iv) The conclusion for the Poincaré–Wirtinger inequality, for any $1 \leq p \leq \infty$,

$$\|u - \widetilde{u}\|_{L^p(M)} \leq C\|\nabla u\|_{L^p(M)}\,,$$

where $\widetilde{u} = \fint_M u \, d\mu$ follows by the one in $\mathbb{R}^n$ (Theorem 5.8.1 in [14]) and arguing exactly as for the Sobolev inequalities.

(v) As shown in Section 5.8.1 of [14], applying Poincaré–Wirtinger with $p=1$, we get

$$\int_M |u-\widetilde{u}|\,d\mu \leq C(p,n,\delta,M_0)\int_M |\nabla u|\,d\mu \leq C(p,n,\delta,M_0)\Big(\int_M |\nabla u|^n\Big)^{\frac{1}{n}}\,,$$

hence, the embedding $W^{1,n}(M)\hookrightarrow BMO(M)$ holds with auniform constant.

(vi) As for the "usual" (with integer order) Sobolev spaces, all the constants in the embeddings of the fractional Sobolev spaces are also uniform for this family. The proof is along the same line, localizing with a partition of unity and using the inequalities holding in $\mathbb{R}^n$ (see [20] and [21]).

(vii) Finally, we want to show that for any q,r real numbers $1\leq q\leq\infty, 1\leq r\leq\infty$ and j,m integers $0\leq j<m$, there exists a constant C depending only on n,j,m,r,q,θ,M_0 and δ such that the following interpolation inequalities hold

$$\|\nabla^j u\|_{L^p(M)} \leq C\big(\|\nabla^m u\|_{L^r(M)}+\|u\|_{L^r(M)}\big)^{\theta}\|u\|^{1-\theta}_{L^q(M)}, \tag{2.7}$$

where

$$\frac{1}{p}=\frac{j}{n}+\theta\Big(\frac{1}{r}-\frac{m}{n}\Big)+\frac{1-\theta}{q}$$

for all $\theta\in[j/m,1]$ such that p is nonnegative, with the exception of the case $r=\frac{n}{m-j}\neq 1$ for which the inequality is not valid for $\theta=1$.
Moreover, if $u:\partial F\to\mathbb{R}$ is a smooth function with $⨍_{\partial F} u\,d\mu=0$, inequality (2.7) simplifies to

$$\|\nabla^j u\|_{L^p(\partial F)} \leq C\|\nabla^m u\|^{\theta}_{L^r(\partial F)}\|u\|^{1-\theta}_{L^q(\partial F)}\,.$$

We can obtain these inequalities by following the proof of Theorem 3.70 of [5] (see also Proposition 5.1 of [19]), noticing that in such proof one only needs a bound on the volume, the Sobolev embedding theorems (that in our case hold with uniform constants, as we saw above) and some "universal" inequalities in which the constants do not depend on the hypersurfaces at all [5, Theorem 3.69]. □

Remark 2.2 (The fractional Sobolev spaces $W^{s,p}(M)$).
At point (vi) of the theorem above we considered the fractional Sobolev space $W^{s,p}$ on the hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$, which are usually defined via local charts for M and partitions of unity, that is, getting back to the definition with the Gagliardo $W^{s,p}$–seminorms in $\mathbb{R}^n$ (we refer to [4,12,20,21], for details). They can be also defined equivalently by considering directly on M the Gagliardo $W^{s,p}$–seminorm of a function $f\in L^p(M)$, for $s\in(0,1)$, as follows

$$[f]^p_{W^{s,p}(M)}=\int_M\int_M \frac{|f(x)-f(y)|^p}{|x-y|^{2+sp}}\,d\mu(x)d\mu(y)$$

and setting $\|f\|_{W^{s,p}(M)}=\|f\|_{L^p(M)}+[f]_{W^{s,p}(M)}$. Moreover, the constants giving the equivalence of the two norms obtained by localization or by this direct definition are uniform for all $M\in\mathfrak{C}^1_\delta(M_0)$. Indeed, the localization method of Section 1, is "uniform" for all $M\in\mathfrak{C}^1_\delta(M_0)$, meaning that the number of necessary local charts is fixed and the diffeomorphisms between $\mathbb{R}^n$ and "corresponding" (associated to correlated local charts, that is, being a graph on the same piece of M_0, as in our construction) local "pieces" of any different hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$, are uniformly close each other in C^1–norm.

3. Geometric Calderón–Zygmund inequalities

Theorem 3.1. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $1 < p < +\infty$. Then, if $\delta > 0$ is small enough, there exists a constant $C = C(M_0, p, \delta)$ such that the following geometric Calderon–Zygmund inequality holds,*

$$\|\mathrm{B}\|_{L^p(M)} \leq C\big(1 + \|\mathrm{H}\|_{L^p(M)}\big) \tag{3.1}$$

for every $M \in \mathfrak{C}^1_\delta(M_0)$.

Proof. In order to show inequality (3.1), we use the graph representation of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0, for δ small enough, introduced in the first section. Thus, since M is locally the graph of a function $f : U \subseteq T_{p_i}M \to \mathbb{R}$ on the tangent hyperplane to M_0 at the point $p_i \in M_0$, we can assume that $T_{p_i}M = \mathbb{R}^n$ and M is given by $M \cap Q_i = \{(x, f(x)) \,:\, x \in U \subset \mathbb{R}^n\}$, moreover, we can also assume that U is a ball $B_{2R} \subseteq \mathbb{R}^n$ of radius $2R > 0$, centered at the origin. The second fundamental form B and mean curvature H of M are then expressed by formulas (1.2) and (1.3),

$$\mathrm{B} = \frac{\mathrm{Hess}\, f}{\sqrt{1 + |\nabla f|^2}}$$

$$\mathrm{H} = \frac{\Delta f}{\sqrt{1 + |\nabla f|^2}} - \frac{\mathrm{Hess}\, f(\nabla f, \nabla f)}{\big(\sqrt{1 + |\nabla f|^2}\big)^3}\,.$$

We let $\rho : \mathbb{R}^n \to [0, 1]$ a cut–off function with compact support in B_{2R} and equal to 1 on B_R and we set $A_R = \{(x, f(x)) \,:\, x \in B_R\}$, $A_{2R} = \{(x, f(x)) \,:\, x \in B_{2R}\}$, then

$$\|\mathrm{B}\|^p_{L^p(A_R)} = \int_{B_R} |\mathrm{B}|^p \sqrt{1 + |\nabla f|^2}\, dx \leq C \int_{B_R} \rho^p |\mathrm{Hess}\, f|^p\, dx \leq C \int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f|^p\, dx \tag{3.2}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f|^p\, dx &\leq C \int_{\mathbb{R}^n} |\mathrm{Hess}(\rho f)|^p\, dx + C \int_{\mathbb{R}^n} |2\langle \nabla \rho | \nabla f\rangle|^p\, dx + C \int_{\mathbb{R}^n} |f \mathrm{Hess} \rho|^p\, dx \\ &\leq C \int_{\mathbb{R}^n} |\mathrm{Hess}(\rho f)|^p\, dx + Q(f, \nabla f, \rho, \nabla \rho, \mathrm{Hess}\, \rho)\,,\end{aligned}$$

where Q is the sum of second and third integral. Hence, by the standard Calderon–Zygmund estimates in $\mathbb{R}^n$ (see [16], for instance), we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f|^p &\leq C \int_{\mathbb{R}^n} |\Delta(\rho f)|^p\, dx + Q \leq \int_{\mathbb{R}^n} |\rho \Delta f|^p\, dx + Q \\ &\leq C \int_{\mathbb{R}^n} \Big| \rho \mathrm{H} \sqrt{1 + |\nabla f|^2} + \frac{\rho \mathrm{Hess}\, f(\nabla f, \nabla f)}{1 + |\nabla f|^2} \Big|^p\, dx + Q \\ &\leq C \int_{\mathbb{R}^n} |\rho \mathrm{H}|^p\, dx + C \int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f(\nabla f, \nabla f)|^p\, dx + Q \\ &\leq C \int_{\mathbb{R}^n} |\rho \mathrm{H}|^p\, dx + C |\nabla f|^{2p} \int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f|^p\, dx + Q\end{aligned}$$

where the constant C depends only on M_0 and δ and $Q = Q(f, \nabla f, \rho, \nabla \rho, \mathrm{Hess}\, \rho)$, thus this latter is also depending only M_0 and δ.

If $\delta > 0$ is small enough, then $C|\nabla f|^{2p} < 1/2$ and we get

$$\int_{\mathbb{R}^n} |\rho \mathrm{Hess}\, f|^p \leq 2C \int_{\mathbb{R}^n} |\rho \mathrm{H}|^p \, dx + 2Q \leq 2C \int_{B_{2R}} |\mathrm{H}|^p \, dx + 2Q \leq 2C \|\mathrm{H}\|^p_{L^p(A_R)} + 2Q$$

which clearly implies, by formula (3.2),

$$\|\mathrm{B}\|_{L^p(A_R)} \leq C\big(1 + \|\mathrm{H}\|_{L^p(M)}\big)\,.$$

Since the number of sets like A_R covering M is fixed, we have the thesis. □

We have an analogous theorem for Schauder estimates.

Theorem 3.2. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $\alpha \in (0, 1]$. Then, if $\delta > 0$ is small enough, there exists a constant $C = C(M_0, \alpha, \delta)$ such that the following geometric Schauder estimate holds,*

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leq C\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right)$$

for every $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$.

Proof. Assuming that $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$ with $\alpha \in (0, 1]$, that is $f \in C^{1,\alpha}(B_R)$ (hence $\nabla f \in C^{0,\alpha}(B_R)$), we have

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} \leq C\left\|\frac{\mathrm{Hess}\, f}{\sqrt{1 + |\nabla f|^2}}\right\|_{C^{0,\alpha}(B_R)} \leq C\, \|f\|_{C^{2,\alpha}(B_R)}\,. \tag{3.3}$$

Hence, by the standard Schauder estimates in B_{2R} (see [16], for instance), we get

$$\begin{aligned}
\|f\|_{C^{2,\alpha}(B_R)} &\leq C\,\|\Delta f\|_{C^{0,\alpha}(B_{2R})} + Q\big(\|f\|_{C^{1,\alpha}(B_{2R})}\big)\\
&\leq C\left\|\mathrm{H}\sqrt{1 + |\nabla f|^2} + \frac{\mathrm{Hess}\, f(\nabla f, \nabla f)}{1 + |\nabla f|^2}\right\|_{C^{0,\alpha}(B_{2R})} + Q\big(\|f\|_{C^{1,\alpha}(B_{2R})}\big)\\
&\leq C\,\|\mathrm{H}\|_{C^{0,\alpha}(B_{2R})} + C\,\|\nabla f\|^2_{C^{0,\alpha}(B_{2R})}\|\mathrm{Hess}\, f\|_{C^{0,\alpha}(B_{2R})} + Q\big(\|f\|_{C^{1,\alpha}(B_{2R})}\big)\\
&\leq C\,\|\mathrm{H}\|_{C^{0,\alpha}(B_{2R})} + C\,\|\nabla f\|^2_{C^{0,\alpha}(B_{2R})}\|f\|_{C^{2,\alpha}(B_{2R})} + Q\big(\|f\|_{C^{1,\alpha}(B_{2R})}\big)
\end{aligned}$$

where the constant C depends only on M_0 and δ.
Then, if $\delta > 0$ is small enough, we have $C\,\|\nabla f\|^2_{C^{0,\alpha}(B_{2R})} < 1/2$ and we get

$$\|f\|_{C^{2,\alpha}(B_R)} \leq C\,\|\mathrm{H}\|_{C^{0,\alpha}(B_{2R})} + Q \leq C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + Q$$

which clearly implies, by formula (3.3),

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} \leq C\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right),$$

where the constant C depends only on M_0 and δ. Since the family of sets like A_R covers all M, this inequality holds on all $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$ and we have the thesis. □

We now consider families of n–dimensional graph hypersurfaces in $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0 as above, with a uniform bound $\|\mathrm{H}\|_{L^p(M)} \leq C_\mathrm{H}$ with $p \geq n$, for every M in such family (by Theorem 3.2, if $\delta > 0$ is small enough, this implies $\|\mathrm{B}\|_{L^p(M)} \leq C_\mathrm{B}$) or $\|\mathrm{B}\|_{L^\infty(M)} \leq C_\mathrm{B}$.

As in the previous section, we consider a finite family of hypercubes Q_i centered at $p_i \in M_0$, an associated partition of unity ρ_i and functions $\theta_i : T_{p_i}M_0 \to \mathbb{R}$ such that $\|\theta_i\|_{C^1(T_{p_i}M_0)} \leq$

2δ, for every $i \in \{1,\dots,k\}$ and δ small enough. Hence, for every C^2–function $u : M \to \mathbb{R}$ we have

$$\|\nabla^2 u\|^p_{L^p(M)} \leq C \sum_{i=1}^k \|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\,. \tag{3.4}$$

Then, for every C^2 function $v : M \to \mathbb{R}$, with compact support in $M \cap Q_i$, we have, setting $\nu_i = \nu(p_i)$,

$$\begin{aligned}\int_{M\cap Q_i} |\nabla^2 v(y)|^p\,d\mu(y) &= \int_{T_{p_i}M_0\cap Q_i} \big|(\nabla^2 v)(x+\theta_i(x)\nu_i)\big|^p \sqrt{1+|\nabla\theta_i(x)|^2}\,dx\\ &\leq (1+2\delta)\int_{T_{p_i}M_0} \big|(\nabla^2 v)(x+\theta_i(x)\nu_i)\big|^p\,dx\,.\end{aligned}$$

We now compute the last integral using on $M \cap Q_i$ the coordinates given by $T_{p_i}M_0 \approx \mathbb{R}^n$, noticing that in such coordinates the local embedding of M is simply given by $x \mapsto \varphi(x) = x + \theta_i(x)\nu_i$ and the metric and the Christoffel symbols $\Gamma^s_{\ell m}$ of the connection ∇ can be expressed as

$$g_{\ell m}(x) = \delta_{\ell m} + \frac{\partial\theta_i}{\partial x_\ell}(x)\frac{\partial\theta_i}{\partial x_m}(x) \qquad \text{hence,} \qquad \Gamma^s_{\ell m}(x) = G^s_{\ell m}(\operatorname{Hess}\theta_i(x), \nabla\theta_i(x))$$

by formula (1.1), where $G^s_{\ell m}$ are smooth functions which are linear in $\operatorname{Hess}\theta_i(x)$. It is then easy to see, by recalling the first formula (1.4), that we can bound the Hessian $\operatorname{Hess}\theta_i(x)$ with $\mathrm{B}(x)$ as follows (notice that by the same formula, immediately holds also $|\mathrm{B}(x)| \leq C(|\nabla\theta_i|)|\operatorname{Hess}\theta_i(x)|$),

$$\Big|\frac{\partial^2\theta_i}{\partial x_\ell \partial x_m}\Big| = \Big|\frac{\partial^2\varphi}{\partial x_\ell \partial x_m}\Big| = \Big|\Gamma^s_{\ell m}\frac{\partial\varphi}{\partial x_s} - h_{\ell m}\nu\Big| \leq |G^s_{\ell m}(\operatorname{Hess}\theta_i, \nabla\theta_i)||\nabla\theta_i| + |\mathrm{B}|\,,$$

hence, being $|\nabla\theta_i| \leq \delta$, we conclude

$$|\operatorname{Hess}\theta_i(x)| \leq C|\operatorname{Hess}\theta_i(x)||\nabla\theta_i(x)| + C|\mathrm{B}(x)|$$

with a constant C depending only on M_0 and δ, which implies, if δ is small enough such that $C|\nabla\theta_i| < 1/2$,

$$|\operatorname{Hess}\theta_i(x)| \leq C|\mathrm{B}(x)|\,.$$

This clearly implies the estimate

$$|\Gamma^s_{\ell m}(x)| \leq C|\mathrm{B}(x)|\,,$$

then, computing schematically, we have

$$(\nabla^2 v)(x+\theta_i(x)\nu_i) = \operatorname{Hess}^{T_{p_i}M_0}_x v(x+\theta_i(x)\nu_i) - \Gamma(x) * \nabla^{T_{p_i}M_0}_x v(x+\theta_i(x)\nu_i) \tag{3.5}$$

hence,

$$\begin{aligned}|(\nabla^2 v)(x+\theta_i(x)\nu_i)| &\leq C|\operatorname{Hess}^{T_{p_i}M_0}_x v(x+\theta_i(x)\nu_i)| + C|\mathrm{B}(x)||\nabla v(x+\theta_i(x)\nu_i)||\nabla\theta_i(x)|\\ &\leq C|\operatorname{Hess}^{T_{p_i}M_0}_x v(x+\theta_i(x)\nu_i)| + C\delta|\mathrm{B}(x)||\nabla v(x+\theta_i(x)\nu_i)|\,.\end{aligned}$$

Applying the Calderón–Zygmund inequality in $T_{p_i}M_0 \approx \mathbb{R}^n$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \big|(\nabla^2 v)(x+\theta_i(x)\nu_i)\big|^p\,dx \leq{}& C\int_{\mathbb{R}^n} |\mathrm{Hess}_x^{T_{p_i}M_0} v(x+\theta_i(x)\nu_i)|^p\,dx\\ &+ C\delta^p\int_{\mathbb{R}^n} |\mathrm{B}(x)|^p|\nabla v(x+\theta_i(x)\nu_i)|^p\,dx\\ \leq{}& C\int_{\mathbb{R}^n} |\Delta_x^{T_{p_i}M_0} v(x+\theta_i(x)\nu_i)|^p\,dx\\ &+ C\int_{M\cap Q_i}|v(y)|^p\,\mu(y) + C\delta^p\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla v(y)|^p\,d\mu(y)\,.\end{aligned}$$

Contracting equation (3.5) with the inverse of the metric and estimating, we have

$$|\Delta_x^{T_{p_i}M_0} v(x+\theta_i(x)\nu_i)| \leq C|(\Delta v)(x+\theta_i(x)\nu_i)| + C|\mathrm{B}(x)||\nabla v(x+\theta_i(x)\nu_i)|$$

thus, by the previous inequality

$$\begin{aligned}\int_{M\cap Q_i}|\nabla^2 v(y)|^p\,d\mu(y) \leq{}& C\int_{\mathbb{R}^n}|(\Delta v)(x+\theta_i(x)\nu_i)|^p\,dx + C\int_{M\cap Q_i}|v(y)|^p\,\mu(y)\\ &+ C\delta^p\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla v(y)|^p\,d\mu(y)\\ \leq{}& C\int_{M\cap Q_i}|\Delta v(y)|^p\,d\mu(y) + C\int_{M\cap Q_i}|v(y)|^p\,\mu(y)\\ &+ C\delta^p\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla v(y)|^p\,d\mu(y)\,.\end{aligned}$$

Getting back to inequality (3.4), we obtain

$$\begin{aligned}\|\nabla^2 u\|^p_{L^p(M)} &\leq C\sum_{i=1}^k \|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\\ &\leq C\sum_{i=1}^k \int_{M\cap Q_i}|\Delta(u\rho_i)|^p\,d\mu + C\int_{M\cap Q_i}|u\rho_i|^p\,d\mu + C\delta^p\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla(u\rho_i)|^p\,d\mu\\ &\leq C\sum_{i=1}^k \int_{M\cap Q_i}|\Delta u|^p\,d\mu + C\int_{M\cap Q_i}\big(|u|^p+|\nabla u|^p\big)\,d\mu\\ &\leq C\int_M|\Delta u|^p\,d\mu + C\int_M\big(|u|^p+|\nabla u|^p\big)\,d\mu\,,\end{aligned}\tag{3.6}$$

with $C = C(M_0, \rho_i, \nabla\rho_i, \mathrm{Hess}\,\rho_i, p, \delta, \|\mathrm{B}\|_{L^\infty(M)})$. Interpolating the integral of $|\nabla u|^p$ and taking into account the uniform Sobolev inequalities of the previous section, we conclude that for any $p \in (1, +\infty)$, if $\delta > 0$ is small enough, there hold

$$\|\nabla^2 u\|_{L^p(M)} \leq C\|\Delta^2 u\|_{L^p(M)} + C\|u\|_{L^p(M)} \tag{3.7}$$

and

$$\|u\|_{W^{2,p}(M)} \leq C\|\Delta^2 u\|_{L^p(M)} + C\|u\|_{L^p(M)} \tag{3.8}$$

where the constant C depends only on M_0, p, δ and $\|\mathrm{B}\|_{L^\infty(M)}$.

Remark 3.3. Notice that if $p < n$, we can modify the chain of inequalities (3.6) as follows,

$$
\begin{aligned}
\|\nabla^2 u\|^p_{L^p(M)} &\leq C \sum_{i=1}^k \|\nabla^2 (u\rho_i)\|^p_{L^p(M\cap Q_i)} \\
&\leq C \sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p \, d\mu + C \int_{M\cap Q_i} |u\rho_i|^p \, d\mu + C\delta^p \int_{M\cap Q_i} |\mathrm{B}|^p |\nabla(u\rho_i)|^p \, d\mu \\
&\leq C \sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p \, d\mu + C \int_{M\cap Q_i} |u\rho_i|^p \, d\mu \\
&\quad + C\delta^p \Big(\int_{M\cap Q_i} |\mathrm{B}|^n \, d\mu \Big)^{p/n} \Big(\int_{M\cap Q_i} |\nabla(u\rho_i)|^{np/(n-p)} \, d\mu \Big)^{(n-p)/n} \\
&\leq C \sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p \, d\mu + C \int_{M\cap Q_i} |u\rho_i|^p \, d\mu \\
&\quad + C\delta^p \|\mathrm{B}\|^p_{L^n(M\cap Q_i)} \|\nabla^2 (u\rho_i)\|^p_{L^p(M\cap Q_i)} \, .
\end{aligned}
$$

Hence, arguing as before, it is easy to conclude that inequalities (3.7) and (3.8) hold with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^n(M)})$, if $\delta > 0$ is small enough.

With a similar argument, computing as in Theorem 3.1, we have analogous Schauder estimates for $C^{2,\alpha}$ functions $u : M \to \mathbb{R}$, with $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$ and $\delta > 0$ is small enough,

$$
\|u\|_{C^{2,\alpha}(M)} \leq C \|\Delta^2 u\|_{C^{0,\alpha}(M)} + C\|u\|_{C^{0,\alpha}(M)} \tag{3.9}
$$

where the constant C depends only on M_0, $\alpha \in (0, 1]$, δ and $\|\mathrm{B}\|_{C^{0,\alpha}}$.

Remark 3.4. By localization, computing in coordinates, it is easy to generalize estimates (3.7), (3.8) and (3.9) also to tensors, under the same hypotheses. The same holds also for all the estimates of the previous section (see [19] for an example of how this can be done).

3.1. Geometric higher order Calderón–Zygmund estimates.

We let M_0 as above and $p > 1$, we want to deal with $\|\nabla^k \mathrm{B}\|_{L^p(M)}$, assuming that we have a uniform bound $\|\mathrm{H}\|_{L^q(M)} \leq C_{\mathrm{H}}$ with $q > n$, where M is an n–dimensional graph hypersurfaces over M_0 in $\mathfrak{C}^1_\delta(M_0)$ as above, if $\delta > 0$ is small enough, which implies $\|\mathrm{B}\|_{L^q(M)} \leq C_{\mathrm{B}}$, by Theorem (3.1).

Using (1.5) and taking into account Remark 3.4, we have

$$
\begin{aligned}
\|\nabla^k \mathrm{B}\|_{L^p(M)} &= \|\nabla_{i_1}\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)}\\
&\leq C\|\Delta\nabla_{i_3}\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla_{i_3}\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)}\\
&= C\|g^{\ell m}\nabla_\ell\nabla_m\nabla_{i_3}\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\leq C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_m\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + \|g^{\ell m}\nabla_\ell(\mathrm{Riem}\star\nabla_{i_4}\dots\nabla_{i_k}\mathrm{B})_{i_3 m}\|_{L^p(M)}\\
&\leq C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\nabla_m\dots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + \|g^{\ell m}\nabla_\ell(\mathrm{Riem}\star\nabla_{i_4}\dots\nabla_{i_k}\mathrm{B})_{i_3 m}\|_{L^p(M)}\\
&\quad + \|g^{\ell m}\nabla_\ell\nabla_{i_3}(\mathrm{Riem}\star\nabla_{i_5}\dots\nabla_{i_k}\mathrm{B})_{i_4 m}\|_{L^p(M)}\\
&\qquad\qquad\dots\\
&\leq C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\dots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\leq C\|g^{\ell m}\nabla_{i_3}\nabla_\ell\nabla_{i_4}\dots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\qquad\qquad\dots\\
&\leq C\|g^{\ell m}\nabla_{i_3}\nabla_{i_4}\dots\nabla_{i_k}\nabla_\ell\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&= C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}
\end{aligned}
\tag{3.10}
$$

where the symbol $\star$ means a sum of terms each one given by some contraction with the inverse of the metric g^{ij}.
By the formula (1.7) for the Riemann tensor, we can write

$$
\begin{aligned}
\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leq C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{B}^2\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\leq C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\sum_{\substack{s,r,t=0\\ s+r+t=k-2}}^{k-2}\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)}\,.
\end{aligned}
$$

Now, by Simons' identity (1.6), we have

$$
\nabla^{k-2}\Delta\mathrm{B} = \nabla^k\mathrm{H} + \nabla^{k-2}(\mathrm{H}\mathrm{B}^2) - \nabla^{k-2}(|\mathrm{B}|^2\mathrm{B})
$$

hence

$$\|\nabla^{k-2}\Delta \mathrm{B}\|_{L^p(M)} \leq \|\nabla^k \mathrm{H}\|_{L^p(M)} + C \sum_{\substack{s,r,t=0\\ s+r+t=k-2}}^{k-2} \|\nabla^s \mathrm{B} \star \nabla^r \mathrm{B} \star \nabla^t \mathrm{B}\|_{L^p(M)} \,.$$

Using this estimate in inequality (3.10), we conclude

$$\|\nabla^k \mathrm{B}\|_{L^p(M)} \leq C\|\nabla^k \mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C \sum_{\substack{s,r,t=0\\ s+r+t=k-2}}^{k-2} \|\nabla^s \mathrm{B} \star \nabla^r \mathrm{B} \star \nabla^t \mathrm{B}\|_{L^p(M)} \,.$$

We now estimate any of the term in the last sum as follows: assuming that $p(k+1) > n$, otherwise we estimate every term $\nabla^s \mathrm{B} \star \nabla^r \mathrm{B} \star \nabla^t \mathrm{B}$ in $L^{(n+k)/(k+1)}(M)$ and then bound with this latter its norm in $L^p(M)$ (the volumes are equibounded for all $M \in \mathfrak{C}^1_\delta(M_0)$), we have

$$\|\nabla^s \mathrm{B} \star \nabla^r \mathrm{B} \star \nabla^t \mathrm{B}\|_{L^p(M)} \leq\; C\|\nabla^s \mathrm{B}\|_{L^{\alpha p}(M)} \|\nabla^r \mathrm{B}\|_{L^{\beta p}(M)} \|\nabla^t \mathrm{B}\|_{L^{\gamma p}(M)} \,, \tag{3.11}$$

with

$$\alpha = \frac{k+1}{s+1}\,, \qquad \beta = \frac{k+1}{r+1}\,, \qquad \gamma = \frac{k+1}{t+1}\,,$$

hence, $1/\alpha + 1/\beta + 1/\gamma = 1$. Moreover, using the interpolation estimates (2.7) (extended to tensors – see Remark 3.4), we have

$$\begin{aligned}
\|\nabla^s \mathrm{B}\|_{L^{p\alpha}(M)} &\leq C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\alpha} \|\mathrm{B}\|^{1-\theta_\alpha}_{L^q(M)}\,,\\
\|\nabla^s \mathrm{B}\|_{L^{p\beta}(M)} &\leq C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\beta} \|\mathrm{B}\|^{1-\theta_\beta}_{L^q(M)}\,,\\
\|\nabla^s \mathrm{B}\|_{L^{p\gamma}(M)} &\leq C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\gamma} \|\mathrm{B}\|^{1-\theta_\gamma}_{L^q(M)}\,,
\end{aligned}$$

with

$$\begin{aligned}
\frac{1}{p\alpha} &= \frac{s}{n} + \theta_\alpha\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\alpha}{q}\,,\\
\frac{1}{p\beta} &= \frac{r}{n} + \theta_\beta\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\beta}{q}\,,\\
\frac{1}{p\gamma} &= \frac{t}{n} + \theta_\gamma\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\gamma}{q}\,,
\end{aligned}$$

hence,

$$\frac{1}{p} = \frac{1}{p\alpha} + \frac{1}{p\beta} + \frac{1}{p\gamma} = \frac{k-2}{n} + (\theta_\alpha + \theta_\beta + \theta_\gamma)\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{3 - \theta_\alpha - \theta_\beta - \theta_\gamma}{q}\,,$$

which implies, letting $\Theta = (\theta_\alpha + \theta_\beta + \theta_\gamma)$,

$$\frac{1}{p} = \frac{k-2}{n} + \Theta\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{3-\Theta}{q} < \frac{k-2}{n} + \Theta\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{3-\Theta}{n} = \frac{k+1}{n} + \Theta\Big(\frac{1}{p} - \frac{k+1}{n}\Big)\,.$$

As we assumed $p(k+1) > n$, it follows $\Theta < 1$.
Thus, putting these estimates in inequality (3.11), we conclude

$$\|\nabla^s \mathrm{B}\star\nabla^r \mathrm{B}\star\nabla^t \mathrm{B}\|_{L^p(M)} \leq C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta} \|\mathrm{B}\|^{3-\Theta}_{L^q(M)} \leq C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta}$$

as we said that $\|\mathrm{B}\|_{L^q(M)}$ is uniformly bounded for all $M \in \mathfrak{C}^1_\delta(M_0)$.
Hence, by means of Young inequality, as $\Theta < 1$, we estimate

$$\begin{aligned}\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leq C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^\Theta \\ &\leq C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\varepsilon\|\nabla^k\mathrm{B}\|_{L^p(M)} + C\|\mathrm{B}\|_{L^p(M)} + C\end{aligned}$$

and choosing $\varepsilon > 0$ such that $C\varepsilon < 1/2$, after "absorbing" the term $C\varepsilon\|\nabla^k\mathrm{B}\|_{L^p(M)}$ in the left hand side and estimating $\|\mathrm{B}\|_{L^p(M)}$ with $C(1 + \|\mathrm{H}\|_{L^p(M)})$, we obtain

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leq C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\mathrm{H}\|_{L^p(M)} + C\,.$$

The term $\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}$ can be treated analogously, by interpolation between $\|\nabla^k\mathrm{B}\|_{L^p(M)}$ and $\|\mathrm{B}\|_{L^p(M)}$ (it is actually easier to be dealt with), hence we finally have the estimate

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leq C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\mathrm{H}\|_{L^p(M)} + C\,.$$

In particular, if we have a uniform bound $\|\mathrm{H}\|_{L^\infty(M)} \leq C_\mathrm{H}$, there holds

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leq C\big(1 + \|\nabla^k\mathrm{H}\|_{L^p(M)}\big)$$

and

$$\|\mathrm{B}\|_{W^{k,p}(M)} \leq C\big(1 + \|\mathrm{H}\|_{W^{k,p}(M)}\big)\,.$$

for any $M \in \mathfrak{C}^1_\delta(M_0)$, with $\delta > 0$ small enough.

4. Other inequalities

Let M_0 be a smooth and compact hypersurface embedded in $\mathbb{R}^{n+1}$, bounding a domain E_0 and $\varepsilon > 0$ the width of a tubular neighborhood N_ε of M_0. For any $\delta \in (0, \varepsilon)$, we consider the family of domains

$$\mathcal{C}^1_\delta(E_0) = \left\{ E = \Psi(E_0) \;:\; \begin{array}{l} \Psi : \overline{E_0} \to \overline{E} \text{ is a diffeomorphism with } \|\Psi - \mathrm{Id}\|_{C^1(E_0)} < \delta \\ \Psi(x) = x + \psi(x)\nu_0(x) \text{ for every } x \in M_0 \text{ and } \|\psi\|_{C^1(M_0)} < \delta \end{array} \right\}$$

where ν_0 is the unit normal vector field pointing outward of M_0.
Then, the Jacobian of the map $\Psi : \overline{E_0} \to \overline{E}$ (and also the tangencial one of its restriction to M_0) is bounded from above and from below by some constants which depend only on δ and the second fundamental form of M_0 (see Section 2 for details).

It clearly follows that if $E \in \mathcal{C}^1_\delta(E_0)$, then $M = \partial E = \Psi(M_0) \in \mathfrak{C}^1_\delta(M_0)$. Moreover, if $M \in \mathfrak{C}^1_{\delta'}(M_0)$, then there exists a smooth function $\psi : M_0 \to \mathbb{R}$ with $\|\psi\|_{C^1(M_0)} < \delta'$, such that $M = \big\{x + \psi(x)\nu(x) \;:\; x \in M_0\big\}$, then we can construct a smooth diffeomorphism $\Psi : \overline{E_0} \to \overline{E}$ as follows (E is the domain bounded by M):

$$\Psi(x) = \begin{cases} x & \text{if } x \in E_0 \setminus N_\varepsilon \\ x + \Theta(d_0(x)/\varepsilon)\psi(\pi_0(x))\nabla d_0(x) & \text{if } x \in \overline{E}_0 \cap N_\varepsilon \end{cases}$$

where d_0 is the signed distance function from M_0 (which is negative in E_0) and $t \mapsto \Theta(t)$ is a smooth monotone nonincreasing function, defined on $\mathbb{R}$, such that it is equal to 1 if $t \leq 0$ and to 0 if $t \geq 1/2$, with $|\Theta'(t)| \leq 3$, for every $t \in \mathbb{R}$. So, it follows

$$\|\Psi - \mathrm{Id}\|_{C^1(E_0)} = \|\Theta(d_0(\cdot)/\varepsilon)\psi(\pi_0(\cdot))\nabla d_0(\cdot)\|_{C^1(E_0)} \leq C\big(\varepsilon, M_0, \|\Theta\|_{C^1(\mathbb{R})}\big)\|\psi\|_{C^1(M_0)}\,.$$

Hence, fixed any $\delta \in (0, \varepsilon)$, depending the constant C only on M_0 and ε, possibly choosing δ' small enough, the set E belongs to $\mathcal{C}^1_\delta(E_0)$.

We now discuss some uniform inequalities involving also “the interiors” of the hypersurfaces.

4.1. **Trace inequalities.**

Letting E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and any $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E_0} \to \overline{E}$), it is well known that, for $s > 1/2$, the *trace* of any function $u \in H^s(E)$ (a real function on $M = \partial E$, which we still simply denote by u, that coincides with the restriction of u to M, if $u \in C^0(\overline{E})$) is well defined and that the following *trace inequality* holds (see [24, Proposition 4.4.5]),

$$\|u\|_{H^{s-1/2}(M)} \leq C_E \|u\|_{H^s(E)} \tag{4.1}$$

(see also [14, 18]). In particular, for $s = 1$, we have that for all the function $u \in H^1(E)$, there holds

$$\|u\|^2_{H^{1/2}(M)} \leq C_E \int_E u^2 + |\nabla u|^2 \, dx \,,$$

which implies

$$\|u - \widetilde{u}\|^2_{H^{1/2}(M)} \leq C_E \int_E |\nabla u|^2 \, dx \,,$$

where $\widetilde{u} = \fint_E u \, dx$.
We want to show that these inequalities hold with uniform constants $C = C(M_0, s, n, \delta)$, for every $E \in \mathcal{C}^1_\delta(E_0)$.
As in Section 2, we use the graph representation of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0 in order to pass from a Sobolev norm over M to the same norm over M_0, that is, being $\|\psi\|_{C^1(M_0)}$ bounded by a constant depending on δ and the second fundamental form of M_0, we have

$$\|u\|_{H^{s-1/2}(M)} \leq C(M_0, \mathrm{B}_0, s, \delta) \|u\|_{H^{s-1/2}(M_0)} \,, \tag{4.2}$$

where B_0 is the second fundamental form of M_0. Then, by means of trace inequality (4.1) for E_0 (and M_0), we have

$$\|u\|_{H^{s-1/2}(M_0)} \leq C_{E_0} \|u\|_{H^s(E_0)} \,. \tag{4.3}$$

Finally, by the boundedness of the Jacobian of Ψ (from both sides) and of $\|\Psi - \mathrm{Id}\|_{C^1(E_0)}$ by constants depending on δ, we get

$$\|u\|_{H^s(E_0)} \leq C(E_0, s, \delta) \|u\|_{H^s(E)} \,, \tag{4.4}$$

hence, putting together inequalities (4.2), (4.3) and (4.4), we have that the constant C_E in the trace inequality (4.1) is uniform, for every $E \in \mathcal{C}^1_\delta(E_0)$.

4.2. **Inequalities for harmonic extensions.**

We let E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E_0} \to \overline{E}$), with $M = \partial E \in \mathfrak{C}^1_\delta(M_0)$.
We denote by $u : E \to \mathbb{R}$ the harmonic extension of a function $f : M \to \mathbb{R}$ in $H^s(M)$, for $s \geq 1/2$. We aim to show that the following inequality (see [24, Proposition 5.1.7])

$$\|u\|_{H^{s+1/2}(E)} \leq C_E \|f\|_{H^s(M)} \tag{4.5}$$

holds with a uniform constant $C = C(E_0, s, \delta)$, for every $E \in \mathcal{C}^1_\delta(E_0)$.
Arguing as above, we end up with the following inequalities:

$$\begin{aligned}
\|u\|_{H^{s+1/2}(E)} &\leq C(E_0, s, \delta)\|u\|_{H^{s+1/2}(E_0)}\,, \\
\|u\|_{H^{s+1/2}(E_0)} &\leq C_{E_0}\|f\|_{H^s(M_0)}\,, \\
\|f\|_{H^s(M_0)} &\leq C(M_0, \mathrm{B}_0, s, \delta)\|f\|_{H^s(M)}\,.
\end{aligned}$$

Putting them together, we have that the constant C_E in the "extension" inequality (4.5) is uniform, for every $E \in \mathcal{C}^1_\delta(E_0)$.

We notice that, in the particular case $s = 1/2$, we obtain for all $f \in H^{1/2}(M)$,

$$\int_E |\nabla u|^2\,dx \leq C(E_0, \delta)\|f\|^2_{H^{1/2}(M)}\,,$$

for every $E \in \mathcal{C}^1_\delta(E_0)$.

5. Some remarks

We collect here some remarks about the conclusions of the previous sections.

- All the constants depend on the geometric properties of M_0, in particular on the maximal width of a tubular neighbourhood, its volume and its second fundamental form. Hence, uniformly controlling such quantities gives uniform estimates for larger families of hypersurfaces, see [7–9, 11, 17] for a deeper and detailed discussion).
- Notice that for Sobolev, Poincaré, interpolation, trace and "harmonic extension" inequalities, we do not ask $\delta > 0$ to be small, but just $\delta < \varepsilon$, while for the Calderón–Zygmund–type inequalities, that we worked out in Section 3, a smallness condition on δ is necessary for the conclusions.
- All the inequalities holds uniformly also for families of immersed–only hypersurfaces (non necessarily embedded), if they can be expressed as graphs on a fixed compact, smooth hypersurface, possibly immersed–only too.
- It is easy to see that everything we did still works also if the ambient is a *flat*, complete Riemannian manifold, in particular in any flat torus $\mathbb{T}^n$. With some effort, the results can be generalized to graph hypersurfaces in any complete Riemannian manifold, then the constants also depends on the geometry (in particular, on the curvature) of such an ambient space.

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