UNIQUENESS OF CRITICAL METRICS FOR A QUADRATIC CURVATURE FUNCTIONAL

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Abstract. In this paper we prove new rigidity results for complete, possibly non-compact, critical metrics of the quadratic curvature functionals $\mathfrak{S}^2 = \int R_g^2 dV_g$. We show that critical metrics (M^n, g) with finite energy are always scalar flat, i.e. a global minimum, provided $n \geq 10$ or $7 \leq n \leq 9$ and M^n has at least two ends.

Key Words: Quadratic functionals, critical metrics, rigidity results

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1. Introduction

This paper is devoted to the study of critical metrics for the quadratic curvature functional

$$\mathfrak{S}^2 = \int R_g^2 dV_g.$$

To fix the notation, let M^n, $n \geq 2$, be a n–dimensional smooth manifold without boundary. Given a Riemannian metric g on M^n, we denote with $\mathrm{Riem_g}$, W_g, Ric_g and R_g, respectively, the Riemann curvature tensor, the Weyl tensor, the Ricci tensor and the scalar curvature. It is well known that a basis for the space of quadratic curvature functionals, defined on the space of smooth metrics on M^n, is given by

$$\mathfrak{W}^2 = \int |W_g|^2 dV_g\,, \qquad \mathfrak{r}^2 = \int |\mathrm{Ric}_g|^2 dV_g\,, \qquad \mathfrak{S}^2 = \int R_g^2 dV_g.$$

The only quadratic functional in the case $n = 2$ is given by $\mathfrak{S}^2$, while in dimension $n = 3$ one only has $\mathfrak{S}^2$ and $\mathfrak{r}^2$. From the standard decomposition of the Riemann tensor, for every $n \geq 4$, one has

$$\mathfrak{R}^2 = \int |\,\mathrm{Riem_g}\,|^2 dV_g = \int \left(|W_g|^2 + \frac{4}{n-2} |\mathrm{Ric}_g|^2 - \frac{2}{(n-1)(n-2)} R_g^2 \right) dV_g\,.$$

Such functionals have attracted a lot of attention from the mathematics' and physicists' communities in recent years. In particular in [4], see also references therein, we proved rigidity results for critical metrics of the functional $\mathfrak{S}^2$ and for the functional

$$\mathfrak{F}_t^2 = \int |\mathrm{Ric}_g|^2 dV_g + t \int R_g^2 dV_g\,,$$

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for suitable values of the parameter $t \in \mathbb{R}$. As far as the functional $\mathfrak{S}^2$ is concerned, we showed that in case $n = 2$ all critical metrics are flat, and thus they are global minima of the functional. The same result holds also when $n = 3$, under the additional hypothesis that $R_g \in L^q(M^3)$ for some $q \in (1, \infty)$. The case $n = 4$ is special, as in this case critical metrics turn out to have harmonic scalar curvature. Thus, if M^4 is compact then it is scalar flat, or it has constant scalar curvature and it is Einstein; on the other hand, if M^4 is complete, non-compact and $R_g \in L^q(M^4)$ for some $q \in (1, \infty)$, then a classical result of Yau implies that M^4 has constant scalar curvature, and hence it is scalar flat or Einstein with finite volume. Finally, when $n \geq 5$ we showed that there exists $q^* > 2$ such that a critical metric of $\mathfrak{S}^2$ having scalar curvature R_g which is *bounded from below* and satisfying $R_g \in L^q(M^n)$ for $q \in (1, q^*)$ must be scalar-flat, and thus it is a global minimum of the functional.

We conjecture however that the condition that R_g is bounded from below in the above results, when $n \geq 5$, is indeed not necessary, i.e.

Conjecture 1.1. Let (M^n, g), $n \geq 5$, be a complete critical metric of $\mathfrak{S}^2$ with finite energy. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.

We also raise the question whether a finite energy assumption $R_g \in L^q(M^n)$ is necessary, in order to deduce that a complete non-compact critical metric of $\mathfrak{S}^2$ must be scalar flat, for $n \geq 3$, $n \neq 4$.

We recall that the Euler–Lagrange equation for a critical metric of $\mathfrak{S}^2$ can be computed by using variations with compact support and is given by

$$2R\,\mathrm{Ric} - 2\nabla^2 R + 2\Delta R\, g \,=\, \frac{1}{2}R^2\, g\,,$$

or, equivalently,

$$R\,\mathrm{Ric} - \nabla^2 R \,=\, \frac{3}{4(n-1)}R^2\, g\,, \tag{1.1}$$

$$\Delta R \,=\, \frac{n-4}{4(n-1)}R^2\,, \tag{1.2}$$

where equation (1.2) is just the trace of (1.1). The main results of this paper are the following two Theorems, where we give an affirmative answer to Conjecture 1.1 in two cases: the first when $n \geq 10$ and the second when $7 \leq n \leq 9$ and M has a finite number N of ends, $N \geq 2$. Indeed we have the following

Theorem 1.2. *Let (M^n, g), $n \geq 10$, be a complete critical metric of $\mathfrak{S}^2$ with finite energy, i.e. $R_g \in L^2(M^n)$. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.*

Theorem 1.3. *Let (M^n, g), $7 \leq n \leq 9$, be a complete critical metric of $\mathfrak{S}^2$ with finite energy, i.e. $R_g \in L^2(M^n)$, and with a finite number N of ends, $N \geq 2$. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.*

We actually show our results under the slightly weaker assumption that $R_g \in L^q(M)$ for some $q \in (1, q^*)$, for a suitable explicit $q^* > 2$.

Our proof relies on a preliminary result that guarantees that a critical metric for $\mathfrak{S}^2$ with $R_g \in L^q(M)$ for some $q \in (1, \infty)$ and $n \geq 5$ must have either identically vanishing or strictly negative scalar curvature. We explicitly note that there exist no critical metric for the $\mathfrak{S}^2$ functional with R_g positive and $n \neq 4$; indeed any such metric should have constant scalar curvature, thus it can exist only when $n = 4$ and (M, g) is Einstein, see [3]. The weaker assumption $R_g \geq 0$ is enough to conclude, using the strong maximum principle, in dimensions $n \leq 4$. The case when the scalar curvature may change sign, and thus it must have infinite energy, remains an interesting and completely open problem.

In order to prove our main theorems, then we show that, under our assumptions, there exists no critical metric with negative scalar curvature. Indeed if $R_g < 0$ we can perform the conformal change of the metric

$$\widetilde{g} = |R_g|^{\frac{6}{n-4}} g \tag{1.3}$$

to produce a "steady quasi-Einstein structure" (in particular a steady Ricci soliton if $n = 10$), i.e. it satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{n-10}{4(n-1)} df \otimes df = 0$$

with

$$f = \frac{2(n-1)}{n-4} \log |R|.$$

Note that $\frac{n-10}{4(n-1)} \geq 0$ when $n \geq 10$, while it is negative for $5 \leq n \leq 9$. This technique has been used in the literature for instance by Anderson in the context of stationary space-times [1], by Fischer–Colbrie [6] to study stable minimal surfaces in $\mathbb{R}^3$ and more recently further exploited to study the stable Bernstein problem in $\mathbb{R}^3$ by Catino–Mastrolia–Roncoroni [5].

Using lower bounds on the scalar curvature of $\widetilde{g}$ we are able to deduce a gradient estimate on R_g, which then allows us to conclude that R_g must actually vanish everywhere if it belongs to $L^q(M)$ for $q \in (1, q^*)$, for a suitable $q^* > 2$, similarly as we did in [4]. A key step in this construction is to show completeness of the conformal metric $\widetilde{g}$, that we are able to obtain either when $n \geq 10$ or when $7 \leq n \leq 9$ and M has a finite number N of ends, $N \geq 2$.

We also explicitly comment on the estimates on the scalar curvature of $\widetilde{g}$ that we use to obtain the gradient estimate on R_g: while in case $n \geq 10$ we rely on results which are already available in the literature concerning nonnegativity of the scalar curvature for steady Ricci solitons (when $n = 10$) and for steady quasi-Einstein manifolds (when $n \geq 11$), in this paper we also show a new bound from below on the scalar curvature for such structures, which may have independent interest, that we employ to conclude the proof also in case $7 \leq n \leq 9$.

The full Conjecture 1.1 for $5 \leq n \leq 9$ and the question whether a finite energy assumption on R_g is necessary in order to prove that a critical metric for $\mathfrak{S}^2$ must be scalar flat remain still open.

The rest of the paper is organized as follows. In Section 2 we show that the conformal change of the metric (1.3) gives rise to a quasi-Einstein manifold $(M^n, \widetilde{g})$, while in Section 3 we prove that $\widetilde{g}$ is complete, under the hypotheses of either Theorem 1.2 or Theorem 1.3. In Sections 4 and 6 we provide the proofs of Theorem 1.2 and Theorem 1.3, respectively. Section 5 contains a new estimate on the scalar curvature of quasi-Einstein manifolds, for values of the relevant parameters that do not appear in the literature, which are instrumental in the proof of Theorem 1.3.

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2. Conformal quasi-Einstein manifolds

From now on we will drop the subscript g in the notation of geometric objects. First of all, we recall the following, see Lemma 5.1 in [4].

Lemma 2.1. *Let (M^n, g), $n \geq 5$, be a complete, non-compact, critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < \infty$. Then (M^n, g) has non-positive scalar curvature.*

Now, if $n \geq 5$, then by (1.2) R is subharmonic, therefore Lemma 2.1 and the strong maximum principle imply the following:

Corollary 2.2. *Let (M^n, g), $n \geq 5$, be a complete, non-compact critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < \infty$. Then (M^n, g) is either scalar flat or it has negative scalar curvature.*

From now on we will assume that (M^n, g), $n \geq 5$, is a complete, non-compact, critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < \infty$ and with negative scalar curvature. Let $u := -R > 0$ on M. From the critical equations, we have

$$\mathrm{Ric} = \frac{\nabla^2 u}{u} - \frac{3}{4(n-1)} u\, g\,, \tag{2.1}$$

$$\Delta u = -\frac{n-4}{4(n-1)} u^2\,, \tag{2.2}$$

Proposition 2.3. *Let (M^n, g), $n \geq 5$, be a critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then, for all $\mathbb{R} \ni k \neq 0, \frac{1}{n-2}$, the conformal metric*

$$\widetilde{g} = |R|^{2k} g = u^{2k} g$$

satisfies

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1+2k-(n-2)k^2}{[(n-2)k-1]^2} df \otimes df = \frac{(n-4)k-3}{4(n-1)} e^{\frac{1-2k}{(n-2)k-1} f}\, \widetilde{g} \tag{2.3}$$

with

$$f = [(n-2)k-1]\log|R| = [(n-2)k-1]\log u.$$

Proof. Since $f = [(n-2)k-1]\log|R| = [(n-2)k-1]\log u$, we have

$$df = [(n-2)k-1]\frac{dR}{R} = [(n-2)k-1]\frac{du}{u}$$

and

$$\nabla_g^2 f = [(n-2)k-1]\left(\frac{\nabla_g^2 u}{u} - \frac{du\otimes du}{u^2}\right),$$

which implies

$$\Delta_g f = [(n-2)k-1]\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|_g^2}{u^2}\right).$$

On the other hand, from the standard formulas for a conformal change of the metric $\tilde{g} = e^{2\varphi}g$, $\varphi\in C^\infty(M)$, $\varphi>0$ we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla_g^2\varphi - d\varphi\otimes d\varphi\right) - \left[\Delta_g\varphi + (n-2)|\nabla_g\varphi|_g^2\right]g$$

and

$$\nabla_{\widetilde{g}}^2 f = \nabla_g^2 f - (df\otimes d\varphi + d\varphi\otimes df) + g\left(\nabla f, \nabla\varphi\right)g.$$

Note that, in our case, $\varphi = k\log u$; now we exploit the fact that u satisfies equations (2.1) and (2.2) to conclude that

$$\mathrm{Ric}_{\widetilde{g}} + \nabla_{\widetilde{g}}^2 f - \frac{1+2k-(n-2)k^2}{[(n-2)k-1]^2}df\otimes df = \frac{(n-4)k-3}{4(n-1)}e^{\frac{1-2k}{(n-2)k-1}f}\,\widetilde{g}.$$

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Corollary 2.4. *Let (M^n, g), $n\geq 5$, be a critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then the conformal metric*

$$\widetilde{g} = |R|^{\frac{6}{n-4}}g$$

satisfies

$$Ric_{\widetilde{g}} + \nabla_{\widetilde{g}}^2 f - \frac{n-10}{4(n-1)}df\otimes df = 0 \tag{2.4}$$

with

$$f = \frac{2(n-1)}{n-4}\log|R|.$$

3. Completeness of the conformal metric

In this Section we show that, under the hypotheses of either Theorem 1.2 or Theorem 1.3, if R is negative on M then the conformal metric

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

is complete on M.

We start with the case $n \geq 10$, where we have the following result.

Proposition 3.1. *Let (M^n, g), $n \geq 10$, be a complete, non-compact, critical metric of $\mathfrak{S}^2$ with negative scalar curvature. Then, the conformal metric*

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

is complete.

Proof. Let $u = -R > 0$ and $\frac{3}{n-4} \leq k < 1$. As shown in [6, Theorem 1], given a fixed reference point $o \in M^n$, we can construct a $\widetilde{g}$-minimizing geodesic

$$\gamma(s) : [0, \infty) \to M^n,$$

where s is the g-arclength. For the sake of completeness, we report the argument here. First of all, for every $\rho > 0$, we consider the geodesic ball (of g) centered at o of radius ρ, $B_\rho(o)$. Then, we first claim that there exists a $\widetilde{g}$-minimizing geodesic joining o to the closest (in $\widetilde{g}$) boundary point of $B_\rho(o)$. Indeed, consider $u_\rho := u + \eta$, where η is a non-negative smooth function such that $\eta \equiv 0$ in $B_\rho(o)$ and $\eta \equiv 1$ on $B^c_{\rho+1}(o)$. Since u_ρ is bounded below away from 0, the metric

$$\widetilde{g}_\rho = u_\rho^{2k} g$$

is complete, and thus there exist $\widetilde{g}_\rho$-minimizing geodesics joining o to any boundary point of $B_\rho(o)$. Now let $\rho_i > 0$ be a sequence of radii monotonically diverging to $+\infty$. For every $\rho_i > 0$, since $\partial B_{\rho_i}(o)$ is compact, there exists $x_i \in \partial B_{\rho_i}(o)$ so that x_i is closest (in $\widetilde{g}_{\rho_i}$) to o. Let γ_i be the $\widetilde{g}_{\rho_i}$-minimizing geodesic joining o to x_i. Note that $\gamma_i \subset \overline{B}_{\rho_i}(o)$, and since $u_{\rho_i} = u$ in $\overline{B}_{\rho_i}(o)$, then γ_i is a $\widetilde{g}$-minimizing geodesic. We parametrize γ_i with respect to g-arclength. In particular, since $|\dot{\gamma}_i(s)|_g = 1$ for every s, up to subsequences, the sequence $\dot{\gamma}_i(0)$ converges to a limit vector as $\rho_i \to \infty$. Thus, by ODE theory and Ascoli-Arzelà, γ_i converge on compact sets of $[0, \infty)$ to a limiting curve γ which is a $\widetilde{g}$-minimizing geodesic and is parametrized by g-arclength.

We observe that the completeness of the metric $\widetilde{g} = u^{2k} g$ will follow if we can show that the $\widetilde{g}$-length of γ is infinite, i.e.

$$\int_\gamma d\tilde{s} = \int_\gamma u^k \, ds = +\infty.$$

Indeed, by construction, the $\widetilde{g}$-length of every other divergent geodesic starting from o (i.e. its image does not lie in any ball $B_\rho(o)$) must be greater than or equal to that of γ.

Since γ is $\widetilde{g}$-minimizing, by the second variation formula one has

$$\int_0^{\tilde r}(n-1)\left(\frac{d\varphi}{d\tilde s}\right)^2-\widetilde{R}_{11}\varphi^2\,d\tilde s\geq 0, \tag{3.1}$$

for all $\varphi\in V_{\tilde r}$, where we set

$$V_{\tilde r}=\{\varphi\in C^0([0,\infty))\,|\,\varphi(s)=\varphi(0)=0\ \forall s\geq A,\ \varphi\in C^2([0,A])\ \text{ for some } 0<A<\tilde r\},$$

$V:=V_\infty$ and where $\tilde r$ is the length of γ in the metric $\widetilde{g}$ and

$$\widetilde{R}_{11}=\widetilde{\mathrm{Ric}}\left(\frac{d\gamma}{d\tilde s},\frac{d\gamma}{d\tilde s}\right).$$

From [7, Appendix], we have

$$\widetilde{R}_{11}=u^{-2k}\left\{R_{11}-k(n-2)(\log u)_{ss}-k\frac{\Delta u}{u}+k\frac{|\nabla u|^2}{u^2}\right\}$$

where $R_{11}=\mathrm{Ric}(e_1,e_1)=\mathrm{Ric}\left(\frac{d\gamma}{ds},\frac{d\gamma}{ds}\right)$. Using the critical equation (2.1) and [7, Appendix] we obtain

$$\begin{aligned}R_{11}&=\frac{\nabla^2_{11}u}{u}-\frac{3}{4(n-1)}u\,g_{11}\\&=\nabla^2_{11}\log u+|(\log u)_s|^2-\frac{3}{4(n-1)}u\\&=(\log u)_{ss}-k|(\nabla\log u)^\perp|^2+|(\log u)_s|^2-\frac{3}{4(n-1)}u\end{aligned}$$

where $(\nabla\log u)^\perp$ is the component of $\nabla\log u$ perpendicular to $\frac{d\gamma}{ds}$. Therefore, from (2.2), we get

$$\begin{aligned}\widetilde{R}_{11}&=u^{-2k}\left\{[1-(n-2)k](\log u)_{ss}+\frac{(n-4)k-3}{4(n-1)}u+|(\log u)_s|^2+k|\nabla\log u|^2-k|(\nabla\log u)^\perp|^2\right\}\\&=u^{-2k}\left\{[1-(n-2)k](\log u)_{ss}+\frac{(n-4)k-3}{4(n-1)}u+(1+k)|(\log u)_s|^2\right\}.\end{aligned}$$

From inequality (3.1), since $k\geq\frac{3}{n-4}$, we obtain

$$\begin{aligned}(n-1)&\int_0^{+\infty}(\varphi_s)^2u^{-k}\,ds\\&\geq\int_0^{+\infty}\varphi^2u^{-k}\left\{[1-(n-2)k](\log u)_{ss}+\frac{(n-4)k-3}{4(n-1)}u+(1+k)|(\log u)_s|^2\right\}ds\\&\geq\int_0^{+\infty}\varphi^2u^{-k}\left\{[1-(n-2)k](\log u)_{ss}+(1+k)|(\log u)_s|^2\right\}ds,\end{aligned}$$

for all $\varphi\in V$. Integrating by parts, we obtain

$$\int_0^{+\infty}\varphi^2u^{-k}(\log u)_{ss}\,ds=-2\int_0^{+\infty}\varphi u^{-k-1}\varphi_su_s\,ds+k\int_0^{+\infty}\varphi^2u^{-k-2}(u_s)^2\,ds,$$

and thus

$$(n-1)\int_0^{+\infty}(\varphi_s)^2u^{-k}\,ds \geq -2[1-(n-2)k]\int_0^{+\infty}\varphi u^{-k-1}\varphi_s u_s\,ds + [1+2k-k^2(n-2)]\int_0^{+\infty}\varphi^2u^{-k-2}(u_s)^2\,ds.$$

Let now $\varphi = u^k\psi$, with $\psi \in V$. We have

$$\begin{aligned} \varphi^2u^{-k} &= u^k\psi^2,\\ \varphi_s &= k\psi u^{k-1}u_s + u^k\psi_s,\\ (\varphi_s)^2u^{-k} &= k^2\psi^2u^{k-2}(u_s)^2 + u^k(\psi_s)^2 + 2k\psi\psi_s u^{k-1}u_s, \end{aligned}$$

and substituting in the previous relation we get

$$(n-1)\int_0^{+\infty}(\psi_s)^2u^k\,ds \geq -2(1+k)\int_0^{+\infty}\psi u^{k-1}\psi_s u_s\,ds + [1-k^2]\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds. \tag{3.2}$$

Integration by parts gives

$$I := \int_0^{+\infty}\psi u^{k-1}\psi_s u_s\,ds = -\frac{1}{k}\int_0^{+\infty}(\psi_s)^2u^k\,ds - \frac{1}{k}\int_0^{+\infty}\psi\,\psi_{ss}u^k\,ds$$

Moreover, for every $t > 1$ and, completing the square, for every $\varepsilon > 0$, we have

$$\begin{aligned} 2(1+k)I &= 2(1+k)tI + 2(1+k)(1-t)I\\ &= -\frac{2t(1+k)}{k}\int_0^{+\infty}u^k(\psi_s)^2\,ds - \frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds\\ &\quad + 2(1+k)(1-t)\int_0^{+\infty}\psi\psi_s u^{k-1}u_s\,ds\\ &= -\frac{2t(1+k)}{k}\int_0^{+\infty}u^k(\psi_s)^2\,ds - \frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds \\ &\quad + (1+k)(t-1)\varepsilon\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds + \frac{(1+k)(t-1)}{\varepsilon}\int_0^{+\infty}u^k(\psi_s)^2\,ds\\ &\quad + \frac{(1+k)(1-t)}{\varepsilon}\int_0^{+\infty}u^k\left(\psi_s+\varepsilon u^{-1}u_s\psi\right)^2\,ds. \end{aligned} \tag{3.3}$$

Since $k < 1$, choosing

$$\varepsilon := \frac{1-k}{t-1}$$

we obtain

$$2(1+k)I = -\frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds + (1-k^2)\int_0^{+\infty}\psi^2u^{k-2}(u_s)^2\,ds$$
$$+\left[\frac{(1+k)(t-1)^2}{1-k}-\frac{2t(1+k)}{k}\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds$$
$$-\frac{(1+k)(1-t)^2}{1-k}\int_0^{+\infty}u^k\left(\psi_s+\frac{1-k}{t-1}u^{-1}u_s\psi\right)^2\,ds.$$

Therefore, from (3.2), we obtain

$$0\le\left[\frac{(1+k)(t-1)^2}{1-k}-\frac{2t(1+k)}{k}+(n-1)\right]\int_0^{+\infty}u^k(\psi_s)^2\,ds-\frac{2t(1+k)}{k}\int_0^{+\infty}\psi\psi_{ss}u^k\,ds$$
$$-\frac{(1+k)(1-t)^2}{1-k}\int_0^{+\infty}u^k\left(\psi_s+\frac{1-k}{t-1}u^{-1}u_s\psi\right)^2\,ds \tag{3.4}$$

for every $t>1$. Let

$$P(t):=\frac{(1+k)(t-1)^2}{1-k}-\frac{2t(1+k)}{k}+(n-1)$$

A computation shows that $P(t)\le 0$ for some $t>1$ if and only if

$$(1+k)(1-k)[1+2k-(n-2)k^2]\ge 0.$$

Choose $k=\frac{3}{n-4}$.

If $n>10$, then $(1+k)(1-k)[1+2k-(n-2)k^2]>0$ and thus $P(t)<0$ for some $t>1$. Therefore, we deduce

$$0\le-\int_0^{+\infty}u^k(\psi_s)^2\,ds-C\int_0^{+\infty}u^k\psi\psi_{ss}\,ds$$

for some $C>0$ and every $\psi\in V$. Now we choose $\psi=s\eta$ with η smooth with compact support in $[0,+\infty)$: thus

$$\psi_s=\eta+s\eta_s,\quad \psi_{ss}=2\eta_s+s\eta_{ss},$$

and we get

$$\int_0^{+\infty}u^k\eta^2\,ds\le\int_0^{+\infty}u^k\left(-2(C+1)s\eta\eta_s-Cs^2\eta\eta_{ss}-s^2(\eta_s)^2\right)\,ds.$$

Choose η so that $\eta\equiv 1$ on $[0,R]$, $\eta\equiv 0$ on $[2R,+\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2 respectively for $R\le s\le 2R$. Then

$$\int_0^R u^k\,ds\le\int_0^{+\infty}u^k\eta^2\,ds\le C\int_R^{+\infty}u^k\,ds$$

for some $C>0$ independent of R. We conclude that

$$\int_0^{+\infty}u^k\,ds=+\infty,$$

i.e. $\widetilde{g} = u^{2k} g = u^{\frac{6}{n-4}} g$ is complete, if $n > 10$.

If $n = 10$, then $k = 1/2$ and $(1+k)(1-k)[1 + 2k - (n-2)k^2] = 0$. In this case. it is easy to verify that $P(t) = 3(t-2)^2$. Choose $t = 2$. From (3.4), since $\varepsilon = 1/2$, we obtain

$$\int_0^{+\infty} u^k \left(\psi_s + \frac{1}{2} u^{-1} u_s \psi \right)^2 ds \leq -C \int_0^{+\infty} u^k \psi \psi_{ss} \, ds$$

for some $C > 0$ and for every $\psi \in V$. Assume, by contradiction, that u^k is integrable. Choosing again $\psi = s\eta$ with η smooth so that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by C/R and C/R^2 respectively for $R \leq s \leq 2R$, we get that the right hand side tends to zero as R tends to $+\infty$. By Fatou's lemma we obtain $su^{-1} u_s = -2$. Therefore, $u(s) = Cs^{-2}$, which contradicts the fact that $u^k = u^{1/2}$ is integrable. Therefore $\widetilde{g} = u^{2k} g = ug$ is complete also if $n = 10$. □

In case $7 \leq n \leq 9$ we need an extra topological assumption, concerning the number of ends of the manifold. Indeed we have the following result.

Proposition 3.2. *Let (M^n, g), $7 \leq n \leq 9$, be a complete, non-compact, critical metric of $\mathfrak{S}^2$ with negative scalar curvature and N ends, with $N \geq 2$. Then, the conformal metric*

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g$$

is complete.

Proof. We reason as in the proof of Proposition 3.1. Let $K \subset M$ be a compact subset and let $E_1, \ldots, E_N$ be the (g unbounded) connected components of $M \setminus K$. We fix an end E_1, we show that there exists a second end, say E_2, such that we can construct a $\widetilde{g}$-minimizing geodesic

$$\gamma_1(s) : (-\infty, \infty) \to M^n,$$

where s is the g-arclength, connecting E_1 and E_2.

We consider the geodesic ball (of g) centered at a fixed point $o \in K$ of radius ρ, $B_\rho(o)$. Let $\rho > 0$ be sufficiently large such that $B_\rho(o) \cap E_i \neq \emptyset$ for $i = 1, \ldots, N$. Fix any $i = 2, \ldots, N$ and consider any points $x \in \partial B_\rho(o) \cap E_1$ and $y \in \partial B_\rho(o) \cap E_i$. Let $u = -R$ and consider $u_\rho := u + \eta_\rho$, where η_ρ is a smooth function such that $\eta_\rho \equiv 0$ in $B_\rho(o)$ and $\eta_\rho \equiv 1$ in $M \setminus B_{\rho+1}(o)$. Since u_ρ is uniformly bounded below away from zero, the metric

$$\widetilde{g}_\rho = u_\rho^{\frac{6}{n-4}} g$$

is complete, and thus there exists a $\widetilde{g}_\rho$-minimizing geodesic connecting x and y. Now we choose the shortest among all the $\widetilde{g}_\rho$-minimizing geodesics constructed in this way, connecting a point $x \in \partial B_\rho(o) \cap E_1$ to a point $y \in \partial B_\rho(o) \cap E_i$, thus minimizing the $\widetilde{g}_\rho$-length for $x \in \partial B_\rho(o) \cap E_1$ and $y \in \partial B_\rho(o) \cap E_i$ (this is possible since $\partial B_\rho(o) \cap E_i$ is compact for every i). We denote by $\gamma_{\rho,1,i}$ such $\widetilde{g}_\rho$-minimizing geodesic, for $i = 2, \ldots, N$. Next, we choose a shortest among these $N-1$ $\widetilde{g}_\rho$-minimizing geodesics, thus minimizing for $i = 2, \ldots, N$, and we denote it by $\gamma_{\rho,1}$. Without loss of generality we can assume that $\gamma_{\rho,1} = \gamma_{\rho,1,2}$, and since it has minimal $\widetilde{g}_\rho$-length among the $N-1$ $\widetilde{g}_\rho$-geodesics connecting

$\partial B_\rho(o)\cap E_1$ to $\partial B_\rho(o)\cap E_i$ for every $i=2,\dots,N$, we have $\gamma_{\rho,1}\cap(\overline{B}^c_\rho(o)\cap E_i)=\emptyset$ for every $i=3,\dots,N$. Moreover, since $\gamma_{\rho,1}=\gamma_{\rho,1,2}$ has minimal $\widetilde{g}_\rho$-length among all the $\widetilde{g}_\rho$-geodesics connecting $x\in\partial B_\rho(o)\cap E_1$ and $y\in\partial B_\rho(o)\cap E_2$, we also have $\gamma_{\rho,1}\cap(\overline{B}^c_\rho(o)\cap E_i)=\emptyset$ for $i=1,2$. Thus $\gamma_{\rho,1}\subset\overline{B}_\rho(o)$. Since $\widetilde{g}$ and $\widetilde{g}_\rho$ coincide in $\overline{B}_\rho(o)$, $\gamma_{\rho,1}$ is a minimizing geodesic also with respect to $\widetilde{g}$. Now we choose a sequence of radii $\rho_j>0$ monotonically diverging to $+\infty$, and we repeat the construction for all $j\in\mathbb{N}$. For each $j\in\mathbb{N}$ there exists an end E_j of M different from E_1 and a minimizing $\widetilde{g}$-geodesic $\gamma_{\rho_j,1}$ connecting $\partial B_\rho(o)\cap E_1$ and $\partial B_\rho(o)\cap E_j$, constructed as above. Since there is only a finite number N of ends, up to taking a subsequence, without loss of generality we can assume that $E_j=E_2$ for every $j\in\mathbb{N}$. We parametrize $\gamma_{\rho_j,1}$ with respect to g-arclength. In particular, since $|\dot\gamma_{\rho_j,1}(s)|_g=1$ for every s, up to subsequences, the sequence $\dot\gamma_{\rho_j,1}(0)$ converges to a limit vector as $\rho_j\to\infty$. Thus, by ODE theory and Ascoli-Arzelà, $\gamma_{\rho_j,1}$ converges on compact sets of $(-\infty,\infty)$ to a limiting curve γ_1 which is a $\widetilde{g}$-minimizing geodesic connecting E_1 and E_2, and is parametrized by g-arclength. Of course, this construction can be carried out for every end $E_1,\dots,E_N$.

We observe that the completeness of the metric $\widetilde{g}=u^{\frac{6}{n-4}}g=|R|^{\frac{6}{n-4}}g$ will follow if we can show that the $\widetilde{g}$-length of $\gamma_1,\dots,\gamma_N$ is infinite, i.e.

$$\int_{\gamma_i}d\tilde{s}=\int_{\gamma_i}u^{\frac{3}{n-4}}\,ds=+\infty,\qquad i=1,\dots,N. \tag{3.5}$$

Indeed, by construction, every divergent geodesic starting from o (i.e. its image does not lie in any ball $B_\rho(o)$) must have nonempty intersection with $E_i\cap B^c_\rho(o)$ for a suitable fixed $i=1,\dots,N$ and arbitrarily large $\rho>0$, and thus its $\widetilde{g}$-length other must be greater than or equal to that of $\gamma_i\cap E_i$, which as we will show is infinite.

Let γ be one of the N $\widetilde{g}$-geodesics constructed above. Fix any point o in the image of γ and let $\tilde{r}_1>0$ be the $\widetilde{g}$-length of the portion of γ which is unbounded in the first end and $\tilde{r}_2>0$ be the $\widetilde{g}$-length of the portion of γ which is unbounded in the second end, which are connected by γ. Since γ is $\widetilde{g}$-minimizing, by the second variation formula one has

$$\int_{-\tilde{r}_1}^{\tilde{r}_2}(n-1)\left(\frac{d\varphi}{d\tilde{s}}\right)^2-\widetilde{R}_{11}\varphi^2\,d\tilde{s}\geq 0, \tag{3.6}$$

for all $\varphi\in V_{\tilde{r}_1,\tilde{r}_2}$, where we set

$$V_{\tilde{r}_1,\tilde{r}_2}=\{\varphi\in C^0(\mathbb{R})\,|\,\varphi(s)=0\ \forall s\notin[A,B],\ \varphi\in C^2([A,B])\ \text{for some}\ [A,B]\subset(-\tilde{r}_1,\tilde{r}_2)\}$$

and

$$\widetilde{R}_{11}=\widetilde{\mathrm{Ric}}\left(\frac{d\gamma}{d\tilde{s}},\frac{d\gamma}{d\tilde{s}}\right).$$

Arguing as in Proposition 3.1 we obtain

$$\begin{aligned}(n-1)\int_{-\infty}^{+\infty}(\varphi_s)^2u^{-k}\,ds\geq{}&-2[1-(n-2)k]\int_{-\infty}^{+\infty}\varphi u^{-k-1}\varphi_s u_s\,ds\\&+[1+2k-k^2(n-2)]\int_{-\infty}^{+\infty}\varphi^2u^{-k-2}(u_s)^2\,ds.\end{aligned}$$

for all $\varphi \in V$, with $V := V_{-\infty,\infty}$, and $k \geq \frac{4}{n-3}$. Now let $k = \frac{3}{n-4}$ and $\varphi = u^{\frac{3}{2(n-4)}}\psi$, with $\psi \in V$. We have

$$
\begin{aligned}
\varphi^2 u^{-\frac{3}{n-4}} &= \psi^2, \\
\varphi_s &= \frac{3}{2(n-4)} u^{-\frac{2n-11}{2(n-4)}} u_s \psi + u^{\frac{3}{2(n-4)}} \psi_s, \\
(\varphi_s)^2 u^{-\frac{3}{n-4}} &= \frac{9}{4(n-4)^2} \psi^2 u^{-2} (u_s)^2 + (\psi_s)^2 + \frac{3}{n-4} \psi \psi_s u^{-1} u_s,
\end{aligned}
$$

and substituting in the previous relation and rearranging terms we get

$$
(n-1)\int_{-\infty}^{+\infty} (\psi_s)^2\, ds \geq \frac{4n^2-29n+25}{4(n-4)^2} \int_{-\infty}^{+\infty} \psi^2 u^{-2}(u_s)^2\, ds + \frac{n-1}{n-4}\int_{-\infty}^{+\infty} \psi\psi_s u^{-1} u_s\, ds. \tag{3.7}
$$

Now note that $\frac{4n^2-29n+25}{4(n-4)^2} > 0$ for every $n \geq 7$, while it is negative for $n \leq 6$. Then for $7 \leq n \leq 9$ using Young's inequality on the second term on the right-hand side of (3.7) we find

$$
\int_{-\infty}^{+\infty} \psi^2 u^{-2}(u_s)^2\, ds \leq C \int_{-\infty}^{+\infty} (\psi_s)^2\, ds. \tag{3.8}
$$

for every $\psi \in V$, for some constant $C > 0$. Now for any $R > 0$ let ψ be a smooth function on $\mathbb{R}$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in $[-R, R]$, $\psi \equiv 0$ in $[-2R, 2R]$ and $|\psi'| \leq \frac{C}{R}$ for some constant $C > 0$. From (3.8) we find that for any $R > 0$

$$
\int_{-R}^{R} u^{-2}(u_s)^2\, ds \leq \frac{C}{R} \tag{3.9}
$$

and passing to the limit as R tends to $+\infty$ we conclude that $u_s = 0$ on $\mathbb{R}$. Thus $u > 0$ is constant on γ, and hence (3.5) holds, with both ends of the $\widetilde{g}$-geodesic having infinite $\widetilde{g}$-length. We conclude that the metric $\widetilde{g}$ is complete. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Let (M^n, g), $n \geq 10$, be a complete critical metric of $\mathfrak{S}^2$ with $R \in L^q(M^n)$ for some $1 < q < q^* = \frac{7n-10}{2(n-4)}$. First of all, if M^n is compact, then integrating (1.2) over M^n we get $R \equiv 0$ on M^n. In case M^n is non-compact, from Corollary 2.2, either $R \equiv 0$ or $R < 0$ on M^n. In the latter case we consider the conformal metric

$$
\widetilde{g} = |R|^{\frac{6}{n-4}} g,
$$

which is complete by Proposition 3.1 and satisfies (2.4). In particular $(M^n, \widetilde{g})$ is a complete steady gradient Ricci soliton, if $n = 10$, or a steady quasi-Einstein manifold, if $n > 10$. In both cases, it is well known (see [2] and [9, Theorem 1.4]) that the scalar curvature of $\widetilde{g}$ must be nonnegative. By the formula for the conformal change, we obtain

$$\begin{aligned} 0 \le \widetilde{R} &= e^{-2w}\left(R - 2(n-1)\Delta w - (n-1)(n-2)|\nabla w|^2\right) \\ &= u^{-\frac{6}{n-4}}\left(-u - \frac{6(n-1)}{n-4}\frac{\Delta u}{u} + \frac{6(n-1)}{n-4}\frac{|\nabla u|^2}{u^2} - \frac{9(n-1)(n-2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) \\ &= u^{-\frac{6}{n-4}}\left(\frac{1}{2}u - \frac{3(n-1)(n+2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) \end{aligned} \tag{4.1}$$

where we used $w = \frac{3}{n-4}\log u$ as in the proof of Proposition 2.3, $R = -u$ and (2.2). Thus

$$|\nabla u|^2 \le \frac{(n-4)^2}{6(n-1)(n+2)}u^3. \tag{4.2}$$

Fixing $O \in M^n$, arguing as in [4, Corollary 5.7], from (4.2), we obtain

$$u(x) \ge \frac{c_1}{c_2 + d_g(x,O)^2} \tag{4.3}$$

for every $x \in M^n$ and some positive constants $c_i = c_i(n, u(O))$, i=1,2. Now the result follows as in the proof of [4, Theorem 1.5]. For the sake of completeness we include the proof.

Let η be a smooth cutoff function such that $\eta \equiv 1$ on $B_s(O)$, $\eta \equiv 0$ on $B^c_{2s}(O)$, $0 \le \eta \le 1$ on M^n and $|\nabla \eta| \le \frac{c}{s}$ for every $s \gg 1$ with $c > 0$ independent of s.

Then, using (2.2) and (4.2) we get

$$\begin{aligned} \frac{n-4}{4(n-1)}\int_M u^q\eta^2\,dV_g &= -\int_M \Delta u\, u^{q-2}\eta^2\,dV_g \\ &= (q-2)\int_M |\nabla u|^2 u^{q-3}\eta^2\,dV_g + 2\int_M u^{q-2}\langle\nabla u, \nabla\eta\rangle\eta\,dV_g \\ &\le \frac{(n-4)^2(q-2)}{6(n-1)(n+2)}\int_M u^q\eta^2\,dV_g + \frac{C}{s}\int_{B_{2s}(O)\setminus B_s(O)} u^{q-\frac{1}{2}}\,dV_g, \end{aligned}$$

for some $C > 0$. By (4.3)

$$\frac{n-4}{4(n-1)}\int_M u^q\eta^2\,dV_g \le \frac{(n-4)^2(q-2)}{6(n-1)(n+2)}\int_M u^q\eta^2\,dV_g + C\frac{(1+s^2)^{\frac{1}{2}}}{s}\int_{B^c_s(O)} u^q\,dV_g. \tag{4.4}$$

Thus, if $u \in L^q(M^n)$, we obtain

$$\frac{(n-4)^2}{6(n-1)(n+2)}\left[\frac{7n-10}{2(n-4)} - q\right]\int_M u^q\eta^2\,dV_g \le C\frac{(1+s^2)^{\frac{1}{2}}}{s}\int_{B^c_s(O)} u^q\,dV_g \longrightarrow 0 \quad \text{as} \quad s \to +\infty.$$

This yields $u \equiv 0$, if

$$1 < q < q^* = \frac{7n-10}{2(n-4)} = 2 + \frac{3(n+2)}{2(n-4)}.$$

which is a contradiction. This concludes the proof of Theorem 1.2. □

Remark 4.1. Note that the gradient estimate (4.2) improves the one in [4, Lemma 5.5], since one can see that, for every $n \geq 10$, the constant $\frac{(n-4)^2}{6(n-1)(n+2)}$ is always smaller than the corresponding constant appearing there. As a consequence, we see that the conclusion of Theorem 1.2 follows assuming $R \in L^q(M^n)$ with $1 < q < \frac{7n-10}{2(n-4)}$, thus improving, for $n \geq 10$, [4, Theorem 1.5] also in this respect.

5. Scalar curvature estimates for quasi-Einstein manifolds

In this section we will drop the notation $\widetilde{g}$ and use g instead. Let (M^n, g) be a complete Riemannian manifold satisfying

$$\mathrm{Ric}_f^\tau := \mathrm{Ric} + \nabla^2 f - \frac{1}{\tau} df \otimes df = \lambda g, \qquad \tau \in \mathbb{R} \setminus \{0\}, \lambda \geq 0. \tag{5.1}$$

We begin recalling the following well-known result (see e.g. [9]):

Lemma 5.1. *Let (M, g) satisfy equation (5.1) for some $f \in C^\infty(M)$, $\tau \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$; then there exists $\mu \in \mathbb{R}$ such that the following equations hold:*

$$\begin{aligned} &\frac{1}{2}\Delta R - \frac{\tau+2}{2\tau} g(\nabla f, \nabla R) = \\ &\qquad - \frac{\tau-1}{\tau}\left|\mathrm{Ric} - \frac{1}{n} R g\right|^2 - \frac{n+\tau-1}{n\tau}(R - n\lambda)\left(R - \frac{n(n-1)}{n+\tau-1}\lambda\right), \end{aligned} \tag{5.2}$$

$$R + \frac{\tau-1}{\tau}|\nabla f|^2 + (\tau - n)\lambda = \mu e^{\frac{2f}{\tau}} \tag{5.3}$$

$$\Delta_f f := \Delta f - |\nabla f|^2 = \tau\lambda - \mu e^{\frac{2f}{\tau}}. \tag{5.4}$$

In what follows we focus on the case $\lambda = 0$ (note that the next results can be extended also to the case $\lambda < 0$ as shown in [9]).

Lemma 5.2. *Let (M, g) satisfy equation (5.1) for some $f \in C^\infty(M)$, $\tau \in \mathbb{R} \setminus \{0\}$ and $\lambda = 0$. Let*

$$Q = \varphi\Big(R + 2\delta e^{\frac{2f}{\tau}}\Big),$$

with $\delta \in \mathbb{R}$ and φ a non-negative smooth cutoff function. Then, for all $\tau < 1 - n$, $\varepsilon > 0$ and for all $\delta > \max\left\{0, \frac{n\mu}{2(1-\tau-n)}\right\}$, at every point where $\varphi \neq 0$ we have

$$\begin{aligned} \frac{1}{2}\Delta_f Q \leq &\frac{\Delta_f \varphi}{2\varphi} Q + \frac{g(\nabla\varphi, \nabla Q)}{\varphi} - \frac{|\nabla\varphi|^2}{\varphi^2} Q + \frac{\varphi^2}{4\varepsilon|\tau|}\left|\frac{\nabla Q}{\varphi} - \frac{Q\nabla\varphi}{\varphi^2}\right|^2 \\ &+ \frac{4\delta(n+\tau-1)}{n\tau} Q e^{\frac{2f}{\tau}} - \frac{n+\tau-1}{n\tau\varphi} Q^2 + \varphi A + \frac{\varepsilon}{(\tau-1)\varphi} Q, \end{aligned} \tag{5.5}$$

where $A = \frac{n\tau\varepsilon^2(2\delta+\mu)^2}{(\tau-1)^2[16\delta^2(n+\tau-1)+8n\delta\mu]\varphi^2}$ and μ is as in Lemma 5.1.

Proof. The proof follows essentially as in [9, Lemma 3.2], with some minor modifications due to the fact that, in our case, $n+\tau-1<0$; in particular, if we let

$$G = R + 2\delta e^{\frac{2f}{\tau}},$$

using Lemma 5.1 we deduce

$$\frac{1}{2}\Delta_f G \leq \frac{\varphi}{4\varepsilon|\tau|}|\nabla G|^2 + \frac{4\delta(n+\tau-1)}{n\tau}Ge^{\frac{2f}{\tau}} - \frac{n+\tau-1}{n\tau}G^2 + \frac{\varepsilon}{(\tau-1)\varphi}G$$
$$- \frac{4\delta^2(n+\tau-1)+2\delta n\mu}{n\tau}e^{\frac{4f}{\tau}} - \frac{(2\delta+\mu)\varepsilon}{(\tau-1)\varphi}e^{\frac{2f}{\tau}}$$

for every $\varepsilon > 0$. Moreover we have

$$-\frac{4\delta^2(n+\tau-1)+2\delta n\mu}{n\tau}e^{\frac{4f}{\tau}} - \frac{(2\delta+\mu)\varepsilon}{(\tau-1)\varphi}e^{\frac{2f}{\tau}} \leq A,$$

provided that

$$\frac{4\delta^2(n+\tau-1)+2\delta n\mu}{n\tau} > 0,$$

i.e.

$$\delta > \max\left\{0, \frac{n\mu}{2(1-\tau-n)}\right\}.$$

Now, since $Q = \varphi G$, with a straightforward computation we can conclude. □

We are now in position to prove the following

Theorem 5.3. *Let (M^n, g), $n \geq 3$, be a complete Riemannian manifolds satisfying (5.1) with $\lambda = 0$. If $\tau < 1-n$, then the scalar curvature satisfies*

$$R + 2\delta e^{\frac{2}{\tau}f} \geq 0$$

for all $\delta > \max\left\{0, \frac{n\mu}{1-n-\tau}\right\}$, with μ as in Lemma 5.1.

Proof. Following the notation in [9, Theorem 5.2], we consider a smooth cutoff function $\theta(t)$ such that $\theta(t) = 1$ if $0 \leq t \leq 1$ and $\theta(t) = 0$ for $t \geq 2$ and

$$-10\theta^{\frac{1}{2}} \leq \theta' \leq 0, \qquad \theta'' \geq -10.$$

Fix ρ_0 positive and define the smooth, nonnegative, cutoff function $\varphi \in C^\infty(M)$ by

$$\varphi(x) = \theta\left(\frac{r(x)}{\rho_0}\right),$$

where $r(x)$ is the distance function from a fixed origin $o \in M$. Then

$$\nabla\varphi = \frac{\theta'\nabla r}{\rho_0}.$$

To control the f-Laplacian of φ, we resort to a recent comparison result [8, Theorem 2.2] for complete, weighted Riemannian manifolds with nonnegative τ-Bakry-Emery Ricci tensor,

Ric_f^τ, in the special case $\tau \leq 1-n$ (note that, in their notation, $\tau = m-n$, with $m \leq 1$): for every $x \in M \setminus (\{o\} \cup \mathrm{Cut}(o))$, where $\mathrm{Cut}(o)$ denotes the cut locus of o, we have

$$\Delta_f r(x) \leq \frac{|\tau|}{s(x)} e^{\frac{2f(x)}{\tau}},$$

where $s(x)$ is the re-parameterized distance from o defined as

$$s(x) := \inf\left\{ \int_0^{r(x)} e^{\frac{2f(\gamma(t))}{\tau}}\, dt : \quad \gamma \text{ is a unit-speed geodesic with } \gamma(0)=o,\, \gamma(r(x))=x \right\}.$$

In particular, for every $x \in (B_{2\rho_0}(o) \setminus B_{\rho_0}(o)) \setminus \mathrm{Cut}(o)$, we have

$$|\nabla\varphi|^2 \leq \frac{100}{\rho_0^2}\varphi$$

and

$$\begin{aligned} \Delta_f \varphi &= \frac{\theta'}{\rho_0}\Delta_f r + \frac{\theta''}{\rho_0^2} \\ &\geq -\frac{10}{\rho_0^2} - \frac{10|\tau|}{\rho_0 s} e^{\frac{2f}{\tau}}. \end{aligned} \tag{5.6}$$

Now let $Q = \varphi\Big(R + 2\delta e^{\frac{2f}{\tau}}\Big)$, and let x_0 be a minimum point of Q in $B_{2\rho_0}(o)$: if $Q(x_0) \geq 0$ the claim follows immediately. On the other hand, suppose that $Q(x_0) < 0$: then, at x_0, we have

$$\nabla Q = 0 \qquad \text{and} \qquad \Delta_f Q \geq 0;$$

using Lemma 5.2 we deduce, at x_0,

$$0 \leq \frac{\Delta_f\varphi}{2}Q - \frac{|\nabla\varphi|^2}{\varphi}Q + \frac{|\nabla\varphi|^2}{4\varepsilon|\tau|\varphi}Q^2 + \frac{4\delta(n+\tau-1)}{n\tau}Qe^{\frac{2f}{\tau}} - \frac{n+\tau-1}{n\tau}Q^2 + A + \frac{\varepsilon}{\tau-1}Q$$

for all $\varepsilon > 0$. If $x_0 \in B_{\rho_0}(o) \setminus \mathrm{Cut}(o)$, then, at x_0,

$$0 \leq -\frac{n+\tau-1}{n\tau}Q^2 + \frac{\varepsilon}{\tau-1}Q + A = -\frac{n+\tau-1}{n\tau}Q^2 + \frac{\varepsilon}{\tau-1}Q + C\varepsilon^2,$$

i.e.

$$Q(x_0) \geq -C\varepsilon \quad \forall\, \varepsilon > 0,$$

which is a contradiction. If $x_0 \in (B_{2\rho_0}(o) \setminus B_{\rho_0}(o)) \setminus \mathrm{Cut}(o)$,

$$0 \leq -\frac{105}{\rho^2}Q - \frac{10|\tau|}{\rho_0 s}Qe^{\frac{2f}{\tau}} + \frac{25}{\varepsilon|\tau|\rho_0^2}Q^2 + \frac{4\delta(n+\tau-1)}{n\tau}Qe^{\frac{2f}{\tau}} - \frac{n+\tau-1}{n\tau}Q^2 + \frac{\varepsilon}{\tau-1}Q + A.$$

Since $r(x_0) \geq \rho_0 > 1$, then, from the definition of s, there exists $k > 0$ such that $s(x_0) \geq k > 0$; we choose $\rho_0 > 1$ large enough such that

$$\frac{4\delta(n+\tau-1)}{n\tau} > \frac{10|\tau|}{\rho_0 k} \geq \frac{10|\tau|}{\rho_0 s} \quad \text{and} \quad \frac{25}{\varepsilon|\tau|\rho_0^2} \leq \frac{n+\tau-1}{2n\tau},$$

therefore, at x_0,

$$0 \leq -\frac{n+\tau-1}{2n\tau}Q^2 + \frac{\varepsilon}{\tau-1}Q + A = -\frac{n+\tau-1}{2n\tau}Q^2 + \frac{\varepsilon}{\tau-1}Q + C\varepsilon^2,$$

i.e.

$$Q(x_0) \geq -C\varepsilon \quad \forall\, \varepsilon > 0,$$

which is again a contradiction. Finally, if $x_0 \in \mathrm{Cut}(o)$, one has to argue using the well-known Calabi trick (see e.g. [3]). We deduce that $Q \geq 0$ on $B_{2\rho_0}(0)$, for any $\rho_0 > 1$ large enough. Hence the result follows and the proof is complete. □

Corollary 5.4. *Let (M^n, g), $n \geq 3$, be a complete Riemannian manifolds satisfying (5.1) with $\lambda = 0$. Let μ as in Lemma 5.1 satisfy $\mu \leq 0$. If $\tau < 1 - n$, then the scalar curvature satisfies $R \geq 0$.*

6. Proof of Theorem 1.3

Proof of Theorem 1.3. Let (M^n, g), $7 \leq n \leq 9$, be a complete critical metric of $\mathfrak{S}^2$ with a finite number N of ends, $N \geq 2$. Assume $R \in L^q(M^n)$ for some $1 < q < \infty$. Then, if M^n is compact, integrating (1.2) over M^n we get $R \equiv 0$ on M^n. In case M^n is non-compact, from Corollary 2.2, either $R \equiv 0$ or $R < 0$ on M^n. In the latter case we consider the conformal metric

$$\widetilde{g} = |R|^{\frac{6}{n-4}} g,$$

which is complete by Proposition 3.2 and satisfies (2.4), i.e. (5.1) with $\tau = \frac{4(n-1)}{n-10} < 1 - n$, $\lambda = 0$ and $f = \frac{2(n-1)}{n-4}\log|R|$. By Theorem 5.3 and by the formula for the conformal change of the scalar curvature, we obtain for every $\delta > \max\left\{0, \frac{n\mu}{1-n-\tau}\right\}$

$$\begin{aligned} 0 &\leq \widetilde{R} + 2\delta e^{\frac{2}{\tau}f} = e^{-2w}\left(R - 2(n-1)\Delta w - (n-1)(n-2)|\nabla w|^2\right) + 2\delta e^{\frac{2}{\tau}f} \\ &= u^{-\frac{6}{n-4}}\left(-u - \frac{6(n-1)}{n-4}\frac{\Delta u}{u} + \frac{6(n-1)}{n-4}\frac{|\nabla u|^2}{u^2} - \frac{9(n-1)(n-2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) + 2\delta u^{\frac{n-10}{n-4}} \\ &= u^{-\frac{6}{n-4}}\left(\frac{1}{2}u - \frac{3(n-1)(n+2)}{(n-4)^2}\frac{|\nabla u|^2}{u^2}\right) + 2\delta u^{\frac{n-10}{n-4}} \end{aligned} \tag{6.1}$$

where we used $w = \frac{3}{n-4}\log u$ as in the proof of Proposition 2.3, $R = -u$ and (2.2). Thus

$$|\nabla u|^2 \leq \frac{(n-4)^2(1+4\delta)}{6(n-1)(n+2)}u^3. \tag{6.2}$$

Now we can conclude exactly as in the proof of Theorem 1.2, using the gradient estimate (6.2) in place of (4.2). Thus we find that $u \equiv 0$, if $u \in L^q(M^n)$ with

$$1 < q < q^* = 2 + \frac{3(n+2)}{2(n-4)(1+4\delta)}$$

which is a contradiction. This concludes the proof of Theorem 1.3. □

Remark 6.1. We explicitly note that the conclusion of Theorem 1.3 follows just assuming $R \in L^q(M^n)$ for some $q \in (1, q^*)$, with $q^* = 2 + \frac{3(n+2)}{2(n-4)(1+4\delta)} > 2$, $\delta > \max\left\{0, \frac{n\mu}{1-n-\tau}\right\}$ and μ as in Lemma 5.1.

References

[1] M.T. Anderson, *On Stationary Vacuum Solutions to the Einstein Equations*, Ann. Henri Poincaré 1 (2000), 977–994.

[2] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differ. Geom. 82 (2009), 363–382.

[3] G. Catino, *Critical metric of the L^2–norm of the scalar curvature*, Proc. Amer. Math. Soc. 142 (2014), 3981–3986.

[4] G. Catino, P. Mastrolia and D.D. Monticelli, *Rigidity of critical metrics for quadratic curvature functionals*, J. Math. Pures Appl., 2023.

[5] G. Catino, P. Mastrolia, A. Roncoroni, *Two rigidity results for stable minimal hypersurfaces*, https://arxiv.org/abs/2209.10500, 2022.

[6] D. Fischer-Colbrie, *On complete minimal surfaces with finite Morse index in three-manifolds*, Invent. Math. 82, (1985), 121–132.

[7] M. F. Elbert, B. Nelli, H. Rosenberg, *Stable constant mean curvature hypersurfaces*, Proc. Am. Math. Soc. 135 n. 10 (2007), 3359–3366.

[8] K. Kuwae, X.-D. Li, *New Laplacian comparison theorem and its applications to diffusion processes on Riemannian manifolds*, Bull. London Math. Soc. 54 (2022), 404–427.

[9] L. W. Wang, On non-compact τ-quasi-Einstein metrics, Pac. Journal Math. 254 (2011), 449–464.

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