Vanishing viscosity limit for aggregation-diffusion equations

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Abstract

This article is devoted to the convergence analysis of the diffusive approximation of the measure-valued solutions to the so-called aggregation equation, which is now widely used to model collective motion of a population directed by an interaction potential. We prove, over the whole space in any dimension, a uniform-in-time convergence in Wasserstein distance, in the general framework of Lipschitz potentials, and provide a $O(\sqrt{\varepsilon})$ rate, where ε is the diffusion parameter, when the potential is $\lambda-$convex. We give an extension to some repulsive potentials and prove sharp convergence rates of the steady states towards the Dirac mass, under some uniform attractiveness assumptions.

1 Introduction

This paper addresses the vanishing viscosity limit $\varepsilon \to 0$ for the following aggregation-diffusion problem on the whole space $\mathbb{R}^d$, in any dimension d (probably all the analysis could be performed on a bounded domain with homogeneous Neumann boundary condition):

$$\partial_t \rho^\varepsilon + \nabla \cdot (a[\rho^\varepsilon]\rho^\varepsilon) = \varepsilon \Delta \rho^\varepsilon, \tag{1.1a}$$

$$a[\rho^\varepsilon] = -\nabla W * \rho^\varepsilon, \tag{1.1b}$$

$$\rho^\varepsilon(0, \cdot) = \rho_0^\varepsilon, \tag{1.1c}$$

where $\varepsilon > 0$, $W : \mathbb{R}^d \to \mathbb{R}$ is a given interaction potential and the sequence of initial data $(\rho_0^\varepsilon)_{\varepsilon>0}$ belongs to $\mathcal{P}_2(\mathbb{R}^d)$ the set of probability measures with finite second order moment, and converges as ε goes to 0 towards a given $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$.

Equation (1.1a)–(1.1b) is often used in population dynamics to describe the collective motion of a population subject to Brownian diffusion and interacting through the interaction potential W. The term $\nabla W * \rho^\varepsilon(x)$ models the combined contribution of the interaction of a particle located at point x with particles at all other points. These equations appear in several applications arising from physics and biology to model, for instance, swarming, chemotaxis, crowd motion, bird flocks, or fish schools, see, e.g., [27, 6, 39, 38, 15, 19]. The potential W depends on the model we consider. For example, the celebrated parabolic-elliptic Patlak-Keller-Segel model [23, 24] for chemotaxis with an adequate set of parameters corresponds to the aggregation-diffusion equation in dimension $d = 2$ for the logarithmic potential $W(x) = \frac{1}{2\pi} \ln(|x|)$.

In this work, we assume that the interaction potential W satisfies the following properties:

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(A0) For all $x \in \mathbb{R}^d$, $W(x) = W(-x)$ and $W(0) = 0$,

(A1) $W \in \mathcal{C}^1(\mathbb{R}^d \backslash \{0\})$,

(A2) W is a_∞-Lipschitz continuous, for some constant $a_\infty \geqslant 0$.

In addition, some of our results only hold under one of the following supplementary assumptions:

(A3) W is $\lambda-$convex for some $\lambda \leqslant 0$, that is, $x \longmapsto W(x) - \frac{\lambda}{2}|x|^2$ is convex,

(A4$-p$) There exists a constant $C > 0$ such that, for all $x \in \mathbb{R}^d$, $\nabla W(x) \cdot x \geqslant C|x|^p$,

where $p \geqslant 1$. Potentials satisfying assumptions (A0)-(A1)-(A2)-(A3) but not differentiable at the origin are often referred to as pointy [11, 13, 26].

Remark 1.1. Note that assumption (A2) is incompatible with assumption (A4$-p$) whenever $p > 1$. This is the reason why we only consider $\lambda \leqslant 0$ in (A2), since (A2) with $\lambda > 0$ implies (A4-2). Still, when studying well-posedness of inviscid aggregation equations, the case $\lambda > 0$ can be tackled considering compactly supported data for, in that case, the support decreases in time (see [13] Theorem 2.1 and [10] Remark 2.14). When $\varepsilon > 0$, it is not clear however that we can reproduce this argument.

When the potential is pointy, finite time blowup of weak solutions occurs [2, 3] for the inviscid problem:

$$\partial_t \rho + \nabla \cdot (a[\rho]\rho) = 0, \tag{1.2a}$$

$$a[\rho] = -\nabla W * \rho, \tag{1.2b}$$

$$\rho(0, \cdot) = \rho^{ini}, \tag{1.2c}$$

After blowup time, the solutions being possibly singular measures, the product $a[\rho]\rho$ is no longer well-defined. For $\lambda-$convex potentials, the continuation of weak solutions valued in $\mathcal{P}_2(\mathbb{R}^d)$ has therefore been studied through three different approaches: gradient flow solutions in the Wasserstein space [10], duality solutions *à la* Bouchut-James [20, 19] in one dimension of space and Filippov solutions [11, 26]. These notions of solutions turn out to be equivalent to that of solutions in the sense of distributions provided the velocity field $a[\rho]$ is replaced by:

$$\widehat{a}[\rho](x) = -\int_{y \neq x} \nabla W(x - y)\rho(dy). \tag{1.3}$$

Our objective in this paper is to study the convergence of the viscous solutions $(\rho^\varepsilon)_{\varepsilon>0}$ towards such a weak measure solution to (1.2). When W is $\lambda-$convex, these asymptotics had previously been mentioned in [8], where the authors explain how to use the techniques for the $\Gamma-$convergence of gradient flows developed by Serfaty in [33]. Our method basically relies on the same arguments which actually do not require the $\lambda-$convexity of the potential but only its Lipschitz continuity – along with the standard assumptions (A0)-(A1). Starting from the so-called Energy Dissipation Equality (EDE) for the viscous problem 1.1, we prove lower bounds of lower semicontinuity-type on each term of the EDE. This amounts to verifying the assumptions of Theorem 2 in [33]; if, in addition, the initial data is well-prepared, then we meet all the hypotheses of this theorem. However, we deliberately pass to the limit *by hand*, so as not to invoke abstract gradient flow arguments. Therefore, our proof is self-contained for the reader with minimal background regarding optimal transport. In particular, in our Theorem 3.1 we recover, at the limit $\varepsilon \to 0$, the right definition of the velocity field for (1.2) as defined in (1.3).

We generalize this result in Corollary 3.4 to arbitrary $\mathcal{P}_2(\mathbb{R}^d)$ initial data converging in Wasserstein distance towards the initial datum ρ^{ini} of the inviscid problem, when W is, in addition, $\lambda-$convex. This is done by smoothing out the initial data and estimating the distance to the modified solutions at time t, which is possible since the interaction energy is $\lambda-$geodesically convex. We then provide a convergence rate based on the differentiation formula of the Wasserstein distance between two absolutely continous curves on the Wasserstein space. Note that, for the Newtonian potential, the vanishing viscosity limit had been established in [12] in dimension $d \geqslant 2$ and up to the time of existence of weak solutions in $L^1 \cap L^\infty$ but, to the best of our knowledge, without convergence rates.

This article is structured as follows. We recall in Section 2 some useful results and definitions regarding optimal transport and functionals defined over the Wasserstein spaces.

In Section 3, in the framework of Lipschitz potentials, we begin with the general convergence result of the diffusive solutions $(\rho^\varepsilon)_{\varepsilon>0}$ towards a solution ρ to the inviscid problem (1.2) for well-prepared initial data. We then relax some of our assumptions on the initial data and focus on $\lambda-$convex potentials, for which we prove that convergence still holds for arbitrary initial data $(\rho_0^\varepsilon)_{\varepsilon>0}$ converging towards ρ^{ini}. We then prove that convergence occurs at rate $O(\sqrt{\varepsilon})$ in Wasserstein distance. We give, in addition, an alternate proof based on the convergence estimates of an upwind-type scheme for the inviscid problem due to the first author with Delarue and Vauchelet [14, 13].

In Section 4, we show that convergence (without convergence rate) still holds, up to an extraction, for repulsive potentials that behave like $W(x) = -|x|$. The idea is to estimate, as in the $\lambda-$convex case, the distance between solutions associated with smoothed out initial data and solutions associated with a fixed initial datum ρ^{ini}. This is done by differentiating the Wasserstein distance between solutions and proving appropriate estimates on the aggregation velocity field using an additional integrability assumption on $\nabla^2 W$.

Section 5 is devoted to the study of the stationary problem and, in particular, we provide higher convergence rates for the viscous steady states towards the unique steady state of the aggregation equation, that is, up to translations, the Dirac mass, when the interaction potential satisfies the key assumption (A4-1) but is not necessarily Lipschitz continuous. Under assumption (A4$-p$) for an arbitrary $p \geqslant 1$, estimates are also obtained and proved to be sharp for $p = 2$. We eventually illustrate our convergence results in Section 6 and observe all the proven convergence rates.

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2 Preliminaries

2.1 Notations

We denote by $\mathcal{C}(\mathbb{R}^d)$ the space of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$, and by $\mathcal{C}_0(\mathbb{R}^d)$ (resp. $\mathcal{C}_b(\mathbb{R}^d)$, $\mathcal{C}_c(\mathbb{R}^d)$) the subspace of continuous functions vanishing at ∞ (resp. of bounded continuous functions, of continuous and compactly supported functions). We also denote by $\mathcal{M}_b(\mathbb{R}^d)$ the space of Borel signed measures with finite total variation, equipped with the weak topology $\sigma(\mathcal{M}_b(\mathbb{R}^d), \mathcal{C}_0(\mathbb{R}^d))$. For a sequence $(\rho_n)_{n\in\mathbb{N}} \in \mathcal{M}_b(\mathbb{R}^d)^\mathbb{N}$ and $\rho \in \mathcal{M}_b(\mathbb{R}^d)$, we denote the weak convergence of $(\rho_n)_{n\in\mathbb{N}}$ towards ρ by $\rho_n \underset{n\to\infty}{\overset{*}{\rightharpoonup}} \rho$.

For $\rho \in \mathcal{M}_b(\mathbb{R}^d)$ and $r \in [0, +\infty)$, we also denote by $M_r(\rho)$ the $r-$th moment of ρ, given by $M_r(\rho) = \int_{\mathbb{R}^d} |x|^r \rho(dx)$, where $|\cdot|$ is the Euclidean norm. For $\rho \in \mathcal{M}_b(\mathbb{R}^d)$ and Z a measurable map, we denote by $Z_\#\rho$ the pushforward measure of ρ by Z, which satisfies, for any $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$,

$$\int \varphi(x)\, Z_\#\rho(dx) = \int \varphi(Z(x))\, \rho(dx).$$

Note that, in the above equality as in the whole article, whenever the integration domain is not specified, the integrals are considered over the whole space (which is $\mathbb{R}^d$ here). If $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ is a positive measure, we also note $\rho \ll \mu$ whenever ρ is absolutely continuous with respect to μ.

We call $\mathcal{P}(\mathbb{R}^d)$ the subset of $\mathcal{M}_b(\mathbb{R}^d)$ of probability measures and we denote, for $p \in [1, +\infty)$, $\mathcal{P}_p(\mathbb{R}^d) := \{\rho \in \mathcal{P}(\mathbb{R}^d),\ M_p(\rho) < +\infty\}$. For $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we define the Wasserstein distance of order p between μ and ν by (see [1, 32, 40]):

$$W_p(\mu, \nu) := \inf_{\gamma \in \Gamma(\mu,\nu)} \left\{ \iint |x - y|^p \, \gamma(dx, dy) \right\}^{1/p} \tag{2.1}$$

where $\Gamma(\mu, \nu)$ is the set of measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν, i.e.

$$\Gamma(\mu, \nu) = \Big\{ \gamma \in \mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d);\ \forall\, \xi \in \mathcal{C}_0(\mathbb{R}^d),\ \int \xi(x)\gamma(dx, dy) = \int \xi(x)\mu(dx), \\ \int \xi(y)\gamma(dx, dy) = \int \xi(y)\nu(dy) \Big\}.$$

Any measure that realizes the minimum in the definition (2.1) of W_p is called an optimal plan, and the set of optimal plans is denoted by $\Gamma_0(\mu, \nu)$. The space $\mathcal{P}_p(\mathbb{R}^d)$ equipped with the distance W_p is called Wasserstein space of order p and denoted $\mathbb{W}_p(\mathbb{R}^d)$.

We recall that the Wasserstein distance W_p metrizes the weak convergence of measures in the sense that, for $(\rho_n)_{n\in\mathbb{N}} \in \mathcal{P}_p(\mathbb{R}^d)^{\mathbb{N}}$ and $\rho \in \mathcal{P}_p(\mathbb{R}^d)$, $W_p(\rho_n, \rho) \underset{n\to+\infty}{\longrightarrow} 0$ if and only if $\rho_n \underset{n\to+\infty}{\overset{*}{\rightharpoonup}} \rho$ and $M_p(\rho_n) \underset{n\to+\infty}{\longrightarrow} M_p(\rho)$ (see [40], Theorem 7.12).

We shall also denote the conjugate exponent of p by $p' \in [1, +\infty]$ defined by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, with the usual convention $1' = +\infty$ and $\infty' = 1$. For $\alpha \in \mathbb{R}$, the positive and negative part of α are denoted by $\alpha^+ := \max(0, \alpha)$ and $\alpha^- := \max(0, -\alpha)$. With that convention, both α^+ and α^- are always nonnegative.

Throughout this paper, we will use the same notation C to denote any positive constant.

2.2 Curves and functionals over the Wasserstein space

Let $p \in [1, +\infty)$ and $T > 0$. We call curve on the metric space $\mathbb{W}_p(\mathbb{R}^d)$ any continuous function $\rho \in \mathcal{C}([0, T], \mathbb{W}_p(\mathbb{R}^d))$. We say that ρ is an absolutely continuous curve if there exists $b \in L^1([0, T])$ such that $W_p(\rho_s, \rho_t) \leqslant \int_s^t b(\tau)d\tau$ for every $0 \leqslant s < t \leqslant T$, and we denote $AC([0, T], \mathbb{W}_p(\mathbb{R}^d))$ the set of absolutely continuous curves on $\mathbb{W}_p(\mathbb{R}^d)$. We also define for $t \in [0, T]$, the metric derivative of ρ at time t as:

$$|\rho_t'| := \lim_{h\to 0} \frac{W_p(\rho_{t+h}, \rho_t)}{h}. \tag{2.2}$$

If ρ is a Lipschitz curve on $\mathbb{W}_p(\mathbb{R}^d)$, then the above limit exists for a.e. $t \in [0, T]$. Now, up to a reparametrization in time, any absolutely continuous curve can become Lipschitz continuous and therefore admits a metric derivative for almost every time.

The fundamental property of absolutely continuous curves in $\mathbb{W}_p(\mathbb{R}^d)$ is the link with a continuity equation:

Theorem 2.1 ([1], Theorem 8.3.1)**.** *Let $p \in (1, +\infty)$ and $T > 0$. Let $\rho \in AC([0, T], \mathbb{W}_p(\mathbb{R}^d))$. Then, for a.e. $t \in [0, T]$ there exists a vector field $v_t \in L^p(\rho_t, \mathbb{R}^d)$ such that:*

- *the continuity equation $\partial_t \rho + \nabla \cdot \rho v = 0$ is satisfied in the sense of distributions*
- *for a.e. $t \in [0, T]$, $\|v_t\|_{L^p(\rho_t)} \leqslant |\rho_t'|$.*

Conversely, if we take a curve $\rho \in \mathcal{C}([0,T],\mathbb{W}_p(\mathbb{R}^d))$ *such that, for each* $t \in [0,T]$*, there exists a vector field* $v_t \in L^p(\rho_t,\mathbb{R}^d)$ *with* $\int_0^T \|v_t\|_{L^p(\rho_t)}dt < +\infty$ *solving the continuity equation* $\partial_t\rho + \nabla\cdot\rho v = 0$*, then* $\rho \in AC([0,T],\mathbb{W}_p(\mathbb{R}^d))$ *and for a.e.* $t \in [0,T]$*, we have* $|\rho'_t| \leqslant \|v_t\|_{L^p(\rho_t)}$*.*

As a consequence, the velocity field v *introduced in the first part of the statement actually satisfies* $\|v_t\|_{L^p(\rho_t)} = |\rho'_t|$ *for a.e.* $t \in [0,T]$*.*

We now recall the definition of the first variation of a functional defined over probability measures.

Definition 2.2. *Let* $F : \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R} \cup \{+\infty\}$*. Assume that* $\rho \in \mathcal{P}(\mathbb{R}^d)$ *is such that:*

$$\forall\delta \in [0,1],\ \forall\mu \in \mathcal{P}(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d), \quad F((1-\delta)\rho + \delta\mu) < +\infty,$$

then we call first variation of F *at* ρ*, denoted* $\frac{\delta F}{\delta\rho}(\rho)$*, any measurable function* g *such that:*

$$\frac{dF(\rho+\delta\chi)}{d\delta}\Big|_{\delta=0} = \int g d\chi,$$

whenever $\chi = \mu - \rho$ *for some* $\mu \in \mathcal{P}(\mathbb{R}^d) \cap L^\infty_c(\mathbb{R}^d)$*, where* $L^\infty_c(\mathbb{R}^d)$ *denotes the set of compactly supported functions in* $L^\infty(\mathbb{R}^d)$*. If it exists, the first variation is defined up to an additive constant.*

We now introduce two functionals that are essential to our study, the interaction energy $\mathcal{W}$ and the entropy $\mathcal{U}$, defined on $\mathcal{P}(\mathbb{R}^d)$ by:

$$\mathcal{W}(\rho) = \frac{1}{2}\iint W(x-y)\rho(dx)\rho(dy), \tag{2.3}$$

$$\mathcal{U}(\rho) = \begin{cases} \int \rho\ln(\rho), & \text{if } \rho \ll \text{Leb} \\ +\infty & \text{otherwise,} \end{cases} \tag{2.4}$$

where Leb is the Lebesgue measure on $\mathbb{R}^d$. Note that, under assumption (A2), the interaction energy $\mathcal{W}(\rho)$ is finite whenever $\rho \in \mathcal{P}_1(\mathbb{R}^d)$. For $\varepsilon \geqslant 0$, we shall also define the energy functional as $F^\varepsilon = \mathcal{W} + \varepsilon\mathcal{U}$. One can easily show that $\frac{\delta\mathcal{W}}{\delta\rho}(\rho) = W * \rho$ and $\frac{\delta\mathcal{U}}{\delta\rho}(\rho) = \ln\rho + 1$.

A key point in our proofs will be the lower semicontinuity (l.s.c) of the above functionals so that minimization arguments apply.

Lemma 2.3.

(1) *If* W *is l.s.c on* $\mathbb{R}^d$ *and bounded from below, then the interaction energy* $\mathcal{W}$ *is l.s.c for the weak convergence.*

(2) *If* W *is Lipschitz continuous, then the interaction energy* $\mathcal{W}$ *is Lipschitz continuous for the* W_1 *distance.*

Proof. Let us recall from [32], Proposition 7.2. that if $V : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is l.s.c and bounded from below, then the functional $\rho \in \mathcal{P}(\mathbb{R}^d) \longmapsto \iint V(x,y)\rho(dx)\rho(dy)$ is l.s.c for the weak convergence of measures. This proves the first claim.

For the second claim, we will prove

$$|\mathcal{W}(\rho) - \mathcal{W}\mu)| \leqslant \text{Lip}(W)W_1(\rho,\mu).$$

Indeed, we can write $\mathcal{W}(\rho) = \frac{1}{2}\int(W*\rho)d\rho$, so that we have

$$\mathcal{W}(\rho) - \mathcal{W}(\mu) = \frac{1}{2}\int(W*\rho)d(\rho-\mu) + \frac{1}{2}\int(W*(\mu-\rho))d\mu.$$

We then use

$$\Big|\int (W*\rho)d(\rho-\mu)\Big| \leqslant \mathrm{Lip}(W*\rho)W_1(\rho,\mu)$$

together with $\mathrm{Lip}(W*\rho) \leqslant \mathrm{Lip}(W)$, and

$$|(W*(\mu-\rho))(x)| = \Big|\int W(x-y)(\rho-\mu)(dy)\Big| \leqslant \mathrm{Lip}(W(x-\cdot))W_1(\rho,\mu)$$

together with $\mathrm{Lip}(W(x-\cdot)) = \mathrm{Lip}(W)$.

□

The following lemma is proven in [31], Proposition 2.1.

Lemma 2.4. *There exists a constant C only depending on d such that the entropy functional $\mathcal{U}$ satisfies $\mathcal{U}(\rho) \geqslant -C(M_1(\rho)^{1/2}+1)$. Moreover, if $(\rho_n)_n \in \mathcal{P}(\mathbb{R}^d)$ is a sequence weakly converging towards some $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that $M_1(\rho_n)$ is bounded, then we have $\mathcal{U}(\rho) \leqslant \liminf_{n\to+\infty} \mathcal{U}(\rho_n)$.*

In particular, this means that the entropy is l.s.c for the W_q distance for all $q \geqslant 1$.

In order to obtain convergence of the moments of a weakly converging sequence of probability measures, we will often make use of the following lemma:

Lemma 2.5. *Let $1 \leqslant p < +\infty$ and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathcal{P}_p(\mathbb{R}^d)$ weakly converging towards $\rho \in \mathcal{P}_p(\mathbb{R}^d)$ as $n \to +\infty$. Assume that, for some constant $C > 0$, we have for all $n \in \mathbb{N}$, $M_p(\rho_n) \leqslant C$. Then, for all $q \in (0,p)$, $M_q(\rho_n) \underset{n\to+\infty}{\longrightarrow} M_q(\rho)$. In particular, if $p > 1$ then $(\rho_n)_{n\in\mathbb{N}}$ converges towards ρ in W_q distance for all $q \in [1,p)$.*

Proof. For $R > 0$, we introduce a nonnegative cut-off function $\eta_R \in \mathcal{C}_c(\mathbb{R}^d,\mathbb{R})$ equal to 1 on $B(0,R)$. We write:

$$\int_{\mathbb{R}^d} |x|^q \rho_n(dx) = \int_{\mathbb{R}^d} |x|^q \eta_R(x)\rho_n(dx) + \int_{\mathbb{R}^d\backslash B(0,R)} |x|^q(1-\eta_R(x))\rho_n(dx), \tag{2.5}$$

$$\leqslant \int_{\mathbb{R}^d} |x|^q \eta_R(x)\rho_n(dx) + \int_{\mathbb{R}^d\backslash B(0,R)} |x|^q \rho_n(dx). \tag{2.6}$$

Firstly, $x \longmapsto |x|^q\eta_R(x) \in \mathcal{C}_0(\mathbb{R}^d)$, therefore the weak convergence $\rho_n \underset{n\to+\infty}{\overset{*}{\rightharpoonup}} \rho$ ensures that the first term in the above inequality converges to $\int_{\mathbb{R}^d} |x|^q\eta_R(x)\rho(dx)$ as $n \to +\infty$. Now, $M_q(\rho) < +\infty$ since $M_p(\rho) < +\infty$, hence, using Lebesgue's dominated convergence theorem, we have $\int_{\mathbb{R}^d} |x|^q\eta_R(x)\rho(dx) \underset{R\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} |x|^q\rho(dx)$.

Besides, the uniform bound on the $p-$moment of ρ_n ensures that the second term converges to 0 as $R \to +\infty$ uniformly with respect to n. Indeed, using a Hölder inequality with the exponents $(\frac{p}{q}, \frac{p}{p-q})$, we have:

$$\int_{\mathbb{R}^d\backslash B(0,R)} |x|^q\rho_n(dx) \leqslant M_p(\rho_n)^{q/p}\rho_n(\mathbb{R}^d\backslash B(0,R))^{\frac{p-q}{p}} \leqslant C^{q/p}\rho_n(\mathbb{R}^d\backslash B(0,R))^{\frac{p-q}{p}}.$$

Moreover, one has $R^p\rho_n(\mathbb{R}^d\backslash B(0,R)) \leqslant \int_{\mathbb{R}^d\backslash B(0,R)} |x|^p\rho_n(dx) \leqslant M_p(\rho_n) \leqslant C$. Combining these inequalities and plugging it into (2.5) then gives:

$$\int_{\mathbb{R}^d} |x|^q\rho_n(dx) \leqslant \int_{\mathbb{R}^d} |x|^q\eta_R(x)\rho_n(dx) + \frac{C}{R^{p-q}}.$$

Passing to the $\lim_{R\to+\infty}\limsup_{n\to+\infty}$, we obtain:

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^d}|x|^q\rho_n(dx)\leqslant\int_{\mathbb{R}^d}|x|^q\rho(dx).$$

Since $x\longmapsto|x|^q$ is l.s.c and bounded from below, the functional $\rho\longmapsto M_q(\rho)$ is l.s.c for the weak convergence. Hence, $\int_{\mathbb{R}^d}|x|^q\rho(dx)\leqslant\liminf_{n\to+\infty}\int_{\mathbb{R}^d}|x|^q\rho_n(dx)$, which concludes the proof. □

We also have, as a corollary of the previous lemma, a compactness result:

Lemma 2.6. *Let $1\leqslant p<+\infty$ and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures in $\mathcal{P}_p(\mathbb{R}^d)$ such that $M_p(\rho_n)$ is uniformly bounded with respect to n. Then, there exist a subsequence of $(\rho_n)_{n\in\mathbb{N}}$ converging towards some $\rho\in\mathcal{P}_p(\mathbb{R}^d)$ in W_q distance for all $q\in[1,p)$.*

Proof. As a sequence of probability measures, $(\rho_n)_{n\in\mathbb{N}}$ converges weakly towards some $\rho\in\mathcal{M}_b(\mathbb{R}^d)$. Now, the uniform bound on $M_p(\rho_n)$ implies tightness on $(\rho_n)_{n\in\mathbb{N}}$, hence it converges narrowly towards ρ and therefore ρ is a probability measure. We can then use the l.s.c of the $p-$th order moment to deduce that $M_p(\rho)<+\infty$. Lemma 2.5 finally gives convergence of $(\rho_n)_{n\in\mathbb{N}}$ towards ρ in $\mathbb{W}_q(\mathbb{R}^d)$ for all $q\in[1,p)$. □

We finally define one last functional that will be useful in our proofs. Let $p\in(1,+\infty)$. We set $K_p=\left\{(a,b)\in\mathbb{R}\times\mathbb{R}^d\mid a+\frac{1}{p'}|b|^{p'}\leqslant 0\right\}$ and, for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$,

$$f_p(t,x)=\begin{cases}\frac{1}{p}\frac{|x|^p}{t^{p-1}}, & \text{if } t>0,\\ 0, & \text{if } t=0,\ x=0,\\ +\infty, & \text{if } t=0,\ x\neq 0.\end{cases}$$

Then, for X a measurable space and for $(\rho,E)\in\mathcal{M}_b(X)\times\mathcal{M}_b(X)^d$, we define the $p-$Benamou-Brenier functional by:

$$\mathcal{B}_p(\rho,E)=\sup\left\{\int a d\rho+\int b\cdot dE\,;\,(a,b)\in\mathcal{C}_b(X,K_p)\right\}.$$

The Benamou-Brenier functional satisfies the following properties (see [32], Proposition 5.18):

Lemma 2.7.

(i) $\mathcal{B}_p$ is convex and l.s.c on $\mathcal{P}(X)\times\mathcal{M}(X)^d$ for the weak convergence,

(ii) If ρ and E are absolutely continuous with respect to a positive measure μ, then $\mathcal{B}_p(\rho,E)=\int f_p(\rho,E)d\mu$,

(iii) $\mathcal{B}_p(\rho,E)<+\infty$ only if $E\ll\rho$,

(iv) In that case, if we denote by v the density of E with respect to ρ, that is $E=\rho v$, then $\mathcal{B}_p(\rho,E)=\displaystyle\int\frac{|v|^p}{p}d\rho$.

We also have the following symmetrization lemma, which we will repeatedly use for $V=\nabla W$:

Lemma 2.8. *Let V be a bounded odd vector field on $\mathbb{R}^d$, $\rho\in\mathcal{P}(\mathbb{R}^d)$ and v a vector field on $\mathbb{R}^d$ such that $v\cdot(V*\rho)$ is integrable with respect to ρ. Then, one has:*

$$\int v(x)\cdot(V*\rho)(x)\rho(dx)=\frac{1}{2}\iint V(x-y)\cdot(v(x)-v(y))\rho(dx)\rho(dy).$$

Proof. Using the fact that V is odd, we can write thanks to the change of variables $x \leftrightarrow y$:

$$\iint V(x-y)\cdot v(x)\rho(dx)\rho(dy) = -\iint V(x-y)\cdot v(y)\rho(dx)\rho(dy).$$

Therefore, taking the half sum of the two quantities above:

$$\begin{aligned}\int v(x)\cdot (V*\rho)(x)\rho(dx) &= \iint V(x-y)\cdot v(x)\rho(dx)\rho(dy)\\ &= \frac{1}{2}\left(\iint V(x-y)\cdot v(x)\rho(dx)\rho(dy) - \iint V(x-y)\cdot v(y)\rho(dx)\rho(dy)\right)\\ &= \frac{1}{2}\iint V(x-y)\cdot (v(x)-v(y))\rho(dx)\rho(dy).\end{aligned}$$

□

We finish with a computation of the derivative of $\mathcal{W}$ along a curve satisfying a continuity equation:

Lemma 2.9. *Let ρ be a curve on $\mathcal{P}(\mathbb{R}^d)$ that solves in the weak sense $\partial_t\rho+\nabla\cdot\rho v = 0$ with $v_t \in L^2(\rho_t)$ for a.e. $t\in[0,T]$ and $\int_0^T \|v_t\|^2_{L^2(\rho_t)}dt < +\infty$. Then:*

$$\forall t\in[0,T],\quad \mathcal{W}(\rho_t)-\mathcal{W}(\rho_0) = \int_0^t\int (\widehat{\nabla W}*\rho_s)\cdot v_s d\rho_s. \tag{2.7}$$

Proof. Let $(W^\delta)_{\delta>0}$ be an approximation of W such that $W^\delta \in \mathcal{C}^1(\mathbb{R}^d)$, $W^\delta \underset{\delta\to 0}{\longrightarrow} W$ uniformly on $\mathbb{R}^d$, W^δ is even, ∇W^δ is bounded by a_∞, and $\nabla W^\delta \underset{\delta\to 0}{\longrightarrow} \nabla W$ pointwise on $\mathbb{R}^d\backslash\{0\}$.

We necessarily have $\nabla W^\delta(0) = 0$ for all $\delta>0$ and therefore $\nabla W^\delta \underset{\delta\to 0}{\longrightarrow} \widehat{\nabla W}$ pointwise on $\mathbb{R}^d$. On the other hand, for $\delta>0$, since $W^\delta\in\mathcal{C}^1(\mathbb{R}^d)$ and W^δ is even, we have, for $t\in[0,T]$:

$$\frac{1}{2}\iint W^\delta(x-y)\rho_t(dx)\rho_t(dy) - \frac{1}{2}\iint W^\delta(x-y)\rho_0(dx)\rho_0(dy) = \int_0^t\iint \nabla W^\delta(x-y)\cdot v_s(x)\rho_s(dx)\rho_s(dy)ds. \tag{2.8}$$

Now, we can bound the integrand on the right-hand side writing $|\nabla W^\delta(x-y)\cdot v_s(x)| \leqslant a_\infty |v_s|$. Noting that we have

$$\int_0^t\iint |v_s(x)|\rho_s(dx)\rho_s(dy)ds = \int_0^t \|v_s\|_{L^1(\rho_s)}ds \leqslant \sqrt{T}\Big(\int_0^T \|v_s\|^2_{L^2(\rho_s)}ds\Big)^{1/2} < +\infty,$$

we can then use Lebesgue's dominated convergence theorem w.r.t $\rho_s(dx)\rho_s(dy)ds$ to get that the right-hand side in equation (2.8) converges to $\int_0^t\iint \widehat{\nabla W}(x-y)\cdot v_s(x)\rho_s(dx)\rho_s(dy)ds$, which is equal to $\int_0^t\int(\widehat{\nabla W}*\rho_s)\cdot v_s d\rho_s$. The uniform convergence of W^δ towards W ensures convergence of the left-hand side, which concludes the proof. □

2.3 Preliminary results

We recall the following result of existence of a characteristic flow and well-posedness of measure-valued solutions to (1.2):

Theorem 2.10 ([10] Theorems 2.12 and 2.13, [11] Theorems 2.5 and 2.9)**.** *Assume W satisfies hypotheses (A0)-(A1)-(A2)-(A3) and let ρ^{ini} be given in $\mathcal{P}_2(\mathbb{R}^d)$. Then, there exists a unique solution $\rho \in \mathcal{C}([0,+\infty), \mathbb{W}_2(\mathbb{R}^d))$ satisfying, in the sense of distributions, the aggregation problem* (1.2) *where $a[\rho]$ is replaced by $\widehat{a}[\rho]$ as defined in* (1.3)*.*

This solution may be represented as the family of pushforward measures $(\rho_t := Z_\rho(t,\cdot)_\# \rho^{ini})_{t\geqslant 0}$ where $(Z_\rho(t,\cdot))_{t\geqslant 0}$ is the unique Filippov characteristic flow associated with the one-sided Lipschitz velocity field $\widehat{a}[\rho]$. Besides, if ρ and μ are the respective solutions to (1.2) *with ρ^{ini} and μ^{ini} as initial conditions in $\mathcal{P}_2(\mathbb{R}^d)$, then, for all $t \geqslant 0$,*

$$W_2(\rho_t, \mu_t) \leqslant e^{-\lambda t} W_2(\rho^{ini}, \mu^{ini}). \tag{2.9}$$

In [9], Carrillo, Gómez-Castro, Yao and Zeng proved the following well-posedness and regularity Theorem for aggregation-diffusion equations with Lipschitz symmetric potentials. They prove existence and uniqueness through a fixed-point argument and regularity applying a bootstrap argument in adequate fractional Sobolev spaces. The solutions they define are mild solutions, which are stronger than our definition of solutions, which is in the sense of distributions. We recall the definition of the heat kernel used in the mild formulation:

$$G_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

Theorem 2.11 ([9], Theorems 1.1, 2.1 and 2.2)**.** *Assume that W satisfies assumptions (A0)-(A1)-(A2). Let $\varepsilon > 0$ and $\rho_0^\varepsilon \in \mathcal{P}(\mathbb{R}^d)$.*

(1) For all $T > 0$, there exists a unique solution $\rho^\varepsilon \in \mathcal{C}([0,T], \mathcal{P}(\mathbb{R}^d))$ to the aggregation-diffusion problem (1.1) *in the sense that:*

$$\forall t \in [0,T], \qquad \rho_t^\varepsilon = G_{\varepsilon t} * \rho_0^\varepsilon + \int_0^t \big(\nabla G_{\varepsilon(t-s)}\big) * \big((\nabla W * \rho_s^\varepsilon)\rho_s^\varepsilon\big) ds.$$

(2) This solution is actually a classical solution that belongs, for all $T > 0$, to $\mathcal{C}((0,T], W^{k,p}(\mathbb{R}^d))$ for all $k \in \mathbb{N}$ and $p \in [1,+\infty]$ in the general case, and to $\mathcal{C}((0,T], W^{s,p}(\mathbb{R}^d))$ for all $s \geqslant 0$ and $p \in [1,+\infty]$ if we assume that $\rho_0^\varepsilon \in W^{s,p}(\mathbb{R}^d)$.

Remark 2.12. In [9], the authors state the second item of the above Theorem under the assumption that $W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ and assuming that the initial datum belongs to $L^1_+(\mathbb{R}^d)$ with total unit mass instead of $\mathcal{P}(\mathbb{R}^d)$. It seems to us that $W \in L^\infty$ is only required to obtain sharp decay of the energy functional and that the L^1 assumption on ρ_0^ε is only useful to simplify the notations.

In the above Theorem, we actually have $\rho^\varepsilon \in \mathcal{C}([0,+\infty[, \mathbb{W}_2(\mathbb{R}^d))$. Indeed, as we will see in the proof of our Theorem 3.1 (see equation (3.7)), $\frac{1}{2}$-Hölder continuity in time follows automatically from a uniform bound with respect to $t \in [0,T]$ on $M_2(\rho_t^\varepsilon)$. This in turn comes from the following computations, where we use, first, integration by parts, and, second, the symmetrization Lemma 2.8:

$$\begin{cases} \dfrac{d}{dt} M_2(\rho_t^\varepsilon) = \displaystyle\int |x|^2 \partial_t \rho_t^\varepsilon = \int |x|^2 \nabla \cdot \big((\nabla W * \rho_t^\varepsilon)\rho_t^\varepsilon\big) + \varepsilon \int |x|^2 \Delta \rho_t^\varepsilon = -2 \int x \cdot (\nabla W * \rho_t^\varepsilon) d\rho_t^\varepsilon + 2\varepsilon d, \\ -2 \displaystyle\int x \cdot (\nabla W * \rho_t^\varepsilon) d\rho_t^\varepsilon = -\iint \nabla W(x-y)\cdot(x-y) \rho_t^\varepsilon(dx)\rho_t^\varepsilon(dy) \leqslant 2a_\infty M_1(\rho_t^\varepsilon) \leqslant 2a_\infty \sqrt{M_2(\rho_t^\varepsilon)}. \end{cases}$$

We thus get $\dfrac{d}{dt} M_2(\rho_t^\varepsilon) \leqslant 2a_\infty \sqrt{M_2(\rho_t^\varepsilon)} + 2\varepsilon d$ which implies, using a nonlinear Grönwall Lemma, that $M_2(\rho_t^\varepsilon)$ is bounded over a finite horizon.

We finish by mentioning the special case of the dimension $d = 1$, with potentials of the form $W(x) = a|x|$ for $a \in \mathbb{R}\backslash\{0\}$, for which the vanishing viscosity limit can be obtained using the correspondence with Burgers' equation. Indeed, let us set, for $\varepsilon \geqslant 0$, $u^\varepsilon(t,x) = a\big(1-2f^\varepsilon(t)\big)$, where $f^\varepsilon(t)$ is the cumulative distribution function of ρ^ε_t. One can show that ρ^ε solves (1.1a) if and only if u^ε solves the viscous Burgers equation:

$$\partial_t u^\varepsilon + \partial_x \frac{(u^\varepsilon)^2}{2} = \varepsilon \partial_{xx} u^\varepsilon, \tag{2.10}$$

and, similarly, ρ solves the aggregation equation (1.2a) with the correct velocity field $\widehat{a}[\rho]$ if and only if u solves Burgers' equation (see [5, 16, 21]). Using the fact that, in dimension $d = 1$, we have the representation $W_1(\rho^\varepsilon_t, \rho_t) = \|f^\varepsilon(t) - f(t)\|_{L^1(\mathbb{R})}$ and combining with Kuznetsov's estimate hereafter for the viscous Burgers equation (see [25]):

$$\|u^\varepsilon(t) - u(t)\|_{L^1(\mathbb{R})} \leqslant CTV(u_0)\sqrt{\varepsilon t},$$

where C is a positive constant, we deduce the following proposition:

Proposition 2.13. *Assume $d = 1$ and $W(x) = a|x|$ for some constant $a \in \mathbb{R}\backslash\{0\}$. Let $\rho^{ini} \in \mathcal{P}_2(\mathbb{R})$, set $\rho^\varepsilon_0 = \rho^{ini}$ for all $\varepsilon > 0$ and let $(\rho^\varepsilon)_{\varepsilon>0}$ be the sequence of weak solutions to* (1.1).

Then, for all $T > 0$, $(\rho^\varepsilon)_{\varepsilon>0}$ converges in W_1 distance and uniformly on $[0,T]$, towards a solution $\rho \in \mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}))$ to (1.2) *with the velocity field $a[\rho]$ being replaced by $\widehat{a}[\rho]$ as defined in* (1.3). *More precisely, we have:*

$$\forall t \in [0,T], \quad W_1(\rho^\varepsilon_t, \rho_t) \leqslant C\sqrt{\varepsilon t},$$

where the constant $C > 0$ depends on a_∞ only.

In the case of one initial Dirac mass $\rho^{ini} = \delta_0$, one can even obtain convergence of ρ^ε towards ρ at order 1 with respect to ε using simple scaling arguments. The initial data to the Burgers problem is $u^{ini} = 1 - 2H_0(x)$, and the solution to the inviscid Burgers problem is stationary, given by $u(t) = u^{ini}$. One can also show that there exists a stationary solution to equation (2.10) of the form $v^\varepsilon(t,x) = V\big(\frac{x}{\varepsilon}\big)$, with $V(-\infty) = 1$, $V(+\infty) = -1$ and $V'(\pm\infty = 0)$. We then have using a contraction property of the viscous Burgers equation and the stationarity of v^ε and u:

$$\|u^\varepsilon(t) - u(t)\|_{L^1} \leqslant \underbrace{\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1}}_{\leqslant \|u^{ini} - v^\varepsilon(0)\|_{L^1}} + \underbrace{\|v^\varepsilon(t) - u(t)\|_{L^1}}_{= \|v^\varepsilon(0) - u^{ini}\|_{L^1}} \leqslant 2\int \left|u^{ini}\Big(\frac{x}{\varepsilon}\Big) - V\Big(\frac{x}{\varepsilon}\Big)\right| dx \leqslant 2\varepsilon \int \left|u^{ini} - V\right|,$$

which gives $W_1(\rho^\varepsilon_t, \rho_t) \leqslant C\varepsilon$ with $C > 0$ independent of time. This result can be extended to the case of a finite sum of Dirac masses as initial datum, using the arguments of Teng and Zhang [37] to compare shocks with traveling waves. We also refer to [35, 34, 36] for generalizations of this result.

3 $O(\varepsilon^{1/2})$ convergence rate when the potential is $\lambda-$convex

In this section, we assume that W satisfies assumptions (A0)-(A1)-(A2)-(A3).

3.1 Method 1: computing $\frac{d}{dt}W_2^2(\rho^\varepsilon_t, \rho_t)$

So as to make integration by parts rigorous, we actually compute $\frac{d}{dt}W_2^2(\rho^\varepsilon_t, \rho^\delta_t)$ for $\varepsilon, \delta > 0$ so that ρ^ε and ρ^δ are regular (see Theorem 2.11), and then we let $\delta \to 0$. We therefore need to know that ρ^δ_t converges in the sense of measures towards ρ_t.

3.1.1 Convergence in $\mathcal{C}([0,T],\mathbb{W}_1(\mathbb{R}^d))$ without convergence rate for W satisfying (A0)-(A1)-(A2)

Let $T>0$ and let $\rho^\varepsilon \in \mathcal{C}([0,T],\mathbb{W}_2(\mathbb{R}^d))$ be the solution to the aggregation-diffusion problem (1.1) on $[0,T]\times\mathbb{R}^d$, as given by Theorem 2.11. Let us denote $v^\varepsilon = -\nabla W * \rho^\varepsilon - \varepsilon\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}$ so that the continuity equation $\partial_t\rho^\varepsilon + \nabla\cdot\rho^\varepsilon v^\varepsilon = 0$ is satisfied in the sense of distributions. We formally have, by definition of the first variation and then by integration by parts:

$$\frac{d}{dt}F^\varepsilon(\rho_t^\varepsilon) = \int \frac{\delta F^\varepsilon}{\delta\rho}(\rho_t^\varepsilon)\partial_t\rho_t^\varepsilon = \int \nabla\frac{\delta F^\varepsilon}{\delta\rho}(\rho_t^\varepsilon)\cdot v_t^\varepsilon d\rho_t^\varepsilon = -\int\left|\nabla\frac{\delta F^\varepsilon}{\delta\rho}(\rho_t^\varepsilon)\right|^2 d\rho_t^\varepsilon, \tag{3.1}$$

where, in the last equality, we used the identity $\frac{\delta F^\varepsilon}{\delta\rho}(\rho) = W*\rho + \varepsilon(\ln\rho+1)$ to deduce that v_t^ε is nothing else than $-\nabla\dfrac{\delta F^\varepsilon}{\delta\rho}(\rho_t^\varepsilon)$. Proving rigorously (3.1) can be made using an easy adaptation of Lemma 2.9. Integrating (3.1) over time then yields:

$$\forall t\in[0,T],\quad F^\varepsilon(\rho_0^\varepsilon) = F^\varepsilon(\rho_t^\varepsilon) + \int_0^t\int\left|\nabla\frac{\delta F^\varepsilon}{\delta\rho}(\rho_s^\varepsilon)\right|^2 d\rho_s^\varepsilon ds.$$

Let us only use this equality as an inequality as it will turn out sufficient for passing to the limit, and let us write $\left|\nabla\dfrac{\delta F^\varepsilon}{\delta\rho}(\rho_s^\varepsilon)\right|^2$ as the half-sum $\dfrac{1}{2}\left(|v_s^\varepsilon|^2 + \left|\nabla\dfrac{\delta F^\varepsilon}{\delta\rho}(\rho_s^\varepsilon)\right|^2\right)$ so as to recover a link between the velocity v and the functional F at the limit $\varepsilon\to 0$. This way, we recover the so-called energy dissipation equality (EDE, that we use as an inequality in our paper):

$$\forall t\in[0,T],\quad F^\varepsilon(\rho_0^\varepsilon) \geqslant F^\varepsilon(\rho_t^\varepsilon) + \frac{1}{2}\int_0^t\int|v_s^\varepsilon|^2 d\rho_s^\varepsilon ds + \frac{1}{2}\int_0^t\int\left|\nabla\frac{\delta F^\varepsilon}{\delta\rho}(\rho_s^\varepsilon)\right|^2 d\rho_s^\varepsilon ds, \tag{3.2}$$

Showing a sort of lower semicontinuity, when $\varepsilon\to 0$, of each term in (3.2), we will prove that up to successive extractions, $(\rho^\varepsilon)_{\varepsilon>0}$ converges towards a measure ρ that satisfies a continuity equation and an EDE. Combining both, we will prove that ρ solves the aggregation problem (1.2). In case the solution to such a Cauchy problem is unique, the whole sequence $(\rho^\varepsilon)_{\varepsilon>0}$ converges towards ρ. This method does not require the $\lambda-$convexity but only the Lipschitz continuity of the potential W.

Theorem 3.1. *Assume W satisfies assumptions (A0)-(A1)-(A2). Let $\rho^{ini}\in\mathcal{P}_2(\mathbb{R}^d)$, and let $(\rho^\varepsilon)_{\varepsilon>0}$ be a sequence of weak solutions to* (1.1).

Assume that the sequence of initial data $(\rho_0^\varepsilon)_{\varepsilon>0}$ satisfies the following assumptions:

$$\limsup_{\varepsilon\to 0} F^\varepsilon(\rho_0^\varepsilon) \leqslant F(\rho^{ini}), \tag{3.3a}$$

$$\forall\varepsilon>0,\quad M_2(\rho_0^\varepsilon)\leqslant C, \tag{3.3b}$$

$$\lim_{\varepsilon\to 0} W_2(\rho_0^\varepsilon,\rho^{ini}) = 0, \tag{3.3c}$$

for some constant $C>0$ independent of ε. Then, for all $T>0$, $(\rho^\varepsilon)_{\varepsilon>0}$ converges up to a subsequence, in W_1 distance and uniformly on $[0,T]$, towards a solution $\rho\in\mathcal{C}([0,T],\mathbb{W}_2(\mathbb{R}^d))$ to (1.2) *with the velocity field $a[\rho]$ being replaced by $\widehat{a}[\rho]$ as defined in* (1.3)*:*

$$\sup_{t\in[0,T]} W_1(\rho_t^\varepsilon,\rho_t) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

If the solution to (1.2) *is unique, then the whole sequence $(\rho^\varepsilon)_{\varepsilon>0}$ converges towards ρ.*

Remark 3.2. Note that assumptions (3.3) are automatically satisfied if the entropy $\mathcal{U}(\rho_0^\varepsilon)$ is uniformly bounded w.r.t $\varepsilon>0$. In case we take $\rho_0^\varepsilon = \rho^{ini}$, this corresponds to ρ^{ini} having finite entropy.

The following lemma shows that it is possible to construct such a sequence of initial data:

Lemma 3.3. *Recall that ρ^{ini} is given in $\mathcal{P}_2(\mathbb{R}^d)$. For all $p \geqslant 1$ such that $\rho^{ini} \in \mathcal{P}_p(\mathbb{R}^d)$ and for all $\alpha \in (-1,0)$, there exists a sequence $(\mu_0^\varepsilon)_{\varepsilon>0}$ in $\mathcal{P}_p(\mathbb{R}^d)$ satisfying:*

$$\liminf_{\varepsilon\to 0} F^\varepsilon(\mu_0^\varepsilon) \leqslant F(\rho^{ini}), \tag{3.4a}$$

$$\forall \varepsilon > 0, \quad M_p(\mu_0^\varepsilon) \leqslant C e^{-C\varepsilon^\alpha}, \tag{3.4b}$$

$$\lim_{\varepsilon\to 0} W_p(\mu_0^\varepsilon, \rho^{ini}) = 0 \tag{3.4c}$$

where the constant $C > 0$ depends on p but not on ε. Actually, we can be more specific in (3.4c)*:*

$$\forall \varepsilon > 0, \quad W_p(\mu_0^\varepsilon, \rho^{ini}) \leqslant C e^{-\varepsilon^\alpha}.$$

Proof. Let $\alpha \in (-1,0)$ and let $p \geqslant 1$ such that $\rho^{ini} \in \mathcal{P}_p(\mathbb{R}^d)$. Let $(r_\varepsilon)_{\varepsilon>0}$ be a sequence of positive real numbers to be specified later in the proof. Let $\eta \in L^1(\mathbb{R}^d)$ be a nonnegative function supported on $B(0,1)$, with unit total mass, such that $\eta \ln \eta$ and $|x|^p\eta(x)$ are integrable on $\mathbb{R}^d$. We then set $\eta^\varepsilon(x) = r_\varepsilon^{-d}\eta\left(\frac{x}{r_\varepsilon}\right)$ and $\mu_0^\varepsilon = \eta^\varepsilon * \rho^{ini}$. Because of the compact support of η we have $M_p(\eta^\varepsilon * \rho^{ini}) \leqslant C(M_p(\rho^{ini}) + M_p(\eta^\varepsilon)) \leqslant C$, so that, in particular, $\mu_0^\varepsilon \in \mathcal{P}_p(\mathbb{R}^d)$ for all $\varepsilon > 0$.

Firstly, let us choose r_ε so that $\varepsilon\mathcal{U}(\eta^\varepsilon)$ goes to 0 as $\varepsilon \to 0$. Since $\eta^\varepsilon \ll$ Leb, we have $\mathcal{U}(\eta^\varepsilon) = \int \eta^\varepsilon \ln \eta^\varepsilon$. Therefore, using the change of variables $x = r_\varepsilon y$, one has:

$$\mathcal{U}(\eta^\varepsilon) = r_\varepsilon^{-d}\int \eta\left(\frac{x}{r_\varepsilon}\right)\ln\left(r_\varepsilon^{-d}\eta\left(\frac{x}{r_\varepsilon}\right)\right)dx = \int \eta(y)\ln\left(r_\varepsilon^{-d}\eta(y)\right)dy = \int \eta(y)\ln\eta(y)dy - d\ln r_\varepsilon. \tag{3.5}$$

Based on the above computation, we choose $r_\varepsilon = e^{-h_\varepsilon/\varepsilon}$ for some positive sequence $(h_\varepsilon)_{\varepsilon>0}$ such that $\lim_{\varepsilon\to 0} h_\varepsilon = 0$. More precisely, we set $h_\varepsilon = \varepsilon^{\alpha+1}$, that is $r_\varepsilon = e^{-\varepsilon^\alpha}$.

Now, using the convexity and the invariance under translation of $\mathcal{U}$, we have $\mathcal{U}(\eta^\varepsilon * \rho^{ini}) \leqslant \mathcal{U}(\eta^\varepsilon)$, and therefore $F^\varepsilon(\mu_0^\varepsilon) \leqslant \mathcal{W}(\mu_0^\varepsilon) + \varepsilon\mathcal{U}(\eta^\varepsilon)$. Since $\mathcal{W}$ is continuous on $\mathbb{W}_1(\mathbb{R}^d)$ thanks to Lemma 2.3, we just need the convergence $\mu_0^\varepsilon \to \rho^{ini}$ in $\mathbb{W}_1(\mathbb{R}^d)$ in order to have $\mathcal{W}(\mu_0^\varepsilon) \to \mathcal{W}(\rho^{ini})$ and hence $\lim_{\varepsilon\to 0}\mathcal{W}(\mu_0^\varepsilon) + \varepsilon\mathcal{U}(\eta^\varepsilon) = \mathcal{W}(\rho^{ini}) = F(\rho^{ini})$. Then, (3.4a) will immediately follow.

We now use

$$W_p^p(\mu_0^\varepsilon, \rho^{ini}) = W_p^p(\eta^\varepsilon * \rho^{ini}, \delta_0 * \rho^{ini}) \leqslant W_p^p(\eta^\varepsilon, \delta_0) = M_p(\eta^\varepsilon) \to 0,$$

where the last limit is justified by $M_p(\eta^\varepsilon) = r_\varepsilon^p M_p(\eta) = Ce^{-p\varepsilon^\alpha}$. This proves (3.4b) and (3.4c) since $\alpha < 0$, and this in turn proves (3.4a). □

Relaxing assumption 3.3a can only be done under additional assumptions on the potential. In the case W satisfies assumption (A3), replacing the original initial data ρ_0^ε by a smoothed out initial data μ_0^ε that verifies assumptions (3.3) and using the $\lambda-$convexity of the potential to estimate the distance between ρ^ε and the new sequence of viscous solutions μ^ε, we obtain as a byproduct of Theorem 3.1 the following corollary:

Corollary 3.4. *Assume W satisfies assumptions (A0)-(A1)-(A2)-(A3). Let $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$, and let $(\rho^\varepsilon)_{\varepsilon>0}$ be the sequence of weak solutions to* (1.1)*. Assume that the sequence of initial data $(\rho_0^\varepsilon)_{\varepsilon>0}$ converges in $\mathbb{W}_2(\mathbb{R}^d)$ to ρ^{ini} as $\varepsilon \to 0$.*

Then, for all $T > 0$, the whole sequence $(\rho^\varepsilon)_{\varepsilon>0}$ converges in W_1 distance, uniformly on $[0,T]$, towards the unique solution $\rho \in \mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$ of (1.2) *with the velocity field $a[\rho]$ being replaced by $\widehat{a}[\rho]$ as defined in* (1.3)*.*

Proof of Theorem 3.1. First of all, let us extract from $(\rho_\varepsilon)_{\varepsilon>0}$ a converging subsequence. For $\varepsilon > 0$, recall that the continuity equation $\partial_t \rho^\varepsilon + \nabla \cdot \rho^\varepsilon v^\varepsilon = 0$ is satisfied. Moreover, let us rewrite equation (3.2) using the identity $\nabla \frac{\delta F^\varepsilon}{\delta \rho}(\rho) = \nabla W * \rho + \varepsilon \frac{\nabla \rho}{\rho}$ and split it into three terms:

$$\forall t \in [0,T], \quad F^\varepsilon(\rho_0^\varepsilon) \geqslant F^\varepsilon(\rho_t^\varepsilon) + \frac{1}{2}\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds + \frac{1}{2}\int_0^t \int \left|\nabla W * \rho_s^\varepsilon + \varepsilon \frac{\nabla \rho_s^\varepsilon}{\rho_s^\varepsilon}\right|^2 d\rho_s^\varepsilon ds =: T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon. \tag{3.6}$$

Note that, if $M_2(\rho_t^\varepsilon)$ is uniformly bounded, then T_1^ε is uniformly bounded from below thanks to the estimate in Lemma 2.4. In that case, using the fact that T_3^ε is nonnegative and the fact that $F^\varepsilon(\rho_0^\varepsilon)$ is bounded from above thanks to assumption (3.3a) on the initial data, we can deduce that $\int_0^T \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds \leqslant C$ for some constant $C > 0$ independent of ε and t. In particular, for all $t \in [0,T]$, $v_t^\varepsilon \in L^2(\rho_t^\varepsilon)$ and $\int_0^T \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds < +\infty$. Using Theorem 2.1, we obtain that $\rho^\varepsilon \in AC([0,T], \mathbb{W}_2(\mathbb{R}^d))$ and that its metric derivative exists and is bounded by the L^2 norm of v_s^ε: $|(\rho^\varepsilon)'_s| \leqslant \|v_s^\varepsilon\|_{L^2(\rho_s^\varepsilon)}$ for all $s \in [0,T]$. We deduce the following bound, that is uniform with respect to ε, by integration over time:

$$\int_0^T |(\rho^\varepsilon)'_s|^2 ds \leqslant C.$$

Then, using a Cauchy-Schwarz inequality, we get:

$$\forall 0 \leqslant s \leqslant t \leqslant T, \quad W_2(\rho_t^\varepsilon, \rho_s^\varepsilon) \leqslant \int_s^t |(\rho^\varepsilon)'_\tau| d\tau \leqslant \left(\int_s^t |(\rho^\varepsilon)'_\tau|^2 d\tau\right)^{1/2} \sqrt{t-s} \leqslant \sqrt{C(t-s)}, \tag{3.7}$$

which gives equicontinuity of $(\rho^\varepsilon)_{\varepsilon>0}$ in W_2 distance (and therefore in W_1 distance). If we still assume that $M_2(\rho_t^\varepsilon)$ is uniformly bounded, then the set $\{\rho_t^\varepsilon, \varepsilon > 0\}$ is relatively compact in $\mathbb{W}_1(\mathbb{R}^d)$ in virtue of Lemma 2.6. We can therefore apply Ascoli-Arzelà theorem in the space $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$ to extract from $(\rho^\varepsilon)_{\varepsilon>0}$ a subsequence converging in $\mathbb{W}_1(\mathbb{R}^d)$, uniformly in $t \in [0,T]$, towards some $\rho \in \mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$. We still denote this subsequence $(\rho^\varepsilon)_{\varepsilon>0}$. Moreover, the l.s.c of the W_2 distance along with the weak convergence $\rho_t^\varepsilon \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \rho_t$ for all $t \in [0,T]$ allows to pass to the $\liminf_{\varepsilon\to 0}$ in (3.7) to show that $\rho \in \mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$. The limit ρ is actually $1/2-$Hölder in time and satifies the same estimate as ρ^ε:

$$\forall 0 \leqslant s \leqslant t \leqslant T, \quad W_2(\rho_t, \rho_s) \leqslant \sqrt{C(t-s)}.$$

Let us come back to the boundedness of $M_2(\rho_t^\varepsilon)$. This bound can actually be obtained from inequality (3.6). Indeed, from (3.6) and assumption (3.3a), we get, since $T_3^\varepsilon \geqslant 0$:

$$F^\varepsilon(\rho_t^\varepsilon) + \frac{1}{2}\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds \leqslant C. \tag{3.8}$$

Let us show that the second term controls $M_2(\rho_t^\varepsilon)$ if $t \in [0,T]$. Differentiating $M_2(\rho_t^\varepsilon)$ in time and integrating by parts, we have:

$$\frac{d}{dt} M_2(\rho_t^\varepsilon) = 2\int x \cdot v_t^\varepsilon(x) \rho_t^\varepsilon(dx) \leqslant 2 M_2(\rho_t^\varepsilon)^{1/2} \left(\int |v_t^\varepsilon|^2 d\rho_t^\varepsilon\right)^{1/2},$$

using Cauchy-Schwarz inequality. Applying a Grönwall Lemma, this implies, for all $t \in [0,T]$,

$$M_2(\rho_t^\varepsilon)^{1/2} \leqslant M_2(\rho_0^\varepsilon)^{1/2} + \int_0^t \left(\int |v_s^\varepsilon|^2 d\rho_s^\varepsilon\right)^{1/2} ds \leqslant M_2(\rho_0^\varepsilon)^{1/2} + \sqrt{T}\left(\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds\right)^{1/2},$$

where we used Jensen's inequality w.r.t the measure $\frac{ds}{t}$. Finally, we get:

$$\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon \geqslant \frac{1}{T}\big(M_2(\rho_t^\varepsilon) - M_2(\rho_0^\varepsilon)\big).$$

Plugging this inequality into (3.8) and using the estimate in Lemma 2.4 one obtains:

$$-a_\infty M_2(\rho_t^\varepsilon)^{1/2} - \varepsilon(M_2(\rho_t^\varepsilon)^{1/4} + C) + \frac{1}{2T}\big(M_2(\rho_t^\varepsilon) - M_2(\rho_0^\varepsilon)\big) \leqslant C,$$

which provides a uniform bound on $M_2(\rho_t^\varepsilon)$.

The point is now, for every $t \in [0,T]$, to show l.s.c of each term T_i^ε, $i = 1,2,3$, with respect to the W_1 convergence of $(\rho_t^\varepsilon)_{\varepsilon>0}$ towards ρ_t that we just proved.

$*$ **Dealing with $T_1^\varepsilon = F^\varepsilon(\rho_t^\varepsilon)$.**
Using Lemma 2.3, the W_1-convergence of $(\rho_t^\varepsilon)_{\varepsilon>0}$ towards ρ_t ensures that $\lim_{\varepsilon\to 0} \mathcal{W}(\rho_t^\varepsilon) = \mathcal{W}(\rho_t)$. Besides, thanks to Lemma 2.4, we have for the entropy $\liminf_{\varepsilon\to 0} \mathcal{U}(\rho_t^\varepsilon) \geqslant \mathcal{U}(\rho_t)$, and we deduce in turn $\liminf_{\varepsilon\to 0} \varepsilon\mathcal{U}(\rho_t^\varepsilon) \geqslant 0$. Therefore:

$$\liminf_{\varepsilon\to 0} F^\varepsilon(\rho_t^\varepsilon) \geqslant F(\rho_t).$$

$*$ **Dealing with $T_2^\varepsilon = \frac{1}{2}\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds$.**

For $\varepsilon > 0$, letting $E^\varepsilon = \rho^\varepsilon v^\varepsilon$, a Cauchy-Schwarz inequality shows that the total variation of E^ε is uniformly bounded with respect to $\varepsilon > 0$:

$$|E^\varepsilon|([0,t]\times\mathbb{R}^d) = \int_0^t \int |v_s^\varepsilon| d\rho_s^\varepsilon ds \leqslant \sqrt{t}\left(\int_0^t \int |v_s^\varepsilon|^2 d\rho_s^\varepsilon ds\right)^{1/2} \leqslant \sqrt{CT},$$

Thus, up to another extraction, we can assume that $E^\varepsilon \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} E$ for some $E \in \mathcal{M}_b([0,t]\times\mathbb{R}^d)^d$. Now, since ρ^ε and E^ε are absolutely continuous with respect to the Lebesgue measure on $[0,t]\times\mathbb{R}^d$ as long as $\varepsilon > 0$, Lemma 2.7 ensures that T_2^ε rewrites as follows:

$$T_2^\varepsilon = \int_0^t \int f_2(\rho_s^\varepsilon, E_s^\varepsilon) dx ds = \mathcal{B}_2(\rho^\varepsilon, E^\varepsilon).$$

Then the lower semicontinuity of $\mathcal{B}_2$ on $\mathcal{M}_b([0,t]\times\mathbb{R}^d) \times \mathcal{M}_b([0,t]\times\mathbb{R}^d)^d$ yields:

$$\liminf_{\varepsilon\to 0} T_2^\varepsilon \geqslant \mathcal{B}_2(\rho, E),$$

which, in turn, implies that $\mathcal{B}_2(\rho, E)$ is finite and therefore gives the existence of a vector-valued density v verifying $E = \rho v$. Using Lemma 2.7 (iv), the above inequality rewrites:

$$\liminf_{\varepsilon\to 0} T_2^\varepsilon \geqslant \frac{1}{2}\int_0^t \int |v_s|^2 d\rho_s ds.$$

In addition, this transformation also allows to pass to the limit in the continuity equation $\partial_t \rho^\varepsilon + \nabla\cdot E^\varepsilon = 0$, which is now linear. Indeed, letting $\varepsilon \to 0$ in the weak formulation, one easily gets $\partial_t\rho + \nabla\cdot(\rho v) = 0$. This shows that the limit density ρ is still a solution to a continuity equation, and the link between the velocity field v and the functional F will be made thorough when passing to the limit $\varepsilon \to 0$ in the EDE (3.2).

$*$ **Dealing with** $T_3^\varepsilon = \dfrac{1}{2}\displaystyle\int_0^t \int \Big|\nabla W * \rho_s^\varepsilon + \varepsilon \frac{\nabla \rho_s^\varepsilon}{\rho_s^\varepsilon}\Big|^2 d\rho_s^\varepsilon ds$**.**

As it is standard when dealing with terms belonging to $L^2(\rho_s^\varepsilon)$, we set $G^\varepsilon = (\nabla W * \rho^\varepsilon)\rho^\varepsilon + \varepsilon \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon}\rho^\varepsilon$, so that $T_3^\varepsilon = \mathcal{B}_2(\rho^\varepsilon, G^\varepsilon)$.

We deduce from (3.6) that T_3^ε is uniformly bounded w.r.t ε, which implies that G^ε is uniformly bounded in $\mathcal{M}_b([0,t] \times \mathbb{R}^d)^d$. Therefore, up to another extraction, we can assume that $G^\varepsilon \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} G$ for some $G \in \mathcal{M}_b([0,t] \times \mathbb{R}^d)^d$. Since W is Lipschitz, we have $\displaystyle\int_0^t \int |\nabla W * \rho_s^\varepsilon| d\rho_s^\varepsilon ds \leqslant a_\infty t$ thus $(\nabla W * \rho^\varepsilon)\rho^\varepsilon$ is uniformly bounded too in $\mathcal{M}_b([0,t] \times \mathbb{R}^d)^d$.

As a consequence, the difference $\varepsilon \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon}\rho^\varepsilon$ is also uniformly bounded in $\mathcal{M}_b([0,t] \times \mathbb{R}^d)^d$. Now, its limit when $\varepsilon \to 0$ is 0 in the sense of distributions. Indeed, for $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\langle \varepsilon \nabla \rho^\varepsilon, \xi \rangle = -\varepsilon \displaystyle\int_0^t \int \nabla \xi d\rho^\varepsilon$ which can be bounded, for instance, by $\varepsilon t \|\nabla \xi\|_{L^\infty}$ and therefore goes to 0 as $\varepsilon \to 0$. We deduce that $\varepsilon \frac{\nabla \rho^\varepsilon}{\rho^\varepsilon}\rho^\varepsilon$ actually converges in the sense of measures towards 0, hence the limit, in the sense of measures, of G^ε is that of $(\nabla W * \rho^\varepsilon)\rho^\varepsilon$.

† **Limit in the sense of measures of** $(\nabla W * \rho^\varepsilon)\rho^\varepsilon$.

Let $\xi \in \mathcal{C}_0([0,t] \times \mathbb{R}^d)$ and let us consider the duality bracket $\langle (\nabla W * \rho^\varepsilon)\rho^\varepsilon, \xi \rangle$ as ε goes to 0. That quantity equals, using Lemma 2.8 applied to the even vector field ∇W:

$$\int_0^t \iint \nabla W(x-y) \cdot \xi(s,x) \rho_s^\varepsilon(dx)\rho_s^\varepsilon(dy) ds = \frac{1}{2}\int_0^t \iint \nabla W(x-y) \cdot (\xi(s,x) - \xi(s,y))\, \rho_s^\varepsilon(dx)\rho_s^\varepsilon(dy) ds. \tag{3.9}$$

Now, since W is Lipschitz, ∇W is bounded, therefore the map

$$(s,x,y) \longmapsto \nabla W(x-y) \cdot \big(\xi(s,x) - \xi(s,y)\big)$$

is continuous and the weak convergence $\rho^\varepsilon \otimes \rho^\varepsilon \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \rho \otimes \rho$ (which is equivalent to narrow convergence since we deal with probability measures) allows to pass to the limit $\varepsilon \to 0$ in the above quantity to obtain:

$$\lim_{\varepsilon\to 0} \int_0^t \iint \nabla W(x-y) \cdot \xi(s,x) \rho_s^\varepsilon(dx)\rho_s^\varepsilon(dy) ds = \frac{1}{2}\int_0^t \iint \nabla W(x-y) \cdot (\xi(s,x) - \xi(s,y))\, \rho_s(dx)\rho_s(dy) ds. \tag{3.10}$$

Note that, until now, the value of $\nabla W(0)$ does not matter. Actually, all the integrals when $\varepsilon > 0$ hold w.r.t to the Lebesgue measure and therefore the diagonal $\{x = y\}$ can be avoided. We therefore only need $\nabla W(z) = -\nabla W(-z)$ for nonzero z to apply Lemma 2.8, and this do not impose any value to $\nabla W(0)$.

Now, to come back to some duality bracket tested against ξ, one needs to unsymmetrize the resulting expression by writing:

$$\begin{aligned}
&\frac{1}{2}\int_0^t \iint \nabla W(x-y) \cdot (\xi(s,x) - \xi(s,y))\, \rho_s(dx)\rho_s(dy) ds \qquad (3.11)\\
&\quad = \frac{1}{2}\left(\int_0^t \iint \widehat{\nabla W}(x-y) \cdot \xi(s,x) \rho_s(dx)\rho_s(dy) ds - \int_0^t \iint \widehat{\nabla W}(x-y) \cdot \xi(s,y) \rho_s(dx)\rho_s(dy) ds \right)\\
&\quad = \frac{1}{2}\left(\int_0^t \iint \widehat{\nabla W}(x-y) \cdot \xi(s,x) \rho_s(dx)\rho_s(dy) ds + \int_0^t \iint \widehat{\nabla W}(x-y) \cdot \xi(s,x) \rho_s(dx)\rho_s(dy) ds \right)\\
&\quad = \int_0^t \iint \widehat{\nabla W}(x-y) \cdot \xi(s,x) \rho_s(dx)\rho_s(dy) ds,
\end{aligned}$$

where we used the fact that $\widehat{\nabla W}(z) = -\widehat{\nabla W}(-z)$ for all $z \in \mathbb{R}^d$, which now imposes $\widehat{\nabla W}(0) = 0$.

Remark 3.5. These computations could hold against a test function ξ that is only Lipschitz on $[0,t] \times \mathbb{R}^d$ provided $\nabla W(z) \leqslant C/|z|^{1-\beta}$ for some $\beta > 0$. Indeed, the map $(s,x,y) \longmapsto \nabla W(x-y) \cdot (\xi(s,x) - \xi(s,y))$ would be continuous on the diagonal and hence everywhere on $[0,t] \times (\mathbb{R}^d)^2$. This could provide a way to deal with the non Lipschitz potentials $W(x) = |x|^\beta$ for $0 < \beta < 1$, that are more singular than the Lipschitz potentials but are still less singular than the logarithmic potential. However, extra difficulties arise for the limit analysis when W is not Lipschitz.

We finally get that $G = (\widehat{\nabla W} * \rho)\rho$ and therefore $\mathcal{B}_2(\rho, G) = \frac{1}{2}\int_0^t \int |\widehat{\nabla W} * \rho_s|^2 d\rho_s ds$. Using the l.s.c of $\mathcal{B}_2$ we finally get:

$$\liminf_{\varepsilon \to 0} T_3^\varepsilon \geqslant \int_0^t \int |\widehat{\nabla W} * \rho_s|^2 d\rho_s ds.$$

$*$ **Passing to the $\liminf_{\varepsilon\to 0}$ to recover a limit EDE.**

We can now pass to the $\liminf_{\varepsilon\to 0}$ in (3.2) using the assumption (3.3a) for the left-hand side to get the following EDE (which, once again is written as an inequality):

$$F(\rho^{ini}) \geqslant F(\rho_t) + \frac{1}{2}\int_0^t \int |v_s|^2 d\rho_s ds + \frac{1}{2}\int_0^t \int \left|\widehat{\nabla W} * \rho_s\right|^2 d\rho_s ds. \tag{3.12}$$

Recall that ρ still solves the continuity equation $\partial_t \rho + \nabla \cdot \rho v = 0$ in the sense of distributions. Identifying the velocity v is made through Lemma 2.9 which gives:

$$\forall t \in [0,T], \quad F(\rho_t) - F(\rho_0) = \int_0^t \int (\widehat{\nabla W} * \rho_s) \cdot v_s d\rho_s.$$

Since $(\rho_0^\varepsilon)_{\varepsilon>0}$ converges to both ρ_0 and ρ^{ini} in $\mathbb{W}_1(\mathbb{R}^d)$, we have $\rho_0 = \rho^{ini}$. Plugging the above identity into (3.12) then yields:

$$\frac{1}{2}\int_0^t \int \left|v_s + \widehat{\nabla W} * \rho_s\right|^2 d\rho_s ds \leqslant 0,$$

so that $v = -\widehat{\nabla W} * \rho = \widehat{a}[\rho]$ almost everywhere. We deduce that ρ solves the aggregation equation (1.2) in the sense of distributions with the correct velocity field $\widehat{a}[\rho]$, which concludes the proof. Incidentally, the identity $v = -\widehat{\nabla W} * \rho$ confirms that the limit EDE (3.12) is actually an equality. □

Proof of Corollary 3.4. We now come back to the case of arbitrary initial data ρ_0^ε i.e. we do not assume anymore that assumptions (3.3) hold. However, we still assume that $W_2(\rho_0^\varepsilon, \rho^{ini}) \underset{\varepsilon\to 0}{\longrightarrow} 0$ and in addition, we now assume W to be $\lambda-$convex.

Let $(\mu_0^\varepsilon)_{\varepsilon>0}$ be a sequence of smoothed out initial data for which $W_2(\mu_0^\varepsilon, \rho^{ini}) \underset{\varepsilon\to 0}{\longrightarrow} 0$ and the assumptions (3.3) hold on $(\mu_0^\varepsilon)_{\varepsilon>0}$. We denote by μ^ε a solution to (1.1) for the modified initial data μ_0^ε. Applying Theorem 3.1, we know that μ^ε converges in $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$ towards ρ solution to (1.2) as $\varepsilon \to 0$, up to a subsequence. But since W satisfies the assumptions of Theorem 2.10, such a solution is unique and we deduce that the whole sequence $(\mu^\varepsilon)_{\varepsilon>0}$ converges towards ρ.

It remains to show that $W_2(\rho_t^\varepsilon, \mu_t^\varepsilon)$ goes to 0 as $\varepsilon \to 0$ by estimating this quantity thanks to the $\lambda-$convexity of W:

Lemma 3.6. *Assume W satisfies assumptions (A0)-(A1)-(A2)-(A3). Let $\rho, \mu \in \mathcal{P}_2(\mathbb{R}^d)$ and denote (φ, ψ) a pair of Kantorovitch potentials from ρ to μ for the quadratic cost $c(x,y) = \frac{1}{2}|x-y|^2$. In addition, we assume that ρ or μ is an absolutely continuous measure. Then,*

$$\int \nabla\varphi \cdot a[\rho] d\rho + \int \nabla\psi \cdot a[\mu] d\mu \leqslant -\lambda W_2^2(\rho, \mu). \tag{3.13}$$

Remark 3.7.

(1) In particular, we recover the last estimate in Theorem 2.10: if $\rho, \mu \in AC_{loc}([0,+\infty), \mathbb{W}_2(\mathbb{R}^d))$ are solution to (1.2) with initial data $\rho^{ini}, \mu^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ and if ρ_t or μ_t is an absolutely continuous measure, the following inequality holds:

$$\frac{d}{dt} W_2^2(\rho_t, \mu_t) \leqslant -2\lambda W_2^2(\rho_t, \mu_t). \tag{3.14}$$

Indeed, this is a direct consequence of Lemma 3.6 and of the following computation (see [32] Theorem 5.25 or [1] Theorem 8.4.7)

$$\frac{d}{dt}\frac{1}{2} W_2^2(\rho_t, \mu_t) = \int \nabla\varphi_t \cdot v_t d\rho_t + \int \nabla\psi_t \cdot w_t d\mu_t, \tag{3.15}$$

whenever ρ, μ satisfy the continuity equations $\partial_t \rho + \nabla \cdot \rho v = 0$, $\partial_t \mu + \nabla \cdot \mu w = 0$. Inequality (3.14) then yields the aforementioned estimate using a Grönwall Lemma:

$$W_2(\rho_t, \mu_t) \leqslant e^{-\lambda t} W_2(\rho^{ini}, \mu^{ini}). \tag{3.16}$$

Relaxing the assumptions that either ρ_t or μ_t is an absolutely continuous measure can be done replacing ρ_t by ρ_t^ε for instance, and passing to the limit $\varepsilon \to 0$ in the resulting estimate, thanks to Corollary 3.4.

(2) Another way of proving Lemma 3.6 can be found in [30], Lemma 4.12.

Proof. Assume ρ is an absolutely continuous measure. Then, there exists an optimal map from ρ to μ for the cost $c(x,y) = \frac{1}{2}|x-y|^2$, which we denote T. Since $\nabla\psi \circ T = -\nabla\varphi$, using $\mu = T_\# \rho$ yields:

$$\begin{aligned}
\int \nabla\varphi \cdot a[\rho] d\rho + \int \nabla\psi \cdot a[\mu] d\mu &= \int \nabla\varphi \cdot (a[\rho] - a[\mu] \circ T) d\rho \\
&= -\iint \nabla\varphi(x) \cdot \nabla W(x-y) \rho(dy)\rho(dx) + \iint \nabla\varphi(x) \cdot \nabla W(T(x)-y) \mu(dy)\rho(dx) \\
&= -\iint \nabla\varphi(x) \cdot \Big(\nabla W(x-y) - \nabla W\big(T(x)-T(y)\big)\Big)\rho(dy)\rho(dx),
\end{aligned}$$

where we used once more $\mu = T_\#\rho$. Symmetrizing the above integral as in Lemma 2.8, since ∇W is odd, and using $\nabla\varphi = \mathrm{id} - T$, we get:

$$\begin{aligned}
\int \nabla\varphi \cdot a[\rho] d\rho + \int \nabla\psi \cdot a[\mu] d\mu &= -\frac{1}{2}\iint \big(\nabla\varphi(x) - \nabla\varphi(y)\big) \cdot \Big(\nabla W(x-y) - \nabla W\big(T(x)-T(y)\big)\Big)\rho(dy)\rho(dx) \\
&= -\frac{1}{2}\iint \Big(x - y - \big(T(x) - T(y)\big)\Big) \cdot \Big(\nabla W(x-y) - \nabla W\big(T(x)-T(y)\big)\Big)\rho(dy)\rho(dx) \\
&\leqslant -\frac{\lambda}{2}\iint |x - T(x) - (y - T(y))|^2 \rho(dy)\rho(dx),
\end{aligned}$$

where we used the $\lambda-$convexity of W. We then expand the square to obtain:

$$\iint |x-T(x)-(y-T(y))|^2 \rho(dy)\rho(dx) = 2\int |x-T(x)|^2\rho(dx) - 2\left(\iint \big(x - T(x)\big)\rho(dx)\right)^2 \leqslant 2W_2^2(\rho,\mu),$$

which concludes the proof, as we assumed in (A3) that $\lambda \leqslant 0$. □

We now come back to the proof of Corollary 3.4. Denoting $(\varphi_t^\varepsilon, \psi_t^\varepsilon)$ a pair of Kantorovitch potentials from ρ_t^ε to μ_t^ε, and using Lemma 3.6 along with equation (3.15), we get:

$$\frac{d}{dt}\frac{1}{2}W_2^2(\rho_t^\varepsilon, \mu_t^\varepsilon) \leqslant -\lambda W_2^2(\rho_t^\varepsilon, \mu_t^\varepsilon) - \varepsilon \int (\nabla\varphi_t^\varepsilon \cdot \nabla\rho_t^\varepsilon + \nabla\psi_t^\varepsilon \cdot \nabla\mu_t^\varepsilon).$$

The last term above being nonpositive (see [32] exercise 66 for instance), we obtain, using a Grönwall lemma, that $W_2(\rho_t^\varepsilon, \mu_t^\varepsilon) \leqslant e^{-\lambda t} W_2(\rho_0^\varepsilon, \mu_0^\varepsilon)$. We then write, for $t \in [0, T]$,

$$W_1(\rho_t^\varepsilon, \rho_t) \leqslant W_1(\rho_t^\varepsilon, \mu_t^\varepsilon) + W_1(\mu_t^\varepsilon, \rho_t) \leqslant e^{-\lambda T} W_2(\rho_0^\varepsilon, \mu_0^\varepsilon) + \sup_{s\in[0,T]} W_1(\mu_s^\varepsilon, \rho_s),$$

where we used the fact that $W_1 \leqslant W_2$. Since both sequences $(\rho_0^\varepsilon)_{\varepsilon>0}$ and $(\mu_0^\varepsilon)_{\varepsilon>0}$ converge in $\mathbb{W}_2(\mathbb{R}^d)$ to the same limit, $W_2(\rho_0^\varepsilon, \mu_0^\varepsilon)$ goes to 0 as $\varepsilon \to 0$. Moreover, $(\mu^\varepsilon)_{\varepsilon>0}$ converges to ρ in W_1 distance uniformly in $[0, T]$. These two facts along with the above inequality show that $(\rho^\varepsilon)_{\varepsilon>0}$ also converges to ρ in $\mathcal{C}([0, T], \mathbb{W}_1(\mathbb{R}^d))$. □

3.1.2 Convergence rate

We are now in position to prove the following theorem:

Theorem 3.8. *Assume W satisfies assumptions (A0)-(A1)-(A2)-(A3). Let $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$, and let $(\rho^\varepsilon)_{\varepsilon>0}$ be the sequence of weak solutions to* (1.1). *Here, we assume that $(\rho_0^\varepsilon)_{\varepsilon>0}$ is an arbitrary sequence in $\mathcal{P}_2(\mathbb{R}^d)$.*

Denoting $\rho \in \mathcal{C}([0, +\infty), \mathbb{W}_2(\mathbb{R}^d))$ the unique solution of (1.2) *with $a[\rho]$ being replaced by $\widehat{a}[\rho]$ as defined in* (1.3)*, we have the following estimate:*

$$\forall t > 0, \qquad W_2(\rho_t^\varepsilon, \rho_t) \leqslant e^{-\lambda t} W_2(\rho_0^\varepsilon, \rho^{ini}) + \sqrt{\frac{1 - e^{-2\lambda t}}{\lambda}}\sqrt{d\varepsilon}. \tag{3.17}$$

Please note that in the above estimate $\lambda \leqslant 0$. If $\lambda < 0$, $1 - e^{-2\lambda t}$ and λ are negative numbers so the ratio is positive and for $\lambda = 0$ the expression should be extended by continuity.

Remark 3.9. In dimension $d = 1$ with the Newtonian potential $W(x) = |x|$, the correspondence with Burgers' equation stated in Proposition 2.13, gives convergence at rate $\sqrt{\varepsilon t}$ in W_1 distance. Due to W being $0-$convex, our estimate leads to the same estimate but in W_2 distance, since taking $\lambda = 0$ in (3.17) gives $W_2(\rho_t^\varepsilon, \rho_t) \leqslant \sqrt{2d\varepsilon t}$ for any $t > 0$.

If assumption (A4$-p$) is satified for some $p \geqslant 1$ instead of assumption (A3) and if $\rho_0^\varepsilon = \delta_0$ for all $\varepsilon > 0$, one can also obtain the exact same estimate using a direct computation. Indeed, in that case, $\rho_t = \delta_0$ for all $t \geqslant 0$ and we have, using integration by parts and Lemma 2.8:

$$\begin{aligned}
\frac{d}{dt}W_2^2(\rho_t^\varepsilon, \delta_0) = \frac{d}{dt}\int |x|^2 \rho_t^\varepsilon(dx) &= -\iint \nabla W(x-y)\cdot(x-y)\rho_t^\varepsilon(dx)\rho_t^\varepsilon(dy) + 2\varepsilon \int \rho_t^\varepsilon(dx) \\
&\leqslant -C \iint |x-y|^p \rho_t^\varepsilon(dx)\rho_t^\varepsilon(dy) + 2\varepsilon d, \quad \text{using assumption (A4} - p), \\
&\leqslant 2\varepsilon d.
\end{aligned}$$

Hence $W_2(\rho_t^\varepsilon, \delta_0) \leqslant \sqrt{2d\varepsilon t}$ for all $t \geqslant 0$.

Proof. Take a sequence of initial data $(\mu_0^\varepsilon)_{\varepsilon>0}$ converging in $\mathbb{W}_2(\mathbb{R}^d)$ to ρ^{ini} as $\varepsilon \to 0$ and denote $(\mu_\varepsilon)_{\varepsilon>0}$ the sequence of solutions to (1.1) with such initial data. Let $\varepsilon > 0$. For all $\delta > 0$, using

Lemma 3.6 along with equation (3.15), we have, denoting (φ_t, ψ_t) a pair of Kantorovitch potentials for the quadratic cost from ρ_t^ε to ρ_t^δ and integrating by parts:

$$\frac{d}{dt}\frac{1}{2}W_2^2(\rho_t^\varepsilon, \mu_t^\delta) \leqslant -\lambda W_2^2(\rho_t^\varepsilon, \mu_t^\delta) - \varepsilon \int \nabla\varphi_t \cdot \nabla\rho_t^\varepsilon - \delta \int \nabla\psi_t \cdot \nabla\mu_t^\delta \leqslant -\lambda W_2^2(\rho_t^\varepsilon, \mu_t^\delta) + \varepsilon \int \Delta\varphi_t\, \rho_t^\varepsilon + \delta \int \Delta\psi_t\, \mu_t^\delta.$$

The map $x \longmapsto \varphi_t(x) - \frac{|x|^2}{2}$ being concave, $\nabla^2\varphi_t \leqslant I_d$, hence $\Delta\varphi_t \leqslant d$ and the same holds for ψ_t. Therefore:

$$\frac{d}{dt}W_2^2(\rho_t^\varepsilon, \mu_t^\delta) \leqslant -2\lambda W_2^2(\rho_t^\varepsilon, \mu_t^\delta) + 2(\varepsilon + \delta)d,$$

which gives the result after using a Grönwall lemma and passing to the limit $\delta \to 0$ thanks to Corollary 3.4. □

3.2 Method 2: using a numerical scheme

We now turn to a different proof of the previous result. This alternate proof will also allow to illustrate the results and the behavior of solutions with numerical results. Our main idea is to let, for a fixed $\varepsilon > 0$, $\rho_{\Delta x}^\varepsilon$ be a suitable numerical approximation of the viscous solution ρ^ε to the problem (1.1) with fixed initial data $\rho_0^\varepsilon = \rho^{ini}$, and then use the formalism of [13] to estimate the distance from this discretized solution to the solution ρ to the aggregation problem (1.2) in terms of ε:

$$\forall n \in \mathbb{N}, \quad W_2(\rho_{\Delta x}^{\varepsilon,n}, \rho_{t^n}) \leqslant C(t^n)\sqrt{\Delta x + \varepsilon},$$

under suitable stability conditions for the numerical scheme, and where $\Delta t > 0$ is the time step, $t^n = n\Delta t$ and $\Delta x > 0$ denotes the maximal space step. Proving the convergence of the scheme with fixed ε beforehand using compactness arguments and a Lax-Wendroff-type theorem, then letting $\Delta x \to 0$, we shall deduce:

$$\forall t > 0, \quad W_2(\rho_t^\varepsilon, \rho_t) \leqslant C(t)\sqrt{\varepsilon},$$

where we shall specify the constant $C(t)$. Note that our method also allows to deal with the case of arbitrary $\mathcal{P}_2(\mathbb{R}^d)$ initial data ρ_0^ε as in Theorem 3.8, but we choose to present it with initial data not depending on ε for the sake of clarity.

Let us be more specific. We consider a Cartesian mesh of $\mathbb{R}^d$ where the space step in the ith direction is denoted by $\Delta x_i > 0$. The nodes of the mesh are denoted by $x_J = (J_1\Delta x_1, \ldots, J_d\Delta x_d)$ for any $J = (J_1, \ldots, J_d) \in \mathbb{Z}^d$, and the cell centered on x_J is denoted by $C_J := [(J_1 - \frac{1}{2})\Delta x_1, (J_1 + \frac{1}{2})\Delta x_1] \times \ldots \times [(J_d - \frac{1}{2})\Delta x_d, (J_d + \frac{1}{2})\Delta x_d]$. We also denote by e_i the ith vector of the canonical basis of $\mathbb{R}^d$. We initialize our discretization with:

$$\rho_J^0 := \int_{C_J} \rho^{ini}(dx) \geqslant 0, \qquad J \in \mathbb{Z}^d, \tag{3.18}$$

and we consider an upwind type discretization for the aggregative part [14, 26, 13] and an explicit discretization for the diffusive part. It writes, for $n \in \mathbb{N}$,

$$\begin{aligned} \rho_J^{n+1} &= \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i}\Big((a_{iJ}^n)^+ \rho_J^n - (a_{iJ+e_i}^n)^- \rho_{J+e_i}^n - (a_{iJ-e_i}^n)^+ \rho_{J-e_i}^n + (a_{iJ}^n)^- \rho_J^n \Big) \\ &\quad + \varepsilon \sum_{i=1}^d \frac{\Delta t}{\Delta x_i^2}\left(\rho_{J+e_i}^n - 2\rho_J^n + \rho_{J-e_i}^n \right), \end{aligned} \tag{3.19}$$

where the macroscopic velocity is defined by:

$$a_{iJ}^n := -\sum_{K \in \mathbb{Z}^d} \rho_K^n\, D_i W_J^K, \qquad \text{where} \qquad D_i W_J^K := \widehat{\partial_{x_i} W}(x_J - x_K). \tag{3.20}$$

Note that, for the sake of simplicity, we drop, in this section, the superscripts ε when it comes to the discrete unknowns $(\rho_J^n)_{J\in\mathbb{Z}^d,n\in\mathbb{N}}$ but these unknowns always solve numerical schemes for the aggregation equation with viscosity $\varepsilon > 0$.

Since W is even, we also have $D_iW_J^K = -D_iW_K^J$ for all $J, K \in \mathbb{Z}^d$ and $i = 1, \ldots, d$. Using a symmetrization argument as in the continuous setting, we deduce the discrete equivalent of Lemma 2.8:

Lemma 3.10. *Denote, for $J, K \in \mathbb{Z}^d$, $DW_J^K = (D_1W_j^K, \ldots, D_dW_J^K)$ and whenever $(v_J)_{J\in\mathbb{Z}^d}$ is a discrete vector field on the mesh, $v_J = (v_{1J}, \ldots, v_{dJ}) \in \mathbb{R}^d$. For any $(v_J)_{J\in\mathbb{Z}^d}$, we have:*

$$\forall i = 1, \ldots, d, \qquad \sum_{J\in\mathbb{Z}^d} v_{iJ}\, a_{iJ}^n\, \rho_J^n = \frac{1}{2}\sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} D_iW_J^K\,(v_{iJ} - v_{iK})\,\rho_J^n\,\rho_K^n,$$

and therefore:

$$\sum_{J\in\mathbb{Z}^d} v_J \cdot a_J^n\, \rho_J^n = \frac{1}{2}\sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} DW_J^K \cdot (v_J - v_K)\,\rho_J^n\,\rho_K^n.$$

Proof. Using the definition of the macroscopic velocity and the fact that $D_iW_J^K = -D_iW_K^J$, we have:

$$\begin{aligned}\sum_{J\in\mathbb{Z}^d} v_{iJ}\, a_{iJ}^n\, \rho_J^n = -\sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} D_iW_J^K\, v_{iJ}\,\rho_J^n\,\rho_K^n &= \sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} D_iW_K^J\, v_{iJ}\,\rho_J^n\,\rho_K^n \\ &= \sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} D_iW_J^K\, v_{iK}\,\rho_J^n\,\rho_K^n,\end{aligned}$$

thanks to exchanging K and J in the latter sum. Taking the half sum of the first sum and the latter, we obtain:

$$\sum_{J\in\mathbb{Z}^d} v_{iJ}\, a_{iJ}^n\, \rho_J^n = \frac{1}{2}\sum_{J\in\mathbb{Z}^d}\sum_{K\in\mathbb{Z}^d} D_iW_J^K\,(v_{iJ} - v_{iK})\,\rho_J^n\,\rho_K^n.$$

Summing over $i = 1, \ldots, d$ concludes the proof. □

It is also natural to consider, instead of the explicit discretization of the Laplacian, an implicit discretization:

$$\begin{aligned}\rho_J^{n+1} &= \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i}\Big((a_{iJ}^n)^+\rho_J^n - (a_{iJ+e_i}^n)^-\rho_{J+e_i}^n - (a_{iJ-e_i}^n)^+\rho_{J-e_i}^n + (a_{iJ}^n)^-\rho_J^n\Big) \\ &\quad + \varepsilon\sum_{i=1}^d \frac{\Delta t}{\Delta x_i^2}\left(\rho_{J+e_i}^{n+1} - 2\rho_J^{n+1} + \rho_{J-e_i}^{n+1}\right),\end{aligned} \tag{3.21}$$

However, for the sake of simplicity, we only provide the proof of our convergence estimate for the explicit scheme (3.19), although our method would also works for the implicit discretization (3.21) but the computations are a bit more involved. Naturally, both schemes are asymptotic-preserving since they degenerate towards the upwind-type scheme of [13] when ε goes to 0.

One could also consider the $\theta-$scheme, for $\theta \in [0, 1]$, defined by:

$$\begin{aligned}\rho_J^{n+1} &= \rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i}\Big((a_{iJ}^n)^+\rho_J^n - (a_{iJ+e_i}^n)^-\rho_{J+e_i}^n - (a_{iJ-e_i}^n)^+\rho_{J-e_i}^n + (a_{iJ}^n)^-\rho_J^n\Big) \\ &\quad + \varepsilon(1-\theta)\sum_{i=1}^d \frac{\Delta t}{\Delta x_i^2}\left(\rho_{J+e_i}^n - 2\rho_J^n + \rho_{J-e_i}^n\right) + \varepsilon\theta\sum_{i=1}^d \frac{\Delta t}{\Delta x_i^2}\left(\rho_{J+e_i}^{n+1} - 2\rho_J^{n+1} + \rho_{J-e_i}^{n+1}\right),\end{aligned} \tag{3.22}$$

The point of defining such as scheme comes from the fact that, for the heat equation $\partial_t\rho = \varepsilon\Delta\rho$, under a parabolic CFL condition $\varepsilon\sum_{i=1}^{d}\frac{\Delta t}{\Delta x_i^2} \leqslant \frac{1}{2(1-2\theta)}$ if $\theta \in [0, 1/2)$ and unconditionally if $\theta \in [1/2, 1]$, the $\theta-$scheme is known to be convergent in L^2 norm at rate $O(\Delta t+\Delta x^2)$. Moreover, for $\theta = 1/2$, one obtains the so-called Crank-Nicolson scheme, which is convergent at rate $O(\Delta t^2 + \Delta x^2)$. However, the convergence order of the $\theta-$scheme (3.22) for the aggregation-diffusion equation (1.1a) will anyway be limited by the order of the upwind scheme. Also, the positivity of the density can only be guaranteed if the more restrictive parabolic CFL condition $a_\infty\sum_{i=1}^{d}\frac{\Delta t}{\Delta x_i} + 2\varepsilon(1-\theta)\sum_{i=1}^{d}\frac{\Delta t}{\Delta x_i^2} \leqslant 1$ holds. Preserving a hyperbolic CFL condition thus imposes taking $\theta = 1$, which corresponds to the implicit scheme (3.21).

Proposition 3.11. *Assume W satisfies assumptions (A0)-(A1)-(A2)-(A3) and let $\rho \in \mathcal{C}([0, +\infty), \mathbb{W}_2(\mathbb{R}^d))$ be the unique measure solution to the aggregation equation* (1.2) *with initial data $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ as given by Theorem 2.10. Assume in addition that the following strict CFL condition holds:*

$$\sum_{i=1}^{d}\left(a_\infty\frac{\Delta t}{\Delta x_i} + 2\varepsilon\frac{\Delta t}{\Delta x_i^2}\right) < \frac{1}{2}. \tag{3.23}$$

Denote also the reconstruction:

$$\rho_{\Delta x}^{\varepsilon,n} := \sum_{J\in\mathbb{Z}^d}\rho_J^n\delta_{x_J}, \quad n \in \mathbb{N}.$$

where $(\rho_J^n)_{J\in\mathbb{Z}^d,n\in\mathbb{N}}$ is defined through the explicit discretization (3.18)–(3.19)–(3.20). *Then, there exists a constant $C > 0$, depending only on λ, a_∞ and d, such that, for all $n \in \mathbb{N}^*$,*

$$W_2(\rho_{t^n}, \rho_{\Delta x}^{\varepsilon,n}) \leqslant C\sqrt{\frac{1-e^{-4\lambda t^n}}{\lambda}}\sqrt{\Delta x+\varepsilon} + e^{-2\lambda t^n}\Delta x. \tag{3.24}$$

Remark 3.12. In estimate (3.24), the $\sqrt{\Delta x+\varepsilon}$ term corresponds to the error induced by the scheme (3.19) and the Δx term corresponds to the finite volume discretization of the initial data (3.18). As in [13], one can also improve the prefactor in the exponentials to get the slightly better estimate:

$$W_2(\rho_{t^n}, \rho_{\Delta x}^{\varepsilon,n}) \leqslant C\sqrt{\frac{1-e^{-2\lambda t^n}}{\lambda}}\sqrt{\Delta x+\varepsilon} + e^{-\lambda t^n}\Delta x.$$

which is similar to the estimates of the continuous setting, for instance (2.9), when Δt is small.

In the above proposition as in the whole paper, we do as if our discrete reconstructions $(\rho_{\Delta x}^{\varepsilon})_{\Delta x>0}$ depended only on Δx. Rigorously speaking, they also depend on Δt, but under the CFL condition (3.23) Δt goes to 0 as Δx goes to 0. Setting Δt to be an adequate function of Δx, we can therefore consider $(\rho_{\Delta x}^{\varepsilon})_{\Delta x>0}$ as sequence labeled by Δx only.

Theorem 3.13. *Assume W satisfies assumptions (A0)-(A1)-(A2)-(A3). Let $\rho \in \mathcal{C}([0, +\infty), \mathbb{W}_2(\mathbb{R}^d))$ be the unique measure solution to the aggregation equation* (1.2) *with initial data $\rho^{ini} \in \mathcal{P}_2(\mathbb{R}^d)$ as given by Theorem 2.10 and let $(\rho^\varepsilon)_{\varepsilon>0}$ be the sequence of weak solutions to* (1.1) *with initial data $\rho_0^\varepsilon = \rho^{ini}$.*

Then, there exists a constant $C > 0$, depending only on λ, a_∞ and d, such that, for all $t > 0$ the following estimate holds:

$$W_2(\rho_t^\varepsilon, \rho_t) \leqslant C\sqrt{\frac{1-e^{-4\lambda t}}{\lambda}}\sqrt{\varepsilon}, \tag{3.25}$$

Remark 3.14. The estimate above is slightly worse than the estimate (3.17) that we obtain using gradient flow arguments. Although, as in the previous remark, the exponential factor can be improved to $e^{-2\lambda t}$ a bit more technical computations, we do not manage to obtain the same constant $C = \sqrt{d}$.

3.2.1 Properties of the scheme

Lemma 3.15. *As in the continuous setting, our discretization* (3.19) *preserves*

(1) total mass:

$$\forall n \in \mathbb{N}, \quad \sum_{J\in\mathbb{Z}^d} \rho_J^n = 1; \tag{3.26}$$

(2) positivity of the density and the bound on the velocity field:

$$\forall (n,J) \in \mathbb{N}\times\mathbb{Z}^d,\ \forall i=1,\dots,d, \quad \rho_J^n \geqslant 0, \qquad |a_{iJ}^{\,n}| \leqslant a_\infty,$$

under the CFL condition:

$$\sum_{i=1}^d \left(a_\infty \frac{\Delta t}{\Delta x_i} + 2\varepsilon \frac{\Delta t}{\Delta x_i^2} \right) \leqslant 1; \tag{3.27}$$

(3) the center of mass:

$$\forall n \in \mathbb{N}^*, \quad \sum_{J\in\mathbb{Z}^d} x_J \rho_J^n = \sum_{J\in\mathbb{Z}^d} x_J \rho_J^0.$$

Proof. The first item comes from summing equation (3.19) over $J \in \mathbb{Z}^d$. Moreover, using the following rewriting of ρ_J^{n+1} as a positive combination of ρ_J and $\rho_{J\pm e_i}$, $i=1,\dots,d$:

$$\begin{aligned}\rho_J^{n+1} = \rho_J^n \left[1 - \sum_{i=1}^d \left(\frac{\Delta t}{\Delta x_i} |a_{iJ}^{\,n}| + \frac{2\varepsilon\Delta t}{\Delta x_i^2} \right) \right] + \sum_{i=1}^d \rho_{J+e_i}^n \left(\frac{\Delta t}{\Delta x_i} (a_{iJ+e_i}^{\,n})^- + \frac{\varepsilon \Delta t}{\Delta x_i^2} \right) \\ + \sum_{i=1}^d \rho_{J-e_i}^n \left(\frac{\Delta t}{\Delta x_i} (a_{iJ-e_i}^{\,n})^+ + \frac{\varepsilon\Delta t}{\Delta x_i^2} \right),\end{aligned} \tag{3.28}$$

it is classical to prove the second item by induction on $n \in \mathbb{N}$, under the CFL condition (3.27).

Using the discretization (3.19) together with a discrete integration by parts, we have:

$$\begin{aligned}\sum_{J\in\mathbb{Z}^d} x_J \rho_J^{n+1} = & \sum_{J\in\mathbb{Z}^d} x_J\rho_J^n - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J\in\mathbb{Z}^d} \left((a_{iJ}^{\,n})^+ \rho_J^n (x_J - x_{J+e_i}) - (a_{iJ}^{\,n})^- \rho_J^n (x_{J-e_i} - x_J) \right) \\ & + \varepsilon \sum_{i=1}^d \frac{\Delta t}{\Delta x_i^2} \sum_{J\in\mathbb{Z}^d} (x_{J-e_i} - x_J)(\rho_{J+e_i}^n - \rho_J^n).\end{aligned}$$

By definition of x_J, we have $x_{J-e_i} - x_J = -\Delta x_i$. Hence, we deduce:

$$\sum_{J\in\mathbb{Z}^d} x_J\rho_J^{n+1} = \sum_{J\in\mathbb{Z}^d} x_J\rho_J^n + \Delta t \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} a_{iJ}^{\,n} \rho_J^n - \varepsilon \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J\in\mathbb{Z}^d} (\rho_{J+e_i}^n - \rho_J^n) = \sum_{J\in\mathbb{Z}^d} x_J\rho_J^n + \Delta t \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} a_{iJ}^{\,n}\rho_J^n.$$

Applying the symmetrization Lemma 3.10 to the constant vector field given by $v_J = (1,\dots,1) \in \mathbb{R}^d$ for all $J \in \mathbb{Z}^d$, we have $\sum_{J\in\mathbb{Z}^d} a_{iJ}^{\,n}\rho_J^n = 0$ for all $i=1,\dots,d$, hence the result. □

The following lemma ensures that $M_2(\rho_{\Delta x}^{\varepsilon,n})$ remains bounded over finite time. It turns out necessary for the proof of convergence of the scheme by compactness, in order to extract a converging subsequence.

Lemma 3.16 (Bound on the second moment)**.** *For all $n \in \mathbb{N}^*$, the following estimate holds:*

$$M_{2,\Delta x}^n := \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n \leqslant e^{-4\lambda t^n}\Big(M_{2,\Delta x}^0 + a_\infty t^n \sum_{i=1}^d \Delta x_i + 2d\varepsilon t^n\Big).$$

Proof. Using (3.19) and a discrete integration by parts, one can write:

$$\begin{aligned}\sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} &= \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n \\ &- \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J\in\mathbb{Z}^d} \Big[(a_{iJ}^n)^+ \rho_J^n \big(|x_J|^2 - |x_{J+e_i}|^2\big) - (a_{iJ}^n)^- \rho_J^n \big(|x_{J-e_i}|^2 - |x_J|^2\big)\Big] \\ &+ \varepsilon \sum_{i=1}^d \frac{\Delta t}{\Delta x_i} \sum_{J\in\mathbb{Z}^d} \big(|x_{J-e_i}|^2 - |x_J|^2\big)\big(\rho_{J+e_i} - \rho_J\big).\end{aligned}$$

By definition of x_J, $|x_J|^2 - |x_{J+e_i}|^2 = -2J_i\,\Delta x_i^2 - \Delta x_i^2$ and $|x_{J-e_i}|^2 - |x_J|^2 = -2J_i\,\Delta x_i^2 + \Delta x_i^2$. Therefore, we get:

$$\begin{aligned}\sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} = \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n + 2\Delta t \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} J_i \Delta x_i\, a_{iJ}^n\, \rho_J^n + \Delta t \sum_{i=1}^d \Delta x_i \sum_{J\in\mathbb{Z}^d} \rho_J^n |a_{iJ}^n| \\ + \varepsilon \Delta t \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} (-2J_i + 1)\Delta x_i \big(\rho_{J+e_i} - \rho_J\big).\end{aligned}$$

As a consequence of Lemma 3.15, we have $|a_{iJ}^n| \leqslant a_\infty$. Using, in addition, the mass conservation, we deduce that the penultimate term is bounded by $a_\infty \Delta t \sum_{i=1}^d \Delta x_i$. As for the last term, another integration by parts shows that the last term equals $2d\varepsilon\Delta t$. Finally, Lemma 3.10 applied to the discrete vector field given by $v_J = x_J$ yields:

$$\begin{aligned}2\Delta t \sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} J_i \Delta x_i\, a_{iJ}^n\, \rho_J^n = 2\Delta t \sum_{J\in\mathbb{Z}^d} x_J \cdot a_j^n\, \rho_J^n &= -\Delta t \sum_{J,K\in\mathbb{Z}^d} DW_J^K \cdot (x_J - x_K)\rho_J^n \rho_K^n \\ &\leqslant -\lambda \Delta t \sum_{J,K\in\mathbb{Z}^d} |x_J - x_K|^2 \rho_J^n \rho_K^n \\ &\leqslant -4\lambda\Delta t \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n,\end{aligned}$$

where we used the $\lambda-$convexity of W and the inequality $|x_J - x_K|^2 \leqslant 2(|x_J|^2 + |x_K|^2)$ along with the fact that λ is nonpositive. We obtain

$$\sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^{n+1} \leqslant \Big(1 - 4\lambda\Delta t\Big) \sum_{J\in\mathbb{Z}^d} |x_J|^2 \rho_J^n + a_\infty \Delta t \sum_{i=1}^d \Delta x_i + 2d\varepsilon\Delta t.$$

We conclude the proof using a discrete version of Grönwall's lemma. □

3.2.2 Proof of Proposition 3.11

Before going through the proof of Proposition 3.11, let us introduce, for $J \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$ the following coefficients:

$$\alpha_J(y) = \begin{cases} 1 - \sum_{i=1}^{d} \left(\dfrac{|\langle y - x_J, e_i \rangle|}{\Delta x_i} - \dfrac{2\varepsilon \Delta t}{\Delta x_i^2} \right) & \text{when } y \in C_J, \\ \dfrac{1}{\Delta x_i} \big(\langle y - x_{J-e_i}, e_i \rangle \big)^+ + \dfrac{\varepsilon \Delta t}{\Delta x_i^2} & \text{when } y \in C_{J-e_i}, \quad \text{for } i = 1, \dots, d, \\ \dfrac{1}{\Delta x_i} \big(\langle y - x_{J+e_i}, e_i \rangle \big)^- + \dfrac{\varepsilon \Delta t}{\Delta x_i^2} & \text{when } y \in C_{J+e_i}, \quad \text{for } i = 1, \dots, d, \\ 0 & \text{otherwise.} \end{cases} \tag{3.29}$$

It then holds that, for any $J, L \in \mathbb{Z}^d$,

$$\alpha_J\big(x_L + \Delta t a_L^n\big) = \begin{cases} 1 - \sum_{i=1}^{d} \left(|a_{i}{}_J^n| \dfrac{\Delta t}{\Delta x_i} - \dfrac{2\varepsilon \Delta t}{\Delta x_i^2} \right) & \text{when } L = J, \\ \dfrac{\Delta t}{\Delta x_i} \big(a_{i}{}_{J-e_i}^n \big)^+ + \dfrac{\varepsilon \Delta t}{\Delta x_i^2} & \text{when } L = J - e_i, \quad \text{for } i = 1, \dots, d, \\ \dfrac{\Delta t}{\Delta x_i} \big(a_{i}{}_{J+e_i}^n \big)^- + \dfrac{\varepsilon \Delta t}{\Delta x_i^2} & \text{when } L = J + e_i, \quad \text{for } i = 1, \dots, d, \\ 0 & \text{otherwise,} \end{cases} \tag{3.30}$$

so that we have the key identity:

$$\forall\, J \in \mathbb{Z}^d, \quad \rho_J^{n+1} = \sum_{L \in \mathbb{Z}^d} \rho_L^n \alpha_J\big(x_L + \Delta t a_L^n\big), \tag{3.31}$$

Lemma 3.17. *For any $y \in \mathbb{R}^d$, we have*

$$\sum_{L \in \mathbb{Z}^d} \alpha_L(y) = 1 \quad \text{and} \quad \sum_{L \in \mathbb{Z}^d} x_L \alpha_L(y) = y.$$

Proof. Let $J \in \mathbb{Z}^d$ such that $y \in C_J$. To prove the first claim, we just use the definition of $\alpha_L(y)$:

$$\begin{aligned} \sum_{L \in \mathbb{Z}^d} \alpha_L(y) &= \alpha_J(y) + \sum_{i=1}^{d} \big(\alpha_{J+e_i}(y) + \alpha_{J-e_i}(y) \big) \\ &= 1 - \sum_{i=1}^{d} \left(\frac{|\langle y - x_L, e_i \rangle|}{\Delta x_i} - \frac{2\varepsilon \Delta t}{\Delta x_i^2} \right) + \sum_{i=1}^{d} \frac{1}{\Delta x_i} \big(\langle y - x_J, e_i \rangle \big)^+ + \big(\langle y - x_J, e_i \rangle \big)^- + 2 \sum_{i=1}^{d} \frac{\varepsilon \Delta t}{\Delta x_i^2} = 1. \end{aligned}$$

As for the preservation of the barycenter, we once again using the definition of the coefficients $\alpha_L(y)$:

$$
\begin{aligned}
\sum_{L\in\mathbb{Z}^d} x_L\alpha_L(y) &= x_J\alpha_J(y) + \sum_{i=1}^d x_{J+e_i}\alpha_{J+e_i}(y) + \sum_{i=1}^d x_{J-e_i}\alpha_{J-e_i}(y) \\
&= x_J\left[1-\sum_{i=1}^d\left(\frac{|\langle y-x_J,e_i\rangle|}{\Delta x_i}-\frac{2\varepsilon\Delta t}{\Delta x_i^2}\right)\right] + \sum_{i=1}^d x_J\left(\frac{1}{\Delta x_i}(\langle y-x_J,e_i\rangle)^+ + \frac{\varepsilon\Delta t}{\Delta x_i^2}\right) \\
&+ \sum_{i=1}^d x_J\left(\frac{1}{\Delta x_i}(\langle y-x_J,e_i\rangle)^- + \frac{\varepsilon\Delta t}{\Delta x_i^2}\right) \\
&= x_J\left[1+\sum_{i=1}^d\left(-\frac{|\langle y-x_J,e_i\rangle|}{\Delta x_i}+\frac{1}{\Delta x_i}(\langle y-x_J,e_i\rangle)^+ + \frac{1}{\Delta x_i}(\langle y-x_J,e_i\rangle)^-\right)\right] \\
&+ \sum_{i=1}^d e_i\left((\langle y-x_J,e_i\rangle)^+ - (\langle y-x_J,e_i\rangle)^-\right) \\
&= x_J + \sum_{i=1}^d \langle y-x_J,e_i\rangle e_i \\
&= y.
\end{aligned}
$$

□

We now turn to the proof of Proposition 3.11.

For $n\in\mathbb{N}^*$, we denote $D_n := W_2(\rho_{t^n},\rho^{\varepsilon,n}_{\Delta x})$. The point is, roughly speaking, to obtain an estimate of the type $D_{n+1}^2 \leqslant D_n^2 + C\Delta t(\Delta t+\Delta x+\varepsilon)$ and then use a discrete Grönwall lemma to obtain estimate (3.24).

Let γ be an optimal transport plan between ρ_{t^n} and $\rho^{\varepsilon,n}_{\Delta x}$, so that $D_n^2 = \iint |x-y|^2\gamma(dx,dy)$. We also let $a^n_{\Delta x}$ be any continuous reconstruction of the velocity, for instance piecewise affine, such that $a^n_{\Delta x}(x_J) = a^n_J$ for all $J\in\mathbb{Z}^d$.

To construct an adequate coupling $\gamma'\in\Gamma(\rho_{t^{n+1}},\rho^{\varepsilon,n+1}_{\Delta x})$, recall that Theorem 2.10 gives $\rho_{t^{n+1}} = Z(t^{n+1},t^n,\cdot)\#\rho_{t^n}$, where Z is the Filippov characteristic flow associated to $\widehat{a}[\rho]$ given by Theorem 2.10, except that here the second variable of Z denotes the time of the Cauchy data (which is the third variable) whereas in Theorem 2.10 it was omitted as it was 0. If the discrete measure $\rho^{\varepsilon,n+1}_{\Delta x}$ was a pushforward measure of $\rho^{\varepsilon,n}_{\Delta x}$, we would also define γ' as a pushforward of γ, but it is not the case. Instead, if we denote by ν the kernel on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ given by:

$$
\forall (y,B)\in\mathbb{R}^d\times\mathcal{B}(\mathbb{R}^d),\qquad \nu(y,B) = \sum_{J\in\mathbb{Z}^d}\alpha_J(y+\Delta t a^n_{\Delta x}(y))\delta_{x_J}(B),
$$

we have the kernel representation:

$$
\forall B\in\mathcal{B}(\mathbb{R}^d),\qquad \rho^{\varepsilon,n+1}_{\Delta x}(B) = \int \nu(y,B)\rho^{\varepsilon,n}_{\Delta x}(dy).
$$

The pushforward $\rho_{t^{n+1}} = Z(t^{n+1},t^n,\cdot)\#\rho_{t^n}$ can also be seen as a kernel representation. Indeed, setting $\mu(x,A) = \delta_{Z(t^{n+1},t^n,x)}(A)$ for $(x,A)\in\mathbb{R}^d\times\mathcal{B}(\mathbb{R}^d)$, we have:

$$
\forall A\in\mathcal{B}(\mathbb{R}^d),\quad \rho_{t^{n+1}}(A) = \int \mathbf{1}_A(Z(t^{n+1},t^n,x))\rho_{t^n}(dx) = \int \delta_{Z(t^{n+1},t^n,x)}(A)\rho_{t^n}(dx) = \int \mu(x,A)\rho_{t^n}(dx).
$$

We now define the product kernel $\mathcal{K}$ on $\big(\mathbb{R}^d \times \mathbb{R}^d\big) \times \mathcal{B}\big(\mathbb{R}^d \times \mathbb{R}^d\big)$ by:

$$\mathcal{K}\big((x,y), A \times B\big) = \mu(x,A)\nu(y,B) = \delta_{Z(t^{n+1},t^n,x)}(A) \sum_{L\in\mathbb{Z}^d} \alpha_L\big(y + \Delta t a^n_{\Delta x}(y)\big)\delta_{x_L}(B)$$

and then set $\gamma'(A \times B) = \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{K}\big((x,y), A \times B\big)\gamma(dx,dy)$. Equivalently, for any $\theta \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{R}^d)$, we have:

$$\begin{aligned}\iint \theta(x,y)\gamma'(dx,dy) &= \iiiint \theta(x',y')\mu(x,dx')\nu(y,dy')\gamma(dx,dy)\\ &= \iint\Big[\sum_{L\in\mathbb{Z}^d} \theta\big(Z(t^{n+1};t^n,x),x_L\big)\alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\Big]\gamma(dx,dy).\end{aligned}$$

One can show as in [13] that the marginals of γ' are $\rho_{t^{n+1}}$ and $\rho^{\varepsilon,n+1}_{\Delta x}$. In particular, we have:

$$D^2_{n+1} \leqslant \iint |x-y|^2\gamma'(dx,dy).$$

Using the definition of γ', we get:

$$D^2_{n+1} \leqslant \iint \sum_{L\in\mathbb{Z}^d} \big|Z(t^{n+1};t^n,x) - x_L\big|^2 \alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\gamma(dx,dy). \tag{3.32}$$

Writing $Z(t^{n+1};t^n,x) - x_L = Z(t^{n+1};t^n,x) - (y+\Delta t a^n_{\Delta x}(y)) - \big(x_L - (y+\Delta t a^n_{\Delta x}(y))\big)$ and expanding the square, we obtain:

$$\begin{aligned}&\sum_{L\in\mathbb{Z}^d} \big|Z(t^{n+1};t^n,x) - x_L\big|^2 \alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\\ &= \big|Z(t^{n+1};t^n,x) - y - \Delta t a^n_{\Delta x}(y)\big|^2 + \sum_{L\in\mathbb{Z}^d} \big|x_L - y - \Delta t a^n_{\Delta x}(y)\big|^2 \alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\\ &\quad - 2\Big(Z(t^{n+1};t^n,x) - y - \Delta t a^n_{\Delta x}(y)\Big)\cdot\Big(\sum_{L\in\mathbb{Z}^d}\big(x_L - y - \Delta t a^n_{\Delta x}(y)\big)\alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\Big).\end{aligned} \tag{3.33}$$

Now, the last term in scalar product vanishes as we have, using Lemma 3.17,

$$\sum_{L\in\mathbb{Z}^d}\big(x_L - y - \Delta t a^n_{\Delta x}(y)\big)\alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big) = y + \Delta t a^n_{\Delta x}(y) - \big(y+\Delta t a^n_{\Delta x}(y)\big) = 0.$$

Plugging (3.33) into (3.32), we therefore deduce, using the fact that $\rho^{\varepsilon,n}_{\Delta x}$ is the second marginal of γ:

$$\begin{aligned}D^2_{n+1} \leqslant &\iint\big|Z(t^{n+1};t^n,x) - y - \Delta t a^n_{\Delta x}(y)\big|^2\gamma(dx,dy)\\ &+ \int\sum_{L\in\mathbb{Z}^d}\big|x_L - y - \Delta t a^n_{\Delta x}(y)\big|^2\alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\rho^{\varepsilon,n}_{\Delta x}(dy),\end{aligned} \tag{3.34}$$

Let us deal with the last term in the above inequality. We have $\rho^{\varepsilon,n}_{\Delta x}(y) = \sum_{J\in\mathbb{Z}^d}\rho^n_J\delta_{x_J}(y)$, therefore:

$$\begin{aligned}&\sum_{L\in\mathbb{Z}^d}\int\big|x_L - y - \Delta t a^n_{\Delta x}(y)\big|^2\alpha_L\big(y+\Delta t a^n_{\Delta x}(y)\big)\rho^{\varepsilon,n}_{\Delta x}(dy)\\ &\qquad= \sum_{J\in\mathbb{Z}^d}\sum_{L\in\mathbb{Z}^d}\big|x_L - x_J - \Delta t a^n_J\big|^2\alpha_L\big(x_J+\Delta t a^n_J\big)\rho^n_J.\end{aligned}$$

Moreover, using the definition of α_L in (3.29), we compute:

$$\sum_{L\in\mathbb{Z}^d} \big|x_L - x_J - \Delta t a_J^n\big|^2 \alpha_L\big(x_J + \Delta t a_J^n\big) = \Delta t^2 |a_J^n|^2 \left(1 - \sum_{i=1}^d \frac{\Delta t}{\Delta x_i}|a_{iJ}^n| - \sum_{i=1}^d \frac{2\varepsilon\Delta t}{\Delta x_i^2}\right)$$
$$+ \sum_{i=1}^d \left(\big|\Delta x_i e_i - \Delta t a_J^n\big|^2 \Big(\frac{\Delta t}{\Delta x_i}(a_{iJ}^n)^+ + \frac{\varepsilon\Delta t}{\Delta x_i^2}\Big) + \big|\Delta x_i e_i + \Delta t a_J^n\big|^2 \Big(\frac{\Delta t}{\Delta x_i}(a_{iJ}^n)^- + \frac{\varepsilon\Delta t}{\Delta x_i^2}\Big)\right)$$
$$\leqslant C\Delta t(\Delta t + \Delta x + \varepsilon),$$

where we used, for the last inequality, the CFL condition (3.23) and the fact that the velocity a_J^n is uniformly bounded. Multiplying by ρ_J^n, summing over $J \in \mathbb{Z}^d$, and injecting into (3.34) yields:

$$D_{n+1}^2 \leqslant \iint \big|Z(t^{n+1}; t^n, x) - y - \Delta t a_{\Delta x}^n(y)\big|^2 \gamma(dx, dy) + C\Delta t(\Delta t + \Delta x + \varepsilon). \tag{3.35}$$

Dealing with the first term amounts to estimating the distance between the exact characteristics $Z(t^{n+1}; t^n, x)$ and the forward Euler discretization $y + \Delta t a_{\Delta x}^n(y)$. To this end, we write, on the one hand, using the definition of the Filippov characteristics [17, 29]:

$$Z(t^{n+1}; t^n, x) = x + \int_{t^n}^{t^{n+1}} \widehat{a}_\rho\big(s, Z(s; t^n, x)\big) ds$$
$$= x - \int_{t^n}^{t^{n+1}} \int \widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big)\rho_{t^n}(d\xi) ds.$$

On the other hand, we have, whenever y is a node of the mesh,

$$y + \Delta t a_{\Delta x}^n(y) = y - \Delta t \int \widehat{\nabla W}(y - \zeta)\rho_{\Delta x}^n(d\zeta).$$

Thus, still for y a node of the mesh, we have:

$$\big|Z(t^{n+1}; t^n, x) - y - \Delta t a_{\Delta x}^n(y)\big|^2 \leqslant |x - y|^2$$
$$- 2\int_{t^n}^{t^{n+1}} \iint \Big(x - y\Big) \cdot \Big(\widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big) - \widehat{\nabla W}(y - \zeta)\Big)\rho_{t^n}(d\xi)\rho_{\Delta x}^{\varepsilon,n}(d\zeta) + C\Delta t^2.$$

Since $\gamma \in \Gamma(\rho_{t^n}, \rho_{\Delta x}^{\varepsilon,n})$ and since the above integral can be decoupled using the linearity of the scalar product, we also have:

$$\iint \Big(x - y\Big)\cdot\Big(\widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big) - \widehat{\nabla W}(y - \zeta)\Big)\rho_{t^n}(d\xi)\rho_{\Delta x}^{\varepsilon,n}(d\zeta)$$
$$= \iint \Big(x - y\Big) \cdot \Big(\widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big) - \widehat{\nabla W}(y - \zeta)\Big)\gamma(d\xi, d\zeta).$$

Injecting into (3.35), we get:

$$D_{n+1}^2 \leqslant D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon)$$
$$- 2\int_{t^n}^{t^{n+1}} \iiiint \Big(x - y\Big) \cdot \Big(\widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big) - \widehat{\nabla W}(y - \zeta)\Big)\gamma(d\xi, d\zeta)\gamma(dx, dy).$$

Decomposing $x - y = x - Z(s; t^n, x) + Z(s; t^n, x) - y$ and using the fact that $|Z(s; t^n, x) - x| \leqslant a_\infty|s - t^n|$, we get:

$$D_{n+1}^2 \leqslant D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon)$$
$$- 2\int_{t^n}^{t^{n+1}} \iiiint \Big(Z(s; t^n, x) - y\Big) \cdot \Big(\widehat{\nabla W}\big(Z(s; t^n, x) - Z(s; t^n, \xi)\big) - \widehat{\nabla W}(y - \zeta)\Big)\gamma(d\xi, d\zeta)\gamma(dx, dy).$$

Using the fact that W is even to symmetrize the last term as in Lemma 2.8, we obtain:

$$\begin{aligned} D_{n+1}^2 \leqslant\ & D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon) \\ & - \int_{t^n}^{t^{n+1}} \iiiint \Big(Z(s;t^n,x) - Z(s;t^n,\xi) - y + \zeta \Big) \cdot \\ & \Big(\widehat{\nabla W}\big(Z(s;t^n,x) - Z(s;t^n,\xi)\big) - \widehat{\nabla W}(y-\zeta) \Big)\, \gamma(d\xi, d\zeta)\gamma(dx,dy). \end{aligned}$$

The λ-convexity of W then yields:

$$D_{n+1}^2 \leqslant D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon) - \lambda \int_{t^n}^{t^{n+1}} \iiiint \big| Z(s;t^n,x) - y - Z(s;t^n,\xi) + \zeta \big|^2 \gamma(d\xi, d\zeta)\gamma(dx,dy).$$

Expanding the last term gives:

$$\begin{aligned} D_{n+1}^2 \leqslant D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon) - 2\lambda \int_{t^n}^{t^{n+1}} \iint \big| Z(s;t^n,x) - y \big|^2 \gamma(dx,dy) \\ + 2\lambda \int_{t^n}^{t^{n+1}} \left| \iint \big(Z(s;t^n,x) - y \big)\, \gamma(dx,dy) \right|^2. \end{aligned} \tag{3.36}$$

Now, since $\lambda \leqslant 0$, the last term above is nonpositive. It remains to estimate the penultimate term. Writing:

$$\big|Z(s;t^n,x) - y\big| \leqslant \big|Z(s;t^n,x) - x\big| + \big|x - y\big| \leqslant a_\infty \big|s - t^n\big| + \big|x-y\big|,$$

we deduce:

$$\big|Z(s;t^n,x) - y\big|^2 \leqslant 2\Big(a_\infty^2 \big|s-t^n\big|^2 + \big|x-y\big|^2\Big) \leqslant 2a_\infty^2 \Delta t^2 + 2\big|x-y\big|^2,$$

whenever $s \in [t^n, t^{n+1}]$. Integrating in space with respect to $\gamma(dx,dy)$ and integrating over $s \in [t^n, t^{n+1}]$, we obtain:

$$-2\lambda \int_{t^n}^{t^{n+1}} \iint \big| Z(s;t^n,x) - y \big|^2 \gamma(dx,dy) \leqslant -4\lambda a_\infty^2 \Delta t^3 - 4\lambda \Delta t D_n^2.$$

Together with (3.36), this yields:

$$D_{n+1}^2 \leqslant (1 - 4\lambda\Delta t) D_n^2 + C\Delta t(\Delta t + \Delta x + \varepsilon).$$

Using a discrete Grönwall lemma, we finally get:

$$D_n^2 \leqslant e^{-4\lambda t^n} D_0^2 + C\frac{1 - e^{-4\lambda t^n}}{4\lambda}(\Delta t + \Delta x + \varepsilon).$$

Now, one can easily prove that $D_0^2 = W_2^2(\rho^{ini}, \rho^0_{\Delta x}) \leqslant \Delta x^2$. This, along with the CFL condition (3.23), which implies that $\Delta t \leqslant C\Delta x$, gives the desired result.

3.2.3 Proof of Theorem 3.8

We are now in position to prove Theorem 3.8 using estimate (3.24) and passing to the limit $\Delta x \to 0$. To do so, we must verify that, for a given $\varepsilon > 0$, the approximate solutions given by the numerical scheme (3.19)–(3.18) converge, say uniformly in time (over a finite horizon) and weakly, in the sense of measures, in space, towards the solution ρ^ε to the aggregation-diffusion problem (1.1) with initial datum ρ^{ini}, as $\Delta x \to 0$. In all this section, ε is a fixed positive real number.

Let $T > 0$ and let $N \in \mathbb{N}$ be such that $N\Delta t = T$ where Δt satisfies the CFL condition. We consider the following piecewise affine reconstruction in time, defined for $t \in [0,T]$ by

$$\rho^{\varepsilon}_{\Delta x,t} := \sum_{n=0}^{N} \left(\frac{t^{n+1}-t}{\Delta t} \rho^{\varepsilon,n}_{\Delta x} + \frac{t-t^n}{\Delta t} \rho^{\varepsilon,n+1}_{\Delta x} \right) \mathbf{1}_{[t^n,t^{n+1}[}(t), \tag{3.37a}$$

$$\rho^{\varepsilon,n}_{\Delta x} := \sum_{J\in\mathbb{Z}^d} \rho^n_J \delta_{x_J}, \quad n = 0,\dots,N, \tag{3.37b}$$

where, once again, $(\rho^n_J)^{n=0,\dots,N}_{J\in\mathbb{Z}^d}$ is given by the explicit discretization (3.19)–(3.18) (it actually depends on ε but we drop this dependence for convenience). Lemmas 3.15 and 3.16 show that, for all $n \in \{0,\dots,N\}$, $\rho^{\varepsilon,n}_{\Delta x} \in \mathcal{P}_2(\mathbb{R}^d)$, hence $(\rho^{\varepsilon}_{\Delta x})_{\Delta x>0}$ is a collection, indexed by Δx, of curves in $\mathcal{C}([0,T],\mathbb{W}_1(\mathbb{R}^d))$ (they are actually curves on $\mathbb{W}_2(\mathbb{R}^d)$ but compactness arguments require to work in a smaller space).

Outline of the proof. We begin with assuming that $\rho^{ini} \in L^2(\mathbb{R}^d)$. Then, from $(\rho^{\varepsilon}_{\Delta x})_{\Delta x>0}$, we shall extract a subsequence, that we still denote $(\rho^{\varepsilon}_{\Delta x})_{\Delta x>0}$, converging in the $\mathcal{C}([0,T],\mathcal{M}_b(\mathbb{R}^d))$ topology towards a limit $\rho \in \mathcal{C}([0,T],\mathbb{W}_2(\mathbb{R}^d))$. To do so, we apply the Ascoli-Arzelà Theorem: the relative compactness assumption follows quite directly from the uniform bound on $M_2(\rho^{\varepsilon,n}_{\Delta x})$ that we proved in Lemma 3.16 ; the equicontinuity assumption, however, is more involved and requires discrete H^1 estimates (Lemma 3.18) in order to control the diffusive term. Then, using classical discrete integration by parts, we show that ρ solves the aggregation-diffusion initial value problem, the solution of which is unique, hence the whole sequence actually converges. Passing to the limit $\Delta x \to 0$ in estimate (3.24) will give us the desired estimate (3.25) for $L^2(\mathbb{R}^d)$ initial datum, and it will only remain to use a regularization argument to conclude in the case of arbitrary $\mathcal{P}_2(\mathbb{R}^d)$ initial datum.

Lemma 3.18. *For all $m \in \{0,\dots,N\}$, we have:*

$$\Delta t \sum_{n=0}^{m-1} \sum_{J\in\mathbb{Z}^d} \sum_{i=1}^{d} \left| \frac{\rho^n_{J+e_i} - \rho^n_J}{\Delta x_i} \right|^2 \leqslant C(a_\infty, d, \varepsilon, T) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho^0_J\right)^2}{2},$$

with $C(a_\infty,d,\varepsilon,T) = \frac{1}{2\varepsilon}\left(1 + \frac{8dTa_\infty^2}{\varepsilon} \sum_{J\in\mathbb{Z}^d} \exp\left(\frac{4(1+d)Ta_\infty^2}{\varepsilon}\right)\right)$.

Proof. The idea is to perform a discrete version of the following rationale. If ρ^ε solves (1.1) with $L^2(\mathbb{R}^d)$ initial data, we have:

$$\frac{d}{dt}\int \frac{\left(\rho^\varepsilon_t\right)^2}{2} = -\int \nabla\rho^\varepsilon_t \cdot (\nabla W * \rho^\varepsilon_t)\rho^\varepsilon_t - \varepsilon \int |\nabla \rho^\varepsilon_t|^2. \tag{3.38}$$

First, using an adequate Young inequality on the first term along with the fact that ∇W is bounded allows to absorb the $|\nabla\rho^\varepsilon_t|^2$ term into the last one, getting:

$$\frac{d}{dt}\int \frac{\left(\rho^\varepsilon_t\right)^2}{2} \leqslant -\frac{\varepsilon}{2}\int |\nabla\rho^\varepsilon_t|^2 + \frac{a_\infty^2}{\varepsilon}\int \frac{\left(\rho^\varepsilon_t\right)^2}{2} \leqslant \frac{a_\infty^2}{\varepsilon}\int \frac{\left(\rho^\varepsilon_t\right)^2}{2}.$$

A Grönwall Lemma then ensures that $\displaystyle\int \frac{\left(\rho^\varepsilon_t\right)^2}{2}$ remains bounded over finite time, where the bound depends on ε, but ε is fixed. Second, plugging back this bound into the above estimate gives a bound on $\int_0^T |\nabla\rho^\varepsilon_t|^2_{H^1(\mathbb{R}^d)}dt$. Let us reproduce these computations in the discrete setting.

Step 1: bound on $\sum_{J\in\mathbb{Z}^d}\frac{\left(\rho_J^n\right)^2}{2}$.

For the sake of compactness, let us note $F^n_{J+\frac{e_i}{2}} = ({a_i}^n_J)^+\rho^n_J - ({a_i}^n_{J+e_i})^-\rho^n_{J+e_i}$. Using twice the definition of the explicit scheme (3.19), we have:

$$\begin{aligned}
\sum_{J\in\mathbb{Z}^d}\frac{\left(\rho_J^{n+1}\right)^2-\left(\rho_J^n\right)^2}{2} &= \sum_{J\in\mathbb{Z}^d}\frac{\rho_J^{n+1}+\rho_J^n}{2}\left(\rho_J^{n+1}-\rho_J^n\right)\\
&= \sum_{J\in\mathbb{Z}^d}\frac{\rho_J^{n+1}+\rho_J^n}{2}\left(-\sum_{i=1}^d\frac{\Delta t}{\Delta x_i}\left(F^n_{J+\frac{e_i}{2}}-F^n_{J-\frac{e_i}{2}}\right)+\varepsilon\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left(\rho^n_{J+e_i}-2\rho^n_J+\rho^n_{J-e_i}\right)\right)\\
&= -\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i}\left(F^n_{J+\frac{e_i}{2}}-F^n_{J-\frac{e_i}{2}}\right)\rho^n_J+\varepsilon\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left(\rho^n_{J+e_i}-2\rho^n_J+\rho^n_{J-e_i}\right)\rho^n_J\\
&+\frac12\sum_{J\in\mathbb{Z}^d}\left(-\sum_{i=1}^d\frac{\Delta t}{\Delta x_i}\left(F^n_{J+\frac{e_i}{2}}-F^n_{J-\frac{e_i}{2}}\right)+\varepsilon\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left(\rho^n_{J+e_i}-2\rho^n_J+\rho^n_{J-e_i}\right)\right)^2\\
&:= S_1^n+S_2^n.
\end{aligned}$$

Performing discrete integrations by parts and using Young's inequality $|ab|\leqslant\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2}$ with $a=F^n_{J+\frac{e_i}{2}}$ and $b=\frac{\rho^n_{J+e_i}-\rho^n_J}{\Delta x_i}$, we can estimate S_1^n as follows:

$$\begin{aligned}
S_1^n &= \sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i}F^n_{J+\frac{e_i}{2}}\left(\rho^n_{J+e_i}-\rho^n_J\right)-\varepsilon\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2\\
&\leqslant \sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\Delta t\left(\frac{\left(F^n_{J+\frac{e_i}{2}}\right)^2}{2\varepsilon}+\frac{\varepsilon}{2}\left|\frac{\rho^n_{J+e_i}-\rho^n_J}{\Delta x_i}\right|^2\right)-\varepsilon\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2\\
&\leqslant \sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\Delta t\frac{\left(F^n_{J+\frac{e_i}{2}}\right)^2}{2\varepsilon}-\frac{\varepsilon}{2}\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2
\end{aligned}$$

As for S_2^n, straightforward computations and the repeated use of $(a\pm b)^2\leqslant 2a^2+2b^2$ lead to:

$$S_2^n\leqslant\sum_{i=1}^d\frac{4d\Delta t^2}{\Delta x_i^2}\sum_{J\in\mathbb{Z}^d}\left(F^n_{J+\frac{e_i}{2}}\right)^2+\sum_{i=1}^d 4d\left(\frac{\varepsilon\Delta t}{\Delta x_i^2}\right)^2\sum_{J\in\mathbb{Z}^d}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2.$$

Using the fact that:

$$\left(F^n_{J+\frac{e_i}{2}}\right)^2\leqslant\left(a_\infty\rho^n_J+a_\infty\rho^n_{J+e_i}\right)^2\leqslant 2a_\infty^2\left(\left(\rho^n_J\right)^2+\left(\rho^n_{J+e_i}\right)^2\right)$$

we deduce that $\sum_{J\in\mathbb{Z}^d}\left(F^n_{J+\frac{e_i}{2}}\right)^2\leqslant 4a_\infty^2\sum_{J\in\mathbb{Z}^d}\left(\rho^n_J\right)^2$. Reporting in both estimates we found on S_1^n

and S_2^n, and summing both, we obtain:

$$\sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^{n+1}\right)^2 - \left(\rho_J^n\right)^2}{2} \leqslant \left(\frac{4d\Delta t a_\infty^2}{\varepsilon} + \sum_{i=1}^d \frac{32 d a_\infty^2 \Delta t^2}{\Delta x_i^2}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} + \sum_{i=1}^d \left(-\frac{\varepsilon\Delta t}{2\Delta x_i^2} + 4d\left(\frac{\varepsilon\Delta t}{\Delta x_i^2}\right)^2\right) \sum_{J\in\mathbb{Z}^d} \left|\rho_{J+e_i}^n - \rho_J^n\right|^2. \tag{3.39}$$

Under the Courant-Friedrichs-Lewy condition

$$\varepsilon d\frac{\Delta t}{\Delta x_i^2} \leqslant \frac{1}{8} \quad \text{for any } i,$$

the last term in the above estimate is nonpositive, thus we get

$$\sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^{n+1}\right)^2 - \left(\rho_J^n\right)^2}{2} \leqslant \frac{4d\Delta t a_\infty^2}{\varepsilon}\left(1 + \sum_{i=1}^d \frac{8\varepsilon\Delta t}{\Delta x_i^2}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} \leqslant \frac{4d\Delta t a_\infty^2}{\varepsilon}\left(1+\frac{1}{d}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} = \frac{4\Delta t a_\infty^2}{\varepsilon}(1+d) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2}.$$

Using a discrete Grönwall Lemma, we deduce the following bound on the discrete L^2 norm of $\rho_{\Delta x}^{\varepsilon,n}$:

$$\sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} \leqslant \exp\left(\frac{4(1+d)t^n a_\infty^2}{\varepsilon}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^0\right)^2}{2}.$$

Step 2: discrete H^1 bound.
Assume a stricter CFL condition: there exists δ such that

$$\varepsilon d\frac{\Delta t}{\Delta x_i^2} \leqslant \delta < \frac{1}{8} \quad \text{for any } i. \tag{3.40}$$

Then, for any i,

$$4d\left(\frac{\varepsilon\Delta t}{\Delta x_i^2}\right)^2 - \frac{\varepsilon\Delta t}{2\Delta x_i^2} = \frac{\varepsilon\Delta t}{2\Delta x_i^2}\left(8d\frac{\varepsilon\Delta t}{\Delta x_i^2} - 1\right) \leqslant \frac{\delta}{d}(8\delta - 1) < 0.$$

Thus, thanks to 3.39,

$$\sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} \left|\rho_{J+e_i}^n - \rho_J^n\right|^2 \leqslant \frac{d}{\delta(1-8\delta)}\left(\left(\frac{4d\Delta t a_\infty^2}{\varepsilon} + \sum_{i=1}^d \frac{32 d a_\infty^2 \Delta t^2}{\Delta x_i^2}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} - \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^{n+1}\right)^2 - \left(\rho_J^n\right)^2}{2}\right) \leqslant \frac{d}{\delta(1-8\delta)}\left(\left(\frac{4d\Delta t a_\infty^2}{\varepsilon} + \sum_{i=1}^d \frac{4a_\infty^2\Delta t}{\varepsilon}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^n\right)^2}{2} - \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^{n+1}\right)^2 - \left(\rho_J^n\right)^2}{2}\right)$$

which implies, thanks to the L^2 estimate,

$$\sum_{i=1}^d \sum_{J\in\mathbb{Z}^d} \left|\rho_{J+e_i}^n - \rho_J^n\right|^2 \leqslant \frac{d}{\delta(1-8\delta)}\left(\frac{8d\Delta t a_\infty^2}{\varepsilon}\exp\left(\frac{4(1+d)t^n a_\infty^2}{\varepsilon}\right) \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^0\right)^2}{2} - \sum_{J\in\mathbb{Z}^d} \frac{\left(\rho_J^{n+1}\right)^2 - \left(\rho_J^n\right)^2}{2}\right)$$

Summing over $n = 0, \dots, m-1$ yields

$$\sum_{n=0}^{m-1}\sum_{i=1}^{d}\sum_{J\in\mathbb{Z}^d}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2 \leqslant \frac{d}{\delta(1-8\delta)}\left(\frac{8dTa_\infty^2}{\varepsilon}\exp\left(\frac{4(1+d)Ta_\infty^2}{\varepsilon}\right)\sum_{J\in\mathbb{Z}^d}\frac{\left(\rho^0_J\right)^2}{2} - \sum_{J\in\mathbb{Z}^d}\frac{\left(\rho^m_J\right)^2}{2} + \sum_{J\in\mathbb{Z}^d}\frac{\left(\rho^0_J\right)^2}{2}\right).$$

Finally

$$\sum_{n=0}^{m-1}\sum_{i=1}^{d}\sum_{J\in\mathbb{Z}^d}\left|\rho^n_{J+e_i}-\rho^n_J\right|^2 \leqslant \frac{d}{\delta(1-8\delta)}\left(1+\frac{8dTa_\infty^2}{\varepsilon}\sum_{J\in\mathbb{Z}^d}\exp\left(\frac{4(1+d)Ta_\infty^2}{\varepsilon}\right)\right)\sum_{J\in\mathbb{Z}^d}\frac{\left(\rho^0_J\right)^2}{2}.$$

This is the desired result choosing $\delta = 1/16$. □

We now resume the proof of Theorem 3.13. From now on, we always assume condition (3.40) to hold.

Step 1: Ascoli-Arzelà Theorem. Let us denote, for $K \subset \mathbb{R}^d$ any compact set, $Lip_K := \mathcal{C}_c(K) \cap W^{1,\infty}(\mathbb{R}^d)$ the space of Lipschitz continuous functions supported in K and $\|\cdot\|_{Lip}$ the Lipschitz seminorm. We then introduce the pseudo-distance defined in duality with $\|\cdot\|_{Lip}$ by:

$$\forall \mu,\nu \in \mathcal{P}_1(\mathbb{R}^d), \quad W_{1,K}(\mu,\nu) := \sup_{\phi\in Lip_K,\,\|\phi\|_{Lip}\leqslant 1}\int \phi\, d(\mu-\nu),$$

For $0 \leqslant s < t \leqslant T$, we have, thanks to the Cauchy-Schwarz inequality:

$$W_{1,K}\big(\rho^\varepsilon_{\Delta x,t},\rho^\varepsilon_{\Delta x,s}\big) = \int_s^t \left|(\rho^\varepsilon_{\Delta x,\tau})'\right|d\tau \leqslant \sqrt{t-s}\sqrt{\int_s^t \left|(\rho^\varepsilon_{\Delta x,\tau})'\right|^2 d\tau}. \tag{3.41}$$

Here, the metric derivative is the one associated to the pseudo-distance $W_{1,K}$. Since we chose $\rho^\varepsilon_{\Delta x}$ to be the piecewise affine reconstruction of the $\rho^{\varepsilon,n}_{\Delta x}$ for $n = 0,\dots N$, we have, for $\tau \in [t^n, t^{n+1}[$, $\left|(\rho^\varepsilon_{\Delta x,\tau})'\right| = \frac{1}{\Delta t}W_{1,K}(\rho^{\varepsilon,n}_{\Delta x},\rho^{\varepsilon,n+1}_{\Delta x})$. Indeed, $\rho^\varepsilon_{\Delta x}$ is a constant-speed geodesic in $\mathbb{W}_1(K)$ from $\rho^{\varepsilon,n}_{\Delta x}$ to $\rho^{\varepsilon,n+1}_{\Delta x}$, hence its length on $[t^n, t^{n+1}[$ equals $\Delta t\left|(\rho^\varepsilon_{\Delta x,\tau})'\right|$ by definition and $W_{1,K}(\rho^{\varepsilon,n}_{\Delta x},\rho^{\varepsilon,n+1}_{\Delta x})$ by the Benamou-Brenier formula. Therefore:

$$\int_s^t \left|(\rho^\varepsilon_{\Delta x,\tau})'\right|^2 d\tau \leqslant \int_0^T \left|(\rho^\varepsilon_{\Delta x,\tau})'\right|^2 d\tau = \sum_{k=0}^{N-1}\int_{t^n}^{t^{n+1}}\left|(\rho^\varepsilon_{\Delta x,\tau})'\right|^2 d\tau = \sum_{k=0}^{N-1}\frac{W^2_{1,K}(\rho^{\varepsilon,n}_{\Delta x},\rho^{\varepsilon,n+1}_{\Delta x})}{\Delta t}. \tag{3.42}$$

Now, let $\phi \in Lip_K$ such that $\|\phi\|_{Lip} \leqslant 1$. We have, denoting $\phi_J = \phi(x_J)$ and using the definition of the scheme (3.19) along with discrete integrations by parts in space:

$$\begin{aligned}
\int \phi\, d\big(\rho^{\varepsilon,n+1}_{\Delta x} - \rho^{\varepsilon,n}_{\Delta x}\big) &= \sum_{J\in\mathbb{Z}^d}\phi_J\left(\rho^{n+1}_J - \rho^n_J\right)\\
&= \sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d \frac{\Delta t}{\Delta x_i}F^n_{J+\frac{e_i}{2}}\Big(\phi_{J+e_i}-\phi_J\Big) - \varepsilon\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\frac{\Delta t}{\Delta x_i^2}\left(\rho^n_{J+e_i}-\rho^n_J\right)\Big(\phi_{J+e_i}-\phi_J\Big)\\
&\leqslant 2da_\infty\Delta t + \varepsilon\Delta t\sum_{J\in\mathbb{Z}^d}\sum_{i=1}^d\left|\frac{\rho^n_{J+e_i}-\rho^n_J}{\Delta x_i}\right|.
\end{aligned}$$

Taking the supremum over ϕ and using $(a+b)^2 \leqslant 2a^2 + 2b^2$, we get:

$$W_{1,K}^2(\rho_{\Delta x}^{\varepsilon,n}, \rho_{\Delta x}^{\varepsilon,n+1}) \leqslant 8d^2 a_\infty^2 \Delta t^2 + 2\varepsilon^2 \Delta t^2 \left(\sum_{J\in\mathbb{Z}^d} \sum_{i=1}^d \left| \frac{\rho_{J+e_i}^n - \rho_J^n}{\Delta x_i} \right| \right)^2$$
$$\leqslant 8d^2 a_\infty^2 \Delta t^2 + 2\varepsilon^2 \Delta t^2 \frac{d Leb(K)}{\prod_{i=1}^d \Delta x_i} \sum_{J\in\mathbb{Z}^d} \sum_{i=1}^d \left| \frac{\rho_{J+e_i}^n - \rho_J^n}{\Delta x_i} \right|^2,$$

where we used a discrete Cauchy-Schwarz inequality so as to use the discrete H^1 estimate we proved in Lemma 3.18: indeed, summing for $n = 0, \dots, N-1$ and plugging into (3.42), we obtain, using the aforementioned Lemma:

$$\int_s^t \left|(\rho_{\Delta x,\tau}^\varepsilon)'\right|^2 d\tau \leqslant 8d^2 a_\infty^2 T + 2d\varepsilon^2 \frac{Leb(K)}{\prod_{i=1}^d \Delta x_i} \Delta t \sum_{n=0}^{N-1} \sum_{J\in\mathbb{Z}^d} \sum_{i=1}^d \left| \frac{\rho_{J+e_i}^n - \rho_J^n}{\Delta x_i} \right|^2$$
$$\leqslant C(a_\infty, d, \varepsilon, T, K) \left(1 + \frac{1}{\prod_{i=1}^d \Delta x_i} \sum_{J\in\mathbb{Z}^d} \left(\rho_J^0\right)^2 \right). \tag{3.43}$$

Now, since we assumed that $\rho^{ini} \in L^2(\mathbb{R}^d)$, the term $\frac{1}{\prod_{i=1}^d \Delta x_i} \sum_{J\in\mathbb{Z}^d} \left(\rho_J^0\right)^2$ is bounded with respect to Δx. Indeed, a Cauchy-Schwarz inequality along with our initialization of the scheme (3.18) yield:

$$\sum_{J\in\mathbb{Z}^d} \left(\rho_J^0\right)^2 = \sum_{J\in\mathbb{Z}^d} \left(\int_{C_J} \rho^{ini} \right)^2$$
$$\leqslant \sum_{J\in\mathbb{Z}^d} Leb(C_J) \int_{C_J} \left(\rho^{ini}\right)^2 = \left(\prod_{i=1}^d \Delta x_i \right) \sum_{J\in\mathbb{Z}^d} \int_{C_J} \left(\rho^{ini}\right)^2 = \left(\prod_{i=1}^d \Delta x_i \right) \|\rho^{ini}\|_{L^2}.$$

Reporting into (3.43), we obtain a bound on $\int_s^t \left|(\rho_{\Delta x,\tau}^\varepsilon)'\right|^2 d\tau$ that is uniform with respect to s, t and Δx. Combining with (3.41), we deduce that $(\rho_{\Delta x}^\varepsilon)_{\Delta x>0}$ is equi-$\frac{1}{2}$-Hölder and in particular, equicontinuous in $\mathcal{C}([0,T], (Lip_K)')$. Lemma 3.16 ensures, in addition, that $M_2(\rho_{\Delta x,t}^\varepsilon)$ is uniformly bounded with respect to $t \in [0,T]$ and $\Delta x > 0$. Using Lemma 2.6, we deduce that $(\rho_{\Delta x,t}^\varepsilon)_{\Delta x>0}$ lies in a relatively compact set for all $t \in [0,T]$ and $\Delta x > 0$. We can therefore apply the Ascoli-Arzelà Theorem along with a diagonal extraction to extract a subsequence, that we still denote $(\rho_{\Delta x}^\varepsilon)_{\Delta x>0}$, converging in $\mathcal{C}([0,T], \mathcal{M}_b(\mathbb{R}^d))$ topology towards some ρ^ε. The uniform bound on $M_2(\rho_{\Delta x,t}^\varepsilon)$ combined with Lemma 2.6 ensures that ρ^ε actually belongs to $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$.

Step 2: ρ^ε solves (1.1)**.**

Using discrete integrations by parts as in [11, 26], we can prove that $\rho_{\Delta x}^\varepsilon$ satisfies the following approximate weak form of (1.1), for any $\phi \in \mathcal{C}([0,T[\times\mathbb{R}^d)$:

$$\int_0^T \int \partial_t \phi(t,x) \rho_{\Delta x,t}^\varepsilon(dx) dt + \int_0^t \int \widehat{a}[\rho_{\Delta x,t}^\varepsilon] \cdot \nabla \phi(t,x) \rho_{\Delta x,t}^\varepsilon(dx) dt + \int \phi(0,x) \rho^{ini}(dx)$$
$$= \varepsilon \int_0^T \int \Delta \phi(t,x) \rho_{\Delta x,t}^\varepsilon(dx) + O(\Delta x + \Delta t). \tag{3.44}$$

Passing to the limit $\Delta x \to 0$ in (3.44) is straightforward for the linear terms since $\rho_{\Delta x,t}^\varepsilon \xrightarrow[\Delta x\to 0]{*} \rho_t^\varepsilon$ uniformly in time. For the nonlinear term, this convergence also ensures that $\rho_{\Delta x,t}^\varepsilon \otimes \rho_{\Delta x,t}^\varepsilon \xrightarrow[\Delta x\to 0]{*} \rho_t^\varepsilon \otimes \rho_t^\varepsilon$. Then, passing to the limit is done using a symmetrization argument as in equations (3.9)-(3.10)-(3.11) using the fact that W is Lipschitz and even.

We deduce that ρ^ε solves in the sense of distributions the aggregation-diffusion problem (1.1) with initial datum $\rho_0^\varepsilon = \rho^{ini}$. Since such a solution is unique (see Theorem 2.11), we deduce that the whole initial sequence $(\rho^\varepsilon_{\Delta x})_{\Delta x>0}$ actually converges towards ρ^ε.

Step 3: passing to the limit in (3.24) and relaxing the assumption $\rho^{ini} \in L^2(\mathbb{R}^d)$.
Now, let $t > 0$ and let $n \in \{0, \dots, N\}$ such that $t \in [t^n, t^{n+1}[$. Estimate (3.24) gives:

$$W_2(\rho_t, \rho^\varepsilon_{\Delta x,t}) \leqslant C\sqrt{\frac{1-e^{-4\lambda t}}{\lambda}}\sqrt{\Delta x + \varepsilon} + e^{-2\lambda t}\Delta x.$$

Passing to the limit $\Delta x \to 0$ in the above estimate using the semicontinuity of W_2 then gives the desired estimate (3.25), hence proving Theorem 3.13 in case of $L^2(\mathbb{R}^d)$ initial datum.

Remark 3.19. As a byproduct of this proof, we obtain uniform in time convergence in W_1 distance in space of the numerical scheme (3.19)–(3.18) towards the $\mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$ distributional solution to the aggregation-diffusion initial value problem, in case of $L^2(\mathbb{R}^d)$ initial datum, and under 1/6-CFL condition. In fact, we expect this convergence result to hold for arbitrary $\mathcal{P}_2(\mathbb{R}^d)$ initial datum and under the standard CFL condition:

$$\sum_{i=1}^d \left(a_\infty \frac{\Delta t}{\Delta x_i} + 2\varepsilon\frac{\Delta t}{\Delta x_i^2}\right) \leqslant \frac{1}{6}.$$

4 Convergence for repulsive potentials such that $\Delta W \leqslant 0$ and $\nabla^2 W \in L^{p_0}(\mathbb{R}^d)$

For any Lipschitz potential satifying assumptions (A0)-(A1)-(A2), Theorem 3.1 guarantees the convergence of ρ^ε towards a solution ρ to the aggregation equation up to a subsequence if the initial data satisfies the assumptions (3.3). Then, Corollary 3.4 extended this result to arbitrary initial data by an approximation procedure, and using $\lambda-$convexity to estimate the distance between two solutions. The goal of this section is to proceed similarly in the case of repulsive potentials, typically $W(x) = -|x|$, where $\lambda-$convexity will be replaced by some intergability of the Hessian. More precisely, we focus on initial data equal to ρ^{ini}, for which we only assume finiteness of moments.

The outline of the proof is the same as that of Corollary 3.4. However, we can no more use the $\lambda-$convexity of W but, using the additional assumption $\nabla^2 W \in L^{p_0}(\mathbb{R}^d)$ for a suitable p_0, we still manage to estimate the distance between ρ_t^ε and a sequence of viscous solutions associated with smoothed out initial data. More precisely, we obtain the following result:

Theorem 4.1. *Let W be an interaction potential satisfying assumptions (A0)-(A1)-(A2) along with the additional assumption:*

$$\text{(A5)}: \qquad \Delta W \leqslant 0 \quad \textit{and} \quad \nabla^2 W \in L^{p_0}(\mathbb{R}^d) \textit{ for some } p_0 > \max\left(\frac{d}{2}, 1\right),$$

and let ρ^{ini} be an initial datum belonging to $\in \mathcal{P}_2(\mathbb{R}^d)$. Denote $(\rho^\varepsilon)_{\varepsilon>0}$ the sequence of weak solutions to (1.1) where the initial data is set to $\rho_0^\varepsilon := \rho^{ini}$ for all $\varepsilon > 0$.

Then, for all $T > 0$, the sequence $(\rho^\varepsilon)_{\varepsilon>0}$ converges in $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$, up to an extraction, towards a solution $\rho \in \mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$ to equation (1.2) with the velocity field $a[\rho]$ being replaced by $\widehat{a}[\rho]$ as defined in (1.3).

If, in addition, $\rho^{ini} \in L^{p_0'}(\mathbb{R}^d) \cap L^{\frac{p_0}{p_0-p}}(\mathbb{R}^d)$, then there exists a unique solution in $\mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d)) \cap L^\infty([0,T], L^{p_0'}(\mathbb{R}^d) \cap L^{\frac{p_0}{p_0-p}}(\mathbb{R}^d))$ to (1.2) and the whole sequence $(\rho^\varepsilon)_{\varepsilon>0}$ actually converges.

Remark 4.2.

(1) For $W(x) = -|x|$, this result cannot be applied in dimension $d = 1$, since $\nabla^2 W = -\delta_0$ is not integrable. When $d > 1$, we have $\nabla^2 W(x) = \frac{\frac{x}{|x|} \otimes \frac{x}{|x|} - I_d}{|x|} \sim \frac{1}{|x|}$, hence $\nabla^2 W \in L^{p_0}$ if and only if $p_0 < d$ (up to cutting off the potential at infinity) and therefore we can find $p_0 \in \left(\frac{d}{2}, d\right)$ so as to apply our result.

(2) In dimension $d = 1$, for $W(x) = -|x|$, Proposition 2.13 shows that the whole sequence $(\rho^\varepsilon)_{\varepsilon>0}$ converges in $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}))$ towards a solution to the aggregation equation that can be obtained as the derivative of the entropy solution to a Burgers-type equation since entropy solutions and viscosity solutions coincide for scalar conservation laws.

(3) As a byproduct of our result, one obtains existence of a solution in $\mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$ to the aggregation problem (1.2) for potentials satisfying (A0)-(A1)-(A2)-(A5).

Proof. Let $T > 0$. As in the proof of Corollary 3.4, for $\varepsilon > 0$, we introduce $\mu^\varepsilon \in \mathcal{C}([0,T], \mathbb{W}_2(\mathbb{R}^d))$ solution to (1.1) with smoothed out initial data μ_0^ε, that we now assume to satisfy assumptions (3.4) for some $\alpha \in (-1, 0)$. In particular, $(\mu_0^\varepsilon)_{\varepsilon>0}$ satisfies assumptions (3.3) and Theorem 3.1 applies to $(\mu^\varepsilon)_{\varepsilon>0}$ and guarantees convergence of a subsequence, in $\mathcal{C}([0,T], \mathbb{W}_1(\mathbb{R}^d))$, towards a solution to the aggregation equation (1.2). As for Corollary 3.4, the key ingredient is now to prove that the distance $W_p(\rho_t^\varepsilon, \mu_t^\varepsilon)$ goes to 0 as $\varepsilon \to 0$, for some $p > 1$ that will be specified later.

For the sake of clarity, let us drop the superscripts ε for the remaining of this section.

Denoting (φ_t, ψ_t) a pair of Kantorovitch potentials from ρ_t to μ_t for the cost $\frac{1}{p}|x - y|^p$, we can formally write (see Theorem 5.24 in [32] or Theorem 8.4.7. in [1])

$$\frac{1}{p}\frac{d}{dt} W_p^p(\rho_t, \mu_t) = \int \nabla\varphi_t \cdot a[\rho_t] d\rho_t + \int \nabla\psi_t \cdot a[\mu_t] d\mu_t - \varepsilon \int \Big(\nabla\varphi_t \cdot \nabla\rho_t + \nabla\psi_t \cdot \nabla\mu_t\Big) dx.$$

The last term above is nonnegative thanks to the so-called five (actually four) gradients inequality proven in [7] for the W_p case with $p > 1$. Actually, [7] proves the inequality in a compact setting and a full treatment of this last term would require a suitable approximation procedure. Yet, the inequality we need, i.e.

$$\frac{1}{p}\frac{d}{dt} W_p^p(\rho_t, \mu_t) \leqslant \int \nabla\varphi_t \cdot a[\rho_t] d\rho_t + \int \nabla\psi_t \cdot a[\mu_t] d\mu_t$$

can also be justified in many different ways, for instance by the stochastic interpretation of ρ_t and μ_t as laws of the solutions of suitable SDE where the choice of a common Brownian motion would allow to get rid of the term coming from diffusion (see, for instance, [4]); since the diffusion effect of the Laplacian in the equation could also be handled using convolution with the heat kernel, another possible way to prove the same inequality would be to approximate the solutions by a splitting method, alternating convolutions (which decrease the W_p distance) and transport (which lets the other term appear).

We thus get, using a triangle inequality along with the fact that $\nabla\varphi_t(x) = |x - T_t(x)|^{p-1}(\widehat{x - T_t(x)}) = -\nabla\psi_t(x)$, where T_t is the optimal transport map from ρ_t to μ_t (which exists since $\rho_t \ll \text{Leb}$ whenever $\varepsilon > 0$):

$$\frac{1}{p}\frac{d}{dt} W_p^p(\rho_t, \mu_t) \leqslant |I_1| + |I_2|, \tag{4.1a}$$

$$I_1 = \int |x - T_t(x)|^{p-1}(\widehat{x - T_t(x)}) \cdot (a[\rho_t](x) - a[\rho_t] \circ T_t(x)) \rho_t(dx), \tag{4.1b}$$

$$I_2 = \int |x - T_t(x)|^{p-1}(\widehat{x - T_t(x)}) \cdot (a[\rho_t] \circ T_t(x) - a[\mu_t] \circ T_t(x)) \rho_t(dx). \tag{4.1c}$$

To estimate I_1, we use the following bound on the Lipschitz constant of $a[\rho_t]$:

$$\mathrm{Lip}(a[\rho_t]) = \|\nabla^2 W * \rho_t\|_{L^\infty} \leqslant \|\nabla^2 W\|_{L^{p_0}} \|\rho_t\|_{L^{p_0'}}.$$

We deduce:

$$|I_1| \leqslant \mathrm{Lip}(a[\rho_t]) \int |x - T_t(x)|^p \rho_t(dx) \leqslant \|\nabla^2 W\|_{L^{p_0}} \|\rho_t\|_{L^{p_0'}} W_p^p(\rho_t, \mu_t).$$

To estimate I_2, we first apply a Hölder inequality w.r.t the measure $\rho_t(dx)$ and with the exponents (p', p). We get, since $p'(p-1) = p$:

$$|I_2| \leqslant \left(\int |x - T_t(x)|^p \rho_t(dx) \right)^{1/p'} \left(\int \left| a[\rho_t] \circ T_t(x) - a[\mu_t] \circ T_t(x) \right|^p \rho_t(dx) \right)^{1/p}. \tag{4.2}$$

We recognize that the first factor equals $W_p^{p-1}(\rho_t, \mu_t)$ since $\frac{p}{p'} = p - 1$.

Let us deal with the second one. We consider $\nu_s := ((1-s)\mathrm{id} + sT_t)_\# \rho_t$ the constant-speed geodesic from ρ_t to μ_t. Note that this curve implicitly depends on t. We also denote by $b_s \in L^p(\nu_s)$ the velocity field associated with $\nu \in AC([0,1], \mathbb{W}_p(\mathbb{R}^d))$, as given by Theorem 2.1. We have as a consequence of the Benamou-Brenier formula $\partial_s \nu_s + \nabla \cdot (b_s \nu_s) = 0$ and $\|b_s\|_{L^p(\nu_s)} = |(\nu_s)'| = W_p(\rho_t, \mu_t)$ for a.e. $s \in [0,1]$. Therefore, for any $y \in \mathbb{R}^d$, one has:

$$\begin{aligned} a[\rho_t](y) - a[\mu_t](y) &= -\int \nabla W(y-z)(\rho_t(z) - \mu_t(z)) dz \\ &= -\int_0^1 \int \nabla W(y-z) \partial_s \nu_s(z) dz ds \\ &= \int_0^1 \int \nabla W(y-z) \nabla \cdot (b_s(z) \nu_s(z)) dz ds \\ &= \int_0^1 \int \nabla^2 W(y-z) b_s(z) \nu_s(dz) ds, \end{aligned}$$

so that the inequality (4.2) rewrites:

$$|I_2| \leqslant W_p^{p-1}(\rho_t, \mu_t) \left(\int \left| \int_0^1 ds \int \nabla^2 W(T_t(x) - z) b_s(z) \nu_s(dz) ds \right|^p \rho_t(dx) \right)^{1/p}.$$

Besides, using a Jensen inequality w.r.t the measure $\nu_s(dz)ds$ for the convex function $|\cdot|^p$, we have:

$$\begin{aligned} \int \left| \int_0^1 ds \int \nabla^2 W(T_t(x) - z) b_s(z) \nu_s(dz) ds \right|^p \rho_t(dx) &\leqslant \int \int_0^1 \int |\nabla^2 W(T_t(x) - z)|^p |b_s(z)|^p \nu_s(dz) ds \rho_t(dx) \\ &\leqslant \int_0^1 \int |b_s(z)|^p \int |\nabla^2 W(T_t(x) - z)|^p \rho_t(dx) \nu_s(dz) ds \end{aligned}$$

Now, since $\mu_t = T_{t\#}\rho_t$, we have $\int |\nabla^2 W(T_t(x) - z)|^p \rho_t(dx) = \int |\nabla^2 W(y-z)|^p \mu_t(y) dy$. Applying a Hölder inequality w.r.t dy and the exponents (q, q'), where we will specify q right afterwards, we get:

$$\int |\nabla^2 W(y-z)|^p d\mu(y) \leqslant \left(\int |\nabla^2 W(y-z)|^{pq'} dy \right)^{1/q'} \left(\int |\mu_t(y)|^q dy \right)^{1/q} = \|\nabla^2 W\|_{L^{pq'}}^p \|\mu_t\|_{L^q}.$$

We therefore have to take q such that $pq' = p_0$, so that $\|\nabla^2 W\|_{L^{pq'}}$ remains finite. This requires that we choose p such that $p \leqslant p_0$, which imposes $p_0 > 1$ since we also needed $p > 1$. We also need to choose p such that $\rho^{ini} \in \mathcal{P}_p$, which means $p \leqslant 2$. Using $\int_0^1 \int |b_s(z)|^p \nu_s(dz) ds = W_p^p(\rho_t, \mu_t)$, we finally obtain:

$$|I_2| \leqslant \|\nabla^2 W\|_{L^{p_0}} \|\mu_t\|_{L^q}^{1/p} W_p^p(\rho_t, \mu_t), \qquad \text{for } q = \frac{p_0}{p_0 - p},$$

where the value of q is computed so that we have $q' = \frac{p_0}{p}$. We therefore have the following Grönwall inequality on $W_p^p(\rho_t, \mu_t)$:

$$\frac{1}{p}\frac{d}{dt} W_p^p(\rho_t, \mu_t) \leqslant \|\nabla^2 W\|_{L^{p_0}} \left(\|\rho_t\|_{L^{p_0'}} + \|\mu_t\|_{L^q}^{1/p} \right) W_p^p(\rho_t, \mu_t), \tag{4.3}$$

Now, we need a bound on $\|\rho_t\|_{L^r}$. The following lemma implies that, if the interaction potential W satisfies $\Delta W \leqslant 0$, then the bound on ρ_t is not worse than the one we would obtain if ρ solved the sole heat equation and does not depend on the initial datum.

Lemma 4.3. *Let $p \in (1, +\infty)$, $\varepsilon > 0$ and let ρ solve the following Fokker-Planck equation on the whole space $\mathbb{R}^d$:*

$$\partial_t \rho + \nabla \cdot (\rho \nabla V) = \varepsilon \Delta \rho, \tag{4.4}$$

where the potential V might depend on ρ and satisfies $\Delta V \geqslant 0$. Assume that ρ_t is smooth for any $t > 0$, and that is has unit total mass. Then one has:

$$\|\rho_t\|_{L^p} \leqslant C(\varepsilon t)^{-d/2p'},$$

for a positive constant $C = C(p,d)$ depending on p only and not on the initial datum ρ_0.

Proof. In the following, $C(p)$ stands for any positive constant depending only on p. For $t > 0$, testing equation (4.4) against ρ_t^{p-1} and integrating by parts yields:

$$\frac{d}{dt}\frac{1}{p}\int \rho_t^p = -\frac{p-1}{p}\int \rho_t^p \Delta V - 4\varepsilon \frac{p-1}{p^2}\int |\nabla \rho_t^{p/2}|^2 \leqslant -4\varepsilon \frac{p-1}{p^2}\int |\nabla \rho_t^{p/2}|^2,$$

since $\Delta V \geqslant 0$. Using the following Gagliardo-Nirenberg-Sobolev inequality [18, 28]:

$$\int \rho^{p+\frac{2}{d}} \leqslant C(p) \int |\nabla \rho_t^{p/2}|^2,$$

and interpolating the L^p norm between the L^1 and $L^{p+\frac{2}{d}}$ norms, we deduce that $y_t := \int \rho_t^p$ verifies the following nonlinear Grönwall inequality:

$$y' - \varepsilon C(p) y^{1+\frac{2}{d(p-1)}} \leqslant 0.$$

Integrating this inequality on $[s,t]$ for $0 < s < t$, we get:

$$y_t^{-2/d(p-1)} \geqslant y_s^{-2/d(p-1)} + \varepsilon C(p) \geqslant \varepsilon C(p),$$

and therefore $\|\rho_t\|_{L^p} = y_t^{1/p} \leqslant C(p)(\varepsilon t)^{-d(p-1)/2} = (\varepsilon t)^{-d/2p'}$. This is the bound one would obtain using a $L^p \times L^1$ convolution inequality if ρ solved the sole heat equation on the whole space, that is, if we had $\rho_t = G_{\varepsilon t} * \rho_0$ where G_t denotes the heat kernel. ☐

Using Lemma 4.3 with the potential $V = -W * \rho$ which has a positive Laplacian under the assumption $\Delta W \leqslant 0$, we get $\|\rho_t\|_{L^{p_0'}} + \|\mu_t\|_{L^q}^{1/p} \leqslant C(d, p_0)(\varepsilon t)^{-d/2p_0}$ which, in turn, yields the Grönwall inequality:

$$\frac{d}{dt} W_p^p(\rho_t, \mu_t) \leqslant C(\varepsilon t)^{-d/2p_0} W_p^p(\rho_t, \mu_t),$$

where C is a positive constant that depends on p, p_0 and $\|\nabla^2 W\|_{L^{p_0}}$ only. We deduce:

$$W_p^p(\rho_t, \mu_t) \leqslant W_p^p(\rho_0, \mu_0) e^{\int_0^t C(\varepsilon\tau)^{-d/2p_0} d\tau},$$

provided $p_0 > \frac{d}{2}$ so that $\tau^{-d/2p_0}$ is integrable on $(0, t]$. Under this assumption, using Lemma 3.3 along with the fact that $\rho_0 = \rho^{ini}$, we get, for some constant $C > 0$ depending on d, p, p_0 and $\|\nabla^2 W\|_{L^{p_0}}$ only:

$$\forall t \in [0, T], \qquad W_p^p(\rho_t, \mu_t) \leqslant C e^{-C\left(\varepsilon^\alpha - \varepsilon^{-d/2p_0}\right)} e^{Ct^{1-d/2p_0}} \leqslant C e^{-C\left(\varepsilon^\alpha + \varepsilon^{-d/2p_0}\right)} e^{CT^{1-d/2p_0}},$$

which goes to 0 uniformly in $t \in [0, T]$, as $\varepsilon \to 0$, provided $\alpha < -d/2p_0$. Since $-d/2p_0 > -1$, it is possible make such a choice while guaranteeing $\alpha \in (-1, 0)$. To finish, we conclude the proof as in that of Corollary 3.4.

Now, note that $\Delta W \leqslant 0$ ensures that any L^p norm of solutions to (1.2) is nonincreasing in time. Therefore, when the initial datum belong to $L^{p_0'}(\mathbb{R}^d) \cap L^{\frac{p_0}{p_0 - p}}(\mathbb{R}^d)$, estimate (4.3) still holds for $\varepsilon = 0$ between any two solutions to (1.2) and gives uniqueness of the solution among the class of $\mathcal{C}([0, T], \mathbb{W}_2(\mathbb{R}^d)) \cap L^\infty([0, T], L^{p_0'}(\mathbb{R}^d) \cap L^{\frac{p_0}{p_0 - p}}(\mathbb{R}^d))$ solutions. □

5 Higher convergence rate for steady states under assumptions (A0)-(A1)-(A4-p)

In this section, we compare stationary solutions to the aggregation-diffusion equation (1.1a) for a given $\varepsilon > 0$ with stationary solutions to the aggregation equation (1.2). We discard, in this section, the assumptions of $\lambda-$convexity and Lipschitz continuity on W but still assume that assumptions (A0) and (A1) hold. In addition, we require the potential to satisfy assumption (A4$-p$), that is, to be at least as attractive as $|x|^p$, for some $p \in [1, \infty)$.

Note that this assumption along with (A0) implies $W(x) \geqslant C\frac{|x|^p}{p}$ for all $x \in \mathbb{R}^d$. If, in addition, W satisfies assumption (A1) then W is l.s.c on $\mathbb{R}^d$ and this implies that $\mathcal{W}$ is l.s.c for the weak convergence thanks to Lemma 2.3.

Also, without loss of generality, we only consider measures with 0 center of mass, that is, measures $\rho \in \mathcal{P}(\mathbb{R}^d)$ verifying:

$$\int x\rho(dx) = 0.$$

We define steady states for the aggregation-diffusion equation in the spirit of [22]:

Definition 5.1. *Let $\varepsilon \geqslant 0$. A steady state for the aggregation-diffusion equation* (1.1a) *is a probability measure $\rho \in \mathcal{P}_1(\mathbb{R}^d)$ such that:*

$$\textit{if } \varepsilon = 0, \quad \widehat{\nabla W} * \rho = 0, \quad \textit{on supp}(\rho),$$

and, if $\varepsilon > 0$:

$$\begin{cases} \nabla W * \rho + \varepsilon \dfrac{\nabla \rho}{\rho} = 0 \quad \textit{on } \mathbb{R}^d, \\ \rho > 0 \quad \textit{on } \mathbb{R}^d. \end{cases}$$

One can prove that this definition is equivalent to that of stationary solutions, in the sense of distributions, to equation (1.1). Besides, if $\varepsilon > 0$, one can show that a distributional solution to the elliptic problem $-\nabla\cdot(\nabla W * \rho)\rho = \varepsilon\Delta\rho$ is necessarily regular and positive on $\mathbb{R}^d$ (see Theorem 2.11).

The following lemma justifies why we compare steady states for the aggregation equation to the Dirac mass.

Lemma 5.2. *Under assumptions (A0)-(A1)-(A4−p) for $p \geqslant 1$, the unique steady state for the aggregation equation* (1.2a) *is, up to a translation, the Dirac mass δ_0.*

Proof. Let ρ be a steady state for (1.2) and assume that ρ is centered. Since $\widehat{\nabla W} * \rho = 0$ on the support of ρ, testing against ρx and using Lemma 2.8 with the odd vector field $\widehat{\nabla W}$ yields:

$$\iint \widehat{\nabla W}(x-y)\cdot(x-y)\rho(dx)\rho(dy) = 0.$$

Under assumption (A4−p), we therefore have $\iint |x-y|^p\rho(dx)\rho(dy) = 0$. In particular $\rho\otimes\rho$ is concentrated on the diagonal. Now, if ρ is not a Dirac mass, then there exists disjoint Borel sets A et B with $\rho(A) > 0$ and $\rho(B) > 0$. Then we have, since $A\times B$ is disjoint from the diagonal

$$0 = \rho\otimes\rho(A\times B) = \rho(A)\rho(B) > 0,$$

and this contradiction concludes the proof. □

Note that the Dirac mass is actually the only minimizer of the interaction energy $\mathcal{W}$ under these assumptions. Conversely, Proposition 7.20 in [32] ensures that minimizers of the energy F^ε are actually steady states. This provides a way to prove existence of steady states for (1.1a) when $\varepsilon > 0$.

5.1 Existence of minimizers of F^ε for $\varepsilon > 0$

Proposition 5.3. *Assume that W satisfies assumptions (A0)-(A1)-(A4-p) for some $p \geqslant 1$ and let $\varepsilon \geqslant 0$ be fixed. The functional $F^\varepsilon = \mathcal{W} + \varepsilon\mathcal{U}$ admits a minimizer over $\mathcal{P}(\mathbb{R}^d)$ that actually has finite p-th order moment.*

Remark 5.4. We were not able to prove uniqueness of the minimizer under such assumptions on W but it is likely to hold. Moreover, numerical illustrations will show that, if we remove assumption (A4-p), multiple steady states can coexist even though $\varepsilon > 0$ (in case $\varepsilon = 0$, it is easy to build explicit counterexamples).

To prove this proposition, we will use that under assumptions (A0) and (A4-p), controlling $\mathcal{W}(\rho)$ gives control on $\iint |x-y|^p\rho(dx)\rho(dy)$, and this latter quantity is equivalent to $M_p(\rho)$ whenever ρ is centered, thanks to the following lemma:

Lemma 5.5. *Let $p \in [1,\infty)$ and $\rho \in \mathcal{P}_p(\mathbb{R}^d)$. Assume that the center of mass of ρ is 0. Then:*

$$M_p(\rho) \leqslant \iint |x-y|^p\rho(dx)\rho(dy) \leqslant 2^{p-1}M_p(\rho).$$

Proof. Let $u(x) = \int |x-y|^p\rho(dy)$. Since $p \geqslant 1$, u is a convex function and therefore, using a Jensen inequality, we get:

$$M_p(\rho) = u(0) = u\Big(\int x\rho(dx)\Big) \leqslant \int u(x)\rho(dx).$$

In other terms, $M_p(\rho) \leqslant \iint |x-y|^p\rho(dx)\rho(dy)$. The upper bound comes from the inequality $|x-y|^p \leqslant 2^{p-1}(|x|^p + |y|^p)$. □

Proof. Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures that minimize F^ε. We can assume that these measures are centered because F^ε is invariant under translation. Up to an extraction, we can assume that $(\rho_n)_{n\in\mathbb{N}}$ converges weakly towards some $\rho \in \mathcal{M}_b(\mathbb{R}^d)$. To ensure that $\rho \in \mathcal{P}(\mathbb{R}^d)$, we need to prove tightness of $(\rho_n)_{n\in\mathbb{N}}$. To do so, let us find a bound on $M_p(\rho_n)$.

Since $(\rho_n)_{n\in\mathbb{N}}$ is a minimizing sequence, $F^\varepsilon(\rho_n) = \mathcal{W}(\rho_n) + \varepsilon\mathcal{U}(\rho_n)$ is bounded from above by some constant that we still denote $C > 0$. Moreover, using assumption (A0) and (A4-p) and Lemma 5.5, since ρ_n is centered, we have:

$$\mathcal{W}(\rho_n) \geqslant \frac{C}{2p}\iint |x-y|^p \rho_n(dx)\rho_n(dy) \geqslant \frac{C}{2p}M_p(\rho_n).$$

In order to get a lower bound involving $M_p(\rho_n)$ on the entropy term, recall that, using a Legendre transform, $y\ln y + e^{z-1} \geqslant yz$ for all $y \geqslant 0$ and $z \in \mathbb{R}$. Setting, for $x \in \mathbb{R}^d$, $y = \rho_n(x)$ and $z = -|x|^{\alpha p}$ for some exponent $\alpha > 0$ to be specified later, and integrating over $x \in \mathbb{R}^d$, we get:

$$\int \rho_n \ln \rho_n \geqslant -\int (|x|^p)^\alpha \rho_n(dx) + \int e^{-|x|^{\alpha p}-1}dx$$

Choosing $\alpha \in (0,1)$ so that $x \longmapsto |x|^\alpha$ is concave, and using a Jensen inequality, we deduce $\mathcal{U}(\rho_n) \geqslant -M_p(\rho_n)^\alpha + C(p,\alpha)$, where $C(p,\alpha)$ depends on α and p only. Finally, we obtain:

$$\frac{C}{2p}M_p(\rho_n) - \varepsilon M_p(\rho_n)^\alpha + \varepsilon C(p,\alpha) \leqslant C,$$

which implies, since $\alpha < 1$, that $M_p(\rho_n)$ is uniformly bounded with respect to n.

On the one hand, this implies that $(\rho_n)_{n\in\mathbb{N}}$ is tight, hence $\rho \in \mathcal{P}(\mathbb{R}^d)$. Since M_p is l.s.c on $\mathcal{P}(\mathbb{R}^d)$ and $\rho_n \underset{n\to+\infty}{\overset{*}{\rightharpoonup}} \rho$, we also get $\rho \in \mathcal{P}_p(\mathbb{R}^d)$. On the other hand, the uniform bound on $M_p(\rho_n)$ along with Lemma 2.5 ensures that $M_q(\rho_n) \underset{n\to+\infty}{\longrightarrow} M_q(\rho)$ for any $q \in (0,p)$. Lemma 2.4 then gives $\mathcal{U}(\rho) \leqslant \liminf_{n\to+\infty} \mathcal{U}(\rho_n)$, and, since $\mathcal{W}$ is l.s.c for the weak convergence, we get $F^\varepsilon(\rho) \leqslant \liminf_{n\to+\infty} F^\varepsilon(\rho_n)$. This proves that ρ minimizes F^ε since $(\rho_n)_{n\in\mathbb{N}}$ is a minimizing sequence. □

5.2 $O(\varepsilon)$ convergence rate in W_p for potentials such that $\nabla W(x)\cdot x \geqslant C|x|$

In this section, we focus on assumption (A4-1) under which the potential is "really pointy" and the aggregation compensates the diffusion so that convergence occurs at rate $O(\varepsilon)$:

Theorem 5.6. *Assume that W satifies assumptions (A0)-(A1)-(A4-1). There exists a constant $C > 0$ depending on d, such that for any $\varepsilon > 0$ and ρ^ε steady state for* (1.1a) *which center of mass is 0, the following estimate holds:*

$$W_1(\rho^\varepsilon, \delta_0) \leqslant C\varepsilon. \tag{5.1}$$

Proof of Theorem 5.6. Let $\varepsilon > 0$ and let ρ^ε be a steady state for (1.1), that is:

$$\nabla W * \rho^\varepsilon + \varepsilon\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon} = 0. \tag{5.2}$$

Testing the above equation against $\rho^\varepsilon x$ we obtain:

$$\int \rho^\varepsilon x \cdot \nabla W * \rho^\varepsilon dx + \varepsilon\int x\cdot\nabla\rho^\varepsilon dx = 0$$

Integrating by parts and using Lemma 2.8 with the odd vector field ∇W yields:

$$\frac{1}{2}\iint \nabla W(x-y)\cdot(x-y)\rho^\varepsilon(dx)\rho^\varepsilon(dy) = \varepsilon d.$$

The desired result then follows from assumption (A4-1) and Lemma 5.5 with $p=1$, since $W_1(\rho^\varepsilon,\delta_0) = M_1(\rho^\varepsilon)$. □

Note that, from equation (5.2), one has $\rho^\varepsilon = C(\varepsilon)e^{-W*\rho^\varepsilon/\varepsilon}$. The value of the constant $C(\varepsilon)$ can be computed by imposing a total mass 1, so that we get $\rho^\varepsilon = \frac{e^{-W*\rho^\varepsilon/\varepsilon}}{\int e^{-W*\rho^\varepsilon/\varepsilon}}$. Using this equality along with estimate (5.1), we obtain a bound in W_p distance for $p \in [1,\infty)$ provided W is also Lipschitz continuous:

Theorem 5.7. *Assume that W satisfies assumptions (A0)-(A1)-(A2)-(A4-1). There exists a constant $C > 0$ depending on d and a_∞, such that for any $p \in [1,\infty)$, $\varepsilon > 0$ and ρ^ε steady state for* (1.1a) *which center of mass is 0, the following estimate holds:*

$$W_p(\rho^\varepsilon, \delta_0) \leqslant C\varepsilon. \tag{5.3}$$

Remark 5.8. At least in dimension one, this result is optimal. Indeed, we can take for W the Newtonian potential $W(x) = |x|$, for which, using the correspondence with Burgers' equation, ρ^ε can be written as $\rho^\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right)$, where $\rho(x) = \frac{1-\tanh^2\left(\frac{x}{2}\right)}{4}$, and a scaling argument then gives $W_p^p(\rho^\varepsilon,\delta_0) = \varepsilon^p M_p(\rho)$.

Proof. Since

$$\rho^\varepsilon = \frac{e^{-W*\rho^\varepsilon/\varepsilon}}{\int e^{-W*\rho^\varepsilon/\varepsilon}},$$

we have:

$$W_p^p(\rho^\varepsilon,\delta_0) = \frac{\int |x|^p e^{-W*\rho^\varepsilon(x)/\varepsilon}dx}{\int e^{-W*\rho^\varepsilon/\varepsilon}},$$

Now, since W is Lipschitz continuous, one has

$$|W*\rho^\varepsilon - W*\delta_0| \leqslant a_\infty \sup_{Lip(\varphi)\leqslant 1} \int \varphi d(\rho^\varepsilon - \delta_0) = a_\infty W_1(\rho^\varepsilon,\delta_0) \leqslant C\varepsilon,$$

because of Theorem 5.6. Thus, $-W*\rho^\varepsilon \leqslant C\varepsilon - W$ and therefore:

$$\int |x|^p e^{-W*\rho^\varepsilon(x)/\varepsilon}dx \leqslant C\int |x|^p e^{-W(x)/\varepsilon}dx \leqslant C\varepsilon^{p+d}\int |y|^p e^{-W(\varepsilon y)/\varepsilon}dy,$$

using the change of variables $x = \varepsilon y$. Recall that Assumption (A4-1) ensures $W(x) \geqslant C|x|$ for all $x \in \mathbb{R}^d$. This allows us to bound $\int |y|^p e^{-W(\varepsilon y)/\varepsilon}dy$ uniformly with respect to ε.

On the other hand, since W is $a_\infty-$Lipschitz continuous, we have $W(x) \leqslant a_\infty|x| + W(0) = a_\infty|x|$. Integrating with respect to $\rho^\varepsilon(dx)$, we deduce $W*\rho^\varepsilon(0) \leqslant a_\infty W_1(\rho^\varepsilon,\delta_0)$. Besides, $W*\rho^\varepsilon$ is also $a_\infty-$Lipschitz continuous. Hence,

$$W*\rho^\varepsilon(x) \leqslant W*\rho^\varepsilon(0) + a_\infty|x| \leqslant a_\infty W_1(\rho^\varepsilon,\delta_0) + a_\infty|x| \leqslant C\varepsilon + a_\infty|x|,$$

thanks again to estimate (5.1). After another rescaling, we deduce:

$$\int e^{-W*\rho^\varepsilon/\varepsilon} \geqslant C\varepsilon^d,$$

thus getting $W_p^p(\rho^\varepsilon,\delta_0) \leqslant C\frac{\varepsilon^{p+d}}{\varepsilon^d} = C\varepsilon^p$, which concludes the proof. □

5.3 $O(\varepsilon^{1/p})$ convergence rate in W_p for potentials such that $\nabla W(x) \cdot x \geqslant C|x|^p$

Assume W satisfies assumptions (A0), (A1) and (A4-p) for some $p \in [1, \infty)$. Under this assumption, a straightforward adaptation of the proof of Theorem 5.6 provides an estimate on $W_p(\rho^\varepsilon, \delta_0)$:

Theorem 5.9. *Assume that W satifies assumptions (A0)-(A1)-(A4-p) for some $p \in [1, \infty)$. There exists a constant $C > 0$ depending on d, such that for any $\varepsilon > 0$ and ρ^ε steady state for* (1.1a) *which is centered, the following estimate holds:*

$$W_p(\rho^\varepsilon, \delta_0) \leqslant C\varepsilon^{1/p}. \tag{5.4}$$

Remark 5.10. It is possible to prove optimality of this rate for $p = 2$. Let us consider the quadratic potential $W(x) = |x|^2$, that satisfies assumption (A4-2). Recall that $\rho^\varepsilon = \frac{e^{-W*\rho^\varepsilon/\varepsilon}}{\int e^{-W*\rho^\varepsilon/\varepsilon}}$. Expanding $W(x-y) = |x-y|^2$ and using both facts that the total mass of ρ is 1 and that ρ^ε is centered, one has:

$$\begin{aligned} e^{-W*\rho^\varepsilon/\varepsilon} &= \exp\left\{-\frac{1}{\varepsilon}\left(\int |x|^2\rho^\varepsilon(y)dy - 2x \cdot \int y\rho^\varepsilon(y)dy + \int |y|^2\rho^\varepsilon(y)dy\right)\right\} \\ &= e^{-|x|^2/\varepsilon} e^{-W_2^2(\rho^\varepsilon,\delta_0)/\varepsilon}. \end{aligned}$$

Hence, $\rho^\varepsilon(x) = \dfrac{e^{-|x|^2/\varepsilon}}{\int e^{-|x|^2/\varepsilon}dx}$, which in turn yields:

$$W_2^2(\rho^\varepsilon, \delta_0) = \frac{\int |x|^2 e^{-|x|^2/\varepsilon}dx}{\int e^{-|x|^2/\varepsilon}dx}.$$

A change of variables in both integrals then gives $W_2^2(\rho^\varepsilon, \delta_0) = C\varepsilon$.

6 Numerical illustrations

This sections aims to illustrate our convergence results both in the evolutive case and in the stationary case. The implementation of the schemes has been done in Python and the code is available at github.com/strantien/aggregation. Tests are conducted on $[-1, 1]$, with $2J + 1$ cells, and the velocity field is always discretized by (3.20). Wasserstein distances between two arbitrary probability measures are computed using the POT package.

6.1 Evolutive solutions

We begin with the convergence rate in Wasserstein distance of the viscous solutions ρ^ε associated with a fixed initial datum ρ^{ini} (not depending on ε). In this subsection $\rho^\varepsilon_{\Delta x}$ is computed using the implicit discretization (3.21), for which the CFL condition is less restrictive than the parabolic CFL condition of the explicit scheme. We also implemented no-flux boundary conditions so as to preserve total mass. In the absence of a reference solution, the convergence rate w.r.t ε is estimated taking Δx small enough so that $\rho^\varepsilon_{\Delta x}$ approximates ρ^ε, and computing $W_p(\rho^{\varepsilon_{i+1}}_{\Delta x,T}, \rho^{\varepsilon_i}_{\Delta x,T})$.

In Theorems 3.8 and 3.13, when W satisfies assumptions (A0)-(A1)-(A2)-(A3), we proved convergence at rate $O(\varepsilon^{1/2})$ in W_2 distance, which is what we recover when W is smooth, as shows Figure 1. In practice, for this test case, we observe $O(\varepsilon^{1/2})$ convergence rate in W_p distance for any $p \in [1, +\infty[$. However, in case W has a Lipschitz discontinuity at the origin (Figure 2) we observe convergence at order 1 in W_1 distance. This is the superconvergence phenomenon investigated by Tang, Teng and Zhang [34, 37] in the framework of scalar conservation laws. In terms of aggregation,

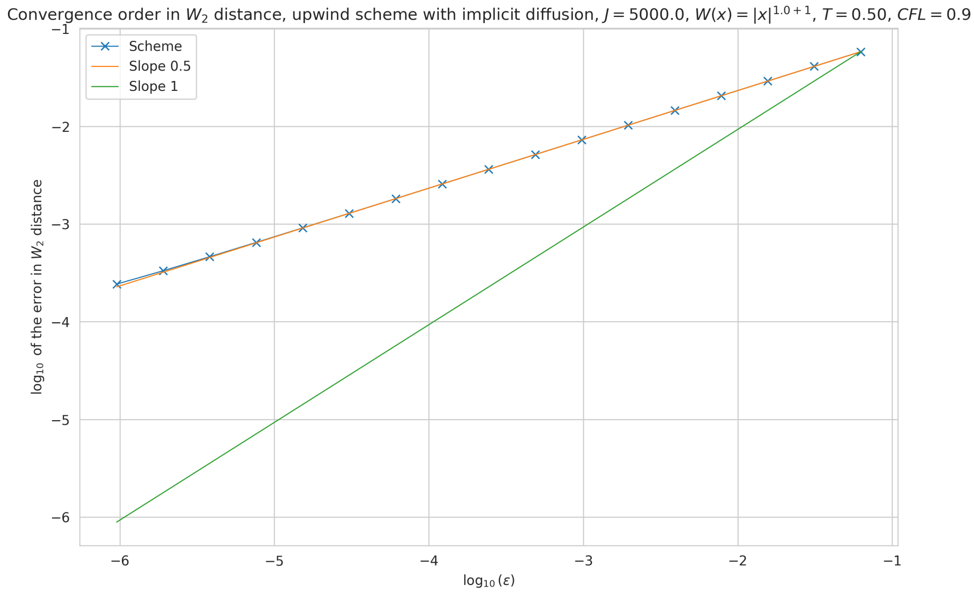

Figure 1: Order 1/2 convergence in W_2 distance of ρ_T^ε towards ρ_T for $\rho^{ini}(x) = 2\sqrt{\frac{5}{\pi}}e^{-20x^2}$, $W(x) = |x|^2$.

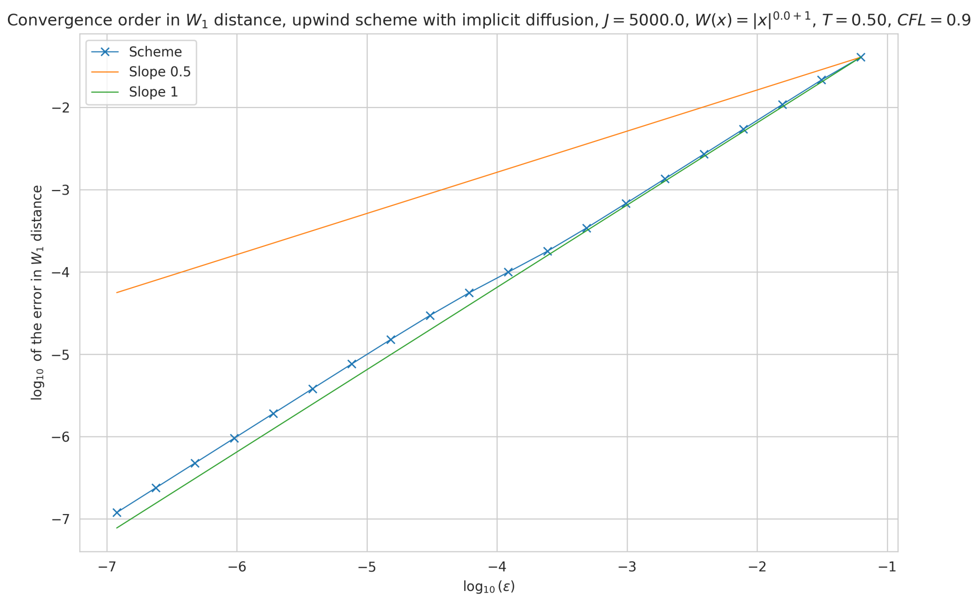

Figure 2: Order 1 convergence in W_1 distance of ρ_T^ε towards ρ_T for $\rho^{ini}(x) = 2\sqrt{\frac{5}{\pi}}e^{-20x^2}$, $W(x) = |x|$.

the interpretation is that, when W is singular, the concentration is strong enough to compensate part of the diffusion. In other W_p distances, converges seems to occur at order 1 when ε is not too small, and then degenerates quite clearly towards order $1/p$ for any $p \in [1, +\infty[$ (see Figure 3 for

$p = 3$). Note that, in every case, the convergence order is robust with respect to the test case (be it for smooth or singular initial data, e.g. Dirac masses).

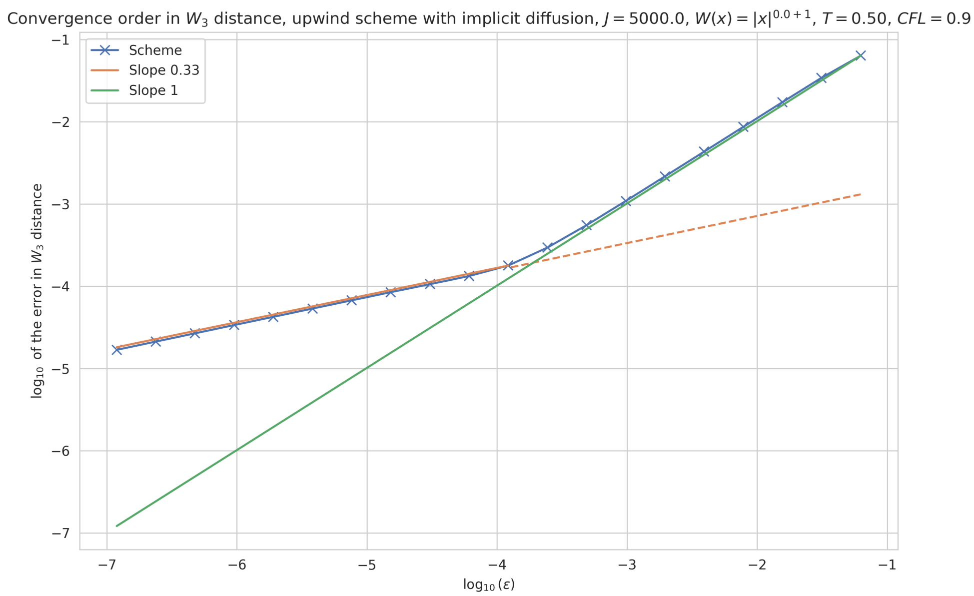

Figure 3: Order 1/3 convergence in W_3 distance, for small ε, of ρ_T^ε towards ρ_T for $\rho^{ini}(x) = 2\sqrt{\frac{5}{\pi}}e^{-20x^2}$, $W(x) = |x|$.

6.2 Steady states

In order to simulate the steady states for $\varepsilon > 0$, recall that they are characterized, over the whole space, by the following equation:

$$\rho^\varepsilon = \frac{e^{-W*\rho^\varepsilon/\varepsilon}}{\int e^{-W*\rho^\varepsilon/\varepsilon}}. \tag{6.1}$$

We therefore use a fixed-point method on equation (6.1), which stops as soon as the W_p distance between two iterations exceeds some tolerance. Numerically, we observe that this method turns two symmetric Gaussian bumps almost immediately (after the first iteration) into a centered Gaussian whenever W is attractive and Lipschitz.

We first investigate the convergence rate towards the Dirac mass, for centered steady states. The error is estimated computing the integral $\int |x|^p \rho(dx) = W_p^p(\rho, \delta_0)$. When W satisfies assumptions (A0)-(A1)-(A4-1), we proved $O(\varepsilon)$ convergence rate in W_1 distance, which we do recover in Table 1 for $W(x) = |x|$. We also explore the case when W verifies (A0)-(A1)-(A4-1) but is not Lipschitz continuous, which is the case of $W(x) = \sqrt{|x|} + |x|$. For this potential, we obtain, in Figure 4 convergence at order 1.82264413 which is slightly less than 2, in W_1 distance. This can be linked to the fact that W satisfies a sort of assumption (A4-$\frac{1}{2}$) when $|x| \leqslant 1$. Under assumptions (A0)-(A1)-(A2)-(A4-3), we observe convergence at rate 1/3 in W_3 distance as we proved in (5.4), as shows Figure 5. More generally, under assumptions (A0)-(A1)-(A2)-(A4-p), convergence at rate $1/p$ seems to occur in any W_q distance, $q \in [1, +\infty[$, which is what we proved in for $p = 1$ or for $p = q$. To illustrate this latter case, we compute the convergence order in W_p distance for $W(x) = |x|^p$, which

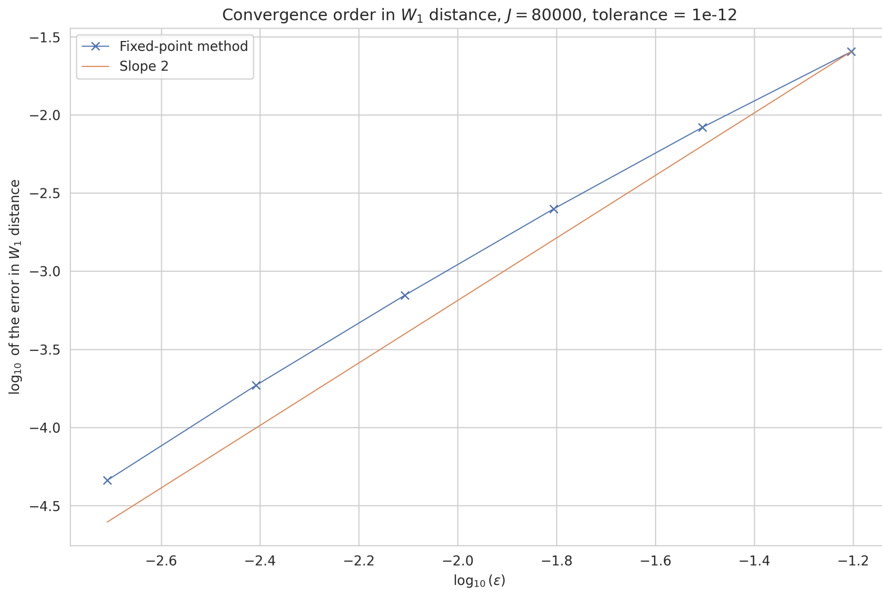

Figure 4: Order of convergence in W_1 distance of ρ^ε towards δ_0, for the non-Lipschitz potential $W(x) = \sqrt{|x|} + |x|$. The initial density is the centered Gaussian $2\sqrt{\frac{5}{\pi}}e^{-20x^2}$.

seems indeed to be $1/p$, see Table 1 (when $p = 1$, since the potential is pointy, one has to refine the mesh so as to observe proper convergence at order 1).

p	Order	J
1	1.00205259	50000
2	0.49999997	2000
3	0.33333333	2000
4	0.25000000	2000
5	0.20000000	2000

Table 1: Convergence order $\simeq \frac{1}{p}$ of ρ^ε towards δ_0 for $W(x) = |x|^p$, $tol = 10^{-6}$, $\varepsilon_i = 2^{-i}$, $i = 4, \ldots, 16$, initial density $2\sqrt{\frac{5}{\pi}}e^{-20x^2}$

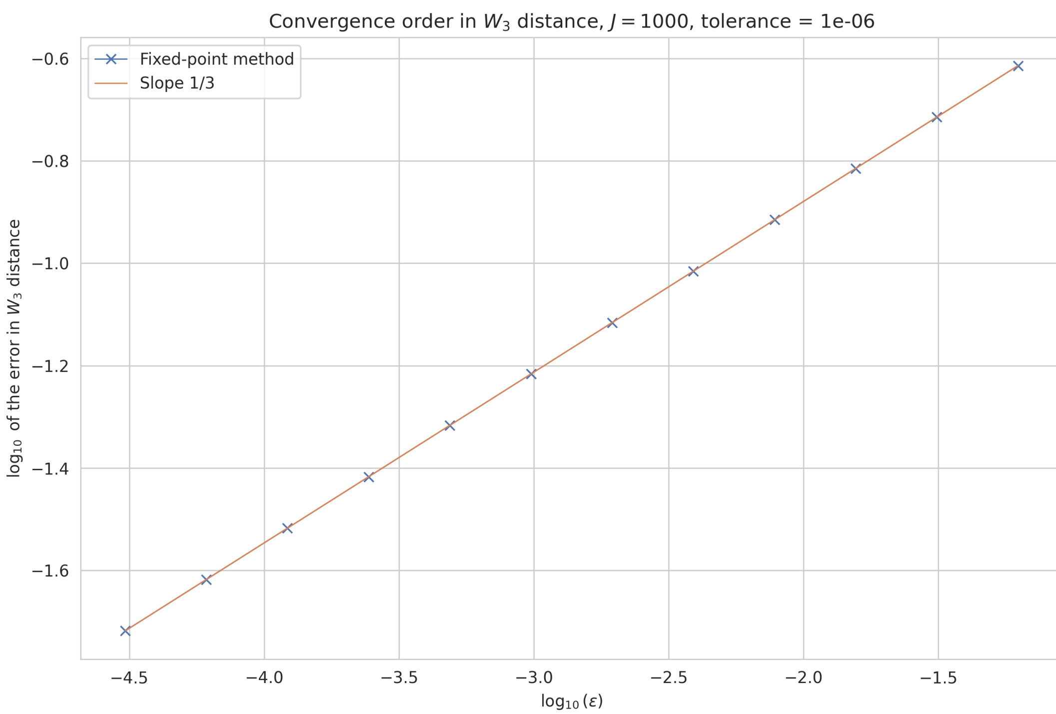

Figure 5: $O(\varepsilon^{1/3})$ convergence in W_3 distance of ρ^ε towards δ_0, $W(x) = |x|^3$. The initial density is the centered Gaussian $2\sqrt{\frac{5}{\pi}}e^{-20x^2}$.

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