Homogenization in perforated domains at the critical scale

Giuseppe Cosma Brusca
SISSA, Via Bonomea 265, Trieste, Italy
gbrusca@sissa.it

Abstract

We describe the asymptotic behaviour of the minimal heterogeneous d-capacity of a small set, which we assume to be a ball for simplicity, in a fixed bounded open set $\Omega \subseteq \mathbb{R}^d$, with $d \geq 2$. Two parameters are involved: ε, the radius of the ball, and δ, the length scale of the heterogeneity of the medium. We prove that this capacity behaves as $C|\log \varepsilon|^{1-d}$, where $C = C(\lambda)$ is an explicit constant depending on the parameter $\lambda := \lim_{\varepsilon\to 0} |\log \delta|/|\log \varepsilon|$.

We determine the Γ-limit of oscillating integral functionals subjected to Dirichlet boundary conditions on periodically perforated domains. Our first result is used to study the behaviour of the functionals near the perforations which, in this instance, are balls of radius ε. We prove that an additional *strange term* arises involving $C(\lambda)$.

Keywords: capacity, homogenization, Γ-convergence, perforated domains.
AMS Class: 49J45, 35B27, 31A15.

1 Introduction

A prototypical variational problem in Sobolev spaces involving scaling-invariant functionals concerns the d-capacity of a set E contained in a bounded open set $\Omega \subseteq \mathbb{R}^d$ with $d \geq 2$. If we assume E having diameter of size $\varepsilon \ll 1$, an explicit computation proves that the asymptotic behaviour of the capacity equals $|\log \varepsilon|^{1-d}$, up to a dimensional factor.

In this paper we introduce a dependence on x, which in the model describes the heterogeneity of a medium, and we analyse the asymptotic behaviour as $\varepsilon \to 0$ of the minimum

$$m_{\varepsilon,\delta} := \min\Big\{ \int_\Omega f\left(\frac{x}{\delta}, \nabla u(x)\right)\, dx : u \in W_0^{1,d}(\Omega), u = 1 \text{ on } B(z,\varepsilon), z \in \Omega \Big\}, \qquad (1)$$

where $\delta = \delta(\varepsilon)$ is positive and vanishing as $\varepsilon \to 0$, and $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is a Borel function with the following properties:

(P) (periodicity) $f(\cdot,\xi)$ is 1-periodic for every $\xi \in \mathbb{R}^d$, i.e., denoting by e_k an element of the canonical basis

$$f(x+e_k,\xi) = f(x,\xi) \text{ for every } x \text{ and } \xi \text{ in } \mathbb{R}^d, \text{ and } k = 1,...,d;$$

(H) (positive d-homogeneity)

$$f(x,t\xi) = t^d f(x,\xi) \text{ for every } x \text{ and } \xi \text{ in } \mathbb{R}^d \text{ and } t > 0;$$

(GC) (standard growth conditions of order d) there exist α, β such that

$$0 < \alpha < \beta \quad \text{and} \quad \alpha|\xi|^d \leq f(x,\xi) \leq \beta|\xi|^d \text{ for every } x \text{ and } \xi \text{ in } \mathbb{R}^d.$$

In light of the assumptions (P) and (H), the minimum defined in (1) stands for the minimal *heterogeneous capacity* of a small set (which is not restrictive to assume to be a ball) of size ε, while δ is the period of the heterogeneity modelled by oscillating terms. The assumption (GC) is technical since it allows to apply a classical homogenization result. By a relaxation argument, we may also assume f being convex in the second variable so that the associated functional is $W^{1,d}(\Omega)$-weakly lower semicontinuous and (1) actually is a minimum.

The first result we achieve is the asymptotic estimate of (1). To this end, we work along subsequences (not relabeled) for which it exists

$$\lambda := \lim_{\varepsilon\to 0} \frac{|\log\delta|}{|\log\varepsilon|} \wedge 1 \in [0,1]. \tag{2}$$

We introduce a function describing the asymptotic concentration of the heterogeneous capacity at a point $z \in \mathbb{R}^d$ given by

$$\begin{aligned}\Phi(z) := \lim_{R\to+\infty} (\log R)^{d-1} \min\Big\{\int_{B(0,R)\setminus B(0,1)} f(z,\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,R)),\\ u = 1 \text{ on } B(0,1)\Big\};\end{aligned} \tag{3}$$

then we define a constant portraying the effect of homogenization

$$\begin{aligned}C_{\rm hom} := \lim_{R\to+\infty} (\log R)^{d-1} \min\Big\{\int_{B(0,R)\setminus B(0,1)} f_{\rm hom}(\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,R)),\\ u = 1 \text{ on } B(0,1)\Big\},\end{aligned} \tag{4}$$

where $f_{\rm hom}$ is the positively d-homogeneous continuous function determined by the above mentioned homogenization result as

$$f_{\rm hom}(\xi) = \min\Big\{\int_{(0,1)^d} f(y,\xi+\nabla\varphi(y))\,dy : \varphi \in W_{loc}^{1,d}(\mathbb{R}^d), \varphi \text{ 1-periodic}\Big\}. \tag{5}$$

Note that the terms (3) and (4) are well defined as a consequence of [17, Proposition 5.1]. A simplified statement of this fact is presented in this work (Lemma 1.1).
The main result (Theorem 2.3) is the following. Assume there exists a point $x_0 \in \Omega$ such that the following hold:

(i) $f(x,\xi) \geq f(x_0,\xi)$ for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;

(ii) for every $\nu > 0$, there exists $r_\nu > 0$ such that

$$f(x,\xi) \leq f(x_0,\xi) + \nu|\xi|^d \text{ for every } x \in B(x_0,r_\nu), \xi \in \mathbb{R}^d.$$

Then

$$\lim_{\varepsilon\to 0} |\log\varepsilon|^{d-1} m_{\varepsilon,\delta} = \Phi(x_0) C_{\rm hom}\Big[\lambda\Phi(x_0)^{\frac{1}{d-1}} + (1-\lambda)C_{\rm hom}^{\frac{1}{d-1}}\Big]^{1-d} =: C(\lambda). \tag{6}$$

As an example, we refer to the quadratic case already treated in [5]. If $d = 2$ and $f(x,\xi) = a(x)|\xi|^2$, where $a(x)$ is a 1-periodic continuous function bounded from below by a constant α, we can pick x_0 so that $\Phi(x_0) = 2\pi\alpha$. Denoting the homogenized matrix by $A_{\rm hom}$, we obtain $C_{\rm hom} = 2\pi\sqrt{\det A_{\rm hom}}$ and we eventually find

$$\lim_{\varepsilon\to 0} |\log\varepsilon| m_{\varepsilon,\delta} = 2\pi\frac{\alpha\sqrt{\det A_{\rm hom}}}{\lambda\alpha + (1-\lambda)\sqrt{\det A_{\rm hom}}}.$$

Our argument relies on a method elaborated by De Giorgi which allows to impose boundary conditions on functions with finite energy. In this work, this tool is presented in a version (Lemma 2.2) which is suitable for our purposes and that is similar to the one proposed in [2].
The proof of Theorem 2.3 is obtained as an adaptation of the argument leading to the intermediate result Proposition 2.1. The latter concerns the asymptotic behaviour of

$$\mu_{\varepsilon,\delta} = \min\Big\{\int_\Omega f\left(\frac{x}{\delta}, \nabla u(x)\right)\,dx : u \in W_0^{1,d}(\Omega), u = 1 \text{ on } B(z_\varepsilon,\varepsilon)\Big\},$$

where the centres z_ε are of the form $\delta z + \delta i_\varepsilon$ with z being fixed and $i_\varepsilon \in \mathbb{Z}^d$. The outcome of the analysis is the same of Theorem 2.3 for the minimum (1), with the constant $C(\lambda)$ which is now given by

$$\Phi(z) C_{\rm hom}\Big[\lambda\Phi(z)^{\frac{1}{d-1}} + (1-\lambda)C_{\rm hom}^{\frac{1}{d-1}}\Big]^{1-d};$$

that is, the same constant of (6) with z in place of x_0 as a consequence of the periodicity of the centres of the inclusions.

The second result concerns homogenization on perforated domains. Denoting by B the open unit ball, and by $d(\varepsilon)$ a positive vanishing function that is the period of the perforations, we define a periodically perforated domain as

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i\in\mathbb{Z}^d} id(\varepsilon) + \varepsilon B$$

and we describe the asymptotic behaviour of the functionals $F_\varepsilon : L^d(\Omega) \to [0, +\infty]$ given by

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\delta(\varepsilon)}, \nabla u(x)\right) dx & \text{if } u \in W^{1,d}(\Omega) \text{ and } u = 0 \text{ on } \Omega \setminus \Omega_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

Dirichlet problems in varying domains have been originally studied from the point of view of the equations, e.g., by Marchenko and Khruslov in [16] and by Cioranescu and Murat in [9]. In these works, it is analysed the homogeneous case $f(x, \xi) = |\xi|^p$ for $p > 1$, and it is provided a critical choice of the period for which the limit is non trivial in the case $p = d$, which is $d(\varepsilon) = |\log \varepsilon|^{(1-d)/d}$. Moreover, recasting their result in terms of Γ-convergence with respect to the strong convergence in $L^d(\Omega)$, it is proved that

$$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \int_\Omega |\nabla u(x)|^d \, dx + \kappa_d \int_\Omega |u(x)|^d \, dx$$

for every $u \in W^{1,d}(\Omega)$, with κ_d a dimensional constant. This shows that internal boundary conditions disappear with the arising of a so-called *strange term* obtained by the analysis of the energy 'near the perforations'.
Afterwards, a compactness result has been achieved by Dal Maso and Murat in [12] for the family of solutions $(u_\varepsilon) \subseteq W_0^{1,p}(\Omega_\varepsilon, \mathbb{R}^m)$ of the problems

$$-\text{div}\, a(x, \nabla u_\varepsilon(x)) = h \quad \text{in } \mathcal{D}'(\Omega_\varepsilon, \mathbb{R}^m),$$

where $a : \Omega \times \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$ is a Carathéodory function satisfying a growth condition of order $p - 1$ that defines a monotone operator on $W_0^{1,p}(\Omega, \mathbb{R}^m)$, $h \in W^{-1,p'}(\Omega, \mathbb{R}^m)$ with p' the conjugate exponent of p and $\Omega_\varepsilon \subset \Omega$ is a general open subset.
They proved that, up to subsequences, $(u_\varepsilon)_\varepsilon$ converges to the solution of the problem

$$-\text{div}\, a(x, \nabla u(x)) + (|u|^{p-2}u)\mu = h \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)$$

being μ a nonnegative Borel measure not charging sets of null p-capacity in Ω.
The inhomogeneous variant was studied by Calvo-Jurado and Casado-Díaz in [8] who considered a more general equation

$$-\text{div}\, a_\varepsilon(x, \nabla u_\varepsilon(x)) + F_\varepsilon(x, u_\varepsilon)\mu_\varepsilon = h_\varepsilon \quad \text{in } \mathcal{D}'(\Omega_\varepsilon, \mathbb{R}^m)$$

being $a_\varepsilon, h_\varepsilon, \mu_\varepsilon$ as above and $F_\varepsilon : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ a suitable Carathéodory function. The same kind of result is achieved for a limit of the same form

$$-\text{div}\, a(x, \nabla u(x)) + F(x, u)\mu = h \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m).$$

An example of the occurrance of separation of scales was provided by Conca, Murat and Timofte in [10]. They studied a Signorini's type problem; that is, a free boundary-value problem consisting in determining a function u_ε and two subsets S_0^ε and S_+^ε (disjoint components of the boundary of the perforations S^ε), such that

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = h \text{ in } \mathcal{D}'(\Omega_\varepsilon), \\ u_\varepsilon = 0 \text{ on } S_0^\varepsilon,\ A_\varepsilon \nabla u_\varepsilon \cdot \nu \geq 0 \text{ on } S_0^\varepsilon, \\ u_\varepsilon > 0 \text{ on } S_+^\varepsilon,\ A_\varepsilon \nabla u_\varepsilon \cdot \nu = 0 \text{ on } S_+^\varepsilon, \end{cases}$$

where ν is the outer unit normal to S^ε and $A_\varepsilon(x) = A(x/\varepsilon)$ is a $d \times d$ matrix satisfying a uniform quadratic growth condition with continuous entries. They detected a different behaviour at the limit according to the size of the holes: if the radius of the perforation is infinitesimal compared with the critical value $\varepsilon^{d/(d-2)}, d > 2$, the problem converges to a homogenized Dirichlet problem on Ω; otherwise, if the holes are considerably large, the limit is seen to be an obstacle problem as a positivity condition is spread over the domain.

In the past decades, the literature on these problems has been enriched by the description of the variational counterpart. The nonlinear (vector-valued) homogeneous case has been studied by Ansini and Braides in [2], while the version at the critical exponent is due to Sigalotti [17]. In these papers, the Γ-limit of the functionals $G_\varepsilon : L^d(\Omega, \mathbb{R}^m) \to [0, +\infty]$ defined by

$$G_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f(\nabla u(x))\, dx & \text{if } u \in W^{1,d}(\Omega, \mathbb{R}^m) \text{ and } u = 0 \text{ on } \Omega \setminus \Omega_\varepsilon, \\ +\infty & \text{otherwise,} \end{cases}$$

for f a quasiconvex energy density with p-growth, is proved to be

$$\int_\Omega f(\nabla u(x))\, dx + \int_\Omega \varphi(u(x))\, dx,$$

where φ is obtained by a capacitary or a homogenization formula, respectively.

In [1], Ansini and Braides take into account a inhomogeneity considering functionals as in (7) in the subcritical case $d \geq 3$, f with quadratic growth and $f(x, \cdot)$ being 2-homogeneous, and prove that a separation of scales occurs depending on the rate of vanishing of the parameters $\varepsilon, \delta(\varepsilon)$ and $d(\varepsilon)$.
If $\delta(\varepsilon) \ll \varepsilon$ or $\delta(\varepsilon) \gg d(\varepsilon)$, the Γ-limit is given by

$$\int_\Omega f_{\mathrm{hom}}(\nabla u(x))\, dx + C \int_\Omega |u(x)|^2\, dx$$

being f_{hom} as in (5) and C a constant depending on the considered regime; while in the intermediate cases the Γ-limit may exist only along proper subsequences and the strange term may have the more general form $\displaystyle\int_\Omega \varphi |u|^2 dx$.

We prove an analogous statement (Theorem 3.1) at the critical scale $p = d$ for $d(\varepsilon) \gg \delta(\varepsilon)$: for simplicity, we assume that $d(\varepsilon)$ is an integer multiple of $\delta(\varepsilon)$ so that the periodicity of the perforation is 'compatible' with that of the energy. More specifically, we suppose that

$$\text{for every } \varepsilon > 0 \text{ there exists a natural number } m(\varepsilon) \text{ such that } d(\varepsilon) = m(\varepsilon)\delta(\varepsilon)$$

and that

$$\frac{\delta(\varepsilon)}{d(\varepsilon)} \to 0 \text{ as } \varepsilon \to 0;$$

we also assume that for every $\nu > 0$, there exists $r_\nu > 0$ such that

$$|f(0,\xi) - f(x,\xi)| \leq \nu|\xi|^d \text{ for every } x \in B(0,r_\nu), \xi \in \mathbb{R}^d.$$

Then we prove that

$$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx + C(\lambda)\int_\Omega |u(x)|^d\,dx$$

for every $u \in W^{1,d}(\Omega)$, where $C(\lambda)$ is given by

$$\Phi(0)C_{\text{hom}}\Big[\lambda\Phi(0)^{\frac{1}{d-1}} + (1-\lambda)C_{\text{hom}}^{\frac{1}{d-1}}\Big]^{1-d}$$

and λ is defined as in (2).
To study the contribution to the energy due to the regions 'near the perforations', we perform the asymptotic analysis of the problems

$$\min\Big\{\int_B f\left(\frac{x}{\delta(\varepsilon)/d(\varepsilon)}, \nabla u(x)\right)\,dx : u \in W_0^{1,d}(B), u = 1 \text{ on } B\Big(0, \frac{\varepsilon}{d(\varepsilon)}\Big)\Big\}.$$

Such asymptotic analysis is made possible by the assumptions $d(\varepsilon)/\delta(\varepsilon) \in \mathbb{N}$ and $\delta(\varepsilon)/d(\varepsilon) \to 0$: in particular, the former exploits the periodicity of $f(\cdot,\xi)$ and it combines with the latter in order to apply Proposition 2.1 with the centres of the perforations fixed at 0 for every ε.
Note that, if the hypothesis $d(\varepsilon)/\delta(\varepsilon) \in \mathbb{N}$ is removed, the Γ-limit may exist only upon possibly passing to a subsequence, and, in some special cases, a more involved description is provided.
As an example, we consider the case studied in [1, Theorem 5.1]; see Remark 3.4 for a more detailed discussion. Assume that $d(\varepsilon) = (m(\varepsilon)/T)\delta(\varepsilon)$ with $m(\varepsilon) \in \mathbb{N}$ prime and $T \in \mathbb{N}$, then it can be proved that

$$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx + C\int_\Omega |u(x)|^d\,dx,$$

where

$$C = \frac{1}{T^d} \sum_{h \in \{0,\dots,T-1\}^d} C^\lambda\Big(\frac{h}{T}\Big)$$

and $C^\lambda(z)$ is given by

$$\lim_{\varepsilon\to 0} |\log\varepsilon|^{d-1} \min\Big\{\int_B f\left(z+\frac{x}{\delta(\varepsilon)}, \nabla u(x)\right) dx : u \in W_0^{1,d}(B), u = 1 \text{ on } B(0,\varepsilon)\Big\}.$$

Note that this function is well defined by Proposition 2.1 applied to the energy density $g(x,\xi) := f(x+z,\xi)$.
We highlight the consistence with the case $d(\varepsilon)/\delta(\varepsilon) \in \mathbb{N}$ as we get $T = 1$ so that $C = C^\lambda(0) = C(\lambda)$.

To sketch a complete picture we also comment the cases $\delta(\varepsilon) \ll \varepsilon$ and $d(\varepsilon)/\delta(\varepsilon) \to q \in [0,\infty)$ without further assumptions on the form of $d(\varepsilon)/\delta(\varepsilon)$. For these we essentially recover the critical version of the above mentioned results presented in [1, Sections 4,5] to which we refer for more details:

- $\delta(\varepsilon) \ll \varepsilon$. This instance forces $\lambda = 1$; our result is obtained with $C(\lambda) = C_{\text{hom}}$ even if the periodicity (P) may not be immediately exploited. To give a glimpse on how this is achieved, we observe that for every perforation $id(\varepsilon)$ there exists a unique point $y_i(\varepsilon)$ on the lattice $\delta(\varepsilon)\mathbb{Z}^d$ such that $id(\varepsilon) \in y_i(\varepsilon) + [0,\delta(\varepsilon))^d$. Since $\delta(\varepsilon) \ll \varepsilon$, the energy on the ball $B(id(\varepsilon), \varepsilon N)$ for fixed $N > 1$, is asymptotically equivalent to the one on the larger ball $B(y_i(\varepsilon), \varepsilon(N + 1/N))$. Hence, we can reason as if our perforations are centred on the lattice $\delta(\varepsilon)\mathbb{Z}^d$, so that the same argument we present in Section 3 works with some minor adaptations.

- $d(\varepsilon) \ll \delta(\varepsilon)$. Several perforations are included in the same cell of periodicity; thus, an averaging effect on the function Φ defined in (3) occurs. Specifically, if we further assume that for every $x \in \Omega$ and for every $\nu > 0$, there exists $r_\nu > 0$ such that

  $$|f(x,\xi) - f(y,\xi)| \le \nu|\xi|^d \text{ for every } y \in B(x,r_\nu), \xi \in \mathbb{R}^d,$$

  then Φ is continuous and we get

  $$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx + \Big(\int_{(0,1)^d} \Phi(y)\,dy\Big) \int_\Omega |u(x)|^d\,dx$$

  for every $u \in W^{1,d}(\Omega)$.

- $\frac{d(\varepsilon)}{\delta(\varepsilon)} \to q \in (0,\infty)$. In this case $\lambda = 0$ and the Γ-limit does not exist in general. Assuming Φ continuous and again

  $$d(\varepsilon) = \frac{m(\varepsilon)}{T}\delta(\varepsilon)$$

being $m(\varepsilon) \in \mathbb{N}$ prime and $T \in \mathbb{N}$, we recover the above example, and since $C^0(x) = \Phi(x)$ we obtain

$$\Gamma\text{-}\lim_{\varepsilon} F_\varepsilon(u) = \int_\Omega f_{\text{hom}}(\nabla u(x))\,dx + C\int_\Omega |u(x)|^d\,dx$$

with

$$C = \frac{1}{T^d} \sum_{h\in\{0,\dots,T-1\}^d} \Phi\Big(\frac{h}{T}\Big).$$

More recent works on the asymptotic behaviour of Dirichlet problems in varying domains are, e.g., [3, 7], or also [15] for the numerical perspective.

1.1 Preliminaries

In this section and the following ones, let $d \geq 2$, $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and $\lambda := \lim_{\varepsilon\to 0} |\log\delta|/|\log\varepsilon|$.

We start by justifying the definitions given in (3) and (4) through the following lemma which takes advantage of a scaling invariance argument; see [17, Proposition 5.1] for a more general statement.

Lemma 1.1. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Borel function which is positively homogeneous of degree d and assume there exist positive constants $C_1 < C_2$ such that $C_1|\xi|^d \leq g(\xi) \leq C_2|\xi|^d$ for every $\xi \in \mathbb{R}^d$. Define*

$$m_R := \min\Big\{\int_{B(0,R)\setminus B(0,1)} g(\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,R)), u = 1 \text{ on } B(0,1)\Big\},$$

then it exists $\lim_{R\to+\infty} (\log R)^{d-1}\, m_R$ and this limit is finite.

Proof. Fix $S > R$ and put $T := \lfloor \log S/\log R\rfloor$ so that the annuli $B(0,R^k)\setminus B(0,R^{k-1})$ are contained in $B(0,S)\setminus B(0,1)$ for every $k = 1,\dots,T$.
Let u be a solution of the problem

$$\min\Big\{\int_{B(0,R)\setminus B(0,1)} g(\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,R)), u = 1 \text{ on } B(0,1)\Big\},$$

for $k = 1,\dots,T$ define functions $u^k \in W^{1,d}(B(0,R^k)\setminus \overline{B}(0,R^{k-1}))$ as

$$u^k(x) := \frac{1}{T} u\left(\frac{x}{R^{k-1}}\right) + \frac{T-k}{T}\,,$$

then put $u_S \in W_0^{1,d}(B(0,S))$ as

$$u_S(x) := \begin{cases} 1 & \text{if } x \in B(0,1) \\ u^k(x) & \text{if } x \in B(0,R^k)\setminus B(0,R^{k-1}),\ k = 1,\dots,T \\ 0 & \text{if } x \in B(0,S)\setminus B(0,R^T). \end{cases}$$

We have

$$\begin{aligned}
(\log S)^{d-1} m_S &\leq (\log S)^{d-1} \int_{B(0,S)\setminus B(0,1)} g(\nabla u_S(x))\,dx \\
&= (\log S)^{d-1} \sum_{k=1}^{T} \int_{B(0,R^k)\setminus B(0,R^{k-1})} g(\nabla u^k(x))\,dx \\
&= (\log S)^{d-1} \sum_{k=1}^{T} \frac{1}{T^d} \int_{B(0,R)\setminus B(0,1)} g(\nabla u(x))\,dx \\
&= (\log S)^{d-1} \frac{1}{T^{d-1}} m_R \\
&\leq (\log S)^{d-1} \Big(\frac{\log R}{\log S - \log R} \Big)^{d-1} m_R,
\end{aligned}$$

hence, if we pass to the lim sup as $S \to +\infty$, and then we pass to the lim inf as $R \to +\infty$, we obtain

$$\limsup_{S\to+\infty} (\log S)^{d-1} m_S \leq \liminf_{R\to+\infty} (\log R)^{d-1} m_R.$$

In order to check that the limit is finite, consider the function

$$u(x) := 1 - \frac{\log|x|}{\log R}, \qquad x \in B(0,R)\setminus \overline{B}(0,1),$$

and note that the estimate

$$\begin{aligned}
(\log R)^{d-1} m_R &\leq \int_{B(0,R)\setminus B(0,1)} g(\nabla u(x)) = (\log R)^{d-1} \int_{B(0,R)\setminus B(0,1)} g\Big(\frac{x}{-|x|^2 \log R}\Big)\,dx \\
= (\log R)^{-1} \int_{B(0,R)\setminus B(0,1)} g\Big(-\frac{x}{|x|^2}\Big)\,dx &\leq (\log R)^{-1} C_2 \int_{B(0,R)\setminus B(0,1)} \frac{1}{|x|^d}\,dx = C_2 \sigma_{d-1}
\end{aligned}$$

holds, completing the proof. □

We state a slightly modified version of a classical homogenization result (see [4, 6, 11, 14]).

Theorem 1.2. *Let A be a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary and $(\tau_\eta)_{\eta>0} \subseteq \mathbb{R}^d$. Then*

$$\Gamma\text{-}\lim_{\eta\to 0} \int_A f\left(\frac{x}{\eta} + \tau_\eta, \nabla u(x)\right)\,dx = \int_A f_{\mathrm{hom}}(\nabla u(x))\,dx\,,$$

for every $u \in W^{1,d}(A)$, where the Γ-limit is computed with respect to the strong convergence in $L^d(A)$ and f_{hom} is the function given by (5).

In particular, for every $\phi \in W^{1,d}(A)$ we have

$$\lim_{\eta\to 0} \inf\Big\{\int_A f\left(\frac{x}{\eta}+\tau_\eta, \nabla u(x)\right)\,dx : u \in \phi + W_0^{1,d}(A)\Big\}$$
$$= \min\Big\{\int_A f_{\mathrm{hom}}(\nabla u(x))\,dx : u \in \phi + W_0^{1,d}(A)\Big\}.$$

Translations $(\tau_\eta)_\eta$ can be taken into account since the function f is periodic in the first variable. Indeed, this implies that we may assume $(\tau_\eta)_\eta$ to be bounded so that $\eta\tau_\eta \to 0$ and we recover the known case $\tau_\eta = 0$.

At this point, assumptions (H), (GC) and Lemma 1.1 make well defined the function

$$\Phi(z) := \lim_{R\to+\infty} (\log R)^{d-1} \min\Big\{\int_{B(0,R)\setminus B(0,1)} f(z, \nabla u(y))\,dy : u \in W_0^{1,d}(B(0,R)),$$
$$u = 1 \text{ on } B(0,1)\Big\};$$

the constant

$$C_{\mathrm{hom}} := \lim_{R\to+\infty} (\log R)^{d-1} \min\Big\{\int_{B(0,R)\setminus B(0,1)} f_{\mathrm{hom}}(\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,R)),$$
$$u = 1 \text{ on } B(0,1)\Big\},$$

is also well defined by Theorem 1.2: this, combined with the fact that (H) and (GC) are inherited by the function f_{hom}, ensures that the above lemma applies.

2 Asymptotic analysis of minima

We aim at estimating the asymptotic behaviour of the minima with fixed centres modulo a translation. More precisely, we fix z in Ω and for every $\varepsilon > 0$ sufficiently small, we consider $(z_\varepsilon)_\varepsilon$ a family of points in Ω of the form $z_\varepsilon = \delta z + \delta i_\varepsilon$ with $(i_\varepsilon)_\varepsilon \subseteq \mathbb{Z}^d$ such that $\inf_\varepsilon \operatorname{dist}(z_\varepsilon, \partial\Omega) > 0$. We put

$$\mu_{\varepsilon,\delta} = \min\Big\{\int_\Omega f\left(\frac{x}{\delta}, \nabla u(x)\right)\,dx : u \in W_0^{1,d}(\Omega), u = 1 \text{ on } B(z_\varepsilon, \varepsilon)\Big\}. \tag{8}$$

and we prove what follows.

Proposition 2.1. *Let $z \in \Omega$ be a fixed point, and let $(z_\varepsilon)_\varepsilon$ be a family of points equal to z modulo δ as above. Assume that for every $\nu > 0$, there exists $r_\nu > 0$ such that*

$$|f(z,\xi) - f(x,\xi)| \le \nu|\xi|^d \text{ for every } x \in B(z, r_\nu),\ \xi \in \mathbb{R}^d. \tag{9}$$

Then

$$\lim_{\varepsilon\to 0} |\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} = \Phi(z) C_{\mathrm{hom}}\Big[\lambda\Phi(z)^{\frac{1}{d-1}} + (1-\lambda)C_{\mathrm{hom}}^{\frac{1}{d-1}}\Big]^{1-d}.$$

The proof is divided in two parts, the bound from below and the construction of an optimal sequence. In the first one, the main tool we use is the following lemma which allows to modify a function in order to attain constant values (in the sense of the trace) on the boundary of a thin annulus keeping control on the value of the associated energy.

Lemma 2.2. *Let $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Borel function satisfying* (GC). *Let $z \in \mathbb{R}^d$, $R > 0$ and define*

$$F(u, A) := \int_A g(x, \nabla u(x))\, dx$$

with $u \in W^{1,d}(B(z, R))$ and $A \subseteq B(z, R)$ a Borel subset.
Let $\eta > 0$, $S := \max\{s \in \mathbb{N} : \eta 2^s \leq R\}$ and assume $S \geq 3$. Take N natural number such that $2 \leq N < S$ and r positive real number such that $r \leq \eta 2^{S-N}$.
Then, for every $u \in W^{1,d}(B(z, R))$ there exists a function v with the following properties:
(i) *$v \in W^{1,d}(B(z, R) \setminus \overline{B}(z, r))$;*
(ii) *there exists $j \in \{1, ..., N - 1\}$ such that*

$$v = u \ \text{ on } B(z, \eta 2^{S-j-1}) \setminus \overline{B}(z, r)) \cup B(z, R) \setminus \overline{B}(z, \eta 2^{S-j+1});$$

(iii) *for the same j, the function v is constant on $\partial B(z, \eta 2^{S-j})$. In particular*

$$v = \frac{1}{|A_j|} \int_{A_j} u\, dx \qquad \text{on } \partial B(z, \eta 2^{S-j}),$$

where $A_j := B(z, \eta 2^{S-j+1}) \setminus B(z, \eta 2^{S-j-1})$. Moreover $\|v\|_\infty \leq \|u\|_\infty$;
(iv) *there exists a positive constant C depending on α, β and the dimension d such that*

$$F(v, B(z, R) \setminus B(z, r)) \leq \Big(1 + \frac{C}{N - 1}\Big) F(u, B(z, R) \setminus B(z, r)).$$

Proof. Assume $z = 0$, if not, center the construction around z and repeat the argument.
For $k = 1, ..., N - 1$, we define annuli $A_k := B(0, \eta 2^{S-N+k+1}) \setminus B(0, \eta 2^{S-N+k-1})$ and radial cutoff functions

$$\phi_k(\rho) := \begin{cases} 0 & \text{if } \rho \in [0, \eta 2^{S-N+k-1}] \\ \frac{\rho - \eta 2^{S-N+k-1}}{\eta 2^{S-N+k-1}} & \text{if } \rho \in (\eta 2^{S-N+k-1}, \eta 2^{S-N+k}] \\ \frac{\eta 2^{S-N+k+1} - \rho}{\eta 2^{S-N+k}} & \text{if } \rho \in (\eta 2^{S-N+k}, \eta 2^{S-N+k+1}] \\ 0 & \text{if } \rho \in (\eta 2^{S-N+k+1}, R], \end{cases}$$

then we put $\psi_k := 1 - \phi_k$ and define $v_k := \psi_k u + (1 - \psi_k) u_{A_k}$, where we denote by u_{A_k} the integral average of u on A_k. Note that, for every k, the functions v_k satisfies the properties (i), (ii), (iii) with $j = N - k$.

At each fixed k, taking into account that $|\psi_k| \leq 1$ and that

$$|\nabla\psi_k|^d = |\nabla\phi_k|^d \leq \Big(\frac{1}{\eta 2^{S-N+k-1}}\Big)^d,$$

we exploit (GC) to have

$$\begin{aligned}\int_{A_k} g(x, \nabla v_k(x))\,dx &\leq \beta \int_{A_k} |\nabla v_k(x)|^d\,dx \\ &= \beta \int_{A_k} |\psi_k \nabla u(x) + (u - u_{A_k})\nabla\psi_k(x)|^d\,dx \\ &\leq \beta 2^{d-1}\Big[\int_{A_k} |\nabla u|^d\,dx + \Big(\frac{1}{\eta 2^{S-N+k-1}}\Big)^d \int_{A_k} |u(x) - u_{A_k}|^d\,dx\Big].\end{aligned} \tag{10}$$

Consider now the following well known scaling property of the Poincaré-Wirtinger inequality: given A open, bounded, connected, with Lipschitz boundary and $\lambda > 0$, it holds

$$\frac{1}{\lambda^d}\int_{\lambda A} |u - u_{\lambda A}|^d\,dx \leq P(A)\int_{\lambda A} |\nabla u|^d\,dx,$$

where $u_{\lambda A}$ is the integral average of u on λA and $P(A)$ is the Poincaré-Wirtinger constant related to A.
We apply this result with $A = B(0,4) \setminus \overline{B}(0,1)$ and $\lambda = \eta 2^{S-N+k-1}$, obtaining

$$\Big(\frac{1}{\eta 2^{S-N+k-1}}\Big)^d \int_{A_k} |u(x) - u_{A_k}|^d\,dx \leq P^d \int_{A_k} |\nabla u|^d\,dx,$$

being $P := P(A)$ a constant which does not depend on k.
As a consequence (10) turns into

$$\begin{aligned}\int_{A_k} g(x, \nabla v_k(x))\,dx &\leq \beta 2^{d-1}\big(1 + P^d\big)\int_{A_k} |\nabla u|^d\,dx \\ &\leq \frac{\beta}{\alpha} 2^{d-1}\big(1 + P^d\big)\int_{A_k} g(x, \nabla u(x))\,dx,\end{aligned}$$

and summing over k, we deduce

$$\sum_{k=1}^{N-1}\int_{A_k} g(x, \nabla v_k(x))\,dx \leq C\int_{B(0,R)\setminus B(0,r)} g(x, \nabla u(x))\,dx,$$

where we put $C := \beta 2^{d-1}\big(1 + P^d\big)/\alpha$. It follows that there exists $l \in \{1, ..., N-1\}$ such that

$$\int_{A_l} g(x, \nabla v_l(x))\,dx \leq \frac{C}{N-1}\int_{B(0,R)\setminus B(0,r)} g(x, \nabla u(x))\,dx,$$

and then it holds

$$\int_{B(0,R)\setminus B(0,r)} g(x,\nabla v_l(x))\,dx = \int_{(B(0,R)\setminus B(0,r))\setminus A_l} g(x,\nabla u(x))\,dx + \int_{A_l} g(x,\nabla v_l(x))\,dx$$
$$\leq \left(1+\frac{C}{N-1}\right)\int_{B(0,R)\setminus B(0,r)} g(x,\nabla u(x))\,dx,$$

which concludes the proof picking $v = v_l$ and $j = N - l$. □

In the proof of Proposition 2.1, and in particular in the estimate from below, we combine the application of this lemma with a preliminary construction: first we subdivide Ω in homothetic annuli having small inner and outer radii, each of order ε^η; then we modify $u \in W_0^{1,d}(\Omega)$ on each annulus using the lemma to achieve constant Dirichlet boundary conditions by (iii). This way, the lower bound is expressed in terms of a sum of minimum problems that we further estimate with some care in dealing with possibly different exponential scales described by η.
The error introduced by the modifications will be negligible since the estimate in (iv) gets more precise as N tends to ∞, i.e., as $\varepsilon \to 0$.

2.1 Lower bound

In what follows, we systematically identify a function $u \in W_0^{1,d}(\Omega)$ with the the extension obtained by setting $u = 0$ on $\mathbb{R}^d \setminus \Omega$, which belongs to $W^{1,d}(\mathbb{R}^d)$.
For simplicity of notation, given A Borel subset of $\mathbb{R}^d$ and $u \in W^{1,d}(\mathbb{R}^d)$, we put

$$F_\varepsilon(u,A) := \int_A f\left(\frac{x}{\delta},\nabla u(x)\right)\,dx$$

and denote by R_Ω the maximum among the diameter of Ω and 1, just to ensure that $\log R_\Omega$ is non negative.

We separately consider the cases $\lambda = 0$, $\lambda \in (0,1)$ and $\lambda = 1$; we obtain for each instance the same kind of estimate and then we conclude by the same argument.

Estimate from below for $\boldsymbol{\lambda = 0}$. If $\lambda = 0$, fix $\lambda_2 \in (\lambda, 1)$ so that

$$\frac{\varepsilon^{\lambda_2}}{\delta} \to 0 \text{ as } \varepsilon \to 0.$$

For every $u \in W_0^{1,d}(\Omega)$ such that $u = 1$ on $B(z_\varepsilon,\varepsilon)$, the inclusion $\Omega \subseteq B(z_\varepsilon, R_\Omega)$ leads to the equality

$$F_\varepsilon(u,\Omega) = F_\varepsilon(u, B(z_\varepsilon, R_\Omega)),$$

then we apply Lemma 2.2 to the function $u \in W_0^{1,d}(B(z_\varepsilon, R_\Omega))$, with

$$f(x,\xi) = f\left(\frac{x}{\delta},\xi\right),\ \eta = \varepsilon,\ R = \varepsilon^{\lambda_2},\ N \in \mathbb{N} \cap \left(1, \left\lfloor \frac{(1-\lambda_2)|\log\varepsilon|}{\log 2}\right\rfloor = S\right) \text{ and } r = \varepsilon.$$

We get a function $v \in W_0^{1,d}(B(z_\varepsilon, R_\Omega))$ such that $v = 1$ on $B(z_\varepsilon, \varepsilon)$, $v = c$ on $\partial B(z_\varepsilon, \varepsilon 2^{S-j})$ for some constant c and some index $j \in \{1, ..., N-1\}$, $v = u$ on $B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon^{\lambda_2})$.
By the estimate provided by (iv) in Lemma 2.2, it holds

$$\begin{aligned}\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, \Omega) &= \left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, B(z_\varepsilon, R_\Omega)) \\ &\geq F_\varepsilon(v, B(z_\varepsilon, R_\Omega)) \\ &= F_\varepsilon(v, B(z_\varepsilon, \varepsilon 2^{S-j})) + F_\varepsilon(v, B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon 2^{S-j})).\end{aligned}$$

Now we set

$$w^1 := \begin{cases} v & \text{on } B(z_\varepsilon, \varepsilon 2^{S-j}) \\ c & \text{on } B(z_\varepsilon, \varepsilon^{\lambda_2}) \setminus B(z_\varepsilon, \varepsilon 2^{S-j}) \end{cases} \qquad w^2 := \begin{cases} c & \text{on } B(z_\varepsilon, \varepsilon 2^{S-j}) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N}) \\ v & \text{on } B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon 2^{S-j}), \end{cases}$$

and we note that

$$F_\varepsilon(w^1, B(z_\varepsilon, \varepsilon^{\lambda_2})) = F_\varepsilon(v, B(z_\varepsilon, \varepsilon 2^{S-j}))$$

and

$$F_\varepsilon(w^2, B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon 2^{S-N})) = F_\varepsilon(v, B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon 2^{S-j})).$$

Thus

$$\begin{aligned}\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, \Omega) &\geq F_\varepsilon(v, B(z_\varepsilon, R_\Omega)) \\ &= F_\varepsilon(w^1, B(z_\varepsilon, \varepsilon^{\lambda_2})) \\ &\quad + F_\varepsilon(w^2, B(z_\varepsilon, R_\Omega) \setminus B(z_\varepsilon, \varepsilon 2^{S-N}))\end{aligned}$$

$$\begin{aligned}&\geq \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_2})) : \zeta \in W^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon), \zeta = c \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_2})\} \\ &\quad + \min\{F_\varepsilon(\zeta, B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})) : \zeta \in W^{1,d}(B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})), \\ &\qquad \zeta = c \text{ on } \partial B(z_\varepsilon, \varepsilon 2^{S-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, R_\Omega)\},\end{aligned}$$

where in the last inequality we took advantage of the Dirichlet boundary conditions satisfied by w^1 and w^2. Taking into account the transformations

$$\zeta(x) \mapsto \frac{\zeta(x) - c}{1 - c}, \qquad \zeta(x) \mapsto \frac{\zeta(x)}{c},$$

and the property (H), we have that the last expression equals

$$\min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon)\}|1 - c|^d \tag{11}$$

$$\begin{aligned}&+ \min\{F_\varepsilon(\zeta, B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})) : \zeta \in W^{1,d}(B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})), \\ &\qquad \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon 2^{S-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, R_\Omega)\}|c|^d.\end{aligned} \tag{12}$$

We separately treat the minima (11) and (12).
As $z_\varepsilon = \delta z + \delta i_\varepsilon$, by (P) we get

$$f\left(\frac{x}{\delta}\varepsilon + \frac{z_\varepsilon}{\delta}, \xi\right) = f\left(\frac{x}{\delta}\varepsilon + z, \xi\right) \text{ for every } \xi \in \mathbb{R}^d.$$

Note that if $x \in B(0, \varepsilon^{\lambda_2 - 1})$, then $\frac{\varepsilon}{\delta}|x| < \frac{\varepsilon^{\lambda_2}}{\delta} \to 0$ as $\varepsilon \to 0$. Hence, for every $\nu > 0$, given r_ν as in (9), assuming ε sufficiently small it holds that $\frac{\varepsilon}{\delta} B(0, \varepsilon^{\lambda_2 - 1}) \subseteq B(0, r_\nu)$, so that we have

$$f\Big(\frac{x}{\delta}\varepsilon + z, \xi\Big) \geq f(z, \xi) - \nu|\xi|^d \text{ for every } \xi \in \mathbb{R}^d.$$

Combining these observations with (GC) we get

$$\int_{B(0,\varepsilon^{\lambda_2-1})} f\Big(\frac{x}{\delta}\varepsilon + \frac{z_\varepsilon}{\delta}, \nabla v(x)\Big)\, dx \geq \left(1 - \frac{\nu}{\alpha}\right) \int_{B(0,\varepsilon^{\lambda_2-1})} f(z, \nabla v(x))\, dx.$$

Then, also considering the change of variables $x := (y - z_\varepsilon)/\varepsilon$ and Lemma 1.1 (which applies since $\lambda_2 < 1$), we obtain the estimate for (11)

$$\begin{aligned}
& \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon)\}|1-c|^d \\
= \;& \min\Big\{\int_{B(0,\varepsilon^{\lambda_2-1})} f\Big(\frac{x}{\delta}\varepsilon + \frac{z_\varepsilon}{\delta}, \nabla\zeta(x)\Big) : \zeta \in W_0^{1,d}(B(0, \varepsilon^{\lambda_2-1})), \zeta = 1 \text{ on } B(0,1)\Big\}|1-c|^d \\
\geq \;& \min\Big\{\int_{B(0,\varepsilon^{\lambda_2-1})} f(z, \nabla\zeta(x)) : \zeta \in W_0^{1,d}(B(0, \varepsilon^{\lambda_2-1})), \zeta = 1 \text{ on } B(0,1)\Big\}|1-c|^d \left(1 - \frac{\nu}{\alpha}\right) \\
= \;& \frac{\Phi(z) + o_\varepsilon(1)}{(1-\lambda_2)^{d-1}|\log\varepsilon|^{d-1}}|1-c|^d \left(1 - \frac{\nu}{\alpha}\right).
\end{aligned} \tag{13}$$

To deal with the minimum in (12), we apply once more property (GC) to get

$$\begin{aligned}
& \min\{F_\varepsilon(\zeta, B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})) : \zeta \in W^{1,d}(B(z_\varepsilon, R_\Omega) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S-N})), \\
& \qquad\qquad \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon 2^{S-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, R_\Omega)\}|c|^d \\
\geq \;& \alpha \min\Big\{\int_{B(z_\varepsilon, R_\Omega)} |\nabla\zeta(x)|^d dx : \zeta \in W_0^{1,d}(B(z_\varepsilon, R_\Omega)), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon 2^{S-N})\Big\}|c|^d \\
= \;& \alpha \mathrm{Cap}_d(B(z_\varepsilon, \varepsilon 2^{S-N}), B(z_\varepsilon, R_\Omega))|c|^d \\
= \;& \frac{\alpha\sigma_{d-1}}{[\log R_\Omega + |\log\varepsilon| - (S-N)\log 2]^{d-1}}|c|^d \\
\geq \;& \frac{\alpha\sigma_{d-1}}{[\log R_\Omega + \lambda_2|\log\varepsilon| + (N+2)\log 2]^{d-1}}|c|^d,
\end{aligned} \tag{14}$$

where in the last inequality we used that $S = \lfloor \frac{(1-\lambda_2)|\log\varepsilon|}{\log 2} \rfloor$.

Gathering (13) and (14), and multiplying by $|\log\varepsilon|^{d-1}$, we get

$$\left(1+\frac{C}{N-1}\right)|\log\varepsilon|^{d-1}F_\varepsilon(u,\Omega) \geq \frac{\Phi(z)+o_\varepsilon(1)}{(1-\lambda_2)^{d-1}}|1-c|^d\left(1-\frac{\nu}{\alpha}\right) + \frac{\alpha\sigma_{d-1}|\log\varepsilon|^{d-1}}{[\log R_\Omega+\lambda_2|\log\varepsilon|+(N+2)\log 2]^{d-1}}|c|^d\,.$$

We recall that, as specified by (iii) in Lemma 2.2, the boundary value c actually depends on ε being the mean value of the function u on an annulus whose radii are ε-dependent. Observe that we can assume that $c(\varepsilon)\to c\in\mathbb{R}$. Indeed, by the estimate

$$F_\varepsilon(u,\Omega)\geq F_\varepsilon((u\vee 0)\wedge 1,\Omega) \text{ for every } u\in W_0^{1,d}(\Omega),$$

we may assume that u takes values in $[0,1]$ so that $(c(\varepsilon))_\varepsilon\subseteq[0,1]$ as well and it admits a convergent subsequence to $c\in[0,1]$.
Finally, since u is arbitrary among the admissible functions for the minimization and since ν may be picked arbitrarily small, we pass to the limit as $\varepsilon\to 0$ and as $N\to+\infty$ to conclude that

$$\liminf_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta}\geq\frac{\Phi(z)}{(1-\lambda_2)^{d-1}}(1-c)^d+\frac{\alpha\sigma_{d-1}}{\lambda_2^{d-1}}c^d\,, \tag{15}$$

for every $\lambda_2\in(0,1)$.

Estimate from below for $\boldsymbol{\lambda\in(0,1)}$. If $\lambda\in(0,1)$, we introduce a further parameter $\lambda_1\in(0,\lambda)$ so that

$$\frac{\delta}{\varepsilon^{\lambda_1}}\to 0 \text{ as } \varepsilon\to 0.$$

Our construction relies on the definition of several concentric annuli. To this end, let

$$T:=\max\{t\in\mathbb{N}:\varepsilon^{\lambda_1}2^t\leq R_\Omega\}=\lfloor\frac{\lambda_1|\log\varepsilon|+\log R_\Omega}{\log 2}\rfloor$$

and assume in particular that T is larger than 4 as ε is small enough. Then pick a natural number $M\in(2,T)$ and define annuli centered in z_ε having radii $\varepsilon^{\lambda_1}2^{kM}$, with $k=0,1,...,\lfloor\frac{T}{M}\rfloor+1$.
This way we have $\Omega\subseteq B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M})$; hence, for every $u\in W_0^{1,d}(\Omega)$ such that $u=1$ on $B(z_\varepsilon,\varepsilon)$, it holds that

$$\begin{aligned}F_\varepsilon(u,\Omega)&=F_\varepsilon(u,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M}))\\&=F_\varepsilon(u,B(z_\varepsilon,\varepsilon^{\lambda_2}))+F_\varepsilon(u,B(z_\varepsilon,\varepsilon^{\lambda_1})\setminus B(z_\varepsilon,\varepsilon^{\lambda_2}))\\&\quad+\sum_{k=1}^{\lfloor T/M\rfloor+1}F_\varepsilon(u,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M})).\end{aligned}$$

Apply Lemma 2.2 to the first summand with

$$f(x,\xi)=f\left(\frac{x}{\delta},\xi\right),\ \eta=\varepsilon,\ R=\varepsilon^{\lambda_2},\ N\in\mathbb{N}\cap\left(1,\left\lfloor\frac{(1-\lambda_2)|\log\varepsilon|}{\log 2}\right\rfloor\right)\ \text{and}\ r=\varepsilon;$$

apply Lemma 2.2 to the second summand with

$$f(x,\xi)=f\left(\frac{x}{\delta},\xi\right),\ \eta=\varepsilon^{\lambda_2},\ R=\varepsilon^{\lambda_1},\ N\in\mathbb{N}\cap\left(1,\left\lfloor\frac{(\lambda_2-\lambda_1)|\log\varepsilon|}{\log 2}\right\rfloor\right)\ \text{and}\ r=\varepsilon^{\lambda_2};$$

apply Lemma 2.2 to the terms of the third summand for $k=1,...,\lfloor T/M\rfloor$ with

$$f(x,\xi)=f\left(\frac{x}{\delta},\xi\right),\ \eta=\varepsilon^{\lambda_1},\ R=\varepsilon^{\lambda_1}2^{kM},\ N\in\mathbb{N}\cap(1,kM)\ \text{and}\ r=\varepsilon^{\lambda_1}2^{(k-1)M}.$$

We set for simplicity of notation

$$S':=\left\lfloor\frac{(1-\lambda_2)|\log\varepsilon|}{\log 2}\right\rfloor\qquad\text{and}\qquad S'':=\left\lfloor\frac{(\lambda_2-\lambda_1)|\log\varepsilon|}{\log 2}\right\rfloor,$$

and we note that since S', S'' and M will get arbitrarily large, we may assume we fixed the same N in each of the above applications of the lemma.
We get functions $v^{-1}\in W^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_2}))$ attaining the constant value c_{-1} on $\partial B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}})$, $v^0\in W^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_1})\setminus\overline{B}(z_\varepsilon,\varepsilon^{\lambda_2}))$ attaining the constant value c_0 on $\partial B(z_\varepsilon,\varepsilon 2^{S''-j_0})$ and $v^k\in W^{1,d}(B(z,\varepsilon^{\lambda_1}2^{kM})\setminus\overline{B}(z,\varepsilon^{\lambda_1}2^{(k-1)M}))$ for $k=1,...,\lfloor T/M\rfloor$ attaining the constant value c_k on $\partial B(z_\varepsilon,\varepsilon 2^{kM-j_k})$ with $j_k\in\{1,...,N-1\}$ for $k=-1,0,1,...,\lfloor T/M\rfloor$.
Then we put

$$v:=\begin{cases} v^{-1} & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_2})\\ v^0 & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_1})\setminus B(z_\varepsilon,\varepsilon^{\lambda_2})\\ v^k & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M}),\ k=1,...,\lfloor T/M\rfloor\\ u & \text{otherwise,}\end{cases}$$

and note that $v\in W_0^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M}))$ since the modifications provided by the lemma occur far from the boundary of each annulus; moreover it holds

$$\begin{aligned}\left(1+\frac{C}{N-1}\right)F_\varepsilon(u,\Omega)&=\left(1+\frac{C}{N-1}\right)F_\varepsilon(u,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M}))\\ &\geq F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M})).\end{aligned}$$

To point out that v attains constant values on proper spheres centered in z_ε, we write

$$
\begin{aligned}
\left(1+\frac{C}{N-1}\right)F_\varepsilon(u,\Omega) &\geq \left(1+\frac{C}{N-1}\right)F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M}))\\
&= F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M}))\\
&= F_\varepsilon(v,B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}}))\\
&\quad + F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_2}2^{S''-j_0})\setminus B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}}))\\
&\quad + F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{M-j_1})\setminus B(z_\varepsilon,\varepsilon^{\lambda_2}2^{S''-j_0}))\\
&\quad + \sum_{k=2}^{\lfloor T/M\rfloor} F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM-j_k})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M-j_{k-1}}))\\
&\quad + F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(\lfloor T/M\rfloor+1)M})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{\lfloor T/M\rfloor M-j_{\lfloor T/M\rfloor}})),
\end{aligned}
\tag{16}
$$

Then we define functions w^k, $k=-1,0,1,...\lfloor T/M\rfloor+1$ as follows: $w^{-1}\in W^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_2}))$ is defined as

$$
w^{-1}:=\begin{cases} v & \text{on } B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}})\\ c_{-1} & \text{otherwise,}\end{cases}
$$

so that

$$
F_\varepsilon(w^{-1},B(z_\varepsilon,\varepsilon^{\lambda_2}))=F_\varepsilon(v,B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}})).
$$

Similarly, set

$$
w^0:=\begin{cases} c_{-1} & \text{on } B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}}))\setminus \overline{B}(z_\varepsilon,\varepsilon 2^{S'-N}))\\ v & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_2}2^{S''-j_0})\setminus B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}}))\\ c_0 & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_1})\setminus B(z_\varepsilon,\varepsilon^{\lambda_2}2^{S''-j_0}),\end{cases}
$$

so that

$$
F_\varepsilon(w^0,B(z_\varepsilon,\varepsilon^{\lambda_1})\setminus B(z_\varepsilon,\varepsilon 2^{S'-N})))=F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_2}2^{S''-j_0})\setminus B(z_\varepsilon,\varepsilon 2^{S'-j_{-1}}))
$$

and

$$
w^1:=\begin{cases} c_0 & \text{on } B(z_\varepsilon,\varepsilon 2^{S''-j_0})\setminus \overline{B}(z_\varepsilon,\varepsilon 2^{S''-N}))\\ v & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_1}2^{M-j_1})\setminus B(z_\varepsilon,\varepsilon 2^{S''-j_0}))\\ c_1 & \text{on } B(z_\varepsilon,\varepsilon^{\lambda_1}2^{M}))\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{M-j_1}),\end{cases}
$$

so that

$$
F_\varepsilon(w^1,B(z_\varepsilon,\varepsilon^{\lambda_1}2^M)\setminus B(z_\varepsilon,\varepsilon 2^{S''-N})))=F_\varepsilon(v,B(z_\varepsilon,\varepsilon^{\lambda_1}2^{M-j_1})\setminus B(z_\varepsilon,\varepsilon 2^{S''-j_0})).
$$

For $k = 2, ..., \lfloor T/M \rfloor + 1$, we define annuli

$$A^N_{M,k} := B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM}) \setminus \overline{B}(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-N}).$$

For $k = 2, ..., \lfloor T/M \rfloor$, we define functions $w^k \in W^{1,d}(A^N_{M,k})$ as

$$w^k := \begin{cases} c_{k-1} & \text{on } B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-j_{k-1}}) \setminus \overline{B}(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-N}) \\ v & \text{on } B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM-j_k}) \setminus B(z, \varepsilon^{\lambda_1} 2^{(k-1)M-j_{k-1}}) \\ c_k & \text{on } B(z, \varepsilon^{\lambda_1} 2^{kM}) \setminus B(z, \varepsilon^{\lambda_1} 2^{kM-j_k}), \end{cases}$$

and for $k = \lfloor T/M \rfloor + 1$,

$$w^{\lfloor T/M \rfloor+1} := \begin{cases} c_{\lfloor T/M \rfloor} & \text{on } B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-j_{k-1}}) \setminus \overline{B}(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-N}) \\ v & \text{otherwise,} \end{cases}$$

so that

$$F_\varepsilon(w^k, A^N_{M,k}) = F_\varepsilon(v, B(z, \varepsilon^{\lambda_1} 2^{kM-j_k}) \setminus B(z, \varepsilon^{\lambda_1} 2^{(k-1)M-j_{k-1}}))$$

for all $k = 2, ..., \lfloor T/M \rfloor + 1$.
Once we set $A^N_{M,-1} := B(z_\varepsilon, \varepsilon^{\lambda_2})$, $A^N_{M,0} := B(z_\varepsilon, \varepsilon^{\lambda_1}) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S'-N})$ and $A^N_{M,1} := B(z_\varepsilon, \varepsilon^{\lambda_1} 2^M) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S''-N})$, we can rewrite (16) simply as

$$\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, \Omega) \geq \sum_{k=-1}^{\lfloor T/M \rfloor+1} F_\varepsilon(w^k, A^N_{M,k}).$$

As the functions $w^{-1}, ..., w^{\lfloor T/M \rfloor+1}$ attain constant value on the components of their annuli of definition, we exploit (H) and suitable affine transformations (as in the case $\lambda = 0$) to get

$$\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, \Omega) \geq$$

$$\begin{aligned} &\geq \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_2})) : \zeta \in W^{1,d}_0(B(z_\varepsilon, \varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon)\}|1 - c_{-1}|^d & (17) \\ &\quad + \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_1}) \setminus B(z_\varepsilon, \varepsilon 2^{S'-N})) : \zeta \in W^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_1}) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S'-N})), \\ &\qquad \zeta = 1 \text{ on } \partial B(z_\varepsilon, \varepsilon 2^{S'-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1})\}|c_{-1} - c_0|^d & (18) \\ &\quad + \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_1} 2^M) \setminus B(z_\varepsilon, \varepsilon 2^{S''-N})) : \zeta \in W^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_1} 2^M) \setminus \overline{B}(z_\varepsilon, \varepsilon 2^{S''-N})), \\ &\qquad \zeta = 1 \text{ on } \partial B(z_\varepsilon, \varepsilon 2^{S''-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1} 2^M)\}|c_0 - c_1|^d & (19) \\ &\quad + \sum_{k=2}^{\lfloor T/M \rfloor+1} \min\{F_\varepsilon(\zeta, A^N_{M,k}) : \zeta \in W^{1,d}(A^N_{M,k}),\ \zeta = 1 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M-N}), \\ &\qquad \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM})\}|c_{k-1} - c_k|^d, & (20) \end{aligned}$$

where we put $c_{\lfloor \frac{T}{M} \rfloor+1} := 0$.
Since $\lambda_2 < \lambda$, the minimum in (17) is estimated as (13) in the case $\lambda = 0$, thus it is greater than or equal to

$$\frac{\Phi(z) + o_\varepsilon(1)}{(1-\lambda_2)^{d-1}|\log\varepsilon|^{d-1}}|1 - c_{-1}|^d \left(1 - \frac{\nu}{\alpha}\right), \tag{21}$$

where ν may be taken arbitrarily small as $\varepsilon \to 0$.
The bounds for (18) and (19) follow again by (GC); in particular, recalling how we defined S' and S'', we have that (18) is larger than or equal to

$$\begin{aligned}\alpha\mathrm{Cap}_d(B(z_\varepsilon, \varepsilon 2^{S'-N}), B(z_\varepsilon, \varepsilon^{\lambda_1})) &\geq \frac{\alpha\sigma_{d-1}}{[(1-\lambda_1)|\log\varepsilon| - (S'-N)\log 2]^{d-1}} \\ &\geq \frac{\alpha\sigma_{d-1}}{[(\lambda_2-\lambda_1)|\log\varepsilon| + (N+1)\log 2]^{d-1}},\end{aligned} \tag{22}$$

while (19) is larger than or equal to

$$\begin{aligned}\alpha\mathrm{Cap}_d(B(z_\varepsilon, \varepsilon 2^{S''-N}), B(z_\varepsilon, \varepsilon^{\lambda_1}2^M)) &\geq \frac{\alpha\sigma_{d-1}}{[M\log 2 + (1-\lambda_1)|\log\varepsilon| - (S''-N)\log 2]^{d-1}} \\ &\geq \frac{\alpha\sigma_{d-1}}{[M\log 2 + (1-\lambda_2)|\log\varepsilon| + (N+1)\log 2]^{d-1}}.\end{aligned} \tag{23}$$

Concerning the summands in (20), fix $k = 2, ..., \lfloor T/M \rfloor + 1$ and apply the change of variables $x := (y - z_\varepsilon)/\varepsilon^{\lambda_1}2^{(k-1)M-N}$, we get

$$\begin{aligned}&\min\{F_\varepsilon(\zeta, A^N_{M,k}) : \zeta \in W^{1,d}(A^N_{M,k}),\ \zeta = 1 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1}2^{(k-1)M-N}), \\ &\qquad\qquad \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1}2^{kM})\}|c_{k-1} - c_k|^d \\ = &\min\Big\{\int_{B(0,2^{M+N})\setminus B(0,1)} f\Big(\frac{x}{\delta}\varepsilon^{\lambda_1}2^{(k-1)M-N} + \frac{z_\varepsilon}{\delta}, \nabla\zeta(x)\Big)\,dx : \\ &\qquad\qquad \zeta \in W^{1,d}_0(B(0,2^{M+N})), \zeta = 1 \text{ on } B(0,1)\Big\}|c_{k-1} - c_k|^d.\end{aligned}$$

By $\lambda_1 < \lambda$ it follows that

$$\frac{\delta}{\varepsilon^{\lambda_1}2^{(k-1)M-N}} \to 0 \text{ as } \varepsilon \to 0;$$

hence, we can apply Theorem 1.2 with

$$A = B(0, 2^{M+N}) \setminus \overline{B}(0,1)\,, \qquad \eta = \frac{\delta}{\varepsilon^{\lambda_1}2^{(k-1)M-N}}\,, \qquad \tau_\eta = \frac{z_\varepsilon}{\delta}$$

and ϕ any function in $W^{1,d}(B(0,2^{M+N})\setminus\overline{B}(0,1))$ such that $\phi=1$ on $\partial B(0,1)$ and $\phi=0$ on $\partial B(0,2^{M+N})$. We get that each of the above minima equals

$$\Big[\min\Big\{\int_{B(0,2^{M+N})\setminus B(0,1)} f_{\mathrm{hom}}(\nabla\zeta(x))\,dx:\,\zeta\in W^{1,d}(B(0,2^{M+N})),\\ \zeta=1 \text{ on } B(0,1),\zeta=0 \text{ on } \partial B(0,2^{M+N})\Big\}+o_\varepsilon(1)\Big]|c_{k-1}-c_k|^d, \tag{24}$$

where f_{hom} is the d-homogeneous function given by (5), which does not depend on k. Recalling the definition of the constant C_{hom} given in (4), (24) turns into

$$\left[\frac{C_{\mathrm{hom}}+o_M(1)}{((M+N)\log 2)^{d-1}}+o_\varepsilon(1)\right]|c_{k-1}-c_k|^d.$$

By the convexity of $x\mapsto|x|^d$ and the facts that $\sum_{k=2}^{\lfloor T/M\rfloor+1}(c_{k-1}-c_k)=c_1$ and $T\le\frac{\lambda_1|\log\varepsilon|+\log R_\Omega}{\log 2}$, we obtain

$$\sum_{k=2}^{\lfloor T/M\rfloor+1}|c_{k-1}-c_k|^d\ge\frac{(M\log 2)^{d-1}}{(\lambda_1|\log\varepsilon|+\log R_\Omega+M\log 2)^{d-1}}|c_1|^d,$$

and in turn

$$\sum_{k=2}^{\lfloor T/M\rfloor+1}\min\{F_\varepsilon(\zeta,A^N_{M,k}):\zeta\in W^{1,d}(A^N_{M,k}),\,\zeta=1 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M-N}),\\ \zeta=0 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\}|c_{k-1}-c_k|^d\\ \ge\left[\frac{C_{\mathrm{hom}}+o_M(1)}{((M+N)\log 2)^{d-1}}+o_\varepsilon(1)\right]\frac{(M\log 2)^{d-1}}{(\lambda_1|\log\varepsilon|+\log R_\Omega+M\log 2)^{d-1}}|c_1|^d. \tag{25}$$

Gathering (21), (22), (23) and (25), and multiplying by $|\log\varepsilon|^{d-1}$, we get

$$\left(1+\frac{C}{N-1}\right)|\log\varepsilon|^{d-1}F_\varepsilon(u,\Omega)\ge\frac{\Phi(z)+o_\varepsilon(1)}{(1-\lambda_2)^{d-1}}|1-c_{-1}|^d\left(1-\frac{\nu}{\alpha}\right)\\ +\frac{\alpha\sigma_{d-1}|\log\varepsilon|^{d-1}}{[(\lambda_2-\lambda_1)|\log\varepsilon|+(N+1)\log 2]^{d-1}}|c_{-1}-c_0|^d\\ +\frac{\alpha\sigma_{d-1}|\log\varepsilon|^{d-1}}{[M\log 2+(1-\lambda_2)|\log\varepsilon|+(N+1)\log 2]^{d-1}}|c_0-c_1|^d\\ +\left[\frac{C_{hom}+o_M(1)}{((M+N)\log 2)^{d-1}}+o_\varepsilon(1)\right]\frac{(M\log 2)^{d-1}|\log\varepsilon|^{d-1}}{(\lambda_1|\log\varepsilon|+\log R_\Omega+M\log 2)^{d-1}}|c_1|^d.$$

We remark that c_{-1}, c_0 and c_1 depend on ε and, arguing as before, they can be picked inside the interval $[0,1]$. This fact leads us to assume that each converges to some finite limit, say c_{-1}, c_0 and c_1, respectively. Moreover, these limits have to coincide; otherwise, letting $\lambda_1, \lambda_2 \to \lambda$ or $\lambda_2 \to 1$, we get a contradiction to the fact

$$\sup_{\varepsilon} |\log \varepsilon|^{d-1} F_\varepsilon(u, \Omega) < \infty.$$

Eventually, the following estimate holds true:

$$\left(1 + \frac{C}{N-1}\right) \liminf_{\varepsilon \to 0} |\log \varepsilon|^{d-1} F_\varepsilon(u, \Omega) \geq \frac{\Phi(z)}{(1-\lambda_2)^{d-1}}(1-c)^d + \left[\frac{C_{\hom} + o_M(1)}{((M+N)\log 2)^{d-1}}\right] \frac{(M \log 2)^{d-1}}{\lambda_1^{d-1}} c^d,$$

and letting $M, N \to +\infty$, by the arbitrariness of u we achieve

$$\liminf_{\varepsilon \to 0} |\log \varepsilon|^{d-1} \mu_{\varepsilon,\delta} \geq \frac{\Phi(z)}{(1-\lambda_2)^{d-1}}(1-c)^d + \frac{C_{\hom}}{\lambda_1^{d-1}} c^d \,. \tag{26}$$

Estimate from below for $\boldsymbol{\lambda = 1}$. If $\lambda = 1$, keeping the notation introduced throughout the proof, define annuli centered in z_ε having radii $\varepsilon^{\lambda_1} 2^{kM}$, with $k = 1, ..., \lfloor \frac{T}{M} \rfloor + 1$. For every function $u \in W_0^{1,d}(\Omega), u = 1$ on $B(z_\varepsilon, \varepsilon)$, we have

$$\begin{aligned} F_\varepsilon(u, \Omega) &= F_\varepsilon(u, B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(\lfloor T/M \rfloor + 1)M})) \\ &= F_\varepsilon(u, B(z_\varepsilon, \varepsilon^{\lambda_1})) \\ &\quad + \sum_{k=1}^{\lfloor T/M \rfloor + 1} F_\varepsilon(u, B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM}) \setminus B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M})). \end{aligned} \tag{27}$$

Apply Lemma 2.2 to the terms of the second summand for $k = 1, ..., \lfloor T/M \rfloor$ with

$$f(x, \xi) = f\left(\frac{x}{\delta}, \xi\right), \, \eta = \varepsilon^{\lambda_1}, \; R = \varepsilon^{\lambda_1} 2^{kM}, \; N \in \mathbb{N} \cap (1, kM) \text{ and } r = \varepsilon^{\lambda_1} 2^{(k-1)M}.$$

Arguing as in the previous instances, with $\lambda \in [0,1)$, we get

$$\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u, \Omega) \geq$$

$$\geq \min\{F_\varepsilon(\zeta, B(z_\varepsilon, \varepsilon^{\lambda_1})) : \zeta \in W^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_1})), \zeta = 1 \text{ on } B(z_\varepsilon, \varepsilon), \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1})\} |1 - c_0|^d \tag{28}$$

$$+ \sum_{k=1}^{\lfloor T/M \rfloor + 1} \min\{F_\varepsilon(\zeta, A_{M,k}^N) : \zeta \in W^{1,d}(A_{M,k}^N), \zeta = 1 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M - N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM})\} |c_{k-1} - c_k|^d \,, \tag{29}$$

where $c_{\lfloor \frac{T}{M} \rfloor +1} := 0$.
Making use of (GC), (28) is bounded from below by

$$\alpha \mathrm{Cap}_d(B(z_\varepsilon, \varepsilon), B(z_\varepsilon, \varepsilon^{\lambda_1}))|1-c_0|^d = \frac{\alpha \sigma_{d-1}}{[(1-\lambda_1)|\log \varepsilon|]^{d-1}}|1-c_0|^d$$

as we did for (22) or (23); while (29) can be estimated as in (25) since $\delta/\varepsilon^{\lambda_1} \to 0$.
At the end, we get the inequality

$$\begin{aligned}\left(1+\frac{C}{N-1}\right)|\log\varepsilon|^{d-1}F_\varepsilon(u,\Omega) &\geq \frac{\alpha\sigma_{d-1}}{(1-\lambda_1)^{d-1}}|1-c_0|^d \\ &+ \left[\frac{C_{\mathrm{hom}}+o_M(1)}{((M+N)\log 2)^{d-1}}+o_\varepsilon(1)\right]\frac{|\log\varepsilon|^{d-1}(M\log 2)^{d-1}}{(\lambda_1|\log\varepsilon|+\log R_\Omega+M\log 2)^{d-1}}|c_0|^d\,.\end{aligned}$$

Recall that we may assume that $c_0 = c_0(\varepsilon)$ converges to a finite value $c \in [0,1]$, hence we let $\varepsilon \to 0, M \to +\infty$ and $N \to +\infty$ to obtain

$$\liminf_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \geq \frac{\alpha\sigma_{d-1}}{(1-\lambda_1)^{d-1}}(1-c)^d + \frac{C_{\mathrm{hom}}}{\lambda_1^{d-1}}c^d\,. \tag{30}$$

Once we gather (15), (26), (30), we have

$$\liminf_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \geq \begin{cases} \frac{\Phi(z)}{(1-\lambda_2)^{d-1}}(1-c)^d + \frac{\alpha\sigma_{d-1}}{\lambda_2^{d-1}}c^d & \text{if } \lambda = 0, \\ \frac{\Phi(z)}{(1-\lambda_2)^{d-1}}(1-c)^d + \frac{C_{\mathrm{hom}}}{\lambda_1^{d-1}}c^d & \text{if } \lambda \in (0,1), \\ \frac{\alpha\sigma_{d-1}}{(1-\lambda_1)^{d-1}}(1-c)^d + \frac{C_{\mathrm{hom}}}{\lambda_1^{d-1}}c^d & \text{if } \lambda = 1 \end{cases}$$

for every $\lambda_1 \in (0,\lambda)$ and $\lambda_2 \in (\lambda,1)$.
These expressions can be estimated by the same argument concerning the minimization of the function $a(1-x)^d + bx^d$ with $x \in [0,1]$ and $a,b > 0$. Indeed, the minimum is attained at

$$x = \left[\left(\frac{b}{a}\right)^{\frac{1}{d-1}}+1\right]^{-1}$$

with minimum value

$$b\left[\left(\frac{b}{a}\right)^{\frac{1}{d-1}}+1\right]^{1-d}.$$

In (15), we set

$$a = \frac{\Phi(z)}{(1-\lambda_2)^{d-1}} \qquad \text{and} \qquad b = \frac{\alpha\sigma_{d-1}}{\lambda_2^{d-1}}\,,$$

to achieve

$$\liminf_{\varepsilon\to 0} |\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \geq \frac{\alpha\sigma_{d-1}}{\lambda_2^{d-1}}\left[\left(\frac{\alpha\sigma_{d-1}/\lambda_2^{d-1}}{\Phi(z)/(1-\lambda_2)^{d-1}}\right)^{\frac{1}{d-1}}+1\right]^{1-d}$$
$$= \Phi(z)\alpha\sigma_{d-1}\left[(1-\lambda_2)(\alpha\sigma_{d-1})^{\frac{1}{d-1}}+\lambda_2\Phi(z)^{\frac{1}{d-1}}\right]^{1-d}.$$

We conclude passing to the limit as $\lambda_2 \to 0$.

In (26), put

$$a = \frac{\Phi(z)}{(1-\lambda_2)^{d-1}} \qquad \text{and} \qquad b = \frac{C_{\rm hom}}{\lambda_1^{d-1}}\,, \tag{31}$$

and let $\lambda_1, \lambda_2 \to \lambda$ getting

$$\liminf_{\varepsilon\to 0} |\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \geq \frac{C_{\rm hom}}{\lambda^{d-1}}\left[\left(\frac{C_{\rm hom}/\lambda^{d-1}}{\Phi(z)/(1-\lambda)^{d-1}}\right)^{\frac{1}{d-1}}+1\right]^{1-d}$$
$$= \Phi(z)C_{\rm hom}\Big[\lambda\Phi(z)^{\frac{1}{d-1}}+(1-\lambda)C_{\rm hom}^{\frac{1}{d-1}}\Big]^{1-d}.$$

Finally, in (30) let

$$a = \frac{\alpha\sigma_{d-1}}{(1-\lambda_1)^{d-1}} \qquad \text{and} \qquad b = \frac{C_{\rm hom}}{\lambda_1^{d-1}}\,,$$

to have

$$\liminf_{\varepsilon\to 0} |\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \geq \frac{C_{\rm hom}}{\lambda_1^{d-1}}\left[\left(\frac{C_{\rm hom}/\lambda_1}{\alpha\sigma_{d-1}/(1-\lambda_1)}\right)^{\frac{1}{d-1}}+1\right]^{1-d}$$
$$= \alpha\sigma_{d-1}C_{\rm hom}\Big[\lambda_1(\alpha\sigma_{d-1})^{\frac{1}{d-1}}+(1-\lambda_1)C_{\rm hom}^{\frac{1}{d-1}}\Big]^{1-d}.$$

Then, conclude letting $\lambda_1 \to 1$.

2.2 Construction of optimal sequences

To finish the proof we find minimizing sequences by suitable capacitary profiles.

Optimal construction for $\boldsymbol{\lambda = 0}$. If $\lambda = 0$, take $\lambda_2 \in (\lambda, 1)$ and let v_ε^0 be a solution of the minimum problem

$$\min\Big\{\int_{B(0,\varepsilon^{\lambda_2-1})} f(z,\nabla u(x))\,dx : u \in W_0^{1,d}(B(0,\varepsilon^{\lambda_2-1})), u = 1 \text{ on } B(0,1)\Big\}.$$

For $\varepsilon \ll 1$, the function $u_\varepsilon^0(x) := v_\varepsilon^0\left(\frac{x-z_\varepsilon}{\varepsilon}\right)$ belongs to $W_0^{1,d}(\Omega)$ and it is admissible for the minimum problem defining (8), thus, taking advantage of (H) and (P), we get

$$\begin{aligned}\mu_{\varepsilon,\delta} \leq F_\varepsilon(u_\varepsilon^0, \Omega) &= F_\varepsilon(u_\varepsilon^0, B(z_\varepsilon, \varepsilon^{\lambda_2}))\\ &= \int_{B(z_\varepsilon,\varepsilon^{\lambda_2})} f\Big(\frac{x}{\delta}, \nabla u_\varepsilon^0(x)\Big)\, dx\\ &= \int_{B(z_\varepsilon,\varepsilon^{\lambda_2})} f\Big(\frac{x}{\delta}, \nabla v_\varepsilon^0\Big(\frac{x-z_\varepsilon}{\varepsilon}\Big)\Big)\frac{1}{\varepsilon^d}\, dx\\ &= \int_{B(0,\varepsilon^{\lambda_2-1})} f\Big(\frac{x}{\delta}\varepsilon + \frac{z_\varepsilon}{\delta}, \nabla v_\varepsilon^0(x)\Big)\, dx\\ &= \int_{B(0,\varepsilon^{\lambda_2-1})} f\Big(\frac{x}{\delta}\varepsilon + z, \nabla v_\varepsilon^0(x)\Big)\, dx.\end{aligned}$$

With the same reasoning used in the bound from below, note that if $x \in B(0, \varepsilon^{\lambda_2-1})$, then $\frac{\varepsilon}{\delta}|x| < \frac{\varepsilon^{\lambda_2}}{\delta} \to 0$ as $\varepsilon \to 0$. Hence, for every $\nu > 0$, given r_ν as in (9), it holds that $\frac{\varepsilon}{\delta}B(0, \varepsilon^{\lambda_2-1}) \subseteq B(0, r_\nu)$, so that for every ε sufficiently small we have

$$f(x,\xi) \leq f(z,\xi) + \nu|\xi|^d \text{ for every } x \in B(0,\varepsilon^{\lambda_2-1}) \text{ and } \xi \in \mathbb{R}^d.$$

As a consequence

$$\int_{B(0,\varepsilon^{\lambda_2-1})} f\Big(\frac{x}{\delta}\varepsilon + z, \nabla v_\varepsilon^0(x)\Big)\, dx \leq \int_{B(0,\varepsilon^{\lambda_2-1})} f(z, \nabla v_\varepsilon^0(x))dx + \nu \int_{B(0,\varepsilon^{\lambda_2-1})} |\nabla v_\varepsilon^0(x)|^d dx$$

which, by (GC), is bounded above by

$$\Big(1 + \frac{\nu}{\alpha}\Big) \int_{B(0,\varepsilon^{\lambda_2-1})} f(z, \nabla v_\varepsilon^0(x))\, dx.$$

Since $\varepsilon^{\lambda_2-1} \to \infty$ as $\varepsilon \to 0$, we apply Lemma 1.1 to deduce

$$\mu_{\varepsilon,\delta} \leq \frac{\Phi(z) + o_\varepsilon(1)}{(1-\lambda_2)^{d-1}|\log\varepsilon|^{d-1}}\left(1 + \frac{\nu}{\alpha}\right), \tag{32}$$

thus, by the arbitrariness of $\nu > 0$ and $\lambda_2 \in (0,1)$, we conclude that

$$\limsup_{\varepsilon\to 0} |\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \leq \inf_{\lambda_2\in(0,1)} \frac{\Phi(z)}{(1-\lambda_2)^{d-1}} = \Phi(z).$$

Optimal construction for $\boldsymbol{\lambda \in (0,1)}$. If $\lambda \in (0,1)$, let $\lambda_1 \in (0,\lambda)$, put

$$T := \max\{t \in \mathbb{N} : \varepsilon^{\lambda_1}2^t \leq \operatorname{dist}(z_\varepsilon, \partial\Omega)\} = \left\lfloor \frac{\lambda_1|\log\varepsilon| + \log\operatorname{dist}(z_\varepsilon,\partial\Omega)}{\log 2}\right\rfloor$$

and take $M \in \mathbb{N} \cap (0,T)$. Since the family of points $\{z_\varepsilon,\, \varepsilon > 0\}$ is contained in a ball, say B, whose closure lays inside Ω, we have that $\mathrm{dist}(z_\varepsilon, \partial\Omega) \geq \mathrm{dist}(\partial B, \partial\Omega) > 0$ so that T is well defined and can be assumed to be greater than 2 for every ε.

Let v_η be a solution of the minimum problem

$$m_\eta := \min\Big\{ \int_{B(0,2^M)} f\Big(\frac{x}{\eta} + \tau_\eta, \nabla u(x)\Big)\, dx : u \in W_0^{1,d}(B(0,2^M)), u = 1 \text{ on } B(0,1)\Big\}$$

with η a positive vanishing parameter and τ_η to be specified, and set

$$m_0 := \min\Big\{ \int_{B(0,2^M)} f_{\mathrm{hom}}(\nabla u(x))\, dx : u \in W_0^{1,d}(B(0,2^M)), u = 1 \text{ on } B(0,1)\Big\}.$$

By Theorem 1.2, there exists an increasing non negative function ω such that

$$|m_\eta - m_0| \leq \omega(\eta) \quad \text{and} \quad \omega(\eta) \to 0 \text{ as } \eta \to 0;$$

thus, for $k = 1, ..., \lfloor T/M \rfloor$, define $u_\varepsilon^k \in W^{1,d}(B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM}) \setminus \overline{B}(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M}))$ as

$$u_\varepsilon^k(x) := \frac{c}{\lfloor T/M \rfloor} v_\eta \left(\frac{x - z_\varepsilon}{\varepsilon^{\lambda_1} 2^{(k-1)M}} \right) + \frac{\lfloor T/M \rfloor - k}{\lfloor T/M \rfloor} c$$

for some constant c to be properly selected. Then, considering the same u_ε^0 introduced in the case $\lambda = 0$, define

$$u_\varepsilon(x) := \begin{cases} (1-c)u_\varepsilon^0(x) + c & \text{if } x \in B(z_\varepsilon, \varepsilon^{\lambda_2}) \\ c & \text{if } x \in B(z_\varepsilon, \varepsilon^{\lambda_1}) \setminus B(z_\varepsilon, \varepsilon^{\lambda_2}) \\ u_\varepsilon^k(x) & \text{if } x \in B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{kM}) \setminus B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{(k-1)M}), k = 1, ..., \lfloor T/M \rfloor \\ 0 & \text{if } x \in \Omega \setminus B(z_\varepsilon, \varepsilon^{\lambda_1} 2^{\lfloor T/M \rfloor M}). \end{cases}$$

Since the boundary conditions match, $u_\varepsilon \in W_0^{1,d}(\Omega)$ and $u_\varepsilon = 1$ on $B(z_\varepsilon, \varepsilon)$; therefore, it is an admissible function for the minimum problem.

We separately estimate $F_\varepsilon(u_\varepsilon, B(z_\varepsilon, \varepsilon^{\lambda_1}))$ and $F_\varepsilon(u_\varepsilon, \Omega \setminus B(z_\varepsilon, \varepsilon^{\lambda_1}))$.
Making use of (H), by the same computation which led to (32) we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon, B(z_\varepsilon, \varepsilon^{\lambda_1})) &= F_\varepsilon(u_\varepsilon, B(z_\varepsilon, \varepsilon^{\lambda_2})) \\ &= F_\varepsilon(u_\varepsilon^0, B(z_\varepsilon, \varepsilon^{\lambda_2}))|1-c|^d \\ &\leq \frac{\Phi(z) + o_\varepsilon(1)}{(1-\lambda_2)^{d-1} |\log \varepsilon|^{d-1}} |1-c|^d \left(1 + \frac{\nu}{\alpha}\right), \end{aligned} \tag{33}$$

for an arbitrarily small ν.
To estimate $F_\varepsilon(u_\varepsilon, \Omega \setminus B(z_\varepsilon, \varepsilon^{\lambda_1}))$, note that, if we set

$$\eta = \frac{\delta}{\varepsilon^{\lambda_1} 2^{(k-1)M}} \qquad \text{and} \qquad \tau_\eta = \frac{z_\varepsilon}{\delta},$$

it holds

$$
\begin{aligned}
&F_\varepsilon(u_\varepsilon^k, B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M})) = \\
&= \int_{B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M})} f\left(\frac{x}{\delta}, \nabla u_\varepsilon^k(x)\right)\, dx \\
&= \left|\frac{c}{\lfloor T/M\rfloor}\right|^d \int_{B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M})} f\left(\frac{x}{\delta}, \nabla v_\eta\Big(\frac{x-z_\varepsilon}{\varepsilon^{\lambda_1}2^{(k-1)M}}\Big)\right) \frac{dx}{|\varepsilon^{\lambda_1}2^{(k-1)M}|^d} \\
&= \left|\frac{c}{\lfloor T/M\rfloor}\right|^d \int_{B(0,2^M)} f\left(\frac{x}{\delta}\varepsilon^{\lambda_1}2^{(k-1)M} + \frac{z_\varepsilon}{\delta}, \nabla v_\eta(x)\right)\, dx \\
&= \left|\frac{c}{\lfloor T/M\rfloor}\right|^d m_\eta \\
&\le \left|\frac{c}{\lfloor T/M\rfloor}\right|^d (m_0 + \omega(\eta)) \\
&\le \left|\frac{c}{\lfloor T/M\rfloor}\right|^d \left(m_0 + \omega\left(\frac{\delta}{\varepsilon^{\lambda_1}}\right)\right).
\end{aligned} \tag{34}
$$

Hence, by (34) we get

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon, \Omega\setminus B(z_\varepsilon,\varepsilon^{\lambda_1})) &= \sum_{k=1}^{\lfloor T/M\rfloor} F_\varepsilon(u_\varepsilon^k, B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M})) \\
&\le \left|\frac{c}{\lfloor T/M\rfloor}\right|^d \sum_{k=1}^{\lfloor T/M\rfloor} \left(m_0 + \omega\left(\frac{\delta}{\varepsilon^{\lambda_1}}\right)\right) \\
&= \frac{|c|^d}{\lfloor T/M\rfloor^{d-1}} \left(m_0 + \omega\left(\frac{\delta}{\varepsilon^{\lambda_1}}\right)\right).
\end{aligned} \tag{35}
$$

As $f_{\hom}(0)=0$, by the definition of $C_{\hom}$ we have

$$
m_0 = \frac{C_{\hom} + o_M(1)}{(M\log 2)^{d-1}}\,;
$$

thus, we substitute in (35) obtaining

$$
F_\varepsilon(u_\varepsilon, \Omega\setminus B(z_\varepsilon,\varepsilon^{\lambda_1})) \le \frac{|c|^d}{\lfloor T/M\rfloor^{d-1}} \left(\frac{C_{\hom}+o_M(1)}{(M\log 2)^{d-1}} + \omega\left(\frac{\delta}{\varepsilon^{\lambda_1}}\right)\right).
$$

Since $T \ge (\lambda_1|\log\varepsilon| + \log \operatorname{dist}(\partial B, \partial\Omega) - \log 2)/\log 2$, it holds

$$
F_\varepsilon(u_\varepsilon, \Omega\setminus B(z_\varepsilon,\varepsilon^{\lambda_1})) \le \frac{C_{\hom} + o_M(1) + (M\log 2)^{d-1}\omega(\delta/\varepsilon^{\lambda_1})}{(\lambda_1|\log\varepsilon| + \log\operatorname{dist}(\partial B,\partial\Omega) - \log 2)^{d-1}}|c|^d. \tag{36}
$$

We gather the estimates (33) and (36) to get

$$
\begin{aligned}
|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta} \quad \le \quad & \frac{\Phi(z)+o_\varepsilon(1)}{(1-\lambda_2)^{d-1}}|1-c|^d\left(1+\frac{\nu}{\alpha}\right) \\
& + \frac{|\log\varepsilon|^{d-1}[C_{\rm hom}+o_M(1)+(M\log 2)^{d-1}\omega(\delta/\varepsilon^{\lambda_1})]}{(\lambda_1|\log\varepsilon|+\log \operatorname{dist}(\partial B,\partial\Omega)-\log 2)^{d-1}}|c|^d.
\end{aligned}
$$

Since $\frac{\delta}{\varepsilon^{\lambda_1}}\to 0$, we let $\varepsilon\to 0$ and then $M\to+\infty$ to deduce

$$
\limsup_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta}\le \frac{\Phi(z)}{(1-\lambda_2)^{d-1}}|1-c|^d+\frac{C_{\rm hom}}{\lambda_1^{d-1}}|c|^d\,;
$$

then, we let $\lambda_1,\lambda_2\to\lambda$ to conclude that

$$
\limsup_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta}\le \frac{\Phi(z)}{(1-\lambda)^{d-1}}|1-c|^d+\frac{C_{\rm hom}+o_M(1)}{\lambda^{d-1}}|c|^d.
$$

Finally, put $c:=\left[\left(\frac{b}{a}\right)^{\frac{1}{d-1}}+1\right]^{-1}$, with $a=\Phi(z)/(1-\lambda)^{d-1}$, $b=C_{\rm hom}/\lambda^{d-1}$. As we are exactly in the case discussed in (31) with $\lambda=\lambda_1=\lambda_2$, the same computation holds, leading to

$$
\limsup_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta}\le \Phi(z)C_{\rm hom}\left[\lambda\Phi(z)^{\frac{1}{d-1}}+(1-\lambda)C_{\rm hom}^{\frac{1}{d-1}}\right]^{1-d}.
$$

Optimal construction for $\boldsymbol{\lambda=1}$. If $\lambda=1$ we just set

$$
u_\varepsilon(x):=\begin{cases}1 & \text{if } x\in B(z_\varepsilon,\varepsilon^{\lambda_1})\\ u_\varepsilon^k(x) & \text{if } x\in B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M}), k=1,...,\lfloor T/M\rfloor\\ 0 & \text{if } x\in\Omega\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}2^{\lfloor T/M\rfloor M}).\end{cases}
$$

Now u_ε is an admissible function for the original problem, so the conclusion follows by (36); in particular

$$
F_\varepsilon(u_\varepsilon,\Omega)=F_\varepsilon(u_\varepsilon,\Omega\setminus B(z_\varepsilon,\varepsilon^{\lambda_1}))\le\frac{C_{\rm hom}+o_M(1)+(M\log 2)^{d-1}\omega(\delta/\varepsilon^{\lambda_1})}{(\lambda_1|\log\varepsilon|+\log d-\log 2)^{d-1}}\,;
$$

hence

$$
\limsup_{\varepsilon\to 0}|\log\varepsilon|^{d-1}\mu_{\varepsilon,\delta}\le\inf_{\lambda_1\in(0,1)}C_{\rm hom}/\lambda_1^{d-1}=C_{\rm hom}.
$$

2.3 Proof of the main result about the convergence of minima

As a consequence of the previous section, we prove the main result on the asymptotic behaviour of the minima defined in (1) by

$$m_{\varepsilon,\delta} := \min\Big\{ \int_\Omega f\left(\frac{x}{\delta}, \nabla u(x)\right)\, dx : u \in W_0^{1,d}(\Omega), u = 1 \text{ on } B(z,\varepsilon), z \in \Omega \Big\},$$

where also the centre of the small inclusion (a ball) is an argument of the minimization.

Theorem 2.3. *Assume there exists a point $x_0 \in \Omega$ such that the following hold:*
(i) *$f(x,\xi) \geq f(x_0,\xi)$ for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$;*
(ii) *for every $\nu > 0$, there exists $r_\nu > 0$ such that*

$$f(x,\xi) \leq f(x_0,\xi) + \nu|\xi|^d \text{ for every } x \in B(x_0, r_\nu), \xi \in \mathbb{R}^d.$$

Then

$$\lim_{\varepsilon\to 0} |\log\varepsilon|^{d-1} m_{\varepsilon,\delta} = \Phi(x_0) C_{\text{hom}} \Big[\lambda \Phi(x_0)^{\frac{1}{d-1}} + (1-\lambda) C_{\text{hom}}^{\frac{1}{d-1}}\Big]^{1-d}.$$

Proof. We use the same argument presented in the proof of Proposition 2.1, thus, we focus on highlighting the main differences, keeping the same notation.

Bound from below. In the case $\lambda = 0$, we introduce $\lambda_2 > 0$, then we apply Lemma 2.2 to get the inequality

$$\begin{aligned}\left(1 + \frac{C}{N-1}\right) F_\varepsilon(u,\Omega) \geq \\ \geq \min\{F_\varepsilon(\zeta, B(z,\varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z,\varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z,\varepsilon)\}|1-c|^d \\ + \min\{F_\varepsilon(\zeta, B(z,R_\Omega)\setminus \overline{B}(z,\varepsilon 2^{S-N})) : \zeta \in W^{1,d}(B(z,R_\Omega)\setminus \overline{B}(z,\varepsilon^{\lambda_2}2^{S-N})), \\ \zeta = 1 \text{ on } B(z,\varepsilon 2^{S-N}), \zeta = 0 \text{ on } \partial B(z,R_\Omega)\}|c|^d.\end{aligned}$$

Note that the second summand is estimated exactly as (12); while, for the first summand we cannot exploit the periodicity (P) since the minimization also involves the centre of the inclusion. To deal with this term, we consider a minimizer u and we simply apply (i) to get

$$\begin{aligned}\min\{F_\varepsilon(\zeta, B(z,\varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z,\varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z,\varepsilon)\}|1-c|^d \\ \geq \int_{B(z,\varepsilon^{\lambda_2})} f(x_0, \nabla u(x))\, dx\, |1-c|^d \\ = \frac{\Phi(x_0) + o_\varepsilon(1)}{(1-\lambda_2)^{d-1}|\log\varepsilon|^{d-1}}|1-c|^d.\end{aligned} \tag{37}$$

This is the same estimate we obtained in (13), with the point x_0 in place of the fixed centre z. Analogously to Proposition 2.1, we conclude that $|\log\varepsilon|^{d-1} m_{\varepsilon,\delta} \to \Phi(x_0)$.

If $\lambda \in (0,1)$, we further introduce $\lambda_1 \in (0,\lambda)$ and we achieve the inequality

$$\left(1+\frac{C}{N-1}\right)F_\varepsilon(u,\Omega) \geq$$

$$\geq \min\{F_\varepsilon(\zeta, B(z_\varepsilon,\varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z_\varepsilon,\varepsilon)\}|1-c_{-1}|^d \tag{38}$$

$$+ \min\{F_\varepsilon(\zeta, B(z_\varepsilon,\varepsilon^{\lambda_1}) \setminus B(z_\varepsilon,\varepsilon 2^{S'-N})) : \zeta \in W^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_1}) \setminus \overline{B}(z_\varepsilon,\varepsilon 2^{S'-N})),$$
$$\zeta = 1 \text{ on } \partial B(z_\varepsilon,\varepsilon 2^{S'-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1})\}|c_{-1}-c_0|^d \tag{39}$$

$$+ \min\{F_\varepsilon(\zeta, B(z_\varepsilon,\varepsilon^{\lambda_1}2^M) \setminus B(z_\varepsilon,\varepsilon 2^{S''-N})) : \zeta \in W^{1,d}(B(z_\varepsilon,\varepsilon^{\lambda_1}2^M) \setminus \overline{B}(z_\varepsilon,\varepsilon 2^{S''-N})),$$
$$\zeta = 1 \text{ on } \partial B(z_\varepsilon,\varepsilon 2^{S''-N}), \zeta = 0 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1}2^M)\}|c_0-c_1|^d \tag{40}$$

$$+ \sum_{k=2}^{\lfloor T/M\rfloor+1} \min\{F_\varepsilon(\zeta, A^N_{M,k}) : \zeta \in W^{1,d}(A^N_{M,k}),\ \zeta = 1 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1}2^{(k-1)M-N}),$$
$$\zeta = 0 \text{ on } \partial B(z_\varepsilon,\varepsilon^{\lambda_1}2^{kM})\}|c_{k-1}-c_k|^d\,, \tag{41}$$

where we put $c_{\lfloor \frac{T}{M}\rfloor+1} := 0$.

The estimates for the terms (39), (40), (41) are achieved precisely as in (18), (19), (20) respectively, while (38) is estimated exploiting (i) as in (37). Once more, the outcome is the same of Proposition 2.1, with x_0 in place of z.

The case $\lambda = 1$ is treated exactly as in Proposition 2.1 starting by the estimate in (27); this might be expected since, at this scale, the only effect in the minimization is due to the homogenization (and then it does not involve the point in which we concentrate our inclusion).

Bound frome above. Take $z_\varepsilon = \delta x_0$ modulo δ in such a way that this family of points is contained in a ball $B \subset\subset \Omega$. Condition (ii) suffices to apply the bound from above given by Proposition 2.1, then we conclude observing that $m_{\varepsilon,\delta} \leq \mu_{\varepsilon,\delta}$. □

We remark that assumption (i) may be weakened. Note indeed that the key estimate we need to carry out our proof, and more specifically the bound from below, is

$$\min\{F_\varepsilon(\zeta, B(z,\varepsilon^{\lambda_2})) : \zeta \in W_0^{1,d}(B(z,\varepsilon^{\lambda_2})), \zeta = 1 \text{ on } B(z,\varepsilon)\}$$
$$\geq \int_{B(z,\varepsilon^{\lambda_2})} f(x_0, \nabla u(x))\,dx,$$

where u is a minimizer for fixed $\lambda_2 \in (\lambda, 1)$.

A plausible sufficient condition might seem to be that Φ attains its minimum at the point x_0. Yet, note that this requirement is inadequate if Φ is not continuous at a minimum point. For instance, consider the function defined on $(0,1)^d$ as

$$f(x,\xi) := \begin{cases} \frac{1}{2}|\xi|^d & \text{if } x = x_0 := \left(\frac{1}{2},\dots,\frac{1}{2}\right) \\ |\xi|^d & \text{otherwise} \end{cases}$$

and then extended by periodicity. We see that (1) reduces to the homogeneous problem, then λ is not involved and $|\log\varepsilon|^{d-1}m_{\varepsilon,\delta}\to\sigma_{d-1}$ as $\varepsilon\to 0$; while $\Phi(x_0)=\sigma_{d-1}/2$, and plugging this in (6) with $\lambda=0$ we get $|\log\varepsilon|^{d-1}m_{\varepsilon,\delta}=\Phi(x_0)=\sigma_{d-1}/2$.

3 Application to perforated domains

In this final section we maintain the setting and notation introduced in the previous ones. We will make use of Proposition 2.1 to compute the Γ-limit of a family of functionals defined with boundary conditions related to varying domains.

Given $(\varepsilon_k)_{k\in\mathbb{N}}$ a positive sequence converging to 0, define the corresponding sequence of critical periods as $d_k:=|\log\varepsilon_k|^{\frac{1-d}{d}}$ and put $x_k^i:=id_k$ for every $i\in\mathbb{Z}^d$.
Consider $\delta=\delta(\varepsilon)$ the scale which rules the periodic structure of the energy, and define $\delta_k:=\delta(\varepsilon_k)$ for every $k\in\mathbb{N}$, obtaining a further positive sequence vanishing as $k\to+\infty$. In accordance with the previous sections, we will always assume that it exists

$$\lambda:=\lim_{k\to+\infty}\frac{|\log\delta_k|}{|\log\varepsilon_k|}\wedge 1. \tag{42}$$

Assuming that Ω is a bounded open subset of $\mathbb{R}^d$ such that $|\partial\Omega|=0$, we define a periodically perforated domain as

$$\Omega_k:=\Omega\setminus\bigcup_{i\in\mathbb{Z}^d}B(x_k^i,\varepsilon_k),$$

and we consider functionals $F_k:L^d(\Omega)\to[0,+\infty]$ given by

$$F_k(u):=\begin{cases}\displaystyle\int_\Omega f\left(\frac{x}{\delta_k},\nabla u(x)\right)dx & \text{if } u\in W^{1,d}(\Omega) \text{ and } u=0 \text{ on } \Omega\setminus\Omega_k\\ +\infty & \text{otherwise.}\end{cases}$$

To prove our result, we assume that the perforations are related to the periodic structure of the heterogeneous medium, in particular we suppose that

$$\text{for every } k \text{ there exists a natural number } m_k \text{ such that } d_k=m_k\delta_k \tag{43}$$

and that

$$\frac{\delta_k}{d_k}\to 0 \text{ as } k\to+\infty. \tag{44}$$

Note that conditions (43) and (P) lead to the identity

$$f\left(\frac{x_k^i}{\delta_k}+y,\xi\right)=f\left(\frac{id_k}{\delta_k}+y,\xi\right)=f(y,\xi) \text{ for every } i\in\mathbb{Z}^d,\ y\in\mathbb{R}^d,\xi\in\mathbb{R}^d. \tag{45}$$

In order to apply Proposition 2.1, we add suitable regularity assumptions on f at the point 0. Our statement reads as follows.

Theorem 3.1. *Assume that for every $\nu > 0$, there exists $r_\nu > 0$ such that*

$$|f(0,\xi) - f(x,\xi)| \leq \nu|\xi|^d \text{ for every } x \in B(0,r_\nu), \xi \in \mathbb{R}^d. \tag{46}$$

Then

$$\Gamma\text{-}\lim_k F_k(u) = F(u) := \int_\Omega f_{\hom}(\nabla u(x))\,dx + C(\lambda)\int_\Omega |u(x)|^d\,dx,$$

for every $u \in W^{1,d}(\Omega)$, where the Γ-limit is computed with respect to the strong convergence in $L^d(\Omega)$ and $C(\lambda)$ is given by

$$\Phi(0)C_{\hom}\left[\lambda\Phi(0)^{\frac{1}{d-1}} + (1-\lambda)C_{\hom}^{\frac{1}{d-1}}\right]^{1-d},$$

with $\Phi, C_{\hom}, f_{\hom}$ and λ defined as in (3), (4), (5) and (42), respectively.

We basically prove that, in the Γ-limit, internal boundary conditions imposed on the perforations vanish, being replaced by the additional term $C(\lambda)\int_\Omega |u|^d dx$.

3.1 The main construction and some auxiliary results

In our proof we will make wide use of Lemma 2.2. We perform the modifications on homothetic annuli with inner and outer radii proportional to the period d_k.

We introduce

$$Z_k := \{i \in \mathbb{Z}^d : \mathrm{dist}(x_k^i, \partial\Omega) > d_k\},$$

namely, the set of the centres of those perforations which are uniformly far from the boundary.
Let $M \in \mathbb{N}$, $\theta > 0$ be such that $\theta 2^{M+1} < 1/2$. Given a sequence $(u_k)_k$ in $W^{1,d}(\Omega)$, fix k, and around each point x_k^i with $i \in Z_k$ apply Lemma 2.2 to the function u_k with

$$f(x,\xi) = f\left(\frac{x}{\delta},\xi\right),\ \eta = \theta d_k,\ R = \theta 2^{M+1} d_k,\ N = M \text{ and } r = \theta d_k. \tag{47}$$

We obtain a function v_k attaining constant values u_k^i on the boundary of the ball centered at x_k^i with radius $\theta 2^{j_i} d_k$ for some $j_i \in \{1,...,M\}$ and $i \in Z_k$.

We take advantage of the following result which is a simplified version of the discretization argument proved by Sigalotti (see [17, Proposition 3.3]).

Proposition 3.2. *Let $(u_k)_k$ be a sequence in $W^{1,d}(\Omega) \cap L^\infty(\Omega)$ strongly converging to u in $L^d(\Omega)$ and such that $(\nabla u_k)_k \subseteq L^d(\Omega)$ is bounded. For every $i \in Z_k$, let u_k^i be the mean values described above and put*

$$Q_k^i := x_k^i + \left(-\frac{d_k}{2}, \frac{d_k}{2}\right)^d.$$

Then
$$\lim_{k\to\infty}\int_\Omega\left|\sum_{i\in Z_k}|u_k^i|^d\chi_{Q_k^i}(x)-|u(x)|^d\right|dx=0.$$

A useful tool to proceed will be the following convergence result which is an application of the Riemann-Lebesgue lemma.

Lemma 3.3. *The sequence*
$$\chi_k(x):=\chi_{\Omega\setminus\bigcup_{i\in Z_k}B(x_k^i,d_k/2)}(x)\,,\qquad k\in\mathbb{N}$$
weakly converges to a strictly positive constant c in $L^\infty(\Omega)$.*

3.2 Liminf inequality

We prove that for every $u\in W^{1,d}(\Omega)$ and for every sequence $(u_k)_k$ in $L^d(\Omega)$ such that $u_k\to u$ in $L^d(\Omega)$, it holds $\liminf_k F_k(u_k)\geq F(u)$. Without loss of generality we may assume that $(u_k)_k\subseteq W^{1,d}(\Omega)$ and $\sup_k F_k(u_k)<\infty$. Note that the last condition, combined with the equi-coerciveness of the functionals $(F_k)_k$, implies that $\sup_k\|\nabla u_k\|_{L^d(\Omega)}<\infty$, and therefore, that $u_k\rightharpoonup u$ in $W^{1,d}(\Omega)$.

The first step of the proof consists in applying the modification lemma as in (47). To simplify the notation here, we limit ourselves to denote the radii on which the modified function v_k attains the constant values u_k^i by ρ_k^i in place of $\theta 2^{j_i}d_k$.

In a first instance, we also assume that $(u_k)_k$ is bounded in $L^\infty(\Omega)$.
We aim at estimating
$$F_k(v_k)=\int_{\Omega\setminus\bigcup_{i\in Z_k}B(x_k^i,\rho_k^i)}f\left(\frac{x}{\delta_k},\nabla v_k(x)\right)dx+\sum_{i\in Z_k}\int_{B(x_k^i,\rho_k^i)}f\left(\frac{x}{\delta_k},\nabla v_k(x)\right)dx.\quad(48)$$

To treat the first term in (48), we perform another modification putting
$$w_k:=\begin{cases}v_k & \text{on }\Omega\setminus\bigcup_{i\in Z_k}B(x_k^i,\rho_k^i),\\ u_k^i & \text{on }B(x_k^i,\rho_k^i),\ i\in Z_k.\end{cases}$$

Note that, according to (iii) of Lemma 2.2, $\|v_k\|_{L^\infty(\Omega)}\leq\|u_k\|_{L^\infty(\Omega)}$, hence $\|w_k\|_{L^\infty(\Omega)}\leq\|u_k\|_{L^\infty(\Omega)}$, so that $(w_k)_k$ is bounded in $L^\infty(\Omega)$ and then also bounded in $L^d(\Omega)$.
By the fact that
$$\left(1+\frac{C}{M-1}\right)F_k(u_k)\geq F_k(v_k)\geq F_k(w_k),\qquad(49)$$
we deduce that $(w_k)_k$ is bounded in $W^{1,d}(\Omega)$; thus, we may extract a subsequence $(w_{k_j})_j$ weakly converging to a certain w in $W^{1,d}(\Omega)$.

As $u_k \rightharpoonup u$ in $W^{1,d}(\Omega)$, we have that $w_{k_j} - u_{k_j} \rightharpoonup w - u$ in $W^{1,d}(\Omega)$; moreover, $w_k - u_k \in W_0^{1,d}(\Omega)$ for every k so that, by Rellich's Theorem, $w_{k_j} - u_{k_j} \to w - u$ in $L^d(\Omega)$. Since $u_k \to u$ in $L^d(\Omega)$, we deduce that $(w_{k_j})_j$ actually converges strongly to w in $L^d(\Omega)$.

We claim that such w does not depend on the subsequence and that it coincides with u. To prove this, note that for every k

$$w_k \chi_{\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, d_k/2)} = u_k \chi_{\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, d_k/2)}$$

and also that, by Lemma 3.3 and the previous observations, the following hold

$$\begin{cases} \chi_{\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, d_k/2)} \overset{*}{\rightharpoonup} c & \text{in } L^\infty(\Omega), \\ u_k \to u & \text{in } L^d(\Omega), \\ w_{k_j} \to w & \text{in } L^d(\Omega). \end{cases}$$

These facts imply

$$\begin{cases} \chi_{\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, d_k/2)} u_k \rightharpoonup cu & \text{in } L^d(\Omega), \\ \chi_{\Omega \setminus \bigcup_{i \in Z_{k_j}} B(x_{k_j}^i, d_{k_j}/2)} w_{k_j} \rightharpoonup cw & \text{in } L^d(\Omega), \end{cases}$$

hence, it follows that $u = w$ in $L^d(\Omega)$ for every subsequence, proving that $w_k \to u$ in $L^d(\Omega)$.

Since $v_k = w_k$ on $\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, \rho_k^i)$, and since w_k is constant on $B(x_k^i, \rho_k^i)$ for every $i \in Z_k$, the liminf inequality provided by a classical homogenization theorem (see, e.g., [4]); we have

$$\begin{aligned} \liminf_k \int_{\Omega \setminus \bigcup_{i \in Z_k} B(x_k^i, \rho_k^i)} f\left(\frac{x}{\delta_k}, \nabla v_k(x)\right) dx &= \liminf_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla w_k(x)\right) dx \\ &\geq \int_\Omega f_{\text{hom}}(\nabla u(x))\, dx. \end{aligned} \tag{50}$$

To estimate the second contribution in (48), fix $i \in Z_k$ and let φ_k^i be a function solving

$$\min\Big\{ \int_{B(x_k^i, \rho_k^i)} f\left(\frac{x}{\delta_k}, \nabla \zeta(x)\right) dx : \zeta \in u_k^i + W_0^{1,d}(B(x_k^i, \rho_k^i)), \zeta = 0 \text{ on } B(x_k^i, \varepsilon_k) \Big\}.$$

Up to extending the function φ_k^i to the constant u_k^i on $B(x_k^i, d_k/2) \setminus B(x_k^i, \rho_k^i)$, we have

$$\int_{B(x_k^i, \rho_k^i)} f\left(\frac{x}{\delta_k}, \nabla v_k(x)\right) dx \geq \int_{B(x_k^i, \rho_k^i)} f\left(\frac{x}{\delta_k}, \nabla \varphi_k^i(x)\right) dx$$

$$\geq \min\Big\{\int_{B\left(x_k^i,\frac{d_k}{2}\right)} f\left(\frac{x}{\delta_k},\nabla\zeta\right)\,dx : \zeta\in u_k^i+W_0^{1,d}(B(x_k^i,d_k/2)), \zeta=0 \text{ on } B(x_k^i,\varepsilon_k)\Big\}$$

$$= \min\Big\{\int_{B\left(0,\frac{1}{2}\right)} f\left(\frac{d_k x}{\delta_k},\nabla\zeta\right)\,dx : \zeta\in 1+W_0^{1,d}(B(0,1/2),\zeta=0 \text{ on } B(0,\varepsilon_k/d_k)\Big\}|u_k^i|^d,$$

where the last equality follows by the change of variables $x:=(y-x_k^i)/d_k$, the identity (45) and the combined application of the transformation

$$\zeta(x)\mapsto\frac{\zeta(x)}{u_k^i}$$

with (H), assuming without loss of generality that u_k^i is different from 0.
Now put

$$\delta_k':=\frac{\delta_k}{d_k}\,,\qquad \varepsilon_k':=\frac{\varepsilon_k}{d_k}\,,\qquad \lambda':=\lim_k\frac{|\log\delta_k'|}{|\log\varepsilon_k'|},$$

and rewrite the previous inequality as

$$\int_{B(x_k^i,\rho_k^i)} f\left(\frac{x}{\delta_k},\nabla v_k(x)\right)\,dx$$
$$\geq\min\Big\{\int_{B\left(0,\frac{1}{2}\right)} f\Big(\frac{x}{\delta_k'},\nabla\zeta\Big)dx\colon \zeta\in 1+W_0^{1,d}\Big(B\Big(0,\frac{1}{2}\Big)\Big),\zeta=0 \text{ on } B(0,\varepsilon_k')\Big\}|u_k^i|^d. \quad (51)$$

Note that $\varepsilon_k'=\varepsilon_k|\log\varepsilon_k|^{1-1/d}\to 0$, while $\delta_k'=\delta_k|\log\varepsilon_k|^{1-1/d}\to 0$ as $k\to\infty$ by assumption (44); also observe that

$$\lambda'=\lim_k\frac{|\log\delta_k+\log|\log\varepsilon_k|^{1-1/d}|}{|\log\varepsilon_k+\log|\log\varepsilon_k|^{1-1/d}|}=\lim_k\frac{|\log\delta_k|}{|\log\varepsilon_k|}=\lambda.$$

In light of the assumption (46), we are in position to apply Proposition 2.1 (up to the transformation $u\mapsto 1-u$) to (51) with $\Omega=B(0,1/2)$ and $z_\varepsilon=0$ for every ε; we get

$$\min\Big\{\int_{B(0,1/2)} f\left(\frac{x}{\delta_k'},\nabla\zeta(x)\right)\,dx : \zeta\in 1+W_0^{1,d}(B(0,1/2)),\zeta=0 \text{ on } B(0,\varepsilon_k')\Big\}$$
$$=\frac{C(\lambda)+o_k(1)}{|\log\varepsilon_k'|^{d-1}}=\frac{C(\lambda)+o_k(1)}{|\log\varepsilon_k|^{d-1}}.$$

Summing over k and applying Proposition 3.2, we conclude that

$$\begin{aligned}\liminf_k\sum_{i\in Z_k}\int_{B(x_k^i,\rho_k^i)} f\left(\frac{x}{\delta_k},\nabla v_k(x)\right)\,dx &\geq \liminf_k\frac{C(\lambda)}{|\log\varepsilon_k|^{d-1}}\sum_{i\in Z_k}|u_k^i|^d+o_k(1)\\ &=C(\lambda)\int_\Omega|u(x)|^d\,dx\,.\end{aligned}\quad (52)$$

Finally, by (49), (50) and (52), we deduce

$$\left(1+\frac{C}{M-1}\right)\liminf_k F_k(u_k) \geq \liminf_k F_k(v_k) \geq \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,dx + C(\lambda)\int_\Omega |u(x)|^d\,dx.$$

Recall that θ and M have been chosen so that $\theta 2^{M+1} < 1/2$ and, since the reasoning leading to the above estimate holds true for every $\theta > 0$, we may let $M \to +\infty$ getting the liminf inequality.

We conclude removing the boundedness assumption on $(u_k)_k \subseteq L^\infty(\Omega)$ by a truncation argument: put $u^T := ((-T) \vee u) \wedge T$ for fixed $T \in \mathbb{N}$ and assume that $u_k \to u$ in $L^d(\Omega)$. Since $f(\cdot, 0) = 0$, it holds

$$\int_\Omega f\left(\frac{x}{\delta_k}, \nabla u_k\right)\,dx \geq \int_\Omega f\left(\frac{x}{\delta_k}, \nabla u_k^T\right)\,dx$$

for every $k, T \in \mathbb{N}$; hence, by the previous instance we have

$$\begin{aligned}\liminf_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla u_k\right)\,dx &\geq \liminf_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla u_k^T\right)\,dx \\ &\geq \int_\Omega f_{\mathrm{hom}}(\nabla u^T(x))\,dx + C(\lambda)\int_\Omega |u^T(x)|^d\,dx.\end{aligned}$$

for every $T \in \mathbb{N}$. Since $u^T \to u$ in $W^{1,d}(\Omega)$ as $T \to +\infty$, we conclude by dominated convergence and the continuity of f_{hom}.

3.3 Limsup inequality

The goal of this section is to define a recovery sequence converging in $L^d(\Omega)$ to a fixed function $u \in W^{1,d}(\Omega)$. First we assume that $u \in L^\infty(\Omega)$.

Start by a recovery sequence $u_k \to u$ in $L^d(\Omega)$ related to the functionals

$$F_k^0(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\delta_k}, \nabla u(x)\right)\,dx & \text{if } u \in W^{1,d}(\Omega), \\ +\infty & \text{if } u \in L^d(\Omega) \setminus W^{1,d}(\Omega) \end{cases}$$

which are known to Γ-converge to

$$F^0(u) := \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\,dx$$

for every $u \in W^{1,d}(\Omega)$ as stated in the already used homogenization theorem. By the equi-coerciveness of the functionals $(F_k^0)_k$, we also deduce that $u_k \rightharpoonup u$ in $W^{1,d}(\Omega)$. It is a known fact that, up to extract a subsequence, we can further assume that $(|\nabla u_k|^d)_k$ is an equi-integrable family (see [13] and [6, Remark C.6]).

We claim that we can make our recovery sequence bounded in $L^\infty(\Omega)$. Let $T := \|u\|_{L^\infty(\Omega)}$ and define $u'_k := (-(T+1)\vee u_k)\wedge(T+1)$. We get a bounded sequence in $L^\infty(\Omega)$ which converges to u in $L^d(\Omega)$ with the property that $(|\nabla u'_k|^d)_k$ is still equi-integrable as it is obtained by truncation.
Note that

$$\begin{aligned}\left|\int_\Omega f_{\rm hom}(\nabla u_k(x))\,dx - \int_\Omega f_{\rm hom}(\nabla u'_k(x))\,dx\right| &\le \int_{\{|u_k|>T+1\}} |f_{\rm hom}(\nabla u_k(x))|\,dx\\ &\le \beta\int_{\{|u_k|>T+1\}} |\nabla u_k(x))|^d\,dx \le \beta\int_{\{|u_k-u|>1\}} |\nabla u_k(x))|^d\,dx\,;\end{aligned}$$

but since $u_k \to u$ in measure and $(|\nabla u_k|^d)_k$ is equi-intergable, the last term tends to 0 and the claim is proved.

For every k, define modifications v_k by transformations around every point x_k^i with $i \in Z_k$ as we did in (47). We recall the construction for clarity: fix $M \in \mathbb{N}$ and let $\theta > 0$ be such that $\theta 2^{M+1} < 1/2$, then apply Lemma 2.2 with

$$f(x,\xi) = f\left(\frac{x}{\delta},\xi\right),\ \eta = \theta d_k,\ R = \theta 2^{M+1} d_k,\ N = M \text{ and } r = \theta d_k.$$

We have that

$$\int_\Omega f\left(\frac{x}{\delta_k},\nabla v_k(x)\right)\,dx \le \left(1+\frac{C}{M-1}\right)\int_\Omega f\left(\frac{x}{\delta_k},\nabla u_k(x)\right)\,dx\,, \tag{53}$$

and that the function v_k attains the constant value u_k^i on $\partial B(x_k^i,\rho_k^i)$, where ρ_k^i is of the form $\theta 2^{j_i} d_k$ for some $j_i \in \{1,...,M\}$ and $i \in Z_k$.
Since $\varepsilon_k/d_k \to 0$ as $k \to +\infty$, we can also assume $\varepsilon_k < \theta d_k$ for every k; hence, we define

$$w_k := \begin{cases} v_k & \text{on } \Omega\setminus\bigcup_{i\in Z_k} B(x_k^i,\rho_k^i)\\ u_k^i & \text{on } B(x_k^i,\rho_k^i)\setminus B(x_k^i,\theta d_k), i\in Z_k\\ \varphi_k^i & \text{on } B(x_k^i,\theta d_k),\ i \in Z_k,\end{cases}$$

where φ_k^i solves the minimum problem

$$\begin{aligned}&\min\Big\{\int_{B(x_k^i,\theta d_k)} f\left(\frac{x}{\delta_k},\nabla\zeta(x)\right)\,dx : \zeta \in u_k^i + W_0^{1,d}(B(x_k^i,\theta d_k)), \zeta = 0 \text{ on } B(x_k^i,\varepsilon_k)\Big\}\\ &= \min\Big\{\int_{B(0,\theta)} f\left(\frac{x}{\delta'_k},\nabla\zeta(x)\right)\,dx : \zeta \in 1 + W_0^{1,d}(B(0,\theta)), \zeta = 0 \text{ on } B(0,\varepsilon'_k)\Big\}|u_k^i|^d\\ &\qquad = \frac{C(\lambda)+o_k(1)}{|\log\varepsilon_k|^{d-1}}|u_k^i|^d\,,\end{aligned}$$

and the last equality follows by Proposition 2.1 applied to $\Omega = B(0,\theta)$ and $z_\varepsilon = 0$ for every ε.
Introduce the set of indices

$$Z'_k := \{i \in \mathbb{Z}^d : B(x^i_k, \varepsilon_k) \cap \Omega \neq \emptyset, i \notin Z_k\},$$

define radii

$$r_k := \theta 2^{M+1} d_k,$$

and for every $i \in Z'_k$, let ψ^i_k be the solution to the homogeneous capacitary problem

$$\min\Big\{\int_{B(x^i_k, r_k)} |\nabla\zeta(x)|^d \, dx : \zeta \in 1 + W^{1,d}_0(B(x^i_k, r_k)), \zeta = 0 \text{ on } B(x^i_k, \varepsilon_k)\Big\}$$

which is (known to be) equal to $\sigma_{d-1}|\log r_k - \log \varepsilon_k|^{1-d}$.
Up to extending ψ^i_k with value 1 on $\mathbb{R}^d \setminus B(x^i_k, r_k)$, we set as recovery sequence

$$w'_k := w_k \prod_{i \in Z'_k} \psi^i_k \qquad \text{on } \Omega;$$

then we put

$$A_k := \bigcup_{i \in Z_k} B(x^i_k, \rho^i_k),$$

and

$$A'_k := \bigcup_{i \in Z'_k} B(x^i_k, r_k).$$

It holds

$$\begin{aligned}
\limsup_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right) dx \leq \quad & \limsup_k \int_{A_k} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right) dx && (54)\\
+ \quad & \limsup_k \int_{\Omega \cap A'_k} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right) dx && (55)\\
+ \quad & \limsup_k \int_{\Omega \setminus (A_k \cup A'_k)} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right) dx. && (56)
\end{aligned}$$

We estimate (54) using Proposition 3.2,

$$\begin{aligned}
\limsup_k \int_{A_k} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right) dx &= \limsup_k \sum_{i \in Z_k} \int_{B(x^i_k, \theta d_k)} f\left(\frac{x}{\delta_k}, \nabla \varphi^i_k(x)\right) dx \\
&= \limsup_k \sum_{i \in Z_k} |u^i_k|^d \frac{C(\lambda) + o_k(1)}{|\log \varepsilon_k|^{d-1}} \\
&= C(\lambda) \int_\Omega |u(x)|^d \, dx. \qquad (57)
\end{aligned}$$

To estimate (55), we set

$$Q_k^i := x_k^i + \left(-\frac{d_k}{2}, \frac{d_k}{2}\right),$$

and we preliminarily see that

$$|\Omega \cap A_k'| \le \sum_{i \in Z_k'} (r_k)^d \le \# Z_k' (d_k)^d = \left| \bigcup_{i \in Z_k'} Q_k^i \right| \to |\partial \Omega| = 0 \tag{58}$$

by assumption.
Now we prove that

$$\limsup_k \int_{\Omega \cap A_k'} f\left(\frac{x}{\delta_k}, \nabla w_k'(x)\right) dx = 0. \tag{59}$$

For every $i \in Z_k'$, we have

$$\begin{aligned}
\int_{\Omega \cap B(x_k^i, r_k)} f\left(\frac{x}{\delta_k}, \nabla w_k'(x)\right) &\le \beta \int_{\Omega \cap B(x_k^i, r_k)} |\nabla w_k'(x)|^d \, dx \\
&= \beta \int_{\Omega \cap B(x_k^i, r_k)} |\nabla (w_k \psi_k^i)(x)|^d \, dx \\
&= \beta \int_{\Omega \cap B(x_k^i, r_k)} |\psi_k^i(x) \nabla w_k(x) + w_k(x) \nabla \psi_k^i(x)|^d \, dx \\
&\le 2^{d-1} \beta (1 + \|u\|_{L^\infty(\Omega)})^d \left[\int_{B(x_k^i, r_k)} |\nabla \psi_k^i(x)|^d \, dx + \int_{\Omega \cap B(x_k^i, r_k)} |\nabla w_k(x)|^d \, dx \right] \\
&\le C \left[|\log r_k - \log \varepsilon_k|^{1-d} + \int_{\Omega \cap B(x_k^i, r_k)} |\nabla w_k(x)|^d \, dx \right]
\end{aligned}$$

for a positive constant C which depends only on $\|u\|_{L^\infty(\Omega)}, \beta$ and the dimension d.
Note that, since $i \in Z_k'$, by definition of w_k we have

$$\int_{\Omega \cap B(x_k^i, r_k)} |\nabla w_k(x)|^d \, dx = \int_{\Omega \cap B(x_k^i, r_k)} |\nabla v_k(x)|^d \, dx \,,$$

and by the property (ii) of Lemma 2.2, which ensures that the modifications on the starting function occur very close to the prescribed radius, it also holds

$$\int_{\Omega \cap B(x_k^i, r_k)} |\nabla v_k(x)|^d \, dx = \int_{\Omega \cap B(x_k^i, r_k)} |\nabla u_k(x)|^d \, dx.$$

Exploiting the equi-integrability of $(|\nabla u_k|^d)_k$, by (58) we infer that

$$\limsup_k \sum_{i \in Z_k'} \int_{\Omega \cap B(x_k^i, r_k)} |\nabla w_k(x)|^d \, dx = 0,$$

and then

$$\limsup_k \int_{\Omega\cap A'_k} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right)\, dx \le C \limsup_k \sum_{i\in Z'_k} |\log r_k - \log \varepsilon_k|^{1-d};$$

but since $|\log r_k| \ll |\log \varepsilon_k|$, we conclude that

$$\begin{aligned}\limsup_k \sum_{i\in Z'_k} |\log r_k - \log \varepsilon_k|^{1-d} &= \limsup_k \sum_{i\in Z'_k} |\log \varepsilon_k|^{1-d} \\ &= \limsup_k \# Z'_k (d_k)^d = 0\end{aligned}$$

again by (58).
Finally, we deal with (56) taking advantage of (53); it holds

$$\begin{aligned}\limsup_k \int_{\Omega\setminus(A_k\cup A'_k)} f\left(\frac{x}{\delta_k}, \nabla w'_k(x)\right)\, dx &= \limsup_k \int_{\Omega\setminus(A_k\cup A'_k)} f\left(\frac{x}{\delta_k}, \nabla v_k(x)\right)\, dx \\ &\le \limsup_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla v_k(x)\right)\, dx \\ &\le \left(1+\frac{C}{M-1}\right) \limsup_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla u_k(x)\right)\, dx \\ &\le \left(1+\frac{C}{M-1}\right) \int_\Omega f_{\mathrm{hom}}(\nabla u(x))\, dx,\end{aligned} \tag{60}$$

where the last inequality is due to the fact that $(u_k)_k$ was originally picked as a recovery sequence to u for the functionals $(F^0_k)_k$.

Gathering (57), (59) and (60), we get

$$\limsup_k \int_\Omega f\left(\frac{x}{\delta_k}, \nabla w'_k\right) dx \le \left(1+\frac{C}{M-1}\right) \int_\Omega f_{\mathrm{hom}}(\nabla u) dx + C(\lambda) \int_\Omega |u|^d dx,$$

and since we can repeat the argument for every $\theta > 0$, we are free to set M arbitrarily large completing the proof of the (approximate) limsup inequality.

We still have to check that $w'_k \to u$ in $L^d(\Omega)$, i.e., it actually is a (approximate) recovery sequence.
Note that $\lim_k |\{w'_k \neq w_k\}| = 0$ and $\sup_k \|w'_k - w_k\|_{L^\infty(\Omega)} \le \|u_k\|_{L^\infty(\Omega)} \le 1 + \|u\|_{L^\infty(\Omega)}$ imply that $w'_k - w_k \to 0$ in $L^d(\Omega)$, hence, it suffices to prove that $w_k \to u$ in $L^d(\Omega)$.
Since $\lim_k |\{w_k \neq v_k\}| = 0$ and $\sup_k \|w_k - v_k\|_{L^\infty(\Omega)} \le \|u_k\|_{L^\infty(\Omega)} \le 1 + \|u\|_{L^\infty(\Omega)}$, it holds that $w_k - v_k \to 0$ in $L^d(\Omega)$, moreover $v_k \to u$ in $L^d(\Omega)$ by the same argument we used in the proof of the liminf inequality based on Lemma 3.3; hence, $w_k \to u$ in $L^d(\Omega)$.

To conclude, we remove the assumption $u \in L^\infty(\Omega)$. Recall that the Γ-limsup of $(F_k)_k$ is defined as

$$F''(u) := \inf\{\limsup_k F_k(u_k) : u_k \to u \in L^d(\Omega)\}$$

for every $u \in W^{1,d}(\Omega)$. F'' is sequentially lower semicontinuous with respect to the strong convergence in $L^d(\Omega)$ and by what we have already shown, it coincides with F on $W^{1,d}(\Omega) \cap L^\infty(\Omega)$.
Hence, given a sequence $(u_k)_k \subseteq W^{1,d}(\Omega) \cap L^\infty(\Omega)$ converging to u in $W^{1,d}(\Omega)$, it holds

$$F''(u) \leq \liminf_k F''(u_k) = \liminf_k F(u_k) = F(u)$$

by the continuity of F with respect to the strong convergence in $W^{1,d}(\Omega)$, and this concludes the proof of the Γ-convergence.

Remark 3.4. We comment on dropping the integer condition $d_k/\delta_k \in \mathbb{N}$. As mentioned in the introduction, the Γ-limit may not exist; we exhibit an instance in which a limit exists and is explicitly computed along proper subsequences. We focus on the computation of the strange term coming from the behaviour near the perforations.
We introduce the function C^λ defined at the point z by

$$\lim_{\varepsilon \to 0} |\log \varepsilon|^{d-1} \min\Big\{ \int_B f\left(z + \frac{x}{\delta(\varepsilon)}, \nabla\zeta(x)\right) dx : u \in W^{1,d}_0(B), u = 1 \text{ on } B(0,\varepsilon)\Big\},$$

so that the computation of the strange term reduces to studying the limit as $k \to \infty$ of

$$\sum_{i \in Z_k} \min\Big\{ \int_{B\left(0,\frac{1}{2}\right)} f\left(\frac{x_k^i + d_k x}{\delta_k}, \nabla\zeta\right) dx : \\ \zeta \in 1 + W^{1,d}_0(B(0,1/2), \zeta = 0 \text{ on } B(0,\varepsilon_k/d_k)\Big\} |u_k^i|^d = \sum_{i \in Z_k} |u_k^i|^d C^\lambda\Big(\frac{x_k^i}{\delta_k}\Big). \tag{61}$$

Assuming

$$d_k = \frac{m_k}{T}\delta_k$$

with $m_k \in \mathbb{N}$ prime and $T \in \mathbb{N}$, we have

$$\frac{x_k^i}{\delta_k} = z + \frac{h}{T} \quad \text{for some } z \in \mathbb{Z}^d,$$

and since C^λ is 1-periodic, as we put

$$I_h = \frac{h}{T} + \mathbb{Z}^d \cap Z_k,$$

we obtain that (61) equals

$$\sum_{h\in\{0,...,T-1\}^d}\sum_{i\in I_h}|u_k^i|^d C^\lambda\Big(\frac{h}{T}\Big).$$

We observe that, for fixed $h\in\{0,...,T-1\}^d$, the sequence

$$\sum_{i\in I_h} C^\lambda\Big(\frac{h}{T}\Big)\chi_{Q_k^i}(x),\qquad k\in\mathbb{N}$$

is (Td_k)-periodic, hence it weakly* converges to its mean value on the unit cube

$$\frac{1}{T^d}C^\lambda\Big(\frac{h}{T}\Big)$$

in $L^\infty(\mathbb{R}^d)$. Combining this fact with Proposition 3.2, we deduce

$$\lim_{k\to\infty}\sum_{i\in Z_k}|u_k^i|^d C^\lambda\Big(\frac{x_k^i}{\delta_k}\Big)=C\int_\Omega|u(x)|^d$$

with

$$C=\frac{1}{T^d}\sum_{h\in\{0,...,T-1\}^d}C^\lambda\Big(\frac{h}{T}\Big).$$

Acknowledgments

The author is grateful to Andrea Braides for having suggested the problem and for helpful discussions about the subject.

References

[1] N. Ansini and A. Braides. Separation of scales and almost-periodic effects in the asymptotic behaviour of perforated periodic media. *Acta Appl. Math.* **65** (2001), 59–81.

[2] N. Ansini and A. Braides. Asymptotic analysis of periodically perforated nonlinear media. *J. Math. Pures Appl.* **81** (2002), 439–451.

[3] X. Blanc and C. Le Bris. *Homogénéisation en milieu périodique... ou non: une introduction*. Springer, 2022.

[4] A. Braides. Γ-*convergence for Beginners*. Oxford University Press, Oxford, 2002.

[5] A. Braides and G. C. Brusca. Asymptotic behaviour of the capacity in two-dimensional heterogeneous media. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* (2022).

[6] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals.* Oxford University Press, Oxford, 1998.

[7] D. Cioranescu, A. Damlamian and G. Griso. *The Periodic Unfolding Method.* Springer, 2018.

[8] C. Calvo-Jurado and J. Casado-Díaz. The limit of Dirichlet systems for variable monotone operators in general perforated domains. *J. Math. Pures Appl.* **81** (2002), 471–493.

[9] D. Cioranescu and F. Murat. Un term étrange venu d'ailleurs, I and II. *Nonlinear Partial Differential Equations and Their Applications.* Collège de France Seminar. Vol. II, 98-138, and Vol. III, 154-178, Res. Notes in Math., 60 and 70, Pitman, London, 1982 and 1983, translated in: A strange term coming from nowhere. *Topics in the Mathematical Modelling of Composite Materials* (Cherkaev, A.V. and Kohn, R.V. eds.), Birkhäuser, Boston, 1994.

[10] C. Conca, F. Murat, C. Timofte. A generalized strange term in Signorini's type problem. *ESAIM: Mathematical Modelling and Numerical Analysis* **37** (2003), 773–805,

[11] G. Dal Maso. *An Introduction to Γ-convergence.* Birkhäuser, Boston, 1993.

[12] G. Dal Maso and F. Murat. Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **24** (1997), 239–290.

[13] I. Fonseca, S. Müller and P. Pedregal. Analysis of concentration and oscillation effects generated by gradients. *SIAM J. Math. Anal.* **29** (1998), 736–756.

[14] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals.* Springer-Verlag, Berlin, 1994.

[15] C. Le Bris, F. Legoll, and A. Lozinski. An MsFEM Type Approach for Perforated Domains. *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal* **12** (2014), 1046–1077.

[16] A.V. Marchenko and E.Ya. Khruslov, *Boundary Value Problems in Domains with Fine-Granulated Boundaries* (in Russian), Naukova Dumka, Kiev, 1974.

[17] L. Sigalotti. Asymptotic analysis of periodically perforated nonlinear media at the critical exponent. *Comm. Cont. Math.* **11** (2009), 1009–1033.