ON SETS WITH FINITE DISTRIBUTIONAL FRACTIONAL PERIMETER

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Abstract. We continue the study of the fine properties of sets having locally finite distributional fractional perimeter. We refine the characterization of their blow-ups and prove a Leibniz rule for the intersection of sets with locally finite distributional fractional perimeter with sets with finite fractional perimeter. As a byproduct, we provide a description of non-local boundaries associated with the distributional fractional perimeter.

1. Introduction

1.1. The distributional fractional perimeter.

Given $\alpha \in (0,1)$, we let

$$\nabla^\alpha f(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(y-x)}{|y-x|^{n+\alpha+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.1}$$

be the *fractional α-gradient* of $f \in \mathrm{Lip}_c(\mathbb{R}^n)$ and, analogously,

$$\mathrm{div}^\alpha \varphi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.2}$$

be the *fractional α-divergence* of $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, where

$$\mu_{n,\alpha} = 2^\alpha \pi^{-\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)} > 0$$

is a renormalization constant. The operators (1.1) and (1.2) are *dual*, in the sense that they satisfy the fractional integration-by-parts formula

$$\int_{\mathbb{R}^n} f\,\mathrm{div}^\alpha \varphi\,\mathrm{d}x = -\int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha f\,\mathrm{d}x. \tag{1.3}$$

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For a more detailed account on the operators in (1.1) and (1.2) and on the formula (1.3), we refer the reader to [15]. In our previous papers [2–8], starting from formula (1.3), we developed a new theory of distributional fractional Sobolev and BV spaces.

In the present note, we continue the study of fractional BV functions. Let us introduce the following definition (see [3, Sec. 3.1] for example). Given $p \in [1, +\infty]$ and an open set $\Omega \subset \mathbb{R}^n$, we say that $f \in BV^{\alpha,p}_{\mathrm{loc}}(\Omega)$ if $f \in L^p(\mathbb{R}^n)$ and

$$\sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \,\mathrm{d}x : \varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1,\ \operatorname{supp}\varphi \subset A\right\} < +\infty$$

for any open set $A \Subset \Omega$. A simple application of Riesz's Representation Theorem (see [3, Th. 3]) yields that $f \in BV^{\alpha,p}_{\mathrm{loc}}(\Omega)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there is a vector-valued Radon measure $D^\alpha f \in \mathscr{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, called *fractional α-variation measure* of f, such that

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \,\mathrm{d}x = -\int_\Omega \varphi \cdot \mathrm{d}D^\alpha f \tag{1.4}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \Omega$, with

$$|D^\alpha f|(A) = \sup\left\{\int_{\mathbb{R}^n} f \operatorname{div}^\alpha \varphi \,\mathrm{d}x : \varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n),\ \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \le 1,\ \operatorname{supp}\varphi \subset A\right\}$$

for any open set $A \Subset \Omega$. If $|D^\alpha f|(\Omega) < +\infty$, then we write $f \in BV^{\alpha,p}(\Omega)$. We warn the reader that the subscript 'loc' in $BV^{\alpha,p}_{\mathrm{loc}}$ always refers to the local finiteness of the fractional variation measure only, as $BV^{\alpha,p}_{\mathrm{loc}}$ functions are in $L^p(\mathbb{R}^n)$ by default.

If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, then the measure $|D^\alpha \chi_E| \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n)$ is called the *distributional fractional* (*Caccioppoli*) *α-perimeter* of $E \subset \mathbb{R}^n$ (see [4, Def. 4.1]). We recall that $W^{\alpha,1}(\mathbb{R}^n) \subset BV^{\alpha,1}(\mathbb{R}^n)$ with strict inclusion, see [4, Ths. 3.18 and 3.31], so $|D^\alpha\chi_E|$ must not be confused with the *fractional α-perimeter* $P_\alpha(E;\,\cdot\,)$ relative to $W^{\alpha,1}$ sets, defined as

$$P_\alpha(E;\Omega) = \int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \tag{1.5}$$

for $E, \Omega \subset \mathbb{R}^n$ (in particular, if $\Omega = \mathbb{R}^n$, then $P_\alpha(E) = P_\alpha(E;\mathbb{R}^n) = [\chi_E]_{W^{\alpha,1}(\mathbb{R}^n)}$). In fact, if $P_\alpha(E;\Omega) < +\infty$, then $\chi_E \in BV^{\alpha,\infty}(\Omega)$ with $D^\alpha\chi_E = \nabla^\alpha\chi_E \mathscr{L}^n$ and $\|\nabla^\alpha\chi_E\|_{L^1(\Omega;\mathbb{R}^n)} \le \mu_{n,\alpha}P_\alpha(E;\Omega)$, see [4, Prop. 4.8], but currently we do not know if there exists $\chi_E \in BV^{\alpha,\infty}(\Omega)$ such that $P_\alpha(E;\Omega) = +\infty$.

Mimicking the classical theory (see [4, Sec. 4.5]), given $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, we say that $x \in \mathbb{R}^n$ belongs to the *fractional reduced boundary* of E, and we write $x \in \mathscr{F}^\alpha E$, if

$$x \in \operatorname{supp}|D^\alpha\chi_E| \quad \text{and} \quad \exists \lim_{r\to0^+} \frac{D^\alpha\chi_E(B_r(x))}{|D^\alpha\chi_E|(B_r(x))} \in \mathbb{S}^{n-1}.$$

Consequently, we let $\nu^\alpha_E \colon \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}$,

$$\nu^\alpha_E(x) = \lim_{r\to0^+} \frac{D^\alpha\chi_E(B_r(x))}{|D^\alpha\chi_E|(B_r(x))}, \quad x \in \mathscr{F}^\alpha E,$$

be the *(measure theoretic) inner unit fractional normal* of E. Hence, (1.4) implies that

$$\int_E \operatorname{div}^\alpha\varphi\,\mathrm{d}x = -\int_{\mathscr{F}^\alpha E} \varphi\cdot\nu^\alpha_E \,\mathrm{d}|D^\alpha\chi_E| \tag{1.6}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$.

1.2. **Leibniz rules.** A large part of our preceding works [2–8] is dedicated to the study of fractional Leibniz rules involving the operators (1.1) and (1.2). For $f, g \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, we have

$$\nabla^\alpha (fg) = f\,\nabla^\alpha g + g\,\nabla^\alpha f + \nabla^\alpha_{\mathrm{NL}}(f,g) \tag{1.7}$$

and, analogously,

$$\mathrm{div}^\alpha_{\mathrm{NL}}(f\varphi) = f\,\mathrm{div}^\alpha \varphi + \varphi \cdot \nabla^\alpha f + \mathrm{div}^\alpha_{\mathrm{NL}}(f,\varphi), \tag{1.8}$$

where

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.9}$$

and

$$\mathrm{div}^\alpha_{\mathrm{NL}}(f,\varphi)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(\varphi(y)-\varphi(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.10}$$

are the *non-local fractional α-gradient* and the *non-local fractional α-divergence*, respectively. The fractional Leibniz rules (1.7) and (1.8), as well as the operators (1.9) and (1.10), can be extended to less regular functions and vector fields in several ways, see [2–8].

The first main aim of the present note is to achieve some new fractional Leibniz rules. On the one side, we provide the following product rule for the intersection of $BV^{\alpha,\infty}_{\mathrm{loc}}$ sets with $W^{\alpha,1}$ sets, generalizing [6, Th. 1.1]. In fact, the locality property (1.17) was inspired by an observation recently made in [14, Rem. 3.4]. Here and in the following, $D^\alpha_s f$ denotes the singular part of the fractional variation measure $D^\alpha f$ of $f \in BV^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n)$. We also let

$$u^\star(x) = \lim_{r\to 0^+} \fint_{B_r(x)} u(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

be the *precise representative* of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, whenever the limit exists in $\mathbb{R}$. We hence define the Borel set

$$\mathscr{R}_u = \{x \in \mathbb{R}^n : u^\star(x) \text{ exists in } \mathbb{R}\}.$$

Let us recall that $x \in \mathbb{R}^n$ is a *Lebesgue point* of u if $x \in \mathscr{R}_u$ and

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y) - u^\star(x)|\,\mathrm{d}y = 0.$$

We note that, if $E \subset \mathbb{R}^n$ is a measurable set and $x \in \mathbb{R}^n$ is a Lebesgue point of χ_E, then $\chi_E^\star(x) = \chi_{E^1}(x)$. Here and below, for $t \in [0,1]$, we let

$$E^t = \left\{ x \in \mathbb{R}^n : \exists \lim_{r\to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\}. \tag{1.11}$$

Theorem 1.1 (Intersection with $W^{\alpha,1}$ set)**.** *If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $P_\alpha(F) < +\infty$, then $\chi_{E\cap F} \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, with*

$$D^\alpha \chi_{E\cap F} = \chi_{F^1} D^\alpha \chi_E + \chi_E \nabla^\alpha \chi_F \,\mathscr{L}^n + \nabla^\alpha_{\mathrm{NL}}(\chi_E, \chi_F)\,\mathscr{L}^n \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n), \tag{1.12}$$

$$\max\Big\{ \|\chi_E \nabla^\alpha \chi_F\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}, \|\nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \Big\} \le \mu_{n,\alpha} P_\alpha(F) \tag{1.13}$$

and

$$\int_{\mathbb{R}^n} \nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\,\mathrm{d}x = 0. \tag{1.14}$$

Consequently, we have

$$D^\alpha \chi_{E\cap F} - \chi_{F^1} D^\alpha \chi_E \in \mathscr{M}(\mathbb{R}^n;\mathbb{R}^n), \tag{1.15}$$

$$|D^\alpha \chi_{E\cap F} - \chi_{F^1} D^\alpha \chi_E|(\mathbb{R}^n) \le \mu_{n,\alpha} P_\alpha(F) \tag{1.16}$$

and

$$D^\alpha_{\mathrm{s}} \chi_{E\cap F} = \chi_{F^1} D^\alpha_{\mathrm{s}} \chi_E \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n). \tag{1.17}$$

In addition, if F is also bounded, then $\chi_{E\cap F} \in BV^{\alpha,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and

$$\int_{F^1} \mathrm{d}D^\alpha \chi_E = -\int_E \nabla^\alpha \chi_F \,\mathrm{d}x. \tag{1.18}$$

We expect that Theorem 1.1 may be extended to $BV^{\alpha,\infty}_{\mathrm{loc}}$ (and even $BV^{\alpha,p}_{\mathrm{loc}}$) functions, but we do not pursue this direction here and leave it to future works.

On the other side, we generalize [6, Rem. 4.6], see Theorem 1.3 below. To state our result, we need to introduce some notation. In analogy with [6, Def. 4.5], we exploit [6, Lem. 2.9] to define the measure version of the non-local fractional gradient (1.9) (actually, we already used this object in [6, Th. 5.1] without providing its explicit definition).

Definition 1.2 (Non-local fractional α-gradient measure)**.** Let $p, q \in [1, +\infty]$ be such that $\frac{1}{p} + \frac{1}{q} \le 1$. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. We say that $D^\alpha_{\mathrm{NL}}(f,g) \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is a *non-local fractional α-gradient measure* of the pair (f,g) if

$$\int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(g,\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi \cdot \mathrm{d}D^\alpha_{\mathrm{NL}}(f,g) \quad \text{for all } \varphi \in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n).$$

Arguing as in [6, Sec. 4.4], we can exploit [6, Cor. 2.7 and Lems. 2.9 and 2.10] to infer that Definition 1.2 is well posed and that $D^\alpha_{\mathrm{NL}}(f,g)$, if it exists, is unique and symmetric, and extends the operator (1.9).

Our second result on the Leibniz rule for $BV^{\alpha,\infty}_{\mathrm{loc}}$ functions can be stated as follows.

Theorem 1.3 (Conditional Leibniz rule in $BV^{\alpha,\infty}_{\mathrm{loc}}$)**.** *If $f, g \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$fg \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n) \iff \exists\, D^\alpha_{\mathrm{NL}}(f,g) \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n), \tag{1.19}$$

and there exist $\bar f \in L^\infty(\mathbb{R}^n, |D^\alpha g|)$ and $\bar g \in L^\infty(\mathbb{R}^n, |D^\alpha f|)$, with

$$\|\bar f\|_{L^\infty(\mathbb{R}^n,|D^\alpha g|)} \le \|f\|_{L^\infty(\mathbb{R}^n)} \quad \text{and} \quad \|\bar g\|_{L^\infty(\mathbb{R}^n,|D^\alpha f|)} \le \|g\|_{L^\infty(\mathbb{R}^n)}, \tag{1.20}$$

$$\bar f = f^\star \ |D^\alpha g|\text{-a.e. in } \mathscr{R}_f \quad \text{and} \quad \bar g = g^\star \ |D^\alpha f|\text{-a.e. in } \mathscr{R}_g, \tag{1.21}$$

such that, provided that $fg \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$D^\alpha(fg) = \bar f\, D^\alpha g + \bar g\, D^\alpha f + D^\alpha_{\mathrm{NL}}(f,g) \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n). \tag{1.22}$$

At the present moment, we do not know if the measure $D^\alpha_{\mathrm{NL}}(f,g)$ is well defined even in the case $f, g \in BV^{\alpha,\infty}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $|D^\alpha f|, |D^\alpha g| \ll \mathscr{L}^n$, see [6, Rem. 4.6]. However, in the simplest case $f = g = \chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, we have the following result.

Corollary 1.4 (The measure $D^\alpha_{\mathrm{NL}}(\chi_E, \chi_E)$)**.** *If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, then $D^\alpha_{\mathrm{NL}}(\chi_E,\chi_E) \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ is well defined and satisfies*

$$D^\alpha_{\mathrm{NL}}(\chi_E,\chi_E) = (1 - 2\,\overline{\chi_E})\, D^\alpha \chi_E \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n), \tag{1.23}$$

where $\overline{\chi_E} \in L^\infty(\mathbb{R}^n, |D^\alpha \chi_E|)$, with

$$0 \le \overline{\chi_E} \le 1 \quad \text{and} \quad \overline{\chi_E} = \chi_E^\star \quad |D^\alpha \chi_E|\text{-a.e. in } \mathscr{R}_{\chi_E}. \tag{1.24}$$

In addition, if $\chi_E \in BV^{\alpha,1}(\mathbb{R}^n)$*, then*

$$D^\alpha_{\mathrm{NL}}(\chi_E,\chi_E)(\mathbb{R}^n) = \int_{\mathbb{R}^n} \overline{\chi_E}\,\mathrm{d}D^\alpha\chi_E = 0. \tag{1.25}$$

In the limiting case $\alpha = 1$, the non-local gradient disappears, hence (1.23) reduces to $(1-2\,\overline{\chi_E})\,D\chi_E = 0$, coherently with the fact that $\overline{\chi_E} = \chi_E^\star = \frac{1}{2}$ $|D\chi_E|$-a.e. in $\mathbb{R}^n$.

1.3. **Analysis of blow-ups.** The main result of our first paper, see [4, Th. 5.8 and Prop. 5.9], provides the following fractional counterpart of De Giorgi's Blow-up Theorem for sets with locally finite perimeter (see [12, Part Two] for a detailed exposition). Here and in the rest of the paper, for any measurable $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we let $\mathrm{Tan}(E,x)$ be the set of all *tangent sets to* E *at* x, i.e., all limit points of $\left\{\frac{E-x}{r} : r>0\right\}$ with respect to the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $r \to 0^+$.

Theorem 1.5 (Existence and rigidity of blow-ups)**.** *If* $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $x \in \mathscr{F}^\alpha E$*, then* $\mathrm{Tan}(E,x) \neq \emptyset$ *and any* $F \in \mathrm{Tan}(E,x)$ *is such that* $\chi_F \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\nu^\alpha_F(y) = \nu^\alpha_E(x)$ *for* $|D^\alpha\chi_F|$*-a.e.* $y \in \mathscr{F}^\alpha F$*.*

The second main aim of this note is to refine Theorem 1.5. On the one hand, we provide the following convergence result, which was somehow implicit in the proof of [4, Prop. 5.9].

Theorem 1.6 (Refined convergence)**.** *Let* $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $x \in \mathscr{F}^\alpha E$*. If* $F \in \mathrm{Tan}(E,x)$ *with* $\chi_{\frac{E-x}{r_k}} \to \chi_F$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *as* $r_k \to 0^+$*, then, up to extracting a subsequence:*

(i) $D^\alpha\chi_{\frac{E-x}{r_k}} \rightharpoonup D^\alpha\chi_F$ *in* $\mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ *as* $r_k \to 0^+$*;*
(ii) $|D^\alpha\chi_{\frac{E-x}{r_k}}| \rightharpoonup |D^\alpha\chi_F|$ *in* $\mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n)$ *as* $r_k \to 0^+$*;*
(iii) $D^\alpha\chi_{\frac{E-x}{r_k}}(B_R) \to D^\alpha\chi_F(B_R)$ *and* $|D^\alpha\chi_{\frac{E-x}{r_k}}|(B_R) \to |D^\alpha\chi_F|(B_R)$ *for all* $R>0$ *such that* $|D^\alpha\chi_F|(\partial B_R) = 0$ *(in particular, for a.e.* $R>0$*).*

On the other hand, we provide the following characterization of blow-ups, which can be seen as a first step towards a fractional counterpart of De Giorgi's Structure Theorem for sets with locally finite perimeter, see [12, Part 2] for instance. Here and in the following, we let $\partial^\alpha = D^\alpha_{\mathbb{R}}$ denote the fractional variation measure in dimension $n=1$ and we let P be the standard De Giorgi's perimeter.

Theorem 1.7 (Characterization of blow-ups)**.** *Let* $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$*,* $x \in \mathscr{F}^\alpha E$ *and* $\nu^\alpha_E(x) = \mathrm{e}_n$*. If* $F \in \mathrm{Tan}(E,x)$*, then* $F = \mathbb{R}^{n-1} \times M$ *with* $M \subset \mathbb{R}$ *such that:*

(i) $\chi_M \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R})$ *with* $\partial^\alpha\chi_M \geq 0$*;*
(ii) $|M|, |M^c| \in \{0,+\infty\}$*;*
(iii) if $|M| = +\infty$*, then* $\operatorname{ess\,sup} M = +\infty$*;*
(iv) if $M \neq \emptyset, \mathbb{R}$ *is such that* $P(M) < +\infty$*, then* $M = (m,+\infty)$ *for some* $m \in \mathbb{R}$*.*

Theorem 1.7 shows a quite surprising similarity between the reduced boundary $\mathscr{F}E$ for $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ and the fractional reduced boundary $\mathscr{F}^\alpha E$ for $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, going in the same direction of De Giorgi's Blow-up Theorem, see [12, Th. 15.5].

A simple consequence of Theorem 1.7 is that a cone cannot be a blow-up set on the fractional reduced boundary, unless it is a half-space, $\mathbb{R}^n$ or $\emptyset$. Such a rigidity of blow-ups for example implies that no vertex of the square $E = [0,1]^2 \subset \mathbb{R}^2$ belongs to $\mathscr{F}^\alpha E$, for any $\alpha \in (0,1)$, in analogy with the case $\alpha = 1$, see [12, Exam. 15.4]. Similarly, no vertex of the *Koch snowflake* (see [11, Sec. 3.3]) belongs to its fractional reduced boundary.

Theorem 1.7 is a consequence of the following two results, which may be interesting on their own. The first one characterizes $BV^{\alpha,\infty}_{\mathrm{loc}}$ functions with zero fractional derivative.

Proposition 1.8 (Null derivative). *Let $f \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and let $i \in \{1,\dots,n\}$. Then, $D^\alpha_i f = 0$ if and only if $D_i f = 0$.*

The second result allows to factorize the fractional variation measure of a $BV^{\alpha,\infty}_{\mathrm{loc}}$ function which does not depend on certain coordinates.

Proposition 1.9 (Splitting). *Let $f \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. If $D_1 f = 0$, then there exists $g \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ such that $f((t,x)) = g(x)$ for a.e. $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^{n-1}$ and*

$$(D^\alpha_{\mathbb{R}^n})_i f = \mathscr{L}^1 \otimes (D^\alpha_{\mathbb{R}^{n-1}})_i g \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n) \text{ for all } i = 2,\dots,n.$$

Propositions 1.8 and 1.9 may hold for $BV^{\alpha,p}_{\mathrm{loc}}$ functions as well, but we leave this line of research for forthcoming works, being out of the scopes of the present note.

1.4. **Analysis of non-local boundaries.** The third and last main aim of this note is to exploit the above results on blow-ups and Leibniz rules for $BV^{\alpha,\infty}_{\mathrm{loc}}$ sets to infer some properties of non-local boundaries linked with the distributional fractional perimeter.

On the one side, given $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, we may decompose the fractional variation measure of E as $D^\alpha \chi_E = D^\alpha_{\mathrm{ac}} \chi_E + D^\alpha_{\mathrm{s}} \chi_E$, where $|D^\alpha_{\mathrm{ac}} \chi_E| \ll \mathscr{L}^n$ and $D^\alpha_{\mathrm{s}} \chi_E \perp \mathscr{L}^n$. In virtue of (1.17) in Theorem 1.1, $D^\alpha_{\mathrm{s}} \chi_E$ has a local nature, in contrast with the non-local and thus 'diffuse' behavior of the measure $D^\alpha_{\mathrm{ac}} \chi_E$. The following result, which is a simple consequence of Theorem 1.1, gives an idea of the size of the support of the measure $D^\alpha_{\mathrm{s}} \chi_E$. Here and in the following, we let

$$\partial^- E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < |B_r(x)| \text{ for all } r > 0\}.$$

Theorem 1.10 (Support of $D^\alpha_{\mathrm{s}} \chi_E$). *If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$|D^\alpha_{\mathrm{s}} \chi_E|(F^1) = 0 \text{ whenever } \chi_F \in W^{\alpha,1}(\mathbb{R}^n) \text{ with either } |E \cap F| = 0 \text{ or } |E^c \cap F| = 0.$$

In particular, $\operatorname{supp} |D^\alpha_{\mathrm{s}} \chi_E| \subset \partial^- E$.

Theorem 1.10 has several analogies with classical results. Indeed, it is well-known that, if $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, then $\operatorname{supp} |D\chi_E| = \partial^- E$ (see [12, Prop. 12.19] for instance), while, if E has locally finite fractional perimeter, then

$$\partial^- E = \left\{x \in \mathbb{R}^n : P^L_\alpha(E; B_r(x)) > 0 \text{ for all } r > 0\right\},$$

where

$$P^L_\alpha(E; A) = \int_{E \cap A} \int_{A \setminus E} \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^{n+\alpha}}, \quad A \subset \mathbb{R}^n,$$

is the *local part* of the fractional perimeter $P_\alpha(E; \cdot)$, see [11, Lem. 3.1]. Furthermore, Theorem 1.10 can be combined with [4, Cor. 5.4] to get the estimate

$$|D^\alpha_{\mathrm{s}} \chi_E| \le c_{n,\alpha} \mathscr{H}^{n-\alpha} \llcorner (\mathscr{F}^\alpha E \cap \partial^- E).$$

In particular, if $\mathscr{H}^{n-\alpha}(\mathscr{F}^\alpha E \cap \partial^- E) = 0$, then $|D^\alpha \chi_E| \ll \mathscr{L}^n$. However, we warn the reader that $\partial^- E$ may have positive Lebesgue measure, even for a set with finite perimeter (see [12, Exam. 12.25] for instance).

On the other side, in view of Theorems 1.5 and 1.7, we wish to study the set of points $x \in \mathscr{F}^\alpha E$ for which fractional blow-ups are non-trivial, i.e., such that $\operatorname{Tan}(E,x) \neq \{\emptyset\}$ and $\operatorname{Tan}(E,x) \neq \{\mathbb{R}^n\}$. As well-known, for any $E \subset \mathbb{R}^n$ measurable and $x \in \mathbb{R}^n$, we have

$$\operatorname{Tan}(E,x) = \{\emptyset\} \iff x \in E^0, \quad \operatorname{Tan}(E,x) = \{\mathbb{R}^n\} \iff x \in E^1, \tag{1.26}$$

where E^0 and E^1 are as in (1.11) for $t = 0, 1$. In order to avoid such cases, we consider

$$\partial^* E = \mathbb{R}^n \setminus (E^0 \cup E^1).$$

On the other hand, it is also worth recalling that blow-up limits for $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ may be not half-spaces and yet not trivial only when considering points $x \in \partial^* E \setminus \mathscr{F}E$. Moreover, at such points, $\operatorname{Tan}(E,x)$ can be very wild. Indeed, as proved in [10, Prop. 2.7], for $n \geq 2$ there exists a measurable set $E \subset \mathbb{R}^n$ with $P(E) < \infty$ and such that

$$\operatorname{Tan}(E,0) \supset \{F \subset \mathbb{R}^n \text{ measurable} : P(F) < +\infty\}.$$

Note that, for such set E, it holds $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$ due to [4, Prop. 4.8], so that $0 \in \partial^* E \setminus \mathscr{F}^\alpha E$ by Theorem 1.7.

The equivalences in (1.26) and the example above motivate the following definition.

Definition 1.11 (Effective fractional reduced boundary)**.** Given $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, we let $\mathscr{F}^\alpha_{\mathrm{e}} E = \mathscr{F}^\alpha E \cap \partial^* E$ be the *effective fractional reduced boundary* of E.

We recall that $\dim_{\mathscr{H}}(\mathscr{F}^\alpha E) \geq n - \alpha$, see [4, Prop. 5.5], possibly with strict inequality. In fact, from (1.6), we immediately deduce that, if $|D^\alpha \chi_E| \ll \mathscr{L}^n$ and $|D^\alpha \chi_E|(\Omega) > 0$, then $|\Omega \cap \mathscr{F}^\alpha E| > 0$ as well. Indeed, in this case, we have

$$\int_E \operatorname{div}^\alpha \varphi \, \mathrm{d}x = \int_{\Omega \cap \mathscr{F}^\alpha E} \varphi \cdot \mathrm{d}D^\alpha_{\mathrm{ac}} \chi_E \quad \text{for all } \varphi \in C^\infty_c(\Omega; \mathbb{R}^n),$$

so that $|\Omega \cap \mathscr{F}^\alpha E| = 0$ implies $|D^\alpha \chi_E|(\Omega) = 0$. In particular, this applies to the case $P_\alpha(E, \Omega) < +\infty$, see [4, Rem. 4.9]. The following result refines such statement for the effective fractional reduced boundary. Here and below, given $\nu \in \mathbb{S}^{n-1}$ and $x_0 \in \mathbb{R}^n$,

$$H^+_\nu(x_0) = \{y \in \mathbb{R}^n : (y - x_0) \cdot \nu \geq 0\}, \quad H_\nu(x_0) = \{y \in \mathbb{R}^n : (y - x_0) \cdot \nu = 0\},$$

with the shorthands $H^+_\nu = H^+_\nu(0)$ and $H_\nu = H_\nu(0)$.

Theorem 1.12 (Properties of $\mathscr{F}^\alpha_{\mathrm{e}} E$)**.**

(i) If $\chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathscr{F}^\alpha_{\mathrm{e}} E$ and $\operatorname{Tan}(E,x) = \{F\}$, then $F = H^+_{\nu^\alpha_E(x)}$, and

$$\Sigma_E = \{x \in \mathscr{F}^\alpha_{\mathrm{e}} E : E \text{ admits a unique blow-up at } x\}$$

can be covered by countably many $(n-1)$-dimensional Lipschitz graphs.

(ii) If $\chi_E \in W^{\alpha,1}_{\mathrm{loc}}(\mathbb{R}^n)$, then $\mathscr{H}^{n-\alpha}(\mathscr{F}^\alpha_{\mathrm{e}} E) = 0$.

(iii) If $\chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)$, then $\mathscr{F}E \subset \mathscr{F}^\alpha_{\mathrm{e}} E$, $\mathscr{H}^{n-1}(\mathscr{F}^\alpha_{\mathrm{e}} E \setminus \mathscr{F}E) = 0$ and $\nu^\alpha_E = \nu_E$ on $\mathscr{F}E$.

The proof of Theorem 1.12 combines Definition 1.11 and the properties of the fractional reduced boundary established so far with several known results concerning blow-ups. Point (i) is a consequence of Theorem 1.7, [10, Prop. 2.1] and [9, Th. 1.2]. Point (ii) exploits [13, Prop. 3.1]. Finally, point (iii) follows from well-known properties of sets with locally finite perimeter as soon as the inclusion $\mathscr{F}E \subset \mathscr{F}^\alpha E$ is proved.

In the proof of Theorem 1.12 we exploit the following result, which can be easily deduced from [7, Prop. 1.8] and in fact provides a notable example for point (iii) in Theorem 1.12.

Proposition 1.13 (Half-space)**.** *If $x_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$, then*

$$\nabla^\alpha \chi_{H_\nu^+(x_0)}(x) = \frac{\mu_{1,\alpha}}{\alpha} \frac{\nu}{|(x - x_0) \cdot \nu|^\alpha} \quad \text{for all } x \notin H_\nu(x_0),$$

with $\mathscr{F}^\alpha H_\nu^+(x_0) = \mathbb{R}^n$, $\mathscr{F}_{\mathrm{e}}^\alpha H_\nu^+(x_0) = H_\nu(x_0)$ and $\nu_{H_\nu^+(x_0)}^\alpha = \nu$ on $\mathbb{R}^n$.

Proposition 1.13, combined with Theorems 1.6 and 1.12(i), implies the following result.

Corollary 1.14 (Refined convergence on Σ_E)**.** *Let $\chi_E \in BV_{\mathrm{loc}}^{\alpha,\infty}(\mathbb{R}^n)$ and $x \in \Sigma_E$. If $\chi_{\frac{E-x}{r_k}} \to \chi_{H^+_{\nu_E^\alpha(x)}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $r_k \to 0^+$, then, up to extracting a subsequence, as $r_k \to 0^+$,*

$$D^\alpha \chi_{\frac{E-x}{r_k}} \rightharpoonup \frac{\mu_{1,\alpha}}{\alpha} \frac{\nu_E^\alpha(x)}{|(\,\cdot\,) \cdot \nu_E^\alpha(x)|^\alpha} \mathscr{L}^n \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n),$$

$$|D^\alpha \chi_{\frac{E-x}{r_k}}| \rightharpoonup \frac{\mu_{1,\alpha}}{\alpha} \frac{1}{|(\,\cdot\,) \cdot \nu_E^\alpha(x)|^\alpha} \mathscr{L}^n \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n).$$

We conclude our paper with the following result, which is a simple consequence of [4, Th. 3.18] and provides another important example for point (iii) in Theorem 1.12.

Proposition 1.15 (Ball)**.** *If $x_0 \in \mathbb{R}^n$ and $r > 0$, then*

$$\nabla^\alpha \chi_{B_r(x_0)}(x) = -\frac{\mu_{n,\alpha}}{n + \alpha - 1} \int_{\partial B_r(x_0)} \frac{y}{|x - y|^{n+\alpha-1}} \, \mathrm{d}\mathscr{H}^{n-1}(y) \quad \text{for all } x \notin \partial B_r(x_0),$$

with $\mathscr{F}^\alpha B_r(x_0) = \mathbb{R}^n \setminus \{x_0\}$, $\mathscr{F}_{\mathrm{e}}^\alpha B_r(x_0) = \partial B_r(x_0)$ and $\nu_{B_r(x_0)}^\alpha(x) = -\frac{x - x_0}{|x - x_0|}$ for all $x \neq x_0$.

2. Proofs of the results

The rest of the paper is devoted to the proofs of our results. Throughout this section, we let $(\varrho_\varepsilon) \subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers, that is, we let $\varrho_\varepsilon(x) = \varepsilon^{-n} \varrho(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^n$ and $\varepsilon > 0$, where

$$\varrho \in C_c^\infty(\mathbb{R}^n), \quad \varrho \geq 0, \quad \varrho \text{ is radial}, \quad \operatorname{supp} \varrho \subset B_1 \quad \text{and} \quad \int_{B_1} \varrho \, \mathrm{d}x = 1. \tag{2.1}$$

In addition, for $x \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$, we let $\hat{x}_i \in \mathbb{R}^{n-1}$ be defined as

$$\hat{x}_i = \begin{cases} (x_2, \dots, x_n) & \text{if } i = 1, \\ (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) & \text{if } i \in \{2, \dots, n-1\}, \\ (x_1, \dots, x_{n-1}) & \text{if } i = n. \end{cases}$$

In some of the proofs below, we will invoke the following result (only for $p = +\infty$).

Lemma 2.1 (Smoothing)**.** *Let $p \in [1, +\infty]$. If $f \in BV_{\mathrm{loc}}^{\alpha,p}(\mathbb{R}^n)$ and $\varrho \in C_c^\infty(\mathbb{R}^n)$, then $\varrho * f \in BV_{\mathrm{loc}}^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with $\nabla^\alpha(\varrho * f) = \varrho * D^\alpha f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$.*

Proof. Clearly, $\varrho * f \in W^{1,p}(\mathbb{R}^n)$, hence $\nabla^\alpha(\varrho * f) \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ by [5, Prop. 3.3]. Moreover, $\varrho * D^\alpha f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. Given $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, we have $\varrho * \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and thus

$$\int_{\mathbb{R}^n} (\varrho * f) \operatorname{div}^\alpha \varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} f \, (\varrho * \operatorname{div}^\alpha \varphi) \, \mathrm{d}x = \int_{\mathbb{R}^n} f \operatorname{div}^\alpha(\varrho * \varphi) \, \mathrm{d}x = -\int_{\mathbb{R}^n} (\varrho * \varphi) \, \mathrm{d}D^\alpha f,$$

thanks to [4, Lem. 3.5], readily yielding $\nabla^\alpha(\varrho * f) = \varrho * D^\alpha f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. □

2.1. **Proof of Theorem 1.1.** Given $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, since $P_\alpha(F) < +\infty$, arguing as in the proof of [6, Lem. 3.2], we can write

$$\operatorname{div}^\alpha(\varphi\chi_F) = \chi_F \operatorname{div}^\alpha\varphi + \varphi \cdot \nabla^\alpha\chi_F + \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_F, \varphi) \quad \text{in } L^1(\mathbb{R}^n). \tag{2.2}$$

Let us observe that

$$\left| \int_{\mathbb{R}^n} \chi_E \, \varphi \cdot \nabla^\alpha \chi_F \,\mathrm{d}x \right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|\nabla^\alpha\chi_F\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}.$$

Moreover, by [6, Lem. 2.9], we have

$$\int_{\mathbb{R}^n} \chi_E \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_F,\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varphi \cdot \nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\,\mathrm{d}x, \tag{2.3}$$

so that

$$\left| \int_{\mathbb{R}^n} \chi_E \operatorname{div}^\alpha_{\mathrm{NL}}(\chi_F,\varphi)\,\mathrm{d}x \right| \le \|\varphi\|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \|\nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}.$$

Recalling the definitions in (1.5) and (1.9), as in [6, Cors. 2.3 and 2.7] (since $|\chi_E(x) - \chi_E(y)| \le 1$ for $x, y \in \mathbb{R}^n$) we can estimate

$$\max\Big\{\|\nabla^\alpha\chi_F\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}, \|\nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)}\Big\} \le \mu_{n,\alpha}\, P_\alpha(F),$$

in particular proving (1.13). A straightforward application of [6, Lem. 2.9] (by taking $p, r = +\infty$, $q = 1$, $f = \chi_E$, $g = \chi_F$ and $\varphi \equiv 1$ in that result, and observing that $b^\alpha_{1,1}(\mathbb{R}^n)$ coincides with the space of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ functions with finite $W^{\alpha,1}$-seminorm, see [6, Sec. 2.1]) gives (1.14). Now, integrating (2.2) and exploiting (2.3), we get

$$\int_{\mathbb{R}^n} \chi_{E\cap F} \operatorname{div}^\alpha\varphi\,\mathrm{d}x = \int_E \operatorname{div}^\alpha(\varphi\chi_F)\,\mathrm{d}x - \int_E \varphi\cdot\nabla^\alpha\chi_F\,\mathrm{d}x - \int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\,\mathrm{d}x$$

Now let $\psi_\varepsilon = \varrho_\varepsilon * (\varphi\chi_F)$ for all $\varepsilon > 0$. Since $\psi \in C_c^\infty(\mathbb{R}^n)$, we can compute

$$\int_E \operatorname{div}^\alpha\psi_\varepsilon\,\mathrm{d}x = -\int_{\mathbb{R}^n} \psi_\varepsilon \cdot \mathrm{d}D^\alpha\chi_E$$

for all $\varepsilon > 0$. Since $\psi_\varepsilon \to \varphi\chi_F$ in $W^{\alpha,1}(\mathbb{R}^n;\mathbb{R}^n)$ as $\varepsilon \to 0^+$ (see [4, App. A]), we have $\operatorname{div}^\alpha\psi_\varepsilon \to \operatorname{div}^\alpha(\varphi\chi_F)$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0^+$ and thus

$$\lim_{\varepsilon\to0^+} \int_E \operatorname{div}^\alpha\psi_\varepsilon\,\mathrm{d}x = \int_E \operatorname{div}^\alpha(\varphi\chi_F)\,\mathrm{d}x.$$

Since $\psi_\varepsilon \to \varphi\chi_{F^1}$ $\mathscr{H}^{n-\alpha}$-a.e. in $\mathbb{R}^n$ as $\varepsilon \to 0^+$ by [13, Prop. 3.1] and the fact that $|D^\alpha\chi_E| \ll \mathscr{H}^{n-\alpha} \llcorner \mathscr{F}^\alpha E$ due to [4, Cor. 5.4], we also get that

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} \psi_\varepsilon \cdot \mathrm{d}D^\alpha\chi_E = \int_{\mathbb{R}^n} \chi_{F^1}\varphi \cdot \mathrm{d}D^\alpha\chi_E.$$

Hence, we get

$$\int_{\mathbb{R}^n} \chi_{E\cap F} \operatorname{div}^\alpha\varphi\,\mathrm{d}x = -\int_{\mathbb{R}^n} \chi_{F^1}\varphi\cdot\mathrm{d}D^\alpha\chi_E - \int_E \varphi\cdot\nabla^\alpha\chi_F\,\mathrm{d}x - \int_{\mathbb{R}^n} \varphi\cdot\nabla^\alpha_{\mathrm{NL}}(\chi_E,\chi_F)\,\mathrm{d}x$$

whenever $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, which easily implies (1.12) via a standard approximation argument. The validity of (1.15), (1.16) and (1.17) is an easy consequence of (1.12). Finally, if F is bounded, then $\chi_{E\cap F} \in BV^{\alpha,1}(\mathbb{R}^n)$, so that [6, Lem. 2.5] implies $D^\alpha\chi_{E\cap F}(\mathbb{R}^n) = 0$, and so (1.18) follows by integrating (1.12) over $\mathbb{R}^n$ and exploiting (1.14). □

2.2. **Proof of Theorem 1.3.** Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and set $g_\varepsilon = \varrho_\varepsilon * g$ for all $\varepsilon > 0$. Since $g_\varepsilon \in \mathrm{Lip}_b(\mathbb{R}^n)$, by [5, Lem. 2.4] we can write

$$\operatorname{div}^\alpha(g_\varepsilon\varphi) = g_\varepsilon \operatorname{div}^\alpha\varphi + \varphi \cdot \nabla^\alpha g_\varepsilon + \operatorname{div}^\alpha_{\mathrm{NL}}(g_\varepsilon, \varphi) \quad \text{in } L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Hence, in virtue of [6, Lem. 2.10] and Lemma 2.1, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f g_\varepsilon \operatorname{div}^\alpha\varphi \,\mathrm{d}x &= \int_{\mathbb{R}^n} f \operatorname{div}^\alpha(g_\varepsilon\varphi)\,\mathrm{d}x - \int_{\mathbb{R}^n} f\varphi\cdot\nabla^\alpha g_\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^n} f \operatorname{div}^\alpha_{\mathrm{NL}}(g_\varepsilon,\varphi)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^n} g_\varepsilon\varphi\cdot \mathrm{d}D^\alpha f - \int_{\mathbb{R}^n} \varrho_\varepsilon * (f\varphi)\cdot \mathrm{d}D^\alpha g - \int_{\mathbb{R}^n} g_\varepsilon \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)\,\mathrm{d}x.\end{aligned} \tag{2.4}$$

Since $\operatorname{div}^\alpha\varphi, \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi) \in L^1(\mathbb{R}^n)$, by the Dominated Convergence Theorem we have

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} f g_\varepsilon \operatorname{div}^\alpha\varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} f g \operatorname{div}^\alpha\varphi\,\mathrm{d}x, \tag{2.5}$$

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} g_\varepsilon \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^n} g \operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)\,\mathrm{d}x. \tag{2.6}$$

Setting $f_\varepsilon = \varrho_\varepsilon * f$ for all $\varepsilon > 0$ and letting $R > 0$ be such that $\operatorname{supp}\varphi \subset B_R$, we also have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \varrho_\varepsilon * (f\varphi)\cdot \mathrm{d}D^\alpha g - \int_{\mathbb{R}^n} f_\varepsilon\varphi\cdot\mathrm{d}D^\alpha g\right| &\le \int_{B_{R+1}} |\varrho_\varepsilon*(f\varphi) - f_\varepsilon\varphi|\,\mathrm{d}|D^\alpha g| \\ &= \int_{B_{R+1}}\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)|f(y)||\varphi(y)-\varphi(x)|\,\mathrm{d}y\,\mathrm{d}|D^\alpha g|(x) \\ &\le \|f\|_{L^\infty(\mathbb{R}^n)} \int_{B_{R+1}}\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)|\varphi(y)-\varphi(x)|\,\mathrm{d}y\,\mathrm{d}|D^\alpha g|(x)\end{aligned}$$

for all $\varepsilon \in (0,1)$. Since φ is uniformly continuous on $\mathbb{R}^n$, we thus get that

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} \varrho_\varepsilon * (f\varphi)\cdot\mathrm{d}D^\alpha g = \lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n} f_\varepsilon\varphi\cdot\mathrm{d}D^\alpha g. \tag{2.7}$$

Now, since $|f_\varepsilon(x)| \le \|f\|_{L^\infty(\mathbb{R}^n)}$ for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, the family $(f_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}^n, |D^\alpha g|)$. Thus, by Banach–Alaoglu Theorem, we can find a sequence $(f_{\varepsilon_k})_{k\in\mathbb{N}}$ and $\bar f \in L^\infty(\mathbb{R}^n, |D^\alpha g|)$ such that $f_{\varepsilon_k} \overset{*}{\rightharpoonup} \bar f$ in $L^\infty(\mathbb{R}^n, |D^\alpha g|)$ as $k \to +\infty$. Similarly (up to subsequences, which we do not relabel), we can also find a sequence $(g_{\varepsilon_k})_{k\in\mathbb{N}}$ and $\bar g \in L^\infty(\mathbb{R}^n, |D^\alpha f|)$ such that $g_{\varepsilon_k} \overset{*}{\rightharpoonup} \bar g$ in $L^\infty(\mathbb{R}^n, |D^\alpha f|)$ as $k \to +\infty$. In particular, (1.20) follows by the lower semicontinuity of the L^∞ norm with respect to the weak* convergence. Therefore, passing to the limit as $k \to +\infty$ in (2.4) along the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ and recalling (2.5), (2.6) and (2.7), we get

$$\int_{\mathbb{R}^n} f g \operatorname{div}^\alpha\varphi\,\mathrm{d}x = -\int_{\mathbb{R}^n} \bar g\varphi\cdot\mathrm{d}D^\alpha f - \int_{\mathbb{R}^n} \bar f\varphi\cdot\mathrm{d}D^\alpha g - \int_{\mathbb{R}^n} g\operatorname{div}^\alpha_{\mathrm{NL}}(f,\varphi)\,\mathrm{d}x$$

whenever $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$, readily yielding (1.19) thanks to Definition 1.2, and therefore (1.22) as long as $fg \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. To conclude, we thus just need to prove (1.21). To this aim, we observe that, by Lemma 2.2 below, $f_{\varepsilon_k}(x) \to f^\star(x)$ as $k \to +\infty$ for all $x \in \mathscr{R}_f$. Hence, by the Dominated Convergence Theorem, we get

$$\int_{\mathscr{R}_f} \bar f\psi\cdot\mathrm{d}D^\alpha g = \lim_{k\to+\infty}\int_{\mathscr{R}_f} f_{\varepsilon_k}\psi\cdot\mathrm{d}D^\alpha g = \int_{\mathscr{R}_f} f^\star\psi\cdot\mathrm{d}D^\alpha g$$

for any $\psi \in C_c(\mathbb{R}^n)$, proving the first half of (1.21). The second half of (1.21) is similar. □

Lemma 2.2. *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathscr{R}_u$, then*

$$u^\star(x) = \lim_{\varepsilon \to 0^+} (\varrho_\varepsilon * u)(x).$$

Proof. In view of (2.1), we can write $\varrho(x) = \eta(|x|)$ for all $x \in \mathbb{R}^n$, where $\eta \in C^\infty_c([0,+\infty))$ is such that $\eta \geq 0$ and $\operatorname{supp}\eta \subset [0,1)$. In particular, η is absolutely continuous on $[0,1]$. Hence, if η is strictly decreasing on its support, we exploit Cavalieri's formula to get

$$\begin{aligned}
(\varrho_\varepsilon * u)(x) &= \varepsilon^{-n} \int_0^{+\infty} \int_{\{y \in B_\varepsilon \,:\, \varrho(y/\varepsilon) > t\}} u(x-y)\,\mathrm{d}y\,\mathrm{d}t \\
&= -\varepsilon^{-n-1} \int_0^\varepsilon \eta'\left(\tfrac{s}{\varepsilon}\right) \int_{\{y \in B_\varepsilon \,:\, \eta(|y|/\varepsilon) > \eta(s/\varepsilon)\}} u(x-y)\,\mathrm{d}y\,\mathrm{d}s \\
&= -\varepsilon^{-n-1} \int_0^\varepsilon \eta'\left(\tfrac{s}{\varepsilon}\right) \int_{B_s} u(x-y)\,\mathrm{d}y\,\mathrm{d}s \\
&= -\int_0^1 \eta'(r)\,\omega_n r^n \fint_{B_{r\varepsilon}(x)} u(y)\,\mathrm{d}y\,\mathrm{d}r.
\end{aligned}$$

If $x \in \mathscr{R}_u$, then we can apply the Dominated Convergence Theorem to get

$$\lim_{\varepsilon \to 0^+} (\varrho_\varepsilon * u)(x) = -\int_0^1 \eta'(r)\,\omega_n r^n \left(\lim_{\varepsilon \to 0^+} \fint_{B_{r\varepsilon}(x)} u(y)\,\mathrm{d}y \right) \mathrm{d}r = u^\star(x),$$

as we easily recognize that

$$-\int_0^1 \eta'(r)\,\omega_n r^n\,\mathrm{d}r = n\omega_n \int_0^1 \eta(r) r^{n-1}\,\mathrm{d}r = \int_{B_1} \varrho(x)\,\mathrm{d}x = 1.$$

In the general case, since η is absolutely continuous on $[0,1]$, we can write $\eta = \eta_1 - \eta_2$ on $[0,1]$, where $\eta_1, \eta_2 \colon [0,1] \to [0,+\infty)$ are strictly decreasing absolutely continuous functions such that $\eta_1(1) = \eta_2(1)$. The conclusion hence follows by performing analogous computations involving Cavalieri's formula on η_1 and η_2 separately, and then by exploiting the linearity of the derivative. □

2.3. **Proof of Corollary 1.4.** The validity of (1.23) immediately follows from (1.22) in Theorem 1.3, since $\chi_E\,\chi_E = \chi_E \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, $\overline{\chi_E} \in L^\infty(\mathbb{R}^n, |D^\alpha \chi_E|)$ given by Theorem 1.3 is uniquely determined by (1.23). The validity of (1.24) follows from (1.21) and by the construction in the proof of Theorem 1.3. Finally, if $\chi_E \in BV^{\alpha,1}(\mathbb{R}^n)$, then (1.23) easily implies that $D^\alpha_{\mathrm{NL}}(\chi_E, \chi_E) \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^n)$. We let $\eta \in C^\infty_c(B_2)$ be such that $\eta \equiv 1$ on B_1 and set $\eta_k(x) = \eta\left(\frac{x}{k}\right)$ for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. By Definition 1.2 with $\varphi_k = \eta_k \mathrm{e}_j$ for $j \in \{1, \dots, n\}$ and [6, Cor. 2.7], we get

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \eta_k\,\mathrm{e}_j \cdot \mathrm{d}D^\alpha_{\mathrm{NL}}(\chi_E, \chi_E) \right| &= \left| \int_{\mathbb{R}^n} \chi_E\,\mathrm{e}_j \cdot \nabla^\alpha_{\mathrm{NL}}(\chi_E, \eta_k)\,\mathrm{d}x \right| \\
&\leq 2\mu_{n,\alpha} |E| \|\nabla^\alpha_{\mathrm{NL}}(\chi_E, \eta_k)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \\
&\leq 2\mu_{n,\alpha} |E| [\eta_k]_{B^\alpha_{\infty,1}(\mathbb{R}^n)} \\
&= 2\mu_{n,\alpha} |E| [\eta]_{B^\alpha_{\infty,1}(\mathbb{R}^n)} k^{-\alpha} \to 0 \quad \text{as } k \to +\infty.
\end{aligned}$$

Thus, by the Dominated Convergence Theorem, we obtain

$$\mathrm{e}_j \cdot D^\alpha_{\mathrm{NL}}(\chi_E, \chi_E)(\mathbb{R}^n) = \lim_{k \to +\infty} \int_{\mathbb{R}^n} \eta_k \mathrm{e}_j \cdot \mathrm{d}D^\alpha_{\mathrm{NL}}(\chi_E, \chi_E) = 0 \quad \text{for all } j \in \{1, \dots, n\},$$

which implies $D^\alpha_{NL}(\chi_E,\chi_E)(\mathbb{R}^n)=0$. Consequently, since $D^\alpha\chi_E(\mathbb{R}^n)=0$ by [6, Lem. 2.5], we can integrate (1.23) over $\mathbb{R}^n$ to get

$$0=D^\alpha_{\mathrm{NL}}(\chi_E,\chi_E)(\mathbb{R}^n)=\int_{\mathbb{R}^n}(1-2\,\overline{\chi_E})\,\mathrm{d}D^\alpha\chi_E=-2\int_{\mathbb{R}^n}\overline{\chi_E}\,\mathrm{d}D^\alpha\chi_E,$$

proving (1.25) and ending the proof. □

2.4. **Proof of Theorem 1.6.** We prove (i), (ii) and (iii) separately.

Proof of (i). Up to extracting a subsequence, we can also assume that $\chi_{\frac{E-x}{r_k}}\to\chi_F$ a.e. in $\mathbb{R}^n$ as $r_k\to0^+$. Hence, by the Dominated Convergence Theorem, we get

$$\int_{\mathbb{R}^n}\chi_{\frac{E-x}{r_k}}\,\mathrm{div}^\alpha\varphi\,\mathrm{d}x\to\int_{\mathbb{R}^n}\chi_F\,\mathrm{div}^\alpha\varphi\,\mathrm{d}x\quad\text{as } r_k\to+\infty$$

for any $\varphi\in\mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, since $\mathrm{div}^\alpha\varphi\in L^1(\mathbb{R}^n)$ by [4, Cor. 2.3]. Thanks to the density of $\mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ into $C_c(\mathbb{R}^n;\mathbb{R}^n)$ with respect to the uniform convergence, we infer (i).

Proof of (ii). Given $\varphi\in C_c(\mathbb{R}^n)$, we can estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\varphi\,\mathrm{d}|D^\alpha\chi_{\frac{E-x}{r_k}}|-\int_{\mathbb{R}^n}\varphi\,\mathrm{d}|D^\alpha\chi_F|\right|&\le r_k^{\alpha-n}\int_{\mathbb{R}^n}\left|\varphi\left(\tfrac{y-x}{r_k}\right)\right||\nu_E^\alpha(y)-\nu_E^\alpha(x)|\,\mathrm{d}|D^\alpha\chi_E|(y)\\&\quad+\left|\int_{\mathbb{R}^n}\varphi\Big(\nu_E^\alpha(x)\cdot\mathrm{d}D^\alpha\chi_{\frac{E-x}{r_k}}-\nu_F^\alpha\cdot\mathrm{d}D^\alpha\chi_F\Big)\right|\end{aligned}\tag{2.8}$$

On the one side, since $x\in\mathscr{F}^\alpha E$, by [4, Th. 5.3] there are $A_{n,\alpha}>0$ and $r_x>0$ such that

$$|D^\alpha\chi_E|(B_r(x))\le A_{n,\alpha}r^{n-\alpha}\quad\text{for all } r\in(0,r_x).\tag{2.9}$$

Hence, letting $R>0$ be such that $\operatorname{supp}\varphi\subset B_R$, we can exploit (2.9) to estimate the first term in the right-hand side of (2.8) as

$$\begin{aligned}r_k^{\alpha-n}&\int_{\mathbb{R}^n}\left|\varphi\left(\tfrac{y-x}{r_k}\right)\right||\nu_E^\alpha(y)-\nu_E^\alpha(x)|\,\mathrm{d}|D^\alpha\chi_E|(y)\\&\le C_{n,\alpha,R}\fint_{B_{r_kR}(x)}\left|\varphi\left(\tfrac{y-x}{r_k}\right)\right||\nu_E^\alpha(y)-\nu_E^\alpha(x)|\,\mathrm{d}|D^\alpha\chi_E|(y)\\&\le\|\varphi\|_{L^\infty(\mathbb{R}^n)}\left(\fint_{B_{r_kR}(x)}|\nu_E^\alpha(y)-\nu_E^\alpha(x)|^2\,\mathrm{d}|D^\alpha\chi_E|(y)\right)^{\frac12}\\&=\|\varphi\|_{L^\infty(\mathbb{R}^n)}\left(\fint_{B_{r_kR}(x)}2(1-\nu_E^\alpha(y)\cdot\nu_E^\alpha(x))\,\mathrm{d}|D^\alpha\chi_E|(y)\right)^{\frac12}\end{aligned}\tag{2.10}$$

for all $r_k>0$ sufficiently small by Jensen's inequality, where $C_{n,\alpha,R}>0$ does not depend on k. Again since $x\in\mathscr{F}^\alpha E$, we have

$$\nu_E^\alpha(x)=\lim_{r\to0^+}\fint_{B_r(x)}\nu_E^\alpha(y)\,\mathrm{d}|D^\alpha\chi_E|(y).\tag{2.11}$$

Therefore, by combining (2.10) with (2.11), we conclude that

$$r_k^{\alpha-n}\int_{\mathbb{R}^n}\left|\varphi\left(\tfrac{y-x}{r_k}\right)\right||\nu_E^\alpha(y)-\nu_E^\alpha(x)|\,\mathrm{d}|D^\alpha\chi_E|(y)\to0\quad\text{as } r_k\to0^+.\tag{2.12}$$

On the other side, by [4, Prop. 5.9], we have $\nu_F^\alpha = \nu_E^\alpha(x)$ $|D^\alpha\chi_F|$-a.e. in $\mathbb{R}^n$. Hence, in virtue of (i), the second term in the right-hand side of (2.8) satisfies

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}\varphi(y)\Big(\nu_E^\alpha(x)\cdot \mathrm{d}D^\alpha\chi_{\frac{E-x}{r_k}}(y)-\nu_F^\alpha(y)\cdot \mathrm{d}D^\alpha\chi_F(y)\Big)\right| \\
&\qquad=\left|\int_{\mathbb{R}^n}\varphi\,\nu_E^\alpha(x)\cdot\Big(\mathrm{d}D^\alpha\chi_{\frac{E-x}{r_k}}-\mathrm{d}D^\alpha\chi_F\Big)\right|\to 0^+\quad\text{as } r_k\to 0^+
\end{aligned}
\tag{2.13}
$$

possibly passing to a further subsequence. Thus (ii) follows from (2.8), (2.12) and (2.13).

Proof of (iii). Points (i) and (ii) implies (iii) for all $R>0$ such that $|D^\alpha\chi_F|(\partial B_R)=0$. Since $|D^\alpha\chi_F|(\partial B_R)=0$ for a.e. $R>0$ (by [1, Exam. 1.63] for instance), the validity of (iii) immediately follows. □

2.5. **Proof of Proposition 1.8.** Assume $D_i^\alpha f=0$. Let $\varphi\in C_c^\infty(\mathbb{R}^n)$ and set $\psi=(-\Delta)^{\frac{1-\alpha}{2}}\varphi$. By [4, Lem. 3.28(ii)], we have $\psi\in S^{\alpha,1}(\mathbb{R}^n)$ (i.e., $\psi\in BV^{\alpha,1}(\mathbb{R}^n)$ with $|D^\alpha\psi|\ll\mathscr{L}^n$, see [4, Sec. 3.9] for an account) with $\nabla^\alpha\psi=\nabla\varphi$. Hence, by [4, Th. 3.23], we can find $(\psi_k)_{k\in\mathbb{N}}\subset C_c^\infty(\mathbb{R}^n)$ such that $\psi_k\to\psi$ in $S^{\alpha,1}(\mathbb{R}^n)$ as $k\to+\infty$. Thus, we have

$$
\int_{\mathbb{R}^n} f\,\partial_i\varphi\,\mathrm{d}x=\int_{\mathbb{R}^n} f\,\partial_i^\alpha\psi\,\mathrm{d}x=\lim_{k\to+\infty}\int_{\mathbb{R}^n} f\,\partial_i^\alpha\psi_k\,\mathrm{d}x=\lim_{k\to+\infty}\int_{\mathbb{R}^n}\psi_k\,\mathrm{d}D_i^\alpha f=0,
$$

from which we readily get $D_i f=0$. Viceversa, assume $D_i f=0$ and suppose $f\in\mathrm{Lip}_b(\mathbb{R}^n)$ at first. Then we can write $f(x)=g(\hat{x}_i)$ for all $x\in\mathbb{R}^n$ for some $g\in\mathrm{Lip}_b(\mathbb{R}^{n-1})$. Thus, by [5, Lem. 2.3], we can compute

$$
\begin{aligned}
\nabla_i^\alpha f(x)&=\mu_{n,\alpha}\lim_{\varepsilon\to0^+}\int_{\{|y|>\varepsilon\}}\frac{y_i f(x+y)}{|y|^{n+\alpha+1}}\,\mathrm{d}y\\
&=\mu_{n,\alpha}\lim_{\varepsilon\to0}\int_{\mathbb{R}^{n-1}} g(\hat{x}_i+\hat{y}_i)\int_{\{|y_i|^2>(\varepsilon^2-|\hat{y}_i|^2)^+\}}\frac{y_i}{(y_i^2+|\hat{y}_i|^2)^{\frac{n+\alpha+1}{2}}}\,\mathrm{d}y_i\,\mathrm{d}\hat{y}_i=0
\end{aligned}
$$

for all $x\in\mathbb{R}^n$, so that $D_i^\alpha f=0$. If now $f\in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$, then $f_\varepsilon=\varrho_\varepsilon*f\in\mathrm{Lip}_b(\mathbb{R}^n)$ for all $\varepsilon>0$. Since $D_i f=0$ by assumption, also $D_i f_\varepsilon=\varrho_\varepsilon*D_i f=0$ for all $\varepsilon>0$, and thus

$$
\int_{\mathbb{R}^n}\varphi\,\mathrm{d}D_i^\alpha f=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n}\varrho_\varepsilon*\varphi\,\mathrm{d}D_i^\alpha f=\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n}\varphi\,\mathrm{d}D_i^\alpha f_\varepsilon=0
$$

by the Dominated Convergence Theorem for all $\varphi\in C_c^\infty(\mathbb{R}^n)$, so that $D_i^\alpha f=0$. □

2.6. **Proof of Proposition 1.9.** Since $D_1 f=0$, there exists $g\in L^\infty(\mathbb{R}^{n-1})$ such that $f((t,x))=g(x)$ for a.e. $t\in\mathbb{R}$ and a.e. $x\in\mathbb{R}^{n-1}$. Now let us assume that $f\in\mathrm{Lip}_b(\mathbb{R}^n)$ at first, so that also $g\in\mathrm{Lip}_b(\mathbb{R}^{n-1})$. By [5, Lem. 2.3], we can write

$$
\begin{aligned}
(\nabla^\alpha_{\mathbb{R}^{n-1}})_i g(x)&=\mu_{n-1,\alpha}\int_{\mathbb{R}^{n-1}}\frac{(g(z)-g(x))\,(z_i-x_i)}{|z-x|^{n+\alpha}}\,\mathrm{d}z\\
&=\mu_{n,\alpha}\frac{\Gamma\left(\frac{n+\alpha}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+\alpha+1}{2}\right)}\int_{\mathbb{R}^{n-1}}\frac{(g(z)-g(x))\,(z_i-x_i)}{|z-x|^{n+\alpha}}\,\mathrm{d}z\\
&=\mu_{n,\alpha}\int_{\mathbb{R}^{n-1}}(g(z)-g(x))\,(z_i-x_i)\int_{\mathbb{R}}\frac{1}{(t^2+|z-x|^2)^{\frac{n+\alpha+1}{2}}}\,\mathrm{d}t\,\mathrm{d}z\\
&=\mu_{n,\alpha}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\frac{(f((t+s,z))-f((s,x)))\,(z_i-x_i)}{(t^2+|z-x|^2)^{\frac{n+\alpha+1}{2}}}\,\mathrm{d}t\,\mathrm{d}z
\end{aligned}
$$

$$= \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y) - f((s,x)))\,(y_i - (s,x)_i)}{|y-(s,x)|^{n+\alpha+1}}\,\mathrm{d}y = (\nabla^\alpha_{\mathbb{R}^n})_i f((s,x))$$

for all $x \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$ and $i = 2, \dots, n$, where $\nabla^\alpha_{\mathbb{R}^m}$ denotes the operator (1.1) taken in the ambient space $\mathbb{R}^m$, $m \in \mathbb{N}$. In the above chain of equalities, we exploited the fact that, by the properties of the Gamma and Beta functions, for $u = |z - x|$ and $s = n + \alpha$,

$$u^s \int_{\mathbb{R}} \frac{\mathrm{d}t}{(t^2+u^2)^{\frac{s+1}{2}}} = 2\int_0^{+\infty} \frac{\mathrm{d}t}{(t^2+1)^{\frac{s+1}{2}}} \overset{[1+t^2=\frac{1}{r}]}{=} \int_0^1 r^{\frac{s}{2}-1}(1-r)^{\frac{1}{2}-1}\,\mathrm{d}r = \frac{\Gamma\left(\frac{s}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{s+1}{2}\right)}.$$

By Proposition 1.8, $D^\alpha_{1,\mathbb{R}^n} f = 0$, and so $D_1 \nabla^\alpha_{\mathbb{R}^n} f = 0$. Now let $f \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and set $f_\varepsilon = \varrho_\varepsilon * f$ for all $\varepsilon > 0$. Then $f_\varepsilon \in \mathrm{Lip}_b(\mathbb{R}^n)$ with $\nabla^\alpha_{\mathbb{R}^n} f_\varepsilon = \varrho_\varepsilon * D^\alpha_{\mathbb{R}^n} f$ for all $\varepsilon > 0$. Since $D_1 f_\varepsilon = 0$, there is $g_\varepsilon \in \mathrm{Lip}_b(\mathbb{R}^{n-1})$ such that $f_\varepsilon((t,x)) = g_\varepsilon(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^{n-1}$ and $g_\varepsilon \to g$ a.e. in $\mathbb{R}^{n-1}$ as $\varepsilon \to 0^+$. Thus

$$(\nabla^\alpha_{\mathbb{R}^{n-1}})_i g_\varepsilon(x) = (\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon((s,x))$$

for all $x \in \mathbb{R}^{n-1}$, $s \in \mathbb{R}$, $i \in \{2, \dots, n\}$ and $\varepsilon > 0$, thanks to Lemma 2.1. Note that $D_1 D^\alpha_{\mathbb{R}^n} f = 0$. Indeed, since $D_1 \nabla^\alpha_{\mathbb{R}^n} f_\varepsilon = 0$ for all $\varepsilon > 0$, we have

$$0 = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} D_1\psi\, \nabla^\alpha_{\mathbb{R}^n} f_\varepsilon\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \varrho_\varepsilon * D_1\psi\,\mathrm{d}D^\alpha_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} D_1\psi\,\mathrm{d}D^\alpha_{\mathbb{R}^n} f$$

for all $\psi \in C^\infty_c(\mathbb{R}^n)$ by the Dominated Convergence Theorem. Now, given $\varphi \in C^\infty_c(\mathbb{R}^{n-1})$ and $\sigma \in C^\infty_c(\mathbb{R})$, setting $\psi(y) = \sigma(y_1)\,\varphi(\hat{y}_1)$ for all $y \in \mathbb{R}^n$, we have $\psi \in C^\infty_c(\mathbb{R}^n)$ and so

$$\begin{aligned}
\left(\int_{\mathbb{R}} \sigma\,\mathrm{d}t\right) \int_{\mathbb{R}^{n-1}} \varphi\,(\nabla^\alpha_{\mathbb{R}^{n-1}})_i g_\varepsilon\,\mathrm{d}x &= \left(\int_{\mathbb{R}} \sigma\,\mathrm{d}t\right) \int_{\mathbb{R}^{n-1}} \varphi(x)\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon((0,x))\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \sigma(t) \int_{\mathbb{R}^{n-1}} \varphi(x)\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon((0,x))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \sigma(t) \int_{\mathbb{R}^{n-1}} \varphi(x)\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon((t,x))\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} \sigma(y_1)\,\varphi(\hat{y}_1)\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \psi\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon\,\mathrm{d}y
\end{aligned}$$

for all $\varepsilon > 0$ and $i \in \{2, \dots, n\}$. By the Dominated Convergence Theorem, we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{n-1}} \varphi\,(\nabla^\alpha_{\mathbb{R}^{n-1}})_i g_\varepsilon\,\mathrm{d}x = -\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{n-1}} g_\varepsilon\,(\nabla^\alpha_{\mathbb{R}^{n-1}})_i \varphi\,\mathrm{d}x = -\int_{\mathbb{R}^{n-1}} g\,(\nabla^\alpha_{\mathbb{R}^{n-1}})_i \varphi\,\mathrm{d}x$$

and, similarly,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \psi\,(\nabla^\alpha_{\mathbb{R}^n})_i f_\varepsilon\,\mathrm{d}y = \int_{\mathbb{R}^n} \psi\,\mathrm{d}(D^\alpha_{\mathbb{R}^n})_i f.$$

We thus conclude that

$$-\left(\int_{\mathbb{R}} \sigma\,\mathrm{d}t\right) \int_{\mathbb{R}^{n-1}} g\,(\nabla^\alpha_{\mathbb{R}^{n-1}})_i \varphi\,\mathrm{d}x = \int_{\mathbb{R}^n} \sigma(y_1)\,\varphi(\hat{y}_1)\,\mathrm{d}(D^\alpha_{\mathbb{R}^n})_i f(y)$$

for all $i \in \{2, \dots, n\}$, $\varphi \in C^\infty_c(\mathbb{R}^n)$ and $\sigma \in C^\infty_c(\mathbb{R})$. Hence $g \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^{n-1})$, with

$$\int_{\mathbb{R}^n} \sigma(y_1)\,\varphi(\hat{y}_1)\,\mathrm{d}(D^\alpha_{\mathbb{R}^n})_i f(y) = \left(\int_{\mathbb{R}} \sigma\,\mathrm{d}t\right)\left(\int_{\mathbb{R}^{n-1}} \varphi\,\mathrm{d}(D^\alpha_{\mathbb{R}^{n-1}})_i g\right)$$

for all $i \in \{2, \dots, n\}$, $\varphi \in C^\infty_c(\mathbb{R}^n)$ and $\sigma \in C^\infty_c(\mathbb{R})$, yielding the conclusion. □

2.7. **Proof of Theorem 1.7.** By [4, Prop. 5.9], we have $\chi_F \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\nu^\alpha_F = \mathrm{e}_n$ $|D^\alpha\chi_F|$-a.e. in $\mathbb{R}^n$. Hence $D^\alpha_i\chi_F = 0$ for all $i = 1,\dots,n-1$ and $D^\alpha_n\chi_F \ge 0$. By Proposition 1.8, we infer that also $D_i\chi_F = 0$ for all $i = 1,\dots,n-1$. Consequently, $F = \mathbb{R}^{n-1} \times M$ for some measurable $M \subset \mathbb{R}$. We now prove the properties of the set M.

Proof of (i). By repeatedly applying Proposition 1.9, we get $\chi_M \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R})$, with

$$(D^\alpha_{\mathbb{R}^n})_n\chi_F = \mathscr{L}^{n-1} \otimes D^\alpha_{\mathbb{R}}\chi_M = \mathscr{L}^{n-1} \otimes \partial^\alpha\chi_M \quad \text{in } \mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n),$$

from which we readily deduce that $\partial^\alpha\chi_M \ge 0$.

Proof of (ii). If $|M| \in (0,+\infty)$ by contradiction, then $\chi_M \in BV^{\alpha,1}_{\mathrm{loc}}(\mathbb{R})$ and thus we obtain $u = I_{1-\alpha}\chi_M \in BV_{\mathrm{loc}}(\mathbb{R})$ with $\partial u = \partial^\alpha\chi_M$, arguing exactly as in [4, Lem. 3.28(i)]. Hence $\partial u \ge 0$ and thus u is a non-negative and non-decreasing function. Moreover, since $|M| < +\infty$, we have $u \in L^p(\mathbb{R})$ for all $p \in \left(\frac{1}{\alpha}, +\infty\right)$, which immediately yields $u \equiv 0$, so that $|M| = 0$, a contradiction. Hence $|M| \in \{0,+\infty\}$ and, since $F^c \in \mathrm{Tan}(E^c, x)$, we also get that $|M^c| \in \{0,+\infty\}$ by a symmetrical argument.

Proof of (iii). Let $|M| = +\infty$ and assume $b = \operatorname{ess\,sup} M < +\infty$ by contradiction. Let $I_b = (b, b+1)$. By (i) and (1.18), we can compute

$$0 \le \partial^\alpha\chi_M(I_b) = -\int_M \nabla^\alpha\chi_{I_b}\,\mathrm{d}t.$$

By [4, Exam. 4.11], $\nabla^\alpha\chi_{I_b}(t) > 0$ for all $t < b$, forcing $|M| = 0$, which is a contradiction.

Proof of (iv). Let $M \neq \emptyset, \mathbb{R}$ be such that $P(M) < +\infty$. Then, up to negligible sets, $M = \cup_{k=1}^N I_k$ for $N \in \mathbb{N}$ closed intervals $I_k \subset \mathbb{R}$ with $a_k = \inf I_k < \sup I_k = b_k$, $a_k, b_k \in [-\infty,+\infty]$ and $\sup I_k < \inf I_{k+1}$ for all $k = 1,\dots,N-1$. Let us assume that $N \ge 2$. Since $|M| = +\infty$ by (ii), we must have $b_N = +\infty$ by (iii). Since also $|M^c| = +\infty$ by (ii), we must have $a_1 > -\infty$. In particular, I_k is a compact interval for all $k = 1,\dots,N-1$. By linearity and in virtue of [4, Exam. 4.11], we have $\partial^\alpha\chi_M = \nabla^\alpha\chi_M\,\mathscr{L}^1$, with

$$\nabla^\alpha\chi_M(t) = \sum_{k=1}^N \nabla^\alpha\chi_{I_k} = c_\alpha \sum_{k=1}^{N-1}\left(|t-a_k|^{-\alpha} - |t-b_k|^{-\alpha}\right) + c_\alpha|t-a_N|^{-\alpha}$$

for all $t \in \mathbb{R}$ with $t \neq a_1, b_1, \dots, a_{N-1}, b_{N-1}, a_N$, where $c_\alpha > 0$ depends on α only. Since $N \ge 2$, we have $\nabla^\alpha\chi_M(t) < 0$ in an open neighborhood of b_1, contradicting (i). We thus must have $N = 1$ and so $M = (a_1,+\infty)$ for some $a_1 \in \mathbb{R}$, concluding the proof. □

2.8. **Proof of Theorem 1.10.** Let $\chi_F \in W^{\alpha,1}(\mathbb{R}^n)$. By Theorem 1.1, we have $\chi_{E\cap F} \in BV^{\alpha,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ with $D^\alpha_{\mathrm{s}}\chi_{E\cap F} = \chi_{F^1}D^\alpha_{\mathrm{s}}\chi_E$ in $\mathscr{M}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. If $|E\cap F| = |F|$, then $\chi_{E\cap F} = \chi_F$ and so $\chi_{F^1}|D^\alpha_{\mathrm{s}}\chi_E| = |D^\alpha_{\mathrm{s}}\chi_{E\cap F}| = |D^\alpha_{\mathrm{s}}\chi_F| = 0$. If instead $|E\cap F| = 0$, then $\chi_{E\cap F} = 0$ and so again $\chi_{F^1}|D^\alpha_{\mathrm{s}}\chi_E| = |D^\alpha_{\mathrm{s}}\chi_{E\cap F}| = 0$. Therefore $|D^\alpha_{\mathrm{s}}\chi_E|(F^1) = 0$ whenever $|E^c\cap F| = 0$ or $|E\cap F| = 0$. Taking $F = B_r(x)$ for $x \in \mathbb{R}^n$ and $r > 0$, we get $\operatorname{supp}|D^\alpha_{\mathrm{s}}\chi_E| \subset \partial^- E$. □

2.9. **Proof of Theorem 1.12.** We prove each statement separately.

Proof of (i). Without loss of generality, we may assume that $x = 0$ and $\nu^\alpha_E(0) = \mathrm{e}_n$. By [10, Prop. 2.1], we must have $F = \lambda F$ for all $\lambda > 0$. Due to Theorem 1.7, this implies that $F = \mathbb{R}^{n-1}\times M$ for some $M \subset \mathbb{R}$ such that $M = \lambda M$ for all $\lambda > 0$. Since $0 \in \mathscr{F}^\alpha_{\mathrm{e}}E$, we must have $M \neq \emptyset, \mathbb{R}$. As a consequence, $|M| = +\infty$ by Theorem 1.7(ii). It is now plain to see that either $M = (0,+\infty)$ or $M = (-\infty, 0)$, but the latter case is automatically excluded by points (iii) and (iv) of Theorem 1.7. We thus get that $F = H^+_{e_n}(0)$, as claimed. The remaining part of the statement is a simple application of [9, Def. 1.1 and Th. 1.2].

Proof of (ii). By [13, Prop. 3.1], we know that $\mathscr{H}^{n-\alpha}(\partial^* E) = 0$. Since $\mathscr{F}^\alpha_{\mathrm{e}} E \subset \partial^* E$ by Definition 1.11, we thus get that $\mathscr{H}^{n-\alpha}(\mathscr{F}^\alpha_{\mathrm{e}} E) = 0$ as well.

Proof of (iii). Let $x \in \mathscr{F} E$. By (iii) in Theorem 1.6 and by Proposition 1.13, we have

$$\lim_{r\to 0^+} \frac{|D^\alpha \chi_E|(B_{rR}(x))}{(rR)^{n-\alpha}} = \lim_{r\to 0^+} |D^\alpha \chi_{\frac{E-x}{r}}|(B_R) = |D^\alpha \chi_{H^+_{\nu_E(x)}}|(B_R) > 0$$

for all $R > 0$. Hence, there exists $r_x > 0$ such that

$$|D^\alpha \chi_E|(B_\varrho(x)) > 0 \quad \text{for any } \varrho \in (0, r_x). \tag{2.14}$$

Moreover, by (i) and (ii) in Theorem 1.6 and by Proposition 1.13, we have

$$\lim_{r\to 0^+} \frac{D^\alpha \chi_E(B_r(x))}{|D^\alpha \chi_E|(B_r(x))} = \frac{\displaystyle\int_{B_1} \nabla^\alpha \chi_{H^+_{\nu_E(x)}}\,\mathrm{d}y}{\displaystyle\int_{B_1} |\nabla^\alpha \chi_{H^+_{\nu_E(x)}}|\,\mathrm{d}y} = \nu_E(x). \tag{2.15}$$

The limits in (2.14) and (2.15) thus imply that $x \in \mathscr{F}^\alpha E$ with $\nu^\alpha_E = \nu_E$ on $\mathscr{F} E$. Since $\mathscr{F} E \subset \partial^* E$, we get $\mathscr{F} E \subset \mathscr{F}^\alpha_{\mathrm{e}} E$. The conclusion thus follows by recalling that $\mathscr{H}^{n-1}(\partial^* E) < +\infty$ and $\mathscr{H}^{n-1}(\partial^* E \setminus \mathscr{F} E) = 0$, see [12, Th. 16.2] for instance. □

2.10. **Proof of Proposition 1.15.** Without loss of generality, we can assume $x_0 = 0$ and $r = 1$. By [4, Th. 3.18, Eq. (3.26)] (applied to $f = \chi_{B_1}$), we have

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n+\alpha-1} \int_{\partial B_1} \frac{y}{|x-y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y)$$

for all $x \in \mathbb{R}^n \setminus \partial B_1$. Changing variables, we easily get that

$$\int_{\partial B_1} \frac{y}{|x-y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y) = \frac{x}{|x|} \int_{\partial B_1} \frac{y_1}{||x|\mathrm{e}_1 - y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y)$$

for all $x \in \mathbb{R}^n$ with $|x| \neq 0, 1$, since $\displaystyle\int_{\partial B_1} \frac{y_i}{||x|\mathrm{e}_1 - y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y) = 0$ for $i \in \{2, \dots, n\}$ by symmetry. We also notice that

$$\begin{aligned}&\int_{\partial B_1} \frac{y_1}{||x|\mathrm{e}_1 - y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y)\\ &\quad= \int_{\partial B_1 \cap \{y_1>0\}} y_1 \left(\frac{1}{((|x|-y_1)^2+|\hat{y}_1|^2)^{\frac{n+\alpha-1}{2}}} - \frac{1}{((|x|+y_1)^2+|\hat{y}_1|^2)^{\frac{n+\alpha-1}{2}}} \right) \mathrm{d}\mathscr{H}^{n-1}(y) > 0\end{aligned}$$

for all $x \in \mathbb{R}^n$ with $|x| \neq 0, 1$. Hence, we can write

$$\nabla^\alpha \chi_{B_1}(x) = -\frac{\mu_{n,\alpha}}{n+\alpha-1}\, g_{n,\alpha}(|x|)\, \frac{x}{|x|} \tag{2.16}$$

for all $x \in \mathbb{R}^n$ with $|x| \neq 0, 1$, where

$$g_{n,\alpha}(t) = \int_{\partial B_1} \frac{y_1}{|t\mathrm{e}_1 - y|^{n+\alpha-1}}\,\mathrm{d}\mathscr{H}^{n-1}(y) > 0$$

for all $t \geq 0$. We now claim that

$$\nu^\alpha_{B_1}(x) = -\frac{x}{|x|} \quad \text{for all } x \neq 0. \tag{2.17}$$

Indeed, since

$$\left| \frac{\displaystyle\int_{B_r(x)} \left(\frac{y}{|y|} - \frac{x}{|x|} \right) g_{n,\alpha}(|y|)\,\mathrm{d}y}{\displaystyle\int_{B_r(x)} g_{n,\alpha}(|y|)\,\mathrm{d}y} \right| \le \sup_{y \in B_r(x)} \left| \frac{y}{|y|} - \frac{x}{|x|} \right|,$$

by (2.16) we have

$$\lim_{r \to 0^+} \frac{\displaystyle\int_{B_r(x)} \nabla^\alpha \chi_{B_1}(y)\,\mathrm{d}y}{\displaystyle\int_{B_r(x)} |\nabla^\alpha \chi_{B_1}(y)|\,\mathrm{d}y} = -\lim_{r \to 0} \frac{\displaystyle\int_{B_r(x)} \frac{y}{|y|}\, g_{n,\alpha}(|y|)\,\mathrm{d}y}{\displaystyle\int_{B_r(x)} g_{n,\alpha}(|y|)\,\mathrm{d}y} = -\frac{x}{|x|},$$

proving (2.17). If $x = 0$, then $\displaystyle\int_{B_r} \frac{y}{|y|}\, g_{n,\alpha}(|y|)\,\mathrm{d}y = 0$ by symmetry, so that

$$\lim_{r \to 0^+} \frac{\displaystyle\int_{B_r} \nabla^\alpha \chi_{B_1}(y)\,\mathrm{d}y}{\displaystyle\int_{B_r} |\nabla^\alpha \chi_{B_1}(y)|\,\mathrm{d}y} = \lim_{r \to 0^+} \frac{\displaystyle\int_{B_r} \frac{y}{|y|} g_{n,\alpha}(|y|)\,\mathrm{d}y}{\displaystyle\int_{B_r} g_{n,\alpha}(|y|)\,\mathrm{d}y} = 0.$$

Consequently, $\mathscr{F}^\alpha B_1 = \mathbb{R}^n \setminus \{0\}$, $\mathscr{F}^\alpha_{\mathrm{e}} B_1 = \partial B_1 = \mathscr{F} B_1$ and $\nu^\alpha_{B_1} = \nu_{B_1}$ on ∂B_1. □

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