Uniqueness of critical metrics for a quadratic curvature functional

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Abstract

We prove that complete, non-compact critical metrics of the quadratic curvature functional $\mathfrak{S}^2 = \int R_g^2 dV_g$ with finite energy are always scalar flat in dimension $n \geq 10$.

Résumé

Nous démontrons que les métriques critiques complètes et non compactes de la fonctionnelle quadratique de courbure $\mathfrak{S}^2 = \int R_g^2\, dV_g$ à énergie finie, sont toujours à courbure scalaire nulle en dimension $n \geq 10$.

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1. Introduction

This paper is devoted to the study of critical metrics for the quadratic curvature functional

$$\mathfrak{S}^2 = \int R_g^2 dV_g.$$

To fix the notation, let M^n, $n \geq 2$, be a n–dimensional smooth manifold without boundary. Given a Riemannian metric g on M^n, we denote by $\mathrm{Riem_g}$, W_g, Ric_g and R_g the Riemann curvature tensor, the Weyl tensor, the Ricci tensor and the scalar curvature, respectively. It is well known that, in dimension greater than or equal to four, a basis for the space of quadratic curvature functionals, defined on the space of smooth metrics on M^n, is given by

$$\mathfrak{W}^2 = \int |W_g|^2 dV_g\,, \qquad \mathfrak{r}^2 = \int |\mathrm{Ric}_g|^2 dV_g\,, \qquad \mathfrak{S}^2 = \int R_g^2 dV_g.$$

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Note that, when $n = 2$, the space is one-dimensional, and one can take $\mathfrak{S}^2$ as a basis, while when $n = 3$ the space is two-dimensional, and one can take $\mathfrak{S}^2$ and $\mathfrak{r}^2$ as a basis. From the standard decomposition of the Riemann tensor, for every $n \geq 4$, one has

$$\mathfrak{R}^2 = \int |\operatorname{Riem_g}|^2 dV_g = \int \left(|W_g|^2 + \frac{4}{n-2} |\mathrm{Ric}_g|^2 - \frac{2}{(n-1)(n-2)} R_g^2 \right) dV_g \, .$$

Such functionals have attracted a lot of attention from the mathematics and physicists communities in recent years. In particular, in [5] (see also references therein) we proved rigidity results for critical metrics of the functional $\mathfrak{S}^2$ and for the functional

$$\mathfrak{F}_t^2 = \int |\mathrm{Ric}_g|^2 dV_g + t \int R_g^2 dV_g \, ,$$

for suitable values of the parameter $t \in \mathbb{R}$ (with $t = -\infty$ formally corresponding to the functional $\mathfrak{S}^2$).

As far as the functional $\mathfrak{S}^2$ is concerned, we showed the following:

- in case $n = 2$ all critical metrics are flat, and thus they are global minima of the functional;
- in case $n = 3$, again all critical metrics are flat (and thus they are global minima of the functional) under the additional hypothesis that $R_g \in L^q(M^3)$ for some $q \in (1, \infty)$;
- the dimension $n = 4$ is special, due to the differing behaviour of the quadratic functionals under constant rescaling of the metric when $n = 4$ and $n \neq 4$; note also that, in this case, critical metrics turn out to have harmonic scalar curvature. Thus, if M^4 is compact, then it is scalar flat, or it has constant scalar curvature and it is Einstein; on the other hand, if M^4 is complete, non-compact and $R_g \in L^q(M^4)$ for some $q \in (1, \infty)$, then a classical result of Yau implies that M^4 has constant scalar curvature, and hence it is scalar flat or Einstein with finite volume;
- finally, when $n \geq 5$, there exists $q^* > 2$ such that a critical metric of $\mathfrak{S}^2$ having scalar curvature R_g which is *bounded from below* and satisfying $R_g \in L^q(M^n)$ for $q \in (1, q^*)$ must be scalar-flat, and thus it is a global minimum of the functional.

We recall that the Euler–Lagrange equation for a critical metric of $\mathfrak{S}^2$ is given by

$$2R\,\mathrm{Ric} - 2\nabla^2 R + 2\Delta R\, g = \frac{1}{2} R^2\, g \, ,$$

or, equivalently,

$$R\,\mathrm{Ric} - \nabla^2 R = \frac{3}{4(n-1)} R^2\, g \, , \tag{1.1}$$

$$\Delta R = \frac{n-4}{4(n-1)} R^2 \, , \tag{1.2}$$

where equation (1.2) is just the trace of (1.1) (see Proposition 4.66 in Besse's book [2]).

The main result of this paper is the following Theorem, where we improve our [5, Theorem 1.5] in dimension $n \geq 10$:

Theorem 1.1. *Let (M^n, g), $n \geq 10$, be a complete, non-compact critical metric of $\mathfrak{S}^2$ with* finite energy*, i.e. $R_g \in L^2(M^n)$. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.*

We actually show our result under the slightly weaker assumption that $R_g \in L^q(M)$ for some $q \in (1, q^*)$, for a suitable explicit $q^* > 2$.

We conjecture, however, that the above result should also hold in the range of dimensions $5 \leq n \leq 9$:

Conjecture 1.2. Let (M^n, g), $n \geq 5$, be a complete, non-compact critical metric of $\mathfrak{S}^2$ with finite energy. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.

Remark 1.3. Of course, the result of Theorem 1.1 holds also in the compact case, but when M is compact a volume constraint is needed and the relevant Euler-Lagrange equation changes; in this case, the existence and the rigidity of critical metrics are still completely open problems.

Our proof relies on a preliminary result that guarantees that a critical metric for $\mathfrak{S}^2$ with $R_g \in L^q(M)$ for some $q \in (1, \infty)$ and $n \geq 5$ must have either identically vanishing or strictly negative scalar curvature. We explicitly note that there exists no critical metric for the $\mathfrak{S}^2$ functional with R_g positive and $n \neq 4$; indeed, any such metric must have constant scalar curvature, thus it can exist only when $n = 4$ and (M, g) is Einstein, see [4]. In dimension $n \leq 4$, using the strong maximum principle, the weaker assumption $R_g \geq 0$ is enough to conclude that critical metrics for $\mathfrak{S}^2$ are scalar flat. The case when the scalar curvature may change sign, and thus it must have infinite energy, remains an interesting and completely open problem.

In order to prove our main theorem we show that, under our assumptions, there exists no critical metric with negative scalar curvature. Indeed, if $R_g < 0$ we can perform the conformal change of the metric

$$\widetilde{g} = |R_g|^{\frac{6}{n-4}} g; \tag{1.3}$$

this specific choice of the exponent is the only one giving stronger information on the curvature of the deformed metric, since it produces a "steady quasi-Einstein structure" (in particular a steady Ricci soliton if $n = 10$), i.e. it satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{n-10}{4(n-1)} df \otimes df = 0$$

with

$$f = \frac{2(n-1)}{n-4} \log |R|.$$

Note that $\frac{n-10}{4(n-1)} \geq 0$ when $n \geq 10$. This conformal technique has been used in the literature, for instance by Anderson in the context of stationary space-times [1], by Fischer–Colbrie [8] to study stable minimal surfaces in $\mathbb{R}^3$ and, more recently, further exploited to study the stable Bernstein problem in $\mathbb{R}^3$ by Catino–Mastrolia–Roncoroni [7], minimally immersed submanifolds in the sphere by Magliaro–Mari–Roing–Savas-Halilaj [9], minimal stable (or, more generally, δ-stable) hypersurfaces $M^n \to N^{n+1}$ immersed into a non-negatively curved space by Catino–Mari–Mastrolia–Roncoroni [6].

Using the fact that the scalar curvature of $\widetilde{g}$ is non-negative (see [3] and [**?** , Theorem 1.4]), we are able to deduce a gradient estimate on R_g, which then allows us to conclude that R_g must actually vanish everywhere if it belongs to $L^q(M)$ for $q \in (1, q^*)$, for a suitable $q^* > 2$ (possibly non-sharp), similar to [5]. Note that a key step in this construction is to show completeness of the conformal metric $\widetilde{g}$, that we are able to obtain when $n \geq 10$.

Note that Conjecture 1.1 for $5 \leq n \leq 9$ and the question whether a finite energy assumption on R_g is necessary in order to prove that a critical metric for $\mathfrak{S}^2$ must be scalar flat remain still open.

We explicitly remark that the conformal technique that we use in the present paper allows us to substantially improve our previous rigidity result [5, Theorem 1.5], removing the strong assumption that R_g is bounded below, in the range $n \geq 10$; unfortunately, in the range $5 \leq n \leq 9$, the conformal metric does not necessarily satisfy *a priori* bounds on the scalar curvature, which are a key ingredient in obtaining the aforementioned gradient estimate.

The rest of the paper is organized as follows. In Section 2 we show that the conformal change of the metric (1.3) gives rise to a quasi-Einstein manifold $(M^n, \widetilde{g})$, while in Section 3 we prove that $\widetilde{g}$ is complete under the hypotheses of Theorem 1.1. In Section 4 we provide the proof of Theorem 1.1

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2. Conformal quasi-Einstein manifolds

From now on we will drop the subscript g in the notation of geometric objects. First of all we recall that, if $n \geq 5$, then by (1.2) R is subharmonic; then, Lemma 5.1 in [5] and the strong maximum principle imply that a complete, non-compact critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < \infty$ is either scalar flat or it has negative scalar curvature.

From now on we will assume that (M^n, g), $n \geq 5$, is a complete, non-compact, critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < \infty$ and with negative scalar curvature.

To simplify the writing, let

$$u := -R > 0$$

on M. From the critical equations, we have

$$\mathrm{Ric} = \frac{\nabla^2 u}{u} - \frac{3}{4(n-1)} u\, g\,, \tag{2.1}$$

$$\Delta u \;= -\frac{n-4}{4(n-1)} u^2\,, \tag{2.2}$$

Proposition 2.1. *Let (M^n, g), $n \geq 5$, be a critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then, for all $\mathbb{R} \ni k \neq 0, \frac{1}{n-2}$, the conformal metric*

$$\widetilde{g} = u^{2k} g$$

satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1+2k-(n-2)k^2}{[(n-2)k-1]^2} df \otimes df = \frac{(n-4)k-3}{4(n-1)} e^{\frac{1-2k}{(n-2)k-1}f}\, \widetilde{g} \tag{2.3}$$

with

$$f = [(n-2)k - 1] \log u.$$

Proof. Since $f = [(n-2)k-1]\log u$, we have

$$df = [(n-2)k-1] \frac{du}{u}$$

and

$$\nabla^2_g f = [(n-2)k-1] \left(\frac{\nabla^2_g u}{u} - \frac{du \otimes du}{u^2} \right),$$

which implies

$$\Delta_g f = [(n-2)k-1] \left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|_g^2}{u^2} \right).$$

From the standard formulas for a conformal change of the metric $\tilde{g} = e^{2\varphi} g$, $\varphi \in C^\infty(M)$, $\varphi > 0$ we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla^2_g \varphi - d\varphi \otimes d\varphi\right) - \left[\Delta_g \varphi + (n-2)|\nabla_g \varphi|_g^2\right] g$$

and

$$\nabla^2_{\widetilde{g}} f = \nabla^2_g f - (df \otimes d\varphi + d\varphi \otimes df) + g\left(\nabla f, \nabla \varphi\right) g.$$

Note that, in our case, $\varphi = k \log u$; now we exploit the fact that u satisfies equations (2.1) and (2.2) to conclude that

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1+2k-(n-2)k^2}{[(n-2)k-1]^2} df \otimes df = \frac{(n-4)k-3}{4(n-1)} e^{\frac{1-2k}{(n-2)k-1}f}\, \widetilde{g}. \qquad \square$$

Corollary 2.2. *Let (M^n, g), $n \geq 5$, be a critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then the conformal metric*

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{n-10}{4(n-1)} df \otimes df = 0 \tag{2.4}$$

with

$$f = \frac{2(n-1)}{n-4} \log |R|.$$

3. Completeness of the conformal metric

In this Section we show that, under the hypotheses of Theorem 1.1, if R is negative on M then the conformal metric

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

is complete on M. This is a consequence of the following more general result:

Proposition 3.1. *Let (M^n, g), $n \geq 10$, be a complete, non-compact, critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then, if $n = 10$ and $k = \frac{1}{2}$, or $n > 10$ and $\frac{3}{n-4} \leq k < \frac{1+\sqrt{n-1}}{n-2}$, the conformal metric*

$$\widetilde{g} = |R|^{2k} g$$

is complete. In particular,

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

is complete for every $n \geq 10$.

Proof. Let $u = -R > 0$ and $\frac{3}{n-4} \leq k < 1$. Following the idea in [8, Theorem 1], given a fixed reference point $o \in M^n$, we can construct a $\widetilde{g}$-minimizing geodesic

$$\gamma(s) : [0, \infty) \to M^n,$$

where s is the g-arclength. For the sake of completeness, we report the argument here. First of all, for every $\rho > 0$, we consider the geodesic ball (of g) centered at o of radius ρ, $B_\rho(o)$. Then, we first claim that there exists a $\widetilde{g}$-minimizing geodesic joining o to the closest (in $\widetilde{g}$) boundary point of $B_\rho(o)$. Indeed, consider $u_\rho := u + \eta$, where η is a non-negative smooth function such that $\eta \equiv 0$ in $B_\rho(o)$ and $\eta \equiv 1$ on $B^c_{\rho+1}(o)$. Since u_ρ is bounded below away from 0, the metric

$$\widetilde{g}_\rho = u_\rho^{2k} g$$

is complete, and thus there exist $\widetilde{g}_\rho$-minimizing geodesics joining o to any boundary point of $B_\rho(o)$. Now let $\rho_i > 0$ be a sequence of radii monotonically diverging to $+\infty$. For every $\rho_i > 0$, since $\partial B_{\rho_i}(o)$ is compact (by the completeness of g), there exists $x_i \in \partial B_{\rho_i}(o)$ so that x_i is closest (in $\widetilde{g}_{\rho_i}$) to o. Let γ_i be the $\widetilde{g}_{\rho_i}$-minimizing geodesic joining o to x_i. Note that $\gamma_i \subset \overline{B}_{\rho_i}(o)$, and since $u_{\rho_i} = u$ in $\overline{B}_{\rho_i}(o)$, then γ_i is a $\widetilde{g}$-minimizing geodesic. We parametrize γ_i with respect to g-arclength. In particular, since $|\dot\gamma_i(s)|_g = 1$ for every s, up to subsequences, the sequence $\dot\gamma_i(0)$ converges to a limit vector as $\rho_i \to \infty$. Thus, by ODE theory and Arzelà–Ascoli theorem, γ_i converge on compact sets of $[0,\infty)$ to a limiting curve γ which is a $\widetilde g$-minimizing geodesic and is parametrized by g-arclength.

We observe that the completeness of the metric $\widetilde g = u^{2k} g$ will follow if we can show that the $\widetilde g$-length of γ is infinite, i.e.

$$\int_\gamma d\tilde s = \int_\gamma u^k \, ds = +\infty.$$

Indeed, by construction, the $\widetilde g$-length of every other divergent geodesic starting from o (i.e. its image does not lie in any ball $B_\rho(o)$) must be greater than or equal to that of γ.

Since γ is $\widetilde g$-minimizing, by the second variation formula one has

$$\int_0^{+\infty} (n-1)\left(\frac{d\varphi}{d\tilde s}\right)^2 - \widetilde R_{11}\varphi^2 \, d\tilde s \geq 0, \tag{3.1}$$

for all $\varphi \in V$, where V is the set of all compactly supported C^2-functions in $[0,+\infty)$ and

$$\widetilde R_{11} = \widetilde{\mathrm{Ric}}\left(\frac{d\gamma}{d\tilde s}, \frac{d\gamma}{d\tilde s}\right).$$

From [10, Appendix], we have

$$\widetilde R_{11} = u^{-2k}\left\{R_{11} - k(n-2)(\log u)_{ss} - k\frac{\Delta u}{u} + k\frac{|\nabla u|^2}{u^2}\right\}$$

where $R_{11} = \mathrm{Ric}(e_1,e_1) = \mathrm{Ric}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)$ (and the subscript s means derivative in the direction of $\frac{d\gamma}{ds}$) . Using the critical equation (2.1) and [10, Appendix] we obtain

$$\begin{aligned} R_{11} &= \frac{\nabla^2_{11}u}{u} - \frac{3}{4(n-1)}u\, g_{11} \\ &= \nabla^2_{11}\log u + |(\log u)_s|^2 - \frac{3}{4(n-1)}u \\ &= (\log u)_{ss} - k|(\nabla \log u)^\perp|^2 + |(\log u)_s|^2 - \frac{3}{4(n-1)}u \end{aligned}$$

where $(\nabla \log u)^\perp$ is the component of $\nabla \log u$ perpendicular to $\frac{d\gamma}{ds}$. Therefore, from (2.2), we get

$$\begin{aligned} \widetilde R_{11} &= u^{-2k}\left\{[1-(n-2)k](\log u)_{ss} + \frac{(n-4)k-3}{4(n-1)}u + |(\log u)_s|^2 + k|\nabla \log u|^2 - k|(\nabla \log u)^\perp|^2\right\} \\ &= u^{-2k}\left\{[1-(n-2)k](\log u)_{ss} + \frac{(n-4)k-3}{4(n-1)}u + (1+k)|(\log u)_s|^2\right\}. \end{aligned}$$

From inequality (3.1), since $k \geq \frac{3}{n-4}$, we obtain

$$
\begin{aligned}
&(n-1)\int_0^{+\infty}(\varphi_s)^2 u^{-k}\,ds \\
&\qquad \geq \int_0^{+\infty}\varphi^2 u^{-k}\left\{[1-(n-2)k](\log u)_{ss} + \frac{(n-4)k-3}{4(n-1)}u + (1+k)|(\log u)_s|^2\right\}ds \\
&\qquad \geq \int_0^{+\infty}\varphi^2 u^{-k}\left\{[1-(n-2)k](\log u)_{ss} + (1+k)|(\log u)_s|^2\right\}ds,
\end{aligned}
$$

for all $\varphi \in V$. Integrating by parts, we obtain

$$
\int_0^{+\infty}\varphi^2 u^{-k}(\log u)_{ss}\,ds = -2\int_0^{+\infty}\varphi u^{-k-1}\varphi_s u_s\,ds + k\int_0^{+\infty}\varphi^2 u^{-k-2}(u_s)^2\,ds,
$$

and thus

$$
\begin{aligned}
(n-1)\int_0^{+\infty}(\varphi_s)^2 u^{-k}\,ds \geq& -2[1-(n-2)k]\int_0^{+\infty}\varphi u^{-k-1}\varphi_s u_s\,ds \\
&+ [1+2k-k^2(n-2)]\int_0^{+\infty}\varphi^2 u^{-k-2}(u_s)^2\,ds.
\end{aligned}
$$

Let now $\varphi = u^k\psi$, with $\psi \in V$. We have

$$
\begin{aligned}
\varphi^2 u^{-k} &= u^k\psi^2, \\
\varphi_s &= k\psi u^{k-1}u_s + u^k\psi_s, \\
(\varphi_s)^2 u^{-k} &= k^2\psi^2 u^{k-2}(u_s)^2 + u^k(\psi_s)^2 + 2k\psi\psi_s u^{k-1}u_s,
\end{aligned}
$$

and substituting in the previous relation we get

$$
\begin{aligned}
(n-1)\int_0^{+\infty}(\psi_s)^2 u^k\,ds \geq& -2(1+k)\int_0^{+\infty}\psi u^{k-1}\psi_s u_s\,ds \\
&+ [1-k^2]\int_0^{+\infty}\psi^2 u^{k-2}(u_s)^2\,ds.
\end{aligned} \tag{3.2}
$$

Integration by parts gives

$$
I := \int_0^{+\infty}\psi u^{k-1}\psi_s u_s\,ds = -\frac{1}{k}\int_0^{+\infty}(\psi_s)^2 u^k\,ds - \frac{1}{k}\int_0^{+\infty}\psi\,\psi_{ss}u^k\,ds
$$

Moreover, for every $t > 1$ and, completing the square, for every $\varepsilon > 0$, we have

$$
\begin{aligned}
2(1+k)I &= 2(1+k)tI + 2(1+k)(1-t)I \\
&= -\frac{2t(1+k)}{k}\int_0^{+\infty} u^k(\psi_s)^2\,ds - \frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds \\
&\quad + 2(1+k)(1-t)\int_0^{+\infty}\psi\psi_s u^{k-1}u_s\,ds \\
&= -\frac{2t(1+k)}{k}\int_0^{+\infty} u^k(\psi_s)^2\,ds - \frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds \qquad (3.3)\\
&\quad + (1+k)(t-1)\varepsilon\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds + \frac{(1+k)(t-1)}{\varepsilon}\int_0^{+\infty}u^k(\psi_s)^2\,ds \\
&\quad + \frac{(1+k)(1-t)}{\varepsilon}\int_0^{+\infty}u^k\left(\psi_s+\varepsilon u^{-1}u_s\psi\right)^2\,ds.
\end{aligned}
$$

Choosing

$$\varepsilon := \frac{1-k}{t-1} > 0$$

we obtain

$$
\begin{aligned}
2(1+k)I = &-\frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds + (1-k^2)\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds \\
&+ \left[\frac{(1+k)(t-1)^2}{1-k} - \frac{2t(1+k)}{k}\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds \\
&- \frac{(1+k)(1-t)^2}{1-k}\int_0^{+\infty}u^k\left(\psi_s+\frac{1-k}{t-1}u^{-1}u_s\psi\right)^2\,ds.
\end{aligned}
$$

Exploiting the previous computation, from (3.2) we obtain

$$
\begin{aligned}
0 \le &\left[\frac{(1+k)(t-1)^2}{1-k} - \frac{2t(1+k)}{k} + (n-1)\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds - \frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds \\
&- \frac{(1+k)(1-t)^2}{1-k}\int_0^{+\infty}u^k\left(\psi_s+\frac{1-k}{t-1}u^{-1}u_s\psi\right)^2\,ds \qquad (3.4)
\end{aligned}
$$

for every $t > 1$. Let

$$P(t) := \frac{(1+k)(t-1)^2}{1-k} - \frac{2t(1+k)}{k} + (n-1)$$

Since $k < 1$, a computation shows that $P(t) \le 0$ for some $t > 1$ if and only if

$$[1 + 2k - (n-2)k^2] \ge 0.$$

If $n > 10$, for $\frac{3}{n-4} \leq k < \frac{1+\sqrt{n-1}}{n-2}$ we have $[1 + 2k - (n-2)k^2] > 0$, and thus $P(t) < 0$ for some $t > 1$. Therefore, we deduce

$$0 \leq -\int_0^{+\infty} u^k (\psi_s)^2 \, ds - C \int_0^{+\infty} u^k \psi \psi_{ss} \, ds$$

for some $C > 0$ and every $\psi \in V$. Now we choose $\psi = s\eta$ with η smooth with compact support in $[0, +\infty)$: thus

$$\psi_s = \eta + s\eta_s, \quad \psi_{ss} = 2\eta_s + s\eta_{ss},$$

and we get

$$\int_0^{+\infty} u^k \eta^2 \, ds \leq \int_0^{+\infty} u^k \left(-2(C+1)s\eta\eta_s - Cs^2\eta\eta_{ss} - s^2(\eta_s)^2 \right) \, ds.$$

Choose η so that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2, respectively, for $R \leq s \leq 2R$ and for some C independent of R. Then

$$\int_0^R u^k \, ds \leq \int_0^{+\infty} u^k \eta^2 \, ds \leq C \int_R^{+\infty} u^k \, ds$$

for some $C > 0$ independent of R. We conclude that

$$\int_0^{+\infty} u^k \, ds = +\infty,$$

i.e. $\widetilde{g} = u^{2k} g$ is complete, if $n > 10$.

If $n = 10$ and $k = \frac{1}{2}$, then $[1 + 2k - (n-2)k^2] = 0$. In this case, it is easy to verify that $P(t) = 3(t-2)^2$. Choose $t = 2$. From (3.4), since $\varepsilon = 1/2$, we obtain

$$\int_0^{+\infty} u^k \left(\psi_s + \frac{1}{2} u^{-1} u_s \psi \right)^2 \, ds \leq -C \int_0^{+\infty} u^k \psi \psi_{ss} \, ds$$

for some $C > 0$ and for every $\psi \in V$. Assume, by contradiction, that u^k is integrable. Choosing again $\psi = s\eta$ with η smooth so that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2, respectively, for $R \leq s \leq 2R$ and for some C independent of R, we get that the right hand side tends to zero as R tends to $+\infty$. By Fatou's lemma we obtain $su^{-1}u_s = -2$. Therefore, $u(s) = Cs^{-2}$, which contradicts the fact that $u^k = u^{1/2}$ is integrable. Therefore $\widetilde{g} = u^{2k} g = ug$ is complete also for $n = 10$. $\square$

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Let (M^n, g), $n \geq 10$, be a complete, non-compact critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < q^* = \frac{7n-10}{2(n-4)}$. As already recalled, we know from [5] that either $R \equiv 0$ or $R < 0$ on M^n; in the latter case we consider the conformal metric

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g,$$

which is complete by Proposition 3.1 and satisfies (2.4). In particular $(M^n, \widetilde{g})$ is a complete steady gradient Ricci soliton, if $n = 10$, or a complete steady quasi-Einstein manifold, if $n > 10$. In both cases, it is well known (see [3] and [**?** , Theorem 1.4]) that the scalar curvature of $\widetilde{g}$ must be nonnegative. By the formula for the conformal change, we obtain

$$\begin{aligned} 0 \leq \widetilde{R} &= e^{-2w}\left(R - 2(n-1)\Delta w - (n-1)(n-2)|\nabla w|^2\right) \\ &= u^{-\frac{6}{n-4}}\left(-u - \frac{6(n-1)}{n-4}\frac{\Delta u}{u} + \frac{6(n-1)}{n-4}\frac{|\nabla u|^2}{u^2} - \frac{9(n-1)(n-2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) \\ &= u^{-\frac{6}{n-4}}\left(\frac{1}{2}u - \frac{3(n-1)(n+2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) \end{aligned} \tag{4.1}$$

where we used $w = \frac{3}{n-4}\log u$ as in the proof of Proposition 2.1, $R = -u$ and (2.2). Thus

$$|\nabla u|^2 \leq \frac{(n-4)^2}{6(n-1)(n+2)} u^3. \tag{4.2}$$

Fixing $O \in M^n$, arguing as in [5, Corollary 5.7], from (4.2), we obtain

$$u(x) \geq \frac{c_1}{c_2 + d_g(x, O)^2} \tag{4.3}$$

for every $x \in M^n$ and some positive constants $c_i = c_i(n, u(O))$, $i = 1, 2$. Now the result follows as in the proof of [5, Theorem 1.5]. □

Remark 4.1. Note that the gradient estimate (4.2) improves the one in [5, Lemma 5.5] (see also Remark 5.6 there for the explicit expression of the constant), since it is possible to show that, for every $n \geq 10$, the constant $\frac{(n-4)^2}{6(n-1)(n+2)}$ is always smaller than the corresponding constant appearing there. As a consequence, we see that the conclusion of Theorem 1.1 follows assuming $R \in L^q(M^n)$ with $1 < q < \frac{7n-10}{2(n-4)}$, thus improving, for $n \geq 10$, [5, Theorem 1.5] also in this respect.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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