

# Relaxed area of graphs of piecewise Lipschitz maps in the strict $BV$ -convergence

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## Abstract

We compute the relaxed Cartesian area in the strict  $BV$ -convergence on a class of piecewise Lipschitz maps from the plane to the plane, having jump made of several curves allowed to meet at a finite number of junction points. We show that the domain of this relaxed area is strictly contained in the domain of the classical  $L^1$ -relaxed area.

**Key words:** Area functional, relaxation, strict convergence, Cartesian currents, total variation of the Jacobian, Plateau problem.

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## 1 Introduction

Let  $\Omega \subset \mathbb{R}^2$  be an open bounded set. Given  $v \in C^1(\Omega; \mathbb{R}^2)$ , the area functional is defined as

$$\mathcal{A}(v, \Omega) := \int_{\Omega} \sqrt{1 + |\nabla v|^2 + |Jv|^2} \, dx = \int_{\Omega} |\mathcal{M}(\nabla v)| \, dx, \quad (1.1)$$

where  $\mathcal{M}(\nabla v) = (1, \nabla v_1, \nabla v_2, Jv)$  and  $Jv = \frac{\partial v_1}{\partial x_1} \frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \frac{\partial v_1}{\partial x_2}$  is the Jacobian determinant of  $v$ . The value  $\mathcal{A}(v, \Omega)$  is the 2-dimensional Hausdorff measure of the graph

$$G_v := \{(x, y) \in \Omega \times \mathbb{R}^2 : y = v(x)\}$$

of  $v$ . In order to extend the area functional to a more general class of maps one is led to consider the relaxation of (1.1): Namely, for all  $u \in L^1(\Omega; \mathbb{R}^2)$  one chooses a convergence, for instance the  $L^1$ -convergence, and sets

$$\overline{\mathcal{A}}_{L^1}(u, \Omega) := \inf \left\{ \liminf_{k \rightarrow +\infty} \mathcal{A}(u_k, \Omega), u_k \in C^1(\Omega; \mathbb{R}^2), u_k \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^2) \right\}. \quad (1.2)$$

In contrast with the case of real valued maps, for which the  $L^1$ -relaxed area is well-understood, in higher dimension, including the case of  $\mathbb{R}^2$ -valued maps considered here, the analysis of  $\overline{\mathcal{A}}_{L^1}$

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has been shown to be very challenging and a lot of questions remain open. For instance, it is known that the domain  $\text{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot, \Omega))$  of  $\overline{\mathcal{A}}_{L^1}(\cdot, \Omega)$  is strictly included in  $BV(\Omega; \mathbb{R}^2)$ , but its complete description is, so far, not available. The main difficulty to treat  $\overline{\mathcal{A}}_{L^1}$  is due to its non-local behaviour: Indeed, for general maps  $u$  with the only exception of very trivial cases, the set function  $E \subseteq \Omega \mapsto \overline{\mathcal{A}}_{L^1}(u, E)$  is not subadditive, and this excludes to represent (1.2) in integral form. As a consequence, the explicit value of  $\overline{\mathcal{A}}_{L^1}(u, \Omega)$  is, at the moment, known only for very specific non-smooth maps  $u$  enjoying a high degree of symmetry [5, 7, 25].

A useful simplification in the relaxation analysis of  $\mathcal{A}$  is to consider some variants of (1.2), for example modifying the convergence of  $v_k$  to  $u$  (see [8, 9, 13, 17]). Even if the  $L^1$ -convergence seems to be natural also with respect to the application to the non-parametric Plateau problem, one can replace the  $L^1$ -topology with different ones. In some recent works [3, 22], instead of relaxing with respect to the  $L^1$ -topology, the authors have considered relaxation with respect to the strict convergence in  $BV(\Omega; \mathbb{R}^2)$  (shortly  $BV$ -relaxed area). Namely, one defines

$$\overline{\mathcal{A}}_{BV}(u, \Omega) := \inf \left\{ \liminf_{k \rightarrow +\infty} \mathcal{A}(u_k, \Omega), u_k \in C^1(\Omega; \mathbb{R}^2), u_k \rightarrow u \text{ strictly } BV(\Omega; \mathbb{R}^2) \right\}. \quad (1.3)$$

area\_fun

Although the analysis of  $\overline{\mathcal{A}}_{BV}$  seems quite more treatable, a complete picture and description of its behaviour is still missing. It is straightforward that for any  $u \in BV(\Omega; \mathbb{R}^2)$

$$\overline{\mathcal{A}}_{BV}(u, \Omega) \geq \overline{\mathcal{A}}_{L^1}(u, \Omega),$$

so  $\text{Dom}(\overline{\mathcal{A}}_{BV}(\cdot, \Omega)) \subset \text{Dom}(\overline{\mathcal{A}}_{L^1}(\cdot, \Omega))$ , and the inclusion is strict as Example 4.1 below shows.

Strictly related to the area functional is the Jacobian total variation functional, namely

$$TVJ(v, \Omega) := \int_{\Omega} |Jv| \, dx,$$

valid for all  $v \in C^1(\Omega; \mathbb{R}^2)$ . Also in this case, to extend  $TVJ$  to a larger class of functions, a relaxation procedure is in order. However, the choice of the  $L^1$ -convergence is in some cases not interesting: for instance, if  $u \in W^{1,1}(\Omega; \mathbb{S}^1)$ , with  $\Omega$  simply connected, the corresponding relaxed functional trivializes and becomes constantly null (see [10, Cor. 5]). On the other hand, the notion of strict convergence in  $BV$  gives rise to a nontrivial relaxed functional which shows to play a crucial role in the analysis of  $\overline{\mathcal{A}}_{BV}$ . Specifically, for  $u \in BV(\Omega; \mathbb{R}^2)$  we consider

$$\overline{TVJ}_{BV}(u, \Omega) := \inf \left\{ \liminf_{k \rightarrow +\infty} TVJ(v_k, \Omega), v_k \in C^1(\Omega; \mathbb{R}^2), v_k \rightarrow u \text{ strictly } BV(\Omega; \mathbb{R}^2) \right\}. \quad (1.4)$$

TVJ\_fun

In the present paper we compute the value of  $\overline{\mathcal{A}}_{BV}(u, \Omega)$  for some particular piecewise Lipschitz maps  $u$  which are allowed to jump on curves in turn meeting at junction points. We refer to Definition 5.2 for the details on these maps, and we summarize here their features: Let  $\Omega \subset \mathbb{R}^2$  be a bounded open set of class  $C^1$  and  $\{\Omega_k\}_{k=1, \dots, N}$  a finite partition of  $\Omega$  made of Lipschitz sets. Suppose that  $\Sigma := \cup_{k=1}^N \partial\Omega_k$  is the support of a finite family of  $C^2$ -curves  $\alpha_\ell : \overline{I}_\ell \rightarrow \overline{\Omega}$ ,  $\ell = 1, \dots, n$ ,  $I_\ell = (a_\ell, b_\ell)$ . We suppose that the curves  $\alpha_\ell$ , arc-length parametrized on  $\overline{I}_\ell$ , are injective on  $I_\ell$ ,  $\alpha_\ell(I_\ell) \subset \Omega$ , and of class  $C^2$  up to  $a_\ell$  and  $b_\ell$  (namely  $\dot{\alpha}_\ell$  and  $\ddot{\alpha}_\ell$  are continuous on  $\overline{I}_\ell$ ). Furthermore, we assume that  $\alpha_\ell(I_\ell)$  and  $\alpha_h(I_h)$ , for  $\ell \neq h$ , may intersect only at the endpoints. Endpoints of  $\alpha_\ell$  are allowed to belong to  $\partial\Omega$ , and we assume such endpoints to be distinct for different curves.

A map  $u \in BV(\Omega; \mathbb{R}^2)$  is called piecewise Lipschitz if its restriction to any  $\Omega_k$  is Lipschitz. Notice that if  $p_i$  is a junction point and  $\Omega_k^i$  ( $k = 1, \dots, N_i$ ) are the connected components of  $\Omega \setminus \Sigma$  having  $p_i$  as boundary point, then there exists the limit  $\beta_k^i := \lim_{x \in \Omega_k^i, x \rightarrow p_i} u(x)$ . For the sake of simplicity, we assume that the enumeration  $k = 1, \dots, N_i$  respects the counterclockwise order of  $\Omega_k^i$ 's around  $p_i$ .

To introduce our main result, we have to consider also a planar Plateau problem for Lipschitz curves: Given a Lipschitz curve  $\varphi : \mathbb{S}^1 = \partial B_1 \rightarrow \mathbb{R}^2$  we consider the quantity

$$P(\varphi) := \inf \left\{ \int_{B_1} |Jv| \, dx : v \in \text{Lip}(B_1; \mathbb{R}^2) : v|_{\partial B_1} = \varphi \right\}. \quad (1.5) \quad \text{Plateau}$$

For all  $i = 1, \dots, m$  we denote by  $\tilde{\gamma}^i$  a Lipschitz curve which parametrizes on  $\mathbb{S}^1$  the polygon in  $\mathbb{R}^2$  with vertices  $\beta_1^i, \beta_2^i, \dots, \beta_{N_i}^i$ , in the order (see Fig. 3). Notice carefully that this curve may self-intersect. Also,  $P(\varphi)$  is invariant under reparametrizations of  $\varphi$  (Proposition 2.9). Finally, set  $I = [0, 1]$ . The main result of the paper is the following

**Theorem 1.1 (Relaxation for piecewise Lipschitz maps).** *Let  $u : \Omega \rightarrow \mathbb{R}^2$  be a piecewise Lipschitz map. Then*

$$\bar{\mathcal{A}}_{BV}(u, \Omega) = \int_{\Omega \setminus \Sigma} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_{(\ell)}^{\text{aff}} \wedge \partial_s X_{(\ell)}^{\text{aff}}| \, dt ds + \sum_{i=1}^m P(\tilde{\gamma}^i), \quad (1.6) \quad \text{eq:ril}$$

where, for any  $\ell = 1, \dots, n$ ,

$$X_{(\ell)}^{\text{aff}}(t, s) := (t, su_\ell^+(t) + (1-s)u_\ell^-(t)) \quad \forall (t, s) \in [a_\ell, b_\ell] \times I, \quad (1.7) \quad \text{eq:affi}$$

and  $u_\ell^\pm$  are the traces of  $u$  on the support  $\alpha_\ell(I_\ell)$  of  $\alpha_\ell$ .

One of the main features of expression (1.6) is the presence of *two* singular contributions: a 0-dimensional term due to the concentration of the Jacobian determinants of a recovery sequence  $(v_k)$  for  $\bar{\mathcal{A}}_{BV}(u, \Omega)$  around the junction points (namely, the term involving the minimum of the Plateau problems  $P(\tilde{\gamma}^i)$ ), and a 1-dimensional term, which essentially takes into account the concentration of the gradients and of the Jacobian determinants of  $v_k$  along the jump set  $\Sigma$ . So, we can interpret (1.6) as a non-trivial generalization of [3, Theorem 1.3], valid for the triple point map  $u_T$  (see also Theorem 4.4), and of [3, Theorem 1.1], valid for 0-homogeneous maps of the form  $\varphi(x/|x|)$  with  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  Lipschitz. Indeed, in the first case the 1-dimensional term was simply the total variation of  $u_T$  (consisting of the area of three vertical walls over  $\Sigma$ ) and the 0-dimensional one was the area of the target triangle, which is a trivial minimum of (1.5), while in the second case we had no 1-dimensional contribution and the 0-dimensional one was the solution of (1.5) with this special  $\varphi$ , that reduces to  $P(\varphi) = \pi|\deg(\varphi)|$ . In other words, the relaxed area of a more general map  $u$  as in (1.1) is still a measure (if we regard it as a function of  $\Omega$ ), which has the same dimensional structure, but with a more involved and rich expression.

We observe that for this special kind of maps it always holds  $\bar{\mathcal{A}}_{BV}(u, \Omega) < +\infty$ , because the contributions of the Plateau problem  $P(\tilde{\gamma}^i)$  is always finite, since one can construct a Lipschitz competitor for (1.5). On the other hand, the presence of a finite number of junction points is crucial, because, as Example 4.1 shows, we can build a piecewise constant map whose  $BV$ -relaxed area is infinite. It is here remarkable that the same map can be seen to have finite  $L^1$ -relaxed area (compare with (4.2)). This in particular shows the proper inclusion

$$\text{Dom}(\bar{\mathcal{A}}_{BV}(\cdot, \Omega)) \subsetneq \text{Dom}(\bar{\mathcal{A}}_{L^1}(\cdot, \Omega)).$$

We divide the proof of Theorem 1.1 in several steps, and in particular we first focus on the relaxation on piecewise Lipschitz maps  $u$  without junction points. In this case we show in Corollary 3.12 (consequence of Theorems 3.7 and 3.11) that the relaxation provides as singular contribution the integral over the jump set  $S_u$  of  $u$  of the area spanned by the affine map  $X^{\text{aff}}$ . The main issue is the proof of Proposition 3.4, the lower bound for the relaxed area of maps jumping on the

central horizontal segment of the rectangle  $R = [a, b] \times [-1, 1]$ . Here, we need to use some tools from the theory of integer multiplicity currents, in particular slicing arguments and the isoperimetric inequality, in order to show that over the jump segment the graph of the elements of an approximating smooth sequence  $(v_k)$  have area bounded below by the area of  $X^{\text{aff}}$ . The properties of the strict convergence (Lemmas 2.3 and 2.7) enter at the level of vertical slices of the graph of  $v_k$  in a neighbourhood of the jump segment, but these results only are not enough to pass to the limit in the area of the graph of  $v_k$ . For this purpose, the idea is to make a decomposition of the graph of  $v_k$  and of the surface  $X^{\text{aff}}$  in several tiny strips, and notice that, when the number of these strips is very high, the boundaries of these two little pieces of surfaces are uniformly close together, as a consequence of the strict convergence and, at the same time, the strips which decompose  $X^{\text{aff}}$  are very close to a minimal mass current having the same boundary.

In [9], the authors compute the relaxed area  $\overline{\mathcal{A}}_{L^\infty}(u, \Omega)$  with respect to the local uniform convergence out of the jump, for  $u$  as in Proposition 3.4. They obtain, as singular contribution, the area of the minimal semicartesian<sup>1</sup> surface spanning the graphs of the two traces. In particular, since  $X^{\text{aff}}$  is semicartesian and spans  $\text{graph}(u^\pm)$  as well (see [9, Definition 2.4]), we have  $\overline{\mathcal{A}}_{L^\infty}(u, R) \leq \overline{\mathcal{A}}_{BV}(u, R)$ . In general, this inequality holds strictly, even if  $\text{graph}(u^\pm)$  are coplanar. We can find an example in [9, Remark 8.5], where one can notice that in order to minimize the area of the spanning surface, the approximating sequence needs not keep the total variation of the limit map, which instead is forced to be preserved under strict convergence. Moreover, it is important to notice that  $\overline{\mathcal{A}}_{L^\infty}(u, \cdot)$  is not subadditive (see [9, Thm. 8.1]), while  $\overline{\mathcal{A}}_{BV}(u, \cdot)$  is clearly a measure.

In a second step we instead consider the case of maps  $u$  which are piecewise constant but whose jump might have junction points. Specifically, in Theorem 4.4 we see that the relaxation on a  $n$ -uple point map (i.e., whose jump consists of  $n$  radii of the same ball  $B_r(0)$ ) provides as singular contribution, besides the total variation of  $u$ , the number  $P(\tilde{\gamma})$ , where  $\tilde{\gamma}$  is the piecewise affine curve which parametrizes the perimeter of the polygon whose vertices are the values of  $u$  around 0.

Finally, in Section 5, we use Corollary 3.12 and Theorem 4.4 to complete the proof of Theorem 1.1.

We point out that, to our best knowledge, it is not yet known, in general, whether the  $BV$ -relaxed area is subadditive if considered as a set function and, further, if it gives rise to a measure. We expect such a subadditivity for  $BV$ -maps  $u$  from the plane to the plane, being motivated by relevant examples with explicit computations, and also because of the presence of a unique cartesian current with minimal completely vertical lifting associated to  $u$  (as recently shown in [23]). Unfortunately, this uniqueness result fails in higher codimension, where in addition we have less explicit examples.

## 2 Preliminaries

preliminaries

We start by collecting some tools needed in the proof of the main theorems. For an integer  $M \geq 2$ , set  $\mathbb{S}^{M-1} := \{x \in \mathbb{R}^M : |x| = 1\}$ . In what follows,  $\Omega \subset \mathbb{R}^2$  is a bounded open set.

### 2.1 Some consequences of the strict convergence

**Theorem 2.1 (Reshetnyak).** *Let  $\mu_h, \mu$  be (finite) Radon measures in  $\Omega$ , taking values in  $\mathbb{R}^M$ . Suppose that  $\mu_h \xrightarrow{*} \mu$  and  $|\mu_h|(\Omega) \rightarrow |\mu|(\Omega)$ . Then*

Reshetnyak

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f \left( x, \frac{\mu_h}{|\mu_h|}(x) \right) d|\mu_h|(x) = \int_{\Omega} f \left( x, \frac{\mu}{|\mu|}(x) \right) d|\mu|(x)$$

for any continuous bounded function  $f : \Omega \times \mathbb{S}^{M-1} \rightarrow \mathbb{R}$ .

<sup>1</sup>A map having the identity as the first component.

*Proof.* See for instance [2, Theorem 2.39]. □

For any  $u \in BV(\Omega; \mathbb{R}^2)$ , we recall that the distributional derivative  $Du$  is a Radon measure valued in  $\mathbb{R}^{2 \times 2}$ .  $SBV(\Omega)$  stands for the space of special functions of bounded variation on  $\Omega$  [2]. The symbol  $|Du|(\Omega)$  stands for the total variation of  $Du$  (see [2, Definition 3.4, pag. 119]) with  $|\cdot|$  the Frobenius norm. We denote by  $S_u$  the jump set of  $u$ .

**Definition 2.2 (Strict convergence).** *Let  $u \in BV(\Omega; \mathbb{R}^2)$  and  $(u_k) \subset BV(\Omega; \mathbb{R}^2)$ . We say that  $(u_k)$  converges to  $u$  strictly BV, if*

$$u_k \xrightarrow{L^1} u \quad \text{and} \quad |Du_k|(\Omega) \rightarrow |Du|(\Omega).$$

Let  $R = [a, b] \times [-1, 1]$ . For  $(t, \sigma) \in R$ , set

$$R_t^{x_1} := \{(x_1, x_2) \in R : x_1 = t\}, \quad R_\sigma^{x_2} := \{(x_1, x_2) \in R : x_2 = \sigma\}.$$

If  $u \in BV(R; \mathbb{R}^2)$ , by Lebesgue differentiation theorem and Fubini theorem, for almost every  $t \in [a, b]$ , the restriction  $u \llcorner R_t^{x_1}$  of  $u$  on the vertical segment  $R_t^{x_1}$  coincides with the trace of  $u$  on  $\mathcal{H}^1$ -almost every point of  $R_t^{x_1}$ . So, for almost every  $t \in [a, b]$ , the map  $u \llcorner R_t^{x_1}$  is well defined because it is independent of the representative of  $u$ . The same argument holds in  $R_\sigma^{x_2}$  for almost every  $\sigma \in [-1, 1]$ .

inheritance

**Lemma 2.3 (Inheritance of strict convergence to slices).** *Let  $u \in BV(R; \mathbb{R}^2)$ . Suppose that  $(v_k) \subset C^1(R; \mathbb{R}^2)$  is a sequence converging to  $u$  strictly BV( $R; \mathbb{R}^2$ ). Then for almost every  $(t, \sigma) \in R$ , there exists a subsequence  $(v_{k_h}) \subset (v_k)$ , depending on  $t$  and  $\sigma$ , such that*

$$v_{k_h} \llcorner R_t^{x_1} \rightarrow u \llcorner R_t^{x_1} \quad \text{strictly BV}(R_t^{x_1}; \mathbb{R}^2), \quad (2.1) \quad \text{thesis}$$

$$v_{k_h} \llcorner R_\sigma^{x_2} \rightarrow u \llcorner R_\sigma^{x_2} \quad \text{strictly BV}(R_\sigma^{x_2}; \mathbb{R}^2). \quad (2.2)$$

*Proof.* For almost every  $t \in [a, b]$ , in view of the definition of  $R_t^{x_1}$ , we can define the total variation of  $u \llcorner R_t^{x_1}$  as

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \sup \left\{ - \int_{-1}^1 u(t, x_2) \cdot g'(x_2) dx_2; g \in C_c^1((-1, 1); \overline{B}_1(0)) \right\}, \quad (2.3) \quad \text{eq:deri}$$

where  $\overline{B}_1(0) = \{(\xi, \eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 \leq 1\}$ . Let us show that

$$|D_2 u|(R) = \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt, \quad (2.4) \quad \text{claim}$$

where  $D_2 u := Du e_2$  is a Radon measure on  $R$  valued in  $\mathbb{R}^2$  with finite total variation. Since, for almost every  $t \in [a, b]$ ,  $v_k \llcorner R_t^{x_1} \rightarrow u \llcorner R_t^{x_1}$  in  $L^1(R_t^{x_1}; \mathbb{R}^2)$ , we have, using (2.3),

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) \leq \liminf_{k \rightarrow +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2. \quad (2.5) \quad \text{lsc}$$

Then, using Fatou lemma and Fubini theorem,

$$\begin{aligned} \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt &\leq \int_a^b \liminf_{k \rightarrow +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2 dt \\ &\leq \liminf_{k \rightarrow +\infty} \int_R |\partial_2 v_k(t, x_2)| dt dx_2 = |D_2 u|(R), \end{aligned} \quad (2.6) \quad \text{ineq}$$

where in the last equality we used Theorem 2.1 with  $f(x, \nu) = \sqrt{\nu_3^2 + \nu_4^2}$ , for every  $x \in R$ ,  $\nu \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ , with

$$\nu = \begin{pmatrix} \nu_1 & \nu_3 \\ \nu_2 & \nu_4 \end{pmatrix}.$$

The converse inequality in (2.4) is standard<sup>2</sup>. So, (2.4) is proved and (2.6) holds as an equality, which implies that also (2.5) holds as an equality, namely

$$|D(u \llcorner R_t^{x_1})|(R_t^{x_1}) = \liminf_{k \rightarrow +\infty} \int_{R_t^{x_1}} |\partial_2 v_k(t, x_2)| dx_2.$$

Extracting a subsequence  $(v_{k_h}) \subset (v_k)$  depending on  $t$ , we get

$$v_{k_h} \llcorner R_t^{x_1} \rightarrow u \llcorner R_t^{x_1} \quad \text{strictly } BV(R_t^{x_1}; \mathbb{R}^2).$$

Finally, repeating the same argument for  $v_{k_h}$  on the horizontal slices  $\{R_\sigma^{x_2}\}$ , we get (2.1) for a (not relabeled) sub-subsequence.  $\square$

Now, let  $B_l$  be the disk of  $\mathbb{R}^2$  centered at the origin of radius  $l > 0$ . We want to prove the analogue of Lemma 2.3 in  $B_l$ , by slicing with concentric circumferences. If  $u \in BV(B_l; \mathbb{R}^2)$ , as in the previous case, for almost every  $r \in (0, l)$  the restriction  $u \llcorner \partial B_r$  is well-defined and independent of the representative of  $u$ . In particular, for almost every  $r \in (0, l)$ , we can define the total variation of  $u \llcorner \partial B_r$  as

$$|D(u \llcorner \partial B_r)|(\partial B_r) := \sup \left\{ - \int_0^{2\pi} \bar{u}(r, \theta) \cdot f'(\theta) d\theta; f \in C^1([0, 2\pi]; \overline{B_1(0)}), f(0) = f(2\pi), f'(0) = f'(2\pi) \right\} \quad (2.7)$$

which turns out to be finite (see Lemma 2.5), giving that  $u \llcorner \partial B_r \in BV(\partial B_r; \mathbb{R}^2)$ , for almost every  $r \in (0, l)$ . Here

$$\bar{u}(r, \theta) := u(r \cos \theta, r \sin \theta), \quad r \in (0, l], \theta \in [0, 2\pi).$$

We want to relate this quantity with the notion of tangential total variation.

**def:annulus** **Definition 2.4.** For  $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , set  $\tau(x) = \frac{1}{|x|}(-x_2, x_1)$ . Let  $0 < l < L$  and  $A_{L,l} := B_L(0) \setminus \overline{B_l(0)}$  be an annulus around 0. We define the tangential total variation of  $u \in BV(A_{L,l}; \mathbb{R}^2)$  as the total variation of the Radon measure  $D_\tau u := Du\tau$ , namely

$$|D_\tau u|(A_{L,l}) = |Du\tau|(A_{L,l}) = \sup \left\{ - \int_{A_{L,l}} u \cdot (\nabla g \tau) dx : g \in C_c^1(A_{L,l}; \overline{B_1(0)}) \right\}. \quad (2.8)$$

The last equality in (2.8) is justified since  $\tau \in C^\infty(A_{L,l}; \mathbb{R}^2)$  satisfies  $\operatorname{div} \tau = 0$  everywhere, so for any  $g = (g^1, g^2) \in C_c^1(A_{L,l}; \mathbb{R}^2)$  we have

$$\begin{aligned} - \int_{A_{L,l}} u \cdot (\nabla g \tau) dx &= - \int_{A_{L,l}} u^1 \nabla g^1 \cdot \tau dx - \int_{A_{L,l}} u^2 \nabla g^2 \cdot \tau dx \\ &= - \int_{A_{L,l}} u^1 \operatorname{div}(g^1 \tau) dx - \int_{A_{L,l}} u^2 \operatorname{div}(g^2 \tau) dx \\ &= \int_{A_{L,l}} g^1 \tau \cdot dDu^1 + \int_{A_{L,l}} g^2 \tau \cdot dDu^2 = \int_{A_{L,l}} g \cdot (dDu)\tau = \langle Du\tau, g \rangle. \end{aligned}$$

<sup>2</sup>We recall that

$$|D_2 u|(R) = \sup \left\{ - \int_R u \cdot \partial_{x_2} g dx : g \in C_c^1(R; \overline{B_1(0)}) \right\}.$$

Now, for  $g \in C_c^1(R; \overline{B_1(0)})$ ,  $\int_R u \cdot \partial_y g dx = \int_a^b \left( \int_{-1}^1 u(t, x_2) \cdot \partial_{x_2} g(t, x_2) dx_2 \right) dt \leq \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt$ , so  $|D_2 u|(R) \leq \int_a^b |D(u \llcorner R_t^{x_1})|(R_t^{x_1}) dt$ .

This computation shows that  $|D_\tau u|(A_{L,l}) \leq |Du|(A_{L,l})$ , since  $|\tau| \leq 1$ , and also that (2.8) is compatible with the case  $u \in W^{1,1}(A_{L,l}; \mathbb{R}^2)$ , where simply  $|D_\tau u|(A_{L,l}) = \int_{A_{L,l}} |\nabla u \tau| dx$ . Moreover,  $Du = \frac{Du}{|Du|} |Du|$  by polar decomposition, so that

$$\langle Du \tau, g \rangle = \int_{A_{L,l}} g \cdot (dDu) \tau = \int_{A_{L,l}} g \cdot \left( \frac{Du}{|Du|} d|Du| \right) \tau = \int_{A_{L,l}} g \cdot \left( \frac{Du}{|Du|} \tau \right) d|Du| \quad \forall g \in C_c^1(B_l; \mathbb{R}^2),$$

giving that

$$D_\tau u = Du \tau = \frac{Du}{|Du|} \tau |Du|. \quad (2.9)$$

**Lemma 2.5 (Inheritance of strict convergence to circumferences).** *Let  $u \in BV(B_R; \mathbb{R}^2)$  and  $(v_k) \subset C^1(B_R; \mathbb{R}^2)$  be a sequence converging to  $u$  strictly  $BV(B_R; \mathbb{R}^2)$ . Then, for almost every  $r \in (0, R)$ , there exists a subsequence  $(v_{k_h}) \subset (v_k)$ , depending on  $r$ , such that*

$$v_{k_h} \llcorner \partial B_r \rightarrow u \llcorner \partial B_r \quad \text{strictly } BV(\partial B_r; \mathbb{R}^2) \quad \text{as } h \rightarrow +\infty. \quad (2.10)$$

*Proof.* For almost every  $r \in (0, R)$ , by Fatou lemma and Fubini theorem, the restriction  $v_k \llcorner \partial B_r$  has equi-bounded variation w.r.t.  $k$ . Moreover, we may assume that  $(v_k)$  converges to  $u$  almost everywhere in  $B_R$ , so that, for almost every  $r \in (0, R)$ ,

$$v_k \llcorner \partial B_r \rightarrow u \llcorner \partial B_r \quad \mathcal{H}^1\text{-a.e. in } \partial B_r. \quad (2.11)$$

Now, let  $r \in (0, R)$  be such that  $v_k \llcorner \partial B_r$  has equi-bounded variation and (2.11) holds. Then, there exists a subsequence  $(v_{k_h}) \subset (v_k)$  depending on  $r$  such that

$$v_{k_h} \llcorner \partial B_r \xrightarrow{*} u \llcorner \partial B_r \quad \text{w}^* - BV(\partial B_r; \mathbb{R}^2).$$

By lower semicontinuity of the variation, we infer that for almost every  $r \in (0, R)$ ,  $u \llcorner \partial B_r \in BV(\partial B_r; \mathbb{R}^2)$  and

$$|D(u \llcorner \partial B_r)|(\partial B_r) \leq \liminf_{h \rightarrow +\infty} \int_{\partial B_r} |\nabla v_{k_h} \tau| d\mathcal{H}^1. \quad (2.12)$$

Let  $0 < l < L \leq R$  be such that  $v_k \rightarrow u$  strictly  $BV(A_{L,l}, \mathbb{R}^2)$  where, as in Definition 2.4,  $A_{L,l} := B_L(0) \setminus \overline{B_l(0)}$  (notice that this holds for a.e.  $l$  and  $L$ ); by integration, we get

$$\begin{aligned} \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) dr &\leq \int_l^L \left( \liminf_{h \rightarrow +\infty} \int_{\partial B_r} |\nabla v_{k_h} \tau| d\mathcal{H}^1 \right) dr \\ &\leq \liminf_{h \rightarrow +\infty} \int_l^L \int_{\partial B_r} |\nabla v_{k_h} \tau| d\mathcal{H}^1 dr = \liminf_{h \rightarrow +\infty} \int_{A_{L,l}} |\nabla v_{k_h} \tau| dx. \end{aligned} \quad (2.13)$$

Thanks to Theorem 2.1, with the choices  $M = 4$ ,  $\mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{R}^{2 \times 2}$ ,  $f \in C_b(A_{L,l} \times \mathbb{S}^3)$ ,

$$f(x, \nu) := \sqrt{|\nu_{\text{hor}} \cdot \tau(x)|^2 + |\nu_{\text{vert}} \cdot \tau(x)|^2},$$

where  $\nu \in \mathbb{S}^3$  and  $\nu_{\text{hor}} := (\nu_1, \nu_3)$ ,  $\nu_{\text{vert}} := (\nu_2, \nu_4)$ , we obtain

$$\lim_{k \rightarrow +\infty} \int_{A_{L,l}} |\nabla v_k \tau| dx = \int_{A_{L,l}} \left| \frac{Du}{|Du|} \tau \right| d|Du| = |D_\tau u|(A_{L,l}), \quad (2.14)$$

where in the last equality we have used (2.9). So we get

$$|D_\tau u|(B_l) \geq \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) dr.$$

In order to prove the converse inequality, let  $g \in C_c^1(A_{L,l}; \overline{B_1}(0))$ . Then, in polar coordinates, by definition (2.7),

$$\int_{A_{L,l}} u \cdot \nabla g \tau dx = \int_l^L \int_0^{2\pi} \bar{u}(\rho, \theta) \cdot \partial_\theta \bar{g}(\rho, \theta) d\rho d\theta \leq \int_l^L |D(u \llcorner \partial B_\rho)|(\partial B_\rho) d\rho.$$

So, we have proved that

$$|D_\tau u|(A_{L,l}) = \int_l^L |D(u \llcorner \partial B_r)|(\partial B_r) dr.$$

In particular, we deduce that (2.13) is a chain of equalities. Then, (2.12) holds as an equality and there exists a subsequence  $(v_{k_h}) \subset (v_k)$ , depending on  $r$ , which achieves the full limit. Since  $l$  and  $L$  are arbitrary, we get the thesis.  $\square$

## 2.2 Further properties in dimension 1

In [3, Proposition 2.4] the following is proved:

**Lemma 2.6.** *Let  $(\gamma_k) \subset W^{1,1}((a,b); \mathbb{R}^2)$  be a sequence converging strictly  $BV((a,b); \mathbb{R}^2)$  to  $\gamma \in W^{1,1}((a,b); \mathbb{R}^2)$ . Then  $\gamma_k \rightarrow \gamma$  uniformly in  $(a,b)$ .*

For our purposes, we need an improvement of Lemma 2.6, where discontinuous functions  $\gamma$  at a single point, or at a finite number of points, are allowed; we start with one point discontinuity.

**Lemma 2.7.** *Let  $I^- := [-1, 0), I^+ := (0, 1]$ . Suppose that  $(\gamma_k) \subset W^{1,1}([-1, 1]; \mathbb{R}^2)$  is a sequence converging strictly  $BV([-1, 1]; \mathbb{R}^2)$  to  $\gamma \in BV([-1, 1]; \mathbb{R}^2) \cap W^{1,1}(I^-; \mathbb{R}^2) \cap W^{1,1}(I^+; \mathbb{R}^2)$ , with  $\gamma^+(0) \neq \gamma^-(0)$ . Let  $S : [-1/3, 1/3] \rightarrow \mathbb{R}^2$  be defined by*

$$S(\tau) := \frac{3}{2} \left( (1/3 + \tau) \gamma^+(0) + (1/3 - \tau) \gamma^-(0) \right), \quad \tau \in [-1/3, 1/3].$$

Let  $\tilde{\gamma}^-$  (resp.  $\tilde{\gamma}^+$ ) be the reparametrization of  $\gamma|_{I^-}$  (resp.  $\gamma|_{I^+}$ ) on  $[-1, -1/3]$  (resp.  $[1/3, 1]$ ) defined by the composition with the increasing linear function taking  $[-1, -1/3]$  onto  $[-1, 0]$  (resp.  $[1/3, 1]$  onto  $[0, 1]$ ). Define

$$\tilde{\gamma} : [-1, 1] \rightarrow \mathbb{R}^2, \quad \tilde{\gamma} := \begin{cases} \tilde{\gamma}^- & \text{in } [-1, -1/3] \\ S & \text{in } [-1/3, 1/3] \\ \tilde{\gamma}^+ & \text{in } [1/3, 1]. \end{cases} \quad (2.15)$$

Then there exist:

- (a) a Lipschitz strictly increasing surjective function  $h : [-1, 1] \rightarrow [-1, 1]$ ,
- (b) a subsequence  $(k_j)$  and Lipschitz strictly increasing surjective functions  $h_{k_j} : [-1, 1] \rightarrow [-1, 1]$  for any  $j \in \mathbb{N}$ , with  $\sup_j \|\dot{h}_{k_j}\|_\infty < +\infty$ ,

such that

$$\lim_{j \rightarrow +\infty} \gamma_{k_j} \circ h_{k_j} = \tilde{\gamma} \circ h \quad \text{uniformly in } [-1, 1]. \quad (2.16)$$



*Proof.* The lengths  $L_k$  of  $\gamma_k$  and  $L$  of  $\gamma$  are given by

$$\begin{aligned} L_k &= \int_{-1}^1 |\dot{\gamma}_k| \, d\tau, \\ L &= |\dot{\gamma}|([-1, 1]) = \int_{-1}^0 |\dot{\gamma}| \, d\tau + |\gamma^+(0) - \gamma^-(0)| + \int_0^1 |\dot{\gamma}| \, d\tau. \end{aligned}$$

Since, by assumption,  $\gamma_k \rightarrow \gamma$  strictly  $BV([-1, 1]; \mathbb{R}^2)$ , we have that  $L_k \rightarrow L$  as  $k \rightarrow +\infty$ . Fix  $\eta > 0$  and for all  $k \in \mathbb{N}$  define the function<sup>3</sup>

$$s_k : [-1, 1] \rightarrow [0, L + \eta], \quad s_k(t) := \frac{L + \eta}{L_k + \eta} \int_{-1}^t \left( |\dot{\gamma}_k(\tau)| + \frac{\eta}{2} \right) \, d\tau, \quad (2.17) \quad \boxed{\text{eq:s\_k}}$$

with Lipschitz inverse  $\alpha_k := s_k^{-1} : [0, L + \eta] \rightarrow [-1, 1]$ . Define

$$\hat{\gamma}_k : [0, L + \eta] \rightarrow \mathbb{R}^2, \quad \hat{\gamma}_k(s) := \gamma_k(\alpha_k(s)) \quad \forall s \in [0, L + \eta]. \quad (2.18)$$

Since from (2.17)

$$\left| \frac{d\hat{\gamma}_k}{ds}(s) \right| \leq \frac{|\dot{\gamma}_k(\alpha_k(s))|}{|\dot{s}_k(\alpha_k(s))|} \leq \frac{L_k + \eta}{L + \eta} \leq C \quad \text{for a.e. } s \in [0, L + \eta],$$

for some constant  $C > 0$  independent of  $k$ , the sequence  $(\hat{\gamma}_k)$  is bounded in  $W^{1,\infty}([0, L + \eta]; \mathbb{R}^2)$ . Thus, up to a (not relabeled) subsequence, we may assume that there exists  $\hat{\gamma} \in W^{1,\infty}([0, L + \eta]; \mathbb{R}^2)$  such that

$$\hat{\gamma}_k \rightharpoonup \hat{\gamma} \text{ weakly* in } W^{1,\infty}([0, L + \eta]; \mathbb{R}^2) \text{ and uniformly in } [0, L + \eta]. \quad (2.19) \quad \boxed{\text{thesis1}}$$

We observe that for any open interval  $J \subseteq [0, L + \eta]$ ,

$$\int_J |\dot{\hat{\gamma}}| \, ds \leq \liminf_{k \rightarrow +\infty} \int_J |\dot{\hat{\gamma}}_k| \, ds \leq |J| \liminf_{k \rightarrow +\infty} \frac{L_k + \eta}{L + \eta} = |J|,$$

and thus

$$|\dot{\hat{\gamma}}| \leq 1 \text{ a.e. in } [0, L + \eta]. \quad (2.20) \quad \boxed{\text{eq:subu}}$$

Now, in order to conclude the proof, we need to show that  $\hat{\gamma}$  is a reparametrization of  $\tilde{\gamma}$ . Then the thesis of the lemma will follow by reparametrizing both  $\hat{\gamma}_k$  and  $\hat{\gamma}$  on  $[-1, 1]$ .

Using that  $(\gamma_k)$  strictly converges  $BV([-1, 1]; \mathbb{R}^2)$  to  $\gamma \in W^{1,1}(I^-; \mathbb{R}^2) \cap W^{1,1}(I^+; \mathbb{R}^2)$ , by Lemma 2.6 and a diagonal process, we can find an infinitesimal sequence  $(\delta_{k_j}) \subset (0, 1]$  such that

$$\|\gamma_{k_j} - \gamma\|_{L^\infty([-1, 1] \setminus (-\delta_{k_j}, \delta_{k_j}))} \rightarrow 0 \quad (2.21) \quad \boxed{\text{uniform}}$$

and

$$\int_{-1}^{-\delta_{k_j}} |\dot{\gamma}_{k_j}(\tau)| \, d\tau \rightarrow \int_{-1}^0 |\dot{\gamma}(\tau)| \, d\tau, \quad \int_{\delta_{k_j}}^1 |\dot{\gamma}_{k_j}(\tau)| \, d\tau \rightarrow \int_0^1 |\dot{\gamma}(\tau)| \, d\tau$$

as  $j \rightarrow +\infty$ . In particular,

$$\lim_{j \rightarrow +\infty} \gamma_{k_j}(\pm \delta_{k_j}) = \gamma^\pm(0) \quad (2.22) \quad \boxed{\text{limit d}}$$

<sup>3</sup>We need  $\eta$ , since in principle  $\dot{\gamma}_k$  could vanish somewhere.

and, setting

$$\begin{aligned} r_{k_j}^- &:= s_{k_j}(-\delta_{k_j}) = \frac{L + \eta}{L_{k_j} + \eta} \int_{-1}^{-\delta_{k_j}} (|\dot{\gamma}_{k_j}| + \frac{\eta}{2}) d\tau, \\ r_{k_j}^+ &:= s_{k_j}(\delta_{k_j}) = \frac{L + \eta}{L_{k_j} + \eta} \left[ \int_{-1}^1 (|\dot{\gamma}_{k_j}| + \frac{\eta}{2}) d\tau - \int_{\delta_{k_j}}^1 (|\dot{\gamma}_{k_j}| + \frac{\eta}{2}) d\tau \right], \end{aligned}$$

we have

$$\begin{aligned} \lim_{j \rightarrow +\infty} r_{k_j}^- &= \frac{\eta}{2} + \int_{-1}^0 |\dot{\gamma}| d\tau =: r^-, \\ \lim_{j \rightarrow +\infty} r_{k_j}^+ &= \frac{\eta}{2} + \int_{-1}^0 |\dot{\gamma}| d\tau + |\gamma^+(0) - \gamma^-(0)| =: r^+. \end{aligned} \tag{2.23} \quad \text{eq:conv}$$

As a consequence of (2.19), (2.22), and (2.23) we get

$$\gamma_{k_j}(\alpha_{k_j}(r_{k_j}^\pm)) = \widehat{\gamma}_{k_j}(r_{k_j}^\pm) \rightarrow \widehat{\gamma}(r^\pm) = \gamma^\pm(0).$$

Therefore the curve  $\widehat{\gamma}$  maps the segment  $[r^-, r^+]$  into a curve joining  $\gamma^-(0)$  and  $\gamma^+(0)$ . Now, since  $r^+ - r^- = |\gamma^+(0) - \gamma^-(0)|$ , from (2.20) we conclude that  $\widehat{\gamma}$  coincides with the unit-speed parametrization of the segment joining  $\gamma^-(0)$  and  $\gamma^+(0)$  on  $[r^-, r^+]$ . Hence we have shown that

$$\gamma_{k_j} \circ \alpha_{k_j} \rightarrow S \circ \widetilde{\alpha} \text{ uniformly in } [r^-, r^+] \text{ as } j \rightarrow +\infty, \tag{2.24} \quad \text{ali}$$

for the affine increasing reparametrization  $\widetilde{\alpha} : [r^-, r^+] \rightarrow [-1/3, 1/3]$ .

We now check that  $\widehat{\gamma} = \gamma \circ \alpha$  on  $[0, r^-]$  for some increasing bijection  $\alpha : [0, r^-] \rightarrow [-1, 0]$ , and similarly  $\widehat{\gamma} = \gamma \circ \beta$  on  $[r^+, L + \eta]$  for some increasing bijection  $\beta : [r^+, L + \eta] \rightarrow [0, 1]$ .

Indeed, the functions  $\alpha_k : [0, L + \eta] \rightarrow [-1, 1]$  are strictly increasing and satisfy

$$|\dot{\alpha}_k(s_k(t))| = \frac{L_k + \eta}{(L + \eta)(|\dot{\gamma}_k(t)| + \frac{\eta}{2})} \leq \frac{C}{\eta},$$

so that we may assume (up to extracting a further not relabeled subsequence) that

$$\alpha_{k_j} \rightharpoonup \alpha \text{ weakly}^* \text{ in } W^{1,\infty}([0, L + \eta]) \text{ and uniformly in } [0, L + \eta],$$

for some nondecreasing map  $\alpha \in W^{1,\infty}([0, L + \eta])$ . Hence, using (2.21), we find out

$$\widehat{\gamma}_{k_j}(s) = \gamma_{k_j}(\alpha_{k_j}(s)) \rightarrow \gamma(\alpha(s)) \text{ for all } s \in [0, r^-].$$

This, together with (2.19), implies

$$\widehat{\gamma}(s) = \gamma \circ \alpha(s) \text{ for all } s \in [0, r^-].$$

A similar argument shows that this also holds for all  $s \in (r^+, L + \eta]$ .

Finally, we observe that  $\alpha$  is strictly increasing on  $[0, r^-] \cup (r^+, L + \eta]$ . For, if  $\alpha$  is constant on some interval  $[s_1, s_2] \subset [0, r^-]$ , we have  $\lim_{j \rightarrow +\infty} \alpha_{k_j}(s_1) = \lim_{h \rightarrow +\infty} \alpha_{k_j}(s_2)$  and hence

$$0 = \lim_{j \rightarrow +\infty} \int_{s_1}^{s_2} \dot{\alpha}_{k_j}(s) ds = \lim_{j \rightarrow +\infty} \int_{t_{k_j,1}}^{t_{k_j,2}} d\tau = \lim_{j \rightarrow +\infty} (t_{k_j,2} - t_{k_j,1}), \tag{2.25} \quad \text{eq:0}$$

where  $t_{k_j,i}$  are defined by  $s_{k_j}(t_{k_j,1}) = s_1$  and  $s_{k_j}(t_{k_j,2}) = s_2$ . By definition (2.17) of  $s_{k_j}$  we have

$$0 < s_2 - s_1 = \int_{t_{k_j,1}}^{t_{k_j,2}} (|\dot{\gamma}_{k_j}(\tau)| + \frac{\eta}{2}) d\tau. \tag{2.26} \quad \text{eq:mag-}$$

Possibly passing to a (not relabeled) subsequence and using (2.25), let  $\bar{t} \in [-1, 0]$  be the limit of  $(t_{k_j,1})$  and  $(t_{k_j,2})$ . If  $\bar{t} \neq 0$ , for any open neighborhood  $J \subset (-1, 0)$  of  $\bar{t}$ , using (2.26), we get

$$\int_J |\dot{\gamma}| \, d\tau = \lim_{h \rightarrow +\infty} \int_J |\dot{\gamma}_{k_j}| \, d\tau \geq s_2 - s_1,$$

which contradicts the inclusion  $\dot{\gamma} \in L^1((-1, 0); \mathbb{R}^2)$ . The same argument holds if  $\bar{t} = 0$ , for  $J$  a left neighbourhood of 0 in  $(-1, 0)$ . We conclude that  $\alpha$  is strictly increasing.

Let  $h_{k_j}$  be a rescaling of  $\alpha_{k_j}$  on  $[-1, 1]$ ; rescaling also  $\alpha$  from  $[0, r^-]$  to  $[-1, -1/3]$ , and then from  $[r^+, L + \eta]$  to  $[1/3, 1]$ , using also  $\tilde{\alpha}$  in (2.24), we construct a reparametrization  $h : [-1, 1] \rightarrow [-1, 1]$  such that (2.16) holds, and the lemma is proved.  $\square$

Lemma 2.7 can be readily extended to curves  $\gamma$  with finitely many jump points:

**Corollary 2.8.** *Let  $(\gamma_k) \subset W^{1,1}([0, 2\pi]; \mathbb{R}^2)$  be a sequence converging strictly BV  $([0, 2\pi]; \mathbb{R}^2)$  to a map  $\gamma \in SBV([0, 2\pi]; \mathbb{R}^2)$  having finitely many jump points  $0 < z_1 < z_2 < \dots < z_n < 2\pi$ . Let  $\theta_0 > 0$  be such that the intervals  $(z_i - \theta_0, z_i + \theta_0) \subset (0, 2\pi)$  are disjoint, and for all  $i = 1, \dots, n$  let  $S_i : [z_i - \theta_0, z_i + \theta_0] \rightarrow \mathbb{R}^2$  be defined by*

$$S_i(\tau) := \frac{1}{2\theta_0} \left( (\tau - z_i + \theta_0) \gamma^+(z_i) + (z_i + \theta_0 - \tau) \gamma^-(z_i) \right), \quad \tau \in [z_i - \theta_0, z_i + \theta_0].$$

Setting  $z_0 := 0$  and  $z_{n+1} := 2\pi$ , for all  $i = 0, \dots, n$  let  $\tilde{\gamma}_i : [z_i + \theta_0, z_{i+1} - \theta_0] \rightarrow \mathbb{R}^2$  be a rescaled reparametrization of  $\gamma : [z_i, z_{i+1}]$ . Finally, let  $\tilde{\gamma} : [0, 2\pi] \rightarrow \mathbb{R}^2$  be the Lipschitz curve defined as

$$\tilde{\gamma} := \tilde{\gamma}_0 \star S_1 \star \tilde{\gamma}_1 \star S_2 \star \tilde{\gamma}_2 \star \dots \star S_n \star \tilde{\gamma}_n, \quad (2.27)$$

where  $\star$  denotes the arc composition. Then there exist a subsequence  $(k_j)$  and Lipschitz increasing surjective functions  $h, h_{k_j} : [0, 2\pi] \rightarrow [0, 2\pi]$  such that

$$\lim_{j \rightarrow +\infty} \gamma_{k_j} \circ h_{k_j} = \tilde{\gamma} \circ h \quad \text{uniformly in } [0, 2\pi]. \quad (2.28)$$

*Proof.* We sketch the proof which is a direct consequence of the arguments used to prove Lemma 2.7. Choose points  $w_i, i = 1, \dots, n-1$  so that  $z_i + \theta_0 < w_i < z_{i+1} - \theta_0$ , and let  $w_0 = 0$  and  $w_n = 2\pi$ . Then we can apply Lemma 2.7 to any interval  $[w_i, w_{i+1}]$ , and taking a suitable subsequence and concatenating the obtained maps one can easily construct the desired parametrizations.  $\square$

### 2.3 Planar Plateau-type problem

Let  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}^2$  be a possibly self-intersecting Lipschitz curve. Let us consider, as in [24] (see also [14]), the planar Plateau-type problem (1.5) spanning  $\varphi$ . Notice that the class of competitors is non-empty, since it contains the map  $v(x) = |x|\varphi\left(\frac{x}{|x|}\right)$  for  $x \neq 0$ , and  $v(0) = 0$ . We first observe that  $P$  is independent of the radius of the domain of integration. Specifically, for any  $r > 0$ , let

$$\varphi_r(y) := \varphi\left(\frac{y}{r}\right) \quad \text{for all } y \in \partial B_r. \quad (2.29)$$

Setting  $y := rx, y \in \overline{B_r}$  and  $v_r(y) := v\left(\frac{y}{r}\right)$ , we have

$$\int_{B_1} |Jv| \, dx = \int_{B_r} |Jv_r| \, dy \quad \forall v \in \text{Lip}(B_1; \mathbb{R}^2). \quad (2.30)$$

In particular, for any  $r > 0$ ,

$$P(\varphi) = \inf \left\{ \int_{B_r} |Jv| \, dx : v \in \text{Lip}(B_r; \mathbb{R}^2), v|_{\partial B_r} = \varphi_r \right\}. \quad (2.31)$$

In the next proposition we show that  $P(\cdot)$  is invariant under Lipschitz reparameterizations of  $\varphi$ .

**Proposition 2.9 (Invariance).** *Let  $\varphi \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  and  $h$  be a Lipschitz homeomorphism of  $\mathbb{S}^1$ . Then*

$$P(\varphi \circ h) = P(\varphi).$$

*Proof.* Since  $h$  and the identity map  $\text{id} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  have the same degree, they are homotopic in  $\mathbb{S}^1$  by Hopf Theorem (see [21, pag. 51]), namely there exists a Lipschitz map<sup>4</sup>  $K : [0, 1] \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that

$$K(0, \cdot) = \text{id}, \quad K(1, \cdot) = h.$$

Define  $H : [0, 1] \times \mathbb{S}^1 \rightarrow \mathbb{R}^2$  as  $H(t, \nu) = \varphi(K(t, \nu))$ . Then,  $H$  is Lipschitz and

$$H(0, \cdot) = \varphi, \quad H(1, \cdot) = \varphi \circ h.$$

Now, suppose  $v_k \in \text{Lip}(B_1; \mathbb{R}^2)$  is such that  $v_k = \varphi$  on  $\partial B_1$  and

$$\lim_{k \rightarrow +\infty} \int_{B_1} |Jv_k| \, dx \rightarrow P(\varphi).$$

Define the map  $\tilde{v}_k : B_1 \rightarrow \mathbb{R}^2$  as

$$\tilde{v}_k(x) = \begin{cases} v_k(kx) & \text{for } x \in B_{\frac{1}{k}}, \\ H\left(k|x| - 1, \frac{x}{|x|}\right) & \text{for } x \in B_{\frac{2}{k}} \setminus B_{\frac{1}{k}}, \\ \varphi \circ h\left(\frac{x}{|x|}\right) & \text{for } x \in B_1 \setminus B_{\frac{2}{k}}. \end{cases} \quad (2.32)$$

Then  $\tilde{v}_k \in \text{Lip}(B_1; \mathbb{R}^2)$  and  $\tilde{v}_k = \varphi \circ h$  on  $\partial B_1$ . Moreover, since  $H$  and  $\varphi \circ h$  take values in  $\varphi(\mathbb{S}^1)$  which is 1-dimensional, by the area formula and (2.30) we have

$$\int_{B_1} |J\tilde{v}_k(x)| \, dx = \int_{B_{\frac{1}{k}}} |Jv_k(kx)| \, dx = \int_{B_1} |Jv_k| \, dx \rightarrow P(\varphi)$$

as  $k \rightarrow +\infty$ . In particular  $P(\varphi \circ h) \leq P(\varphi)$ . Exchanging the role of  $\varphi$  and  $\varphi \circ h$ , we obtain the converse inequality.  $\square$

**Lemma 2.10.** *Let  $\varphi_1, \varphi_2 \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ . Then*

$$|P(\varphi_1) - P(\varphi_2)| \leq 2\|\varphi_1 - \varphi_2\|_\infty (\|\dot{\varphi}_1\|_1 + \|\dot{\varphi}_2\|_1). \quad (2.33)$$

*Proof.* Let  $v \in \text{Lip}(B_1; \mathbb{R}^2)$  be such that  $v = \varphi_2$  on  $\mathbb{S}^1$ . We define

$$w(x) = \begin{cases} v_{\frac{1}{2}}(x) = v(2x) & \text{if } |x| < \frac{1}{2}, \\ 2(1 - |x|)\varphi_2\left(\frac{x}{|x|}\right) + 2\left(|x| - \frac{1}{2}\right)\varphi_1\left(\frac{x}{|x|}\right) & \text{if } \frac{1}{2} \leq |x| \leq 1. \end{cases} \quad (2.34)$$

Then  $w \in \text{Lip}(B_1; \mathbb{R}^2)$ ,  $w(x) = \varphi_2(x/|x|)$  if  $x \in \partial B_{\frac{1}{2}}$  and  $w = \varphi_1$  on  $\partial B_1$ . Let us estimate

$$\int_{B_1 \setminus \overline{B_{\frac{1}{2}}}} |Jw| \, dx.$$

<sup>4</sup>The construction of a Lipschitz homotopy between  $h$  and  $\text{id}$  can be done at the level of liftings, by considering the affine interpolation map (for more details, see for instance [3, Proposition 3.4]).

Writing  $w$  in polar coordinates in the annulus  $B_1 \setminus \overline{B_{\frac{1}{2}}}$ ,  $\rho \in (\frac{1}{2}, 1)$ ,  $\theta \in [0, 2\pi)$ ,

$$\bar{w}(\rho, \theta) := w(\rho \cos \theta, \rho \sin \theta) = 2(1 - \rho)\bar{\varphi}_2(\theta) + 2\left(\rho - \frac{1}{2}\right)\bar{\varphi}_1(\theta),$$

where  $\bar{\varphi}_i(\theta) := \varphi_i(\cos \theta, \sin \theta)$ ,  $i = 1, 2$ . Then

$$\begin{aligned} |J\bar{w}(\rho, \theta)| &= 4 \left| (\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)) \wedge \left( (1 - \rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta) \right) \right| \\ &\leq 4 |\bar{\varphi}_1(\theta) - \bar{\varphi}_2(\theta)| \left| (1 - \rho)\dot{\bar{\varphi}}_2(\theta) + \left(\rho - \frac{1}{2}\right)\dot{\bar{\varphi}}_1(\theta) \right| \\ &\leq 4\|\varphi_1 - \varphi_2\|_\infty (|\dot{\bar{\varphi}}_2(\theta)| + |\dot{\bar{\varphi}}_1(\theta)|). \end{aligned}$$

Thus, integrating on  $B_1 \setminus \overline{B_{\frac{1}{2}}}$ ,

$$\int_{B_1 \setminus \overline{B_{\frac{1}{2}}}} |Jw(x)| dx = \int_{\frac{1}{2}}^1 \int_0^{2\pi} |J\bar{w}(\rho, \theta)| d\rho d\theta \quad (2.35)$$

$$\leq 2\|\varphi_1 - \varphi_2\|_\infty \int_0^{2\pi} (|\dot{\bar{\varphi}}_2(\theta)| + |\dot{\bar{\varphi}}_1(\theta)|) d\theta \quad (2.36)$$

$$= 2\|\varphi_1 - \varphi_2\|_\infty (\|\dot{\bar{\varphi}}_1\|_1 + \|\dot{\bar{\varphi}}_2\|_1). \quad (2.37)$$

Hence

$$P(\varphi_1) \leq \int_{B_1} |Jw| dx \leq \int_{B_{\frac{1}{2}}} |Jv_{\frac{1}{2}}| dx + 2\|\varphi_1 - \varphi_2\|_\infty (\|\dot{\bar{\varphi}}_1\|_1 + \|\dot{\bar{\varphi}}_2\|_1). \quad (2.38)$$

Since  $v$  is arbitrary and (with the notation in (2.29))  $v_{\frac{1}{2}} = (\varphi_2)_{\frac{1}{2}}$  on  $\partial B_{\frac{1}{2}}$ , using (2.31) with  $r = \frac{1}{2}$  we can take the infimum on these maps in (2.38) and get

$$P(\varphi_1) - P(\varphi_2) \leq 2\|\varphi_1 - \varphi_2\|_\infty (\|\dot{\bar{\varphi}}_1\|_1 + \|\dot{\bar{\varphi}}_2\|_1).$$

Exchanging the role of  $\varphi_1$  and  $\varphi_2$  we find that also  $P(\varphi_2) - P(\varphi_1)$  is bounded by the right-hand side of the previous expression. This concludes the proof.  $\square$

**Remark 2.11.** With a similar argument used in the proof of Lemma 2.10 it is immediate to obtain that if  $[a, b] \subset \mathbb{R}$  is a bounded interval and  $\gamma_1, \gamma_2 : [a, b] \rightarrow \mathbb{R}^2$  are Lipschitz curves, then the following holds: Let  $\Phi : [a, b] \times [0, 1] \rightarrow \mathbb{R}^2$  be the affine interpolation map  $\Phi(t, s) := s\gamma_1(t) + (1 - s)\gamma_2(t)$ . Then, as in (2.35),

$$\int_{[a, b] \times [0, 1]} |\Phi_t \wedge \Phi_s| dt ds \leq \|\gamma_1 - \gamma_2\|_\infty (\|\dot{\gamma}_1\|_1 + \|\dot{\gamma}_2\|_1). \quad (2.39)$$

Using Lemma 2.10 we readily obtain the following continuity property for the minimum of the Plateau-type problem (1.5).

**Corollary 2.12 (Continuity of  $P$ ).** *Let  $\varphi \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  and suppose that  $(\varphi_k)_k \subset \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  is such that*

$$\varphi_k \rightarrow \varphi \quad \text{uniformly} \quad \text{and} \quad \sup_{k \in \mathbb{N}} \|\dot{\varphi}_k\|_1 < +\infty.$$

*Then  $P(\varphi_k) \rightarrow P(\varphi)$  as  $k \rightarrow +\infty$ .*

In what follows it is convenient to consider the relaxation

$$\overline{P}(\gamma) := \inf \left\{ \liminf_{k \rightarrow +\infty} P(\varphi_k) : \varphi_k \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2), \varphi_k \rightarrow \gamma \text{ strictly } BV(\mathbb{S}^1; \mathbb{R}^2) \right\} \quad \forall \gamma \in BV(\mathbb{S}^1; \mathbb{R}^2) \quad (2.40)$$

of  $P$  with respect to the strict convergence in  $BV$  of the boundary datum. It is well known that the infimum in (2.40) is taken on a non-empty class of approximation maps. Moreover, by (2.30), also  $\overline{P}$  is invariant by rescaling, i.e.  $\overline{P}(\gamma) = \overline{P}(\gamma_r)$ .

**Lemma 2.13.** *Let  $\varphi \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$ . Then  $\overline{P}(\varphi) = P(\varphi)$ .*

*Proof.* If  $(\varphi_k) \subset \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  is a sequence converging to  $\varphi$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$ , then by Lemma 2.6  $\varphi_k \rightarrow \varphi$  uniformly on  $\mathbb{S}^1$  as  $k \rightarrow +\infty$ . Moreover, the strict convergence guarantees that the total variations of  $\varphi_k$  are equibounded. So, thanks to Corollary 2.12,

$$P(\varphi_k) \rightarrow P(\varphi) \quad (2.41)$$

as  $k \rightarrow +\infty$ . Since this holds for any sequence  $(\varphi_k)$  as above, the thesis follows.  $\square$

**Lemma 2.14.** *Let  $\gamma \in SBV(\mathbb{S}^1; \mathbb{R}^2)$  have a finite number of jump points  $z_i \in \mathbb{S}^1$ ,  $i = 1, \dots, n$ . Let  $\tilde{\gamma} : \mathbb{S}^1 \rightarrow \mathbb{R}^2$  be the Lipschitz map in (2.27) (with  $\mathbb{S}^1$  identified with  $[0, 2\pi]$ ). Then*

$$\overline{P}(\gamma) = P(\tilde{\gamma}). \quad (2.42)$$

*Proof.* Let  $(\varphi_k)_k \subset \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  be a sequence converging strictly to  $\gamma$ . Let us consider a not-relabeled subsequence of  $(\varphi_k)_k$ ; by Corollary 2.8 there are a further subsequence  $(\varphi_{k_j})_j$  and Lipschitz reparametrizations  $\gamma_{k_j} = \varphi_{k_j} \circ h_{k_j} \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  of  $\varphi_{k_j}$  such that  $\gamma_{k_j} \rightarrow \tilde{\gamma} \circ h$  uniformly as  $j \rightarrow +\infty$ , for some Lipschitz homeomorphism  $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ . Moreover, since by Lemma 2.7(b) the reparametrization maps  $h_{k_j}$  can be chosen with uniformly bounded Lipschitz constants, it follows that  $\gamma_{k_j}$  have uniformly bounded total variations. Hence it follows from Corollary 2.12 that  $P(\gamma_{k_j}) \rightarrow P(\tilde{\gamma} \circ h)$  as  $j \rightarrow +\infty$ . On the other hand, by Proposition 2.9 we also have  $P(\varphi_{k_j}) \rightarrow P(\tilde{\gamma})$  as  $j \rightarrow +\infty$ . Finally, since this argument holds for any subsequence of  $(\varphi_k)$ , we conclude that the whole sequence satisfies  $P(\varphi_k) \rightarrow P(\tilde{\gamma})$ , and therefore  $\overline{P}(\gamma) = P(\tilde{\gamma})$ .  $\square$

As a consequence of the argument in the proof of Lemma 2.14, we easily infer the following continuity property:

**Corollary 2.15.** *Let  $\gamma \in SBV(\mathbb{S}^1; \mathbb{R}^2)$  and  $\tilde{\gamma}$  be as in Corollary 2.8, and assume that  $(\varphi_k)_k \subset \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  is a sequence converging strictly to  $\gamma$ . Then*

$$\lim_{k \rightarrow +\infty} P(\varphi_k) = \overline{P}(\gamma) = P(\tilde{\gamma}).$$

Furthermore, we can refine the previous corollary as follows:

**Corollary 2.16.** *Let  $\gamma, \gamma_k \in SBV(\mathbb{S}^1; \mathbb{R}^2)$ ,  $k \geq 1$ , be maps as in Corollary 2.8. Assume that  $(\gamma_k)$  converges to  $\gamma$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$ . Then*

$$\lim_{k \rightarrow +\infty} \overline{P}(\gamma_k) = \overline{P}(\gamma).$$

*Proof.* By Corollary 2.15 and the density of  $\text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  in  $BV(\mathbb{S}^1; \mathbb{R}^2)$  with respect to the strict convergence, for all  $k \geq 1$  we can find  $\varphi_k \in \text{Lip}(\mathbb{S}^1; \mathbb{R}^2)$  such that

$$\|\gamma_k - \varphi_k\|_1 + \left| |\dot{\varphi}_k|(\mathbb{S}^1) - |\dot{\gamma}_k|(\mathbb{S}^1) \right| + |P(\varphi_k) - \bar{P}(\gamma_k)| \leq \frac{1}{k}.$$

Hence the sequence  $(\varphi_k)$  converges to  $\gamma$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$ , and by the triangle inequality and Corollary 2.15 we conclude

$$\lim_{k \rightarrow +\infty} \bar{P}(\gamma_k) = \bar{P}(\gamma).$$

□

### 3 Relaxation on piecewise Lipschitz maps jumping on a curve

pschitz\_maps

Recalling that  $R = [a, b] \times [-1, 1]$ , consider  $R^+ = \{(x_1, x_2) \in R : x_2 > 0\}$  and  $R^- = \{(x_1, x_2) \in R : x_2 < 0\}$ .

**Definition 3.1 (Piecewise Lipschitz map).** *We say that a map  $u : R \rightarrow \mathbb{R}^2$  is piecewise Lipschitz if  $u \in BV(R; \mathbb{R}^2)$  and  $u \in \text{Lip}(R^-; \mathbb{R}^2) \cap \text{Lip}(R^+; \mathbb{R}^2)$ .*

Thus  $S_u \subseteq [a, b] \times \{0\}$ ; we define  $u^\pm : [a, b] \times \{0\} \rightarrow \mathbb{R}^2$  the traces of  $u|_{R^\pm}$ , which are Lipschitz maps. Set  $I = [0, 1]$  and define  $X^{\text{aff}} : [a, b] \times I \rightarrow \mathbb{R}^3$  the affine interpolation surface spanning  $\text{graph}(u^\pm) = \{(t, u^\pm(t)) : t \in [a, b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$ , namely

$$X^{\text{aff}}(t, s) = (t, su^+(t) + (1-s)u^-(t)) =: (t, \widehat{X}(t, s)) \quad \forall (t, s) \in [a, b] \times I. \quad (3.1)$$

eq:affi

rt integrand

**Remark 3.2.** For a (semicartesian) map  $\Phi : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$  of the form  $\Phi(t, \sigma) = (t, \phi(t, \sigma)) = (t, \phi_1(t, \sigma), \phi_2(t, \sigma))$ , the area integrand is given by

$$|\partial_t \Phi \wedge \partial_\sigma \Phi| = \sqrt{|\partial_\sigma \phi_1|^2 + |\partial_\sigma \phi_2|^2 + (\partial_t \phi_1 \partial_\sigma \phi_2 - \partial_\sigma \phi_1 \partial_t \phi_2)^2} = \sqrt{|\partial_\sigma \phi|^2 + |J\phi|^2}.$$

The main result of this section is the following:

teo:main\_thm

**Theorem 3.3 (Relaxed area of piecewise Lipschitz maps: straight jump).** *Let  $u : R \rightarrow \mathbb{R}^2$  be a piecewise Lipschitz map. Then*

$$\bar{\mathcal{A}}_{BV}(u, R) = \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.2)$$

eq:ril

Notice that the Lipschitz regularity of  $u$  on  $R^\pm$  ensures that the area functional has the classical expression

$$\mathcal{A}(u, R^\pm) = \int_{R^\pm} \sqrt{1 + |\nabla u|^2 + |\det \nabla u|^2} dx;$$

therefore, the singular contribution produced by the relaxation in (3.2) is given by the area of  $X^{\text{aff}}$ .

We divide the proof of (3.2) in two parts: the lower bound (Proposition 3.4) and the upper bound (Proposition 3.5).

lower\_bound

**Proposition 3.4 (Lower bound for (3.2)).** *Let  $u : R \rightarrow \mathbb{R}^2$  be a piecewise Lipschitz map, and  $(v_k) \subset C^1(R; \mathbb{R}^2) \cap BV(R; \mathbb{R}^2)$  be a sequence converging to  $u$  strictly  $BV(R; \mathbb{R}^2)$ . Then*

$$\liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, R) \geq \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.3)$$

eq:lowe

*Proof.* Fix  $\varepsilon \in (0, 1)$ . We have

$$\begin{aligned} \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, R) &\geq \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) + \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) \\ &\geq \mathcal{A}(u, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) + \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]), \end{aligned}$$

where in the last inequality we used [1, Theorem 3.7]. Sending  $\varepsilon$  to  $0^+$ , by dominated convergence it follows  $\mathcal{A}(u, R \setminus ([a, b] \times [-\varepsilon, \varepsilon])) \rightarrow \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-)$ , so (3.3) will be proven provided we show that

$$\lim_{\varepsilon \rightarrow 0^+} \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) \geq \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.4) \quad \text{eq:main}$$

Consider the maps

$$V_k^\varepsilon : R \rightarrow \mathbb{R}^3, \quad V_k^\varepsilon(t, \sigma) = (t, v_k(t, \varepsilon\sigma)),$$

and the associated integer multiplicity 2-currents in  $\mathbb{R}^3$

$$\mathcal{V}_k^\varepsilon = V_k^\varepsilon \# [R].$$

Notice that, neglecting the term  $1 + |\partial_{x_1} v_k|^2$ , we get

$$\begin{aligned} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) &\geq \int_{[a, b] \times [-\varepsilon, \varepsilon]} \sqrt{|\partial_{x_2} v_k|^2 + |Jv_k|^2} dx \\ &= \int_R |\partial_t V_k^\varepsilon \wedge \partial_\sigma V_k^\varepsilon| dt d\sigma = |\mathcal{V}_k^\varepsilon|, \end{aligned} \quad (3.5) \quad \text{eq:negl}$$

where we used Remark 3.2, and  $|\cdot|$  stands for the mass current. Consider also the maps

$$U_\pm^\varepsilon : R^\pm \rightarrow \mathbb{R}^3, \quad U_\pm^\varepsilon(t, \sigma) = (t, u(t, \varepsilon\sigma)), \quad (3.6) \quad \text{eq:def}$$

and the current

$$S_\varepsilon = X_\#^{\text{aff}} [[a, b] \times I] + U_{+\#}^\varepsilon [R^+] + U_{-\#}^\varepsilon [R^-], \quad (3.7) \quad \text{def } S_\varepsilon$$

see Fig. 1. We want now prove the following crucial inequality:

$$\liminf_{k \rightarrow +\infty} |\mathcal{V}_k^\varepsilon| \geq |S_\varepsilon|. \quad (3.8) \quad \text{eq:semi}$$

To show (3.8) we prove that  $\mathcal{V}_k^\varepsilon$  are close to suitable currents  $\mathcal{M}_n^\varepsilon$  independent of  $k$  (see (3.19)) which converge to  $S_\varepsilon$  as  $n \rightarrow +\infty$ .

For any  $n \in \mathbb{N}$ ,  $n \geq 1$ , consider a partition  $\{t_0 = a, t_1, \dots, t_{n+1} = b\}$  of  $[a, b]$  in  $(n+1)$  intervals  $[t_{i-1}, t_i]$ , with

$$t_i - t_{i-1} \in \left( \frac{b-a}{2n}, 2 \frac{(b-a)}{n} \right). \quad (3.9) \quad \text{scelta}$$

Moreover, set

$$R_i = [t_{i-1}, t_i] \times [-1, 1], \quad R_i^+ = [t_{i-1}, t_i] \times (0, 1], \quad R_i^- = [t_{i-1}, t_i] \times [-1, 0),$$

and define the currents

$$\mathcal{V}_{k,i}^\varepsilon = V_{k,\#}^\varepsilon [R_i], \quad S_{\varepsilon,i} = X_\#^{\text{aff}} [[t_{i-1}, t_i] \times I] + U_{+\#}^\varepsilon [R_i^+] + U_{-\#}^\varepsilon [R_i^-], \quad (3.10) \quad \text{eq:def}$$



see Fig. 1. By definition, we have

$$\begin{aligned} \mathcal{V}_k^\varepsilon &= \sum_{i=1}^{n+1} \mathcal{V}_{k,i}^\varepsilon \quad \text{and} \quad \mathcal{H}^2(\text{spt}\mathcal{V}_{k,i}^\varepsilon \cap \text{spt}\mathcal{V}_{k,j}^\varepsilon) = 0 \quad \text{for } i \neq j, \\ S_\varepsilon &= \sum_{i=1}^{n+1} S_{\varepsilon,i} \quad \text{and} \quad \mathcal{H}^2(\text{spt}S_{\varepsilon,i} \cap \text{spt}S_{\varepsilon,j}) = 0 \quad \text{for } i \neq j. \end{aligned} \tag{3.11}$$

Furthermore,

$$\begin{aligned} \partial S_{\varepsilon,i} &= - \left( U_{-\sharp}^\varepsilon[\{t_{i-1}\} \times [-1, 0]] + X_{\sharp}^{\text{aff}}[\{t_{i-1}\} \times I] + U_{+\sharp}^\varepsilon[\{t_{i-1}\} \times (0, 1]] \right) \\ &\quad - U_{+\sharp}^\varepsilon[(t_{i-1}, t_i) \times \{1\}] \\ &\quad + \left( U_{-\sharp}^\varepsilon[\{t_i\} \times [-1, 0]] + X_{\sharp}^{\text{aff}}[\{t_i\} \times I] + U_{+\sharp}^\varepsilon[\{t_i\} \times (0, 1]] \right) \\ &\quad + U_{-\sharp}^\varepsilon[(t_{i-1}, t_i) \times \{-1\}]. \end{aligned} \tag{3.12}$$

Now, for fixed  $i \in \{1, \dots, n\}$ , set

$$\begin{aligned} \gamma_{-,i}^{u,\varepsilon}(\sigma) &= u(t_i, \varepsilon\sigma) & \forall \sigma \in [-1, 0), \\ \gamma_{+,i}^{u,\varepsilon}(\sigma) &= u(t_i, \varepsilon\sigma) & \forall \sigma \in (0, 1], \\ \gamma_i^0(s) &= su^+(t_i) + (1-s)u^-(t_i) & \forall s \in I, \\ \Lambda_{u,i}^{\pm,\varepsilon}(t) &= (t, u(t, \pm\varepsilon)) & \forall t \in [t_{i-1}, t_i], \end{aligned}$$

and define  $\gamma_i^{u,\varepsilon} : [-1, 1] \rightarrow \mathbb{R}^2$  as in (2.15) where  $\tilde{\gamma}^-$ ,  $S$ , and  $\tilde{\gamma}^+$  are replaced by  $\gamma_{-,i}^{u,\varepsilon}$ ,  $\gamma_i^0$  and  $\gamma_{+,i}^{u,\varepsilon}$  in the order, after a rescaling on  $[-1, -\frac{1}{3}]$ ,  $[-\frac{1}{3}, \frac{1}{3}]$ , and  $[\frac{1}{3}, 1]$ , respectively, as in the statement of Lemma 2.7. Also, define  $\Gamma_i^{u,\varepsilon} : [-1, 1] \rightarrow (\{t_i\} \times \mathbb{R}^2)$  as

$$\Gamma_i^{u,\varepsilon}(\sigma) := (t_i, \gamma_i^{u,\varepsilon}(\sigma)) \quad \forall \sigma \in [-1, 1].$$

Using the definition of  $U_{\pm}^\varepsilon$  and  $X^{\text{aff}}$ , by (3.12) we infer

$$\partial S_{\varepsilon,i} = -\Gamma_{i-1\sharp}^{u,\varepsilon}[[[-1, 1]]] - \Lambda_{u,i\sharp}^{+,\varepsilon}[[[t_{i-1}, t_i]]] + \Gamma_i^{u,\varepsilon}[[[-1, 1]]] + \Lambda_{u,i\sharp}^{-,\varepsilon}[[[t_{i-1}, t_i]]]. \tag{3.13}$$

Moreover, set

$$\begin{aligned} \gamma_{k,i}^\varepsilon(\sigma) &= v_k(t_i, \varepsilon\sigma), & \Gamma_{k,i}^\varepsilon(\sigma) &= (t_i, \gamma_{k,i}^\varepsilon(\sigma)) & \forall \sigma \in [-1, 1], \\ \Lambda_{k,i}^{\pm,\varepsilon}(t) &= (t, v_k(t, \pm\varepsilon)) & & & \forall t \in [t_{i-1}, t_i]. \end{aligned}$$

By definition of  $\mathcal{V}_{k,i}^\varepsilon$  in (3.10), we also have

$$\partial \mathcal{V}_{k,i}^\varepsilon = -\Gamma_{k,i-1\sharp}^\varepsilon[[[-1, 1]]] - \Lambda_{k,i\sharp}^{+,\varepsilon}[[[t_{i-1}, t_i]]] + \Gamma_{k,i\sharp}^\varepsilon[[[-1, 1]]] + \Lambda_{k,i\sharp}^{-,\varepsilon}[[[t_{i-1}, t_i]]]. \tag{3.14}$$

We now define  $F_{k,i}^\varepsilon \in \mathcal{D}_2(\mathbb{R}^3)$  as a suitable affine interpolation between  $\partial \mathcal{V}_{k,i}^\varepsilon$  and  $\partial S_{\varepsilon,i}$ , see Fig. 1. First observe that by Lemma 2.3, we can suppose that, for our choice of  $\varepsilon$  and  $\{t_1, \dots, t_n\}$ , there exists a (not relabeled) subsequence of  $(v_k)_k$ , such that

$$v_k(t_i, \varepsilon \cdot) \rightarrow u(t_i, \varepsilon \cdot) \quad \text{strictly } BV([-1, 1]; \mathbb{R}^2) \quad \forall i = 1, \dots, n, \tag{3.15}$$

$$v_k(\cdot, \pm\varepsilon) \rightarrow u(\cdot, \pm\varepsilon) \quad \text{strictly } BV([a, b]; \mathbb{R}^2). \tag{3.16}$$

In particular, by Lemma 2.7, we know that there are increasing Lipschitz bijections  $h_{k,i}^\varepsilon, h_i^\varepsilon : [-1, 1] \rightarrow [-1, 1]$  such that  $\gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon \rightarrow \gamma_i^{u,\varepsilon} \circ h_i^\varepsilon$  uniformly in  $[-1, 1]$  as  $k \rightarrow +\infty$ .

For  $i = 1, \dots, n$ , we define

$$\begin{aligned}\Phi_{k,i}^\varepsilon(\sigma, s) &:= s(\Gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon(\sigma)) + (1-s)(\Gamma_i^{u,\varepsilon} \circ h_i^\varepsilon(\sigma)), & (\sigma, s) \in [-1, 1] \times I, \\ \Psi_{k,i}^{\pm,\varepsilon}(t, s) &:= s\Lambda_{k,i}^{\pm,\varepsilon}(t) + (1-s)\Lambda_{u,i}^{\pm,\varepsilon}(t), & (t, s) \in [t_{i-1}, t_i] \times I.\end{aligned}$$

Therefore we set

$$\begin{aligned}F_{k,i}^\varepsilon &= -\Phi_{k,i-1}^\varepsilon \llbracket [-1, 1] \times I \rrbracket - \Psi_{k,i}^{+,\varepsilon} \llbracket [t_{i-1}, t_i] \times I \rrbracket \\ &\quad + \Phi_{k,i}^\varepsilon \llbracket [-1, 1] \times I \rrbracket + \Psi_{k,i}^{-,\varepsilon} \llbracket [t_{i-1}, t_i] \times I \rrbracket.\end{aligned}\tag{3.17} \quad \text{eq: def_}$$

In particular, from (3.13) and (3.14), a direct check shows that

$$\partial F_{k,i}^\varepsilon = \partial \mathcal{V}_{k,i}^\varepsilon - \partial S_{\varepsilon,i}.\tag{3.18} \quad \text{boundar}$$

Eventually, we let  $M_{\varepsilon,i}$  be an integer multiplicity 2-current of  $\mathbb{R}^3$  with minimal mass and boundary  $\partial S_{\varepsilon,i}$  (the existence of  $M_{\varepsilon,i}$  is guaranteed, for instance, by [19, Theorem 8.3.3]) and set

$$\mathcal{M}_n^\varepsilon := \sum_{i=2}^n M_{\varepsilon,i}.\tag{3.19} \quad \text{eq: math}$$

Note carefully that we do not sum over  $i$  from 1 to  $n+1$ , but only from 2 to  $n$ . In particular, setting  $S_\varepsilon^n = S_\varepsilon - S_{\varepsilon,1} - S_{\varepsilon,n+1}$ , we have

$$\partial \mathcal{M}_n^\varepsilon = \partial S_\varepsilon^n = -\Gamma_1^{u,\varepsilon} \llbracket [-1, 1] \rrbracket + \Gamma_n^{u,\varepsilon} \llbracket [-1, 1] \rrbracket - \Lambda_u^{+,\varepsilon} \llbracket [t_1, t_n] \rrbracket + \Lambda_u^{-,\varepsilon} \llbracket [t_1, t_n] \rrbracket,\tag{3.20} \quad 314$$

where

$$\Lambda_u^{\pm,\varepsilon}(t) := (t, u(t, \pm\varepsilon)), \quad t \in (t_1, t_n).$$

Thus, we have

$$|\mathcal{V}_{k,i}^\varepsilon| \geq |\mathcal{V}_{k,i}^\varepsilon - F_{k,i}^\varepsilon| - |F_{k,i}^\varepsilon| \geq |M_{\varepsilon,i}| - |F_{k,i}^\varepsilon| \quad \text{for } i = 2, \dots, n,$$

where we used the minimality of  $M_{\varepsilon,i}$  and (3.18). By summing up, using (3.11), we get<sup>5</sup>

$$|\mathcal{V}_k^\varepsilon| = \sum_{i=1}^{n+1} |\mathcal{V}_{k,i}^\varepsilon| \geq \sum_{i=2}^n |\mathcal{V}_{k,i}^\varepsilon| \geq \sum_{i=2}^n |M_{\varepsilon,i}| - \sum_{i=2}^n |F_{k,i}^\varepsilon| \geq |\mathcal{M}_n^\varepsilon| - \sum_{i=2}^n |F_{k,i}^\varepsilon|.\tag{3.21} \quad \text{inequal}$$

Therefore,

$$\liminf_{k \rightarrow +\infty} |\mathcal{V}_k^\varepsilon| \geq |\mathcal{M}_n^\varepsilon| - \sum_{i=2}^n \limsup_{k \rightarrow +\infty} |F_{k,i}^\varepsilon|.\tag{3.22} \quad \text{eq: Tau_}$$

In order to obtain (3.8), we have to prove that:

- (i)  $|F_{k,i}^\varepsilon| \rightarrow 0$  as  $k \rightarrow +\infty$  for every  $i = 2, \dots, n$ ;
- (ii)  $\mathcal{M}_n^\varepsilon \rightarrow S_\varepsilon$  as  $n \rightarrow +\infty$ ,

so that (3.8) would follow by lower semicontinuity of the mass and (3.22).

(i). Since  $\gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon \rightarrow \gamma_i^{u,\varepsilon} \circ h_i^\varepsilon$  uniformly in  $[-1, 1]$  as  $k \rightarrow +\infty$ , also  $\Gamma_{k,i}^\varepsilon \circ h_{k,i}^\varepsilon \rightarrow \Gamma_i^{u,\varepsilon} \circ h_i^\varepsilon$  uniformly; moreover, by Lemma 2.6 and thanks to (3.16),  $v_k(\cdot, \pm\varepsilon) \rightarrow u(\cdot, \pm\varepsilon)$  uniformly on  $[t_{i-1}, t_i]$ , and the same holds for  $\Lambda_{k,i}^{\pm,\varepsilon}$  and  $\Lambda_{u,i}^{\pm,\varepsilon}$ . Finally, by (3.15) and (3.16), and recalling also Lemma 2.7 (b), the

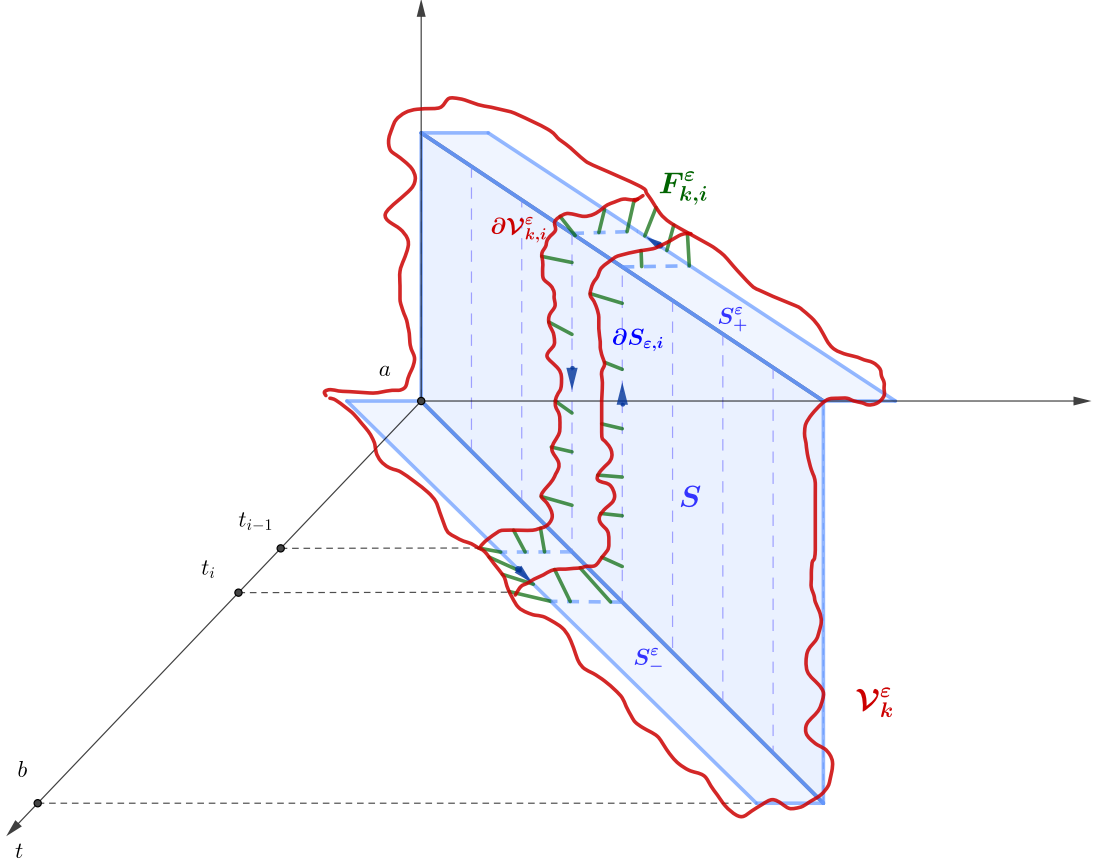


Figure 1: Here  $S = X_{\#}^{\text{aff}}[[a, b] \times I]$ ,  $S_{\pm}^{\epsilon} = U_{\pm\#}^{\epsilon}[[R^{\pm}]]$ . The horizontal and vertical axes span the target space  $\mathbb{R}^2$ . The approximating current  $\mathcal{V}_k^{\epsilon}$  is depicted in bold, as well as the boundary of its restriction to  $R_i$ , i.e. the current  $\partial\mathcal{V}_{k,i}^{\epsilon}$ . The current  $\partial S_{\epsilon,i}$  is depicted with the oriented dotted straight lines, while  $F_{k,i}^{\epsilon}$  is the oriented surface obtained as the union of the short segments connecting  $\partial\mathcal{V}_{k,i}^{\epsilon}$  and  $\partial S_{\epsilon,i}$ . Finally, for simplicity, we depict with straight segments the graph of  $u^{\pm}$  and the (semi)graph of  $u$  on  $\{(t, \sigma) : \sigma = \pm\epsilon\}$ , but it is worth to remember that they are graph of Lipschitz maps.

fig:currents

$L^1$ -norm of the derivative of  $\Gamma_{k,i}^{\epsilon} \circ h_{k,i}^{\epsilon}$  and of  $\Lambda_{k,i}^{\pm,\epsilon}$  is uniformly bounded with respect to  $k$ . Hence (i) readily follows from the definition of  $F_{k,i}^{\epsilon}$  in (3.17) and Remark 2.11.

(ii). First observe that  $\partial\mathcal{M}_n^{\epsilon}$  has mass uniformly bounded with respect to  $n$ . Indeed by (3.20)

$$|\partial\mathcal{M}_n^{\epsilon}| = |\partial S_{\epsilon}^n| \leq |\dot{\gamma}_1^{u,\epsilon}|([-1, 1]) + |\dot{\gamma}_n^{u,\epsilon}|([-1, 1]) + \int_a^b \sqrt{1 + |\partial_t u(t, \epsilon)|^2} dt + \int_a^b \sqrt{1 + |\partial_t u(t, -\epsilon)|^2} dt \\ \leq C(\epsilon, \|u\|_{\infty}, \text{lip}(u|_{R^+}), \text{lip}(u|_{R^-})).$$

Moreover, by minimality of  $\mathcal{M}_n^{\epsilon}$  and (3.11),  $|\mathcal{M}_n^{\epsilon}| \leq |S_{\epsilon}^n| \leq |S_{\epsilon}|$ , hence the sequence  $(\mathcal{M}_n^{\epsilon})_n$  is compactly supported in  $\mathbb{R}^3$  and has bounded mass and bounded boundary mass. Then, by [19, Theorem 8.2.1], we have

$$\mathcal{M}_n^{\epsilon} \rightharpoonup S_{\epsilon} \iff \|\mathcal{M}_n^{\epsilon} - S_{\epsilon}\|_F \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

<sup>5</sup>In (3.21) we had to remove the first and last term of the sum, because condition (i) can be false for  $i = 1$  and  $i = n + 1$ , since the strict convergence is inherited only on almost every line, as stated in Lemma 2.3.

where  $\|\cdot\|_F$  stands for the flat norm. Then, we are reduced to show that  $\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \rightarrow 0$  as  $n \rightarrow +\infty$ . Notice that

$$\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \leq \sum_{i=2}^n \|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F + \|S_{\varepsilon,1}\|_F + \|S_{\varepsilon,n+1}\|_F, \quad (3.23) \quad \text{eq:T\_me}$$

where, by definition of flat norm (see [15, Sec. 5.1.3]),

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \leq \inf\{|G_i^\varepsilon| : G_i^\varepsilon \text{ integer multiplicity 3-current s.t. } \partial G_i^\varepsilon = M_{\varepsilon,i} - S_{\varepsilon,i}\}.$$

Observe that the class of competitors in the above minimum problem is non empty, since it contains the affine interpolation current between  $M_{\varepsilon,i}$  and  $S_{\varepsilon,i}$ . So, pick a 3-current  $G_i^\varepsilon$  such that  $\partial G_i^\varepsilon = M_{\varepsilon,i} - S_{\varepsilon,i}$ ; then

$$|G_i^\varepsilon| \leq C|\partial G_i^\varepsilon|^{\frac{3}{2}}$$

by the isoperimetric inequality [19, Theorem 7.9.1], for an absolute positive constant  $C > 0$ . For  $i = 2, \dots, n$ , we have

$$\|M_{\varepsilon,i} - S_{\varepsilon,i}\|_F \leq |G_i^\varepsilon| \leq C|\partial G_i^\varepsilon|^{\frac{3}{2}} = C|M_{\varepsilon,i} - S_{\varepsilon,i}|^{\frac{3}{2}} \leq C\left(|M_{\varepsilon,i}|^{\frac{3}{2}} + |S_{\varepsilon,i}|^{\frac{3}{2}}\right) \leq 2C|S_{\varepsilon,i}|^{\frac{3}{2}}, \quad (3.24) \quad \text{eq:appl}$$

where in the last inequality we used the minimality of  $M_{\varepsilon,i}$ . Now let us prove that  $|S_{\varepsilon,i}| \leq \frac{C}{n}$  for every  $i = 1, \dots, n+1$ , where  $C$  is a constant independent of  $n$ . We start observing that

$$\begin{aligned} |X_\#^{\text{aff}}[[t_{i-1}, t_i] \times I]| &= \int_{[t_{i-1}, t_i] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds \\ &= \int_{t_{i-1}}^{t_i} \int_I |(1, s\dot{u}^+ + (1-s)\dot{u}^-) \wedge (0, u^+ - u^-)| dt ds \\ &\leq \int_{t_{i-1}}^{t_i} \int_I (|u^+ - u^-| + |(s\dot{u}_1^+ + (1-s)\dot{u}_1^-)(u_2^+ - u_2^-) - (s\dot{u}_2^+ + (1-s)\dot{u}_2^-)(u_1^+ - u_1^-)|) dt ds \\ &\leq \frac{C_1}{n} \|u^+ - u^-\|_{L^\infty(a,b)} + \frac{C_2}{n} \|u^+ - u^-\|_{L^\infty(a,b)} (\|\dot{u}^+\|_{L^\infty(a,b)} + \|\dot{u}^-\|_{L^\infty(a,b)}) \\ &= \frac{C}{n}, \end{aligned}$$

where we used (3.9). Moreover, recalling (3.6), we have

$$\begin{aligned} |U_{\pm\#}^\varepsilon[[R_i^\pm]]| &= \int_{R_i^\pm} |\partial_t U_\pm^\varepsilon \wedge \partial_\sigma U_\pm^\varepsilon| dt d\sigma \\ &= \int_{R_i^\pm} |(1, \partial_t u(t, \varepsilon\sigma)) \wedge (0, \varepsilon \partial_\sigma u(t, \varepsilon\sigma))| dt d\sigma \\ &\leq \varepsilon \int_{R_i^\pm} |\partial_\sigma u(t, \varepsilon\sigma)| dt d\sigma + \varepsilon \int_{R_i^\pm} |\partial_t u_1(t, \varepsilon\sigma) \partial_\sigma u_2(t, \varepsilon\sigma) - \partial_t u_2(t, \varepsilon\sigma) \partial_\sigma u_1(t, \varepsilon\sigma)| dt d\sigma \quad (3.25) \quad \text{stimu U} \\ &\leq \varepsilon \frac{C_3}{n} \left( \|\nabla u\|_{L^\infty(R^\pm)} + \|\nabla u\|_{L^\infty(R^\pm)}^2 \right) \\ &= \frac{C\varepsilon}{n}. \end{aligned}$$

Thus,

$$|S_{\varepsilon,i}| \leq |X_\#^{\text{aff}}[[t_{i-1}, t_i] \times I]| + |U_{+\#}^\varepsilon[[R_i^+]]| + |U_{-\#}^\varepsilon[[R_i^-]]| \leq \frac{C}{n},$$

as claimed. Finally, by definition of flat norm and the isoperimetric inequality,  $\|S_{\varepsilon,i}\|_F \leq |S_{\varepsilon,i}|^{\frac{3}{2}}$  for  $i = 1, \dots, n+1$ , so that, from (3.24) and (3.23), we obtain

$$\|\mathcal{M}_n^\varepsilon - S_\varepsilon\|_F \leq C(n-1)\frac{1}{n^{\frac{3}{2}}} + \frac{C}{n^{\frac{3}{2}}} \leq \frac{C}{n^{\frac{1}{2}}} + \frac{C}{n^{\frac{3}{2}}} \rightarrow 0.$$

This concludes the proof of (ii) and hence of (3.8).

We are now in a position to show (3.4). From (3.5) and (3.8),

$$\liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) \geq \liminf_{k \rightarrow +\infty} |\mathcal{V}_k^\varepsilon| \geq |S_\varepsilon|. \quad (3.26)$$

As in (3.25), we have

$$|U_{\pm\sharp}^\varepsilon[\mathbb{I}R^\pm]| \leq \varepsilon \left( \|\nabla u\|_{L^\infty(R^\pm)} + \|\nabla u\|_{L^\infty(R^\pm)}^2 \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

so, from (3.26) and (3.7), we conclude

$$\lim_{\varepsilon \rightarrow 0^+} \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, [a, b] \times [-\varepsilon, \varepsilon]) \geq \lim_{\varepsilon \rightarrow 0^+} |S_\varepsilon| = |X_\sharp^{\text{aff}}[[a, b] \times I]| = \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds.$$

□

**Proposition 3.5 (Upper bound for (3.2)).** *Let  $u : R \rightarrow \mathbb{R}^2$  be a piecewise Lipschitz map. Then there exists a sequence  $(v_k)_k \subset C^1(R; \mathbb{R}^2)$  converging to  $u$  strictly  $BV(R; \mathbb{R}^2)$  such that*

$$\limsup_{k \rightarrow +\infty} \mathcal{A}(v_k, R) \leq \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.27)$$

*Proof.* Although  $v_k$  needs to be of class  $C^1$ , we claim that it suffices to build  $v_k$  just Lipschitz continuous. Indeed, assume that  $(v_k)_k \subset W^{1,\infty}(R; \mathbb{R}^2) \cap C^1(R; \mathbb{R}^2)$  converges to  $u$  strictly  $BV(R; \mathbb{R}^2)$  and (3.27) holds. Consider, for all  $k \in \mathbb{N}$ , a sequence  $(v_h^k)_h \subset C^1(R; \mathbb{R}^2)$  approaching  $v_k$  in  $W^{1,2}(R; \mathbb{R}^2)$  as  $h \rightarrow +\infty$ . In particular, we get the  $L^1$ -convergence of all minors of  $\nabla v_h^k$  to the corresponding ones of  $\nabla v_k$ . Then, by dominated convergence,

$$\lim_{h \rightarrow +\infty} \mathcal{A}(v_h^k; R) = \mathcal{A}(v_k, R). \quad (3.28)$$

Hence, by a diagonal argument, we find a sequence  $(v_{h_k}^k)_k$  converging to  $u$  strictly  $BV(R; \mathbb{R}^2)$  such that (3.27) holds for  $v_{h_k}^k$  in place of  $v_k$ .

Set for simplicity  $\varepsilon = \varepsilon_k = \frac{1}{k}$ , and define the sequence  $(v_\varepsilon) \subset \text{Lip}(R; \mathbb{R}^2)$  as

$$v_\varepsilon(t, \sigma) := \begin{cases} u(t, \sigma) & (t, \sigma) \in R \setminus ([a, b] \times [-\varepsilon, \varepsilon]), \\ \frac{\varepsilon + \sigma}{2\varepsilon} u(t, \varepsilon) + \frac{\varepsilon - \sigma}{2\varepsilon} u(t, -\varepsilon) & (t, \sigma) \in [a, b] \times (-\varepsilon, \varepsilon). \end{cases} \quad (3.29)$$

First, let us check that  $v_\varepsilon \rightarrow u$  strictly  $BV(R; \mathbb{R}^2)$  as  $\varepsilon \rightarrow 0^+$ . Clearly,  $v_\varepsilon \rightarrow u$  in  $L^1(R; \mathbb{R}^2)$ . Hence, by lower semicontinuity of the total variation, it is enough to show that

$$\limsup_{\varepsilon \rightarrow 0^+} \int_R |\nabla v_\varepsilon| dt d\sigma \leq |Du|(R),$$

which in turn reduces to prove

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\nabla v_\varepsilon| dt d\sigma \leq |Du|([a, b] \times \{0\}),$$

since

$$\int_{R \setminus ([a,b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| dt d\sigma = \int_{R \setminus ([a,b] \times [-\varepsilon, \varepsilon])} |\nabla u| dt d\sigma \rightarrow \int_{R^+} |\nabla u| dt d\sigma + \int_{R^-} |\nabla u| dt d\sigma \quad \text{as } \varepsilon \rightarrow 0^+.$$

For almost every  $t \in [a, b]$  and every  $\sigma \in [-\varepsilon, \varepsilon]$ , one has

$$\partial_t v_\varepsilon(t, \sigma) = \frac{\varepsilon + \sigma}{2\varepsilon} \partial_t u(t, \varepsilon) + \frac{\varepsilon - \sigma}{2\varepsilon} \partial_t u(t, -\varepsilon), \quad \partial_\sigma v_\varepsilon(t, \sigma) = \frac{1}{2\varepsilon} (u(t, \varepsilon) - u(t, -\varepsilon)).$$

Thus, setting  $M := \max\{\text{lip}(u|_{R^-}), \text{lip}(u|_{R^+})\}$ , we get

$$\begin{aligned} \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\nabla v_\varepsilon| dt d\sigma &\leq \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_t v_\varepsilon(t, \sigma)| dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_\sigma v_\varepsilon(t, \sigma)| dt d\sigma \\ &\leq M \int_{[a,b] \times [-\varepsilon, \varepsilon]} dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} \frac{1}{2\varepsilon} |u(t, \varepsilon) - u(t, -\varepsilon)| dt d\sigma \\ &= M(b-a)2\varepsilon + \int_a^b |u(t, \varepsilon) - u(t, -\varepsilon)| dt \\ &\xrightarrow{\varepsilon \rightarrow 0^+} \int_a^b |u^+(t) - u^-(t)| dt = |Du|([a, b] \times \{0\}). \end{aligned}$$

Furthermore, since  $u$  is piecewise Lipschitz, we have

$$\mathcal{A}(v_\varepsilon; R \setminus [a, b] \times [-\varepsilon, \varepsilon]) = \mathcal{A}(u, R \setminus [a, b] \times [-\varepsilon, \varepsilon]) \rightarrow \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) \quad \text{as } \varepsilon \rightarrow 0^+.$$

So it remains to prove that

$$\limsup_{\varepsilon \rightarrow 0^+} \mathcal{A}(v_\varepsilon; [a, b] \times [-\varepsilon, \varepsilon]) \leq \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.30) \quad \boxed{\text{liminf}}$$

Let us linearly reparametrize  $X^{\text{aff}}$  on  $R = [a, b] \times [-1, 1]$ , namely consider  $Y$ , having the same image as  $X^{\text{aff}}$ , given by

$$Y(t, \sigma) = (t, \widehat{Y}(t, \sigma)) = \left( t, \frac{1 + \sigma}{2} u^+(t) + \frac{1 - \sigma}{2} u^-(t) \right), \quad (t, \sigma) \in R.$$

Now, using the trivial inequality  $\sqrt{1 + a^2 + b^2 + c^2} \leq 1 + |a| + \sqrt{b^2 + c^2}$ , we find

$$\begin{aligned} \mathcal{A}(v_\varepsilon; [a, b] \times [-\varepsilon, \varepsilon]) &\leq \int_{[a,b] \times [-\varepsilon, \varepsilon]} dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} |\partial_t v_\varepsilon| dt d\sigma + \int_{[a,b] \times [-\varepsilon, \varepsilon]} \sqrt{|\partial_\sigma v_\varepsilon|^2 + |Jv_\varepsilon|^2} dt d\sigma \\ &= \varepsilon(b-a) + \varepsilon \int_R |\partial_t \tilde{v}_\varepsilon| dt d\sigma + \varepsilon \int_R \sqrt{|\partial_\sigma \tilde{v}_\varepsilon|^2 + |J\tilde{v}_\varepsilon|^2} dt d\sigma, \end{aligned} \quad (3.31) \quad \boxed{\text{estim}}$$

where  $\tilde{v}_\varepsilon : R \rightarrow \mathbb{R}^2$  is defined as  $\tilde{v}_\varepsilon(t, \sigma) = v_\varepsilon(t, \varepsilon\sigma)$ . A direct computation based in (3.29) gives

$$\begin{aligned} \partial_t \tilde{v}_\varepsilon(t, \sigma) &= \frac{1 + \sigma}{2} \partial_t u(t, \varepsilon) + \frac{1 - \sigma}{2} \partial_t u(t, -\varepsilon) \quad \text{for a.e. } t \in [a, b] \quad \forall \sigma \in [-1, 1] \\ \partial_\sigma \tilde{v}_\varepsilon(t, \sigma) &= \varepsilon \partial_\sigma v_\varepsilon(t, \varepsilon\sigma) = \frac{u(t, \varepsilon) - u(t, -\varepsilon)}{2} \quad \text{for a.e. } t \in [a, b] \quad \forall \sigma \in [-1, 1]. \end{aligned}$$

Then we have

$$\begin{aligned} \partial_t \tilde{v}_\varepsilon(t, \sigma) &\rightarrow \frac{1 + \sigma}{2} \dot{u}^+(t) + \frac{1 - \sigma}{2} \dot{u}^-(t) = \partial_t \widehat{Y}(t, \sigma) \quad \text{a.e. in } R, \\ \partial_\sigma \tilde{v}_\varepsilon(t, \sigma) &\rightarrow \frac{u^+(t) - u^-(t)}{2} = \partial_\sigma \widehat{Y}(t, \sigma) \quad \text{a.e. in } R. \end{aligned}$$

Since  $\partial_\sigma \widehat{Y}$  and  $\partial_t \widehat{Y}$  are in  $L^\infty(R; \mathbb{R}^2)$ , by dominated convergence we can pass to the limit in (3.31) as  $\varepsilon \rightarrow 0^+$ , so that, using Remark 3.2, we obtain (3.30).  $\square$

**Remark 3.6.** After having proved the upper bound inequality in Proposition 3.5, we readily infer that  $\overline{\mathcal{A}}_{BV}(u, R) < +\infty$ . Hence Proposition 3.4 can be deduced from an argument independently developed in [23], based on the theory of Cartesian currents [16]. Indeed, consider  $T_u := G_u + S$ , where  $G_u$  is the 2-current on  $R \times \mathbb{R}^2$  carried by the graph of  $u$  and  $S$  is the 2-current on  $R \times \mathbb{R}^2$  given by  $S := \tilde{X}_\# [[a, b] \times I]$ , where

$$\tilde{X}(t, s) := (t, 0, \widehat{X}(t, s)) = (t, 0, su^+(t) + (1-s)u^-(t)), \quad t \in [a, b], s \in I.$$

Clearly, the mass of  $T_u$  is given by

$$\begin{aligned} |T_u| &= |G_u| + |S| = \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a, b] \times I} |\partial_t \tilde{X} \wedge \partial_s \tilde{X}| \, dt ds \\ &= \mathcal{A}(u, R^+) + \mathcal{A}(u, R^-) + \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| \, dt ds. \end{aligned}$$

Now we claim that  $T_u$  is the unique Cartesian current on  $R \times \mathbb{R}^2$  with minimal completely vertical lifting associated to  $u$ , according to [23, Definition 3.1]. Borrowing the notation from [23], this definition is given by imposing that the mixed components of  $T_u$  are the minimal lifting measures  $\mu_i^j[u]$  associated to  $u$  in the sense of Jerrard and Jung [18]. Once the claim is proven, by the lower semicontinuity of the mass and the continuity of the lifting measures with respect to the strict convergence (see [18, Theorem 1.1]), we deduce

$$|T_u| \leq \overline{\mathcal{A}}_{BV}(u; R),$$

i.e., inequality (3.3).

In order to show the claim, we start to prove that  $T_u \in \text{cart}(R, \mathbb{R}^2)$ . For this, it is enough to see that  $(\partial T_u) \llcorner (R \times \mathbb{R}^2) = 0$ : We get

$$(\partial G_u) \llcorner (R \times \mathbb{R}^2) = \widehat{X}_\#^- [[a, b]] - \widehat{X}_\#^+ [[a, b]] = -\partial \tilde{X}_\# [[a, b] \times I] = -(\partial S) \llcorner (R \times \mathbb{R}^2),$$

where  $\widehat{X}^\pm(t) := (t, 0, u^\pm(t))$ ,  $t \in [a, b]$ . Next, what remains to prove is that the vertical component of  $T_u$  is the minimal completely vertical lifting associated to  $u$ . To this purpose, denote by  $x = (x^1, x^2)$  the (horizontal) variable of  $R$ ,  $y = (y^1, y^2)$  the vertical variable of  $\mathbb{R}^2$  and  $u = (u^1, u^2)$  the components of  $u$ . We have to check that

$$\mu_i^j[T_u] = \mu_i^j[u] \quad \forall i, j = 1, 2, \tag{3.32}$$

where  $\mu_i^j[T_u] := T_u \llcorner ((-1)^i dx^{\bar{i}} \wedge dy^j)$ . By [18, Theorem 2.2], for every  $f \in C_c^\infty(R \times \mathbb{R}^2)$ ,

$$\int_{R \times \mathbb{R}^2} f(x, y) d\mu_j^i[T_u] = \int_{R^+ \cup R^-} f(x, u(x)) \partial_i u^j dx + \int_a^b \left( \int_0^1 f(t, 0, \widehat{X}(t, s)) ds \right) (u^{j^+} - u^{j^-}) \delta_{i2} \, dt.$$

On the other hand, setting  $\omega(x, y) := (-1)^i f(x, y) dx^{\bar{i}} \wedge dy^j$ , we have

$$\begin{aligned} \int_{R \times \mathbb{R}^2} f(x, y) d\mu_j^i[T_u] &= \int_{R^+ \cup R^-} f(x, u(x)) \partial_i u^j dx + \int_{\tilde{X}([a, b] \times I)} \omega \\ &= \int_{R^+ \cup R^-} f(x, u(x)) \partial_i u^j dx + \int_{[a, b] \times I} \omega(\tilde{X}(t, s)) d\tilde{X}^{\bar{i}j}, \end{aligned}$$

where, if  $\tilde{X} = (\tilde{X}_1^1, \tilde{X}_2^1, \tilde{X}_1^2, \tilde{X}_2^2)$ , then  $d\tilde{X}^{\bar{i}j} = d\tilde{X}_1^{\bar{i}} \wedge d\tilde{X}_2^j$ . Notice that  $d\tilde{X}^{\bar{i}j} = 0$  if  $\bar{i} = 2$  and  $d\tilde{X}^{1j} = (u^{j+} - u^{j-}) dt \wedge ds$ , so we get

$$\begin{aligned} \int_{[a,b] \times I} \omega(\tilde{X}(t,s)) d\tilde{X}^{\bar{i}j} &= \int_{[a,b] \times I} (-1)^i f(\tilde{X}(t,s)) (u^{j+} - u^{j-}) \delta_{i2} dt \wedge ds \\ &= \int_a^b \left( \int_0^1 f(t,0, \hat{X}(t,s)) ds \right) (u^{j+} - u^{j-}) \delta_{i2} dt, \end{aligned}$$

and (3.32) follows.

### 3.1 Extension of Theorem 3.3

The validity of Theorem 3.3 is guaranteed also when the two traces  $u^\pm$  of  $u$  on  $[a,b] \times \{0\}$  coincide on some subset of  $[a,b] \times \{0\}$ . In particular, (3.2) extends to maps  $u$  whose jump set  $S_u$  is a subset of  $[a,b] \times \{0\}$ . However, the situation is different when the jump set is curvilinear. Specifically, assume  $\Omega \subset \mathbb{R}^2$  is a bounded open and connected set, and:

- (H1)  $\Sigma = \alpha([a,b]) \subset \Omega$  is a simple curve, arc-length parametrized by  $\alpha : [a,b] \rightarrow \Omega$  of class  $C^2$  injective in  $[a,b]$ ;
- (H2) If  $\alpha(a) = \alpha(b)$ , then  $\dot{\alpha}(a^+) = \dot{\alpha}(b^-)$  and  $\ddot{\alpha}(a^+) = \ddot{\alpha}(b^-)$ ;
- (H3)  $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$ ; as usual, we denote by  $u^\pm$  the traces of  $u$  on  $\Sigma$ , satisfying  $u^\pm \in \text{Lip}(\Sigma; \mathbb{R}^2)$ .

Again, we introduce the affine interpolation surface  $X^{\text{aff}} : [a,b] \times I \rightarrow \mathbb{R}^3$  spanning  $\text{graph}(u^\pm) = \{(t, u^\pm(\alpha(t))) : t \in [a,b]\} \subset \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$ , namely

$$X^{\text{aff}}(t,s) = (t, su^+(\alpha(t)) + (1-s)u^-(\alpha(t))) \quad \forall (t,s) \in [a,b] \times I. \quad (3.33)$$

**Theorem 3.7 (Relaxed area of piecewise Lipschitz maps: curved jump).** *Suppose (H1)-(H3). Then*

$$\bar{\mathcal{A}}_{BV}(u, \Omega) = \int_{\Omega \setminus \Sigma} |\mathcal{M}(\nabla u)| dx + \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.34)$$

**Remark 3.8.** The image of the map  $X^{\text{aff}}$  sits in  $\mathbb{R}^3$  and it is not exactly the interpolation surface which closes the holes in the graph of  $u$ , which is instead given by

$$\Psi(t,s) = (\alpha(t), su^+(\alpha(t)) + (1-s)u^-(\alpha(t))) \in \mathbb{R}^4 \quad \forall t \in [a,b] \times I. \quad (3.35)$$

However, since  $|\dot{\alpha}| = 1$ ,

$$\int_{[a,b] \times I} |\partial_t \Psi \wedge \partial_s \Psi| dt ds = \int_{[a,b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.36)$$

To prove Theorem 3.7, we borrow from [8] some notation. We denote by  $x = (x_1, x_2)$  coordinates in  $\Omega$  and by  $(t, \sigma)$  coordinates in  $R = [a,b] \times [-1, 1]$ . Since  $\Sigma$  is simple and of class  $C^2$ , we can find  $\delta > 0$  and a  $C^1$ -diffeomorphism  $\Lambda : R_\delta \rightarrow \Lambda(R_\delta)$ , where  $R_\delta = [a,b] \times [-\delta, \delta]$  and  $\Lambda(R_\delta) \subset \Omega$  is a curvilinear strip containing  $\Sigma$  of width  $2\delta$ . Explicitly we have

$$\Lambda(t, \sigma) = \alpha(t) + \sigma \dot{\alpha}(t)^\perp \quad \forall (t, \sigma) \in R_\delta, \quad (3.37)$$

with  $\dot{\alpha}(t)^\perp$  the counter-clockwise  $\frac{\pi}{2}$ -rotation of  $\dot{\alpha}(t)$ . For  $(x_1, x_2) \in \Lambda(R_\delta)$ , we can write the inverse  $\Lambda^{-1}(x_1, x_2) = (t(x_1, x_2), \sigma(x_1, x_2))$ , where:



- $\sigma(x_1, x_2) = d_\Sigma(x_1, x_2)$  is the signed distance<sup>6</sup> of  $(x_1, x_2)$  from  $\Sigma$ ;
- $t(x_1, x_2)$  is the unique number in  $[a, b]$  such that  $\alpha(t(x_1, x_2)) = \pi_\Sigma(x_1, x_2)$ , where  $\pi_\Sigma(x_1, x_2) = (x_1, x_2) - d_\Sigma(x_1, x_2)\nabla d_\Sigma(x_1, x_2)$  is the orthogonal projection on  $\Sigma$ .

Since  $\alpha$  is of class  $C^2$ , we have that  $\sigma$  is of class  $C^2$  as well and  $t$  is of class  $C^1$  on  $\overline{\Lambda(R_\delta)}$ . Moreover, for  $(x_1, x_2) \in \overline{\Lambda(R_\delta)}$ , we have

$$|\nabla\sigma(x_1, x_2)| = |\nabla d_\Sigma(x_1, x_2)| = 1, \quad (3.38)$$

$$|\nabla t(x_1, x_2)| = 1 + \delta\|\nabla d_\Sigma\|_\infty \leq 1 + C\delta. \quad (3.39)$$

We divide the proof of Theorem 3.3 in two parts, the lower and the upper bound inequalities.

**Proposition 3.9 (Lower bound for (3.34)).** *Let  $u : \Omega \rightarrow \mathbb{R}^2$  as in Theorem 3.7 and  $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$  be a sequence converging to  $u$  strictly BV( $\Omega; \mathbb{R}^2$ ). Then (3.3) holds with  $X^{\text{aff}}$  in (3.33).*

*Proof.* It is enough to show that

$$\lim_{\varepsilon \rightarrow 0^+} \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon])) \geq \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds. \quad (3.40)$$

We start by defining the maps  $\Psi_k^\varepsilon : R \rightarrow \mathbb{R}^4$  and  $\Psi_\pm^\varepsilon : R^\pm \rightarrow \mathbb{R}^4$  given by

$$\Psi_k^\varepsilon(t, \sigma) = (\Lambda(t, \varepsilon\sigma), v_k(\Lambda(t, \varepsilon\sigma))), \quad \Psi_\pm^\varepsilon(t, \sigma) = (\Lambda(t, \varepsilon\sigma), u(\Lambda(t, \varepsilon\sigma))).$$

Introduce the following integer multiplicity 2-currents in  $\mathbb{R}^4$ :

$$\mathcal{V}_k^\varepsilon = \Psi_{k\sharp}^\varepsilon[[R]], \quad S^\varepsilon = \Psi_\sharp[[a, b] \times I] + \Psi_{-\sharp}^\varepsilon[[R^-]] + \Psi_{+\sharp}^\varepsilon[[R^+]],$$

where  $\Psi$  is defined in (3.35). Using that  $Av \wedge Aw = \det A v \wedge w$  for any  $A \in \mathbb{R}^{2 \times 2}$  and  $v, w \in \mathbb{R}^2$ , by direct computation, we have

$$|\partial_t \Psi_k^\varepsilon \wedge \partial_\sigma \Psi_k^\varepsilon|^2 = \varepsilon^2 |\partial_t \Lambda(t, \varepsilon\sigma) \wedge \partial_\sigma \Lambda(t, \varepsilon\sigma)|^2 \left[ 1 + |\nabla v_k(\Lambda(t, \varepsilon\sigma))|^2 + |Jv_k(\Lambda(t, \varepsilon\sigma))|^2 \right].$$

Hence, making the change of variable  $x = \Lambda(t, \varepsilon\sigma)$ , we obtain

$$\mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon])) = \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |\mathcal{M}(\nabla v_k)| dx = \int_R |\partial_t \Psi_k^\varepsilon \wedge \partial_\sigma \Psi_k^\varepsilon| dt d\sigma = |\mathcal{V}_k^\varepsilon|.$$

We notice that  $|\Psi_{\pm\sharp}^\varepsilon[[R^\pm]]| \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ , as in (3.25), where  $\|\nabla u\|_{L^\infty(R^\pm)}$  is replaced with  $\|u\|_{W^{1,\infty}(\Omega)}$  and it is used that  $|\ddot{\alpha}| \leq C$ . Therefore, recalling also (3.36),

$$\lim_{\varepsilon \rightarrow 0^+} |S^\varepsilon| = |\Psi_\sharp[[a, b] \times I]| = \int_{[a, b] \times I} |\partial_t \Psi \wedge \partial_s \Psi| dt ds = \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds.$$

So it is enough to show  $\liminf_{k \rightarrow +\infty} |\mathcal{V}_k^\varepsilon| \geq |S^\varepsilon|$ , which can be proved proceeding as in the proof of Proposition 3.4, once we have checked that  $v_k \circ \Lambda(\cdot, \varepsilon\cdot) \rightarrow u \circ \Lambda(\cdot, \varepsilon\cdot)$  strictly BV( $R; \mathbb{R}^2$ ). This is a straightforward computation, and we omit the details.  $\square$

**Proposition 3.10 (Upper bound for (3.34)).** *Let  $u : \Omega \rightarrow \mathbb{R}^2$  be as in Theorem 3.7. Then, there exists a sequence  $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$  converging to  $u$  strictly BV( $\Omega; \mathbb{R}^2$ ) and such that (3.27) holds with  $X^{\text{aff}}$  in (3.33).*

<sup>6</sup>The sign of  $d_\Sigma$  is determined by the orientation induced on  $\Sigma$  by  $\alpha$ , so that  $d_\Sigma > 0$  in the part of  $\Lambda(R_\delta)$  which is pointed by  $\dot{\alpha}^\perp$ .

*Proof.* For simplicity, we assume that  $\alpha(a) \neq \alpha(b)$  (the case of closed curves is simpler and the following proof can be straightforwardly adapted). We start by fixing  $\eta > 0$  small enough and we extend the curve  $\alpha$  to  $[a - \eta, b + \eta]$  in a  $C^2$ -way, so that  $\Sigma^\eta := \alpha([a - \eta, b + \eta]) \subset \Omega$ , keeping the validity of (H1) on  $\Sigma^\eta$ . With this extension, we can assume (by choosing a different  $\delta$  if necessary) that  $\Lambda$  in (3.37) is defined on  $R^\eta := [a - \eta, b + \eta] \times [-\delta, \delta]$ . We observe that

$$u^+(\alpha(t)) = u^-(\alpha(t)) \quad \text{for all } t \in [a - \eta, a] \cup [b, b + \eta]. \quad (3.41) \quad \text{boundar}$$

Now, set  $\varepsilon = \frac{1}{k}$  and, for  $k$  large enough,

$$\begin{aligned} \Delta_\varepsilon^a &:= \{x \in \Lambda([a - \varepsilon, a] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq t(x) - a + \varepsilon\}, \\ \Delta_\varepsilon^b &:= \{x \in \Lambda([b, b + \varepsilon] \times [-\varepsilon, \varepsilon]) : |\sigma(x)| \leq b + \varepsilon - t(x)\}. \end{aligned}$$

We define the recovery sequence  $(v_\varepsilon) \subset \text{Lip}(\Omega; \mathbb{R}^2)$  as

$$v_\varepsilon(x) = \begin{cases} \frac{\varepsilon + \sigma(x)}{2\varepsilon} u(\Lambda(t(x), \varepsilon)) + \frac{\varepsilon - \sigma(x)}{2\varepsilon} u(\Lambda(t(x), -\varepsilon)) & \text{in } \Lambda([a, b] \times [-\varepsilon, \varepsilon]), \\ u(x) & \text{in } \Omega \setminus \left( \Lambda([a, b] \times [-\varepsilon, \varepsilon]) \cup \overline{\Delta_\varepsilon^a} \cup \overline{\Delta_\varepsilon^b} \right). \end{cases} \quad (3.42) \quad \text{v\_k}$$

In order to define  $v_\varepsilon$  in  $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$  it is sufficient to observe that, by (3.41), the restriction of  $v_\varepsilon$  on  $\partial\Delta_\varepsilon^a$  and  $\partial\Delta_\varepsilon^b$  is Lipschitz continuous with Lipschitz constant bounded by  $\|u\|_{W^{1,\infty}}$ . Hence, we can take a Lipschitz extension of  $v_\varepsilon$  in  $\Delta_\varepsilon^a \cup \Delta_\varepsilon^b$  keeping the Lipschitz constant (up to a dimensional factor independent of  $\varepsilon$ ). Thus

$$\int_{\Delta_\varepsilon^a \cup \Delta_\varepsilon^b} |\mathcal{M}(\nabla v_\varepsilon)| \, dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+. \quad (3.43) \quad \text{cont\_De}$$

Let us check that  $v_\varepsilon \rightarrow u$  strictly  $BV(\Omega; \mathbb{R}^2)$  as  $\varepsilon \rightarrow 0^+$ . Clearly,  $v_\varepsilon \rightarrow u$  in  $L^1(\Omega; \mathbb{R}^2)$ , since  $|\Lambda([a, b] \times [-\varepsilon, \varepsilon])| \rightarrow 0$  and  $|\Delta_\varepsilon^a \cup \Delta_\varepsilon^b| \rightarrow 0$ . So, by (3.43), as in the proof of Proposition 3.5, it is enough to show that

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| \, dx \leq |Du|(\Sigma) = \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))| \, dt.$$

Almost everywhere in  $\Lambda([a, b] \times [-\varepsilon, \varepsilon])$ , we have

$$\begin{aligned} \nabla v_\varepsilon &= \frac{\varepsilon + \sigma}{2\varepsilon} \nabla u(\Lambda(t, \varepsilon)) \partial_t \Lambda(t, \varepsilon) \otimes \nabla t + \frac{\varepsilon - \sigma}{2\varepsilon} \nabla u(\Lambda(t, -\varepsilon)) \partial_t \Lambda(t, -\varepsilon) \otimes \nabla t \\ &\quad + \frac{1}{2\varepsilon} \nabla \sigma \otimes (u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))). \end{aligned}$$

Therefore,

$$\begin{aligned} |\nabla v_\varepsilon| &\leq \frac{1}{2\varepsilon} \left[ (\varepsilon + \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, -\varepsilon))| |\nabla t| + (\varepsilon - \sigma) \|\partial_t \Lambda\|_\infty |\nabla u(\Lambda(t, \varepsilon))| |\nabla t| \right. \\ &\quad \left. + |\nabla \sigma| |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \right] \\ &\leq \frac{1}{2\varepsilon} \left[ 2\varepsilon \|u\|_{W^{1,\infty}} \|\partial_t \Lambda\|_\infty (1 + C\varepsilon) + |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \right], \end{aligned}$$

where we used (3.38) and (3.39) with  $\varepsilon$  in place of  $\delta$ . Thus, we get

$$\begin{aligned} \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |\nabla v_\varepsilon| \, dx &\leq C(\delta)(1 + C\varepsilon) |\Lambda([a, b] \times [-\varepsilon, \varepsilon])| \\ &\quad + \frac{1}{2\varepsilon} \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \, dx \\ &= o_\varepsilon(1) + \frac{1}{2\varepsilon} \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| \, dx. \end{aligned}$$

Consider the last integral and perform the change of variable  $x = (x_1, x_2) = \Lambda(t, \sigma)$ , with

$$|\det \nabla \Lambda(t, \sigma)| = |\partial_t \Lambda \wedge \partial_\sigma \Lambda| = |1 + \sigma \ddot{\alpha} \wedge \dot{\alpha}| = |1 - \kappa_\Sigma \sigma|,$$

where  $\kappa_\Sigma$  is the curvature of  $\Sigma$ . We get

$$\begin{aligned} & \frac{1}{2\varepsilon} \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dx = \frac{1}{2\varepsilon} \int_{[a, b] \times [-\varepsilon, \varepsilon]} |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| |1 - \kappa_\Sigma \sigma| dt d\sigma \\ & \leq \frac{1}{2\varepsilon} \int_a^b \int_{-\varepsilon}^\varepsilon |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dt d\sigma + o_\varepsilon(1) = \int_a^b |u(\Lambda(t, \varepsilon)) - u(\Lambda(t, -\varepsilon))| dt + o_\varepsilon(1) \\ & \longrightarrow \int_a^b |u^+(\alpha(t)) - u^-(\alpha(t))| dt \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

It remains to prove (3.27) with  $X^{\text{aff}}$  in (3.33). To this purpose it is enough to show that

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{A}(v_\varepsilon; \Lambda([a, b] \times [-\varepsilon, \varepsilon])) \leq \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds.$$

Let us define  $\varphi_\varepsilon : R \rightarrow \mathbb{R}^2$  as

$$\varphi_\varepsilon(t, \sigma) := \frac{1 + \sigma}{2} u(\Lambda(t, \varepsilon)) + \frac{1 - \sigma}{2} u(\Lambda(t, -\varepsilon)).$$

Thus, for  $x \in \Lambda([a, b] \times [-\varepsilon, \varepsilon])$

$$v_\varepsilon(x) = \varphi_\varepsilon \left( t(x), \frac{\sigma(x)}{\varepsilon} \right)$$

and, almost everywhere in  $\Lambda([a, b] \times [-\varepsilon, \varepsilon])$ ,

$$\nabla v_\varepsilon = \partial_t \varphi_\varepsilon \nabla t + \frac{1}{\varepsilon} \partial_\sigma \varphi_\varepsilon \nabla \sigma, \quad Jv_\varepsilon = \frac{1}{\varepsilon} |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon| |\nabla t \wedge \nabla \sigma|,$$

where from now on,  $\nabla t$  and  $\nabla \sigma$  are evaluated at  $x$ , while  $\partial_t \varphi_\varepsilon$  and  $\partial_\sigma \varphi_\varepsilon$  are evaluated at  $\left( t(x), \frac{\sigma(x)}{\varepsilon} \right)$ .

Then, we get

$$\begin{aligned} |\mathcal{M}(\nabla v_\varepsilon)|^2 &= 1 + |\partial_t \varphi_\varepsilon|^2 |\nabla t|^2 + \frac{2}{\varepsilon} \partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon \nabla t \cdot \nabla \sigma + \frac{1}{\varepsilon^2} \left[ |\partial_\sigma \varphi_\varepsilon|^2 |\nabla \sigma|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 |\nabla t \wedge \nabla \sigma|^2 \right] \\ &\leq 1 + |\partial_t \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| (1 + o_\varepsilon(1)) \\ &\quad + \frac{1}{\varepsilon^2} \left[ |\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right], \end{aligned}$$

where we used (3.38) and (3.39) with  $\varepsilon$  in place of  $\delta$ . Now, since  $o_\varepsilon(1) \sim \varepsilon$  and  $\varphi_\varepsilon$  is Lipschitz with Lipschitz constant independent of  $\varepsilon$ , we obtain

$$\begin{aligned} & \mathcal{A}(v_\varepsilon; \Lambda([a, b] \times [-\varepsilon, \varepsilon])) \\ & \leq \int_{\Lambda([a, b] \times [-\varepsilon, \varepsilon])} \sqrt{1 + |\partial_t \varphi_\varepsilon|^2 + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| + \frac{1}{\varepsilon^2} \left[ |\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right]} dx + o_\varepsilon(1) \\ & \leq \int_{[a, b] \times [-\varepsilon, \varepsilon]} \sqrt{1 + |\partial_t \varphi_\varepsilon|^2 + \frac{2}{\varepsilon} |\partial_t \varphi_\varepsilon \cdot \partial_\sigma \varphi_\varepsilon| + \frac{1}{\varepsilon^2} \left[ |\partial_\sigma \varphi_\varepsilon|^2 + |\partial_t \varphi_\varepsilon \wedge \partial_\sigma \varphi_\varepsilon|^2 (1 + o_\varepsilon(1)) \right]} |1 - \kappa_\Sigma \sigma| dt d\sigma \\ & \quad + o_\varepsilon(1), \end{aligned}$$

where we made the change of variable  $x = \Lambda(t, \sigma)$ , and so  $\partial_t \varphi_\varepsilon$  and  $\partial_\sigma \varphi_\varepsilon$  are computed at  $(t, \frac{\sigma}{\varepsilon})$ . Finally, by the change of variable  $\frac{\sigma}{\varepsilon} \rightarrow \sigma$ , we get

$$\begin{aligned} & \mathcal{A}(v_\varepsilon; \Lambda([a, b] \times [-\varepsilon, \varepsilon])) \\ & \leq \int_R \sqrt{o_\varepsilon(1) + |\partial_\sigma \varphi_\varepsilon(t, \sigma)|^2 + |\partial_t \varphi_\varepsilon(t, \sigma) \wedge \partial_\sigma \varphi_\varepsilon(t, \sigma)|^2 (1 + o_\varepsilon(1))} |1 - \kappa_\Sigma \varepsilon \sigma| dt d\sigma \\ & \quad + o_\varepsilon(1) \longrightarrow \int_{[a, b] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds, \end{aligned}$$

where, to pass to the limit as  $\varepsilon \rightarrow 0^+$ , we apply the dominated convergence theorem (as in the proof of Proposition 3.5).  $\square$

We observe that Theorem 3.7 can be easily extended to the case of curves with one endpoint or both endpoints on  $\partial\Omega$ . Write:

(H4)  $\Omega$  is of class  $C^1$ ,  $\alpha : [a, b] \rightarrow \bar{\Omega}$  is injective, arc-length parametrized, of class  $C^2$ ,  $\alpha((a, b)) \subset \Omega$ , and  $\alpha$  hits  $\partial\Omega$  transversally at  $\alpha(a), \alpha(b)$ .

**Theorem 3.11.** *Suppose (H3) and (H4). Then (3.34) holds with  $X^{\text{aff}}$  in (3.33).*

*Proof.* Lower bound: let  $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$  be a sequence converging to  $u$  strictly  $BV(\Omega; \mathbb{R}^2)$ . Fix  $0 < \rho < \frac{b-a}{2}$  and notice that  $\Lambda([a + \rho, b - \rho] \times [-\varepsilon, \varepsilon]) \subset \Omega$ , for  $\varepsilon > 0$  small enough. Then it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0^+} \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon]) \cap \Omega) \geq \int_{[a+\rho, b-\rho] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds; \quad (3.44)$$

since the lower bound will follow by the arbitrariness of  $\rho > 0$ . After writing  $\mathcal{A}(v_k, \Lambda([a, b] \times [-\varepsilon, \varepsilon]) \cap \Omega) \geq \mathcal{A}(v_k, \Lambda([a + \rho, b - \rho] \times [-\varepsilon, \varepsilon]))$ , the proof of (3.44) is identical to that of (3.40).

Upper bound: let us fix  $\eta > 0$  small enough so that  $B_{2\eta}(\alpha(a))$  and  $B_{2\eta}(\alpha(b))$  are disjoint, and consider  $\Omega^\eta := \Omega \cup B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))$ . We extend the curve  $\alpha$  (still calling  $\alpha$  the extension) in  $\Omega^\eta \setminus \Omega$  in such a way that it satisfies (H4) in  $\Omega^\eta$ , and so that it reaches the boundary of  $B_{2\eta}(\alpha(a)) \setminus \bar{\Omega}$  and of  $B_{2\eta}(\alpha(b)) \setminus \bar{\Omega}$  splitting both  $B_{2\eta}(\alpha(a)) \setminus \bar{\Omega}$  and  $B_{2\eta}(\alpha(b)) \setminus \bar{\Omega}$  in two connected components. If  $\alpha$  is now defined on an interval of the form  $[a - \delta, b + \delta]$  with  $\delta = \delta(\eta) > \eta$ , and if we set  $\Sigma^\delta = \alpha([a - \delta, b + \delta])$ , we prescribe the traces  $u^+$  and  $u^-$  on  $\Sigma^\delta$  in such a way that they are Lipschitz continuous and  $u^+ \circ \alpha = u^- \circ \alpha$  on  $[a - \delta, a - \eta] \cup [b + \eta, b + \delta]$ . Finally we take a Lipschitz extension  $u^\eta$  of  $u$  on the four connected components of  $B_{2\eta}(\alpha(a)) \setminus \bar{\Omega} \setminus \Sigma^\delta$  and of  $B_{2\eta}(\alpha(b)) \setminus \bar{\Omega} \setminus \Sigma^\delta$ . It turns out that  $u^\eta \in W^{1, \infty}((B_{2\eta}(\alpha(a)) \cup B_{2\eta}(\alpha(b))) \setminus \Sigma^\eta; \mathbb{R}^2)$ , where  $\Sigma^\eta = \alpha([a - \eta, b + \eta]) \subset \Omega^\eta$ . Since the definition of  $(u^\eta)^\pm$  is arbitrary, we can assume that

$$\begin{aligned} (u^\eta)^\pm(\alpha(t)) &= u^\pm(\alpha(a)) \left(1 - \frac{a-t}{\eta}\right) & \text{for } t \in [a - \eta, a], \\ (u^\eta)^\pm(\alpha(t)) &= u^\pm(\alpha(b)) \left(1 - \frac{t-b}{\eta}\right) & \text{for } t \in [b, b + \eta]. \end{aligned}$$

For  $\varepsilon > 0$  small enough, we see that  $\Lambda_\varepsilon := \Lambda([a - \eta, b + \eta] \times [-\varepsilon, \varepsilon]) \subset \Omega^\eta$ . Hence we define  $v_k$  as in the proof of Proposition 3.10 with  $\Omega$  replaced by  $\Omega^\eta$  and  $u$  replaced by  $u^\eta$  (in particular,  $v_\varepsilon = u$  on  $\Omega \setminus \Lambda_\varepsilon$ ). Finally, let us fix  $\rho \in (0, \eta)$ . We can write

$$\begin{aligned} \bar{\mathcal{A}}_{BV}(u, \Omega) &\leq \liminf_{\varepsilon \rightarrow 0^+} \mathcal{A}(v_\varepsilon, \Omega) \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega \setminus \Lambda_\varepsilon} |\mathcal{M}(\nabla u)| dx + \liminf_{\varepsilon \rightarrow 0^+} \int_{\Lambda([a-\rho, b+\rho] \times [-\varepsilon, \varepsilon])} |\mathcal{M}(\nabla v_\varepsilon)| dx \\ &= \int_{\Omega} |\mathcal{M}(\nabla u)| dx + \int_{[a-\rho, b+\rho] \times I} |\partial_t X^{\text{aff}} \wedge \partial_s X^{\text{aff}}| dt ds, \end{aligned}$$

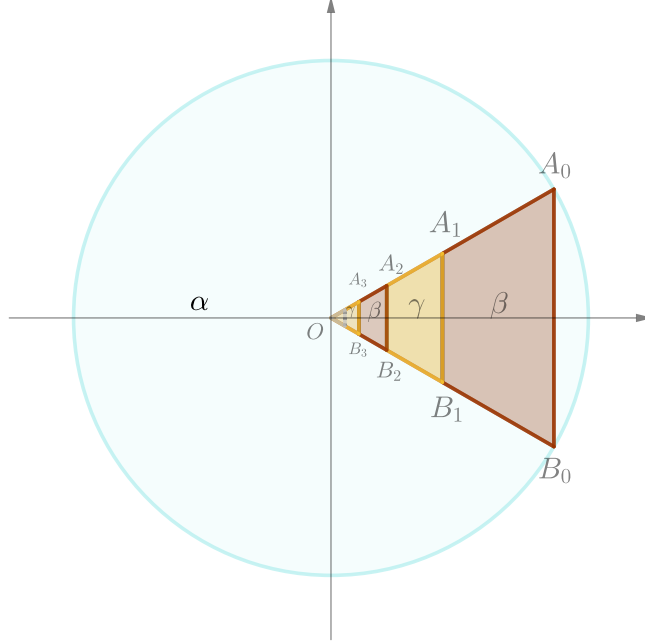


Figure 2: The source disc  $B_1(0)$  and the values  $\{\alpha, \beta, \gamma\}$  of  $u$ , with infinitely many triple points.

where we use that  $\Omega \subset ((\Omega \setminus \Lambda_\varepsilon) \cup \Lambda([a - \rho, b + \rho] \times [-\varepsilon, \varepsilon]))$  for  $\varepsilon > 0$  small enough. The upper bound then follows by the arbitrariness of  $\rho$ .  $\square$

With straightforward modifications of the previous arguments one can show the following:

**Corollary 3.12.** *Let  $\Omega$  have  $C^1$ -boundary, let  $n \in \mathbb{N}$  and  $\alpha_i : [a_i, b_i] \rightarrow \bar{\Omega}$ ,  $i = 1, \dots, n$ , be curves satisfying either (H1)-(H2), or (H4). Assume that  $\Sigma_i := \alpha_i([a_i, b_i]) \subset \bar{\Omega}$  are mutually disjoint, and let  $u \in W^{1,\infty}(\Omega \setminus \Sigma; \mathbb{R}^2)$  satisfy (H3), where  $\Sigma := \cup_{i=1}^n \Sigma_i$ . Then*

$$\bar{\mathcal{A}}_{BV}(u, \Omega) = \int_{\Omega} |\mathcal{M}(\nabla u)| \, dx + \sum_{i=1}^n \int_{[a_i, b_i] \times I} |\partial_t X_{(i)}^{\text{aff}} \wedge \partial_s X_{(i)}^{\text{aff}}| \, dt ds.$$

where  $X_{(i)}^{\text{aff}} : [a_i, b_i] \times I \rightarrow \mathbb{R}^3$  is the map  $X_{(i)}^{\text{aff}}(t, s) = (t, su^+(\alpha_i(t)) + (1-s)u^-(\alpha_i(t)))$ .

## 4 Piecewise constant maps

In this section we study the relaxed area (1.3) and the relaxed total variation (1.4), on certain piecewise constant maps. We start by exhibiting a  $BV$  map taking three values having infinite relaxed total variation of the Jacobian (and hence infinite  $BV$ -relaxed area), but finite  $L^1$ -relaxed area.

**Example 4.1. ( $BV$ -relaxed area and  $L^1$ -relaxed area)** Let  $\alpha, \beta, \gamma \in \mathbb{R}^2$  be three non-collinear vectors. Consider the map  $u : B_1(0) \subset \mathbb{R}^2 \rightarrow \{\alpha, \beta, \gamma\}$  in Fig. 2, obtained by the following

procedure: divide the source equilateral triangle  $T_{A_0OB_0}$  in two regions with a vertical segment connecting  $A_1$  and  $B_1$ , the middle points of the oblique sides of the triangle; assign the value  $\beta$  and  $\gamma$  on the right and on the left as in the figure, and repeat this construction on the equilateral triangle  $T_{A_1OB_1}$ , and then repeat the argument iteratively on all smaller triangles; finally set  $u = \alpha$  in  $B_1(0) \setminus T_{A_0OB_0}$ . In this way we get an infinite collection of triple points located at  $\{A_i, B_i\}_{i \geq 1}$ . Then,  $u \in BV(B_1(0); \{\alpha, \beta, \gamma\})$ , since

$$\begin{aligned} |Du|(B_1(0)) &= \left(1 + 2\left(1 - \sum_{i=1}^{+\infty} 2^{-2i}\right)\right) |\beta - \alpha| + 2 \sum_{i=1}^{+\infty} 2^{-2i} |\alpha - \gamma| + \sum_{i=1}^{+\infty} 2^{-i} |\beta - \gamma| \\ &= \frac{7}{3} |\beta - \alpha| + \frac{2}{3} |\alpha - \gamma| + |\beta - \gamma|. \end{aligned}$$

On the other hand, consider an infinitesimal sequence  $(r_i)_{i \geq 1}$  of radii with  $0 < r_i < 2^{-(i+1)}$ . With an argument similar to [3, Theorem 1.3], we have

$$\overline{TVJ}_{BV}(u, B_{r_i}(A_i)) = |T_{\alpha\beta\gamma}|,$$

$|T_{\alpha\beta\gamma}|$  denoting the Lebesgue measure of the target triangle with vertices  $\alpha, \beta, \gamma$ , and thus, for every  $N \in \mathbb{N}$ ,

$$\overline{TVJ}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, \cup_{i=1}^N B_{r_i}(A_i)) \geq \sum_{i=1}^N |T_{\alpha\beta\gamma}| = N |T_{\alpha\beta\gamma}|.$$

Whence

$$\overline{\mathcal{A}}_{BV}(u, B_1(0)) \geq \overline{TVJ}_{BV}(u, B_1(0)) = +\infty. \quad (4.1) \quad \boxed{\text{TV infi}}$$

On the other hand, we claim that

$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) < +\infty. \quad (4.2) \quad \boxed{\text{eq:L1_r}}$$

Indeed, we can construct a sequence  $(v_\varepsilon)$  of piecewise constant maps on  $B_1(0)$ , taking values in  $\{\alpha, \beta, \gamma\}$ , with uniformly bounded  $L^1$ -relaxed area and converging to  $u$  in  $L^1(B_1(0); \mathbb{R}^2)$ : Let  $\varepsilon \in (0, 1)$  and consider the intersection with  $T_{A_0OB_0}$  of a tubular neighbourhood of the segment  $\overline{A_iB_i}$  of diameter  $\varepsilon 2^{-(i+1)}$ , for every  $i \in \mathbb{N}$ . Then, the map  $v_\varepsilon$  is obtained by modifying  $u$  on these strips in the triangle, by assigning the value  $\alpha$ . Now,  $v_\varepsilon$  is a piecewise constant map valued in  $\{\alpha, \beta, \gamma\}$  without triple points, hence, by [1, Theorem 3.14],

$$\begin{aligned} \overline{\mathcal{A}}_{L^1}(v_\varepsilon, B_1(0)) &= |B_1(0)| + |Dv_\varepsilon|(B_1(0)) \\ &\leq \pi + \frac{7}{3} |\beta - \alpha| + \frac{2}{3} |\alpha - \gamma| + \left(1 + \frac{\varepsilon}{2}\right) \sum_{i=1}^{+\infty} 2^{-i} (|\beta - \alpha| + |\alpha - \gamma|) \\ &\leq \pi + \frac{23}{6} |\beta - \alpha| + \frac{13}{6} |\alpha - \gamma|. \end{aligned}$$

Clearly,  $v_\varepsilon \rightarrow u$  in  $L^1(B_1(0); \mathbb{R}^2)$  as  $\varepsilon \rightarrow 0^+$ , so by lower semicontinuity

$$\overline{\mathcal{A}}_{L^1}(u, B_1(0)) \leq \pi + \frac{23}{6} |\beta - \alpha| + \frac{13}{6} |\alpha - \gamma| < +\infty.$$

In particular

$$\text{Dom}\left(\overline{\mathcal{A}}^{BV}(\cdot, B_1(0))\right) \subsetneq \text{Dom}\left(\overline{\mathcal{A}}^{L^1}(\cdot, B_1(0))\right).$$

**Remark 4.2.** Following the notation of [23], one can show (4.1) also by considering the measure  $\mu_w^J$  defined for every  $w \in BV(B_1(0); \mathbb{R}^2)$  as

$$\langle \mu_w^J, g \rangle = \frac{1}{2} \int_{S_w} (w^{1-} w^{2+} - w^{1+} w^{2-}) \partial_\tau g d\mathcal{H}^1 \quad \forall g \in C_c^\infty(B_1(0)),$$

where  $\tau = \nu^\perp$  and  $\nu$  is the unit normal to  $S_w$ , so that  $Dw \llcorner S_w = (w^+ - w^-) \otimes \nu \mathcal{H}^1 \llcorner S_w$ .

If  $\overline{\mathcal{A}}_{BV}(w, B_1(0))$  is finite, we can consider the unique cartesian current  $T_w \in \text{cart}(B_1(0); \mathbb{R}^2)$  associated to  $w$  defined in [23, Theorem 3.5], whose vertical part is by definition equal to the minimal completely vertical lifting  $\mu_v[w]$  associated to  $w$ , according to [23, Definition 3.1]. Then, since  $|\mu_v[w]|$  is lower semicontinuous with respect to the weak convergence of measures and  $\mu_v[v] = TVJ(v, \cdot)$  for  $v$  smooth (see [23, (3.6)]), we get

$$|\mu_v[w]|(B_1(0) \times \mathbb{R}^2) \leq \overline{TVJ}_{BV}(w, B_1(0)).$$

In particular, if  $w \in BV(B_1(0); \mathbb{R}^2)$  is piecewise constant, we have

$$|\mu_w^J|(B_1(0)) \leq |\mu_v[w]|(B_1(0) \times \mathbb{R}^2) \leq \overline{TVJ}_{BV}(w, B_1(0)), \quad (4.3) \quad \boxed{\text{eq:TVJb}}$$

where the first inequality is a consequence of [23, Corollary 4.3].

Now, if by contradiction  $\overline{\mathcal{A}}_{BV}(u, B_1(0))$  is finite for the map  $u$  in Example 4.1 we have

$$\mu_u^J = \sum_{i=1}^{+\infty} |T_{\alpha\beta\gamma}|(\delta_{A_i} - \delta_{B_i}).$$

In particular  $|\mu_u^J|(B_1(0)) = +\infty$ , and (4.1) follows from (4.3). In Example 4.6, we construct a piecewise constant map  $u \in BV(B_1(0); \mathbb{R}^2)$  taking only five values in  $\mathbb{R}^2$  with  $\overline{TVJ}_{BV}(u, B_1(0)) = +\infty$  and  $\mu_u^J = 0$ . In that case, one can see even that  $\mu_v[u] = 0$ , whence a maximal gap phenomenon occurs between the mass of the current  $T_u$  (which is finite and without a vertical contribution) and  $\overline{\mathcal{A}}_{BV}(u, B_1(0))$  (which is infinite as well).

#### 4.1 Piecewise constant homogeneous maps

We need some tools that allow us to characterize (and compute in some cases) the relaxed functionals for  $n$ -uple point maps with  $n \geq 3$ . Thus, for  $r > 0$ , we consider maps  $u : B_r := B_r(0) \rightarrow \mathbb{R}^2$  of the form

$$u(x) = \gamma \left( \frac{x}{|x|} \right) \quad \text{for a.e. } x \in B_r, \quad (4.4) \quad \boxed{\text{def of}}$$

where  $\gamma : \mathbb{S}^1 \rightarrow \{\alpha_1, \dots, \alpha_n\}$  is piecewise constant and takes the (not necessarily distinct) values  $\alpha_1, \dots, \alpha_n \in \mathbb{R}^2$  on the arcs  $C_1, \dots, C_n$  in the order (see Fig. 3 for  $n = 5$ ). So,  $u$  is an  $n$ -uple point map with one  $n$ -uple junction at the origin. Now, we can consider the broken line curve  $\tilde{\gamma} \subset \mathbb{R}^2$  (an example of which is in Fig. 3) made of the segments connecting  $\alpha_1$  to  $\alpha_2$ ,  $\alpha_2$  to  $\alpha_3$  and so on, closing up by connecting  $\alpha_n$  to  $\alpha_1$ . The curve  $\tilde{\gamma}$  can be parametrized as in (2.27), and the curves  $\tilde{\gamma}_i$  are constant. Denoting by  $L(\gamma)$  the length of  $\tilde{\gamma}$ , we have

$$L(\gamma) = \sum_{i=1}^n |\alpha_{i+1} - \alpha_i| = |\dot{\gamma}|(\mathbb{S}^1) = \sup \left\{ \sum_{i=1}^{m-1} |\gamma(\nu_{i+1}) - \gamma(\nu_i)| : m \in \mathbb{N}, \{\nu_1, \dots, \nu_m\} \subset \mathbb{S}^1 \right\}, \quad (4.5) \quad \boxed{\text{eq:def_}}$$

with the convention  $\alpha_{n+1} := \alpha_1$ . Clearly, by definition of  $u$ , we have

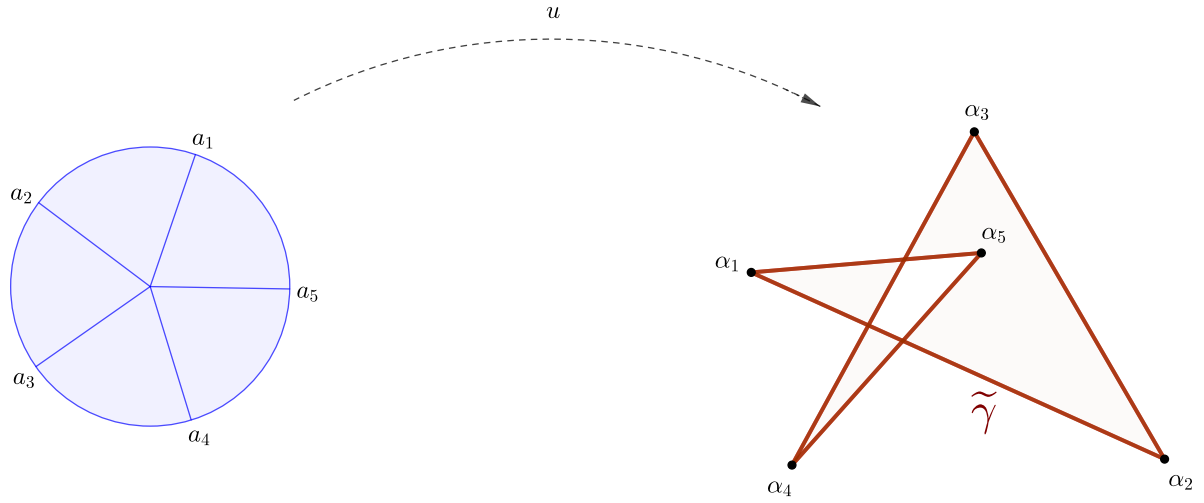


Figure 3: An  $n$ -uple point map and the corresponding curve  $\gamma$ , for  $n = 5$ .

$$|Du|(B_r) = r|\dot{\gamma}|(\mathbb{S}^1) = rL(\gamma).$$

Thanks to Lemma 2.14, for  $\bar{P}(\gamma)$  as in (2.40) we know that

$$\bar{P}(\gamma) = P(\tilde{\gamma}). \quad (4.6)$$

For a general  $\gamma$  the computation of  $\bar{P}(\gamma)$  seems not immediate. For the configuration in Fig. 3, we expect it to be the area of the region enclosed by  $\tilde{\gamma}$ , with the small internal quadrilateral counted twice.

**Theorem 4.3 (Relaxation of TVJ on piecewise constant maps).** *Let  $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}^2$ ,  $\gamma \in BV(\mathbb{S}^1; \{\alpha_1, \dots, \alpha_n\})$  be a function with a finite number of jump points, and let  $u$  be as in (4.4). Then*

$$\overline{TVJ}_{BV}(u, B_r) = \bar{P}(\gamma).$$

*Proof.* Lower bound: Assume that  $(v_k) \subset C^1(B_r; \mathbb{R}^2)$  converges to  $u$  strictly  $BV(B_r; \mathbb{R}^2)$  and

$$\lim_{k \rightarrow +\infty} \int_{B_r} |Jv_k| \, dx = \overline{TVJ}_{BV}(u, B_r).$$

By Lemma 2.5, we can fix  $\varepsilon \in (0, r)$  and a not-relabeled subsequence depending on  $\varepsilon$ , such that  $v_k \llcorner \partial B_\varepsilon \rightarrow u \llcorner \partial B_\varepsilon$  strictly  $BV(\partial B_\varepsilon; \mathbb{R}^2)$ . Thus, using Corollary 2.15 and the rescaling invariance of (2.40), we can estimate

$$\overline{TVJ}_{BV}(u, B_r) \geq \liminf_{k \rightarrow +\infty} \int_{B_\varepsilon} |Jv_k| \, dx \geq \liminf_{k \rightarrow +\infty} P(v_k \llcorner \partial B_\varepsilon) = \bar{P}(u \llcorner \partial B_\varepsilon) = \bar{P}(\gamma). \quad (4.7) \quad \boxed{5.6}$$

Upper bound: By an argument similar to the one at the beginning of the proof of Proposition 3.5, it will be enough to construct a recovery sequence  $(u_k) \subset \text{Lip}(B_r; \mathbb{R}^2)$ . Let  $\tilde{\gamma}$  be as above. We start by building a sequence  $(\gamma_k)_k$  of Lipschitz reparameterizations of  $\tilde{\gamma}$  which converges strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$  to  $\gamma$ . Let us denote by  $a_1, \dots, a_n \in [0, 2\pi)$  the angular coordinates of the extremal points of  $C_1, \dots, C_n$ , and assume without loss of generality  $0 = a_1 < a_2 < \dots < a_n$ . Then

$$\bigcup_{i=1}^n [a_i, a_{i+1}] = [0, 2\pi],$$



with the convention  $a_{n+1} = 2\pi$ . Let  $(\delta_k)_k$  be an infinitesimal sequence with  $0 < \delta_k < \max\{|a_{i+1} - a_i|, i = 1, \dots, n\}$ , for instance  $\delta_k = \frac{2}{k}$ ,  $k$  large enough. We define the piecewise affine map  $\gamma_k : [0, 2\pi] \rightarrow \mathbb{R}^2$  as

$$\gamma_k(t) = \begin{cases} \alpha_i & \text{if } t \in [a_i + \delta_k/2, a_{i+1} - \delta_k/2], \\ \frac{a_{i+1} + \delta_k/2 - t}{\delta_k} \alpha_i + \frac{t - a_{i+1} + \delta_k/2}{\delta_k} \alpha_{i+1} & \text{if } t \in [a_{i+1} - \delta_k/2, a_{i+1} + \delta_k/2], \end{cases} \quad i = 1, \dots, n. \quad (4.8)$$

Then  $\gamma_k \rightarrow \gamma$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$  (a direct computation shows that  $|\dot{\gamma}_k|(\mathbb{S}^1) = |\dot{\gamma}|(\mathbb{S}^1)$ ),  $\gamma_k$  are uniformly bounded in  $L^\infty$ , and converge almost everywhere to  $\gamma$ . As a consequence, from Corollary 2.15,

$$P(\gamma_k) \rightarrow \bar{P}(\gamma) \quad \text{as } k \rightarrow +\infty. \quad (4.9)$$

Therefore, by (1.5) we choose, for all  $k > 1$  large enough, a map  $v_k \in \text{Lip}(B_1; \mathbb{R}^2)$  such that

$$v_k \llcorner \mathbb{S}^1 = \gamma_k, \quad \left| P(\gamma_k) - \int_{B_1} |Jv_k| dx \right| \leq \frac{1}{k}. \quad (4.10)$$

Let  $c_k > 0$  be the Lipschitz constant of  $v_k$ . Defining  $v_{k,\rho} \in \text{Lip}(B_\rho; \mathbb{R}^2)$  as  $v_{k,\rho}(y) := v_k(\frac{y}{\rho})$  for any  $\rho > 0$ , it is straightforward that the Lipschitz constant of  $v_{k,\rho}$  is  $c_k/\rho$ .

We now choose an infinitesimal sequence  $(\rho_k) \subset (0, r)$  in such a way that  $\lim_{k \rightarrow +\infty} c_k \rho_k = 0$ . As a consequence we get

$$\int_{B_{\rho_k}} |\nabla v_{k,\rho_k}| dx \leq \pi c_k \rho_k \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (4.11)$$

We are now in a position to introduce our recovery sequence: We define  $u_k \in \text{Lip}(B_r; \mathbb{R}^2)$  as

$$u_k(x) := \begin{cases} \gamma_k\left(\frac{x}{|x|}\right) & \forall x \in B_r \setminus B_{\rho_k}, \\ v_{k,\rho_k}(x) & \forall x \in B_{\rho_k}. \end{cases} \quad (4.12)$$

Using that  $\gamma_k \rightarrow \gamma$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$  and (4.11) we see that  $u_k \rightarrow u$  strictly  $BV(B_r; \mathbb{R}^2)$ . Finally, since in  $B_r \setminus B_{\rho_k}$  the map  $u_k$  depends only on the angular coordinate, its Jacobian determinant vanishes in  $B_r \setminus B_{\rho_k}$ . Hence

$$\liminf_{k \rightarrow +\infty} \int_{B_r} |Ju_k| dx = \liminf_{k \rightarrow +\infty} \int_{B_{\rho_k}} |Jv_{k,\rho_k}| dx = \bar{P}(\gamma), \quad (4.13)$$

the convergence being a consequence of (2.30), (4.10), and (4.9).  $\square$

As a consequence of Theorem 4.3 we deduce:

**Theorem 4.4 (Relaxation of  $\mathcal{A}$  on piecewise constant maps).** *Let  $\gamma$  and  $u$  as in Theorem 4.3. Then, for any  $r > 0$ , we have*

$$\bar{\mathcal{A}}_{BV}(u, B_r) = \pi r^2 + rL(\gamma) + \bar{P}(\gamma). \quad (4.14)$$

*Proof.* Lower bound: Suppose that  $v_k \in C^1(B_r; \mathbb{R}^2)$  is such that

$$v_k \rightarrow u \quad \text{strictly } BV(B_r; \mathbb{R}^2) \quad \text{and} \quad \lim_{k \rightarrow +\infty} \mathcal{A}(v_k, B_r) = \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, B_r).$$

Now, let  $\varepsilon \in (0, r)$  and write  $\mathcal{A}(v_k, B_r) = \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \mathcal{A}(v_k, B_\varepsilon) \geq \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \int_{B_\varepsilon} |Jv_k| dx$ , so that, by [1, Theorem 3.7],

$$\begin{aligned} \lim_{k \rightarrow +\infty} \mathcal{A}(v_k, B_r) &\geq \liminf_{k \rightarrow +\infty} \mathcal{A}(v_k, B_r \setminus B_\varepsilon) + \liminf_{k \rightarrow +\infty} \int_{B_\varepsilon} |Jv_k| dx \\ &\geq |B_r \setminus B_\varepsilon| + r(1 - \varepsilon)L(\gamma) + \liminf_{k \rightarrow +\infty} \int_{B_\varepsilon} |Jv_k| dx \\ &\geq |B_r \setminus B_\varepsilon| + r(1 - \varepsilon)L(\gamma) + \bar{P}(\gamma), \end{aligned}$$

where in the last line we have applied Theorem 4.3 with  $r$  replaced by  $\varepsilon$ . We now pass to the limit as  $\varepsilon \rightarrow 0^+$  to get the lower bound  $\bar{\mathcal{A}}_{BV}(u, B_r) \geq \pi r^2 + rL(\gamma) + \bar{P}(\gamma)$  in (4.14).

Upper bound: It is sufficient to consider the sequence  $(u_k)_k$  defined in (4.12), for which

$$\begin{aligned} \bar{\mathcal{A}}_{BV}(u, B_r) &\leq \limsup_{k \rightarrow +\infty} \mathcal{A}(u_k, B_1) \leq |B_r| + \lim_{k \rightarrow +\infty} \int_{B_r} |\nabla u_k| dx + \lim_{k \rightarrow +\infty} \int_{B_r} |Ju_k| dx \\ &= \pi r^2 + rL(\gamma) + \bar{P}(\gamma). \end{aligned}$$

□

Now, we are in the position to show an example of a piecewise constant map  $u \in BV(B_1; \mathbb{R}^2)$  with infinite relaxed Jacobian total variation but vanishing associated minimal vertical lifting measure  $\mu_v[u]$ . This map is constructed in Example 4.6, while the Example 4.5 is preparatory.

**Example 4.5.** We want to show here how singular topological phenomena related to the double-eight map [20], [15], [22], [24], [13] arise also among piecewise constant maps. In particular, as pointed out in [23], for the homogeneous extension of the double-eight map, a gap phenomenon occurs between the minimal vertical lifting measure and the relaxed Jacobian total variation. We show now that we find such a gap also among piecewise constant maps, by exhibiting a piecewise constant map with vanishing minimal vertical lifting measure but with finite non-zero  $\overline{TVJ}$ . Namely, we are going to define a map  $u : B_1 \rightarrow \mathbb{R}^2$  assuming five distinct values, for which the resulting closed curve  $\tilde{\gamma}$  has zero degree, but is homotopically trivial, since it is, in fact, homeomorphic to the double-eight curve. Let  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$  be the vertices of two equilateral triangles with a common vertex, say  $\alpha_1$  (see Figure 4). Fix a partition of  $\mathbb{S}^1$  in twelve disjoint non-empty arcs  $C_1, \dots, C_{12}$  (not necessarily of the same length), with extremal points  $a_1, \dots, a_{12}$  in counter-clockwise order. Then, define  $\gamma : \mathbb{S}^1 \rightarrow \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  to be constant on the arcs  $C_1, \dots, C_{12}$ , precisely equal to, in the order,  $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_5, \alpha_4$ . Then, the broken line curve  $\tilde{\gamma}$  runs consecutively the triangles  $T_{123} := T_{\alpha_1\alpha_2\alpha_3}$  and  $T_{145} := T_{\alpha_1\alpha_4\alpha_5}$  twice, and every time with different orientation. Define  $u$  as in (4.4), obtaining a 12-point map. Now, by applying Theorem 4.3 and computing the minimum of the Plateau problem (1.5) for  $\tilde{\gamma}$  as in [24, Theorem 5], we obtain

$$\overline{TVJ}_{BV}(u, B_1) = \bar{P}(\gamma) = P(\tilde{\gamma}) = 2 \min\{|T_{123}|, |T_{145}|\}. \quad (4.15)$$

Moreover, it is not difficult to see that

$$\mu_u^J = (|T_{123}| + |T_{145}| - |T_{123}| - |T_{145}|)\delta_0 = 0.$$

In this case, we have also  $\mu_v[u] = 0$ , indeed we can prove that the unique current  $T_u$  with minimal completely vertical lifting associated to  $u$  is given by

$$T_u = G_u + S = \sum_{l=1}^{12} [\widehat{C}_l] \times [c_l] + \sum_{l=1}^{12} [0, a_l] \times [c_{l-1}, c_l], \quad (4.16)$$

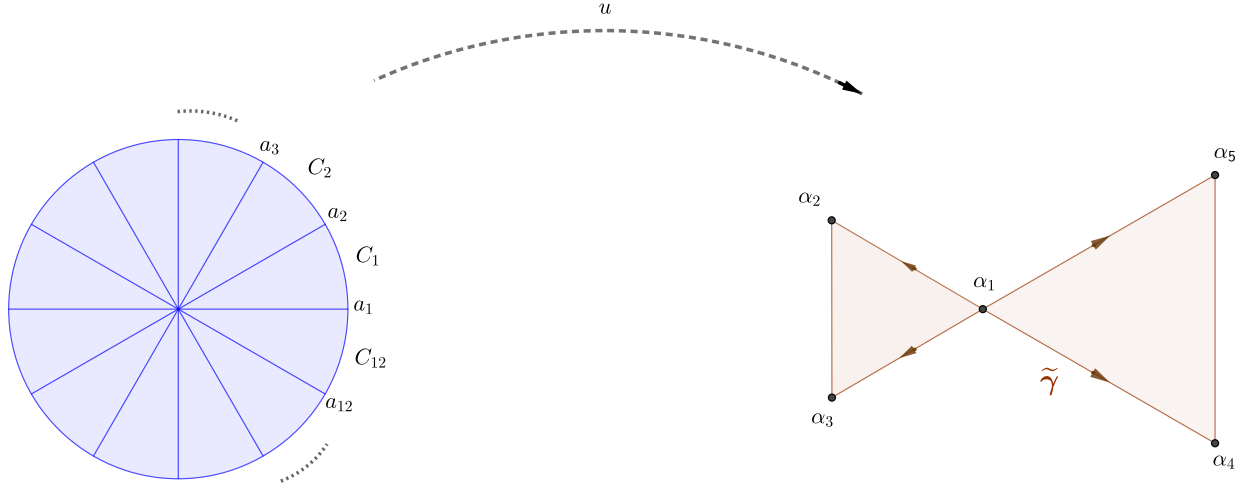


Figure 4: The map  $u$  and the broken line curve  $\tilde{\gamma}$  of Example 4.5.

butterfly

where  $\widehat{C}_l$  is the circular sector corresponding to  $C_l$  and  $c_l$  is the assigned value of  $\gamma$  on  $C_l$  for  $l = 1, \dots, 12$  (we used the convention  $c_0 = c_{12}$ ). Let us show (4.16). One checks that  $\mu_i^j[T_u] = \mu_i^j[u]$  for  $i, j = 1, 2$  by proceeding as in Remark 3.6. So, it remains to prove that  $T_u \in \text{cart}(B_1; \mathbb{R}^2)$ : it is enough to check that  $(\partial T_u) \lrcorner B_1 \times \mathbb{R}^2 = 0$ . Compute

$$\partial S = \sum_{l=1}^{12} \partial([0, a_l] \times [c_{l-1}, c_l]) = \sum_{l=1}^{12} (-[0] \times [c_{l-1}, c_l] + [0, a_l] \times [c_l] - [0, a_l] \times [c_{l-1}]).$$

Now, since by convention  $a_{13} = a_1$ ,

$$\partial G_u = \sum_{l=1}^{12} ([0, a_{l+1}] \times [c_l] - [0, a_l] \times [c_l]) = - \sum_{l=1}^{12} ([0, a_l] \times [c_l] - [0, a_l] \times [c_{l-1}]).$$

Moreover, by the choice of  $\{c_l\}$ ,

$$\sum_{l=1}^{12} [0] \times [c_{l-1}, c_l] = [0] \times [\alpha_1, \alpha_2] + [0] \times [\alpha_2, \alpha_3] + \dots + [0] \times [\alpha_4, \alpha_1] = 0.$$

Therefore,  $\partial G_u = -\partial S$ .

Notice that the action of  $T_u$  against 2-forms with only vertical differentials is 0, which means that  $T_u$  does not have completely vertical part and so  $\mu_v[u] = 0$ . Roughly, due to cancellations in the part of the boundary of  $T_u$  in correspondence to the origin, the current  $T_u$  is not able to detect the hole upon the origin in the graph of  $u$ , generated by the presence of the multiple junction.

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**Example 4.6.** This example is an adaptation of [22, Theorem 1.3] to the case of piecewise constant maps. Indeed, we construct a piecewise constant map  $u$ , taking only five values of  $\mathbb{R}^2$ , such that

$$\mu_v[u] = 0 \quad \text{and} \quad \overline{TVJ}_{BV}(u, B_1) = +\infty.$$

The idea is to replicate the map of Example 4.5 infinitely many times on a sequence  $\{D_i\}_{i \in \mathbb{N}} \subset B_1$  of disjoint balls, whose measures form an infinitesimal sequence (see Figure 5). So, for  $i \in \mathbb{N}$ , set

$$D_i := B_{r_i}(x_i), \quad \text{with } x_i := \left( -1 + \sum_{j=0}^{i-1} 2^{-j}, 0 \right), \quad r_i := 2^{-i-1}.$$

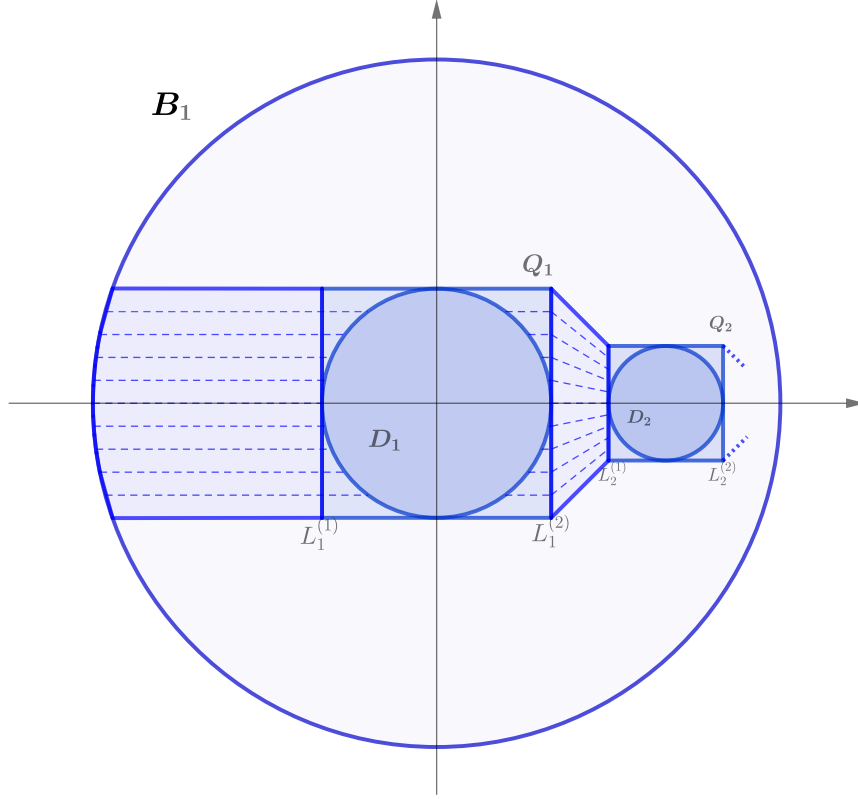


Figure 5: The sequence  $\{D_i\} \subset B_1$  of disks of Example 4.6.

mucci\_domain

Let  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \subset \mathbb{R}^2$  and  $\gamma : \mathbb{S}^1 \rightarrow \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  be as in Example 4.5. Now, define the map  $\widehat{\gamma} : \mathbb{S}^1 \rightarrow \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  in the same way as  $\gamma$ , but with different order of the values, in a symmetric way with respect to the vertical axis through  $\alpha_1$ , namely, in the same arcs  $C_1, \dots, C_{12}$ ,  $\widehat{\gamma}$  is equal to  $\alpha_1, \alpha_5, \alpha_4, \alpha_1, \alpha_3, \alpha_2, \alpha_1, \alpha_4, \alpha_5, \alpha_1, \alpha_2, \alpha_3$ . Then, for  $i \in \mathbb{N}$ , define  $u|_{D_i} := u^{(i)}$  as

$$u^{(i)}(x) = \begin{cases} \gamma \left( \frac{x - x_i}{|x - x_i|} \right) & \text{if } i \text{ is odd,} \\ \widehat{\gamma} \left( \frac{x - x_i}{|x - x_i|} \right) & \text{if } i \text{ is even.} \end{cases}$$

It remains to define  $u$  in  $B_1 \setminus \cup_{i \in \mathbb{N}} D_i$ . Start by considering, for every  $i \in \mathbb{N}$ , the square  $Q_i$  that circumscribes  $D_i$  and extend  $u^{(i)}$  to  $Q_i$  to be constant along horizontal lines. Now, denote by  $L_i^{(1)}$  and  $L_i^{(2)}$  the vertical left and right sides of  $\partial Q_i$ , then extend  $u$  to the convex hull of  $L_i^{(2)}$  and  $L_{i+1}^{(1)}$  to be constant along straight lines which interpolate pointwise the two sides. Finally, extend  $u$  in the strip that connects  $L_1^{(1)}$  to  $\partial B_1$  to be constant along horizontal lines and set  $u = \alpha_1$  in the rest of  $B_1$ . (see Figure 5). It is not difficult to see that  $u \in BV(B_1; \mathbb{R}^2)$ , by the choice of the infinitesimal sequence  $(r_i)$ . Thus, assuming by contradiction that  $\overline{\mathcal{A}}_{BV}(u, B_1)$  be finite, one can define the current  $T_u = G_u + S$  in a similar way as in Example 4.5, that is to say, by setting  $S$  to be the trivial affine interpolation surface on the jump segments of  $u$ . One can prove in the same

way that  $T_u$  is the current with minimal completely vertical lifting associated to  $u$  and  $\mu_v[u] = 0$ . In particular,  $T_u \in \text{cart}(B_1; \mathbb{R}^2)$  and has finite mass. On the other hand,

$$\overline{TVJ}_{BV}(u, B_1) \geq \sum_{i=1}^{+\infty} \overline{TVJ}_{BV}(u, D_i) = \sum_{i=1}^{+\infty} 2 \min\{|T_{\alpha_1 \alpha_2 \alpha_3}|, |T_{\alpha_1 \alpha_4 \alpha_5}|\} = +\infty.$$

In particular  $\overline{A}_{BV}(u, B_1) = +\infty$  as well.

## 5 Piecewise Lipschitz maps

ec:Lip\_npoint

In this last section we combine the results of the previous sections and compute the  $BV$ -relaxed area for an interesting class of maps that we call *piecewise Lipschitz maps*, quickly mentioned in the Introduction. As stated in our main result (Theorem 1.1), the relaxed area turns out to be composed by a regular term and a singular one, that interestingly further splits into two non-trivial pieces, respectively related to the 1-dimensional and 0-dimensional singularities.

Let  $\Omega \subset \mathbb{R}^2$  be a connected bounded open set with boundary of class  $C^1$ . We say that a collection  $\{\Omega_1, \dots, \Omega_N\}$  of disjoint nonempty open sets is a Lipschitz partition of  $\Omega$  if  $\overline{\Omega} = \cup_{k=1}^N \overline{\Omega}_k$  and for each  $k = 1, \dots, N$ ,  $\Omega_k$  is connected and Lipschitz.

For a given Lipschitz partition of  $\Omega$  we can consider its interface  $\Sigma := \cup_{k=1}^N \partial\Omega_k$ . Also, we can define the (possibly empty) set of interior junction points  $\{p_i\}_{i=1}^m$ , i.e. points  $p_i \in \Omega$  such that there exist  $r > 0$  and an integer  $N_i$  with  $3 \leq N_i \leq N$ , such that  $B_r(p_i) \subset \Omega$  and  $B_s(p_i)$  has nonempty intersection with exactly  $N_i$  connected components of  $\Omega$ , for every  $s \in (0, r]$ .

We shall consider Lipschitz partitions whose interface is a *network* in the following sense:

ample\_network

**Definition 5.1 (Network).** *The interface  $\Sigma$  of a Lipschitz partition of  $\Omega$  is a network if*

$$\Sigma := \bigcup_{\ell=1}^n \overline{J}_\ell, \quad J_\ell = \alpha_\ell(I_\ell), \quad I_\ell = (a_\ell, b_\ell), \quad (5.1) \quad \square$$

where the curves  $\alpha_\ell : \overline{I}_\ell := [a_\ell, b_\ell] \rightarrow \overline{\Omega}$ ,  $\ell = 1, \dots, n$ , satisfy the following properties:

- $\alpha_\ell$  is of class  $C^2$ , injective with  $|\dot{\alpha}_\ell| \equiv 1$  on  $I_\ell$ , and  $J_\ell \subset \Omega$ ;
- $\ell_1 \neq \ell_2 \Rightarrow J_{\ell_1} \cap J_{\ell_2} = \emptyset$ ;
- $\alpha_\ell(\{a_\ell, b_\ell\}) \subset \{p_1, \dots, p_m\} \cup \partial\Omega$  for all  $\ell = 1, \dots, n$  such that  $\alpha_\ell(a_\ell) \neq \alpha_\ell(b_\ell)$ ;
- if  $x \in \overline{J}_\ell \cap \partial\Omega$ ,  $\alpha_\ell$  is transversal to  $\partial\Omega$  at  $x$ ;
- $\ell_1 \neq \ell_2 \Rightarrow \overline{J}_{\ell_1} \cap \overline{J}_{\ell_2} \subset \{p_1, \dots, p_m\}$ .

From the last condition it follows that if two curves have endpoints on  $\partial\Omega$ , then these points are distinct.

def\_MLM

**Definition 5.2 (Piecewise Lipschitz map).** *Let  $\{\Omega_k\}_{k=1}^N$  be a Lipschitz partition of  $\Omega$  whose interface  $\Sigma$  is a network. We say that  $u \in BV(\Omega; \mathbb{R}^2)$  is a piecewise Lipschitz map if its jump set  $S_u$  coincides with  $\Sigma$  and  $u \llcorner \Omega_k \in \text{Lip}(\Omega_k; \mathbb{R}^2)$  for any  $k = 1, \dots, N$ .*

Since  $u \llcorner \Omega_k \in \text{Lip}(\Omega_k; \mathbb{R}^2)$ , the trace of  $u$  on  $\partial\Omega_k$  is also Lipschitz. In particular, for any  $i \in \{1, \dots, m\}$  such that  $p_i \in \partial\Omega_k$ ,

$$\exists \lim_{\substack{x \rightarrow p_i \\ x \in \Omega_k}} u(x) =: \beta_i^k \in \mathbb{R}^2.$$

Let  $\rho > 0$  be sufficiently small so that  $B_\rho(p_i) \subset \Omega$  for  $i \in \{1, \dots, m\}$ . Let  $\ell \in \{1, \dots, n\}$  be such that  $p_i$  is an endpoint of  $\bar{J}_\ell$ ; since  $\alpha_\ell$  is of class  $C^2$ , for  $\rho$  small enough the intersection  $\bar{J}_\ell \cap \partial B_\rho(p_i)$  consists either of a single point, or of two points if  $\alpha_\ell(a_\ell) = \alpha_\ell(b_\ell) = p_i$ . Hence, the map  $u \llcorner \partial B_\rho(p_i)$  is piecewise Lipschitz and jumps at any point of  $\Sigma \cap \partial B_\rho(p_i)$ . In particular, the number of these jump points is, by definition of junction point,

$$N_i = \#(\Sigma \cap \partial B_\rho(p_i)) \geq 3, \quad i = 1, \dots, m.$$

For  $i = 1, \dots, m$ , we denote by  $\Omega_1^i, \dots, \Omega_{N_i}^i$  the connected components of  $\Omega \setminus \Sigma$  whose closure contains  $p_i$ , chosen in counterclockwise order around  $p_i$ . Since  $\Omega_k$  is Lipschitz for every  $k = 1, \dots, N$ , any  $\Omega_k^i$  has a corner at  $p_i$  whose aperture is a positive angle  $\theta_k^i \in (0, 2\pi)$ .

emma\_palline

**Lemma 5.3 (Circular slices).** *Let  $i \in \{1, \dots, m\}$  be fixed and let  $\rho > 0$  be as above. Then the maps  $\gamma_\rho^i \in BV(\mathbb{S}^1; \mathbb{R}^2)$  defined by  $\gamma_\rho^i(\nu) := u(p_i + \rho\nu)$  converge strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$ , as  $\rho \rightarrow 0^+$ , to a piecewise constant map  $\gamma^i : \mathbb{S}^1 \rightarrow \mathbb{R}^2$  taking, in counterclockwise order, the values  $\beta_1^i, \beta_2^i, \dots, \beta_{N_i}^i$  on arcs of size  $\theta_1^i, \theta_2^i, \dots, \theta_{N_i}^i$ , respectively.*

The map  $\gamma^i$  has  $N_i$  jumps on  $\mathbb{S}^1$  whose angular coordinates are denoted by  $a_i^1, a_i^2, \dots, a_i^{N_i}$  (where  $a_i^j - a_i^{j-1} = \theta_j^i$ , for  $j = 1, \dots, N_i + 1$ ).

*Proof.* It is easy to see that  $(\gamma_\rho^i)$  converges to  $\gamma^i$  almost everywhere on  $\mathbb{S}^1$  as  $\rho \rightarrow 0^+$ . Moreover,  $\gamma_\rho^i$ , for  $\rho$  small enough, has exactly  $N_i$  jumps at points  $a_{i,\rho}^j$  of amplitude  $|u^+(p_i + \rho a_{i,\rho}^j) - u^-(p_i + \rho a_{i,\rho}^j)|$  which tend, by continuity of  $u$  in  $B_\rho(p_i) \setminus \Sigma$ , to  $|\beta_i^j - \beta_i^{j+1}|$ . Also, on the arcs between  $a_{i,\rho}^j$  and  $a_{i,\rho}^{j+1}$ ,  $|\dot{\gamma}_\rho^i| \leq L\rho$ , where  $L$  is the maximum of the Lipschitz constants of  $u$  on the sectors  $\Omega_k^i$ . Hence  $|\dot{\gamma}_\rho^i|(\mathbb{S}^1) \rightarrow |\dot{\gamma}^i|(\mathbb{S}^1)$  and the thesis follows straightforwardly.  $\square$

For  $\ell = 1, \dots, n$ , we denote by  $u_{(\ell)}^\pm$  the two traces of  $u$  on  $J_\ell$ , and consider the affine interpolation surface  $X_{(\ell)}^{\text{aff}} : [a_\ell, b_\ell] \times I \rightarrow \mathbb{R}^3$  spanning the graphs of  $u_{(\ell)}^-$  and  $u_{(\ell)}^+$ , given by (1.7):

$$X_{(\ell)}^{\text{aff}}(t, s) = (t, su_{(\ell)}^+(t) + (1-s)u_{(\ell)}^-(t)), \quad (t, s) \in [a_\ell, b_\ell] \times I. \quad (5.2)$$

eq:affi

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Lower bound: Consider a sequence  $(v_k) \subset C^1(\Omega; \mathbb{R}^2)$  converging to  $u$  strictly  $BV(\Omega; \mathbb{R}^2)$ . For any  $\rho > 0$  small enough, we take a family of mutually disjoint balls  $B_\rho(p_i) \subset \Omega$ ,  $i = 1, \dots, m$ . By Lemma 2.5, there exists a subsequence  $(v_{k_h}) \subset (v_k)$  depending on  $\rho$  such that for  $i = 1, \dots, m$

$$v_{k_h} \llcorner \partial B_\rho(p_i) \rightarrow u \llcorner \partial B_\rho(p_i) \quad \text{strictly } BV(\partial B_\rho(p_i); \mathbb{R}^2). \quad (5.3)$$

converg

We may also assume that for  $i = 1, \dots, m$

$$\liminf_{k \rightarrow +\infty} \int_{B_\rho(p_i)} |Jv_k| \, dx = \lim_{h \rightarrow +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| \, dx.$$

Then

$$\mathcal{A}(v_{k_h}, \Omega) = \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \bar{B}_\rho(p_i)) + \sum_{i=1}^m \mathcal{A}(v_{k_h}; \bar{B}_\rho(p_i)) \geq \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \bar{B}_\rho(p_i)) + \sum_{i=1}^m \int_{B_\rho(p_i)} |Jv_{k_h}| \, dx.$$

<sup>7</sup>With the convention  $N_i + 1 = 1$ .

By Corollary 3.12, we get

$$\begin{aligned}
\liminf_{h \rightarrow +\infty} \mathcal{A}(v_{k_h}, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) &\geq \overline{\mathcal{A}}_{BV}(u, \Omega \setminus \cup_{i=1}^m \overline{B}_\rho(p_i)) \\
&= \int_{\Omega \setminus \cup_{i=1}^m B_\rho(p_i)} |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a_\ell^\rho, b_\ell^\rho] \times I} |\partial_t X_{(\ell)}^{\text{aff}} \wedge \partial_s X_{(\ell)}^{\text{aff}}| dt ds \\
&\rightarrow \int_{\Omega} |\mathcal{M}(\nabla u)| dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_{(\ell)}^{\text{aff}} \wedge \partial_s X_{(\ell)}^{\text{aff}}| dt ds \quad \text{as } \rho \rightarrow 0^+,
\end{aligned}$$

where  $(a_\ell^\rho), (b_\ell^\rho) \subset [a_\ell, b_\ell]$  are respectively a decreasing and increasing sequence of numbers satisfying  $a_\ell^\rho \rightarrow a_\ell$  and  $b_\ell^\rho \rightarrow b_\ell$  as  $\rho \rightarrow 0^+$  and  $\alpha_\ell([a_\ell^\rho, b_\ell^\rho]) = \alpha_\ell([a_\ell, b_\ell]) \setminus \cup_{i=1}^m B_\rho(p_i)$ .

Let us recall that, by Lemma 2.14,  $P(\tilde{\gamma}^i) = \overline{P}(\gamma^i)$ , with  $\gamma^i$  as in Lemma 5.3. So, it remains to show that

$$\liminf_{\rho \rightarrow 0^+} \lim_{h \rightarrow +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| dx \geq \overline{P}(\gamma^i) \quad \forall i = 1, \dots, m. \quad (5.4)$$

eq:limi

By definition (2.40), using (2.31) and (5.3), we readily conclude that

$$\lim_{h \rightarrow +\infty} \int_{B_\rho(p_i)} |Jv_{k_h}| dx \geq \overline{P}(\gamma_\rho^i),$$

where  $\gamma_\rho^i$  is defined in Lemma 5.3. Then, since  $\gamma_\rho^i$  converge to  $\gamma^i$  strictly  $BV(\mathbb{S}^1; \mathbb{R}^2)$  as  $\rho \rightarrow 0^+$ , (5.4) follows, thanks to Lemma 5.3 and Corollary 2.16.

Upper bound: Fix  $r > 0$  small enough and consider mutually disjoint balls  $B_r(p_i) \subset \Omega$ ,  $i = 1, \dots, m$ , such that, for every  $\ell \in \{1, \dots, n\}$ ,  $J_\ell \cap \partial B_s(p_i)$ , if nonempty, consists either of a single point, or of two points if  $\alpha_\ell(a_\ell) = \alpha_\ell(b_\ell) = p_i$ , for every  $s \in (0, r]$ .

Clearly, the difficulty of the proof is concentrated around the junction points  $p_i$ . The idea is to modify  $u$  on  $\cup_{i=1}^m B_r(p_i)$  by constructing a new map  $u_r$  (see (5.8) and (5.20)), which coincides with  $u$  out of  $\cup_{i=1}^m B_r(p_i)$  and converges to  $u$  strictly  $BV(\Omega; \mathbb{R}^2)$  as  $r$  tends to  $0^+$ . The map  $u_r$  will be again a piecewise Lipschitz map with the same set  $\{p_i\}$  of junction points, but different jump set  $\Sigma_r$ , with  $\Sigma_r \cap B_{r/2}(p_i)$  made of segments, i.e.  $u_r$  is of the form (4.4) in  $B_{r/2}(p_i)$ . The difficult point will be to provide that  $\Sigma_r$  is still a union of (pairwise disjoint up to the endpoints)  $C^2$ -curves  $\hat{\alpha}_\ell$ , in particular that each one hits  $\partial B_{r/2}(p_i)$  with vanishing second derivative. At the end, we will apply Theorem 4.4 to  $u_r$  in  $\cup_{i=1}^m B_{r/2}(p_i)$  and Corollary 3.12 to  $u_r$  in  $\Omega \setminus (\cup_{i=1}^m B_{r/2}(p_i))$ , and conclude by lower semicontinuity of  $\overline{\mathcal{A}}_{BV}(\cdot, \Omega)$ .

We start by considering a smooth strictly increasing surjective function  $\psi_r : [\frac{r}{2}, +\infty) \rightarrow [0, +\infty)$  with <sup>8</sup>

$$\psi_r(\rho) = \rho \quad \forall \rho \geq r, \quad \psi_r(\rho) = \left(\rho - \frac{r}{2}\right)^3 \text{ in a right neighborhood of } \frac{r}{2}, \quad |\psi_r'| \leq C \quad \text{in } \left(\frac{r}{2}, r\right) \quad (5.5)$$

psi

with  $C > 0$  independent of  $r$ . We define the radial map  $\Phi_r : \mathbb{R}^2 \setminus B_{\frac{r}{2}}(0) \rightarrow \mathbb{R}^2 \setminus \{0\}$  as

$$\Phi_r(x) = \psi_r(|x|) \frac{x}{|x|},$$

whose inverse is  $\Phi_r^{-1}(y) = f_r(|y|) \frac{y}{|y|}$ , where  $f_r := \psi_r^{-1}$ , and set

$$\hat{u}_r(x) := u(p_i + \Phi_r(x - p_i)) \quad \text{for } x \in B_r(p_i) \setminus \overline{B}_{\frac{r}{2}}(p_i), \quad i = 1, \dots, m. \quad (5.6)$$

<sup>8</sup>The exponent must be chosen greater than 2 in order to ensure (5.19).

The jump set of  $\widehat{u}_r$  in  $B_r(p_i) \setminus B_{r/2}(p_i)$  is parametrized by the curves

$$\widehat{\alpha}_\ell := p_i + \Phi_r^{-1}(\alpha_\ell - p_i) \quad \forall \ell = 1, \dots, n. \quad (5.7)$$

Notice carefully that  $\widehat{\alpha}_\ell$  is parametrized on the same parameter interval of  $\alpha_\ell$ , but this is not an arc length parametrization for  $\widehat{\alpha}_\ell$ . Moreover, thanks to the regularity of  $\Phi_r$ , the map

$$u_r := \begin{cases} u & \text{in } \Omega \setminus (\cup_{i=1}^m B_r(p_i)) \\ \widehat{u}_r & \text{in } B_r(p_i) \setminus B_{\frac{r}{2}}(p_i) \end{cases} \quad (5.8)$$

has jump set  $\Sigma_r$  which is parametrized by the curves  $\widehat{\alpha}_\ell$ , whose supports  $\widehat{J}_\ell$  are pairwise disjoint and in turn coincide with the ones of  $\alpha_\ell$  in  $\Omega \setminus (\cup_{i=1}^m B_r(p_i))$ .

*Step 1:* Let us first check that the length of  $\widehat{\alpha}_\ell$  in  $\cup_{i=1}^m (B_r(p_i) \setminus B_{r/2}(p_i))$  is controlled, more precisely, we will show that for each  $i$  and  $\ell$ , the length of  $\widehat{\alpha}_\ell$  in  $B_r(p_i) \setminus B_{r/2}(p_i)$  goes to 0 as  $r \rightarrow 0^+$ . We suppose that  $J_\ell \cap \partial B_s(p_i)$ , for every  $s \leq r$ , consists of a single point, because the argument adapts also if  $\alpha_\ell$  has two arcs exiting from  $p_i$ , simply by considering them separately. To this aim, fix  $i$  and  $\ell$  and denote  $\alpha_\ell = \alpha$ ,  $J_\ell = J$ . Without loss of generality, assume  $p_i = 0$ ,  $B_r(0) = B_r$ , and suppose that  $J \cap B_r$  is parametrized by arc length on  $[0, R]$ , with  $\alpha(0) = 0$  and  $\alpha(R) \in \partial B_r$ , where  $R(r) = R = \mathcal{H}^1(J \cap B_r)$ . We can express the gradient of  $\Phi_r^{-1}$  as follows:

$$\nabla \Phi_r^{-1}(y) = f'_r(|y|) \frac{y}{|y|} \otimes \frac{y}{|y|} + f_r(|y|) \nabla \left( \frac{y}{|y|} \right) = f'_r(|y|) \frac{y}{|y|} \otimes \frac{y}{|y|} + \frac{f_r(|y|)}{|y|} \Pi(y), \quad (5.9)$$

where

$$\Pi(y) := \text{Id} - \frac{y \otimes y}{|y|^2},$$

and we used that

$$\nabla \left( \frac{y}{|y|} \right) = \frac{1}{|y|} \Pi(y). \quad (5.10)$$

From (5.7), we have  $\dot{\widehat{\alpha}} = \nabla \Phi_r^{-1}(\alpha) \dot{\alpha}$ , and using (5.9) and  $|\dot{\alpha}| = 1$ ,

$$|\dot{\widehat{\alpha}}| \leq f'_r(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|} |\Pi(\alpha) \dot{\alpha}|. \quad (5.11)$$

Notice that if  $r$  is small, the function  $t \mapsto |\alpha(t)| =: \sigma(t)$  is  $C^1$  and invertible from  $[0, R]$  to  $[0, r]$ . Moreover,  $\sigma'(t) = \frac{\alpha(t)}{|\alpha(t)|} \cdot \dot{\alpha}(t) \rightarrow \frac{\dot{\alpha}(0)}{|\dot{\alpha}(0)|} \cdot \dot{\alpha}(0) = |\dot{\alpha}(0)| = 1$  as  $t \rightarrow 0^+$ . Let us integrate on  $[0, R]$  the term  $f'_r(|\alpha|)$ : performing the change of variable  $\sigma(t) = \rho$ , we get

$$\int_0^R f'_r(|\alpha(t)|) dt = \int_0^R f'_r(\sigma(t)) dt = \int_0^r f'_r(\rho) \frac{d\rho}{\sigma'(\sigma^{-1}(\rho))} \leq 2 \int_0^r f'_r(\rho) d\rho,$$

where in the last inequality we used that, for small  $r$ ,  $\sigma'(\sigma^{-1}(\rho)) \geq \frac{1}{2}$  for every  $\rho \in [0, r]$ . Sending  $r$  to  $0^+$ , we have that  $\int_0^R f'_r(|\alpha(t)|) dt \rightarrow 0$  by integrability of  $f'$  near to the origin.

In order to estimate the second term on the right hand side of (5.11), we can use a Taylor expansion of  $\alpha$  around 0, writing  $\alpha(t) = vt + wt^2 + o(t^2)$ , with  $v = \dot{\alpha}(0)$ ,  $w = \frac{\ddot{\alpha}(0)}{2}$ , and  $\lim_{t \rightarrow 0^+} o(t^p)/t^p = 0$ . We have

$$\Pi(\alpha) \dot{\alpha} = \Pi(vt + wt^2 + o(t^2))(v + 2wt + o_2(t)) = \Pi(v + wt + o_1(t))(v + 2wt + o_2(t)),$$



where  $o_1(t) = o(t^2)/t$  and  $o_2(t) = o(t)$ . Writing  $v + 2wt + o_2(t) = v + wt + o_1(t) + wt + o_2(t) - o_1(t)$ , we get

$$\Pi(\alpha)\dot{\alpha} = \Pi(v + wt + o_1(t))(v + wt + o_1(t)) + \Pi(v + wt + o_1(t))(wt + o_2(t) - o_1(t)).$$

The first term on the right hand side is 0 and the norm of the second term can be estimated from above by  $|w|t + o(t)$ . Now, by definition of arc length parameter,  $R = \mathcal{H}^1(\text{spt}\alpha \cap B_r(0)) \rightarrow 0$  as  $r \rightarrow 0^+$ . Moreover, by Taylor expansion,  $|\alpha(t)| > \frac{t}{2}$  for  $t$  small enough. Therefore, since  $f_r(0) = \frac{r}{2}$ , for  $r$  small enough we have  $\frac{f_r(|\alpha(t)|)}{|\alpha(t)|} \leq \frac{2r}{t}$  on  $[0, R]$ . So, integrating on  $[0, R]$  the second term on the right hand side of (5.11),

$$\int_0^R \frac{f_r(|\alpha(t)|)}{|\alpha(t)|} |\Pi(\alpha(t))\dot{\alpha}(t)| dt \leq \int_0^R \frac{2r}{t} (|w|t + o(t)) dt \rightarrow 0 \quad \text{as } r \rightarrow 0^+.$$

*Step 2:* Let  $\hat{J} = \hat{J}_l$  be the support of  $\hat{\alpha}$ ; let us show that there is a parametrization of  $\hat{J} \cap (B_r \setminus B_{r/2})$  on an interval  $[0, L]$ , which is of class  $C^2$  up to 0 and with vanishing second derivative at 0. Indeed, set  $L := \mathcal{H}^1(\hat{J} \cap (B_r \setminus B_{r/2}))$  and consider the arc-length parameter  $s \in [0, L]$  given by

$$s(t) = \int_0^t |V_r(\alpha(\tau))| d\tau,$$

where

$$V_r(\alpha) := \nabla \Phi_r^{-1}(\alpha)\dot{\alpha}.$$

We compute

$$\frac{d^2}{ds^2} \hat{\alpha}(t) = \frac{d}{ds} \left( \frac{V_r(\alpha)}{|V_r(\alpha)|} \right) = \Pi(V_r(\alpha)) \left( \frac{\nabla^2 \Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha}) + \nabla \Phi_r^{-1}(\alpha) \ddot{\alpha}}{|V_r(\alpha)|^2} \right). \quad (5.12) \quad \text{curvatu}$$

Here and in what follows,  $\alpha$  is evaluated at  $t = t(s)$  and  $\dot{\alpha}$  and  $\ddot{\alpha}$  denote the first and second derivative of  $\alpha$  with respect to  $t$ . The operation  $:$  between a tensor  $T = (T_{ijk}) \in \mathbb{R}^{2 \times 2 \times 2}$  and a matrix  $M = (M_{ij}) \in \mathbb{R}^{2 \times 2}$  is defined as the vector  $T : M \in \mathbb{R}^2$  with components  $(T : M)_k = T_{ijk} M_{ij}$  for  $k = 1, 2$ .

We get

$$\begin{aligned} \left| \frac{d^2}{ds^2} \hat{\alpha}(t) \right| &\leq \left| \Pi(V_r(\alpha)) \left( \frac{\nabla^2 \Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2} \right) \right| + \frac{|\nabla \Phi_r^{-1}(\alpha) \ddot{\alpha}|}{|V_r(\alpha)|^2} \\ &\leq \left| \Pi(V_r(\alpha)) \left( \frac{\nabla^2 \Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2} \right) \right| + C \frac{f'_r(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}. \end{aligned} \quad (5.13) \quad \text{second}$$

where we have used (5.9) and that  $\ddot{\alpha}$  is bounded.

The Hessian of  $\Phi_r^{-1}$  can be computed as

$$\begin{aligned} \nabla^2 \Phi_r^{-1}(y) &= f''_r(|y|) \frac{y}{|y|} \otimes \frac{y}{|y|} \otimes \frac{y}{|y|} + f'_r(|y|) \nabla \left( \frac{y}{|y|} \otimes \frac{y}{|y|} \right) + \\ &\quad + f'_r(|y|) \frac{y}{|y|} \otimes \nabla \left( \frac{y}{|y|} \right) + f_r(|y|) \nabla^2 \left( \frac{y}{|y|} \right) \\ &= f''_r(|y|) \frac{y}{|y|} \otimes \frac{y}{|y|} \otimes \frac{y}{|y|} + f'_r(|y|) \nabla \left( \frac{y}{|y|} \right) \otimes \frac{y}{|y|} + \\ &\quad + 2f'_r(|y|) \frac{y}{|y|} \otimes \nabla \left( \frac{y}{|y|} \right) + f_r(|y|) \nabla \left( \nabla \left( \frac{y}{|y|} \right) \right). \end{aligned}$$

Then, by (5.10), we have

$$\begin{aligned}\nabla^2\Phi_r^{-1}(\alpha) &= f_r''(|\alpha|)\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|} + \left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|} \\ &\quad + \left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha).\end{aligned}$$

So, for  $k = 1, 2$ , we have

$$\begin{aligned}&(\nabla^2\Phi_r^{-1}(\alpha) : (\dot{\alpha} \otimes \dot{\alpha}))_k \\ &= f_r''(|\alpha|)\left(\left(\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k \\ &\quad + \left(\frac{f_r'(|\alpha|)}{|\alpha|} - 2\frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k \tag{5.14} \quad \boxed{\text{term1}} \\ &\quad + \left(2\frac{f_r'(|\alpha|)}{|\alpha|} - \frac{f_r(|\alpha|)}{|\alpha|^2}\right)\left(\left(\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha)\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k.\tag{5.15} \quad \boxed{\text{term2}}\end{aligned}$$

Notice that, since  $\Pi(\alpha)$  is symmetric,

$$\Pi(\alpha)_{ij}\alpha_j = 0, \quad \Pi(\alpha)_{ij}\alpha_i = 0, \tag{5.16} \quad \boxed{\text{property}}$$

where we sum on repeated indices. So, using (5.16) and that, from Taylor expansion,  $\dot{\alpha}(t) = v + 2wt + o(t) = \frac{\alpha(t)}{t} + wt + o(t)$ , we have

$$\begin{aligned}\left(\left(\Pi(\alpha)\otimes\frac{\alpha}{|\alpha|}\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k &= \Pi(\alpha)_{ij}\dot{\alpha}_i\dot{\alpha}_j\frac{\alpha_k}{|\alpha|} = \Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_it + o(t)\right)\dot{\alpha}_j\frac{\alpha_k}{|\alpha|} \\ &= \Pi(\alpha)_{ij}(w_it + o(t))\dot{\alpha}_j\frac{\alpha_k}{|\alpha|}; \\ \left(\left(\frac{\alpha}{|\alpha|}\otimes\Pi(\alpha)\right) : (\dot{\alpha} \otimes \dot{\alpha})\right)_k &= \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\dot{\alpha}_i\dot{\alpha}_j = \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}\left(\frac{\alpha_j}{t} + w_jt + o(t)\right)\dot{\alpha}_i \\ &= \frac{\alpha_i}{|\alpha|}\Pi(\alpha)_{jk}(w_jt + o(t))\dot{\alpha}_i.\end{aligned}$$

So, the norm of the sum of (5.14) and (5.15) can be easily estimated by

$$3\left(\frac{f_r'(|\alpha|)}{|\alpha|} + \frac{f_r(|\alpha|)}{|\alpha|^2}\right)(|w|t + o(t)) \leq C\left(f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}\right),$$

where we used that, for  $t$  small,  $|\alpha(t)| \geq \frac{t}{2}$ .

Therefore, (5.13) becomes

$$\left|\frac{d^2}{ds^2}\hat{\alpha}(t)\right| \leq |f_r''(|\alpha|)|\left|\Pi(V_r(\alpha))\left(\frac{\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}\otimes\frac{\alpha}{|\alpha|}}{|V_r(\alpha)|^2} : (\dot{\alpha} \otimes \dot{\alpha})\right)\right| + C\frac{f_r'(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}. \tag{5.17} \quad \boxed{\text{curvature}}$$

Now we treat the first term of the right hand side of (5.17). For  $j = 1, 2$ , by definition of  $V_r(\alpha)$ , using Taylor expansion and (5.16), we have

$$\begin{aligned}(V_r)_j(\alpha) &= f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\dot{\alpha}_i + f_r(|\alpha|)\Pi(\alpha)_{ij}\dot{\alpha}_i \\ &= f_r'(|\alpha|)\frac{\alpha_i\alpha_j}{|\alpha|^2}\left(\frac{\alpha_i}{t} + w_it + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}\left(\frac{\alpha_i}{t} + w_it + o(t)\right) \\ &= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + \frac{\alpha_i\alpha_j}{|\alpha|^2}w_it + o(t)\right) + f_r(|\alpha|)\Pi(\alpha)_{ij}(w_it + o(t)) \\ &= f_r'(|\alpha|)\left(\frac{\alpha_j}{t} + o(t)\right) + f_r(|\alpha|)O_j(t),\end{aligned} \tag{5.18} \quad \boxed{V_r}$$

where in the last equality we used that  $\alpha_i w_i = o(t)$ , since  $v_i w_i = 0$  because  $|\dot{\alpha}| = 1$ , and we setted  $O_j(t) := \Pi(\alpha)_{ij}(w_i t + o(t))$ , meaning that  $\lim_{t \rightarrow 0^+} |O_j(t)|/t < +\infty$ . Then, we get

$$\alpha = t \left( \frac{V_r(\alpha) - O(t)}{f'_r(|\alpha|)} + o(t) \right).$$

So,

$$\begin{aligned} \Pi(V_r(\alpha)) \frac{\frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2} &= \frac{\alpha_i \alpha_j}{|\alpha|^2} \dot{\alpha}_i \dot{\alpha}_j \Pi(V_r(\alpha)) \frac{\frac{\alpha}{|\alpha|}}{|V_r(\alpha)|^2} \\ &= \frac{\alpha_i \alpha_j}{|\alpha|^2} \dot{\alpha}_i \dot{\alpha}_j \frac{t}{|\alpha|} \Pi(V_r(\alpha)) \frac{\left( \frac{V_r(\alpha) - O(t)}{f'_r(|\alpha|)} + o(t) \right)}{|V_r(\alpha)|^2} \\ &= \frac{\alpha_i \alpha_j}{|\alpha|^2} \dot{\alpha}_i \dot{\alpha}_j \frac{t}{|\alpha|} \Pi(V_r(\alpha)) \frac{\left( \frac{O(t)}{f'_r(|\alpha|)} + o(t) \right)}{|V_r(\alpha)|^2}, \end{aligned}$$

where we used that  $\Pi(V_r(\alpha))V_r(\alpha) = 0$ . For  $t$  small, we get

$$\left| \Pi(V_r(\alpha)) \frac{\frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} \otimes \frac{\alpha}{|\alpha|} : (\dot{\alpha} \otimes \dot{\alpha})}{|V_r(\alpha)|^2} \right| \leq 2 \frac{\frac{O(t)}{f'_r(|\alpha|)} + o(t)}{|V_r(\alpha)|^2}.$$

Finally, from (5.17), we obtain

$$\left| \frac{d^2}{ds^2} \hat{\alpha}(t) \right| \leq |f''_r(|\alpha|)| \frac{\frac{O(t)}{f'_r(|\alpha|)} + o(t)}{|V_r(\alpha)|^2} + C \frac{f'_r(|\alpha|) + \frac{f_r(|\alpha|)}{|\alpha|}}{|V_r(\alpha)|^2}.$$

From the definition of  $f_r$ , we have that  $f_r(|\alpha(t)|) = \frac{r}{2} + t^{\frac{1}{3}} + o(t^{\frac{1}{3}})$  for  $t$  near to 0. So, by (5.18), we have  $|V_r(\alpha(t))| \geq C f'_r(|\alpha(t)|) = Ct^{-\frac{2}{3}} + o(t^{-\frac{2}{3}})$ . Then, since  $|f''_r(|\alpha(t)|)| = Ct^{-\frac{5}{3}} + o(t^{-\frac{5}{3}})$ , a straightforward check shows that

$$\frac{d^2}{ds^2} \hat{\alpha}(t) \rightarrow 0 \quad \text{as } t \rightarrow 0^+. \quad (5.19) \quad \boxed{\text{vanishi}}$$

We conclude that the curve  $\hat{\alpha}$  is  $C^2$  up to 0 with vanishing second derivative, and hence can be extended on the interval  $(-\frac{r}{2}, 0)$  to a (not relabeled) curve  $\hat{\alpha}$  whose support is a straight segment connecting  $\hat{\alpha}(0)$  to 0 (namely a radius of  $B_{r/2}(0)$ ). Going back to the curves  $\hat{\alpha}_\ell$ , we have just proved that we can extend them in  $B_{r/2}(p_i)$  with  $C^2$ -regularity using a segment along a radius, reaching  $p_i$ . In particular, the new supports of  $\hat{\alpha}_\ell$ 's form a  $N^i$ -junction point around  $p_i$  in  $B_{r/2}(p_i)$ , whose circular sectors  $\hat{C}_j^i$  ( $j = 1, \dots, N_i$ ) have amplitudes  $\theta_i^1, \dots, \theta_i^{N_i}$  (according to Lemma 5.3). Up to a reparametrization by arc-length of  $\hat{\alpha}_\ell$ , we will suppose that  $\hat{\alpha}_\ell : [\hat{a}_\ell, \hat{b}_\ell] \rightarrow \mathbb{R}^2$  have always derivative of modulus 1.

*Step 3:* We are ready to extend the map  $u_r$  in  $B_{r/2}(p_i)$ . We eventually observe that, from (5.8),  $u_r(x) = \gamma^i \left( \frac{2}{r}(x - p_i) \right)$  on  $\partial B_{r/2}(p_i)$  (see Lemma 5.3), and hence it is constant on any arc with angular coordinate in  $(a_i^{j-1}, a_i^j)$ . Hence we define

$$u_r(x) := \gamma^i \left( \frac{x - p_i}{|x - p_i|} \right) \quad x \in B_{\frac{r}{2}}(p_i). \quad (5.20) \quad \boxed{\text{def ur}}$$

Now,  $u_r$  satisfies the hypotheses of Corollary 3.12 in  $\Omega_r := \Omega \setminus (\cup_{i=1}^m \bar{B}_{r/4}(p_i))$ , where all the curves  $\hat{\alpha}_j$  satisfy hypotheses (H3), and they run on a straight segment (along a radius of  $B_{r/2}(p_i)$ ) inside

$B_{r/2}(p_i) \setminus B_{r/4}(p_i)$ . Then we introduce a sequence of Lipschitz maps  $\tilde{v}_k : \Omega_r \rightarrow \mathbb{R}^2$  which are defined as in (3.42), where, we recall,  $\varepsilon = \frac{1}{k}$ , with  $u_r$  in place of  $u$  and  $\Lambda = \text{id}$ ; in particular, for  $k$  large enough, the trace of  $\tilde{v}_k$  on  $\partial B_{r/3}(p_i)$  is a piecewise affine map coinciding with  $\gamma_k$  in (4.8), with  $\beta_i$  in place of  $\alpha_i$ . Thus, if we introduce also the sequence of Lipschitz maps  $\hat{v}_k : B_{r/2}(p_i) \rightarrow \mathbb{R}^2$  as in (4.12) (with  $B_r$  replaced by  $B_{r/2}(p_i)$ ) we see that  $\tilde{v}_k = \hat{v}_k$  on  $\partial B_{r/3}(p_i)$ . Therefore we define

$$v_k^r := \begin{cases} \tilde{v}_k & \text{in } \Omega \setminus (\cup_{i=1}^m B_{r/3}(p_i)) \\ \hat{v}_k & \text{in } \cup_{i=1}^m B_{r/3}(p_i), \end{cases} \quad (5.21)$$

and we readily see that  $v_k^r \rightarrow u_r$  strictly  $BV(\Omega; \mathbb{R}^2)$ .

Since the supports of  $\alpha_\ell$  and  $\hat{\alpha}_\ell$  coincide out of  $\cup_i B_r(p_i)$ , there exist  $\hat{a}_\ell^r, \hat{b}_\ell^r \in [\hat{a}_\ell, \hat{b}_\ell]$  and  $a_\ell^r, b_\ell^r \in [a_\ell, b_\ell]$  such that

$$\hat{\alpha}_\ell([\hat{a}_\ell^r, \hat{b}_\ell^r]) = \alpha_\ell([a_\ell^r, b_\ell^r]), \quad \hat{\alpha}_\ell(\hat{a}_\ell^r) = \alpha_\ell(a_\ell^r), \quad \hat{\alpha}_\ell(\hat{b}_\ell^r) = \alpha_\ell(b_\ell^r).$$

In particular,  $\hat{b}_\ell^r - \hat{a}_\ell^r = b_\ell^r - a_\ell^r$ , so up to a translation of the parameter interval of  $[\hat{a}_\ell, \hat{b}_\ell]$ , we can suppose  $\hat{a}_\ell^r = a_\ell^r$  and  $\hat{b}_\ell^r = b_\ell^r$ . Clearly,  $a_\ell^r \rightarrow a_\ell$  non increasingly and  $b_\ell^r \rightarrow b_\ell$  non decreasingly as  $r \rightarrow 0^+$ .

In view of Corollary 3.12 and Theorem 4.4 we conclude

$$\begin{aligned} \overline{\mathcal{A}}_{BV}(u_r, \Omega) &\leq \lim_{k \rightarrow +\infty} \mathcal{A}(v_k^r, \Omega) = \int_{\Omega \setminus (\cup_{i=1}^m B_{r/3}(p_i))} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[\hat{a}_\ell, \hat{b}_\ell] \times I} |\partial_t X_{\ell,r}^{\text{aff}} \wedge \partial_s X_{\ell,r}^{\text{aff}}| \, dt ds \\ &\quad + \int_{\cup_{i=1}^m (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + m \frac{\pi r^2}{9} + \sum_{i=1}^m \overline{P}(\gamma^i) \\ &= \int_{\Omega \setminus (\cup_{i=1}^m B_{r/3}(p_i))} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell^r, b_\ell^r] \times I} |\partial_t X_\ell^{\text{aff}} \wedge \partial_s X_\ell^{\text{aff}}| \, dt ds + \sum_{i=1}^m \overline{P}(\gamma^i) \\ &\quad + \int_{\cup_{i=1}^m (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + \sum_{\ell=1}^n \int_{([\hat{a}_\ell^r/3, a_\ell^r] \cup [b_\ell^r, \hat{b}_\ell^r/3]) \times I} |\partial_t X_{\ell,r}^{\text{aff}} \wedge \partial_s X_{\ell,r}^{\text{aff}}| \, dt ds \\ &\quad + \frac{r}{3} \sum_{i=1}^m \sum_{j=1}^{N_i} |\beta_i^j - \beta_i^{j+1}| + m \frac{\pi r^2}{9}, \end{aligned} \quad (5.22)$$

where for all  $\ell = 1, \dots, n$  we have  $\hat{a}_\ell \leq \hat{a}_\ell^{r/3} \leq a_\ell^r < b_\ell^r \leq \hat{b}_\ell^{r/3} \leq \hat{b}_\ell$ , where  $\hat{\alpha}_\ell(\hat{a}_\ell^{r/3}) \in \partial B_{r/3}(p_i)$ ,  $\hat{\alpha}_\ell(\hat{b}_\ell^{r/3}) \in \partial B_{r/3}(p_j)$  for some  $i, j \in \{1, \dots, m\}$ , unless one of them belongs to  $\partial \Omega$ , and where  $X_{\ell,r}^{\text{aff}}$  is defined as  $X_\ell^{\text{aff}}$  with  $u_r$  replacing  $u$ .

Now, since by (5.5)  $|\psi_r'| \leq C$ ,  $u_r$  is still a piecewise Lipschitz map on  $\Omega$ , hence, by *Step 1*, the last four terms in (2.5) are negligible as  $r \rightarrow 0^+$ . We then conclude, provided that  $u_r \rightarrow u$  strictly  $BV(\Omega; \mathbb{R}^2)$ , that

$$\overline{\mathcal{A}}_{BV}(u, \Omega) \leq \liminf_{r \rightarrow 0^+} \overline{\mathcal{A}}_{BV}(u_r, \Omega) = \int_{\Omega} |\mathcal{M}(\nabla u)| \, dx + \sum_{\ell=1}^n \int_{[a_\ell, b_\ell] \times I} |\partial_t X_\ell^{\text{aff}} \wedge \partial_s X_\ell^{\text{aff}}| \, dt ds + \sum_{i=1}^m \overline{P}(\gamma^i),$$

that is the thesis. In order to check that  $u_r \rightarrow u$  strictly  $BV(\Omega; \mathbb{R}^2)$  it is sufficient to observe that

$u = u_r$  outside  $\cup_{i=1}^m B_r(p_i)$  and that

$$\begin{aligned}
& \limsup_{r \rightarrow 0^+} |Du_r|(\cup_{i=1}^m B_r(p_i)) \\
& \leq \limsup_{r \rightarrow 0^+} \limsup_{k \rightarrow +\infty} \int_{\cup_{i=1}^m B_r(p_i)} \sqrt{1 + |\nabla v_k^r|^2} \, dx \\
& \leq \limsup_{r \rightarrow 0^+} \lim_{k \rightarrow +\infty} \mathcal{A}(v_k^r; \cup_{i=1}^m B_r(p_i)) \\
& = \limsup_{r \rightarrow 0^+} \left( \int_{\cup_{i=1}^m (B_r(p_i) \setminus B_{r/3}(p_i))} |\mathcal{M}(\nabla u_r)| \, dx + m \frac{\pi r^2}{9} \right. \\
& \quad \left. + \sum_{\ell=1}^n \int_{([\widehat{a}_\ell^{r/3}, \widehat{a}_\ell^r] \cup [\widehat{b}_\ell^r, \widehat{b}_\ell^{r/3}]) \times I} |\partial_t X_{\ell,r}^{\text{aff}} \wedge \partial_s X_{\ell,r}^{\text{aff}}| \, dt ds + \frac{r}{3} \sum_{i=1}^m \sum_{j=1}^{N_i} |\alpha_j^i - \alpha_{j+1}^i| \right) = 0.
\end{aligned}$$

The proof is complete.  $\square$

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