ON THE STEINER TREE CONNECTING A FRACTAL SET

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ABSTRACT. We construct an example of an infinite planar embedded self-similar binary tree Σ which is the essentially unique solution to the Steiner problem of finding the shortest connection of a given planar self-similar fractal set C of positive Hausdorff dimension. The set C can be considered the set of *leaves*, or the "boundary", of the tree Σ, so that Σ is an *irreducible* solution to the Steiner problem with datum C (i.e. $\Sigma \setminus C$ is connected).

1. INTRODUCTION

Let C be a subset of $\mathbb{R}^2$. The Steiner problem with datum C consists of finding a set $S \subset \mathbb{R}^2$ such that $S \cup C$ is connected and $\mathcal{H}^1(S)$ is minimal where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure (i.e. the length of the set). Usually this problem is considered for the case when C is the finite set and in this setting it becomes a problem of combinatorial geometry. The Steiner problem goes back to Jarník and Kössler [4] (although some of its particular cases are much more ancient, for instance when C is the set of the three vertices of a triangle the problem was stated already by Fermat and solved by Torricelli). The problem is computationally hard and relatively few explicit solutions are known, especially in the cases when the number of points is large.

However the problem is important for the more general setting of possibly infinite sets C. Moreover finding explicit solutions for an infinite C may shed light on the solutions to finite sets with an arbitrarily large number of points. One natural question that has arosen in this context is whether there exists a planar irreducible Steiner tree S connecting an uncountable and possibly fractal set of points C (we say that S is irreducible if $S \setminus C$ is connected). A very similar question, that of existence of irreducible Steiner sets with the topology of a binary tree for a finite set of points C with arbitrarily large number of points, has been answered positively by Ivanov and Tuzhilin [3] by means of an abstract construction. Further an explicit construction of such a tree has been constructed in [8]. Namely, the set C consists of a *root H* and uncountably many points (*leaves*) which together give a totally disconnected perfect set (i.e. every connected component is a point, but the points are not isolated). No segment of S touches the leaves, while every leaf is an accumulation point of segments of S. The infinite tree S is composed by a *trunk* of some length ℓ which splits at angles of 120 degrees into two branches of length $\lambda_1 \ell$ both of which split further at the same angles into two branches of length $\lambda_1 \lambda_2 \ell$ and so on (see Figure 1).
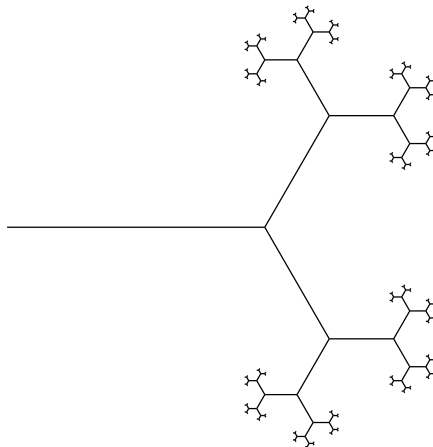
FIGURE 1. The self-similar tree S with a countable number of triple junctions and an uncountable number of leaves which are the limit points of the triple junctions.

In [8] it has been proven that the suggested tree is in fact the solution (and even the unique one) to the Steiner problem when the sequence of coefficients λ_k is small and rapidly decreasing to zero (in particular summable). In such a situation the set C though being uncountable has zero Hausdorff dimension. The natural original question, whether this construction is also valid for fractal self-similar C (i.e. when the coefficients are constant, i.e. $\lambda_j = \lambda$ for all j) was left open. In the attempt to solve it, a completely new and different method, based on special group valued calibrations, to check whether the given set S is a solution to the Steiner problem, has been developed in [6]. In [5] the authors also show an example of an infinite *irrigation* tree which is a solution to a ramified transportation problem similar to the Steiner problem. The method itself is very useful and can be applied in many situations. However it failed in this particular problem. A breakthrough has been provided by the recent preprint [1] where the authors provide a much simpler symmetry based argument to show that for sufficiently small but constant coefficients the above construction gives in fact an irreducible Steiner tree for a set C of positive Hausdorff dimension. However this construction unfortunately does not provide uniqueness of the minimizer leaving open the possibility of having other solutions different from the conjectured one. In this paper we close the original question of both existence and uniqueness. To do so we develop a method which in particular uses also a further simplification and a generalization of the idea developed in [1]. Moreover since we do not use the symmetry argument anymore our method could be applied even to non symmetric trees.

To be more precise, let $\mathbb{R}^2$ be identified with the complex plane $\mathbb{C}$ and given $\lambda > 0$, consider $f_j(z) = 1 + \theta_j z$, where $\theta = \lambda e^{(-1)^j i \frac{\pi}{3}}$, for $j = 1, 2$. The affine map f_j is the composition of a rotation, rescaling and translation. Let A be the only non-empty compact set such that

$$A = f_1(A) \cup f_2(A).$$

One can check that A is the limit, in the sense of Hausdorff distance, of the finite sets A_N where $A_N := \{f_{j_1}(f_{j_2}(\ldots f_{j_N}(0)))\colon j_1, j_2, \ldots, j_N \in \{1, 2\}\}$. We are going to prove that if λ is sufficiently small then for $C := \{H\} \cup A$, with $H = 0$, the closure $\bar{S}$ of any solution S to the Steiner problem is connected and is equal to the binary self-similar tree Σ defined by the recurrence equation

$$\Sigma = [0, 1] \cup f_1(\Sigma) \cup f_2(\Sigma).$$

Equivalently one can define Σ as

$$\Sigma := \overline{\bigcup_{N \in \mathbb{N}} \bigcup \{f_{j_1} \circ \cdots \circ f_{j_N}([0, 1])\colon j_1, \ldots, j_N \in \{1, 2\}\}}.$$

2. Notation and preliminaries

Let $\mathrm{St}(C)$ be the family of all sets S such that $S \cup C$ is connected and let $\mathcal{M}(C)$ be the subset of $\mathrm{St}(C)$ of sets S with minimal (possibly infinite) length $\mathcal{H}^1(S)$.

For a set $S \subset \mathbb{R}^2$ we denote by $\bar{S}$ its closure and by ∂S its topological boundary. If $x \in \mathbb{R}^2$ and $\rho > 0$ then we let $B_\rho(x) \subset \mathbb{R}^2$ be the open ball of radius ρ centered at x. When $x \in \mathbb{R}^2$, S, S_1, S_2 subsets of $\mathbb{R}^2$ we also write

$$\mathrm{dist}(x, S) := \inf\{|x - y| : y \in S\},$$
$$\mathrm{dist}(S_1, S_2) := \inf\{|x - y| : x \in S_1, y \in S_2\}.$$

For $P, Q \in \mathbb{R}^2$ we denote by $[PQ]$ the closed straight line segment with endpoints P and Q and by $|PQ|$ its length.

2.1. Auxiliary lemmata.
In the following proposition we state some known facts about solutions to the Steiner problem collected from [7] and [3].

Proposition 2.1 (known facts about minimizers). *Let $C \subset \mathbb{R}^n$ be a compact set. Then $\mathcal{M}(C) \neq \emptyset$. If $S \in \mathcal{M}(C)$ with $\mathcal{H}^1(S) < +\infty$ then the following assertions hold.*

   (i) *$S \cup C$ is compact, hence $\bar{S} \setminus S \subset C$;*

   (ii) *the closure of every connected component of S is a topological tree (contains no subset homeomorphic to $\mathbb{S}^1$) with endpoints[1] on C and with at most one endpoint on each connected component of C;*

   (iii) *for all $x \in S \setminus C$ and all $\rho < \mathrm{dist}(x, C)$ the set $S \cap \bar{B}_\rho(x)$ is the union of a finite number of straight line segments which meet in triple points with equal angles of 120 degrees and have endpoints in $\partial B_\rho(x)$.*

The following lemma is also taken from [7, lemma 2.6].

Lemma 2.2. *If S is connected and $\mathcal{H}^1(S) < +\infty$ then $\mathcal{H}^1(S) = \mathcal{H}^1(\bar{S})$.*

We also need the following lemmata for the case when the set to be connected is not compact but contains a line.

Lemma 2.3. *Let C be a compact subset of $\mathbb{R}^2$ and ℓ be a straight line. Then $\mathcal{M}(\ell \cup C) \neq \emptyset$. If C is also totally disconnected, $C \cap \ell = \emptyset$ and $S \in \mathcal{M}(\ell \cup C)$ then $\bar{S} \supset C$. If moreover $\mathcal{H}^1(S) < +\infty$ and $\mathcal{H}^1(C) = 0$ then $\bar{S} \in \mathcal{M}(\ell \cup C)$.*

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[1]When S is a compact topological tree, then $x \in S$ is an endpoint of S if and only if $S \setminus \{x\}$ is connected.

Proof. Take a compact connected segment $\sigma \subset \ell$ such that the orthogonal projection of C on ℓ is contained in σ. Then $\sigma \cup C$ is compact and by [7] we know that $\mathcal{M}(\sigma \cup C) \neq \emptyset$. For all $\Sigma \in \mathrm{St}(\ell \cup C)$ we have that its projection $\tilde{\Sigma}$ onto the convex hull of $\sigma \cup C$ is in $\mathrm{St}(\sigma \cup C)$ and $\mathcal{H}^1(\tilde{\Sigma}) \leq \mathcal{H}^1(\Sigma)$. Hence $\mathcal{M}(\ell \cup C) \supset \mathcal{M}(\sigma \cup C) \neq \emptyset$ as claimed.

To prove the second part note that if C is totally disconnected then for every $x \in C$ one has $x \in \bar{S}$ otherwise $S \cup C \cup \ell$ cannot be connected. Therefore $C \subset \bar{S}$.

For the last claim notice that $\bar{S} \cap \ell$ is a finite set. In fact the number of connected components of S arriving at ℓ is finite since every such component must touch also points of C and hence has length at least $\mathrm{dist}(C, \ell)$. Hence these components cannot be infinitely many, since otherwise we would have $\mathcal{H}^1(S) = +\infty$. In view of Proposition 2.1(ii) the closure of every connected component of $S \in \mathcal{M}(C \cup \sigma)$ has at most one point on σ, showing that the set $W := \bar{S} \cap \sigma = \bar{S} \cap \ell$ is finite.

Moreover $\bar{S} \setminus S \subset C \cup W$ because $S \cup C \cup \sigma$ is compact by Proposition 2.1(i). When $\mathcal{H}^1(C) = 0$ this implies $\mathcal{H}^1(\bar{S}) = \mathcal{H}^1(S)$ and hence $\bar{S} \in \mathcal{M}(\ell \cup C)$. $\qquad\square$

Lemma 2.4. *Let ℓ be a line, $\rho > 0$, $r = 2\rho/\sqrt{3}$. Let P be a point such that $\mathrm{dist}(P, \ell) > 3r$ and let C be a compact subset of $\bar{B}_\rho(P)$. Let $S \in \mathcal{M}(\ell \cup C)$ (such S exists in view of Lemma 2.3) and suppose $\mathcal{H}^1(S) < +\infty$. Then there exists a point $Q \in \partial B_r(P)$ and a point $H \in \ell$ such that $\bar{S} \setminus B_r(P) = [HQ]$. Moreover $[HQ]$ is perpendicular to ℓ.*

Proof. We claim that $S \setminus B_r(P)$ belongs to a single connected component of S. In fact, if not, there are two different connected components S_1 and S_2 of S which have some points outside of $B_r(P)$. Their closures must have points in ℓ otherwise they would be contained in the convex hull of $C \subset B_r(P)$. By Proposition 2.1(ii) we know that $\bar{S}_1$ and $\bar{S}_2$ each have at most one endpoint on ℓ (hence exactly one). Let H_1 and H_2 be such endpoints. Note that $S_j \in \mathcal{M}(\{H_j\} \cup C_j)$ with $C_j := \bar{S}_j \cap C$, $j = 1, 2$, compact subsets of C. By Lemma 2.5 below with C_j and H_j in place of C and H respectively, one has that the set $S_j \setminus B_r(P)$ is a straight line segment $[H_j T_j]$ with $T_j \in \partial B_r(P)$, $j = 1, 2$. Since $\bar{S}_1 \cap \bar{S}_2 \subset \ell \cup C$ the line segments $[H_1 T_1]$ and $[H_2 T_2]$ may only intersect at their endpoints. Let $S' := (S \setminus [H_2 T_2]) \cup [T_1 T_2])$ so that $S' \in \mathrm{St}(\ell \cup C)$ and

$$\mathcal{H}^1(S) - \mathcal{H}^1(S') = |H_2 T_2| - |T_1 T_2| \geq (d(P, \ell) - r) - 2r > (3r - r) - 2r = 0.$$

Hence we would have $\mathcal{H}^1(S') < \mathcal{H}^1(S)$ contrary to the minimality of S.

Having proven that $S \setminus B_r(P)$ belongs to a single connected component of S, denoting by H its endpoint on ℓ, and applying Lemma 2.5, we have that $S \setminus B_r(P)$ is a straight line segment $[HT]$. Finally, if $[HT]$ were not perpendicular to ℓ by moving H along ℓ we could decrease the length of $[HT]$ contrary to the minimality of S. $\qquad\square$

In the proof of the above Lemma 2.4 we used the following statement.

Lemma 2.5. *Let $S \in \mathcal{M}(\{H\} \cup C)$ with C a compact set, $H \notin C$. Then there exists a point Q such that $[HQ] \subset \bar{S}$ and either $Q \in C$ or Q is a triple point of S and there are no other branching points of S on $[HQ]$.*

Moreover if $C \subset \bar{B}_\rho(P)$ for some point P and radius $\rho > 0$ then for any $r \geq 2\rho/\sqrt{3}$ there exists a point $T \in [HQ] \cap \partial B_r(P)$ such that $\bar{S} \setminus B_r(P) = [HT]$ (see Figure 2).
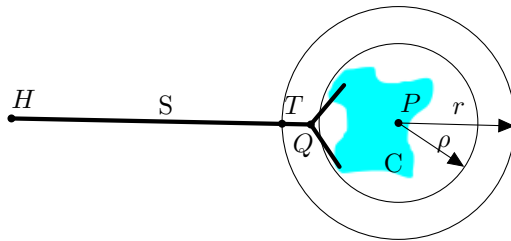
FIGURE 2. The situation in Lemma 2.5.

Proof. Consider the connected component S_1 of S such that $H \in \bar{S}_1$ (see Proposition 2.1(ii)). We have two cases depending on whether S_1 has branching points or not.

Case 1. Suppose $\bar{S}_1$ has no branching points. Then it is itself a line segment $[HQ]$, and thus, necessarily $Q \in C$ (again by Proposition 2.1(ii)). The second claim is then also obvious because every connected component of S different from S_1 touches only connected components of C but not H and hence is contained in the convex hull of C hence in $B_\rho(P) \subset B_r(P)$. Therefore $\bar{S} \setminus B_r(P) = \bar{S}_1 \setminus B_r(P) = [HT]$ where T is the intersection point of $\partial B_r(P)$ and $[HQ]$.

Case 2. If $\bar{S}_1$ has branching points then let Q be the branching point closest to H in the intrinsic (geodesic) distance on $\bar{S}_1$. Such a point exists. In fact otherwise there would exist a sequence of branching points in $\bar{S}_1$ converging to H (branching points can only accumulate on $\{H\} \cup C$ by Proposition 2.1(ii)). but to each of such branching points there would correspond at least one branch of $\bar{S}_1$ containing an endpoint on C (i.e. different from H) and hence having length at least $\text{dist}(H, C) > 0$. Since $\mathcal{H}^1(\bar{S}_1) = \mathcal{H}^1(S_1)$, by Lemma 2.2, we would have $\mathcal{H}^1(S) \geq \mathcal{H}^1(S_1) = +\infty$, contrary to the assumption. Clearly the arc from H to Q in $\bar{S}_1$ is a straight line segment (by Proposition 2.1(iii)).

To prove the second claim in this case, it is enough to show that $Q \in \bar{B}_r(P)$. If not, for $S' := S \setminus [HQ]$ one clearly has $S' \in \mathcal{M}(\{Q\} \cup C)$ and hence S' is contained in the convex hull of $\{Q\} \cup C$ hence in the convex hull of $\{Q\} \cup \bar{B}_\rho(P)$. Since by contradiction $|QP| > r = 2\rho/\sqrt{3}$ we notice that the latter convex hull is contained in an angle of less than 120 degrees with vertex in Q. This is in contradiction with the fact that Q was a triple point which defines equal angles of 120 degrees as claimed in Proposition 2.1(iii). $\square$

3. MAIN RESULT

Let $\lambda > 0$ be fixed, f_j $(j = 1, 2)$, A and Σ be defined as in the Introduction. Let $Y := \{(x, y) \in \mathbb{R}^2 : x = 0\}$ be the y-axis, and $Y_j := f_j(Y)$ for $j = 1, 2$. Consider the point $P := (1 + \frac{\lambda}{2}, 0)$. Clearly $\Sigma \in \text{St}(Y \cup A)$. We need the following lemmata.

Lemma 3.1. *For every $d \in \mathbb{R}$ the family $\mathcal{M}(\{x = d\} \cup A)$ is nonempty. Moreover if $\lambda < 2\sqrt{3} - 3 \approx 0.46$ and $S \in \mathcal{M}(\{x = d\} \cup A)$ one has that $\mathcal{H}^1(S) < \infty$, $\bar{S} \in \mathcal{M}(\{x = d\} \cup A)$ and $\bar{S}$ contains A.*

Proof. Recall that A is a self-similar set satisfying the recurrence relation $A = f_1(A) \cup f_2(A)$ with f_1 and f_2 being compositions of euclidean isometries with a rescaling of factor λ. The *fractal dimension* of A is hence D such that $1 = 2\lambda^D$ i.e.

$D = -1/\log_2 \lambda$. By [2, theorem 5.3(1)] the Hausdorff dimension of A equals D if it is possible to find an open set U such that

(1) $$U \supset f_1(U) \cup f_2(U) \qquad \text{and} \qquad f_1(U) \cap f_2(U) = \emptyset.$$

We can choose $U := B_R(T)$ with $T = (1,0)$ and $R := \lambda + \lambda^2 + \lambda^3 + \cdots = \frac{\lambda}{1-\lambda}$. Using the identity $f_j(B_R(T)) = B_{\lambda R}(f_j(T))$ one finds that $f_j(U) \subset U$ for any $\lambda < 1$ since $|f_j(T) - T| = \lambda$ and $\lambda + \lambda R = R$. For the second requirement in (1) we observe that $|f_1(T) - f_2(T)| = \lambda\sqrt{3} < 2\lambda R$ when $\lambda < 2\sqrt{3} - 3$. Hence under this hypothesis we have that the Hausdorff dimension of A equals D and since also $\lambda < \frac{1}{2}$ we find $D < 1$, so that $\mathcal{H}^1(A) = 0$ and in particular A is totally disconnected.

Now the conclusion follows from Lemma 2.3 with $C := A$ minding that

$$\mathcal{H}^1(S) \le \mathcal{H}^1(\Sigma) = \frac{1}{1 - 2\lambda} < +\infty.$$

$\square$

Lemma 3.2. *Let P and A be defined as above, $\lambda < 1/10$ and $R = \frac{2\lambda}{\sqrt{3}}$. Then $A \subset B_\lambda(P)$ and there exists $T \in \partial B_R(P)$ such that given any $d < \frac{1}{2}$ and any $S \in \mathcal{M}(\{x = d\} \cup A)$ we have that $S \setminus B_R(P)$ is a straight line segment $[HT]$ perpendicular to $\{x = d\}$ with $H \in \{x = d\}$. In particular $\bar{S}$ is connected.*

Proof. The claim $A \subset B_\lambda(P)$ can be easily checked by noticing that any point of A has distance from P not larger than

$$\frac{\sqrt{3}}{2}\lambda + \lambda^2 + \lambda^3 + \cdots = \frac{\sqrt{3}}{2}\lambda + \frac{\lambda^2}{1-\lambda}$$

and this is less than λ if $\lambda < 1/10$. Lemma 2.4 with $C := A$ assures that T exists with the desired properties and that $\bar{S} \cap \{x = d\} = \{H\}$. Since, by Lemma 3.1, $\bar{S} \supset A$ it follows that $\bar{S}$ is connected. $\square$

Lemma 3.3. *Let ν_1, ν_2, ν_3 be three unit vectors in $\mathbb{R}^2$ with $\nu_1 + \nu_2 + \nu_3 = 0$. Let Y_j be a line perpendicular to ν_j for $j = 1, 2, 3$. Let T be any point in $\mathbb{R}^2$ and let H_j be the orthogonal projection of T on Y_j. Define $d_j(T) := (T - H_j) \cdot \nu_j$ be the signed distance of T from Y_j. Then $d_1(T) + d_2(T) + d_3(T)$ is constant (i.e. independent of T).*

In particular suppose that Y_1, Y_2, Y_3 are three lines in $\mathbb{R}^2$ forming angles of 60 degrees so that the three pairwise intersections identify an equilateral triangle with sides of length ℓ. Let T be any point and let $d_i(T)$ be the signed distance of T from Y_i with positive sign when T is inside the triangle. Then $d_1(T) + d_2(T) + d_3(T) = \frac{\sqrt{3}}{2}\ell$.

Proof. If we move one of the lines parallel to itself by an amount δ then the sum $d_1(T) + d_2(T) + d_3(T)$ changes by δ, independent of T. Therefore, without loss of generality, we may assume that the three lines intersect in the origin. In this case $d_j(T) = T \cdot \nu_j$ and hence $d_1(T) + d_2(T) + d_3(T) = T \cdot (\nu_1 + \nu_2 + \nu_3) = 0$ concluding the proof of the first claim.

For the second claim just notice that $d_1(T) + d_2(T) + d_3(T)$ is constant (by the first claim) and hence is equal to the value obtained when T is one of vertices of the triangle, in which case two distances are 0 and the third one is equal to the height of the triangle, which is the claimed value. $\square$

Lemma 3.4. *One has*

$$\mathrm{dist}(Y, f_j(A)) = \mathrm{dist}(Y, A) \geq 1 + \frac{\lambda}{2} - \frac{\lambda^2(1+2\lambda)}{2(1-\lambda^2)}, \qquad j = 1, 2,$$

$$\mathrm{dist}(f_1(A), f_2(A)) \geq \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda}.$$

Proof. If we follow the tree Σ starting from the root at the origin and trying to get as close as possible to the line Y we first must follow the trunk of length 1 then at the first branching nothing changes if we turn left or right and then we go further along the oblique branch of length λ so that the distance from Y increases by $\lambda/2$. After that the path leading towards Y is made of alternating branches with horizontal and oblique directions. The computation gives

$$\mathrm{dist}(Y, f_j(A)) \geq 1 + \frac{\lambda}{2} - \left(\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{2} + \lambda^5 + \frac{\lambda^6}{4} + \dots\right)$$

$$= 1 + \frac{\lambda}{2} - \frac{1}{2}\frac{\lambda^2}{1-\lambda^2} - \frac{\lambda^3}{1-\lambda^2}$$

$$= 1 + \frac{\lambda}{2} - \frac{\lambda^2(1+2\lambda)}{2(1-\lambda^2)}.$$

Analogously,

$$\mathrm{dist}(f_1(A), f_2(A)) \geq 2\,\mathrm{dist}(f_1(A), \{y=0\})$$

$$\geq 2\left(\frac{\sqrt{3}}{2}\lambda - \frac{\sqrt{3}}{2}\left(\lambda^3 + \lambda^4 + \dots\right)\right) = \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda},$$

as claimed. $\square$

Remark 3.5. Notice that if the right hand side of the inequalities in the claim of Lemma 3.4 are positive then in fact those inequalities become equalities as it is easy to see from the proof. This happens for λ sufficiently small, for instance when $\lambda < \frac{1}{2}$.

Lemma 3.6. *Let $S \in \mathcal{M}(\{T\}\cup C)$ for some $T \in \mathbb{R}^2$ and a compact set C contained in a horizontal strip $E = \{(x,y)\colon |y| \leq \delta\}$ of width $2\delta > 0$. Suppose also that there is some $T' \in S$, $T' \neq T$ such that $[T,T']$ is horizontal. Then $T \in E$ and $S \subset E$.*

Proof. Suppose by contradiction that $T = (x_T, y_T)$ is outside E. For example $y_T > \delta$. Then the convex hull of $\{T\} \cup C$ has a single point, which is T, on the line $\{y = y_T\}$. This is in contradiction with the fact that $T' \in S$ has the same y-coordinate as T. This proves $T \in E$ and hence shows that the whole convex hull of $\{T\} \cup C$ is contained in E. Therefore, S must be contained in E as well. $\square$

Lemma 3.7. *Let $Y' := \{x = d\}$ for some $d < \frac{1}{2}$, be a line parallel to $Y = \{x = 0\}$. If $S \in \mathcal{M}(Y' \cup A)$ and $\lambda < \frac{1}{25}$ then there is a branching point $T \in S$ such that $S \setminus \{T\}$ is composed of three connected sets, the closures of which are $[HT]$, S_1 and S_2 respectively, with $H \in Y'$ and $[HT]$ perpendicular to Y'. Moreover,*

$$(2) \qquad\qquad \mathrm{dist}(H, \{y = 0\}) \leq \lambda^2 \qquad and \qquad T \in B_{\lambda^2}(T_0),$$

where $T_0 := (1, 0)$. Finally if Y'_j is the line parallel to $Y_j := f_j(Y)$ passing through T then $S_j \in \mathcal{M}(Y'_j \cup f_j(A))$ and $Y'_j \subset f_j(\{x < \frac{1}{2}\})$, for $j = 1, 2$.

FIGURE 3. Constructions and notation in the proof of Lemma 3.7 and Theorem 3.8. The triangle Δ from the proof of Lemma 3.7 is highlighted.

Proof. Step 1. We first claim that $\bar{S}$ cannot contain two compact connected sets σ and γ with $\mathcal{H}^1(\sigma \cap \gamma) = 0$ such that σ touches both Y' and $f_1(A)$ while γ touches both $f_1(A)$ and $f_2(A)$. In fact, otherwise we would have

$$\mathcal{H}^1(\bar{S}) \geq \mathcal{H}^1(\sigma) + \mathcal{H}^1(\gamma) \geq \text{dist}(Y', f_1(A)) + \text{dist}(f_1(A), f_2(A)),$$

but $\text{dist}(Y', f_1(A)) \geq \text{dist}(Y, f_1(A)) - d$, hence, by Lemma 3.4, we would get

$$(3) \qquad \mathcal{H}^1(\bar{S}) \geq 1 - d + \frac{\lambda}{2} - \frac{\lambda^2}{2}\frac{1}{1-\lambda^2} - \frac{\lambda^3}{2}\frac{1}{1-\lambda^2} + \sqrt{3}\lambda - \sqrt{3}\frac{\lambda^3}{1-\lambda}.$$

Let $O' = (d, 0)$ and $\Sigma' := [O', T_0] \cup f_1(\Sigma) \cup f_2(\Sigma)$ be the tree obtained by adding or removing a segment of length $|d|$ from Σ so that $\Sigma' \in \text{St}(Y' \cup A)$. We have

$$(4) \qquad \mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma) - d = 1 - d + 2\lambda + 4\lambda^2 + \cdots = \frac{1}{1-2\lambda} - d$$

and one can check that the rhs of (3) is strictly greater than the rhs of (4) for $\lambda < \frac{1}{25}$. Hence $\mathcal{H}^1(S) = \mathcal{H}^1(\bar{S}) > \mathcal{H}^1(\Sigma')$ contrary to the optimality of S.

Step 2. By Proposition 2.1(ii) we know that S touches Y' in a single point H. Recall that $\bar{S}$ is connected in view of Lemma 3.2. Consider an injective arc $\theta\colon [0,1] \to \bar{S}$ such that $\theta(0) = H$, $\theta([0,1)) \cap A = \emptyset$ and $\theta(1) \in A$. Let

$$t := \sup\left\{s \in [0,1]\colon \bar{S} \setminus \theta([0,s]) \text{ is connected}\right\}$$

and let $T := \theta(t)$, $S_0 := \theta([0,t])$. Observe that $(\bar{S} \setminus S_0) \cup \{T\}$ is compact and arcwise connected. In fact for every pair of points $P, Q \in (\bar{S} \setminus S_0) \cup \{T\}$ there exists a unique injective arc Γ in $\bar{S}$ connecting P and Q (this follows by Proposition 2.1(ii) since $\bar{S}$ is connected and having finite length is an arcwise connected topological tree). It is enough to note now that $\Gamma \cap \theta([0,s]) = \emptyset$ for all $s < t$ and hence $\Gamma \subset \bar{S} \setminus \theta([0,t)) = (S \setminus S_0) \cup \{T\}$.

Step 3. We are going to define S_1 and S_2 closed connected subsets of $\bar{S}$ such that $\bar{S} = S_0 \cup S_1 \cup S_2$ and S_j, $j = 0, 1, 2$, have T as the only common point, and

moreover, S_j contains $f_j(A)$ for $j = 1, 2$. To this aim we are going to consider two cases which will be excluded.

Case 1. Suppose T is a point of A. Without loss of generality suppose $T \in f_1(A)$. The set $(\bar{S} \setminus S_0) \cup \{T\}$ is connected and contains points of both $f_1(A)$ and $f_2(A)$ because $(S_0 \setminus \{T\}) \cap A = \emptyset$ by construction hence there exists an arc γ in $(\bar{S} \setminus S_0) \cup \{T\}$. The claim of Step 1 with $\sigma := S_0$ implies that this cannot happen.

Case 2. Suppose $T \notin A$. In this case S is a finite tree in a neighbourhood of T by Proposition 2.1(iii) and therefore T is a triple point of S. Hence $\bar{S} \setminus S_0$ has two connected components S_1' and S_2' (recall that $\bar{S}$ contains no loops). Let $S_j := S_j' \cup \{T\}$, $j = 1, 2$. Each S_j must contain at least one point of A because otherwise $\bar{S} \setminus S_j' \in \mathrm{St}(Y \cup A)$ will be strictly shorter than $S \in \mathcal{M}(Y \cup A)$. Moreover each point of A is contained in either S_1 or S_2. Without loss of generality suppose that S_1 contains at least one point of $f_1(A)$.

Case 2a. Suppose that S_1 contains also points of $f_2(A)$. Then we can apply the claim of Step 1 with $\sigma := S_0 \cup S_2$ and $\gamma := S_1$ and exclude that this case can happen.

Case 2b. If S_2 contains points of both $f_1(A)$ and $f_2(A)$ we proceed as in Case 2a with S_1 and S_2 interchanged.

The only remaining possibility is that S_1 only touches points of $f_1(A)$ and S_2 only touches points of $f_2(A)$. Hence $S_1 \supset f_1(A)$ and $S_2 \supset f_2(A)$ since $A \subset \bar{S}$. Therefore, $S_0 \in \mathcal{M}(Y' \cup \{T\})$ and $S_j \in \mathcal{M}(\{T\} \cup f_j(A))$ for $j = 1, 2$: otherwise, by substituting S_0 with an element of $\mathcal{M}(Y' \cup \{T\})$ and S_j with an element of $\mathcal{M}(\{T\} \cup f_j(A))$, we could construct a better competitor to $S \in \mathrm{St}(Y' \cup A)$.

Step 4. Clearly $S_0 \in \mathcal{M}(Y' \cup \{T\})$ implies that S_0 is the straight line segment $[HT]$ perpendicular to Y'.

Step 5. By Lemma 3.2 we have $A \subset B_\lambda(P)$, hence $f_j(A) \subset B_{\lambda^2}(f_j(P))$ for $j = 1, 2$. As noticed in Step 3 we know that S is a regular tripod in a small neighbourhood of T composed by three straight line segments forming equal angles of 120 degrees. Since S_0 is perpendicular to Y it follows that S_j contains a small segment perpendicular to $Y_j = f_j(Y)$. Since $S_j \in \mathcal{M}(\{T\} \cup f_j(A))$ for $j = 1, 2$, by Lemma 3.6 (applied to S_j in place of S, $f(A_j)$ in place of C and coordinate y along the axis Y_j and x perpendicular to Y_j), we obtain that S_j is contained in the strip perpendicular to Y_j, centered in T_0 and containing $f_j(A)$. Since $f_j(A) \subset B_{\lambda^2}(f_j(P))$ such a strip has width λ^2 (notice that $f_j(P)$ lies on the line passing through T_0 and perpendicular to Y_j).

The intersection between the two strips for $j = 1, 2$ is the union of two equilateral triangles each with height λ^2. Hence T is contained in the ball $B_{\lambda^2}(T_0)$, and thus, in particular,

$$\mathrm{dist}(H, \{y = 0\}) = \mathrm{dist}(T, \{y = 0\}) \leq |TT_0| \leq \lambda^2,$$

proving (2). Moreover also the distance of the line Y_j' from T_0 is less than λ^2 hence $Y_j' \subset f_j(\{x < \lambda\}) \subset f_j(\{x < \frac{1}{2}\})$, for $j = 1, 2$.

Step 6. We claim that each S_j, $j = 1, 2$, has no branching points inside the triangle delimited by Y, Y_1 and Y_2 (to avoid confusion, notice that T is not a branching point of S_j).

If T is itself outside of this triangle there is nothing to prove because $S_j \subset \overline{\mathrm{co}}(\{T\} \cup A_j)$ is also outside of the triangle. Otherwise, since $f_j(A) \subset B_{\lambda^2}(f_j(P))$,

then, by Lemma 3.2 all branching points of S_j are inside $B_{\frac{2}{\sqrt{3}}\lambda^2}(f_j(P))$ while $d(f_j(P), Y_j) = \lambda d(P, Y) = \lambda \left(1 + \frac{\lambda}{2}\right) > \frac{2}{\sqrt{3}}\lambda^2$.

Step 7. Let Y_j'' be the line parallel to Y_j and passing through P. Denote with Δ be the triangle delimited by Y, Y_1'' and Y_2'', and notice that T is inside Δ because $T \in B_{\lambda^2}(T_0)$ (as proven in Step 5) while $\operatorname{dist}(T_0, Y_j'') = \frac{\lambda}{4} > \lambda^2$ for $j = 1, 2$. Let T_j'' be the point on Y_j'' such that $[TT_j'']$ is perpendicular to Y_j. Recall that S_j in a neighbourhood of T is a segment perpendicular to Y_j because S, in a neighbourhood of T, is a regular tripod with equal angles of 120 degrees and S_0 is perpendicular to Y. We claim that $[TT_j'']$ is contained in S_j: otherwise there would be a branching point of S_j on such segment, which is not the case because $f_j(A) \subset B_{\lambda^2}(P)$ in view of Lemma 3.2, and hence by Lemma 2.5 with $f_j(A)$ in place of C, λ^2 in place of ρ and T in place of H, one has that all branching points of S_j are inside the ball $B_{\frac{2}{\sqrt{3}}\lambda^2}(f_j(P))$ while

$$\operatorname{dist}(f_j(A), Y_j'') \geq \operatorname{dist}(f_j(A), Y_j) - \operatorname{dist}(Y_j, Y_j'') \geq \lambda \operatorname{dist}(A, Y) - \frac{\lambda}{4}$$
$$\geq \lambda \left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{1 - \lambda}\right) - \frac{\lambda}{4}$$
$$\geq \frac{3}{4}\lambda > \frac{2}{\sqrt{3}}\lambda^2 \qquad \text{for } \lambda < \frac{1}{3}.$$

Hence $\tilde{S}_j := S_j \setminus [TT_j'']$ is connected and $\mathcal{H}^1(\tilde{S}_j) = \mathcal{H}^1(S_j) - \operatorname{dist}(T, Y_j'')$.

Step 8: conclusion. We now show the last claim $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$, $j = 1, 2$. To this aim take any $S_j'' \in \mathcal{M}(Y_j'' \cup f_j(A))$ and let H_j'' be the endpoint of S_j'' on the line Y_j'' (S_j'' touches Y_j'' in a single point in view of Lemma 2.5). Note that H_j'' is on the boundary of the triangle Δ in view of Lemma 3.6 because the orthogonal projection of $f_j(A)$ onto Y_j'' belongs to the respective side of this triangle. Define $S_j' := S_j'' \cup [H_j'' H_j']$ where H_j' is the projection of H_j'' onto the line Y_j', $j = 1, 2$ and let $\Gamma \in \mathcal{M}(Y \cup \{H_1'', H_2''\})$ be the regular tripod with endpoints in H_1'', H_2'' and a third point on the side of Δ contained in Y. Clearly $S_j' \in \mathcal{M}(Y_j' \cup f_j(A))$ and $\mathcal{H}^1(S_j') = \mathcal{H}^1(S_j'') + \left|H_j H_j''\right|$ while $\mathcal{H}^1(\Gamma) = 1 + \frac{\lambda}{2}$ in view of Lemma 3.3.

Therefore, if we consider $\tilde{S} := \Gamma \cup S_1'' \cup S_1''$ we have $\tilde{S} \in \operatorname{St}(Y \cup A)$ hence $\mathcal{H}^1(\tilde{S}) \geq \mathcal{H}^1(S)$. On the other hand, we have

$$\mathcal{H}^1(\tilde{S}) = \mathcal{H}^1(\Gamma) + \mathcal{H}^1(S_1'') + \mathcal{H}^1(S_2'')$$
$$\leq 1 + \frac{\lambda}{2} + \mathcal{H}^1(S_1') - |H_1' H_1''| + \mathcal{H}^1(S_2') - |H_2' H_2''|$$
$$\leq 1 + \frac{\lambda}{2} + \mathcal{H}^1(S_1) - |H_1' H_1''| + \mathcal{H}^1(S_2) - |H_2' H_2''|$$
$$= 1 + \frac{\lambda}{2} + \mathcal{H}^1(S_1) - |TT_1''| + \mathcal{H}^1(S_2) - |TT_2''|$$

but $\mathcal{H}^1(S_0) + |TT_1''| + |TT_2''| = 1 + \frac{\lambda}{2}$ by Lemma 3.3 and hence

$$\mathcal{H}^1(\tilde{S}) \leq \mathcal{H}^1(S_0) + \mathcal{H}^1(S_1) + \mathcal{H}^1(S_2) = \mathcal{H}^1(S).$$

Hence, all the above inequalities are, in fact, equalities, which means in particular that $\mathcal{H}^1(S_j') = \mathcal{H}^1(S_j)$ and thus $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$, $j = 1, 2$ as claimed. $\qquad\square$

Finally we are in the position to prove the main result.

Theorem 3.8. *Suppose* $\lambda < \frac{1}{25}$. *One has then* $\Sigma \in \mathcal{M}(\{0\} \cup A) = \mathcal{M}(Y \cup A)$. *Moreover* $S \in \mathcal{M}(Y \cup A)$ *if and only if* $\Sigma \setminus (\{0\} \cup A) \subset S \subset \Sigma$, *which in particular implies* $\bar{S} = \Sigma$.

Proof. By Lemma 3.1 we know that $\mathcal{M}(\{x = d\} \cup A)$ is nonempty for every $d \in \mathbb{R}$ and that for any $S \in \mathcal{M}(\{x = d\} \cup A)$ one has that $\bar{S} \in \mathcal{M}(\{x = d\} \cup A)$ and $\bar{S} \supset A$. Moreover, by Lemma 3.2, $\bar{S}$ is connected. Thus we may consider an arbitrary closed $S \in \mathcal{M}(\{x = d\} \cup A)$. If $d < \frac{1}{2}$, by Lemma 3.7, we are able to define S_1 and S_2, H, T such that $S = [HT] \cup S_1 \cup S_2$, $[HT]$ is perpendicular to Y. If we define the sets $b_1(S)$ and $b_2(S)$ (the "*branches*" of S) by $b_j(S) := f_j^{-1}(S_j)$, $j = 1, 2$. We notice that $b_j(S) \in \mathcal{M}(\{x = d_j\} \cup A)$ for some $d_j < 1$, as claimed in the Lemma 3.7, $j = 1, 2$. Let $d(S)$ and $\delta(S)$ be the two coordinates of the point H so that $d(S)$ is the distance of H from the line Y and $\delta(S)$ is the distance of the point H (or, which is the same, the point T) from the line $X := \{y = 0\}$. Define $\mathcal{S}^0 := \left\{\tilde{S}\right\}$ where $\tilde{S} \in \mathcal{M}(\{x = 0\} \cup A)$ is fixed and define inductively $\mathcal{S}^k$ as the family of the rescaled branches of $\tilde{S}$ at level k:

$$\mathcal{S}^{k+1} := \bigcup_{S \in \mathcal{S}^k} \{b_1(S), b_2(S)\}.$$

Notice that Lemma 3.7 can be applied to $\tilde{S}$ and hence inductively on the two rescaled branches $b_1(S)$ and $b_2(S)$ for every $S \in \mathcal{S}^k$ since the properties $S \in \mathcal{M}(\{x = d(S)\} \cup A)$ and $d(S) < 1$ are preserved by the operators b_1 and b_2 as stated in Lemma 3.7 itself.

We claim that for all $S \in \mathcal{S}^k$, one has

$$\delta(S) \leq 2\lambda \max\left\{\delta(b_1(S)), \delta(b_2(S))\right\}.$$

We know that $S \in \mathcal{S}^k$ touches the vertical line $\{x = d(S)\}$ in a single point H. By Lemma 3.7 we know that S is composed by a segment $[HT]$ and two trees S_1, S_2 with $S_j \in \mathcal{M}(Y_j' \cup f_j(A))$, $j = 1, 2$ where Y_j' is the line passing through T and parallel to Y_j. Let δ_j be the distance of T from $f_j(X)$, $j = 1, 2$. By Lemma 3.7 one has $\delta_j \leq \lambda^2$. By Lemma 3.3 applied to the three lines X, $f_1(X)$ and $f_2(X)$ (all lines passing through T_0) we know that the sum of the signed distances of T from the three lines passing through T_0 is equal to 0. The absolute values of these three distances are, respectively, $\delta(S)$, δ_1 and δ_2, so that

$$\delta(S) \leq \delta_1 + \delta_2 \leq 2\max\{\delta_1, \delta_2\}$$
$$= 2\max\{\lambda\delta(b_1(S)), \lambda\delta(b_2(S))\}$$

because $b_j(S) = f_j^{-1}(S_j)$, showing the claim.

The proven claim now gives

$$\Delta_k := \max\left\{\delta(S) \colon S \in \mathcal{S}^k\right\} \leq 2\lambda \max\left\{\delta(S) \colon S \in \mathcal{S}^{k+1}\right\} = 2\lambda\Delta_{k+1}.$$

We have $\Delta_0 = 0$ since otherwise for $\Delta_0 > 0$ we would have $\Delta_k \to +\infty$ (because $2\lambda < 1$), while we know, by Lemma 3.7, that $\Delta_k \leq \lambda^2$ for all k. But $\Delta_0 = 0$ implies $\Delta_k = 0$ for all k. In particular for $d = 0$ this implies $\tilde{S} = \Sigma$ (recall that $\tilde{S}$ has been choosen to be closed).

Note that we have proven that for any arbitrary $S \in \mathcal{M}(Y \cup A)$ one has $\bar{S} = \Sigma$ and in particular $S \in \mathcal{M}(\{0\} \cup A)$. But we also have that $\bar{S} \setminus S \subset \{0\} \cup A$ by Proposition 2.1(i), that is $\Sigma \setminus S \subset \{0\} \cup A$. This implies that $S \supset \Sigma \setminus (\{0\} \cup A)$.

It remains to prove that $\mathcal{M}(\{0\} \cup A) = \mathcal{M}(Y \cup A)$. To this aim note that we have already proven that $\mathcal{M}(Y \cup A) \subset \mathcal{M}(\{0\} \cup A)$ but for every $S \in \mathcal{M}(\{0\} \cup A)$ one necessarily has $S \in \mathrm{St}(Y \cup A)$ and $\mathcal{H}^1(S) = \mathcal{H}^1(\Sigma)$ while $\Sigma \in \mathcal{M}(Y \cup A)$ which implies the claim.

$\square$

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