FUNDAMENTAL GROUPS AND THE MILNOR CONJECTURE

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ABSTRACT. It was conjectured by Milnor in 1968 that the fundamental group of a complete manifold with nonnegative Ricci curvature is finitely generated. The main result of this paper is a counterexample, which provides an example M^7 with Ric ≥ 0 such that $\pi_1(M) = \mathbb{Q}/\mathbb{Z}$ is infinitely generated.

There are several new points behind the result. The first is a new topological construction for building manifolds with infinitely generated fundamental groups, which can be interpreted as a smooth version of the fractal snowflake. The ability to build such a fractal structure will rely on a very twisted gluing mechanism. Thus the other new point is a careful analysis of the mapping class group $\pi_0\mathrm{Diff}(S^3 \times S^3)$ and its relationship to Ricci curvature. In particular, a key point will be to show that the action of $\pi_0\mathrm{Diff}(S^3 \times S^3)$ on the standard metric $g_{S^3 \times S^3}$ lives in a path connected component of the space of metrics with Ric > 0.

CONTENTS

1. Introduction

The study of the structure of the fundamental group $\pi_1(M)$ of a manifold with lower Ricci curvature bounds has received a good deal of attention, and at this point its structural properties are very well understood. Before discussing the results of this paper let us focus for a moment on some background about what is known.

One of the earliest results in the analysis of spaces with lower Ricci curvature bounds is by John Milnor [Mi]. Milnor used an early version of volume comparison by Bishop [Bi] in order to show that if M^n has nonnegative Ricci curvature, then any finitely generated subgroup of the fundamental group $\pi_1(M)$ has polynomial growth. These results led Milnor to conjecture that the fundamental group need automatically be finitely generated.

The importance of the polynomial growth condition to the inherent structure of a group became clear when Gromov [Gr2] proved that any finitely generated polynomial growth group must be almost nilpotent, that is, must have a nilpotent subgroup of finite index. Combining this with Milnor's result we see that any finitely generated subgroup of $\pi_1(M)$, when M has nonnegative Ricci curvature, is itself almost nilpotent. Wilking [Wi] gives a form of converse to this statement, where by building on the work of Wei [We] he can show that for any finitely generated almost nilpotent group there exists a manifold with nonnegative Ricci curvature which has this group as its fundamental group.

In the context of lower sectional curvature one could do more. Gromov proved in [Gr1] that the local fundamental group [1] is always generated by a uniformly finite number of generators. This gave the first real hints toward finite generation. The next major breakthrough on relating the structure of the fundamental group with geometry came from Fukaya and Yamaguchi [FG]. They proved that on a space with lower sectional curvature bounds the local fundamental group is almost nilpotent. This influential work gave the first real structure theory for the fundamental group. A subtle point in their work is that the index of the nilpotent subgroup of the local fundamental group was not uniformly controlled. This point was resolved in the work of Kapovitch, Petrunin and Tuschmann [KPT]. Fukaya and Yamaguchi went on to conjecture in [FG] that in the nonnegative sectional context a manifold should have almost abelian fundamental group, not just almost nilpotent. An interesting example of Wei [We] shows this conjecture cannot hold for manifolds with nonnegative Ricci curvature, though the conjecture remains open for spaces with nonnegative sectional curvature.

The results and techniques of Fukaya and Yamaguchi were extended to the context of lower Ricci bounds by Kapovitch and Wilking [KW]. Among the important applications of this extension was to understand that for a manifold with nonnegative Ricci curvature, a finitely generated subgroup of the fundamental group has a dimensionally bounded number of generators. A result by Colding and Naber [CN1] proves that the isometry group of a limit of spaces with lower Ricci curvature bounds is a Lie group, and combining this with their structure, Kapovitch and Wilking [KW] are able to give a fairly comprehensive understanding of the fundamental group in the compact case. In [Wi] Wilking was able to show how a counterexample to the Milnor conjecture must arise from an abelian action.

In low dimensions the Milnor conjecture has been resolved. At its heart this is because one can prove much stronger rigidities in these contexts, and control much more than just the fundamental group. In dimension two Cohn-Vossen [CV] proved that if M^2 satisfies Ric ≥ 0 and is noncompact, then M is flat or diffeomorphic to $\mathbb{R}^2$. In particular, that M^2 has finitely generated fundamental group is an easy consequence. In dimension three the first major result was by Schoen-Yau [SY], where they proved that if Ric > 0 for a noncompact M^3, then it is diffeomorphic to $\mathbb{R}^3$. Their proof was unique in comparison to the techniques used in other papers being cited, and relied heavily on minimal surface theory. Their program was expanded on by Liu [Liu], who was able to prove that if M^3 satisfies Ric ≥ 0 then M^3 is either diffeomorphic to $\mathbb{R}^3$ or its universal cover isometrically splits. The Milnor conjecture is again an easy consequence in this context. Recently Pan [Pa1] has given a distinct proof in the three dimensional case.

In addition to the broad points of progress mentioned above, let us also mention some of the more specific lines of attack which have had success over the years. The most rigid result is in the completely noncollapsed case, that is when $\mathrm{Vol}(B_r(p)) \geq vr^n$ for all large r. In this case Li [Li] showed that the fundamental group is uniformly finite. Anderson [A] generalized this to show that if $b_1(M) \geq k$ and $\mathrm{Vol}(B_r(p)) \geq vr^{n-k}$, then again M^n has finitely generated fundamental group. On the opposite end of rigidity, Sormani [So1] studied manifolds with minimal growth. In particular, if a space satisfies small diameter growth diam $\partial B_r \leq \epsilon(n)r$ for all large r, then she showed that the fundamental group of M^n is finitely generated. More recently,

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[1]The image $\pi_1(B_{\epsilon(n)}(p)) \to \pi_1(B_1(p))$.

Pan [Pa2] has extended these techniques in order to show that if M^n has a unique metric tangent cone at infinity, then the Milnor conjecture holds and M^n has finitely generated fundamental group. See also [So2, SW1, SW2, Wu, Pa3, PW, Wa], for many other interesting directions and related results.

1.1. **Main Results on Fundamental Groups.** The results of Gromov [Gr1], Fukaya-Yamaguchi [FG], Kapovitch-Wilking [KW] and Wilking [Wi] thus tell us that the fundamental group $\pi_1(M)$ of a manifold with nonnegative Ricci curvature is well understood, and locally it is uniformly finitely generated. In particular, even if $\pi_1(M)$ were infinitely generated then necessarily all finitely generated subgroups are $C(n)$-uniformly finitely generated. The first main result of this paper is to build such an example, and in particular we can take the fundamental group to be the rationals:

Theorem 1.1 (Infinitely Generated Fundamental Group)**.** *Let* $\Gamma \leq \mathbb{Q}/\mathbb{Z} \subseteq S^1$ *be any subgroup. Then there exists a smooth complete manifold* (M^7, g) *with* $\pi_1(M) = \Gamma$ *and such that* $Ric \geq 0$.

We will outline the constructions in Sections 2 and 3 more carefully, however let us begin with a very rough picture of the space and its properties. There are several topological methods to build spaces with infinitely generated fundamental groups, with the dyadic solenoid complement being a geometrically popular method. The constructions of this paper are quite distinct.

We will not directly build M, instead we will focus on constructing the universal cover $\tilde{M}$ with the appropriate group action by Γ. The overall structure of $\tilde{M}$, with respect to a basepoint $\tilde{p} \in \tilde{M}$, will in many ways mimic that of a fractal snowflake, see Section 2. The ability to build such a fractal structure will rely on a very twisted gluing mechanism. As we move up in scales we can study the local group $\Gamma_r \equiv \langle \gamma : d(\tilde{p}, \gamma \cdot \tilde{p}) \leq r \rangle \leq \Gamma$, which will jump one generator at a time at scales r_j with $\Gamma_j \equiv \Gamma_{r_j} = \langle \gamma_j, \Gamma_{j-1} \rangle$. Note that the local group will always be generated by a single action, what jumps is what this generator will be. At the scales r_j when the local group increases the space will look very close to $S^3 \times \mathbb{R}^4$ with the generating γ_j acting by a rotation on both the S^3 factor and the $\mathbb{R}^4$ factor.

A major subtlety of the construction of $\tilde{M}$ will occur between two of the scales r_j and r_{j+1}. At the bottom scale the generating γ_j action will rotate both the $\mathbb{R}^4$ factor and the S^3 factor, while at the top scale the same γ_j only rotates the S^3 factor. Geometrically the space may look like $S^3 \times \mathbb{R}^4$ at both the r_j and r_{j+1} scales, however one should view these two copies of $S^3 \times \mathbb{R}^4$ quite distinctly. In particular, the two 3-spheres in $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3)$ will necessarily mix together in order to change the behavior of the action. We will see this behavior is closely connected to the mapping class group of $S^3 \times S^3$.

As this point is of some independent interest it is worth discussing it briefly, we refer the reader to Section 9 for a more in depth discussion. Let $\mathcal{M}_0(S^3 \times S^3) \equiv \left\{ [g] : g \sim \phi^* g : \phi \in \mathrm{Diff}_0(S^3 \times S^3) \right\}$ represent the space of smooth Riemannian metrics modulo diffeomorphisms which are isotopic to the identity. From the perspective of gluing and topology a diffeomorphism which is not isotopic to the identity is a highly twisted object, and thus it is good to distinguish between those which are and are not connected to the identity by a

continuous path. We can let $\mathcal{M}_0^+(S^3 \times S^3) \equiv \{[g] \in \mathcal{M}_0 : \mathrm{Ric} > 0\}$ be the subset of metrics with positive Ricci curvature. Note that there is a canonical action of the mapping class group $\pi_0\mathrm{Diff}(S^3 \times S^3)$ on these spaces given by $[\phi] \cdot [g] = [\phi^*g]$. One of the main technical lemmas of this paper is that this action of the mapping class group $\pi_0\mathrm{Diff}(S^3 \times S^3)$ on the standard metric $g_{S^3 \times S^3}$ lives in a connected component of $\mathcal{M}_0^+(S^3 \times S^3)$:

Lemma 1.2 (Mapping Class Group and Ricci Curvature on $S^3 \times S^3$). *Let $g_0 = g_{S^3 \times S^3}$ be the standard metric on $S^3 \times S^3$. Then given $\phi \in \mathrm{Diff}(S^3 \times S^3)$ there exists a smooth family g_t of metrics with $\mathrm{Ric}_{g_t} > 0$ such that g_0 is the standard metric and $g_1 = \phi^*g_0$. That is, the orbit $\pi_0\mathrm{Diff}(S^3 \times S^3) \cdot [g_{S^3 \times S^3}]$ of the mapping class group lives in a connected component of $\mathcal{M}_0^+(S^3 \times S^3)$, the space of metrics with strictly positive Ricci curvature.*

Remark 1.1. Observe that if $\phi \in \mathrm{Diff}_0(S^3 \times S^3)$ is isotopic to the identity then the above is trivial as one can take $g_t = \phi_t^* g_{S^3 \times S^3}$ to all differ from the standard metric by diffeomorphisms. If $[\phi] \in \pi_0\mathrm{Diff}(S^3 \times S^3)$ is not the trivial element, the above is of course much more subtle.

An equivariant version of the above will be one of the driving mechanisms allowing us to untwist our actions as the scale increases and slide Euclidean rotations of $S^3 \times \mathbb{R}^4$ at scale r_j to spherical rotations of $S^3 \times \mathbb{R}^4$ at scale r_{j+1}. This process will be described in detail in the next Sections.

Topologically we will have that our counterexample $\tilde{M}$ is 2-connected, but that $H_3(\tilde{M})$ is also infinitely generated. We will essentially add one new $H_3(\tilde{M})$ generator each time the local group Γ_r increases in size. That is, every time we stick in more fundamental group we will need some compensating three spheres.

Geometrically we will have at large scales that M typically looks like a cone over a lens space $\approx C(S_s^3/\mathbb{Z}_k)$ for some sphere size $s \le 1$ and $k \in \mathbb{N}$. As the scale increases the size of spheres S_s^3 will decrease until M is close to a ray, and when the ray opens again M will become close to a potentially different lens space $\approx C(S_s^3/\mathbb{Z}_{k'})$. This process will repeat indefinitely, and in the case of a $\mathbb{Q}/\mathbb{Z}$-fundamental group one can arrange it so that a cone over every possible lens space occurs infinitely often. In particular, the tangent cones of M at infinity will include $C(S_s^3/\mathbb{Z}_k)$ for every choice of $k \in \mathbb{N}$ and $0 \le s \le 1$. It is important to note that the basepoint for the tangent cone at infinity may not always be the cone point itself. In addition to these lens space tangents, by blowing up at the scale of the actions when $k \to \infty$ we will also see tangent cones at infinity of the form $\mathbb{R}^3 \times S^1$.

We are left with the following open question:

Question 1.1. If M^n satisfies $\mathrm{Ric} \ge 0$ with $n = 4, 5$, or 6, then is $\pi_1(M)$ finitely generated?

The techniques of this paper need to be extended to work in lowest dimensions, and so the above are important open questions. Additionally, our examples are quite collapsed in nature. The issue of finite generation is still open in the noncollapsed setting:

Question 1.2. If (M^n, g, p) satisfies $\mathrm{Ric} \ge 0$ with the universal cover $\tilde{M}$ noncollapsed, i.e. $\mathrm{Vol}(B_r(\tilde{p})) \ge vr^n > 0$ for all $r > 0$, then is $\pi_1(M)$ finitely generated?

2. Geometric and Topological Outline for Theorem 1.1

The focus of this Section is to describe in broad strokes the Example of Theorem 1.1. In particular, we will begin in Section 2.1 by outlining the main topological ingredients in the construction, and then in Section 2.2 we want to describe the large scale geometry of the construction. Both of these discussions are meant to help draw an intuitive picture as a preamble to the careful construction given in the next Section.

2.0.1. **Decomposing the group** $\Gamma \leq \mathbb{Q}/\mathbb{Z}$. Let us begin by choosing in $\Gamma \leq \mathbb{Q}/\mathbb{Z} \subseteq S^1$ a nested sequence of finitely generated subgroups $\{e\} = \Gamma_{-1} \leq \Gamma_0 \leq \Gamma_1 \leq \cdots$ which generate Γ in the sense that for every $\gamma \in \Gamma$ we have that $\gamma \in \Gamma_j$ for some j sufficiently large. For instance such a sequence of subgroups may be built using that Γ is countable and choosing an enumeration. A finitely generated subgroup $\Gamma_j \leq \mathbb{Q}/\mathbb{Z}$ is necessarily finite and generated by a single element $\gamma_j \in \Gamma_j$. In this way we can write

$$\Gamma_j = \langle \gamma_j, \Gamma_{j-1} \rangle \text{ and } \exists! \text{ minimal } k_j \in \mathbb{N} \text{ such that } \gamma_j^{k_j} = \gamma_{j-1}. \tag{1}$$

It will be convenient to adopt the notation $k_{\leq j} \equiv k_0 \cdot k_1 \cdots k_j$, for $j \in \mathbb{N}$ and we shall denote by $|\gamma|$ the order of any $\gamma \in \Gamma$. Notice that, with this notation, $|\gamma_j| = k_{\leq j}$. There is no harm in assume that $k_j > 1$ for each j, as otherwise $\Gamma_j = \Gamma_{j-1}$.

Example 2.1. Let p be a prime and $\Gamma = \langle 1, p^{-1}, p^{-2}, \ldots \rangle \leq \mathbb{Q}/\mathbb{Z}$ be the set of rationals which can be written as a finite series $\gamma = \sum a_i p^{-i}$ with $0 \leq a_i < p - 1$. In this case we let $\gamma_i = p^{-i}$, so that $k_i = p$ for all i. We have that $\Gamma_j = \{\gamma = \sum_1^j a_i p^{-i}\}$. $\square$

Example 2.2. Let $\Gamma = \mathbb{Q}/\mathbb{Z}$. Let us choose k_j to cyclically evaluate at the primes, that is

$$\{k_j\} = 2; 2, 3; 2, 3, 5; 2, 3, 5, 7 \ldots \tag{2}$$

and let $\gamma_j \equiv \frac{1}{k_{\leq j}}$. Thus Γ_j is the set of all rationals whose denominators are products of primes up to some order and power. We can take subsequences of $\{k_j\}$ which converge to any prime or to ∞. □

Example 2.3. Let $\Gamma = \mathbb{Q}/\mathbb{Z}$. Let us choose k_j to cyclically evaluate at all the integers, that is

$$\{k_j\} = 2; 2, 3; 2, 3, 4; 2, 3, 4, 5 \ldots \tag{3}$$

and let $\gamma_j \equiv \frac{1}{k_{\leq j}}$. We can take subsequences of $\{k_j\}$ which converge to any element of $\mathbb{N}$ or to ∞. □

Let us make a few useful observations about the induced structure. For each $\gamma \in \Gamma$ we can then uniquely write it as

$$\gamma = \prod_j \gamma_j^{a_j} , \text{ such that } a_j < k_j , \tag{4}$$

where at most a finite number of a_j are nonvanishing. This builds for us a natural splitting of the short exact sequence $0 \to \Gamma_j \to \Gamma \to \Gamma/\Gamma_j \to 0$ with

$$\Gamma = \Gamma_j \oplus \Gamma/\Gamma_j , \text{ given by}$$
$$\gamma = \gamma_{\leq j} \cdot \gamma_{>j} = \prod_{i \leq j} \gamma_i^{a_i} \cdot \prod_{i > j} \gamma_i^{a_i} . \tag{5}$$

Remark 2.1. It is possible, and helpful, to include into the discussion the case where Γ is finitely generated, or equivalently $\Gamma = \Gamma_j$ for some j. This is more in line with how our inductive construction in Section 3 will proceed. However our main focus is of course on the case where Γ is not finitely generated.

2.1. **Topological Outline of** $(\tilde{M}, \tilde{p}, \Gamma)$**.** Let us open with the topological construction of $\tilde{M}$ with its group action by Γ. We will not worry in this subsection about geometry or preserving Ricci curvature. Indeed the viewpoint we will take in Section 3 when we carefully construct our space will be quite different, however the point of view we use here is particularly convenient for understanding the global structure of our space.

2.1.1. *Identifying $\tilde{M}$ with a Directed Graph.* In order to visualize the space it is helpful to build the following directed graph (V, E) of vertices and directed edges. We should think of each vertex as a copy of $S^3 \times D^4 \approx S^3 \times \mathbb{R}^4$, and it will also be geometrically convenient to view a vertex as the central $S^3 \times \{0\}$ factor, as then each vertex will represent a generator of $H_3(\tilde{M})$. If a v^a is a given vertex we will sometimes write $S^3 \times D_a^4$ in order to explicitly understand that the copy of $S^3 \times D^4$ we are staring at is the one represented by v^a.

A directed edge E_{ab} will represent for us a gluing. Given a vertex $v^a \approx S^3 \times D_a^4$ note that its boundary is a single $S^3 \times S^3$. This boundary will be glued into the target vertex $v^b \approx S^3 \times D_b^4$ by removing a smaller disk $S^3 \times D_b^4 \setminus (S^3 \times D_{ab}^4)$ with $D_{ab}^4 \subseteq D_b^4$ and choosing a gluing map $\phi_{ab} : \partial(S^3 \times D_a^4) \to \partial(S^3 \times D_{ab}^4)$. Note that we can identify $\phi_{ab} : S^3 \times S^3 \to S^3 \times S^3$. The exact choices of D_{ab} and ϕ_{ab} will be discussed after the enumeration of the vertices and edges is complete, however it is worth pointing out that the ϕ_{ab} will be a nontrivial element of the mapping class group, with the goal of twisting our underlying action. We see from the above that we should expect each vertex to be the base of at most one directed edge, although it may be

the target of multiple edges.

In order to enumerate our vertices $V = \{v^a\}$ it is convenient to decompose them as a disjoint union $V = \cup_j V_j$ as follows. Recall that we will have a global Γ action on the end manifold, and so for each of our subgroups Γ_j let $V_j = \{v_j^a\}$ represent those vertices whose associated $S^3 \times D_{a_j}^4$ will be preserved under the Γ_j action. In this case we will always have that $\Gamma_j \leq S^1$ is induced by the $(1, k_{\leq j-1})$-action, which is to say that the generator γ_j will (left) Hopf rotate the $\mathbb{R}^4$ factor by $2\pi/k_j = 2\pi k_{\leq j-1}/|\gamma_j|$ and will (left) Hopf rotate the S^3 factor by $2\pi/|\gamma_j| = 2\pi/(k_0 \cdots k_j)$. We will use the notation

$$\theta \cdot_{(a,b)} (g_1, g_2) = (a\theta \cdot g_1, b\theta \cdot g_2), \tag{6}$$

where $\theta \cdot g$ denotes the (left) Hopf rotation of S^3 by angle θ. We refer to $\cdot_{(a,b)}$ as the (a, b)-Hopf rotational action. We will often be viewing $\mathbb{R}^4 = C(S^3)$, and hence the Hopf rotation of S^3 naturally induces a rotation on $\mathbb{R}^4$. Analogous considerations hold for D^4, which we view as the ball centered at the origin of $\mathbb{R}^4$.

Identifying the vertices v_j^a with glued copies of $S^3 \times D_{a_j}^4$ in the end manifold $(\tilde{M}, \tilde{p}, \Gamma)$, we see that we should expect an induced Γ/Γ_j action on V_j. In fact, this action will be a transitive and free action. Hence we will enumerate V_j by identifying it directly with

$$V_j \equiv \{v_j^a : a \in \Gamma/\Gamma_j\}. \tag{7}$$

Thus as a set we have identified V_j with Γ/Γ_j, and from the point of view of our construction we will have one Γ_j-preserved $S^3 \times D^4$ neighborhood per element of Γ/Γ_j.

In order to build our directed graph we also need our edges. Each vertex v_j^a will be the base one of edge, and so we can we view the edges as a map $E : V \to V$. We will see that each vertex v_j^a will be the target of k_j edges, and indeed on each $V_j \subset V$ our edge map is given by

$$E : V_j \to V_{j+1} \text{ by } E[v_j^a] \equiv v_j^a/\Gamma_{j+1}. \tag{8}$$

In particular, as $|\Gamma_j/\Gamma_{j-1}| = k_j$ we have that if $v_j^b \in \Gamma/\Gamma_j$ then there are exactly k_j elements $v_{j-1}^a \in \Gamma/\Gamma_{j-1}$ for which $v_j^b = v_{j-1}^a/\Gamma_j$, as claimed.

2.1.2. *The Gluing Maps.* We have now identified our set of vertices

$$V = \bigoplus_j V_j = \bigoplus_j \Gamma/\Gamma_j, \tag{9}$$

and our edges by $E[v_j^a] = v_j^a/\Gamma_{j+1}$.

Let us fix $v_j^b \approx S^3 \times D_b^4$ and let $v_{j-1}^a \in \Gamma/\Gamma_{j-1}$ be the k_j vertices such that $v_{j-1}^a/\Gamma_j = v_j^b$. As discussed in the last subsection, associated to each $v_{j-1}^a \approx S^3 \times D_a^4$ there is a disk $D_{ab}^4 \subseteq D_b^4$ and a gluing map $\phi_{ab} : S^3 \times S^3 \to S^3 \times S^3$ identifying the boundaries $\partial(S^3 \times D_a^4)$ with $\partial(S^3 \times D_{ab}^4)$. Let us discuss these disks and mappings.

To begin, note that in the equivalence class $[v_j^b] \in \Gamma/\Gamma_j$ there is a distinguished element with $v_j^b \in \Gamma/\Gamma_{j-1}$. More precisely, if

$$[v_j^b] = \prod_{i>j} \gamma_i^{a_i} \in \Gamma/\Gamma_j, \tag{10}$$

then we can write

$$v_j^b \equiv e \cdot \prod_{i>j} \gamma_i^{a_i} = \prod_{i>j} \gamma_i^{a_i} \in \Gamma/\Gamma_{j-1}. \tag{11}$$

Then we can identify the elements of $v_{j-1}^a \in V_{j-1}$ for which $v_{j-1}^a/\Gamma_j = v_j^b$ as the collection $\{\gamma_j^a v_j^b\} \in V_{j-1}$ for $a = 0, \ldots, k_j - 1$. Now consider the disk $S^3 \times D_b^4$ and let $D_{0b} = B_r(x^0) \subseteq D_b$ be any ball $0 \notin B_{2r}(x^0)$ which is not too close to the origin, and for which $r << k_j^{-1}$. Let x^a be the rotation of x^0 by angle $2\pi a/k_j$, and hence $D_{ab} = B_r(x^a)$ is the rotation of D_{0b} by angle $2\pi a/k_j$. Note that this is a set $\{D_{ab}\}$ of k_j disjoint balls in D_b^4. By definition the set $\bigcup_a S^3 \times D_{ab}^4$ is invariant under the action by Γ_j.

Now we need to define the gluing maps $\phi_{ab} : \partial(S^3 \times D_a^4) \to \partial(S^3 \times D_{ab}^4)$, that is maps $\phi_{ab} : S^3 \times S^3 \to S^3 \times S^3$. The challenge is that we need the gluing maps to respect the Γ_j actions, and the identity map does not do this. More specifically, if we consider the glued space

$$\left(S^3 \times (D_b^4 \setminus \bigcup D_{ab}^4) \right) \bigcup_{\phi_{ab}} S^3 \times D_a^4, \tag{12}$$

then we want a well defined Γ_j action. The Γ_j action should restrict to the $(1, k_{\leq j-1})$ action on each $S^3 \times (D_b \setminus \bigcup D_{ab})$, so that in particular the Γ_{j-1} action on $S^3 \times (D_b \setminus \bigcup D_{ab})$ should be the $(1, 0)$-action. However, this same Γ_{j-1} action should be the $(1, k_{\leq j-2})$ action on each glued copy of $S^3 \times D_a^4$. If we unwind this, this is telling us we need a diffeomorphism $\phi_j : S^3 \times S^3 \to S^3 \times S^3$ such that

$$\phi_j\left(\theta \cdot_{(1, k_{\leq j-2})} (g_1, g_2) \right) = \theta \cdot_{(1,0)} \phi_j(g_1, g_2). \tag{13}$$

In fact we have such a diffeomorphism, see Section 6.1. This diffeomorphism is not isotopic to the identity, a point which causes some trouble on the geometric side of the gluing procedure, see Section 3.1.2 for more on this. For the topological picture we will now define $\phi_{ab} : \partial(S^3 \times D_a^4) \to \partial(S^3 \times D_{ab}^3)$ by

$$\phi_{ab}(g_1, g_2) = \gamma_j^a \cdot \phi_j(g_1, g_2).$$

Note that the gluing maps are built precisely so that the Γ_{j-1} action on $S^3 \times D_a^4$ extends to an action of Γ_j on the glued space $\left(S^3 \times (D_b^4 \setminus \bigcup D_{ab}^4) \right) \bigcup_{\phi_{ab}} S^3 \times D_a^4$.

2.1.3. *Construction of* $(\tilde{M}, \Gamma)$. Let us build our global space $\tilde{M}$ as follows. Having defined our collection of vertices $\{v^a\} \in V = \bigoplus V_j = \bigoplus \Gamma/\Gamma_j$ let us first consider the disjoint collection

$$\bigcup_j \bigcup_{V_j} S^3 \times D_{b_j}^4, \tag{14}$$

where we have assigned to each vertex $v_j^b \in V_j$ a copy of the disk cross a sphere. Observe that there is a free action of Γ on this space, where if $\gamma \in \Gamma$ then its action on $S^3 \times D_{b_j}^4$ may be understood in the following

manner. Recall that Γ has a natural splitting as in (5), so that each $\gamma \in \Gamma$ can be written $\gamma = \gamma_{\leq j} \cdot \gamma_{>j}$ where

$$\gamma_{\leq j} = \prod_{i \leq j} \gamma_i^{a_i} \in \Gamma_j , \text{ and } \gamma_{>j} = \prod_{i > j} \gamma_i^{a_i} \in \Gamma/\Gamma_j . \tag{15}$$

Note there is the defined action of $\gamma_{\leq j} \in \Gamma_j \leq S^1$ on $S^3 \times D_{b_j}^4$ induced by the $(1, k_{\leq j-1})$ action which Hopf rotates D^4 at speed $2\pi/k_j$ and Hopf rotates S^3 at speed $2\pi/|\gamma_j| = 2\pi/(k_0 \cdots k_j)$. Additionally, $\gamma_{>j} \in \Gamma/\Gamma_j$ naturally acts on $b_j \in \Gamma/\Gamma_j$. Combining we get that $\gamma \cdot : S^3 \times D_{b_j}^4 \to S^3 \times D_{\gamma_{>j} \cdot b_j}^4$ acts by letting $\gamma_{\leq j}$ rotate and $\gamma_{>j}$ identify the two factors.

Now for each directed edge E_{ab_j} let us remove the corresponding ball $D_{ab_j} \subseteq D_{b_j}$ as in the last subsection:

$$\bigcup_j \bigcup_{V_j} S^3 \times (D_{b_j}^4 \setminus \bigcup_{E_{ab_j}} D_{ab_j}^4) . \tag{16}$$

Observe that the action of Γ restricts to an action of the above. Finally let us observe that for each directed edge E_{ab_j} we have defined the corresponding gluing maps $\phi_{ab_j} : \partial(S^3 \times D_a^4) \to \partial(S^3 \times D_{ab_j}^4)$ which were built precisely to commute with the above action of Γ. Thus we arrive at our end space

$$\tilde{M} \equiv \left(\bigcup_j \bigcup_{V_j} S^3 \times (D_{b_j}^4 \setminus \bigcup_{E_{ab_j}} D_{ab_j}^4) \right) \Big|_{\{\phi_{ab_j} \in E\}} , \tag{17}$$

together with its free action by Γ. In (17) it is understood that the boundary components $\partial(S^3 \times D_a^4)$ and $\partial(S^3 \times D_{ab_j}^4)$ are identified according to the directed edges $E_{ab_j} \in E$ and via the diffeomorphisms ϕ_{ab_j}. We refer to Section 3 for an alternative (but equivalent) approach to the definition of the total space $\tilde{M}$ which has a more geometric flavor.

The homology of $\tilde{M}$ may be conveniently computed with a Mayer-Vietoris sequence. Indeed, as $\tilde{M}$ is the gluing of 2-connected spaces whose intersections are all 2-connected, it is clear that $\tilde{M}$ is 2-connected.

2.2. **Geometric Outline of** $(\tilde{M}, \tilde{p}, \Gamma)$.

We described in the previous subsection the topological construction of the universal cover $\tilde{M}$ from Theorem 1.1 together with its free action by Γ. In this subsection we want to understand the broad geometry of $\tilde{M}$. We will mostly concern ourselves with a rough Gromov Hausdorff picture of what is happening, with only some mild comments toward the finer geometric and topological points. In the next subsection we will introduce a precise inductive construction that will put the pictures of this subsection and the last together more comprehensively. The geometric viewpoint will have a different flavor than the topological construction, as we will focus ourselves more locally as we move up in scale. This will also be the convenient viewpoint for the inductive construction in Section 3.

For a chosen basepoint $\tilde{p} \in \tilde{M}$ let us look at the ball $B_r(\tilde{p})$ and consider the local group $\Gamma_r \equiv \langle \gamma \in \Gamma : d, \tilde{p}, \gamma \cdot \tilde{p}) \leq r \rangle$ generated by those actions which move $\tilde{p}$ at most $r > 0$. The local groups $\Gamma_r \subseteq \Gamma$ will then necessarily be monotone increasing, and there will be discrete radii r_j at which the local group jumps. We will have for $r_j \leq r < r_{j+1}$ that $\Gamma_r = \Gamma_j \leq \mathbb{Q}/\mathbb{Z} \subseteq S^1$ as in (1). In particular, at scale r_j we will add one new generator $\Gamma_j = \langle \gamma_j, \Gamma_{j-1} \rangle$ to the local group. As usual we will denote by k_j the minimal integer for which

$\gamma_j^{k_j} = \gamma_{j-1} \in \Gamma_{j-1}$ becomes the generator of Γ_{j-1}. In this way the local group is always generated by a single element, and what is happening at scale r_j is that this element is changing.

2.2.1. *Geometry at Scale r_j.*

Let us then roughly describe what $\tilde{M}$ looks like on the scales r_j, and then next we will even more roughly describe what happens to $\tilde{M}$ between scales r_j and r_{j+1}. At each scale r_j the manifold $\tilde{M}$ will be Gromov-Hausdorff close to a ball in $S^3 \times \mathbb{R}^4$. Indeed the space will be mostly diffeomorphic and nearly isometric to $S^3 \times \mathbb{R}^4$ at scale r_j, however as in the gluing construction of Section 2.1.2 there will be k_j small balls around the local orbit $\Gamma_j \cdot \tilde{p}$ which will contain a good deal of topology at smaller scales. It is worth pointing out that for j large the sphere factor S^3 will have scale invariantly decreasing radius, so that from a Gromov-Hausdorff point of view the space is looking increasingly like $\mathbb{R}^4$

Note that there is a $T^2 = S^1 \times S^1$ action on $S^3 \times \mathbb{R}^4$. The first S^1 acts freely on the S^3 factor by Hopf rotating. The second S^1 acts on the $\mathbb{R}^4 = C(S^3)$ factor by Hopf rotating the unit sphere. For $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ there is an induced S^1 action on $S^3 \times \mathbb{R}^4$ through the homomorphic embedding $S^1 \to S^1 \times S^1$ given by $\theta \mapsto (a\theta, b\theta)$. That is, the (a, b)-action of S^1 will Hopf rotate the spheres of $\mathbb{R}^4 = C(S^3)$ at speed b and will Hopf rotate S^3 at speed a. Note that if a and b are coprime then this is a free action. The size of the 3-sphere will be growing, but go to zero relative to r_j, and so from a pure Gromov-Hausdorff point of view the space will be close to $\mathbb{R}^4$ at the scales r_j.

Now on the scale r_j the action of the generator $\gamma_j \in \Gamma_j$ will look like a rotation of the $\mathbb{R}^4$ factor by $2\pi/k_j$, and a Hopf rotation of the S^3 factor by $2\pi/|\gamma_j| = 2\pi/(k_0 \cdots k_j)$. If we view $\Gamma_j \leq S^1$ then the action of Γ_j is the one induced by the $(1, k_{\leq j-1})$-S^1 action as above, where we recall that we set $k_{\leq j} \equiv k_0 \cdot k_1 \cdots k_j$. Observe that Γ_{j-1} is generated by $\gamma_{j-1} = \gamma_j^{k_j}$, and therefore it looks like a rotation of purely the S^3 factor. The basepoint $\tilde{p}$ should *not* be viewed as the center of the rotation of γ_j in $\mathbb{R}^4$. The center of the rotation $\approx S^3 \times \{0\}$ will be a central 3-sphere, which from the point of view of the topological construction as in Section 2.1 is a $H_3(\tilde{M})$ generator associated to a vertex. The point $\tilde{p}$ should be viewed as a point of distance roughly $k_j r_j$ from the center of this rotation. In this way $d(\tilde{p}, \gamma_j \cdot \tilde{p}) = r_j$ and the size of the orbit of the Γ_j action is roughly $k_j r_j$.

2.2.2. *Geometry between Scales r_j and r_{j+1}.*

We have described that at scale r_j the space is close to $S^3 \times \mathbb{R}^4$ and the local group Γ_j looks primarily like a rotation of the $\mathbb{R}^4$ factor. Let us now discuss very roughly what happens between scales r_j and r_{j+1}. Observe that for the picture of the last paragraphs to hold, something substantial must have happened. Indeed, let us consider the group Γ_j at scales r_j and r_{j+1}. At both of these scales the space looks like $S^3 \times \mathbb{R}^4$, however the action of Γ_j on the bottom r_j-scale rotates both factors, while on the top r_{j+1}-scale it rotates only the second factor. In particular the action of the generator γ_j, which looks mostly like a rotation of $\mathbb{R}^4$ on the bottom scale, has slid in to become just a rotation of S^3 on the top scale.

The topological mechanism for this twisting was described in Section 2.1.2, namely we needed to glue these two copies of $\mathbb{R}^4 \times S^3 \approx D^4 \times S^3$ together by a boundary map $\phi_j : S^3 \times S^3 \to S^3 \times S^3$ which is

homotopically nontrivial, and which commutes with the action by untwisting

$$\phi_j(\theta \cdot_{(1,k_{\leq j-1})} (g_1, g_2)) = \theta \cdot_{(1,0)} \phi_j(g_1, g_2) \,. \tag{18}$$

Let us give a different viewpoint here which is geometrically convenient. Between scales r_j and r_{j+1} our space will be diffeomorphic to an annulus in $S^3 \times \mathbb{R}^4$, or equivalently diffeomorphic to an annulus $A_{r_j, r_{j+1}}(0) \subseteq C(S^3 \times S^3)$. Very roughly, we can view the metric on this annulus as $dr^2 + r^2 g_r$, where g_r is a family of metrics on $S^3 \times S^3$. We know that the top and bottom scales are very close to $S^3 \times \mathbb{R}^4$, and so to first approximation we can say that g_{r_j} and $g_{r_{j+1}}$ are isometrically very close to a product of two spheres $S^3_\delta \times S^3_1$, where the subscript denotes the radius and we are viewing $0 < \delta << 1$. Note that $C(S^3_1) = \mathbb{R}^4$ and the small sphere is playing the role of S^3 cross factor. However we understand from Section 6 that these isometries are very different, and indeed not even isotopic to one another. That is, even if $g_{r_{j+1}}$ and g_{r_j} are isometric, as tensors we do not have $g_{r_{j+1}} \approx g_{r_j}$ but instead have $g_{r_{j+1}} \approx \phi_j^* g_{r_j}$, where $\phi_j : S^3 \times S^3 \to S^3 \times S^3$ is a diffeomorphism as above. So although the geometry at scales r_j and r_{j+1} begins and ends at the same point, we should be interpreting these two copies of $S^3 \times \mathbb{R}^4$ very differently. Step 2 of Section 3 will discuss this in greater detail, and see Section 7 for the precise discussion. Note for precision sake that the metric is not a cone metric at the beginning and end, and that the two product 3-spheres at the top and bottom scales will be of very different size.

Geometrically, the rough description of the geometry of $dr^2 + r^2 g_r$ on the region between scales r_j and r_{j+1} is as follows. The metric g_r on $S^3 \times S^3$ begins at r_j so that the space is isometrically very close to $S^3 \times \mathbb{R}^4 = S^3 \times C(S^3_1)$. As the first sphere is very small, and indeed how small will be scale invariantly going to zero as j increases, this is Gromov-Hausdorff close to $\mathbb{R}^4$. Then slowly in r the metric will shrink the second S^3 factor, so that geometrically our space becomes a ray $\mathbb{R}^+$. Now the complicated twisting of the cross sections from Section 6 will take place, however geometrically the space will look roughly like a ray this whole time. Finally the metric will reexpand to $g_{r_{j+1}}$, which is again isometrically very close to $S^3 \times \mathbb{R}^4$, albeit a very different copy of $S^3 \times \mathbb{R}^4$.

2.2.3. *Transitioning from Scale r_j to Scale r_{j+1}.* After the action has been untwisted between scales r_j and r_{j+1}, let us remark that there is an additional challenge when the next generator γ_{j+1} enters the picture. At scale r_{j+1} we again look close to $S^3 \times \mathbb{R}^3$, however we then suddenly see k_{j+1} copies of our original space appear. Geometrically this will occur on very (scale-invariantly) small balls, and so from a broad geometrical viewpoint the space will still look Gromov Hausdorff close to $\mathbb{R}^4$. These new copies will be identified by the γ_{j+1} action, as our local group has jumped. Step 3 in Section 3 will deal with this issue with more care, and see Section 8 for the precise discussion.

2.2.4. **Tangent Cones at Infinity of $\tilde{M}$ and M.** Let us consider a sequence of radii $s_j \to \infty$ and understand the limits of $(s_j^{-1} \tilde{M}, p, \Gamma)$ and $(s_j^{-1} M, p)$. After passing to subsequences (and reindexing) we can break ourselves down into various cases depending on how s_j compares to our naturally defined scales r_j from before.

2.2.5. *The scales $s_j = r_j$.* Let us begin with the base case of understanding the sequence $(r_j^{-1}\tilde{M}, p, \Gamma)$ on the universal cover. We have determined that $\tilde{M}$ looks very close to $S^3 \times \mathbb{R}^4$ at these scales with (scale invariantly) shrinking sphere factor. In particular, we have that geometrically the tangent cone at infinity along this sequence gives $r_j^{-1}\tilde{M} \to \mathbb{R}^4$. The action of γ_j at scale r_j is visible as a rotation by angle $2\pi/k_j$ of the $\mathbb{R}^4$ factor with respect to a basepoint distance k_j away. Therefore to understand the equivariant limit we need to break ourselves into two cases. Namely, after passing to subsequences either k_j converges or not.

2.2.6. *The scales $s_j = r_j$ with $k_j \to k < \infty$.* In this case the action of γ_j looks like a rotation with respect to a point distance kr_j away from p, and so we have that $(r_j^{-1}\tilde{M}, p, \Gamma) \to (\mathbb{R}^4, p_\infty, \mathbb{Z}_k)$ where $\mathbb{Z}_k$ is acting by rotation around the origin and p_∞ is a point distance k from the origin. We get that the quotient space

$$(r_j^{-1}M, p) \to (C(S_1^3/\mathbb{Z}_k), p_\infty) \tag{19}$$

limits to a cone over a lens space. The basepoint p_∞ of this limit is again a point distance k from the cone point.

2.2.7. *The scales $s_j = r_j$ with $k_j \to \infty$.* In this case the action of γ_j is looking increasingly like a translation by $\mathbb{Z}$, and we get that $(r_j^{-1}\tilde{M}, p, \Gamma) \to (\mathbb{R}^4, 0, \mathbb{Z})$ where $\mathbb{Z}$ acts by unit translation. The quotient space in this case limits

$$r_j^{-1}M \to \mathbb{R}^3 \times S^1. \tag{20}$$

2.2.8. *The scales $r_j < s_j \ll k_j r_j$ with $k_j \to \infty$.* In the case that $k_j \to k$ remains bounded there is no distinction between this case and the last. Therefore, we are only concerned with the case where we have some subsequence for which $k_j \to \infty$. In this situation note with $\frac{s_j}{r_j}, \frac{k_j r_j}{s_j} \to \infty$ that our $\mathbb{Z}$ action is looking increasingly like an $\mathbb{R}$ action. Our limit in this case becomes $(s_j^{-1}\tilde{M}, p, \Gamma) \to (\mathbb{R}^4, 0, \mathbb{R})$, where $\mathbb{R}$ is acting by translation. Our quotient space is therefore limiting

$$s_j^{-1}M \to \mathbb{R}^3. \tag{21}$$

2.2.9. *The scales $s_j \approx k_j r_j$ when $k_j \to \infty$.* Note the action of γ_j at these scales looks like a rotation by angle $2\pi/k_j$. In particular, we get that $(s_j^{-1}\tilde{M}, p, \Gamma) \to (\mathbb{R}^4, p_\infty, S^1)$, where S^1 is a rotation around the origin. Our basepoint is now roughly distance 1 from the center of the rotation. In particular our quotient limit is given by

$$(r_j^{-1}M, p) \to (C(S_{1/2}^2), p_\infty). \tag{22}$$

2.2.10. *The scales $k_j r_j \ll s_j \ll r_{j+1}$ when $k_j \to k < \infty$.* We discussed that at scale $s_j \approx k_j r_j$ we have $s_j^{-1}\tilde{M}$ looks like $\mathbb{R}^4 = C(S_1^3)$. As $\frac{s_j}{k_j r_j}$ increases our cross section sphere S_s^3 begins to decrease in radius until it looks like a half ray. Therefore we get the possible limits $(s_j^{-1}\tilde{M}, p, \Gamma) \to (C(S_s^3), 0, \mathbb{Z}_k)$ for all $0 \le s \le 1$. In the case when $\frac{s_j}{k_j r_j}$ becomes sufficiently large we get that the limit is a half ray with the trivial action. Our quotient limits in this range are therefore

$$(s_j^{-1}M, p) \to (C(S_s^3/\mathbb{Z}_k), p_\infty), \tag{23}$$

for all $0 \le s \le 1$.

2.2.11. *The scales $s_j \to r_{j+1}$.* As the scale s_j continues to increase to r_{j+1}, we have that the half ray reopens up so that we again have $s_j^{-1}\tilde{M} \approx \mathbb{R}^4$. However, as it reopens the Γ_j is now a trivial action. As we approach scale r_{j+1} a new γ_{j+1} action appears and we repeat the above process.   □

Let us make several quick observations about this process. In the case $\Gamma = \mathbb{Q}/\mathbb{Z}$ we can choose k_j so that every $k \in \mathbb{N}$ appears infinitely often, see Example 2.3. Consequently, all of the cones

$$M_\infty \equiv C(S^3_s/\mathbb{Z}_k), \tag{24}$$

appear as tangent cones at infinity for all $s \in [0, 1]$ and $k \in \mathbb{N}$. Geometrically, when we start at a scale for which the space M looks like a cone over a lens space $C(S^3/\mathbb{Z}_k)$, then as the scale increases the cross section shrinks so that the space looks like a half ray. As the scale continues to increase the space expands to again look like a cone over a lens space $C(S^3/\mathbb{Z}_{k'})$. However, when it reopens it may appear to be a different lens space.

The last point to remark on is that though every tangent cone at infinity is a metric cone, the pointed limit does not always have the cone point as the base point. This is in agreement with [So1], where we understand some tangent cones at infinity need to not be polar *with respect to* the base point.

3. INDUCTIVE CONSTRUCTION FOR THEOREM 1.1

Let us now describe our construction for Theorem 1.1 in more technical detail. The proof will be set up in an inductive fashion, where we will build a sequence of pointed manifolds (M_j, p_j, Γ_j) with $\mathrm{Ric}_j \geq 0$ together with free uniformly discrete isometric actions by Γ_j. This Section will begin with a description of the main properties of our inductive sequence M_j, together with how one proves Theorem 1.1 once this sequence has been constructed. The induction criteria will be such that building $(\tilde{M}, \Gamma)$ from the inductive sequence will be relatively straightforward.

The remainder of this Section will then focus on proving the induction, namely on how to construct M_{j+1} from M_j in order to complete the induction proof. The construction will boil down to three major steps, and in each step we will state one of our three main inductive Propositions. These Propositions will be proved in remaining Sections of the paper, and thus the proof of Theorem 1.1 will be complete in this Section modulo these main Propositions.

Let us now set the stage for a precise statement of our induction criteria. Recall that we have chosen as in (1) a sequence of finitely generated subgroups $\Gamma_j \leq \Gamma$ with $\Gamma_j = \langle \gamma_j, \Gamma_{j-1} \rangle$ which are all generated by a single element γ_j such that $\gamma_j^{k_j} = \gamma_{j-1}$. From the point of view of the topological construction of Section 2 we can view the sequence (M_j, p_j, Γ_j) as the manifold obtained under the construction tree with $\Gamma = \Gamma_j$.

Our geometric construction will be based on a sequence of parameters $\epsilon_j \to 0$ and $\delta_j \to 0$. We may begin by choosing any sequence $\epsilon_j \to 0$. Indeed any sequence of constants $\epsilon_j < 1$ will do, but in our description of the tangent cones at infinity of $\tilde{M}$ in Section 2.2.4 we have used that these constants tend to zero, which

gives a slightly cleaner picture. Let $\delta_1 << 1$ also be any small constant, the remaining δ_j will be chosen based on applications of our Inductive Propositions. We shall adopt the notation $A_{s_1,s_2}(p)$ to denote the annuls $B_{s_2}(p) \setminus B_{s_1}(p)$ for any $0 \le s_1 < s_2 \le \infty$.

Our sequence (M_j, p_j, Γ_j) will inductively be assumed to satisfy:

(I1) There exists a free isometric action by Γ_j on M_j with $r_j \equiv d(p_j, \gamma_j \cdot p_j)$ and $\frac{r_j}{k_{j-1}r_{j-1}} >> 1$.

(I2) There exists an isometry $\Phi_j : U_j \subseteq M_j \to M_{j+1}$ with $B_{10k_jr_j}(p_j) \subseteq U_j \subseteq B_{10^3k_jr_j}(p_j)$ with $\Phi_j(p_j) = p_{j+1}$, where U_j is Γ_j invariant with $\Phi_j(x \cdot \gamma) = \Phi_j(x) \cdot \gamma$ for all $\gamma \in \Gamma_j \le \Gamma_{j+1}$.

(I3) $M_j \setminus U_j$ is isometric to $S^3_{\delta_j r_j} \times A_{10^2 k_j r_j, \infty}(0) \subseteq S^3_{\delta_j r_j} \times C(S^3_{1-\epsilon_j})^2$. The action of γ_j in this domain rotates the cross section $S^3_{1-\epsilon_j}$ of the cone factor by $2\pi/k_j$ and the $S^3_{\delta_j r_j}$ factor by $2\pi/|\gamma_j| = 2\pi/(k_0 k_1 \cdots k_j)$.

Remark 3.1. It follows from (I3) that the orbit of the action of Γ_j has diameter roughly $k_j r_j$.

Remark 3.2. It will be clear from the construction that $\frac{r_{j+1}}{k_j r_j} \to \infty$. That is, the scale of the action of the next generator γ_{j+1} relative to the orbit of the previous generator γ_j is tending to infinity.

Before discussing more carefully the structure of the spaces M_j above, let us quickly see that if such an inductive sequence as above can be built, then we are done. Indeed, consider first the Γ_i-equivariant isometries $\Phi_{ji} = \Phi_j \circ \cdots \circ \Phi_i : U_i \to U_{ji} \equiv \Phi_{ji}(U_i) \subseteq M_j$. We can take an abstract equivariant pointed Gromov-Hausdorff limit of the sequence (M_j, p_j, Γ_j). However the setup is such that we can also simply define the direct limit

$$\tilde{M} \equiv \{(x_j, x_{j+1}, \ldots) : x_{k+1} = \Phi_k(x_k) \text{ for all } k \ge j\}/ \sim, \tag{25}$$

where there is an equivalence relation $(x_j, x_{j+1}, \ldots) \sim (y_{j'}, y_{j'+1}, \ldots)$ if there exists $k \ge \max\{j, j'\}$ such that $x_k = y_k$. By the equivariance of the isometries Φ_i we have that Γ_j naturally acts on all sequences $(x_k, x_{k+1}, \ldots)$ with $k \ge j$. In particular there is an induced action of Γ on $\tilde{M}$. Note that $U_j \subseteq M_j$ all embed isometrically into $\tilde{M}$ and exhaust $\tilde{M}$, and the restriction of the Γ_j action to $U_j \subseteq \tilde{M}$ is the expected action. Thus $\tilde{M}$ is a smooth Riemannian manifold with Ric ≥ 0 and a free discrete isometric action by Γ, as claimed.

3.1. The Steps of the Inductive Construction.
We will break down this inductive construction into three steps. Each will involve a Proposition which will form the main constructive ingredient in the step. Our goal in this subsection is then to discuss these steps and state the Propositions. We will then see how to finish the induction given these results. Future sections will then be dedicated to proving each of these Propositions individually.

The first step will build our background model space $\mathcal{B}(\epsilon, \delta) \approx S^3 \times \mathbb{R}^4$. It will form the basis of both our base step of the induction, and also the underlying space for which previous induction manifolds M_j will be glued into in order to form M_{j+1}. From the point of view of the topological construction of Section 2, there will be one copy of a background model space per vertex in our construction tree. The construction of

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[2]Observe that this is isometrically very close to $S^3 \times \mathbb{R}^4$. Indeed, in our setup U_j itself is very Gromov-Hausdorff close to $S^3 \times \mathbb{R}^4$.

the model space $\mathcal{B}(\epsilon, \delta)$ is actually a fairly standard one, but it will help with the exposition to isolate it and discuss the role it plays.

The second step will deal with the action twisting described in Section 2.2.2 . Each M_j looks like $S^3 \times \mathbb{R}^4$ at infinity with the action Γ_j induced by the $(1, k_{\leq j-1})$-Hopf S^1 action. The first step in building M_{j+1} is to equivariantly twist M_j to a new manifold $\hat{M}_j$, so that after our twisting $\hat{M}_j$ again looks like $S^3 \times \mathbb{R}^4$ at infinity but this time the Γ_j action is induced by the $(1, 0)$-Hopf action.

The third step of the inductive construction is to take our twisted $\hat{M}_j$ and glue in k_{j+1} copies into a new base manifold $\mathcal{B}_{j+1}$. The gluing is such that we have now extended the Γ_j action on $\hat{M}_j$ to a $\Gamma_{j+1} = \langle \gamma_{j+1}, \Gamma_j \rangle$ action on M_{j+1} in the appropriate fashion. From the point of view of the topological construction, there will be one gluing per directed edge in our construction tree.

3.1.1. **Step 1: The Background Model Space** $\mathcal{B}(\epsilon, \delta)$.

Our construction will begin by building a background manifold $\mathcal{B}(\epsilon, \delta)$. The space will both play the role of base step in the inductive construction, and additionally when we move from M_j to M_{j+1} the basis for our construction will be to glue in k_{j+1} copies of M_j into the background space $\mathcal{B}_{j+1}$. From the point of view of our construction tree in Section 2.1, there will eventually be one copy of $\mathcal{B}_j$ per vertex $v_j \in V_j = \Gamma/\Gamma_j$.

The construction of $\mathcal{B}(\epsilon, \delta)$ is relatively straightforward, we will simply take $S^3 \times \mathbb{R}^4$ and slightly curve the $\mathbb{R}^4$ factor in order to give it a slight cone angle. The precise setup is the following:

Proposition 3.1 (Step 1: The Model Space). *For each $\delta > 0$ and $1 > \epsilon > 0$, there exists a smooth manifold* $\mathcal{B}^7 = \mathcal{B}(\epsilon, \delta)$ *such that the following hold:*

(1) *$(\mathcal{B}^7, g_B, p)$ is a complete Riemannian manifold with $Ric \geq 0$, it is diffeomorphic to $S^3 \times \mathbb{R}^4$.*
(2) *There exists $B_{10^{-3}}(p) \subseteq U \subseteq B_{10^{-1}}(p)$ such that $\mathcal{B} \setminus U$ is isometric to $S^3_\delta \times A_{10^{-2}, \infty}(0) \subseteq S^3_\delta \times C(S^3_{1-\epsilon})$*
(3) *There is an isometric $T^2 = S^1 \times S^1$ action on $\mathcal{B}$ for which on $\mathcal{B} \setminus U \approx S^3_\delta \times C(S^3_{1-\epsilon})$ the first S^1 acts on the S^3_δ factor by a globally free (left) Hopf rotation and the second S^1 acts on the cross sections $S^3_{1-\epsilon}$ of the cone factor by (left) Hopf rotation.*
(4) *The S^1-action induced by the homomorphic embedding $S^1 \ni \theta \mapsto (a\theta, b\theta) \in T^2$ is free whenever $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ are coprime and $a \neq 0$.*

Remark 3.3. Thus for each $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ we have the induced (a, b)-S^1 action given by the homomorphic embedding $S^1 \ni \theta \mapsto (a\theta, b\theta) \in T^2$.

Base Step: Let us then define the base step of our induction as $M_1 = \mathcal{B}(\epsilon_1, \delta_1)$ as above. We will equip M_1 with the the isometric group action of $\Gamma_1 \leq S^1$, which is induced by the $(1, k_0)$-action as above. In particular, on $S^3_{\delta_1} \times S^3_{1-\epsilon_1}$ we have that the generator γ_1 will act by Hopf rotating $S^3_{\delta_1}$ by $2\pi/|\gamma_1| = 2\pi/(k_1 k_0)$ and by Hopf rotating the cross section of $C(S^3_{1-\epsilon_1})$ by $2\pi/k_1 = 2\pi k_0/|\gamma_1|$.

3.1.2. **Step 2: The Equivariant Mapping Class Group and Twisting the Geometry of M_j at Infinity.**

By condition (I3) of the induction we know that outside some compact set U_j our space $M_j \setminus U_j$ is isometric

to $S^3_{\delta_j r_j} \times A_{10^2 k_j r_j, \infty}(0) \subseteq S^3_{\delta_j r_j} \times C(S^3_{1-\epsilon_j}) \approx S^3 \times \mathbb{R}^4$. Further, we understand that in this region the action of the generator $\gamma_j \in \Gamma_j$ looks primarily like a rotation of the $\mathbb{R}^4$ factor. More precisely, it rotates the $\mathbb{R}^4$ factor by $2\pi/k_j$ and it rotates the $S^3_{\delta_j r_j}$ factor by the much smaller $2\pi/|\gamma_j| = 2\pi/(k_0 \cdots k_j)$.

In Step 3 we will be gluing k_{j+1} copies of M_j into a model space $\mathcal{B}_{j+1}$, and in the gluing region we will again have that $\mathcal{B}_{j+1} \approx S^3 \times \mathbb{R}^4$. However, the action of Γ_j on $\mathcal{B}_{j+1}$ will look like a rotation of just the S^3 factor without any rotational bit on the $\mathbb{R}^4$ factor. Thus to accomplish the gluing we will need to modify M_j at infinity into a new space $\hat{M}_j$, which will again look close to $S^3 \times \mathbb{R}^4$ but for which the action of γ_j is now purely a rotation of the S^3 factor.

In order to address this problem let us first consider $S^3 \times S^3$ with the standard metric $g_{S^3 \times S^3}$, and let us recall that if $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ then we have the S^1-isometric action $\cdot_{(a,b)} : S^3 \times S^3 \times S^1 \to S^3 \times S^3$ which acts by a times the (left) Hopf rotation on the first S^3 and b times the (left) Hopf rotation on the second S^3. The following will provide for us how the cross sections of our new space $\hat{M}_j$ will be twisting. It will be proved in Section 6:

Theorem 3.2 (Equivariant Mapping Class Group on $S^3 \times S^3$). *Let $g_0 = g_{S^3 \times S^3}$ be the standard metric on $S^3 \times S^3$, and let $k \in \mathbb{Z}$. Then there exist a diffeomorphism $\phi : S^3 \times S^3 \to S^3 \times S^3$ and a family of metrics $(S^3 \times S^3, g_t)$ such that*

    (1) *$Ric_t > 0$ for all $t \in [0, 1]$*
    (2) *The S^1-action $\cdot_{(1,k)}$ on $S^3 \times S^3$ is an isometric action for all g_t.*
    (3) *$g_1 = \phi^* g_0$ with $\phi(\theta \cdot_{(1,k)} (s_1, s_2)) = \theta \cdot_{(1,0)} \phi(s_1, s_2)$.*

Remark 3.4. The diffeomorphism $\phi : S^3 \times S^3 \to S^3 \times S^3$ will represent a nontrivial element of the mapping class group.

Remark 3.5. In Section 9 we will discuss how to connect $g_{S^3 \times S^3}$ and $\phi^* g_{S^3 \times S^3}$ for *any* element of the mapping class group $[\phi] \in \pi_0 \mathrm{Diff}(S^3 \times S^3)$. However for an arbitrary element of the mapping class group we cannot necessarily keep track of the behavior of an isometric action.

The above tells us that we can find an S^1-invariant family of metrics with positive Ricci curvature which (from an isometric point of view) start and end at the classical $S^3 \times S^3$, however the beginning and ending S^1 actions are quite distinct. Our main use of the above will be to build the following neck region, which will be used to alter M_j to $\hat{M}_j$:

Proposition 3.3 (Step 2: Twisting the Action). *Let $\epsilon, \hat{\epsilon}, \delta > 0$ with $k \in \mathbb{Z}$. Then there exist $\hat{\delta}(\epsilon, \hat{\epsilon}, \delta, k) > 0$ and $R(\epsilon, \hat{\epsilon}, \delta, k) > 1$ and a metric space X with an isometric and free S^1 action such that*

    (1) *X is smooth away from a single three sphere $S^3_{\delta} \times \{p\} \in X$ with $Ric_X \geq 0$.*
    (2) *There exists $B_{10^{-3}}(p) \subseteq U \subseteq B_{10^{-1}}(p) \subseteq X$ which is isometric to $S^3_{\delta} \times B_{10^{-2}}(0) \subseteq S^3_{\delta} \times C(S^3_{1-\epsilon})$, and under this isometry the S^1 action on U identifies with the $(1, k)$-Hopf action.*
    (3) *There exists $B_{10^{-1}R}(p) \subseteq \hat{U} \subseteq B_{10R}(p) \subseteq X$ s.t. $X \setminus \hat{U}$ is isometric to $S^3_{\hat{\delta}R} \times A_{R,\infty}(0) \subseteq S^3_{\hat{\delta}R} \times C(S^3_{1-\hat{\epsilon}})$, and under this isometry the S^1 action on $X \setminus \hat{U}$ identifies with the $(1, 0)$-Hopf action.*

Constructing $\hat{M}_j$: Before moving on to Step 3, let us see how the above will be used as part of our induction process. Thus let us assume we have constructed M_j as in (I1)-(I3) with sphere radius δ_j. Recall by (I3) that outside of a compact subset we have that M_j is isometric to $S^3_{\delta_j r_j} \times C(S^3_{1-\epsilon_j})$, and the action of γ_j Hopf rotates the $S^3_{1-\epsilon_j}$ factor by $2\pi/k_j$ and the $S^3_{\delta_j r_j}$ factor by $2\pi/|\gamma_j|$. Observe that if we consider the $(1, k_{\leq j-1})$-Hopf S^1-action on $S^3_{\delta_j r_j} \times C(S^3_{1-\epsilon_j})$, then $\Gamma_j \subseteq S^1$ can be viewed as a subaction.

Now with any $\hat{\epsilon}_j > 0$, the precise constant will be chosen later, we have for $R_j = R_j(\epsilon_j, \hat{\epsilon}_j, \delta_j, k_{\leq j-1})$ and $\hat{\delta}_j = \hat{\delta}_j(\epsilon_j, \hat{\epsilon}_j, \delta_j, k_{\leq j-1})$ the existence of X_j as in Proposition 3.3, where we chose $k = k_{\leq j-1} = k_0 \cdot k_1 \cdots k_{j-1}$ in the application of the Proposition. We can rescale $X_j \to r_j X_j$ by r_j so that it is isometric to $S^3_{\delta_j r_j} \times C(S^3_{1-\epsilon_j})$ on a region U containing $B_{r_j}(p)$, and it is isometric to $S^3_{\hat{\delta}_j R_j r_j} \times C(S^3_{1-\hat{\epsilon}_j})$ on a region $X_j \setminus \hat{U}$ containing the annulus $A_{R_j r_j, \infty}(p)$. Further, there is a free isometric S^1 action on X_j which looks like the $(1, k_{\leq j-1})$ action on U and the $(1, 0)$ action on $X_j \setminus \hat{U}$. In particular, by condition (2) in Proposition 3.3 and the inductive assumption (I3) there is an induced Γ_j action on X_j and an open annulus of $U \subseteq X_j$ which is equivariantly isometric to an open annulus in $M_j \setminus U_j$.

We can thus glue X_j to M_j in order to produce the space $\hat{M}_j$. The space $\hat{M}_j$ is now isometric to $S^3_{\hat{\delta}_j R_j r_j} \times C(S^3_{1-\hat{\epsilon}_j})$ outside of some compact set $V_j \subseteq \hat{M}_j$, and the Γ_j action is a pure Hopf rotation on the $S^3_{\hat{\delta}_j R_j r_j}$ factor on $\hat{M}_j \setminus V_j$.

3.1.3. **Step 3: Gluing Construction.** The third step of the construction involves extending the action of Γ_j to an action of Γ_{j+1} in order to move from the manifold M_j to the next step of the induction M_{j+1}. This will occur by taking k_{j+1} copies of the twisted space $\hat{M}_j$, constructed in the second step, and gluing them into a model space $\mathcal{B}_{j+1} \approx \mathcal{B}(\epsilon_{j+1}, \delta_{j+1})$ constructed in the first step. From the point of view of the topological construction in Section 2, there is one gluing per directed edge in our construction tree.

Recall that a model space $\mathcal{B}(\epsilon, \delta)$ is isometric to an annulus in $S^3_\delta \times C(S^3_{1-\epsilon})$ outside of a compact set, and recall that the induction manifolds $\hat{M}$ are isometric to annuli in $S^3_\delta \times C(S^3_{1-\hat{\epsilon}})$ outside of a compact set. We will therefore outline our gluing constructions purely in terms of annuli, which is where the gluing will take place. If we can accomplish this with the correct behaviors, we can then glue our model space $\mathcal{B}_{j+1}$ and inductive manifolds $\hat{M}_j$ directly into our glued space and finish the inductive construction of M_{j+1}.

Let us first outline the gluing strategy without worrying about smoothness or Ricci curvature. We will end with Proposition 3.4, which will state the end construction in a smooth Ricci preserving manner. We describe this in some generality, with the understanding that we will be applying it as above afterwards. So let $\mathcal{A}' \equiv S^3_\delta \times C(S^3_{1-\epsilon})$ and let $\hat{\mathcal{A}} = S^3_\delta \times B_1(0) \subseteq S^3_\delta \times C(S^3_{1-\hat{\epsilon}})$ with $\Gamma \leq S^1$ a finite group generated by a single element γ whose order is divisible by k. Let $\hat{\Gamma}$ be the group generated by $\hat{\gamma} \equiv \gamma^k$. Consider the action of Γ on $\mathcal{A}'$ induced by the $(1, |\gamma|/k)$-Hopf action. Thus γ Hopf rotates the $S^3_{1-\epsilon}$ factor by $2\pi/k$ and the S^3_δ factor by $2\pi/|\gamma|$. Let us also consider the action of $\hat{\Gamma}$ on $\hat{\mathcal{A}}$ obtained by just rotating the S^3_δ factor by $2\pi/|\hat{\gamma}|$.

Consider k copies of the annulus $\hat{\mathcal{A}}^a \equiv \hat{\mathcal{A}} \times \{a\}$ with $a = 0, \ldots, k - 1$, and note that $\partial \hat{\mathcal{A}}^a = S^3_\delta \times S^3_{1-\hat{\epsilon}}$ isometrically. Our goal is to glue in these k copies into $\mathcal{A}'$ such that there is an induced Γ action on the glued space. We will want that $\hat{\Gamma}$ restricts to the usual actions on both $\mathcal{A}'$ and the glued copies of $\hat{\mathcal{A}}$. To be more precise let $x \in C(S^3_{1-\epsilon})$ be a point which is distance is $10^2 k$ from the origin. Let $x^a \in C(S^3_{1-\epsilon})$ with $a = 0, \ldots, k - 1$ be the k points obtained by Hopf rotating $x^0 = x$ by $2\pi a / k$.

Consider each of the domains $S^3_\delta \times B_1(x^a) \subseteq \mathcal{A}'$, and note that their boundaries are diffeomorphic (and nearly isometric) to $S^3_\delta \times S^3_1$. Note that the $\hat{\Gamma}$ action restricts to actions on each of these domains, while the Γ action simply restricts to an isometry between potentially different pairs of domains. We will want to glue $\hat{\mathcal{A}}^0, \ldots, \hat{\mathcal{A}}^{k-1}$ into the space

$$\mathcal{A}' \setminus \left(\bigcup_a S^3_\delta \times B_1(x^a) \right). \tag{26}$$

In order to perform the gluing we need to define the gluing diffeomorphisms

$$\varphi^a : \partial \hat{\mathcal{A}}^a \to S^3_\delta \times \partial B_1(x^a). \tag{27}$$

Recalling that $\partial \hat{\mathcal{A}}^0 = S^3_\delta \times S^3_1$ and $S^3_\delta \times \partial B_1(x)$ is nearly isometric to $S^3_\delta \times S^3_1$, let us first choose an almost isometry $\varphi^0 : \partial \hat{\mathcal{A}}^0 \to S^3_\delta \times \partial B_1(x)$ which is the identity on the first sphere factor. In particular, it follows that φ^0 commutes with the natural $\hat{\Gamma}$ actions on each of these spaces. Let us then define $\varphi^a : \partial \hat{\mathcal{A}}^a \to S^3_\delta \times \partial B_1(x^a)$ by

$$\varphi^a(y, a) = \gamma^a \cdot \varphi^0(y, 0), \quad y \in \hat{\mathcal{A}}, \tag{28}$$

for $a = 0, \ldots, k - 1$. Note that we could naturally extend the above maps for any $a \in \mathbb{Z}$. However, we would have that $\varphi^k : \partial \hat{\mathcal{A}}^0 \to S^3_\delta \times \partial B_1(x^0)$ would not be the same mapping as φ^0. Indeed, we see that $\varphi^k = \gamma^k \cdot \varphi^0 = \hat{\gamma} \cdot \varphi^0$. To understand the implications of this consider the glued space

$$\tilde{\mathcal{A}} \equiv \left(\mathcal{A}' \setminus \bigcup_a S^3_\delta \times B_1(x^a) \right) \bigcup_{\varphi^a} \hat{\mathcal{A}}^a, \tag{29}$$

where we have plucked out the k domains $S^3_\delta \times B_1(x^a)$ and plugged in the new annular regions $\hat{\mathcal{A}}^a$. The new space $\tilde{\mathcal{A}}$ is still isometrically of the form $S^3_\delta \times C(S^3_{1-\epsilon})$ near the origin and infinity. The effect of the gluing maps is that the $\hat{\Gamma}$ action on $\hat{\mathcal{A}}$ extends to a Γ action on $\tilde{\mathcal{A}}$. To understand this action, we need to describe the action of γ on $\bigcup_a \hat{\mathcal{A}}^a$. The latter is given by

$$\gamma \cdot (y, a) = (y, a + 1), \quad a = 0, \ldots, k - 2$$
$$\gamma \cdot (y, k - 1) = (\hat{\gamma} \cdot y, 0) \tag{30}$$

for every $y \in \hat{\mathcal{A}}$. In particular, the action of $\hat{\Gamma}$ restricts to the expected action on each piece of the gluing.

The main Proposition of this step is to show that, up to some altering of constants, the above construction can be smoothed to preserve nonnegative Ricci curvature:

Proposition 3.4 (Step 3: Action Extension). *Let $\epsilon, \epsilon', \delta > 0$ with $0 < \epsilon - \epsilon' \leq \frac{1}{10^2}\epsilon$, and let $\hat{\Gamma} \leq \mathbb{Q}/\mathbb{Z} \subseteq S^1$ be a finite subgroup with $\Gamma = \langle \gamma, \hat{\Gamma} \rangle$ such that $\hat{\gamma} \equiv \gamma^k$ is the generator of $\hat{\Gamma}$. Then for $\hat{\epsilon} \leq \hat{\epsilon}(\epsilon, \epsilon')$ there exists a pointed space $(\tilde{\mathcal{A}}, p)$, isometric to a smooth Riemannian manifold with Ric ≥ 0 away from $k + 1$ three spheres, with an isometric and free action by Γ such that*

(1) *There exists a Γ-invariant set $B_{10^{-1}}(p) \subseteq U' \subseteq B_{10}(p)$ which is isometric to $S^3_\delta \times B_1(0) \subseteq S^3_\delta \times C(S^3_{1-\epsilon'})$ and such that Γ is induced by the $(1, |\gamma|/k)$-Hopf action on $S^3_\delta \times S^3_{1-\epsilon'}$,*

(2) *There exists a Γ-invariant set $B_{10^3 k}(p) \subseteq U \subseteq B_{10^5 k}(p)$ such that $\tilde{\mathcal{A}} \setminus U$ is isometric to $A_{10^4 k, \infty}(0) \times S^3_\delta \subseteq S^3_\delta \times C(S^3_{1-\epsilon})$ and such that Γ is induced by the $(1, |\gamma|/k)$-Hopf action on $S^3_\delta \times S^3_{1-\epsilon}$*

(3) *There exist $\hat{\Gamma}$-invariant sets $S^3_\delta \times B_{2^{-1}}(x^a) \subseteq V^a \subseteq S^3_\delta \times B_2(x^a)$ with $d(S^3_\delta \times \{x^a\}, S^3_\delta \times \{p\}) = 10^2 k$ which are isometric to $S^3_\delta \times B_1(0) \subseteq S^3_\delta \times C(S^3_{1-\hat{\epsilon}})$ and such that $\hat{\Gamma}$ is induced by the $(1, 0)$-Hopf action on $S^3_\delta \times S^3_{1-\hat{\epsilon}}$.*

Remark 3.6. It is important to observe that $\hat{\epsilon}(\epsilon, \epsilon')$ depends on the choices of ϵ and $\epsilon > \epsilon'$, however it does not depend on the choice of δ.

Constructing M_{j+1}: Let us now apply Proposition 3.4 in order to finish the construction of M_{j+1}. Let us take in the above $\Gamma = \Gamma_{j+1}$ and $\hat{\Gamma} = \Gamma_j$, and let us choose $\epsilon = \epsilon_{j+1}$ with $\epsilon' = \epsilon_{j+1} \cdot \frac{99}{100}$. Recall that the construction of $\hat{M}_j$ in Section 2 depended on a choice of $\hat{\epsilon}_j$, which had not yet been fixed. Let us now use Proposition 3.4 in order to choose $\hat{\epsilon}_j = \hat{\epsilon}_j(\epsilon_{j+1})$. From this we now have from Proposition 3.3 a well defined R_j and $\hat{\delta}_j$. Finally let us now choose $\delta = \hat{\delta}_j$ in the application of Proposition 3.3, so that we have built the space $\tilde{\mathcal{A}}_j$. After rescaling $\tilde{\mathcal{A}}_j \rightarrow (R_j r_j)\tilde{\mathcal{A}}_j$ by $R_j r_j$ observe that there exists $U \subseteq \tilde{\mathcal{A}}_j$ which is isometric to $S^3_{\hat{\delta}_j R_j r_j} \times B_{R_j r_j}(0) \subseteq S^3_{\hat{\delta}_j R_j r_j} \times C(S^3_{1-\epsilon'})$, and also observe that the domains V^a are isometric to $S^3_{\hat{\delta}_j R_j r_j} \times B_{R_j r_j}(0) \subseteq S^3_{\hat{\delta}_j R_j r_j} \times C(S^3_{1-\hat{\epsilon}_j})$.

Finally, let us consider the base model $\mathcal{B}_{j+1} = \mathcal{B}(\epsilon', \hat{\delta}_j R_j r_j)$ from Proposition 3.1. We see we can glue it isometrically into $U \subseteq \tilde{\mathcal{A}}_j$. Additionally we can isometrically glue $\hat{M}_j$ into each $V^a \subseteq \tilde{\mathcal{A}}_j$. The resulting space is M_{j+1}. If we define $p_{j+1} = p^0_j$ to be the basepoint of the copy of M_j glued into V^0, then we can define $r_{j+1} \equiv d(p_{j+1}, \gamma_{j+1} \cdot p_{j+1})$ and δ_{j+1} through the formula $\delta_{j+1} r_{j+1} \equiv \hat{\delta}_j R_j r_j$. This completes the induction step of the construction. In particular, we have proved Theorem 1.1 up to the proofs of Propositions 3.1, 3.3, and 3.4. □

4. PRELIMINARIES

In our constructions we will exploit the well known expressions for the Ricci curvature in various setups. We will recall and record some of them here with the relevant sources.

4.1. **Riemannian Submersions.**
The first special case we recall is that of a Riemannian submersion with totally geodesics fibers. Our setup is that we have Riemannian manifolds (M^n, g) and (B, g_b) together with a Riemannian submersion

$$\pi : M \xrightarrow{F} B . \tag{31}$$

We will assume throughout this section that the fibers $F_x \equiv \pi^{-1}(x)$ are totally geodesic submanifolds of M.

Throughout we will let $U, V, ..$ denote vertical vector fields on M, so $U, V \in TF \equiv \mathcal{V} \subseteq TM$, and we will let $X, Y, ..$ denote horizontal vector fields on M, so $X, Y \in T^\perp F \equiv \mathcal{H} \subseteq TM$. Though we have assumed the fibers are totally geodesic, there is still a remaining piece of structure, namely the integrability tensor defined by

$$A_{E_1}E_2 := \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2 \,, \tag{32}$$

where our notation $\mathcal{V}E$ and $\mathcal{H}E$ denote the projections of E to the corresponding subspaces, see [Be, Definition 9.20]. Recall that if X, Y are horizontal vector fields then

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y] \,. \tag{33}$$

For the proposition below we refer the reader to O' Neil [O] (see also [Be, Proposition 9.36]):

Proposition 4.1 (Ricci curvature for Riemannian submersions)**.** *Let* $\pi : (M, g) \to (B, g_B)$ *be a Riemannian submersion with totally geodesic fibers F. Then*

$$Ric_M(U, V) = Ric_F(U, V) + (AU, AV) \,, \tag{34}$$

$$Ric_M(U, X) = (div_B A[X], U) \,, \tag{35}$$

$$Ric_M(X, Y) = Ric_B(X, Y) - 2(A_X, A_Y) \,, \tag{36}$$

where Ric_F stands for the Ricci curvature of the fiber with the induced Riemannian metric and Ric_B is the Ricci curvature of the base, understood as a horizontal tensor on M.

Remark 4.1. Note that in the above proposition we have the explicit expressions

$$(AU, AV) := \sum_i g(A_{X_i}U, A_{X_i}V) \,,$$

$$(A_X, A_Y) := \sum_i g(A_X X_i, A_Y X_i) \,,$$

$$\mathrm{div}_B A := \sum_i (\nabla_{X_i} A)(X_i, \cdot) \,, \tag{37}$$

where $\{X_i\}$ is an orthonormal basis of the horizontal space.

It is helpful to record how the Ricci curvature on the total space of the Riemannian submersion changes when we perform the so called *canonical variation* of the metric, i.e. we define g_t by leaving the horizontal distribution unchanged, the metric on the base unchanged, and scaling the metric on the fibers by a factor t. Below we shall assume again that the fibers are totally geodesic, see [Be, Proposition 9.70].

Corollary 4.2. *Let* $\pi : (M, g) \to (B, g_B)$ *be a Riemannian submersion with totally geodesic fibers and let g_t the Riemannian metric on M obtained by scaling the fibers metrics with a factor t. Then*

$$Ric_t(U, V) = Ric_F(U, U) + t^2(AU, AV) \,, \tag{38}$$

$$Ric_t(X, U) = t(div_B A[X], U) \,, \tag{39}$$

$$Ric_t(X, Y) = Ric_B(X, Y) - 2t(A_X, A_Y) \,. \tag{40}$$

Above, A denotes the integrability tensor of the Riemannian submersion $\pi : (M, g) \to (B, g_B)$.

A particularly natural form of Riemannian submersion is obtained via principal bundles:

Definition 4.3 (Riemannian Principal Bundle). We call a principal G-bundle $P \xrightarrow{G} B$ a Riemannian G-principal bundle if it is equipped with a Riemannian metric g_P which is invariant under the G-action.

Observe that if P is a Riemannian G-principal bundle then it well defines a metric g_B on the base B through the quotient, and as the horizontal distribution $\mathcal{H} = (TG_x)^{\perp}$ is right invariant it well defines a principal connection $\xi \in \Omega^1(P; \mathfrak{g})$. The remaining information is a family of right invariant metrics on the G fibers, which may equivalently be viewed as a metric on the adjoint vector bundle over B. Conversely, this triple of data well defines a metric g_P on P which is invariant under the G action.

Consider the case when G is simple and there exists a unique bi-invariant metric $\langle, \rangle$ on G up to scaling. Then given a metric g_B on the base and ξ a connection one form, we can write a Riemannian principal bundle structure on P as

$$g_P(X, Y) \equiv g_B(\pi_*[X], \pi_*[Y]) + \lambda(x) \langle \xi[X], \xi[Y] \rangle_G , \tag{41}$$

where $\lambda : B \to \mathbb{R}^+$ determines the scaling of the fibers. It follows from Vilms [Vi] that this metric has totally geodesic fibers iff $\lambda(x) = \lambda$ is a constant.

4.2. **Riemannian Submersions and Circle Bundles.** Let us now restrict ourselves to the case of a Riemannian S^1-principal bundle, so that $\pi : M \to B$ is the total space of an S^1-principal bundle over B. Note that if (B, g_B) is a Riemannian manifold, then an S^1-invariant metric on M is well defined by the additional data of a principal connection $\eta \in \Omega^1(M)$ and a smooth $f : B \to \mathbb{R}^+$ which prescribes the length of the S^1 fiber above a point. If ∂_t is the invariant vertical vector field coming from the S^1 action, then we have the expressions

$$\mathcal{H} = \ker \eta ,$$
$$\eta[\partial_t] = 1 ,$$
$$g(\partial_t, \partial_t) = f^2 . \tag{42}$$

In the case of an S^1 bundle we have that $d\eta = \pi^*\omega$ where $\omega \in \Omega^2(B)$ is the curvature 2-form, which relates to the integrability tensor A on M by

$$A(X, Y) = -\frac{1}{2}\omega[X, Y]\,\partial_t . \tag{43}$$

The following proposition is borrowed from [GPT, Lemma 1.3], where it was used to show that any principal S^1 bundle $\pi : M \to B$ admits an S^1-invariant metric of positive Ricci curvature when the base (B, g_B) has positive Ricci curvature and the total space has finite fundamental group.

Proposition 4.4. *Let $M \xrightarrow{S^1} B$ be a Riemannian S^1-principal bundle as above with X a unit horizontal vector and $U = f^{-1}\partial_t$ a unit vertical vector. Then*

$$Ric(U, U) = -\frac{\Delta f}{f^2} + \frac{f^2}{4}|\omega|^2 \, , \tag{44}$$

$$Ric(U, X) = \frac{1}{2}\left(-f\,(div_B\omega)\,(X) + 3\omega[X, \nabla f]\right) \tag{45}$$

$$Ric(X, X) = Ric_B(X, X) - \frac{f^2}{2}|\omega[X]\,|^2 - \frac{\nabla^2 f(X, X)}{f^2} \, , \tag{46}$$

where it is understood, when necessary, that we are identifying the horizontal vector field X with an element of TB.

4.3. **Doubly Warped Products.** A particularly common ansatz is that of the doubly warped factor. Let $(N_1^{n_1}, h_1)$ and $(N_2^{n_2}, h_2)$ be Riemannian manifolds. Consider the space $M \equiv (r_-, r_+) \times N_1 \times N_2 \subseteq \mathbb{R}^+ \times N_1 \times N_2$ with the metric

$$g \equiv dr^2 + f_1(r)^2 g_{N_1} + f_2(r)^2 g_{N_2} \, . \tag{47}$$

Let $X_1 \in TN_1$ and $X_2 \in TN_2$ represent unit directions on N_1 and N_2 respectively. Then the nonzero Ricci curvatures on M are given by

$$\begin{aligned}
\text{Ric}(\partial_r, \partial_r) &= -n_1\frac{f_1''}{f_1} - n_2\frac{f_2''}{f_2} \\
\text{Ric}(X_1, X_1) &= \text{Ric}_{N_1}(X_1, X_1) - \frac{f_1''}{f_1} + \left(\frac{f_1'}{f_1}\right)^2 - \frac{f_1'}{f_1}\left(n_1\frac{f_1'}{f_1} + n_2\frac{f_2'}{f_2}\right), \\
\text{Ric}(X_2, X_2) &= \text{Ric}_{N_2}(X_2, X_2) - \frac{f_2''}{f_2} + \left(\frac{f_2'}{f_2}\right)^2 - \frac{f_2'}{f_2}\left(n_1\frac{f_1'}{f_1} + n_2\frac{f_2'}{f_2}\right),
\end{aligned} \tag{48}$$

see for instance [Pe], dealing with the case where $(N_1^{n_1}, h_1)$ and $(N_2^{n_2}, h_2)$ are isometric to standard spheres.

5. Step 1: The Base Model

In this Section we complete the construction of the base model $\mathcal{B}(\epsilon, \delta)$. It is a straightforward construction, however as we have discussed previously the base models play the role of the $S^3 \times \mathbb{R}^4$ vertices in the topological construction. As such they are the starting point for each new inductive step and it seems worthwhile to record their geometric properties. Our main goal is to prove Proposition 3.1, which we restate below for the ease of readability:

Proposition 5.1 (Step 1: The Model Space). *For each $\delta > 0$ and $1 > \epsilon > 0$, there exists a smooth manifold $\mathcal{B}^7 = \mathcal{B}(\epsilon, \delta)$ such that the following hold:*

(1) *$(\mathcal{B}^7, g_B, p)$ is a complete Riemannian manifold with $Ric \geq 0$, it is diffeomorphic to $S^3 \times \mathbb{R}^4$.*

(2) *There exists $B_{10^{-3}}(p) \subseteq U \subseteq B_{10^{-1}}(p)$ such that $\mathcal{B} \setminus U$ is isometric to $S^3_\delta \times A_{10^{-2}, \infty}(0) \subseteq S^3_\delta \times C(S^3_{1-\epsilon})$*

(3) *There is an isometric $T^2 = S^1 \times S^1$ action on $\mathcal{B}$ for which on $\mathcal{B} \setminus U \approx S^3_\delta \times C(S^3_{1-\epsilon})$ the first S^1 acts on the S^3_δ factor by a globally free (left) Hopf rotation and the second S^1 acts on the cross sections $S^3_{1-\epsilon}$ of the cone factor by (left) Hopf rotation.*

(4) *The S^1-action induced by the homomorphic embedding $S^1 \ni \theta \mapsto (a\theta, b\theta) \in T^2$ is free whenever $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ are coprime and $a \neq 0$.*

To build $\mathcal{B}(\epsilon, \delta)$ let us start with the geometry of $S^3_\delta \times \mathbb{R}^4$. Note that if we view $\mathbb{R}^4 = C(S^3_1)$ then we can write this metric as

$$g_0 \equiv dr^2 + \delta^2 g_{S^3} + r^2 g_{S^3}. \tag{49}$$

We will warp this metric by considering the ansatz

$$g \equiv dr^2 + \delta^2 g_{S^3} + h(r)^2 g_{S^3}. \tag{50}$$

If we let a, b, c denote the directions on the first S^3 factor and i, j, k on the second S^3 factor, then we can compute the nonzero terms of the Ricci curvature of this ansatz as

$$\mathrm{Ric}_{rr} = -3\frac{h''}{h},$$

$$\mathrm{Ric}_{aa} = \frac{2}{\delta^2},$$

$$\mathrm{Ric}_{ii} = 2\frac{1 - (h')^2}{h^2} - \frac{h''}{h}. \tag{51}$$

Let us now consider any smooth function $h(r)$ of the following form:

$$h(r) \equiv \begin{cases} r & \text{if } r \leq 10^{-5}, \\ h'' < 0, & \text{if } 10^{-5} \leq r \leq 10^{-3}, \\ 10^{-4} + (1 - \epsilon)(r - 10^{-4}) & \text{if } r \geq 10^{-3}. \end{cases}$$

We can build a function as above by smoothing out $h(r) = \min\{r, 10^{-4} + (1 - \epsilon)(r - 10^{-4})\}$. Note these two linear functions intersect at $r = 10^{-4}$. Observe that any such function satisfies $|h'| \leq 1$ on $(0, +\infty)$ as $h'' \leq 0$, and so we can compute

$$\mathrm{Ric}_{rr} \geq 0,$$

$$\mathrm{Ric}_{aa} = \frac{2}{\delta^2} > 0,$$

$$\mathrm{Ric}_{ii} \geq 0. \tag{52}$$

In particular, this metric has nonnegative Ricci curvature. The $S^1 \times S^1$ torus Hopf action on $S^3 \times S^3$ induces an isometric action with respect to g from our warping coordinates. On the domain $\{r \geq 10^{-3}\}$ we can write this metric

$$dr^2 + \delta^2 g_{S^3} + (10^{-4} + (1 - \epsilon)(r - 10^{-4}))^2 g_{S^3}. \tag{53}$$

If we consider the domain $U = \{r \leq 10^{-2} - 10^{-4}\}$ then we see that $\mathcal{B} \setminus U$ is isometric to the annulus $S^3_\delta \times A_{10^{-2}, \infty} \subseteq S^3_\delta \times C(S^3_{1-\epsilon})$, as claimed. $\quad\square$

6. Equivariant Mapping Class Group and Ricci Curvature

Recall that we have equipped $S^3 \times S^3$ with the left (a, b)-Hopf action given by

$$\theta \cdot_{(a,b)} (s_1, s_2) = (a\theta \cdot s_1, b\theta \cdot s_2), \quad \theta \in S^1, \tag{54}$$

where $\theta \cdot s$ is the classical (left) Hopf action on S^3. We will consider in this Section $S^3 \times S^3$ with the standard metric $g_{S^3 \times S^3}$ and the distinct $(1, k)$ and $(1, 0)$ actions. Our goal is to connect these two spaces by a family of smooth S^1-invariant Riemannian manifolds $(S^3 \times S^3, g_t)$ with positive Ricci curvature which begin and end with the standard geometry but with these distinct actions. The subtle point is that while g_1 is isometric to g_0, necessarily it cannot be equal to the standard metric as a tensor and we will have $g_1 = \phi^* g_{S^3 \times S^3}$ for some mapping class nontrivial diffeomorphism ϕ. The precise statement is the following, which is the main goal of this Section:

Theorem 6.1 (Equivariant Mapping Class and Ricci). *Let $g_0 = g_{S^3 \times S^3}$ be the standard metric on $S^3 \times S^3$, and let $k \in \mathbb{Z}$. Then there exist a diffeomorphism $\phi : S^3 \times S^3 \to S^3 \times S^3$ and a family of metrics $(S^3 \times S^3, g_t)$ so that*

(1) *$Ric_t > 0$ for all $t \in [0, 1]$*
(2) *The S^1-action $\cdot_{(1,k)}$ on $S^3 \times S^3$ is an isometric action for all g_t.*
(3) *$g_1 = \phi^* g_0$ with $\phi(\theta \cdot_{(1,k)} (s_1, s_2)) = \theta \cdot_{(1,0)} \phi(s_1, s_2)$.*

Remark 6.1. We will show in Section 6.9 that in the case $k = 1$ we can pick $\phi = \phi_1$ to be the diffeomorphism specified in the Examples of Section 6.1.

Remark 6.2. The above Theorem is for the $(1, k)$-actions induced by the left Hopf actions. Of course, one can equally well deal with the right actions or mapping $(k, 1)$-actions to $(0, 1)$-actions.

The proof will be divided into various steps. In Section 6.2 we will begin by studying the geometry of $N \equiv S^1 \backslash S^3 \times S^3$, that is the quotient of $S^3 \times S^3$ by the $(1, k)$-Hopf action. We will see that we can view N itself as a Riemannian S^3-principal bundle over S^2, so that we have the viewpoint

$$S^3 \times S^3 \xrightarrow{S^1} N \xrightarrow{S^3} S^2. \tag{55}$$

Each of these bundle structures are Riemannian submersions, however the connections will be nonstandard and the S^3 fibers will not have the standard geometry. In particular, $N \xrightarrow{S^3} S^2$ is an S^3 bundle over S^2 with twisted connection and totally geodesic fibers. We notice that the induced metric on the fibers will be right invariant but not biinvariant.

Our construction will now proceed as follows. In the first step of the construction in Section 6.3 we will construct a family $(N_t, h_t) = S^1 \backslash (S^3 \times S^3, g_t)$ with positive Ricci which begins at $N_0 \equiv N$ as above, and ends at $N_{1/2}$. Each N_t will itself be the total space of a Riemannian S^3-principal bundle over S^2, where for $t \in [0, 1/2]$ the connection will be fixed but the geometry of the S^3 fibers will round off so that $N_{1/2}$ will become an S^3 bundle over S^2 with small but round S^3 fibers. This construction will essentially take place directly on $(S^3 \times S^3, g_t)$, however it will be crucial in the remaining steps that we emphasize the geometry

of N_t during this process.

The second and third steps of the construction will focus on just changing the geometry of (N_t, h_t). For each $t \in [1/2, 3/4]$ we will have in Section 6.4 that N_t is a Riemannian S^3-principal bundle with totally geodesic fibers which are isometric to spheres. The connection of this bundle on $N_{1/2}$ is highly nontrivial, and in the second step we will vary this connection until we arrive at a flat connection on $N_{3/4}$. The sphere fibers may shrink during this process in order to preserve the positive Ricci condition. It is worth emphasizing that although the connection at the end is flat, it will look highly nontrivial in the original coordinates of $N = S^1 \backslash S^3 \times S^3$. In the third step of Section 6.5, where we now have a flat connection with small round sphere fibers, we will increase the size of the sphere fiber until we arrive at $N_1 = S^2 \times S^3$ isometrically.

The last steps of the construction will view $S^3 \times S^3$ as the total space of a Riemannian S^1-principal bundle over N_t. This S^1 bundle structure will generate our family of actions, and in particular by definition will begin with the $(1, k)$ action on $S^3 \times S^3$. A metric g_t on this total space is then well defined by an S^1-principal connection η_t and fiber size functions $f_t : N_t \to \mathbb{R}$, see Section 4.1. We will choose η_t to be a Yang-Mills connection under the Coulumb gauge, which will allow us to show $\mathrm{Ric}_t \geq 0$. In Step 5 of Section 6.7 we will further vary the fiber size f_t of the circle bundle so that we can push the Ricci curvature up to become strictly positive. Recall the construction of N_t ends with $N_1 = S^2 \times S^3$ isometrically. Thus we will see that the Yang-mills connection must give us a total space which is isometric to $S^3 \times S^3$ with the bundle action being the $(1, 0)$ action. In Section 6.8 we will put all of these ingredients together in order to complete the proof of Theorem 6.1.

6.1. **Examples: Equivariant Mappings** $\phi_k : S^3 \times S^3 \to S^3 \times S^3$**.** In this subsection let us first present some explicit examples of equivariant mappings which satisfy:

$$\phi_k(\theta \cdot_{(1,k)} (s_1, s_2)) = \theta \cdot_{(1,0)} \phi_k(s_1, s_2). \tag{56}$$

We will see in Theorem 6.1 that we will be able to take these explicit mappings as our end diffeomorphisms.

6.1.1. *The Case $k = 1$.* Let us treat $k = 1$ and $k > 1$ separately. In the case of $k = 1$ we will view each S^3 as the corresponding Lie Group SU(2) and so write $(s_1, s_2) \in S^3 \times S^3$. Let us explicitly define the mapping

$$\phi_1(s_1, s_2) = (s_1, s_1^{-1} s_2). \tag{57}$$

It follows that the left $(1, 1)$-action pushes forward to the left $(1, 0)$ action, that is $\phi_1(\theta \cdot_{(1,1)} (s_1, s_2)) = \theta \cdot_{(1,0)} \phi_1(s_1, s_2)$.

6.1.2. *The Case $k > 1$.* The case $k > 1$ is a little trickier. Let $u, z \in S^3$. We write $u = (u_1, u_2)$, $z = (z_1, z_2)$, where $u_1, u_2, z_1, z_2 \in \mathbb{C}$. The diffeomorphism is given by

$$
\begin{aligned}
\phi_k(u_1, u_2, z_1, z_2) &\equiv \left(u_1, u_2, \frac{1}{\sqrt{|u_1|^{2k} + |u_2|^{2k}}} (\bar{u}_1^k, -u_2^k) \cdot (z_1, z_2)\right) \\
&= \left(u_1, u_2, \frac{1}{\sqrt{|u_1|^{2k} + |u_2|^{2k}}} (\bar{u}_1^k z_1 + u_2^k \bar{z}_2, -u_2^k \bar{z}_1 + \bar{u}_1^k z_2)\right).
\end{aligned}
\tag{58}
$$

where $\cdot$ denotes the product of S^3 as Lie-group.

With this choice, we have the claimed equivariance property that

$$
\phi_k(\theta \cdot_{(1,k)} (u_1, u_2, z_1, z_2)) = \theta \cdot_{(1,0)} \phi_k(u_1, u_2, z_1, z_2).
\tag{59}
$$

Moreover, ϕ_k is equivariant with respect to the S^3-action on the second S^3-factor, i.e.

$$
\phi_k(u, z \cdot g) = \phi_k(u, z) \cdot g, \quad \text{for every } g \in S^3.
\tag{60}
$$

Note this corresponds to the previous construction when $k = 1$.

6.2. **The Geometry of $N \equiv S^1 \backslash S^3 \times S^3$.** Let us begin with $S^3 \times S^3$ endowed with the product Lie group structure. Let U_1, U_2, U_3 be an orthonormal basis of right invariant vector fields on the first S^3 factor, and similarly let V_1, V_2, V_3 be an orthonormal basis of right invariant vector fields on the second S^3 factor. Let U_1 and V_1 be the right invariant vector fields induced from the left Hopf actions. We will write U_j^* and V_j^* to denote the dual basis of one forms.

Let us now define $(N, h) \equiv S^1 \backslash S^3 \times S^3$ to be the isometric quotient of $S^3 \times S^3$ by the left $(1, k)$-Hopf action, which is a free and isometric action. We have that $\pi_{(1,k)} : S^3 \times S^3 \to N$ is a principal S^1-bundle, and it is endowed with the Yang-Mills principal connection

$$
\eta_0 := \frac{1}{1 + k^2} \left(U_1^* + k V_1^*\right).
\tag{61}
$$

We shall denote the projection $\pi_{(1,k)}(s_1, s_2) \in N$ as $[s_1, s_2]$, where $(s_1, s_2) \in S^3 \times S^3$. Note that the right S^3 action on the second factor of $S^3 \times S^3$ commutes with the left $(1, k)$ action. The quotient action on N:

$$
[s_1, s_2] \cdot s = [s_1, s_2 \cdot s], \quad s \in S^3, \quad [s_1, s_2] \in N,
\tag{62}
$$

is a free and isometric action, and thus $\pi : N \xrightarrow{S^3} S^2$ admits a structure of principal S^3-bundle over S^2. Note that

$$
\pi([s_1, s_2]) = \pi_{\text{Hopf}}(s_1),
\tag{63}
$$

is the same as the Hopf projection on the first factor.

Remark 6.3. We can understand N in the following manner. Begin with $S^3 \xrightarrow{S^1} S^2$, viewed as an S^1-principal bundle over S^2 with respect to the *left* Hopf action. Consider the homomorphism $\rho : S^1 \to S^3$, $\rho(z) = z^{-k}$ where $z \in S^1$ is identified with a complex number and $z^{-k} \in S^3$ with a unit quaternion. Then we can identify N as the associated S^3 bundle over S^2 under this representation. This point of view is particularly convenient for writing coordinate expressions of the above.

Our first claim is about the isomorphism class of the principal S^3-bundle $\pi : N \to S^2$.

Lemma 6.2. *The principal S^3-bundle $\pi : N \to S^2$ is isomorphic to the trivial bundle $S^2 \times S^3$.*

Proof. It is a classical fact that the isomorphism classes of principal G-bundles over spheres S^n are in bijection with $\pi_{n-1}(G)$, where G is any connected Lie group. See for instance [St, Chapter 18]. As $n = 2$, and $G = S^3$ is simply connected, we infer that any principal S^3-bundle over S^2 is isomorphic to the trivial one. $\square$

Remark 6.4. It was already noted in [PT, Example 4.1] that N is diffeomorphic to $S^2 \times S^3$, with an argument based on the classification of simply connected 5-manifolds, see also [WZ]. The principal S^3-bundle structure of N plays a key role for our purposes.

6.2.1. *Geometry on N.* Our next goal is to understand the metric h on N.

Begin by observing that if we endow the base space S^2 with $\frac{1}{4}g_{S^2}$, the round metric of radius $\frac{1}{2}$, then $\pi : (N, h) \to S^2_{1/2}$ is a Riemannian submersion. To see this let us write $S^2 = N/S^3 = S^1\backslash S^3 \times S^3/S^3$. If we quotient by the S^3 action first then we have $S^2 = S^1\backslash S^3$, where the $(1, k)$ action descends to the $(1, 0)$ action on S^3 as we have quotiented out the second factor. Thus the quotient space is the usual Hopf quotient, which is the sphere of radius $\frac{1}{2}$.

Notice that over $(S^3 \times S^3, g_0)$ we have the orthonormal basis

$$\frac{1}{\sqrt{1 + k^2}}(U_1 + kV_1), \ U_2, \ U_3, \ \frac{1}{\sqrt{1 + k^2}}(-kU_1 + V_1), \ V_2, \ V_3 \,, \tag{64}$$

where as before U_1, U_2, U_3 are the right invariant vector fields in the first S^3 factor, and V_1, V_2, V_3 are the right invariant vector fields in the second S^3 factor. The first vector field is vertical with respect to the $(1, k)$-projection map $\pi_{(1,k)} : (S^3 \times S^3, g_0) \to (N, h)$, while the last five vector fields are horizontal.

The following claims are almost immediate:

  (1) $d\pi_{(1,k)}[V_2]$, $d\pi_{(1,k)}[V_3]$, $d\pi_{(1,k)}\left[\frac{1}{\sqrt{1+k^2}}(-kU_1 + V_1)\right]$ span the vertical directions in N.
  (2) $d\pi_{(1,k)}[U_2]$, $d\pi_{(1,k)}[U_3]$ span the horizontal directions in N.

In order to check (1), it is enough to observe that the three vectors span a three-dimensional subspace of the tangent of N, and belong to the kernel of $d\pi$ since $\pi \circ \pi_{(1,k)}(g_1, g_2) = \pi_{\text{Hopf}}(g_1)$, by definition.

Claim (2) follows immediately since the span of the vectors in (2) is two-dimensional and orthogonal to the span of the vectors in (1). The orthogonality follows from the fact that (64) is an orthonormal frame: $h(d\pi_{(1,k)}[U_2], d\pi_{(1,k)}[V_2]) = g_0(U_2, V_2) = 0$, and similarly for the other vectors.

6.3. **Step 1: Squishing the Fibers.** By looking at the last three vector fields in (64) we see that the metric on the S^3 fibers of N is right invariant, but not left invariant. We do see that it is invariant under the left S^1 action, the action of which just rotates the V_2, V_3 plane. Our first step of the construction will be to round off the metric so that the fibers become bi-invariant round spheres. We will work directly to build a family of

metrics $(S^3 \times S^3, g_t)$ for $t \in [0, 1/2]$ which continue to be invariant by the left $S^1 \times S^1$ Hopf action, and invariant by the right S^3 action. In particular, for each such t we can define $(N_t, h_t) = (N, h_t) \equiv S^1 \backslash (S^3 \times S^3, g_t)$ where we have quotiented out by the $(1, k)$-Hopf action. As there are many properties of this family that we will need in the sequel, let us summarize the end results of our constructions for this step:

Lemma 6.3. *For $t \in [0, 1/2]$ there exists smooth families $(S^3 \times S^3, g_t)$, $(N, h_t) \equiv S^1 \backslash (S^3 \times S^3, g_t)$ which satisfy $Ric_{h_t}, Ric_{g_t} > 0$ and such that:*

(i) *$\pi : (N, h_t) \xrightarrow{S^3} S^2_{1/2} = (S^2, \frac{1}{4} g_{S^2})$ is a Riemannian S^3-principal bundle with totally geodesic fibers. Further the S^3 fibers in $N_{1/2}$ are bi-invariant spheres;*

(ii) *$\pi_{(1,k)} : (S^3 \times S^3, g_t) \to (N_t, h_t)$ is a Riemannian S^1-principal bundle with total geodesic fibers and constant connection $\eta_t = \eta_0$. Further, the connections η_t are Yang-Mills and in Coulomb Gauge.*

Remark 6.5. We will also take our construction so that g_t, h_t are constant metrics near $t = 0, \frac{1}{2}$.

Remark 6.6. Recall that we call a principal G-bundle $P \xrightarrow{G} B$ a Riemannian G-principal bundle if P is equipped with a metric g_P which is invariant under the right G action. Recall such a metric defines a principal connection ξ and a family over B of right invariant metrics on G.

To define our family of metrics on $S^3 \times S^3$ will have as a global orthonormal basis the vector fields

$$T^t \equiv \frac{1}{f_t}(U_1 + kV_1), \quad X_2 \equiv U_2, \quad X_3 \equiv U_3,$$

$$W_1^t \equiv \frac{1}{a_t}(-kU_1 + V_1), \quad W_2^t \equiv \frac{1}{a_t b_t} V_2, \quad W_3^t \equiv \frac{1}{a_t b_t} V_3. \tag{65}$$

Let us begin with some remarks on this basis. Observe that $U_1 + kV_1$ is the direction associated to the $(1, k)$-action, and so T is the unit direction associated to the $(1, k)$ action. In particular, we have the S^1-fibers have length $2\pi f_t$, and in order for the geometry at $t = 0$ to be the standard geometry we will choose $f_t = \sqrt{1 + k^2}$, $a_t = \sqrt{1 + k^2}$ and $b_t = \frac{1}{\sqrt{1 + k^2}}$ for t near zero.

The vector fields X_2, X_3 represent the horizontal directions associated to the base S^2. As the geometry is symmetric with respect to the X_2, X_3 and W_2^t, W_3^t indices we have that the left $S^1 \times S^1$ actions are isometric actions. As X_2, X_3 are time independent we have that $N_t / S^3 = S^2_{1/2}$ remains a sphere of radius $1/2$.

The directions W_1^t, W_2^t, W_3^t represent the directions horizontal with respect to the S^1 action, but will induce vertical vector fields on N with respect to the right S^3 action. The orbit of the $(k, 1)$ action in the torus generated by $\{T^t, W_1^t\}$ is $\frac{1}{\sqrt{1+k^2}}$ dense, and hence the geometry of the S^3 fibers of the Riemannian S^3-bundle $N_t \xrightarrow{S^3} S^2_{1/2}$ can be seen to be determined by the right invariant orthonormal basis $\{\frac{1+k^2}{a_t} V_1, \frac{1}{a_t b_t} V_2, \frac{1}{a_t b_t} V_3\}$.

We will therefore choose our warping function $b_t = b(t)$ as any smooth function such that

$$b(t) = \begin{cases} \frac{1}{\sqrt{1+k^2}} & \text{if } t \text{ is near } 0, \\ \dot{b}_t \geq 0 & \text{if } t \in [0, 1/2], \\ \frac{1}{1+k^2} & \text{if } t \text{ is near } 1/2. \end{cases}$$

We will choose a_t to be nonincreasing shortly. This will play the role of squishing additional positive curvature into the system to compensate for the movement of b_t.

Let us now study some properties about our ansatz. We begin by computing the nonzero brackets of our vector fields. These computations all boil down to using that U_j, V_j are a standard right invariant basis for S^3:

$$[T^t, X_2] = \frac{2}{f_t} X_3, \quad [X_3, T^t] = \frac{2}{f_t} X_2, \quad [X_2, X_3] = \frac{2f_t}{1+k^2} T^t - \frac{2ka_t}{1+k^2} W_1^t,$$

$$[T^t, W_2^t] = \frac{2k}{f_t} X_3, \quad [W_3^t, T^t] = \frac{2k}{f_t} W_2^t,$$

$$[W_1^t, W_2^t] = \frac{2}{a_t} W_3^t, \quad [W_3^t, W_1^t] = \frac{2}{a_t} W_2^t, \quad [W_2^t, W_3^t] = \frac{2kf_t}{(1+k^2)a_t^2 b_t^2} T^t + \frac{2}{(1+k^2)a_t b_t^2} W_1^t,$$

$$[W_1^t, X_2] = -\frac{2k}{a_t} X_3, \quad [X_3, W_1^t] = -\frac{2k}{a_t} X_2. \tag{66}$$

There are several takeaways from the above computations. In order to have an ease in the formulas, let us be abusive in notation and define $W_T^t = T^t$, $W_{X_2}^t = X_2$ and $W_{X_3}^t = X_3$. Observe that the structural and Christofell coefficients

$$c_{ijk} \equiv g_t([W_i^t, W_j^t], W_k^t),$$

$$\Gamma_{ijk} \equiv g_t(\nabla_{W_i^t} W_j^t, W_k^t), \tag{67}$$

are constants and related by $\Gamma_{ijk} = \frac{1}{2}(c_{ijk} + c_{kij} - c_{jki})$. As a first observation note that to be nonvanishing all three of the indices must be distinct, and in particular we can conclude that

$$\nabla_{W_j^t} W_j^t = 0,$$

$$g_t(\nabla_{W_k^t} W_j^t, W_k^t) = 0. \tag{68}$$

To put this into perspective, the first equation will tell us shortly that all the bundles of interest have totally geodesic fibers, and the second equation will tell us that our S^1-connections are Yang-Mills.

6.3.1. *The S^1-Principal Bundle $S^3 \times S^3 \xrightarrow{S^1} N_t$.* Thus let us consider now the S^1-principal bundle $S^3 \times S^3 \xrightarrow{S^1} N$. It follows from (68) that $\nabla_{T^t} T^t = 0$, and hence the S^1-fibers of this principal bundle are totally geodesic. It follows from (65) that the connection 1-form of this bundle is given by the metric dual

$$\eta_t = f_t^{-1} g_t(T^t, \cdot). \tag{69}$$

Note that as $W_j^t \propto W_j^0$ we have that $\eta_t = \eta_0$ is independent of t as a 1-form. We can compute the curvature form $\omega_t = d\eta_t$ of this bundle by

$$\omega_t(W_j^t, W_k^t) = \frac{1}{2f_t}(\langle \nabla_{W_j^t} T, W_k^t \rangle - \langle \nabla_{W_k^t} T, W_j^t \rangle) = \frac{c_{kjT}}{f_t}, \tag{70}$$

so that we can use (66) to write

$$\omega_t = -\frac{2}{1+k^2}(X_2^* \otimes X_3^* - X_3^* \otimes X_2^*) - \frac{2k}{(1+k^2)a_t^2 b_t^2}(W_2^{t,*} \otimes W_3^{t,*} - W_3^{t,*} \otimes W_2^{t,*}). \tag{71}$$

Consequently, it follows from (68) that

$$\mathrm{div}_{N_t}\eta_t = 0\,,$$
$$\mathrm{div}_{N_t}\omega_t = 0\,, \tag{72}$$

and hence η_t is a Yang-Mills connection in Coulomb gauge.

Let us now compute the Ricci curvature Ric_{g_t} of $(S^3 \times S^3, g_t)$ in terms of the Ricci curvature of N_t. We will compute Ric_{N_t} shortly after. Let us use $H \in \mathrm{span}\{X_2, X_3, W_j^t\}$ to denote any unit horizontal vector with respect to the S^1 action. Using Proposition 4.4 we have that

$$\mathrm{Ric}_{g_t}(T^t, T^t) = \frac{1}{4}|\omega_t|_t^2 \geq c(k)\frac{f_t^2}{a_t^2}\,,$$

$$\mathrm{Ric}_{g_t}(T^t, H) = -\mathrm{div}_{N_t}\omega_t(H) = 0\,,$$

$$\mathrm{Ric}_{g_t}(H, H) = \mathrm{Ric}_{h_t}(H, H) - \frac{f_t^2}{2}|\omega_t[H]|^2$$

$$\geq \mathrm{Ric}_{h_t}(H, H) - c(k)\frac{f_t^2}{a_t^2}\,, \tag{73}$$

where we have used that b_t is bounded above and below by functions of k. Our main takeaway is that if $\mathrm{Ric}_{h_t} > 0$ then for $f_t \leq f_t(h_t, a_t)$ we have that the Ricci curvature of $(S^3 \times S^3, g_t)$ is itself positive.

6.3.2. *The Geometry of N_t.* We understand that the right S^3 action on N_t is an isometric action which gives N_t the structure of a Riemannian S^3-principal bundle

$$N_t \xrightarrow{S^3} S^2_{1/2}\,. \tag{74}$$

Let us see that the S^3 fibers of this bundle are totally geodesic. Indeed, on $S^3 \times S^3$ we know that W_1^t, W_2^t, W_3^t span the horizontal subspace induced the S^3 action downstairs. On the other hand, we know by (68) that $\nabla_{W_j^t} W_j^t = 0$. One can compute directly from this that the S^3 fibers are totally geodesic, but let us also explain it geometrically. As $\nabla_{W_j^t} W_j^t = 0$, it follows that the orbits of W_j^t are horizontal geodesics in $S^3 \times S^3$ and thus project to geodesics in N_t which are tangent to the S^3 fibers. As they span the tangent of the S^3 fibers at each point, we see that the fibers are totally geodesic.

To study the Ricci curvature of N_t let us first study the geometry of the S^3 fibers. We again observe from (65) that the geometry of the S^3 fibers are right invariant and invariant under the left S^1 action. Thus we see that geometrically the fibers are a Hopf bundle

$$S^3 \xrightarrow{S^1} S^2_{a_t b_t/2}\,, \tag{75}$$

with fibers of length $\frac{a_t}{1+k^2}$ and the standard Hopf connection. In particular, for $b_t \geq \frac{1}{1+k^2}$ the Ricci curvature of the fibers satisfy

$$\mathrm{Ric}_F \geq \frac{c(k)}{a_t^2}\,. \tag{76}$$

Let us compute the Ricci curvature on N_t. Recall that the vector fields $H \in \text{span}\{X_2, X_3, W_1^t, W_2^t, W_3^t\}$ in $S^3 \times S^3$ horizontally span the tangent space of N_t. Though they do not well define vector fields on N_t (one needs to lift a point $[s_1, s_2] \in N_t$ to $(s_1, s_2) \in S^3 \times S^3$ for such an association), the value of $\text{Ric}_{N_t}(H, H)$ is independent of the lift. Now let $\xi_t \in \Omega^1(N_t; su(2))$ be the connection one form for the bundle $N_t \xrightarrow{S^3} S^2_{1/2}$ with $\Omega_t = d\xi_t + \frac{1}{2}[\xi_t \wedge \xi_t]$ the curvature form. Using Proposition 4.1 we have that the Ricci curvature of N_t can be estimated

$$\text{Ric}_{N_t}(W_i^t, W_i^t) \geq c(k)\left(\frac{1}{a_t^2} - a_t^2|\Omega_t|^2\right),$$

$$|\text{Ric}_{N_t}(W_i^t, X_j)| \leq c(k)a_t|\text{div } \Omega_t|^2,$$

$$\text{Ric}_{N_t}(X_j, X_j) = 4 - c(k)a_t^2|\Omega_t[X_j]|^2.$$

$$(77)$$

It follows that if $a_t \leq a_t(k, \xi_t)$ is sufficiently small then $\text{Ric}_{N_t} > 2$. This finishes Step 1 of the construction. $\square$

6.4. **Step 2: Trivializing the connection on** (N_t, h_t). In the second and third steps of our construction we focus on (N_t, h_t). In this second step we change smoothly the principal connection until we get to the flat one, which exists in view of Lemma 6.2. Along the process, we keep fixed the metric on the base space $(S^2, \frac{1}{4}g_{S^2})$, and we squish the metric on the fibers by a factor $\lambda(t)$, depending smoothly in time. The latter is needed to keep the Ricci curvature positive along the way. In the third step we can increase the fiber size to arrive at $N_1 = S^2_{1/2} \times S^3_1$ isometrically. Precisely, the next two steps will accomplish the following:

Lemma 6.4. *There exists a smooth family* (N, h_t) *for* $t \in [1/2, 1]$ *of Riemannian metrics on* N_t *with* $\text{Ric}_{h_t} > 0$, *constant in a neighborhood of the end points, verifying the following properties:*

   i) $N_t \xrightarrow{S^3} S^2_{1/2}$ *is a Riemannian* S^3-*principal bundle with totally geodesic fibers isometric to* $(S^3, \lambda_t^2 g_{S^3})$
   ii) *The* S^3-*connection* η_t *on* N_t *is flat for* $t \in [3/4, 1]$,
   iii) N_1 *is isometric to* $(S^2 \times S^3, \frac{1}{4}g_{S^2} + g_{S^3})$.

Let $\lambda_{1/2} \equiv \frac{a_{1/2}}{1+k^2}$ be the size of the S^3 fibers of $N_{1/2}$, as in the previous section. Recall that $N_{1/2} \xrightarrow{S^3_{\lambda_{1/2}}} S^2_{1/2}$ has the structure of a Riemannian S^3-principal bundle whose metric is well defined by a principal connection $\xi_{1/2} \in \Omega^1(N; su(2))$, the base metric $S^2_{1/2}$, and the binvariant fiber metric $S^3_{\lambda_{1/2}}$.

For all $t \in [1/2, 1]$ we will construct a (smoothly varying) family $\xi_t \in \Omega^1(N; su(2))$ and λ_t such that N_t is the induced Riemannian S^3-principal bundle

$$N_t \xrightarrow{S^3_{\lambda_t}} S^2_{1/2}. \tag{78}$$

Let us first build the family of connections. By Lemma 6.2, there exists a smooth S^3-equivariant map $\Phi : N \to S^2 \times S^3$, where we view $S^2 \times S^3$ as the trivial principal S^3-bundle over S^2. The flat principal connection on $S^2 \times S^3$ can be identified with the Maurer-Cartan form ξ^{MC} of S^3. Note that $\Phi^*\xi^{MC}$ is then

a flat connection on $N \to S^2$, though clearly its coordinate expression does not mesh well with the earlier constructions. We define ξ_t as the family of connections which come from an affine combination:

$$\xi_t = (1 - \alpha(t)) \, \Phi^* \xi^{\mathrm{MC}} + \alpha(t) \, \xi_{1/2} \, , \tag{79}$$

where $\alpha(t) \geq 0$ is a nonincreasing smooth function with $\alpha \equiv 1$ for t near $1/2$ and $\alpha \equiv 0$ for t near $\frac{3}{4}$. We can then define the metric h_t explicitly as by Vilms [Vi]:

$$h_t(X, Y) := \frac{1}{4} g_{S^2}(\pi_*[X], \pi_*[Y]) + \lambda_t^2 g_{S^3}(\xi_t[X], \xi_t[Y]) \, , \tag{80}$$

where we will specify the smooth function $\lambda(t) : [\frac{1}{2}, 1] \to \mathbb{R}^+$ momentarily. Let us denote $\Omega_t = d\xi_t + \frac{1}{2}[\xi_t \wedge \xi_t]$ the curvature of our connection. Let X denote a unit horizontal direction with respect to the $S^2_{1/2}$ base, and let W denote a unit vertical direction with respect to the S^3 action. Then using Proposition 4.1 we can compute the Ricci curvature of this metric as

$$\mathrm{Ric}_{h_t}[W, W] = \frac{2}{\lambda_t^2} + \frac{\lambda_t^2}{4} |\Omega_t|^2 \, ,$$

$$\mathrm{Ric}_{h_t}[W, X] = \lambda_t \mathrm{div}_{S^2} \Omega_t[X] \, ,$$

$$\mathrm{Ric}_{h_t}[X, X] = 4 - \lambda_t^2 |\Omega_t[X]|^2 \, . \tag{81}$$

It follows for $\lambda_t \leq \lambda_t(\Omega_t)$ that $\mathrm{Ric}_{h_t} > 1$ is uniformly positive for $t \in [1/2, 3/4]$.

6.5. Step 3: Trivializing the Geometry.

For $t = 3/4$ we now have that $\Omega_{3/4} = 0$ vanishes. In particular we have isometrically that

$$N_{3/4} \equiv S^2_{1/2} \times S^3_{\lambda_{3/4}} \, . \tag{82}$$

It follows that over the range $t \in [3/4, 1]$ we may increase λ_t until $\lambda_t \equiv 1$ for t in a neighborhood of 1, at which point we have that $N_1 \equiv S^2_{1/2} \times S^3_1$, finishing the proof of Lemma 6.4. $\square$

6.6. Step 4: Constructing $(S^3 \times S^3, g_t)$ with $\mathrm{Ric}_{g_t} \geq 0$.

We have now built a family of geometries (N, h_t) with $\mathrm{Ric}_{h_t} > 0$ which begin at $N_0 = S^1 \backslash (S^3 \times S^3, g_0)$ and end at $N_1 = S^2 \times S^3$ isometrically. Further, we have explicitly built for $t \in [0, 1/2]$ a family of metrics $(S^3 \times S^3, g_t)$ which are Riemannian S^1-principal bundles over (N_t, h_t) with totally geodesic fibers of constant length $2\pi f_t$. The S^1-connections η_t of these bundles are both Yang-Mills and in Coulomb gauge. That is, the curvature form $\omega_t = d\eta_t$ and η_t both have vanishing horizontal divergence on $S^3 \times S^3$. For ω_t this is equivalent to asking that the curvature 2-form on N_t be divergence free, which is itself equivalent to asking that ω_t be Hodge harmonic.

In the next Step our goal is to extend this construction to a smoothly varying family of Riemannian metrics g_t for $t \in [1/2, 1]$ with nonnegative Ricci curvature. The metrics $(S^3 \times S^3, g_t)$ will also be Riemannian S^1-principal bundles over N_t with respect to connections η_t and with totally geodesic S^1 fibers of constant length f_t. In this Step of the construction we will choose the connections η_t uniquely so that they are Yang-Mills and in Coulomb Gauge. We will see this is sufficient to force $\mathrm{Ric}_{g_t} \geq 0$. In the next Step we will allow the warping function f_t to not be constant and vary as a function of N_t in order to push the Ricci curvature

to become positive.

Now let $[\omega_t] = [\omega_0] \in H^2(N_t)$ be the cohomology class associated to the $(1, k)$-circle bundle $S^3 \times S^3 \to N_t$. Note that $H^2(N_t) = \mathbb{Z}$ as we already understand from Lemma 6.2 that N_t is diffeomorphic to $S^2 \times S^3$, and that $[\omega_t]$ is the generating class of this cohomology. We can view $[\omega_t]$ as the deRham cohomology class generated by the curvature of any connection of this bundle.

For each $t \in [0, 1]$ let ω_t be the unique representative of $[\omega_0]$ which is divergence free with respect to the geometry of N_t. That is, let ω_t be the unique Hodge-harmonic representative. As this class is unique it follows that for $t \in [0, 1/2]$ this choice agrees with our original construction of the curvature, and it smoothly extends this choice to all $t \in [0, 1]$. Thanks to the invariance of h_t by the right action of S^3 on N_t, we automatically have that ω_t is also invariant by the right action of S^3 on N_t.

Now for each such ω_t there is a connection $[\eta_t] \in \Omega^1(S^3 \times S^3)$ whose curvature $d[\eta_t] = \pi^* \omega_t$ is equal to our enforced curvature choice for our $(1, k)$-bundle. We write $[\eta_t]$ to represent that the connection is not uniquely defined by this condition. It is only well defined up to the addition of $d\rho_t$ where ρ_t is a S^1-invariant function on $S^3 \times S^3$. That is, ρ_t is a function on N_t. In order to pick a unique $\eta_t \in [\eta_t]$ from this class we will ask that it minimizes

$$\eta_t = \arg \min_{\zeta_t \in [\eta_t]} \int_{S^3 \times S^3} |\zeta_t|_t^2 \,. \tag{83}$$

It is classical that this minimization exists, and indeed there is a unique solution. The Euler-Lagrange equation for this minimization is given by the Coulomb gauge condition

$$\text{div}_{N_t} \eta_t = \text{div}_{g_t} \eta_t = 0 \,. \tag{84}$$

A Riemannian metric $(S^3 \times S^3, g_t)$ is now well defined by the metrics (N, h_t), the family of connections η_t, and the fiber size $f_t \in \mathbb{R}$ of the circle fibers. If we let T represent a unit vertical direction with respect to the S^1 action and H a unit horizontal direction, then we can compute the Ricci curvature

$$\text{Ric}_{g_t}(T, T) = \frac{f_t^2}{4} |\omega_t|^2 \,,$$

$$\text{Ric}_{g_t}(T, H) = \text{div}_{N_t} \omega_t[H] = 0 \,,$$

$$\text{Ric}_{g_t}(H, H) = \text{Ric}_{h_t}[H, H] - \frac{f_t^2}{2} |\omega_t[H]|^2 \,. \tag{85}$$

Let us now make some observations on the above computations. Recall that for t near 1 we have that $N_t \equiv S^2 \times S^3$. For quantitative sake let us say this holds for $t \geq t_0$ with $t_0 < 1$. Recall that for all t we have $\text{Ric}_{h_t} > 0$, and so in particular for $t \leq t_0$ we see that if $f_t \leq f_t(h_t, \omega_t)$ then $\text{Ric}_{g_t} \geq 0$ and $\text{Ric}_{g_t}(H, H) > \tau$ for some $\tau > 0$. For $t \geq t_0$, where $N_t \equiv S^2_{1/2} \times S^3_1$, we see that ω_t is precisely the volume form on the S^2 factor, as this is the unique Hodge-harmonic representative of the cohomology class generated by S^2. We further have that η_t is the canonical Hopf connection on the first S^3 factor, as it is the unique Coulumb gauge connection representing this curvature. In particular, in the range $t \in [t_0, 1]$ we can increase f_t until $f_t \equiv 1$ for t near 1. We then get that $g_1 \equiv S^3_1 \times S^3_1$.

Observe however that while g_1 is isometrically equivalent to $S^3_1 \times S^3_1$, the S^1 action coming from the S^1-principal bundle structure $S^3 \times S^3 \to N_1$ is now precisely the $(1,0)$ action, as our S^1 bundle was the one coming from the Hopf fiber of the first factor. We have therefore nearly completed the proof of Theorem 6.1.

6.7. **Step 5: Constructing** $(S^3 \times S^3, g_t)$ **with Ric**$_{g_t} > 0$. We have at this stage built a family of metrics $(S^3 \times S^3, g_t)$ with isometric S^1 actions which begin and end isometrically at $S^3_1 \times S^3_1$ with the $(1,k)$ and $(1,0)$ actions, respectively. Further, these metrics all satisfy Ric$_{g_t} \geq 0$.

We have not yet completed the proof however. It follows from (85) that we can choose f_t sufficiently small that

$$\mathrm{Ric}_{g_t}[H, H] > \tau > 0 \,, \tag{86}$$

is uniformly positive for all $t \in [1/2, 1]$. However, if we have $|\omega_t|^2 = 0$ at some point, then $\mathrm{Ric}_{g_t}(T,T) = \frac{f_t^2}{4}|\omega_t|^2 = 0$ at this point. Were this to occur, then we would only have $\mathrm{Ric}_{g_t} \geq 0$, and it will be important in the applications that we get strict positivity.

In order to handle this, we will perturb our S^1 warping functions f_t. Currently f_t is spatially constant, and our perturbation will be by a small function of N_t, cf. with [GPT]. Let us write our new warping function as

$$\tilde{f}_t = f_t + \epsilon_t h_t \,, \tag{87}$$

where $h_t : N \to \mathbb{R}$ will be a smoothly varying collection of smooth functions which vanish for t near $\frac{1}{2}$ and 1, and ϵ_t will be sufficiently small constants depending smoothly on time.

In order to pick $h_t : N_t \to \mathbb{R}$ let us begin with several observations. First as (N, h_t) is invariant under the right S^3 action, we have that $|\omega_t|^2$ is invariant under this right action as well. In particular, we can view $|\omega_t|^2$ as a function of $N/S^3 = S^2_{1/2}$. We will similarly choose h_t to be an S^3 invariant function, which is to say a function of S^2.

As a second observation, let us point out that ω_t is non-trivial in cohomology, hence cannot be flat and so we have

$$\fint_N |\omega_t|^2 \geq c_0 \,, \quad \text{for every } t \in [0,1] \,, \tag{88}$$

for some $c_0 > 0$.

Let us now consider a smooth cutoff function $\phi : \mathbb{R} \to [-1, 0]$ with $\phi(s) = -1$ if $s \leq 10^{-2}c_0$ and $\phi(s) = 0$ if $s \geq 10^{-1}c_0$. Let us define $h_t : S^2 \to \mathbb{R}$ as the solution of

$$\Delta_{S^2} h_t \equiv \phi(|\omega_t|^2) - \fint_{S^2} \phi(|\omega_t|^2) \,. \tag{89}$$

If we take $\fint h_t = 0$ then h_t is uniquely defined, smooth, and smoothly varying in t. As a first observation note that if $|\omega_t|^2 > 10^{-1}c_0$ on N, then we have that $\Delta h_t \equiv 0$ identically vanishes. Let us also see that Δh_t is

uniformly negative when $|\omega_t|$ is small. To begin, as everything is smooth let us define M so that

$$|\nabla h_t| + |\nabla^2 h_t| + |\omega_t| + |\nabla \omega_t| \leq M, \tag{90}$$

uniformly for all $t \in [1/2, 1]$. Using (88) we have that there exists at least one point with $|\omega_t|^2(p) \geq c_0$, and so by (90) we have that $|\omega_t|^2 > 10^{-1}c_0$ and hence $\phi(|\omega_t|^2) = 0$ on $B_{c_0(20M^2)^{-1}}(p)$. Consequently, we have that

$$\fint \phi(|\omega_t|^2) > -1 + c_0^2(20M^2)^{-2}. \tag{91}$$

It follows that if $x \in \{|\omega_t|^2 < 10^{-2}c_0\}$ then

$$\Delta h_t(x) < -c_0^2(20M^2)^{-2}. \tag{92}$$

Now let $\tilde{f}_t = f_t + \epsilon_t h_t$ be our warping function and to begin let $\epsilon_t < (2M)^{-1}f_t$, so that $\frac{1}{2}f_t < |\tilde{f}_t| < 2f_t$. We will further decrease ϵ_t later. Recall that we previously chose f_t small enough in Step 4, in order to guarantee $\mathrm{Ric}_{g_t} \geq 0$ and $\mathrm{Ric}_{g_t} > \tau > 0$ in the horizontal directions when we set $\epsilon_t = 0$.

We will use Proposition 4.4 to compute the Ricci curvature on $(S^3 \times S^3, g_t)$. Let T be a unit direction in the S^1 fiber direction and let H represent any unit direction perpendicular to T. Let us split our computation into two regions. If we are on the region $\{|\omega_t|^2 > 10^{-2}c_0\}$ then we can estimate

$$\mathrm{Ric}(T, T) \geq \frac{f_t^2 c_0}{400} - \frac{\epsilon_t}{f_t^2},$$

$$|\mathrm{Ric}(T, H)| \leq \frac{3}{2}M^2 \epsilon_t,$$

$$\mathrm{Ric}(H, H) \geq \mathrm{Ric}_{h_t}(H, H) - \frac{f_t^2}{2}M^2 - \frac{4M\epsilon_t}{f_t^2}. \tag{93}$$

Here we used that Laplacians and Hessians of S^3-invariant functions on N can be identified with the corresponding objects in the base space S^2, since the fibers are totally geodesic. Then we can choose $f_t < f_t(h_t)$ and $\epsilon_t < \epsilon_t(f_t, M, c_0, |\mathrm{Ric}|_{h_t})$ and obtain $\mathrm{Ric}_{g_t} > 0$ in the region $\{|\omega_t|^2 > 10^{-2}c_0\}$.

On the other hand, let us consider the region $\{|\omega_t|^2 < 10^{-2}c_0\}$, then we can use (92) to estimate

$$\mathrm{Ric}(T, T) \geq \epsilon_t c_0^2(20M^2)^{-2},$$

$$|\mathrm{Ric}(T, H)| \leq \frac{3}{2}M^2 \epsilon_t,$$

$$\mathrm{Ric}(H, H) \geq \mathrm{Ric}_{h_t}(H, H) - \frac{f_t^2}{2}M^2 - \frac{4M\epsilon_t}{f_t^2}. \tag{94}$$

It is the estimate of the first term which has changed. If we now further assume $f_t < f_t(h_t)$ and $\epsilon_t < \epsilon_t(f_t, M, c_0, |\mathrm{Ric}|_{h_t})$, then we can again conclude that $\mathrm{Ric}_{g_t} > 0$.

6.8. **Finishing the Proof of Theorem 6.1.** We have essentially finished the proof of Theorem 6.1 at this stage, however let us put all the ingredients together to see that this is the case.

We already noticed that there is an S^3-equivariant isometry $\Phi : (N, h_1) \to (S^2 \times S^3, \frac{1}{4}g_{S^2} + g_{S^3})$ with respect to the respective right S^3 actions. Moreover, consider the S^1 bundles $\pi_{(1,k)} : S^3 \times S^3 \to N$ and $\pi_{(1,0)} : S^3 \times S^3 \to S^2 \times S^3$. Then the S^1 bundles $\pi_{(1,k)} : S^3 \times S^3 \to N$ and $\Phi^*\pi_{(1,0)} : S^3 \times S^3 \to N$ are necessarily isomorphic as S^1 bundles over N as they arise from the same cohomology class. That is, we can find an S^1-equivariant diffeomorphism

$$\hat{\Phi} : S^3 \times S^3 \to S^3 \times S^3, \tag{95}$$

with $\hat{\Phi}(\theta \cdot_{(1,k)} (s_1, s_2)) = \theta \cdot_{(1,0)} \hat{\Phi}(s_1, s_2)$ whose induced mapping on the quotients is given by the isometry $\Phi : (N, h_1) \to S^2_{1/2} \times S^3_1$.

We claim that, up to composition with a gauge transformation, $\hat{\Phi}$ is an S^1-equivariant isometry between $(S^3 \times S^3, g_1)$ with the $(1, k)$-Hopf action, and $(S^3 \times S^3, g_{S^3} + g_{S^3})$ with the $(1, 0)$-Hopf action.

Begin by observing that as the induced quotient map Φ is an isometry, if we denote by η_1 and η_c the S^1-principal connections of $\pi_{(1,k)} : S^3 \times S^3 \to N_1$ and $\pi_{(1,0)} : S^3 \times S^3 \to S^2 \times S^3$ respectively, it holds

$$d\left(\hat{\Phi}^*\eta_c\right) = d\eta_1, \tag{96}$$

since both connections are Yang-Mills, and the curvature form being Hodge-harmonic is a uniquely defined condition in its cohomology class. As $S^3 \times S^3$ has trivial first cohomology, there exists a smooth function $\phi : S^3 \times S^3 \to \mathbb{R}$ such that

$$\hat{\Phi}^*\eta_c = \eta_1 + d\phi, \tag{97}$$

and by right invariance we can assume that $\phi = \phi' \circ \pi_{(1,k)}$ for some smooth function $\phi' : N \to \mathbb{R}$. In particular, and with a slight abuse of notation, after composing with the gauge transformation induced by ϕ' we can assume that

$$\hat{\Phi}^*\eta_c = \eta_1. \tag{98}$$

Therefore, $\hat{\Phi}$ is an S^1-equivariant diffeomorphism between principal S^1-bundles with totally geodesic and isometric S^1-fibers, it induces and isometry between the base spaces and maps one connection form to the other. Hence it is an S^1-equivariant isometry, and this finishes the proof of Theorem 6.1. $\square$

6.9. **Explicit Diffeomorphism for $k = 1$.** Let us end by addressing Remark 6.1 and show that for $k = 1$ we can explicitly choose the diffeomorphism ϕ in Theorem 6.1 by $\phi = \phi_1$ from Example 6.1.

Our first observation is that ϕ_1, besides pushing forward the $(1, 1)$-action to the $(1, 0)$-action is also equivariant with respect to the right S^3-action. Hence it induces an S^3-principal bundles isomorphism $\Phi_1 : N \to S^2 \times S^3$. It turns out that there is an explicit expression for Φ_1, namely

$$\Phi_1([s_1, s_2]) = (\pi_{\text{Hopf}}(s_1), s_1^{-1} \cdot s_2). \tag{99}$$

By using Φ_1 in place of Φ in Step 2, and arguing as Section 6.8, we get an isometry $\hat{\Phi}_1 : (S^3 \times S^3, g_0) \to (S^3 \times S^3, g_1)$ which coincides with ϕ_1 up to gauge transformation. In particular, $\hat{\Phi}_1$ and ϕ_1 are isotopic.

7. STEP 2: EQUIVARIANT TWISTING

Our main goal in this Section is to prove Proposition 3.3, which is Step 2 of the construction, which we restate for the convenience of the reader:

Proposition 7.1 (Step 2: Twisting the Action)**.** *Let $1 > \epsilon, \hat{\epsilon}, \delta > 0$ with $k \in \mathbb{Z}$. Then there exist $\hat{\delta}(\epsilon, \hat{\epsilon}, \delta, k) > 0$ with $R(\epsilon, \hat{\epsilon}, \delta, k) > 1$ and a metric space X with an isometric and free S^1 action such that*

(1) *X is smooth away from a single three sphere $S_{\hat{\delta}}^3 \times \{p\} \in X$ with $Ric_X \geq 0$.*

(2) *There exists $B_{10^{-3}}(p) \subseteq U \subseteq B_{10^{-1}}(p) \subseteq X$ which is isometric to $S_{\delta}^3 \times B_{10^{-2}}(0) \subseteq S_{\delta}^3 \times C(S_{1-\epsilon}^3)$, and under this isometry the S^1 action on U identifies with the $(1, k)$-Hopf action.*

(3) *There exists $B_{10^{-1}R}(p) \subseteq \hat{U} \subseteq B_{10R}(p) \subseteq X$ s.t. $X \setminus \hat{U}$ is isometric to $S_{\hat{\delta}R}^3 \times A_{R,\infty}(0) \subseteq S_{\hat{\delta}R}^3 \times C(S_{1-\hat{\epsilon}}^3)$, and under this isometry the S^1 action on $X \setminus \hat{U}$ identifies with the $(1, 0)$-Hopf action.*

We will build the space X in pieces throughout this Section. Let us give a rough description of the steps involved and break down the role of each subsection. The starting point for our construction is to take $X_0 \equiv S_{\delta}^3 \times C(S_{1-\epsilon}^3)$, which by condition (2) in Proposition 7.1 is the beginning of our X. In Section 7.1 we will construct X_1 by first bending the $\mathbb{R}^4 = C(S_{1-\epsilon}^3)$ factor down to a sharper cone $C(S_{1/4}^3)$. This will give us the extra curvature we need in Sections 7.2 and 7.3 in order to construct X_3. The goal of X_2 and X_3 will be to take the constant S_{δ}^3 factor and lift it to a linearly growing factor. That is, outside a compact subset X_3 will be isometric to $C(S_{\delta_3}^3 \times S_{1/8}^3)$.

In Section 7.4 we will prove a refinement of Theorem 6.1. Namely we will use Theorem 6.1 in order to construct a family of S^1 invariant metrics $(S^3 \times S^3, g_t)$ which begin at $S_{\delta_3}^3 \times S_{1/8}^3$ with the $(1, k)$-action, end at $S_{\delta_3}^3 \times S_{1/8}^3$ with the $(1, 0)$ action, satisfy $Ric_t > 6$ and which have constant volume form. We will apply this in Section 7.5 to build X_4, which will perform our twisting and end so that outside a compact subset X_4 is isometric to an annulus in $C(S_{\delta_4}^3 \times S_{1/16}^3)$ with the new action.

We still need to take X_4 back to the $S^3 \times \mathbb{R}^4$ geometry, which will require several steps. In Section 7.6 we will construct X_5, which outside of a compact subset will turn the linear growth of the second $S_{\delta_4}^3$ factor into a slow polynomial growth. That slow growth will be useful in Section 7.7 to construct X_6, which will take the $C(S_{1/16}^3)$ factor and increase the size of the cross section until we arrive at $C(S_{1-\hat{\epsilon}}^3)$, which is isometrically very close to $\mathbb{R}^4$. Finally in Section 7.8 we will construct $X = X_7$, which will take the first sphere, which is still growing at a slow polynomial rate, and level it out to constant size $S_{\hat{\delta}}^3$, which will finish the proof of Proposition 7.1.

7.1. **Constructing X_1.** Let us begin with $X_0 \equiv S_{\delta}^3 \times C(S_{1-\epsilon}^3)$, which geometrically has a metric which may be written

$$g_0 \equiv dr^2 + \delta^2 g_{S^3} + (1 - \epsilon)^2 r^2 g_{S^3}. \tag{100}$$

Our first step of the construction is to shrink down the size of the cone cross section. The purpose of this is to increase the curvature sufficiently so that in the second and third steps we can turn our constant S^3 factor into a cone factor as well.

Let $U_0 \equiv \{r \le 1\}$, then we will define the metric g_1 on $X_1 \setminus U_0$ through the ansatz

$$g_1 \equiv dr^2 + \delta^2 g_{S^3} + h(r)^2 g_{S^3} . \tag{101}$$

If we let a, b, c denote the directions on the first S^3 factor and i, j, k on the second S^3 factor (the one that we are currently viewing as a cross section), then we can compute the nonzero terms of the Ricci curvature of this ansatz as

$$\mathrm{Ric}_{rr} = -3\frac{h''}{h} ,$$

$$\mathrm{Ric}_{aa} = \frac{2}{\delta^2} ,$$

$$\mathrm{Ric}_{ii} = 2\frac{1 - (h')^2}{h^2} - \frac{h''}{h} . \tag{102}$$

Now let us define $h(r)$ so that

$$h(r) \equiv \begin{cases} (1 - \epsilon)r & \text{if } r \le 10^{-1} , \\ h'' < 0 & \text{if } 10^{-1} \le r \le 10 , \\ (1 - \epsilon) + (r - 1)/4 & \text{if } r \ge 10 . \end{cases}$$

We can build a function as above by smoothing out $h(r) = \min\{(1 - \epsilon)r, (1 - \epsilon) + (r - 1)/4\}$ near $r = 1$, the intersection point of the two lines. Observe that we always have $|h'| \le 1 - \epsilon$ as $h'' < 0$, and so we can estimate the Ricci curvature

$$\mathrm{Ric}_{rr} \ge 0 ,$$

$$\mathrm{Ric}_{aa} = \frac{2}{\delta^2} \ge 0,$$

$$\mathrm{Ric}_{ii} \ge \frac{2\epsilon - \epsilon^2}{h^2} \ge 0 . \tag{103}$$

Note that outside the region $\{r \le 10\}$ we have that X_1 is isometric to $S^3_\delta \times C(S^3_{1/4})$. Let us change coordinates $r \to r - 1 + 4(1 - \epsilon)$. Set $R_1 \equiv 20$, and $U_1 \equiv \{r \le R_1\}$ in the new coordinates. Then $X_1 \setminus U_1$ is isometric to

$$g_1 \equiv dr^2 + \delta^2 g_{S^3} + (r/4)^2 g_{S^3} . \tag{104}$$

7.2. **Constructing X_2.** Let us now define X_2 by changing the metric on X_1 in the region $X_1 \setminus U_1$. Our goal over the next two steps will be to turn the constant S^3_δ factor into a cone factor. Let us begin by defining the metric g_2 on $X_2 \setminus U_1$ through the ansatz

$$g_2 \equiv dr^2 + f(r)^2 g_{S^3} + h(r)^2 g_{S^3}. \tag{105}$$

We can compute the nonzero terms of the Ricci curvature of this ansatz as

$$\mathrm{Ric}_{rr} = -3\frac{h''}{h} - 3\frac{f''}{f},$$

$$\mathrm{Ric}_{aa} = \frac{2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\left(2\frac{f'}{f} + 3\frac{h'}{h}\right),$$

$$\mathrm{Ric}_{ii} = 2\frac{1-(h')^2}{h^2} - \frac{h''}{h} - 3\frac{h'}{h}\frac{f'}{f}. \tag{106}$$

Let us denote $R_2 \equiv 10^4$, and let $\delta_2 \le \delta_2(\delta)$ be a constant we will choose in the next subsections. Let us choose a smooth $f(r)$ so that

$$f(r) \equiv \begin{cases} \delta & \text{if } r \le 10^{-1}R_2, \\ 0 < R_2 f'' < \delta_2 & \text{if } 10^{-1}R_2 \le r \le 10R_2, \\ \delta + \delta_2(r - R_2) & \text{if } r \ge 10R_2. \end{cases}$$

We can build $f(r)$ by smoothing out $f(r) = \min\{\delta, \delta + \delta_2(r - R_2)\}$ near $r = R_2$, the intersection point.

Let $h(r)$ be a smooth function such that

$$h(r) \equiv \begin{cases} r/4 & \text{if } r \le 10^{-2}R_2, \\ h'' < 0 & \text{if } 10^{-2}R_2 \le r \le 10^4 R_2, \\ R_2 h'' < -10^{-4} & \text{if } 10^{-1}R_2 \le r \le 10R_2, \\ R_2/4 + (r - R_2)/6 & \text{if } r \ge 10^2 R_2. \end{cases}$$

We can build such a function by similarly smoothing out $h(r) = \min\{r/4, R_2/4 + (r - R_2)/6\}$ near $r = R_2$.

Observe that the ending metric will have both sphere factors end with a linear growth, but the center of cones will be different for the first and second spheres. Our construction of X_3 will fix this, for now let us compute the Ricci tensor of the above.

We will choose δ_2 in a future subsection, however we will impose the restriction now that

$$\delta_2 \le 10^{-9}\delta = 10^{-6}\frac{\delta}{R_2}. \tag{107}$$

Note that away from the interval $r \in [10^{-1}R_2, 10R_2]$ the Ricci curvature is nonnegative by similar computations as the last subsection. The interval $r \in [10^{-1}R_2, 10R_2]$ is a little more complicated. We use that for

every $r \in [10^{-2}R_2, 10R_2]$ it holds

$$f'(r) \le 10\delta_2$$
$$\delta \le f(r) \le 2\delta \qquad (108)$$
$$\frac{R_2}{4} \le h(r) \le \frac{7}{4}R_2,$$

to conclude that

$$\text{Ric}_{rr} \ge \frac{3}{R_2}\Big(\frac{4 \cdot 10^{-4}}{7 \cdot R_2} - \frac{\delta_2}{\delta}\Big) > 0,$$

$$\text{Ric}_{aa} \ge \frac{1}{f}\Big(\frac{1}{\delta} - \frac{\delta_2}{R_2} - 200\frac{\delta_2^2}{\delta} - 60\frac{\delta_2}{R_2}\Big) > 0,$$

$$\text{Ric}_{ii} \ge \frac{1}{h}\Big(\frac{15}{14 \cdot R_2} + \frac{10^{-4}}{R_2} - \frac{15}{2} \cdot \frac{\delta_2}{\delta}\Big) > 0. \qquad (109)$$

Note that if we define $U_2 \equiv \{r \le 10R_2\}$ then $X_2 \setminus U_2$ is *almost isometric* to an annulus in $C(S^3_{1/6} \times S^3_{\delta_2})$. Precisely, we have:

$$g_2 \equiv dr^2 + (\delta + \delta_2(r - R_2))^2 g_{S^3} + (R_2/4 + (r - R_2)/6)^2 g_{S^3}, \qquad (110)$$

7.3. Constructing X_3.

Our space X_2 has ended so that it looks like a cone over each of the sphere factors, however the centers of those cone points are not the same. Specifically let us write the metric in the form

$$g_2 = dr^2 + \delta_2^2(r + c_2)^2 g_{S^3} + (R_2/4 + (r - R_2)/6)^2 g_{S^3}, \qquad (111)$$

where $c_2 \equiv \delta_2^{-1}\delta - R_2 \gg R_2$. We will want to change the metric in this subsection so that they are both metric cones with respect to the center c_2. We will change the geometry on $X_2 \setminus U_2$ through the ansatz

$$g_3 \equiv dr^2 + \delta_2^2(r + c_2)^2 g_{S^3} + h(r)^2 g_{S^3}. \qquad (112)$$

We can compute the nonzero terms of the Ricci curvature of this ansatz as

$$\text{Ric}_{rr} = -3\frac{h''}{h},$$

$$\text{Ric}_{aa} = \frac{2}{\delta_2^2(r + c_2)^2} - \frac{1}{r + c_2}\Big(\frac{2}{r + c_2} + 3\frac{h'}{h}\Big),$$

$$\text{Ric}_{ii} = 2\frac{1 - (h')^2}{h^2} - \frac{h''}{h} - \frac{3}{r + c_2}\frac{h'}{h}. \qquad (113)$$

Let $R_3 \equiv 3c_2 - 2R_2 = R_3(\delta, \delta_2)$ be the point where the lines $R_2/4 + (r - R_2)/6$ and $(r + c_2)/8$ intersect. Then we will choose $h(r)$ as a smooth function which satisfies

$$h(r) \equiv \begin{cases} R_2/4 + (r - R_2)/6 & \text{if } r \leq 10^{-1}R_3\,, \\ h'' < 0 & \text{if } 10^{-1}R_3 \leq r \leq 10R_3\,, \\ (r + c_2)/8 & \text{if } r \geq 10R_3\,. \end{cases}$$

Note that we may pick such an h by smoothing out $h(r) = \min\{R_2/4 + (r - R_2)/6, (r + c_2)/8\}$, so that we may also insist that $\frac{1}{2}\min\{r/6, (r + c_2)/8\} \leq h(r) \leq \min\{r/6, (r + c_2)/8\}$. As $h'' < 0$ we also have that $|h'| \leq 1/6$, and so we can compute

$$\text{Ric}_{rr} \geq 0\,,$$

$$\text{Ric}_{aa} = \frac{1}{r + c_2}\Big(\frac{2}{\delta_2^2(r + c_2)} - \frac{2}{r + c_2} - \frac{6}{r}\Big) \geq 0\,,$$

$$\text{Ric}_{ii} \geq 2 \cdot \frac{35}{36}\frac{1}{h^2} - \frac{1}{16}\frac{1}{h^2} \geq 0\,. \tag{114}$$

If we define $\delta_3 \equiv \delta_2$ and consider the domain $U_3 \equiv \{r \leq 10R_3\}$, then we have that $X_3 \setminus U_3$ is isometric to $C(S_{\delta_3}^3 \times S_{1/8}^3)$. After shifting $r \to r + c_2$ this metric is written in coordinates as

$$g_3 \equiv dr^2 + (\delta_3 r)^2 g_{S^3} + (r/8)^2 g_{S^3}\,. \tag{115}$$

Note that in these coordinates we can identify $U_3 = \{r \leq 10R_3 + c_2\} \subseteq \{r \leq 11R_3\}$.

7.4. **Refinement of Theorem 6.1.**

Theorem 6.1 from Section 6 proved the existence of a family of S^1-invariant metrics $(S^3 \times S^3, g_t)$ which all have positive Ricci, begin and end at the standard metric, but for which the beginning and ending isometric S^1 actions are homotopically inequivalent. In this subsection we would like to construct from this a refinement which will keep control for us various geometric quantities. This refinement will be directly used in the next subsection to build X_2.

Lemma 7.2 (Refined Equivariant Twisting). *Let $(S^3 \times S^3, g_0)$ be a product of two spheres with $g_0 = g_{S_\delta^3 \times S_{1/8}^3}$, and take this space to be equipped with the $(1, k)$-S^1 isometric Hopf action. If $\delta \leq \delta(k)$ then there exist a family of metrics g_t, and a diffeomorphism $\phi : S^3 \times S^3 \to S^3 \times S^3$ such that*

(1) *The $(1, k)$-S^1 action on $S^3 \times S^3$ is an isometric action for all t,*
(2) *The Ricci curvature $\text{Ric}_t > 6$ is uniformly positive,*
(3) *The volume form $dv_{g_t} = dv_{g_0}$ is a constant.*
(4) *$g_1 = \phi^* g_{S_\delta^3 \times S_{1/8}^3}$ with $\phi(\theta \cdot_{(k,1)} (s_1, s_2)) = \theta \cdot_{(1,0)} \phi(s_1, s_2)$.*

The construction of the above is in several steps. To begin, let $\hat{g}_r$ be the metric from Theorem 6.1, reparametrized so that $\hat{g}_{1/3} = g_{S^3 \times S^3}$ is the standard metric on $S^3 \times S^3$ and $\hat{g}_{2/3} = \varphi^* g_{S^3 \times S^3}$ is the pullback of the standard metric by our required nontrivial diffeomorphism. Let us first try and normalize this collection. Namely, let us first consider the family of metrics

$$\tilde{g}_r = a_r \varphi_r^* \hat{g}_r\,, \tag{116}$$

where

$$a_r := \frac{\tilde{a}}{\text{Vol}(\hat{g}_r)^{1/6}} \,, \tag{117}$$

$\tilde{a} > 0$ is small enough so that $\text{Ric}_{a_r \hat{g}_r} > 6$ uniformly in $r \in [1/3, 2/3]$, and $\varphi_r : S^3 \times S^3 \to S^3 \times S^3$ is a family of diffeomorphisms with $\varphi_{1/3} = Id$. Observe that, by construction, $\text{Vol}(a_r \hat{g}_r)$ is constant with respect to r.

Let us now build the diffeomorphisms $\varphi_r : S^3 \times S^3 \to S^3 \times S^3$. Consider first the volume forms $\nu_r \equiv dv_{\hat{g}_r}$. Recall that our remaining challenge is to force this family to be a constant. In order to do so we will follow [Mo] in a spirit related to [CN2], with the additional subtlety that we need to work equivariantly.

Even though the family ν_r is not constant, as the action is isometric we do have that ν_r is invariant under the $(1, k)$-Hopf action. Let $\nu \equiv dv_{a_0 \hat{g}_0}$, which (up to multiple) is simply the standard volume form on $S^3 \times S^3$. Let us write $\nu_r \equiv \rho_r \nu_0$, so note that the function ρ_r is invariant under the $(1, k)$-hopf action. For each r let us solve on $S^3 \times S^3$

$$\Delta f_r + \langle \nabla \ln \rho_r, \nabla f_r \rangle = -\frac{\partial}{\partial r} \ln \rho_r \,. \tag{118}$$

Observe first that the above is smoothly solvable because $\int \frac{\partial}{\partial r} \ln \rho_r \cdot \rho_r \nu_0 = \int \frac{\partial}{\partial r} \rho_r \nu_0 = \frac{d}{dr} \int \nu_r = 0$. The solution is a unique up to a constant, so if we assume $\int f_r \rho_r \nu_0 = 0$ then the solution is uniquely defined. As the equation and right hand side commute with this action, as the solution is unique we must have that is also invariant by the $(1, k)$-Hopf action.

Let us now define the family of diffeomorphisms $\varphi_r : S^3 \times S^3 \to S^3 \times S^3$ by $\varphi_{1/3} = Id$ and

$$\frac{d}{dr} \varphi_r(x) = \nabla f_r(\varphi_r(x)) \quad \text{for } r \geq 1/3 \,. \tag{119}$$

Note that as ∇f_r is invariant by the $(1, k)$-Hopf action we get that φ_r commutes with the $(1, k)$-Hopf action. We also get that

$$\frac{d}{dr} \varphi_r^* \nu_r = (\frac{\partial}{\partial r} \ln \rho_r + \Delta f_r + \langle \nabla \ln \rho_r, \nabla f_r \rangle) \nu_r = 0 \,. \tag{120}$$

From this we have built a family of metrics $\tilde{g}_r$ such that

$$\begin{aligned}
\tilde{g}_{1/3} &= a_{1/3} \, g_{S^3 \times S^3} \,, \\
\tilde{g}_{2/3} &= a_{1/3} \, \varphi_{2/3}^* \phi^* g_{S^3 \times S^3} \equiv a_{1/3} \varphi^* g_{S^3 \times S^3} \,, \\
dv_{\tilde{g}_r} &= a_{1/3}^6 dv_{S^3 \times S^3} \,, \\
\text{Ric}_{\tilde{g}_r} &> 6 \,.
\end{aligned} \tag{121}$$

We can now finish the construction with one more interpolation. Let us define the family g_r by

$$g_r = \begin{cases} b_r \, g_{S^3} + c_r \, g_{S^3} & \text{if } 0 \leq r \leq 1/3 \\ \frac{b_{1/3}}{a_{1/3}} \, \tilde{g}_r & \text{if } 1/3 \leq r \leq 2/3 \\ \varphi^* \big(d_r \, g_{S^3} + e_r \, g_{S^3} \big) & \text{if } 2/3 \leq r \leq 1 \,. \end{cases}$$

Here we have that b_r, c_r are smooth functions such that

$$b_0 = 1/8, \quad c_0 = \delta, \quad b_{1/3} = c_{1/3}, \quad b_r \cdot c_r = const = \delta/8,$$

$$d_{2/3} = e_{2/3} = b_{1/3}, \quad d_1 = 1/8, \quad e_1 = \delta, \quad d_r \cdot e_r = const = \delta/8. \tag{122}$$

Note we need $\frac{b_{1/3}}{a_{1/3}} \leq 1$, so that the Ricci curvature of the above satisfies Ric > 6. This becomes a restriction on δ given by

$$\delta \leq 8a_{1/3}^2 \leq \delta(k). \tag{123}$$

This finishes the construction of Lemma 7.2   □

7.5. Constructing X_4.

Recall that in the construction of X_3 we had a variable $\delta_3 = \delta_2$ which had not yet been fixed. Let us now use Lemma 7.2 and Section 7.2 to fix $\delta_2 = \delta_2(k)$. Recall there is a compact subset $U_3 \subseteq X_3$ such that $X_3 \setminus U_3$ is isometric

$$g_2 = dr^2 + (\delta_3 r)^2 g_{S^3} + (r/8)^2 g_{S^3}. \tag{124}$$

In order to construct X_4 let us modify the metric on $X_3 \setminus U_3$ by looking at an ansatz of the form

$$g_4 \equiv dr^2 + h(r)^2 g_r, \tag{125}$$

where g_r will be some family of metrics on $S^3 \times S^3$. Following a line of construction similar to [CN2], if we assume that the volume forms on g_r are constant, then we can compute the Ricci curvature of the above ansatz as

$$\mathrm{Ric}_{rr} = -6\frac{h''}{h} - \frac{1}{4}g^{ab}g^{cd}g'_{ac}g'_{bd},$$

$$\mathrm{Ric}_{ir} = \frac{1}{2}\Big(\partial_a(g^{ab}g'_{bi}) + \frac{1}{2}(g^{ab})'(\partial_i g_{ab} - g_{ib}g^{pq}\partial_a g_{pq})\Big),$$

$$\mathrm{Ric}_{ij} = \mathrm{Ric}_{ij}^g + h^2\Big(-\frac{h''}{h} - 5(\frac{h'}{h})^2\Big)g_{ij} + \Big(-\frac{7}{2}\frac{h'}{h}g'_{ij} + \frac{1}{2}g^{ab}g'_{ai}g'_{bj}\Big), \tag{126}$$

where $g'_{ab} = \frac{\partial}{\partial r}g_{ab}$ and analogously for $(g^{ab})'$.

Let $10^4 R_3 < R_4 < \infty$ be chosen momentarily. To define $h(r)$ let us consider the three functions

$$h_1(r) = r,$$

$$h_2(r) = 10^3 R_3 + \Big(1 - \frac{1}{4} + \frac{1}{16}\frac{\ln(\ln(15R_3))}{\ln(\ln(r))}\Big)(r - 10^3 R_3),$$

$$h_3(r) = 10^3 R_3 + \Big(1 - \frac{1}{4} + \frac{1}{16}\frac{\ln(\ln(15R_3))}{\ln(\ln(10R_4))}\Big)(10R_4 - 10^3 R_3) + (r - 10R_4)/2. \tag{127}$$

Note that h_1 and h_3 are linear functions, while h_2 is almost linear but has a slight amount of convexity added. Observe that all of the normalizing constants are built so that h_1 and h_2 intersect at $10^3 R_3$, while h_2 and h_3

intersect at $10R_4$. We will want to build a smooth $h(r)$ of the form

$$h(r) \equiv \begin{cases} h_1(r) & \text{if } r \leq 10^2 R_3\,, \\ h'' < 0 & \text{if } 10^2 R_3 \leq r \leq 10^2 R_4\,, \\ h_2(r) & \text{if } 10^4 R_3 \leq r \leq R_4\,, \\ h_3(r) & \text{if } r \geq 10^2 R_4\,. \end{cases}$$

We can build such a function h as above by smoothing out the function $h(r) = \min\{h_1(r), h_2(r), h_3(r)\}$ at the relevant intersection points.

We define the family of metrics g_r as follows. Consider first the metrics $\tilde{g}_t$ defined in Lemma 7.2. In the range $r \leq 10^4 R_3$ let $g_r = \tilde{g}_0$, and in the range $r \geq R_4$ let $g_r = \tilde{g}_1$. In these two ranges the nonnegativity of the Ricci curvature follows by computations analogous to the previous subsections.

In the range $r \in [10^4 R_3, R_4]$ we proceed as follows. Let $t : [10^4 R_3, R_4] \to [0, 1]$ be a smooth function such that

$$t(r) = \frac{\ln \ln \ln(r) - \ln \ln \ln(10^4 R_3)}{\ln \ln \ln(R_4) - \ln \ln \ln(10^4 R_3)}\,. \tag{128}$$

Note that $t(r)$ has the property that $t(10^4 R_3) = 0$ and $t(R_4) = 1$. We define $g_r = \tilde{g}_{t(r)}$. Note that we have not yet defined R_4.

Let $M > 0$ be such that

$$|\nabla \tilde{g}|_{\tilde{g}} + |\tilde{g}'|_{\tilde{g}} + |\nabla \tilde{g}'|_{\tilde{g}} \leq M\,, \quad \text{uniformly in } t \in [0, 1]\,. \tag{129}$$

Thus we can estimate

$$|g'|_g + |\nabla g'|_g \leq \frac{1}{r \cdot \ln(r) \cdot \ln \ln(r)} \cdot \frac{1}{\ln \ln \ln(R_4) - \ln \ln \ln(10^4 R_3)}(|\tilde{g}'|_{\tilde{g}} + |\tilde{g}'|_{\tilde{g}})$$

$$\leq \frac{M}{r \cdot \ln(r) \cdot \ln \ln(r)} \cdot \frac{1}{\ln \ln \ln(R_4) - \ln \ln \ln(10^4 R_3)}\,. \tag{130}$$

Notice that $\frac{r}{2} \leq h(r) \leq r$, and $0 \leq h'(r) \leq 1$ for every $r \in [10^4 R_3, R_4]$, hence

$$\frac{h'(r)}{h(r)} \leq \frac{1}{r}$$
$$\frac{h''(r)}{h(r)} \leq -\frac{1}{100} \frac{\ln \ln(15 R_3)}{r^2 \cdot \ln(r) \cdot (\ln \ln(r))^2}\,, \tag{131}$$

for every $r \in [10^4 R_3, R_4]$.

From the expression (126), we can estimate the diagonal Ricci terms

$$\text{Ric}_{rr} \geq -6\frac{h''}{h} - |g'|_g^2 \geq \frac{1}{100} \frac{\ln \ln(15 R_3)}{r^2 \cdot \ln(r) \cdot (\ln \ln(r))^2}\,, \tag{132}$$

$$\text{Ric}_{ii} \geq 6 - hh'' - 5(h')^2 - \frac{7}{2r}|g'|_g - \frac{1}{2}|g'|_g^2 \geq \frac{1}{2}\,, \tag{133}$$

provided $R_4 = R_4(M)$ is chosen big enough.

The cross term has the estimate,

$$|\mathrm{Ric}_{ir}| \le 2(|g'|_g + |\nabla g'|_g)(1 + |\nabla g|_g) \le \frac{2(M+1)^2}{r \cdot \ln(r) \cdot \ln\ln(r)} \cdot \frac{1}{\ln\ln\ln(R_4) - \ln\ln\ln(10^4 R_3)}$$
$$\le \frac{1}{r \cdot \ln(r) \cdot \ln\ln(r)},$$

(134)

again, assuming $R_4 = R_4(M)$ big enough.

Note that the negativity of the cross term Ric_{ir} dominates the positivity of the radial Ric_{rr}. However, it is itself dominated by the positivity of the cross section Ric_{ii}. Consider now a direction $v = a\partial_r + \frac{b}{r}\partial_i$, then we can estimate

$$\mathrm{Ric}_{vv} \ge \frac{a^2}{100} \frac{\ln\ln(15R_3)}{r^2 \cdot \ln(r) \cdot (\ln\ln(r))^2} - 2\frac{ab}{r^2 \cdot \ln(r) \cdot \ln\ln(r)} + \frac{b^2}{2r^2}.$$

(135)

Let us split the above into two cases. If $b \le \frac{a}{10^3 \ln\ln r}$ then the negative term is dominated by the first term when $r \ge R_4(R_3, M)$. On the other hand, if $b \ge \frac{a}{10^3 \ln\ln r}$ then the negative middle term is dominated by the last term when $r \ge R_4(R_3, M)$. In any situation we then see for $r \ge R_4(R_3, M)$ that $\mathrm{Ric} > 0$ is positive.

Note that for $U_4 \equiv \{r \le 10^2 R_4\}$ we have that $X_4 \setminus U_4$ is isometric to $C(S^3_{\delta_3/2} \times S^3_{1/16}) \equiv C(S^3_{\delta_4} \times S^3_{1/16})$:

$$g_3 \equiv dr^2 + (\delta_4 r)^2 g_{S^3} + (r/16)^2 g_{S^3}.$$

(136)

7.6. Constructing X_5.
To construct X_5 we want to modify X_4 on the neighborhood $X_4 \setminus U_4$. The goal will be to end so that the second S^3 factor is growing at a slow polynomial rate. Our ansatz will be of the form

$$dr^2 + f(r)^2 g_{S^3} + (r/16)^2 g_{S^3}.$$

(137)

The Ricci curvature of this ansatz may be computed

$$\mathrm{Ric}_{rr} = -3\frac{f''}{f},$$
$$\mathrm{Ric}_{aa} = \frac{2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\left(2\frac{f'}{f} + \frac{3}{r}\right),$$
$$\mathrm{Ric}_{ii} = 2\frac{16^2 - 1}{r^2} - \frac{3}{r}\frac{f'}{f}.$$

(138)

For $0 < \alpha << 1$, which will be chosen in the next construction, let us consider a function f of the form

$$f(r) \equiv \begin{cases} \delta_4 r & \text{if } r \le 10R_4, \\ f'' < 0 & \text{if } 10R_4 \le r \le 10^3 R_4, \\ 10^2\delta_4 R_4 \left(\frac{r}{10^2 R_4}\right)^\alpha & \text{if } r \ge 10^3 R_4. \end{cases}$$

To build such an f one can smooth the function $f \equiv \min\left\{\delta_4 r, 10^2\delta_4 R_4 \left(\frac{r}{10^2 R_4}\right)^\alpha\right\}$. Note that these two functions agree at $10^2 R_4$ by construction. If we plug this into (138) we see that the resulting space has $\mathrm{Ric} \ge 0$.

If we define $R_5 \equiv 10^4 R_5$, $c_5 \equiv 10^2 \delta_4 R_4 \left(\frac{1}{10^2 R_4} \right)^\alpha$ and $U_5 \equiv \{r \leq R_5\}$, then $X_5 \setminus U_5$ is isometric to the warped product

$$g_5 = dr^2 + (c_5 r^\alpha)^2 g_{S^3} + (r/16)^2 g_{S^3}. \tag{139}$$

7.7. Constructing X_6.

The next step of the construction is dedicated to increasing the size of the cone S^3 factor until we are again geometrically close to $\mathbb{R}^4$. We will construct X_6 by modifying X_5 on the neighborhood $X_5 \setminus U_5$. The ansatz of our new metric will take the form

$$g_6 = dr^2 + (c_5 r^\alpha)^2 g_{S^3} + h(r)^2 g_{S^3}. \tag{140}$$

The nonzero terms of the Ricci tensor may be computed as

$$\mathrm{Ric}_{rr} = -3 \frac{h''}{h} + \frac{3\alpha(1-\alpha)}{r^2},$$
$$\mathrm{Ric}_{aa} = \frac{2}{(c_5 r^\alpha)^2} + \frac{\alpha(1-\alpha)}{r^2} - \frac{\alpha}{r} \left(\frac{2\alpha}{r} + 3 \frac{h'}{h} \right),$$
$$\mathrm{Ric}_{ii} = 2 \frac{1 - (h')^2}{h^2} - \frac{h''}{h} - \frac{3\alpha}{r} \frac{h'}{h}. \tag{141}$$

Recall the construction of X_5 depended on the parameter $\alpha > 0$, let us now choose $\alpha = 10^{-3} \hat{\epsilon}$. Then for $R_6 = R_6(\hat{\epsilon})$ we can choose a smooth function $h(r)$ so that it satisfies

$$h(r) \equiv \begin{cases} r/16 & \text{if } r \leq 10 R_5, \\ |h'| < (1 - 10^{-1}\hat{\epsilon}), \ |r h''| < 10^{-10}\hat{\epsilon} & \text{if } 10 R_5 \leq r \leq 10^{-1} R_6, \\ (1 - \hat{\epsilon}) r & \text{if } r \geq 10^{-1} R_6. \end{cases}$$

If we plug this into (138) then we see that $\mathrm{Ric} \geq 0$. If we let $U_6 \equiv \{r \leq R_6\}$ then we see that $X_6 \setminus U_6$ is isometric to the warped product

$$g_6 = dr^2 + (c_5 r^\alpha)^2 g_{S^3} + (1 - \hat{\epsilon})^2 r^2 g_{S^3}. \tag{142}$$

7.8. Constructing $X = X_7$.

We are now in a position to finish the construction of X and prove Proposition 7.1. The last step of the construction just needs to flatten out the first S^3 factor back into a cross product. Recall that we have built X_6 and that outside of U_6 we have that it is isometric to

$$g_6 = dr^2 + (c_5 r^\alpha)^2 g_{S^3} + (1 - \hat{\epsilon})^2 r^2 g_{S^3}. \tag{143}$$

We will look to alter this metric by looking for an ansatz of the form

$$g_7 = dr^2 + f(r)^2 g_{S^3} + (1 - \hat{\epsilon})^2 r^2 g_{S^3}. \tag{144}$$

The nonzero Ricci curvatures of this ansatz can be computed

$$\text{Ric}_{rr} = -3\frac{f''}{f}\,,$$

$$\text{Ric}_{aa} = \frac{2}{f^2} - \frac{f''}{f} - \frac{f'}{f}\Big(2\frac{f'}{f} + \frac{3}{r}\Big),$$

$$\text{Ric}_{ii} = \Big(\frac{1}{(1-\hat\epsilon)^2} - 1\Big)\frac{2}{r^2} - \frac{3}{r}\frac{f'}{f}\,. \tag{145}$$

We will choose a smooth function $f(r)$ of the form

$$f(r) \equiv \begin{cases} c_5 r^\alpha & \text{if } r \le 10R_6\,, \\ f'' < 0 & \text{if } 10R_5 \le r \le 10^4 R_6\,, \\ \hat\delta & \text{if } r \ge 10^4 R_6\,. \end{cases}$$

If $\hat\delta = \hat\delta(c_5, R_6)$ then we can build such a function f by smoothing the function $f(r) = \min\{c_5 r^\alpha, c_5(10^3 R_6)^\alpha\} \equiv \min\{c_5 r^\alpha, \hat\delta\}$. By plugging this into (145) we see that $\text{Ric} \ge 0$.

If we let $R_7 \equiv 10^5 R_6$ and $U_7 \equiv \{r \le R_7\}$ then we see that $X_7 \setminus U_7$ is isometric to

$$g_7 = dr^2 + \hat\delta^2 g_{S^3} + (1-\hat\epsilon)^2 r^2 g_{S^3}\,. \tag{146}$$

In particular, we have finished the construction of X and proved Proposition 7.1. □

8. STEP 3: EXTENDING THE ACTION

In this Section we focus on the third step of the construction, which was outlined in Section 3.1.3. The primary goal of this step of the construction is to extend the Γ_j action on M_j to a Γ_{j+1} action on M_{j+1}. Recall that Γ_{j+1} is generated by γ_{j+1} such that $\gamma_{j+1}^{k_{j+1}} = \gamma_j \in \Gamma_j$.

In order to accomplish this we will begin with a model space $\mathcal{B}_{j+1} = \mathcal{B}(\epsilon_{j+1}, \delta_{j+1}) \approx S^3 \times \mathbb{R}^4$, pluck out k_{j+1} balls $S^3 \times D^4$ and glue in copies of M_j. To do this precisely, and in order to preserve the geometry in the process, the main technical Proposition we need to prove in this Section is the following:

Proposition 8.1 (Step 3: Action Extension). *Let $\epsilon, \epsilon', \delta > 0$ with $0 < \epsilon - \epsilon' \le \frac{\epsilon}{10^2}$, and let $\hat\Gamma \le \mathbb{Q}/\mathbb{Z} \subseteq S^1$ be a finite subgroup with $\Gamma = \langle \gamma, \hat\Gamma \rangle$ such that $\hat\gamma \equiv \gamma^k$ is the generator of $\hat\Gamma$. Then for $\hat\epsilon \le \hat\epsilon(\epsilon, \epsilon')$ there exists a pointed space $(\tilde{A}, p)$, isometric to a smoooth Riemannian manifold with $\text{Ric} \ge 0$ away from $k + 1$ singular three spheres, with an isometric and free action by Γ such that*

(1) *There exists Γ-invariant set $B_{10^{-1}}(p) \subseteq U' \subseteq B_{10}(p)$ which is isometric to $S_\delta^3 \times B_1(0) \subseteq S_\delta^3 \times C(S_{1-\epsilon'}^3)$ and such that Γ is induced by the $(1, k)$-Hopf action on $S_\delta^3 \times S_{1-\epsilon'}^3$,*

(2) *There exists Γ-invariant set $B_{10^3 k}(p) \subseteq U \subseteq B_{10^5 k}(p)$ such that $\tilde{A} \setminus U$ is isometric to $S_\delta^3 \times A_{10^4 k, \infty}(0) \subseteq S_\delta^3 \times C(S_{1-\epsilon}^3)$ and such that Γ is induced by the $(1, k)$-Hopf action on $S_\delta^3 \times S_{1-\epsilon}^3$,*

(3) *There exists $\hat\Gamma$-invariant sets $S_\delta^3 \times B_{2^{-1}}(x^a) \subseteq V^a \subseteq S_\delta^3 \times B_2(x^a)$ with $d(S_\delta^3 \times \{x^a\}, S_\delta^3 \times \{p\}) = 10^2 k$ which are isometric to $S_\delta^3 \times B_1(0) \subseteq S_\delta^3 \times C(S_{1-\hat\epsilon}^3)$ and such that $\hat\Gamma$ is induced by the $(1, 0)$-Hopf action on $S_\delta^3 \times S_{1-\hat\epsilon}^3$.*

The construction will come in three steps. We will begin with $\tilde{\mathcal{A}}_0 = S_\delta^3 \times C(S_{1-\epsilon'}^3)$, which we see by (1) is what our space should look like on small scales. In Section 8.1 we will construct $\tilde{\mathcal{A}}_1$ by adding a bend to $\tilde{\mathcal{A}}_0$. The effect of this will be that on large scales the space looks like $S_\delta^3 \times C(S_{1-\epsilon}^3)$, however on some middle scale $\tilde{\mathcal{A}}_1$ will be isometric to an annulus in a 4-sphere S_R^4, where $R = R(\epsilon', \epsilon, k)$ is potentially very large.

In Section 8.2 we will construct the gluing pieces $\hat{\mathcal{A}}$ by beginning with $\hat{\mathcal{A}}_0 = S_\delta^3 \times C(S_{1-\hat{\epsilon}}^3)$ and bending it in an analogous manner to which we built $\tilde{\mathcal{A}}_1$. However, $\hat{\mathcal{A}}$ will have boundary and near the boundary will be isometric to a small annulus in the 4-sphere S_R^4.

We see we are now able to glue copies of $\hat{\mathcal{A}}$ into $\tilde{\mathcal{A}}_1$ as an open set near boundary of $\hat{\mathcal{A}}$ is isometric to a region in $\tilde{\mathcal{A}}_1$. We will want to glue k copies of $\hat{\mathcal{A}}^a = \hat{\mathcal{A}}$ into $\tilde{\mathcal{A}}_1$ in order to complete our construction, however as in the discussion in Section 3.1.3 we need to be careful about the choice of gluing maps. This will be done in Section 8.3.

8.1. Constructing $\mathcal{A}_1$.

Let us begin with $\tilde{\mathcal{A}}_0 \equiv S_\delta^3 \times C(S_{1-\epsilon'}^3)$, which geometrically has the metric

$$g_0 \equiv dr^2 + \delta^2 g_{S^3} + (1 - \epsilon')^2 r^2 g_{S^3} \,. \tag{147}$$

Let $U_0 = \{r \leq 1\}$, then we will build $\tilde{\mathcal{A}}_1$ by modifying the above metric on the region $\tilde{\mathcal{A}}_0 \setminus U_0$. We will look for a metric which is of the form

$$g_1 \equiv dr^2 + \delta^2 g_{S^3} + h(r)^2 g_{S^3} \,. \tag{148}$$

The nonzero Ricci curvatures of a warped metric as above are

$$\mathrm{Ric}_{rr} = -3\frac{h''}{h} \,,$$

$$\mathrm{Ric}_{aa} = \frac{2}{\delta^2} \,,$$

$$\mathrm{Ric}_{ii} = 2\frac{1 - (h')^2}{h^2} - \frac{h''}{h} \,. \tag{149}$$

In order to choose our warping function $h(r)$ let us begin by defining the following three functions:

$$h_1(r) \equiv (1 - \epsilon')r \,,$$

$$h_2(r) \equiv R \sin\left(R^{-1}(r - r_R)\right) ,$$

$$h_3(r) \equiv (1 - \epsilon)(r + r_\epsilon) \,. \tag{150}$$

Our goal will be to show for appropriate constants $R, r_R, r_\epsilon, \hat{r}$ and $r_1 \in [10k, 10^3 k]$ that we can choose $h(r)$ in the form

$$h(r) \equiv \begin{cases} h_1(r) & \text{if } r \leq 10, \\ h'' < 0 & \text{if } 10 \leq r \leq 10^4 k, \\ h_2(r) & \text{if } r_1 \leq r \leq r_1 + \hat{r}, \\ h_3(r) & \text{if } r \geq 10^4 k. \end{cases}$$

If we have an $h(r)$ then by (149) we have that

$$\text{Ric}_{rr} \geq 0, \quad \text{Ric}_{aa} = \frac{2}{\delta^2} > 0, \quad \text{Ric}_{ii} \geq \frac{1 - (1 - \epsilon')^2}{h^2} > 0. \tag{151}$$

Recall now that $\epsilon' < \epsilon$ have already been fixed, and the lines $h_1(r)$ and $h_3(r)$ must intersect at a unique point. Let us choose r_ϵ uniquely so that the point of intersection is at $10^2 k$. In particular, we can solve for r_ϵ as

$$r_\epsilon \equiv \frac{\epsilon - \epsilon'}{1 - \epsilon} 10^2 k. \tag{152}$$

With r_ϵ fixed, let us observe that for any $R \geq 0$ there is a unique smallest $r_R \in (0, 2\pi R]$ such that $h_2(r) \leq \min\{h_1(r), h_3(r)\}$ for every $r > 0$. Note for this r_R that $h_2(r)$ intersects $h_1(r)$ and $h_3(r)$ at most once, but must intersect one of them (otherwise $h_2 < \min\{h_1, h_3\}$ and we could have decreased r_R). On the other hand, note that for R small we must have that $h_2(r)$ intersects $h_1(r)$, while for R large we must have that $h_2(r)$ intersects $h_3(r)$. We can then also find a unique value of $R = R(\epsilon, \epsilon', k)$ for which h_2 intersects both h_1 and h_3. Let us fix this as our value of R and hence r_R, and let us call these intersection points $s_1 < 10^2 k < s_2$ respectively.

In order to estimate the value of r_R let us observe that $h_2(r) \leq r - r_R$, and as such we get that

$$(1 - \epsilon')s_1 = h_1(s_1) = h_2(s_1) \leq s_1 - r_R$$
$$\implies r_R \leq \epsilon' s_1 \leq 10^2 k \epsilon'. \tag{153}$$

Let us observe that $\dot{h}_2(s_1) = \dot{h}_1(s_1)$ and $\dot{h}_2(s_3) = \dot{h}_3(s_3)$ to get the relations

$$\cos(R^{-1}(s_1 - r_R)) = 1 - \epsilon',$$
$$\cos(R^{-1}(s_2 - r_R)) = 1 - \epsilon,$$
$$\implies \left| s_1 - r_R - \sqrt{2\epsilon'}R \right| \leq 10\epsilon' R$$
$$\left| s_2 - r_R - \sqrt{2\epsilon}R \right| \leq 10\epsilon R,$$
$$\implies |s_1 - \sqrt{2\epsilon'}R| \leq 10(10k + R)\epsilon',$$
$$|s_2 - \sqrt{2\epsilon}R| \leq 10(10k + R)\epsilon, \tag{154}$$

where we used the Taylor expansion of $\cos(x)$ and the fact that $s_1 - r_R, s_2 - r_R \in (0, 2\pi R]$.

Using that $s_1 \leq 10^2 k \leq s_2$, this gives the estimate on R:

$$|10^2 k - \sqrt{2\epsilon}R| \le \max\{|s_1 - \sqrt{2\epsilon}R|, |s_2 - \sqrt{2\epsilon}R|\} \tag{155}$$

$$\le \sqrt{2\epsilon}R\Big(\frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{\sqrt{\epsilon}}\Big) + 20(10k + R)\epsilon. \tag{156}$$

Hence, if $\sqrt{\epsilon} - \sqrt{\epsilon'} \le 10^{-1}\sqrt{\epsilon}$, we can deduce

$$\Big|R - \frac{10^2 k}{\sqrt{2\epsilon}}\Big| \le 10^3 k\Big(\sqrt{\epsilon} + \frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{\sqrt{\epsilon}}\Big). \tag{157}$$

From this we get the estimate

$$\Big|(s_2 - s_1) - \frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{\sqrt{\epsilon}}10^2 k\Big| \le 10^4 k\sqrt{\epsilon}\Big(1 + \Big(\frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{\sqrt{\epsilon}}\Big)^2\Big). \tag{158}$$

Hence, we can define $\hat{r} := \frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{100\sqrt{\epsilon}}10^2 k \le \frac{1}{10} \cdot 10^2 k$.

Then we may build $h(r)$ by smoothing out

$$h(r) \equiv \begin{cases} h_1(r) & \text{if } r \le s_1, \\ h_2(r) & \text{if } s_1 \le r \le s_2, \\ h_3(r) & \text{if } r \ge s_2. \end{cases}$$

8.2. Construction of $\hat{A}$.

Let us begin with $\hat{A}_0 \equiv S_\delta^3 \times C(S_{1-\hat{\epsilon}}^3)$ on the domain $\hat{U}_0 \equiv \{r \le \hat{r}\}$, where $\hat{r} \equiv \frac{\sqrt{\epsilon} - \sqrt{\epsilon'}}{100\sqrt{\epsilon}}10^2 k$ was defined in the previous section. The metric on $\hat{A}_0$ may be written

$$\hat{g}_0 \equiv dr^2 + \delta^2 g_{S^3} + (1 - \hat{\epsilon})^2 r^2 g_{S^3}. \tag{159}$$

In order to construct $\hat{A}$ we will need to alter the geometry by looking for a metric of the form

$$\hat{g} \equiv dr^2 + \delta^2 g_{S^3} + h(r)^2 g_{S^3}. \tag{160}$$

We will build $h(r)$ in a manner analogous to the previous subsection. Let us start by looking at the functions

$$h_1(r) \equiv (1 - \hat{\epsilon})r,$$
$$h_2(r) \equiv R\sin\big(R^{-1}(r - \hat{r}_R)\big), \tag{161}$$

where $R = R(\epsilon, \epsilon', k) > 0$ has been fixed in the previous subsection. Observe that h_1 and h_2 will intersect twice for $\hat{r}_R$ small, and for $\hat{r}_R$ large they will not intersect. Let us choose $\hat{r}_R$ uniquely so that they intersect at $\hat{s}$ precisely once. Note that at this point of intersection $\hat{s}$ we will then have $\dot{h}_1(\hat{s}) = \dot{h}_2(\hat{s})$, which is the equation

$$\cos(R^{-1}(\hat{s} - \hat{r}_R)) = 1 - \hat{\epsilon},$$
$$\implies \big|\hat{s} - \hat{r}_R - \sqrt{2\hat{\epsilon}}R\big| \le \hat{\epsilon}R. \tag{162}$$

Additionally, using that $h_2(r) \leq r - \hat{r}_R$ we have the inequality

$$(1 - \hat{\epsilon})\hat{s} = h_1(\hat{s}) = h_2(\hat{s}) \leq \hat{s} - \hat{r}_R \,,$$
$$\implies \hat{r}_R \leq \hat{\epsilon}\hat{s} \,. \tag{163}$$

Combining the last two estimates we conclude

$$\left| \hat{s} - \sqrt{2\hat{\epsilon}}R \right| \leq (\hat{\epsilon} + \sqrt{2}\hat{\epsilon}^{3/2})R \,,$$
$$\hat{r}_R \leq 2\hat{\epsilon}R \,. \tag{164}$$

Let us now choose $\hat{\epsilon} \leq \hat{\epsilon}(R) \leq \hat{\epsilon}(\epsilon, \epsilon')$ so that $\hat{s} \leq \frac{1}{2}\hat{r}$. Then we can define $h(r)$ for $r \leq 2\hat{s}$ by smoothing

$$h(r) \equiv \begin{cases} h_1(r) & \text{if } r \leq \hat{s} \,, \\ h_2(r) & \text{if } \hat{s} \leq r \leq 2\hat{s} \,. \end{cases}$$

In particular, for $\hat{A} = \{r \leq 2\hat{s}\}$ we see that a neighborhood of the boundary is isometric to the product of S_δ^3 with an annulus in S_R^4. The verification that $\text{Ric} \geq 0$ with this choice of h is completely analogous to the one discussed in the previous subsection, using (149).

8.3. Constructing $\tilde{A}$.

As our final step let us now denote $\hat{A}^a$ for $a = 0, \ldots, k - 1$ as k copies of our constructed neck from the last subsection. If we denote $\tilde{r} \equiv 2\hat{s} - \hat{r}_R < r_R$ then we have

$$\partial \hat{A}^a = \{r = 2\hat{s}\} \equiv S_\delta^3 \times \partial B_{\tilde{r}} \subseteq S_\delta^3 \times S_R^4 \,, \tag{165}$$

be the boundary of our neck region. The boundary, and indeed all of $\hat{A}^a$ near the boundary, is isometric to a neighborhood in $S_\delta^3 \times S_R^4$.

Recall that $\tilde{A}_1$ has a metric of the form

$$\tilde{g}_1 \equiv dr^2 + \delta^2 g_{S^3} + h(r)^2 g_{S^3} \,, \tag{166}$$

such that the region $\{10^2 k - \hat{r} \leq r \leq 10^2 k + \hat{r}\}$ is isometric to an annulus in $S_\delta^3 \times S_R^4$. In the coordinates from the above description let us choose the point $x^0 = (10^2 k, e, e) \in \tilde{A}_1$ so that $r(x^0) = 10^2 k$, where $e \in S^3$ is the identity. Let $x^a = (10^2 k, (2\pi a/k) \cdot e, e)$ be the Hopf rotation of x^0 by angle $2\pi a/k$. Consider the domains $S_\delta^3 \times B_{\tilde{r}}(x^a)$, and let

$$\varphi^0 : \partial \hat{A}^0 = S_\delta^3 \times \partial B_{\tilde{r}} \rightarrow S_\delta^3 \times \partial B_{\tilde{r}}(x^0) \,, \tag{167}$$

be the canonical isometry which fixes the S_δ^3 factor. Let $\gamma \in \Gamma$ be the action which Hopf rotates the first S^3 factor by $2\pi/|\gamma|$ and Hopf rotates the second S^3 factor by $2\pi/k$. Then we define the mappings

$$\varphi^a : \partial \hat{A}^a \rightarrow S_\delta^3 \times \partial B_{\tilde{r}}(x^a) \,, \text{ by } \varphi^a \equiv \gamma^a \cdot \varphi^0 \,. \tag{168}$$

This allows us to define our space

$$\tilde{A} \equiv \left(\tilde{A}_1 \setminus \bigcup_a S_\delta^3 \times B_{\tilde{r}}(x^a) \right) \bigcup_{\varphi^a} \hat{A}^a \,. \tag{169}$$

Note that this gluing extends isometrically to a small neighborhood, and $\tilde{A}$ is a smooth manifold (away from $(k + 1)$ singular three spheres) with $\text{Ric} \geq 0$. Additionally, we have the required property that $\Gamma = \langle \gamma \rangle$ acts

isometrically on $\tilde{A}$ such that in both the $\tilde{A}_1$ domain and the glued domains $\hat{A}^a$ the action of $\gamma^k = \hat{\gamma}$ is purely by Hopf rotation of the S^3_δ factor. $\quad\square$

9. Geometry of the Mapping Class Group of $S^3 \times S^3$

The primary goal of this Section is to prove Lemma 1.2, which we restate for the readers convenience below:

Lemma 9.1 (Mapping Class Group and Ricci Curvature on $S^3 \times S^3$). *Let $g_0 = g_{S^3 \times S^3}$ be the standard metric on $S^3 \times S^3$. Then given $\phi \in \mathrm{Diff}(S^3 \times S^3)$ there exists a smooth family g_t of metrics with $\mathrm{Ric}_{g_t} > 0$ such that g_0 is the standard metric and $g_1 = \phi^* g_0$. That is, the orbit $\pi_0\mathrm{Diff}(S^3 \times S^3) \cdot [g_{S^3 \times S^3}]$ of the mapping class group lives in a connected component of $\mathcal{M}_0^+(S^3 \times S^3)$, the space of metrics with strictly positive Ricci curvature.*

Let us begin by recalling some basic structure of the mapping class group $\pi_0\mathrm{Diff}(S^3 \times S^3)$. So consider $\mathrm{Diff}(S^3 \times S^3)$, the diffeomorphism group of $S^3 \times S^3$, and let $\pi_0\mathrm{Diff}(S^3 \times S^3)$ denote the connected components of it. This set inherits a group structure, and the mapping class group of $S^3 \times S^3$ is a discrete group. There is a natural surjective mapping

$$\kappa : \pi_0\mathrm{Diff}(S^3 \times S^3) \to \mathrm{SL}(2, \mathbb{Z}), \tag{170}$$

given by looking at the action of $[\phi] \in \pi_0\mathrm{Diff}(S^3 \times S^3)$ on the homology ring $\kappa[\phi] = [\phi_*] : H_3(S^3 \times S^3) \to H_3(S^3 \times S^3)$. It is now well understood, see [Kre],[Kry], that the kernel

$$\mathcal{K} = \ker \kappa \triangleleft \pi_0\mathrm{Diff}(S^3 \times S^3) \tag{171}$$

is a 2-step nilpotent group and obeys the short exact sequence

$$0 \to \mathbb{Z}_{28} \to \mathcal{K} \to \mathbb{Z} \times \mathbb{Z} \to 0. \tag{172}$$

This kernel and its $\mathbb{Z}_{28}$ extension are closely related to the exotic differentiable structures on seven manifolds. The group $\pi_0\mathrm{Diff}(S^3 \times S^3)/\mathcal{K} = \mathrm{SL}(2, \mathbb{Z})$ is generated by the two diffeomorphisms

$$\phi_1(g_1, g_2) = (g_1, g_1 g_2),$$
$$\phi_2(g_1, g_2) = (g_1 g_2^{-1}, g_2), \tag{173}$$

see [Kry]. On the other hand the kernel $\mathcal{K}$, which is the collection of diffeomorphisms whose induced action on the homology is trivial, can be identified as the nilpotent group

$$\mathcal{K} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad a, b \in \mathbb{Z}, \quad c \in \mathbb{Z}_{28}. \tag{174}$$

It is generated by two elements, which are given by the diffeomorphisms

$$\phi_1^{\mathcal{K}}(g_1, g_2) = (g_2 g_1 g_2^{-1}, g_2),$$
$$\phi_2^{\mathcal{K}}(g_1, g_2) = (g_1, g_1 g_2 g_1^{-1}). \tag{175}$$

In particular, if we consider the diffeomorphisms

$$\phi_3(g_1, g_2) = (g_1, g_2 g_1^{-1}),$$
$$\phi_4(g_1, g_2) = (g_2 g_1, g_2),$$ (176)

then we see that $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ generates $\pi_0 \text{Diff}(S^3 \times S^3)$.

Now it follows from Theorem 6.1 and Remark 6.1 that there exist families of metrics $g_{1,t}, g_{2,t}, g_{3,t}, g_{4,t}$ with Ric > 0 such that

$$g_{j,0} = g_{S^3 \times S^3},$$
$$g_{j,1} = \phi_j^* g_{S^3 \times S^3}.$$ (177)

To prove the Theorem it is now enough to show for each $[\phi] \in \pi_0 \text{Diff}(S^3 \times S^3)$ that there exists some representative $\phi \in [\phi]$ for which the Theorem holds, as we can clearly vary the metric within a fixed class by the diffeomorphism action itself. Thus let $\phi = \phi_{j_k} \circ \cdots \circ \phi_{j_1}$ represent any element of the mapping class group. If we denote $j_0 = 0$ with $\phi_0 = Id$, then let us define the family of metrics

$$g_t \equiv \phi_{j_\ell}^* \circ \cdots \circ \phi_{j_0}^* g_{j_{\ell+1}, k(t - \frac{\ell}{k})} \quad \text{if} \quad t \in \left[\frac{\ell}{k}, \frac{\ell+1}{k}\right].$$ (178)

Then we have that $\text{Ric}_t > 0$ with $g_0 = g_{S^3 \times S^3}$ and $g_1 = \phi^* g_{S^3 \times S^3}$, as claimed. $\square$

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