A RELAXATION RESULT FOR A SECOND ORDER ENERGY OF MAPPINGS INTO THE SPHERE

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Abstract. A relaxation problem for maps from n-dimensional domains into the unit 2-sphere is analysed, the energy being given in the smooth case by the integral of the modulus of the Laplacean vector. For second order Sobolev maps, a complete explicit formula of the relaxed energy is obtained. Our proof is based on the following results: minimal energy computation of maps with fixed degree, Dipole-like problems, density of maps with small singular sets, lower semicontinuity of the extended energy, and strong approximation properties on Cartesian currents.

Introduction

First order variational problems for maps taking values into isometrically embedded Riemannian manifolds $\mathcal{N}$ are widely studied, a relevant model being given by the Dirichlet integral

$$\mathbb{D}(u) := \frac{1}{2} \int_{B^n} |Du|^2 \, dx \tag{0.1}$$

of maps from the unit ball B^n into the $\mathfrak{p}$-dimensional unit sphere $\mathcal{N} = \mathbb{S}^{\mathfrak{p}}$.

When e.g. $n = 3$ and $\mathfrak{p} = 2$, unit vector fields minimizing the Dirichlet energy (under prescribed boundary conditions) represent a simplified model for the Ericksen-Leslie theory of liquid crystals, see [23] or [25, Sec. 5.1].

Harmonic maps u with values into the sphere $\mathbb{S}^{\mathfrak{p}}$ satisfy the Euler-Lagrange system $\tau(u) = 0$, where

$$\tau(u) := \Delta u + |Du|^2 u \tag{0.2}$$

is the *intrinsic* Laplacean, or *tension field*, compare [25, Sec. 3.1.1]. More precisely, viewing the $\mathfrak{p}$-sphere as embedded into the Euclidean space $\mathbb{R}^{\mathfrak{p}+1}$, and working with maps $u : B^n \to \mathbb{R}^{\mathfrak{p}+1}$ such that $|u(x)| \equiv 1$, then Δu is the *Laplacean vector* in $\mathbb{R}^{\mathfrak{p}+1}$, and its normal component to $\mathbb{S}^{\mathfrak{p}}$ at $u(x)$ is $(\Delta u)^\perp = -|Du|^2 \, u$, whence $\tau(u)$ is the tangential component of Δu, and

$$|\Delta u|^2 = |Du|^4 + |\tau(u)|^2 \, . \tag{0.3}$$

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In dimension $n = 2$, the Dirichlet integral is conformally invariant. Therefore, its second order analogous is probably given by the *bienergy* functional

$$\mathbb{H}(u) := \int_{B^n} |\Delta u|^2 \, dx \tag{0.4}$$

of maps u from B^n into $\mathbb{S}^{\mathfrak{p}}$. In dimension $n = 4$, in fact, the bienergy functional is conformally invariant. In addition, equation (0.3) implies the lower bound

$$\mathbb{H}(u) \geq \int_{B^n} |Du|^4 \, dx$$

where equality holds when $\tau(u) = 0$, i.e., for harmonic maps.

As a consequence, in any dimension $n \geq \mathfrak{p} \geq 2$, Sobolev maps u from B^n into $\mathbb{S}^{\mathfrak{p}}$ with finite bienergy belong to the Sobolev class $W^{1,4}(B^n, \mathbb{S}^{\mathfrak{p}})$. Moreover, when $n = \mathfrak{p}$, by the parallelogram inequality the Jacobian of a smooth map u from $\mathbb{R}^{\mathfrak{p}}$ into $\mathbb{S}^{\mathfrak{p}}$ satisfies the pointwise upper bound

$$J_{\mathfrak{p}} u \leq \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du|^{\mathfrak{p}} \tag{0.5}$$

where equality holds if and only if u is conformal.

Therefore, when in particular $\mathfrak{p} = 4$, it turns out that in any dimension n the “graph” in $B^n \times \mathbb{S}^4$ of a Sobolev map $u \in W^{2,1}(B^n, \mathbb{S}^4)$ with finite bienergy has finite “area” .

However, in view of analyzing the corresponding *relaxation problem*, a nontrivial open question comes into play: *finding a bienergy minimizer among smooth maps from $\mathbb{R}^4$ into $\mathbb{S}^4$ of* degree *one*, see Sec. 8.

One expects that it is given by the inverse σ_4^{-1} of the stereographic projection map from $\mathbb{S}^4$ to $\mathbb{R}^4$, compare (1.1). In fact, Angelsberg [5] showed that the energy minimum among degree one maps is attained and it is greater than $16 \cdot \mathcal{H}^4(\mathbb{S}^4)$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. Moreover, recalling that $\int_{\mathbb{R}^4} |\Delta \sigma_4^{-1}|^2 \, dx = 24 \cdot \mathcal{H}^4(\mathbb{S}^4)$, Cooper [15] proved that σ_4^{-1} minimizes the bienergy among degree one $O(4)$-equivariant maps from $\mathbb{R}^4$ into $\mathbb{S}^4$.

Laplacean energy. In this paper, we consider the functional

$$\mathbb{L}(u) := \int_{B^n} |\Delta u| \, dx$$

on maps $u : B^n \to \mathbb{S}^2$ taking values into the unit 2-sphere of $\mathbb{R}^3$. It will be called *Laplacean energy.* If u is sufficiently smooth, by (0.3) we get

$$|\Delta u| \geq |Du|^2 \tag{0.6}$$

where equality holds for harmonic maps. In addition, by (0.5), where $\mathfrak{p} = 2$, it turns out that the “graph” of u has finite “area” in $B^n \times \mathbb{S}^2$.

Minimal energy of degree one maps. Differently to the nontrivial case of the bienergy of maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, we now see that *the minimal*

Laplacean energy among degree one maps from $\mathbb{R}^2$ *into* $\mathbb{S}^2$ *is attained by the inverse* σ_2^{-1} *of the stereographic map, where*

$$\int_{\mathbb{R}^2} |\Delta\sigma_2^{-1}|\,dx = \int_{\mathbb{R}^2} |D\sigma_2^{-1}|^2\,dx = 2\int_{\mathbb{R}^2} J_2\sigma_2^{-1}\,dx = 2\,\mathcal{H}^2(\mathbb{S}^2) = 8\pi\,. \tag{0.7}$$

More precisely, we denote

$$W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2) := \{u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n,\mathbb{S}^2) \,:\, \Delta u \in L^1(\mathbb{R}^n,\mathbb{R}^3)\} \tag{0.8}$$

and also

$$\mathbb{L}(u,\mathbb{R}^n) := \int_{\mathbb{R}^n} |\Delta u|\,dx\,,\quad u \in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)\,. \tag{0.9}$$

If $u \in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)$, inequality (0.6) holds $\mathcal{L}^n$-a.e. in B^n, where $\mathcal{L}^n$ is the Lebesgue measure, and hence $|Du| \in L^2(\mathbb{R}^n)$. In particular, in low dimension $n = 2$, by (0.5), with $\mathfrak{p} = 2$, any map $u \in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ satisfies the energy lower bound

$$\mathbb{L}(u,\mathbb{R}^2) \geq \int_{\mathbb{R}^2} |Du|^2\,dx \geq 2\int_{\mathbb{R}^2} J_2 u\,dx \tag{0.10}$$

where both inequalities are equalities if u is harmonic and conformal, and that is the case of the inverse σ_2^{-1} of the stereographic map $\sigma_2 : \mathbb{S}^2 \to \mathbb{R}^2$, compare (1.1). On the other hand, maps $u \in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ have a well-defined integer degree, and degree one maps as e.g. $u = \sigma_2^{-1}$ satisfy inequality $\int_{\mathbb{R}^2} J_2 u\,dx \geq 4\pi$, whence $\mathbb{L}(u,\mathbb{R}^2) \geq 8\pi = \mathbb{L}(\sigma_2^{-1},\mathbb{R}^2)$, by (0.10) and (0.7).

Relaxed energy. Following the classical Lebesgue-Serrin approach, we introduce in any dimension $n \geq 2$ the *relaxed energy*

$$\widetilde{\mathbb{L}}(u) := \inf\Big\{\liminf_{h\to\infty} \mathbb{L}(u_h) \mid \quad \{u_h\} \subset C^\infty(B^n,\mathbb{S}^2),\ u_k \to u \text{ strongly in } L^1(B^n,\mathbb{R}^3)\Big\} \tag{0.11}$$

of maps u in $L^1(B^n,\mathbb{S}^2)$. Our first objective is to analyze the explicit formula of $\widetilde{\mathbb{L}}(u)$ on the class of maps with finite relaxed energy. We thus denote:

$$\mathbb{L}(B^n,\mathbb{S}^2) := \{\, u \in L^1(B^n,\mathbb{S}^2) \mid \widetilde{\mathbb{L}}(u) < \infty\,\} \tag{0.12}$$

and refer to Sec. 2 for details on the following preliminary discussion.

If $u \in \mathbb{L}(B^n,\mathbb{S}^2)$, inequality (0.6) implies that $u \in W^{1,2}(B^n,\mathbb{S}^2)$, whence the *distributional divergence* of the gradient Du is well defined by

$$\langle \mathrm{Div} Du;\varphi\rangle := -\int_{B^n} \mathrm{tr}\,[Du\,(D\varphi)^t]\,dx\,,\quad \varphi \in C^\infty_c(B^n,\mathbb{R}^3) \tag{0.13}$$

where $A \mapsto A^t$ is the transpose operator in $\mathbb{R}^{3\times n}$ and $B \mapsto \mathrm{tr}\,B$ the trace operator in $\mathbb{R}^{3\times 3}$. By lower semicontinuity, we have:

$$\widetilde{\mathbb{L}}(u) \geq |\mathrm{Div} Du|(B^n) \quad \forall\, u \in \mathbb{L}(B^n,\mathbb{S}^2) \tag{0.14}$$

and hence DivDu is a *finite* $\mathbb{R}^3$*-valued regular measure*. Since moreover

$$\mathrm{Div}Du = \Delta u\,\mathcal{L}^n \llcorner B^n \quad \forall\, u \in W^{2,1}(B^n,\mathbb{S}^2)$$

the measure DivDu may be called a *weak Laplacean*.

In the critical dimension $n = 2$, due to the continuous embedding of $W^{1,2}(B^2)$ in the class VMO of functions with *vanishing mean oscillation*, by Schoen-Uhlenbeck density theorem [33] it turns out that there is no gap:

$$\widetilde{\mathbb{L}}(u) = |\mathrm{Div}Du|(B^2) \quad \forall\, u \in \mathbb{L}(B^2,\mathbb{S}^2)\,. \tag{0.15}$$

In high dimension $n \geq 3$, the energy gap is positive, in general, i.e., strict inequality holds in (0.14). However, for a generic map $u \in \mathbb{L}(B^n,\mathbb{S}^2)$, it is an open problem to find the explicit formula of the relaxed energy (0.11).

This is essentially due to a lack of information concerning the structure of the measure DivDu. For that reason, in this paper we shall focus on the more regular subclass of second order Sobolev maps, since

$$|\mathrm{Div}Du|(B^n) = \int_{B^n} |\Delta u|\,dx =: \mathbb{L}(u) \quad \forall\, u \in W^{2,1}(B^n,\mathbb{S}^2)\,. \tag{0.16}$$

Main Result. For maps u in $W^{2,1}(B^n,\mathbb{S}^2)$, when $n \geq 3$, we shall see that the energy gap only depends (up to the constant factor 8π) on the *mass* $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ of a *minimal connection* of the *current of the singularities* of u.

Referring to Sec. 1 for the precise notation, we only mention here that the *relevant singularities* of maps $u \in W^{2,1}(B^n,\mathbb{S}^2)$ are described by an *integral flat* $(n-3)$-chain $\mathbb{P}(u)$ in B^n. This means that the current $\mathbb{P}(u)$ is the boundary in B^n of an integer multiplicity (say i.m.) rectifiable $(n-2)$-current L, and the *integral mass* $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is the mass of a minimizer among $L \in \mathcal{R}_{n-2}(B^n)$ satisfying $(\partial L) \llcorner B^n = -\mathbb{P}(u)$.

If e.g. $n = 3$ and u is the harmonic map $u(x) = x/|x|$, then $|\Delta u| = |Du|^2 = 2/|x|^2$, and on account of (1.7) the current of the singularities is such that $-\mathbb{P}(u) = \delta_O$, the unit Dirac mass at the origin O, whence $\mathbf{m}_{i,B^3}(\mathbb{P}(u))$ is equal to the length of a segment connecting O to a point at the boundary of B^3, a so called *string* in the sense of Brezis-Coron-Lieb [13].

The Main Result of this paper is enclosed in the following theorem, where we are able to give an explicit formula for the relaxed energy (0.11) of maps in the Sobolev class $W^{2,1}(B^n,\mathbb{S}^2)$.

Theorem 0.1. *If $u \in W^{2,1}(B^n,\mathbb{S}^2)$, where $n \geq 3$, then*

$$\widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty\,.$$

In the proof of Theorem 0.1, we rely on the previous observation concerning the minimal Laplacean energy of degree one maps, and we follow a similar strategy to the one exploited in case of the Dirichlet energy (0.1). In particular, we make use of tools from the theory of *Cartesian currents* by Giaquinta-Modica-Souček [24, 25].

Finally, concerning the wider class

$$\mathbb{L}_{BV}(B^n,\mathbb{S}^2) := \big\{u \in \mathbb{L}(B^n,\mathbb{S}^2) \mid Du \in \mathrm{BV}(B^n,\mathbb{R}^{3\times 2})\big\} \tag{0.17}$$

in Sec. 8 we prove in any dimension $n \geq 3$ the lower bound

$$\widetilde{\mathbb{L}}(u) \geq |\mathrm{Div}Du|(B^n) + 8\pi\cdot \mathbf{m}_{i,B^n}(B^n) \quad \forall\, u \in \mathbb{L}_{BV}(B^n,\mathbb{S}^2) \tag{0.18}$$

and explain why equality is expected to hold true in the latter formula.

Content of the paper. In Sec. 1, we collect some notation and background material, focusing in particular on the analogous relaxation problem for the Dirichlet integral (0.1) of mappings into the 2-sphere.

In Sec. 2, we preliminarily discuss some general properties of maps with finite relaxed energy, explaining the difficulties that one encounters in the general case when $u \notin W^{2,1}(B^n,\mathbb{S}^2)$ and $n \geq 3$. We also prove a lower semicontinuity result in dimension $n = 2$, Theorem 2.2.

In Sec. 3, we introduce a suitable modification of the inverse to the stereographic map, Proposition 3.1. We then compute the *minimal Laplacean energy* among maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ with fixed degree, Theorem 3.2, and describe the related bubbling phenomenon.

In Sec. 4, we extend to the Laplacean energy the classical *Dipole problem* of Brezis-Coron-Lieb [13] for the Dirichlet energy in 3D, Theorem 4.1.

In Sec. 5, we find a dense class of maps which are smooth outside a small singular set, Theorem 5.2. We then provide a cohomological criterion for strong density of smooth maps, Theorem 5.4.

In Sec. 6, we introduce the class $\mathrm{cart}^{\mathbb{L}}(B^n\times\mathbb{S}^2)$ of Cartesian currents with underlying functions in $W^{2,1}(B^n,\mathbb{S}^2)$. More precisely, see Definition 6.1, an element T in $\mathrm{cart}^{\mathbb{L}}(B^n\times\mathbb{S}^2)$ is given by

$$T = G_u + L \times [\![\, \mathbb{S}^2 \,]\!]$$

where G_u is the graph current of a map $u \in W^{2,1}(B^n,\mathbb{S}^2)$ and L is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = -\mathbb{P}(u)$, if $n \geq 3$. We then extend the Laplacean energy to a functional $T \mapsto \mathbb{L}(T)$ on Cartesian currents, by letting

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi\cdot\mathbf{M}(L) \quad \text{if} \quad T = G_u + L\times [\![\, \mathbb{S}^2 \,]\!]$$

and prove a weak sequential *lower semicontinuity* property, Theorem 6.3.

In Sec. 7, we deal with the explicit formula for the relaxed energy (0.11). The proof of Theorem 0.1 is based on the lower semicontinuity theorem 6.3 and on a *strong density result*. In Theorem 7.1, in fact, we show that each current in $\mathrm{cart}^{\mathbb{L}}(B^n\times\mathbb{S}^2)$ can be approximated weakly and with energy convergence by a sequence of smooth graphs. A shorter proof of Theorem 7.1 in low dimension $n = 3$ is given in App. A, whereas in high dimension we make use of the *approximation theorem* 7.3, whose proof is reported in Appendices B and C.

Final remarks and open questions are reported in Sec. 8.

1. Background material and preliminary results

In this section, we collect some well-known facts about stereographic maps, divergence-measure fields, and topics from Geometric Measure Theory, degree, Cartesian currents, singularities (for which we refer to the treatise [24, 25] or to [28]). We then describe the strong density and relaxation results for the Dirichlet energy of maps into the 2-sphere.

Let B^n be the open unit ball of dimension $n \geq 2$ centered at the origin, and $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. For $X = L^1$, $W^{k,p}$, or C^∞, we denote

$$X(B^n, \mathbb{S}^2) := \{u \in X(B^n, \mathbb{R}^3) \,:\, |u(x)| = 1 \text{ for } \mathcal{L}^n\text{-a.e. } x \in B^n\}\,.$$

Stereographic projection. For $\mathfrak{p} \geq 2$ integer, setting

$$\mathbb{S}^{\mathfrak{p}} := \{(y,z) \mid y \in \mathbb{R}^{\mathfrak{p}}\,,\ z \in \mathbb{R}\,,\ |(y,z)| = 1\} \subset \mathbb{R}^{\mathfrak{p}+1}$$

the stereographic projection from the "South Pole" $P_S := (0_{\mathbb{R}^{\mathfrak{p}}}, -1)$ is given by $\sigma_{\mathfrak{p}}(y,z) := \dfrac{y}{1+z}$. Its inverse $\sigma_{\mathfrak{p}}^{-1} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ satisfies

$$\sigma_{\mathfrak{p}}^{-1}(x) = \Big(\frac{2x}{1+\rho^2}, \frac{1-\rho^2}{1+\rho^2}\Big)\,, \qquad x \in \mathbb{R}^{\mathfrak{p}}\,, \quad \rho := |x|\,. \tag{1.1}$$

The map $(-1)^{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is an orientation preserving conformal diffeomorphism from $\mathbb{R}^{\mathfrak{p}}$ onto $\mathbb{S}^{\mathfrak{p}} \setminus \{P_S\}$. In fact, denoting by $\bullet$ the scalar product in $\mathbb{R}^{\mathfrak{p}+1}$ and by δ^i_j the Kronecker symbol, the conformality relations

$$\partial_i\sigma_{\mathfrak{p}}^{-1} \bullet \partial_j\sigma_{\mathfrak{p}}^{-1} = \delta^i_j\, U^2 \quad \forall\, i,j = 1,\ldots,\mathfrak{p}$$

hold, with scaling factor $U(x) := \dfrac{2}{1+|x|^2}$, whence in (0.5) one has

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}|D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}} = J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1} = U^{\mathfrak{p}}$$

where $J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is the Jacobian of $\sigma_{\mathfrak{p}}^{-1}$. As a consequence, concerning the *conformal Dirichlet integral*, for any $\mathfrak{p} \geq 2$ integer one obtains:

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{\mathbb{R}^{\mathfrak{p}}} |D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}}\,dx = \int_{\mathbb{R}^{\mathfrak{p}}} J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}\,dx = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \tag{1.2}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.

Most importantly, it turns out that the map $\sigma_{\mathfrak{p}}^{-1}$ is harmonic if and only if $\mathfrak{p} = 2$. Therefore, σ_2^{-1} satisfies the Euler-Lagrange system $\tau(u) = 0$, where $\tau(u)$ is the tension field (0.2). In conclusion, one readily obtains the energy computation (0.7).

Divergence-measure fields. Let $n \geq 2$. The *distributional divergence* of a vector field $F \in L^2(B^n, \mathbb{R}^n)$ is well defined by:

$$\langle \mathrm{Div}F; \phi\rangle := -\int_{B^n} F \cdot D\phi\,dx\,, \quad \phi \in C^\infty_c(B^n)\,.$$

Definition 1.1. We call $F \in L^2(B^n, \mathbb{R}^n)$ a *divergence-measure field*, say $F \in \mathcal{DM}^{1,2}(B^n)$, if $\mathrm{Div}F$ is a real finite Radon measure on B^n.

If $F \in \mathcal{DM}^{1,2}(B^n)$, a decomposition into mutually singular measures

$$\mathrm{Div}F = (\mathrm{Div}F)^a + (\mathrm{Div}F)^s\,, \quad (\mathrm{Div}F)^a = \widetilde{\mathrm{div}F}\,\mathcal{L}^n \llcorner B^n$$

holds, where $\widetilde{\mathrm{div}F} \in L^1(B^n)$ denotes the Radon-Nikodym derivative of $\mathrm{Div}F$ w.r.t. $\mathcal{L}^n$. Referring to [4] for further details on *functions of bounded variations*, we remark that if in addition $F \in \mathrm{BV}(B^n, \mathbb{R}^n)$, the density $\widetilde{\mathrm{div}F}$ agrees with the trace of the approximate gradient matrix ∇F, and that $(\mathrm{Div}F)^s = 0$ if in particular $F \in W^{1,1}(B^n, \mathbb{R}^n)$.

Šilhavý [34, Thm. 3.2] proved the following absolute continuity property:

Proposition 1.2. *If $F \in \mathcal{DM}^{1,2}(B^n)$, then $|\mathrm{Div}F|(B) = 0$ for each Borel set $B \subset B^n$ with σ-finite $\mathcal{H}^{n-2}$-measure. In particular, the measure* $\mathrm{Div}F$ *does not charge any atom.*

By the chain rule formula in BV, cf. [4, Thm. 3.96] and [24, p. 487], it turns out that if $v^1 \in W^{1,2}(B^n) \cap L^\infty(B^n)$ and $v^2 \in \mathrm{BV}(B^n) \cap L^2(B^n)$, then

$$D(v^1 v^2) = v^1 D v^2 + v^2 \nabla v^1\, \mathcal{L}^n \llcorner B^n\,.$$

In this setting, the following version of the Leibnitz-rule is due to Comi [14].

Proposition 1.3. *Let $F \in \mathcal{DM}^{1,2}(B^n)$ and $g \in W^{1,2}(B^n) \cap L^\infty(B^n)$. Then, $gF \in \mathcal{DM}^{1,2}(B^n)$ and*

$$\mathrm{Div}(gF) = \widetilde{g}\,\mathrm{Div}F + F \cdot \nabla g\, \mathcal{L}^n \llcorner B^n$$

where $\widetilde{g}$ is the precise representative of g.

Integer rectifiable currents. For $U \subset \mathbb{R}^m$ an open set, and $k = 0, \ldots, m$, we denote by $\mathcal{D}_k(U)$ the strong dual to the space $\mathcal{D}^k(U)$ of compactly supported smooth k-forms, whence $\mathcal{D}_0(U)$ is the class of distributions in U. For any $T \in \mathcal{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{\langle T; \omega\rangle \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \le 1\}$$

and (for $k \ge 1$) its *boundary* as the $(k-1)$-current ∂T defined by the relation

$$\langle \partial T; \eta\rangle := \langle T; d\eta\rangle \qquad \forall\, \eta \in \mathcal{D}^{k-1}(U)$$

where $d\eta$ is the differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathcal{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h; \omega\rangle = \langle T; \omega\rangle\,, \qquad \forall\, \omega \in \mathcal{D}^k(U)$$

and the mass is sequentially weakly lower semicontinuous, i.e.,

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h) \quad \text{if} \quad T_h \rightharpoonup T\,.$$

For $k \ge 1$, a k-current T with finite mass is called *rectifiable* if

$$\langle T; \omega\rangle = \int_{\mathcal{M}} \theta\, \langle \omega; \xi\rangle\, d\mathcal{H}^k \qquad \forall\, \omega \in \mathcal{D}^k(U)$$

with $\mathcal{M}$ a k-rectifiable set in U, $\xi : \mathcal{M} \to \Lambda_k \mathbb{R}^n$ a $\mathcal{H}^k \llcorner \mathcal{M}$-measurable function such that $\xi(x)$ is a simple unit k-vector in $\Lambda_k \mathbb{R}^n$ orienting the approximate tangent space to $\mathcal{M}$ at $\mathcal{H}^k$-a.e. $x \in \mathcal{M}$, and $\theta : \mathcal{M} \to [0, +\infty)$ a $\mathcal{H}^k \llcorner \mathcal{M}$-summable non-negative function, so that $\mathbf{M}(T) = \int_{\mathcal{M}} \theta \, d\mathcal{H}^k < \infty$. In that case, the short-hand notation $T = [\![\mathcal{M}, \xi, \theta]\!]$ is commonly adopted, and set (T) denotes the set of points in $\mathcal{M}$ with positive multiplicity.

In addition, if θ is integer-valued, the current T is called *integer multiplicity* (in short *i.m.*) *rectifiable* and the corresponding class is denoted by $\mathcal{R}_k(U)$. The usefulness of i.m. rectifiable currents in the Calculus of Variations stems from Federer-Fleming's compactness theorem [18]. It states that if a sequence $\{T_h\} \subset \mathcal{R}_k(U)$ satisfies $\sup_h \big(\mathbf{M}(T_h) + \mathbf{M}((\partial T_h) \llcorner U) \big) < \infty$, where $\llcorner$ denotes restriction, then there exists $T \in \mathcal{R}_k(U)$ and a (not relabeled) subsequence of $\{T_h\}$ such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_k(U)$.

Example 1.4. If $\mathcal{M}$ is a smooth, oriented, compact k-submanifold of U, then $[\![\mathcal{M}]\!]$ is the current in $\mathcal{R}_k(U)$ given by integration of k-forms in the sense of differential geometry, i.e., $\langle [\![\mathcal{M}]\!]; \omega \rangle := \int_{\mathcal{M}} \omega$ for all $\omega \in \mathcal{D}^k(U)$.

Graph currents. If u is a map in $W^{1,1}(B^n, \mathbb{R}^N)$, where $n, N \geq 2$, then u has a Lusin representative on the subset $\widetilde{B}^n$ of Lebesgue points pertaining to both u and the gradient Du, where $\mathcal{L}^n(B^n \setminus \widetilde{B}^n) = 0$. Following [24], the *graph* of u is the countably n-rectifiable subset of $U = B^n \times \mathbb{R}^N$

$$\mathcal{G}_u := \{(x, y) \in B^n \times \mathbb{R}^N \mid x \in \widetilde{B}^n \,, \ y = \widetilde{u}(x)\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. By the area formula, one has $\mathcal{H}^n(\mathcal{G}_u) < \infty$ if in addition all the minors of Du are in $L^1(B^n)$. In that case, u is called a map in $\mathcal{A}^1(B^n, \mathbb{R}^N)$. More precisely, the approximate tangent n-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_i(x) = (\mathbf{e}_i, \partial_i u(x)) \in \mathbb{R}^{n+N}$, for $i = 1, \ldots, n$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the canonical basis in $\mathbb{R}^n$ and the partial derivative $\partial_i u(x)$ is the i-th column vector of the gradient matrix $Du(x)$ given by the Lebesgue value of Du at $x \in \widetilde{B}^n$. Therefore, the unit n-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)}{|\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)|} \in \Lambda_n \mathbb{R}^{n+N} \,, \quad x \in \widetilde{B}^n$$

provides an orientation to $\mathcal{G}_u$, and the *graph current* $G_u = [\![\mathcal{G}_u, \xi, 1]\!]$ is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$.

The action of G_u can be read (in an approximate $\mathcal{L}^n$-a.e. sense) through the pull-back of the *graph map* $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ by:

$$\langle G_u; \omega \rangle = \int_{B^n} (\mathrm{Id} \bowtie u)^{\#} \omega \qquad \forall \, \omega \in \mathcal{D}^n(B^n \times \mathbb{R}^N) \,. \tag{1.3}$$

Therefore, the mass of G_u is equal to the *graph area* $\mathbb{A}(u)$, i.e.,

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = \mathbb{A}(u) := \int_{B^n} |M(Du)| \, dx < \infty \tag{1.4}$$

where $|M(Du)|$ is the Jacobian of $\mathrm{Id} \bowtie u$, so that $|M(Du)|^2$ is equal to 1 plus the sum of the square of all minors of the $N \times n$ matrix Du.

Let now $N=3$ and $u\in W^{1,2}(B^n,\mathbb{S}^2)$. If $n\geq 3$, by the area formula all the 3×3 minors of Du are zero $\mathcal{L}^n$-a.e. in B^n. Therefore, for any $n\geq 2$ the Jacobian $|M(Du)|$ is $\mathcal{L}^n$-essentially bounded (up to an absolute constant factor c_n only depending on the dimension n) by $1+|Du|^2$, where $Du\in L^2(B^n,\mathbb{R}^{3\times n})$. Whence, $u\in\mathcal{A}^1(B^n,\mathbb{R}^3)$ and by (1.4) we get

$$\mathbf{M}(G_u)=\mathbb{A}(u)\leq c_n\int_{B^n}(1+|Du|^2)\,dx<\infty\,. \tag{1.5}$$

In addition, by Federer's flatness theorem, the graph current G_u is an i.m. rectifiable current in $B^n\times\mathbb{S}^2$, say $G_u\in\mathcal{R}_n(B^n\times\mathbb{S}^2)$.

Now, if $u\in W^{1,2}(B^n,\mathbb{S}^2)$ is smooth, we have $G_u=[\![\,\mathcal{G}_u\,]\!]$, see (1.4), where the graph manifold $\mathcal{G}_u$ has no "fractures" or "holes". By Stokes' theorem, such a condition is read in terms of the graph current G_u by the property:

$$\langle\partial G_u;\eta\rangle:=\langle G_u;d\eta\rangle=0\qquad\forall\,\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^2)\,. \tag{1.6}$$

Remark 1.5. Given $u\in W^{1,2}(B^n,\mathbb{S}^2)$, assume that there exists a sequence of smooth maps $\{u_h\}\subset C^\infty(B^n,\mathbb{S}^2)$ such that $u_h\to u$ strongly in $W^{1,2}(B^n,\mathbb{R}^3)$. We recall that this is always the case in the critical dimension $n=2$, by the continuous embedding of $W^{1,2}(B^2)$ in VMO. Strong $W^{1,2}$ convergence implies that $G_{u_h}\rightharpoonup G_u$ weakly as currents in $\mathcal{D}_n(B^n\times\mathbb{S}^2)$, a convergence that preserves the homological property (1.6). Therefore, we conclude that the map u satisfies the *null-boundary condition* (1.6).

Remark 1.6. Condition (1.6) is violated in high dimension $n\geq 3$, in general. If e.g. $n=3$, the 0-homogeneous harmonic map $u(x)=x/|x|$ belongs to the class $W^{1,2}(B^3,\mathbb{S}^2)$, and one has (cf. [24, Sec. 3.2.2, Ex. 1])

$$(\partial G_u)\llcorner B^3\times\mathbb{S}^2=-\delta_O\times[\![\,\mathbb{S}^2\,]\!] \tag{1.7}$$

where δ_O is the unit Dirac mass at the origin O. Therefore, one cannot find a sequence $\{u_h\}\subset C^\infty(B^3,\mathbb{S}^2)$ strongly converging to u in $W^{1,2}(B^3,\mathbb{R}^3)$.

Remark 1.7. For maps $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{S}^2)$ with $|Du|\in L^2(\mathbb{R}^n)$, we denote

$$\mathbb{D}(u,\mathbb{R}^n):=\frac{1}{2}\int_{\mathbb{R}^n}|Du|^2\,dx \tag{1.8}$$

and notice that this time the graph current G_u is locally i.m. rectifiable in $\mathbb{R}^n\times\mathbb{S}^2$, i.e., $G_u\llcorner\Omega\times\mathbb{S}^2\in\mathcal{R}_n(\Omega\times\mathbb{S}^2)$ for each bounded open set $\Omega\subset\mathbb{R}^n$.

DEGREE. In dimension $n=2$, the degree of maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ is well defined provided that $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{S}^2)$ with $|Du|\in L^2(\mathbb{R}^2)$. In fact, by Remark 1.7 the graph current G_u is locally i.m. rectifiable. In addition, it satisfies the null-boundary condition

$$\langle\partial G_u;\omega\rangle=0\qquad\forall\,\omega\in\mathcal{D}^1(\mathbb{R}^2\times\mathbb{S}^2)\,.$$

Therefore, denoting by $\Pi_y(x,y):=y$ the orthogonal projection onto the target space $\mathbb{S}^2\subset\mathbb{R}^3$, the image current $\Pi_{y\#}G_u$ is an *integral* 2-*cycle* in $\mathbb{S}^2$, i.e., $\Pi_{y\#}G_u\in\mathcal{R}_2(\mathbb{S}^2)$ with $\partial(\Pi_{y\#}G_u)=0$. By the constancy theorem, compare [24, Sec. 4.3.1, Thm. 4], we thus have $\Pi_{y\#}G_u = \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!]$ for some integer $\mathrm{d} \in \mathbb{Z}$. Moreover, if ω_2 denotes the volume 2-form on $\mathbb{S}^2$

$$\omega_2 := y^1 dy^2 \wedge dy^3 + y^2 dy^3 \wedge dy^1 + y^3 dy^1 \wedge dy^2 \tag{1.9}$$

by the action (1.3) we have

$$\int_{\mathbb{R}^2} u^{\#}\omega_2 = \langle \Pi_{y\#}G_u ; \omega_2 \rangle = \langle \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!] ; \omega_2 \rangle = \mathrm{d} \int_{\mathbb{S}^2} \omega_2 = \mathrm{d} \cdot 4\pi\,.$$

Definition 1.8. Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ with $|Du| \in L^2(\mathbb{R}^2)$. We call *degree* $\deg u$ of u the integer $\mathrm{d} \in \mathbb{Z}$ given by formula

$$\deg u := \frac{1}{4\pi} \int_{\mathbb{R}^2} u^{\#}\omega_2 = \mathrm{d}\,.$$

Notice that the degree is *strongly continuous*: if $\{u_h\} \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ is a sequence converging to $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ a.e. in $\mathbb{R}^2$, with $Du_h \to Du$ strongly in $L^2(\mathbb{R}^2, \mathbb{R}^{3\times 2})$, by dominated convergence we get

$$\lim_{h\to\infty} \frac{1}{4\pi} \cdot \Big| \int_{\mathbb{R}^2} (u_h^{\#}\omega_2 - u^{\#}\omega_2) \Big| = 0$$

and hence $\deg u_h = \deg u$, for h large enough.

Cartesian currents. Let $n \geq 2$ and $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ be a sequence of smooth maps with equibounded Dirichlet energies, $\sup_h \mathbb{D}(u_h) < \infty$, see (0.1). The graph currents G_{u_h} belong to $\mathcal{R}_n(B^n \times \mathbb{S}^2)$ and satisfy condition (1.6) and $\sup_h \mathbf{M}(G_{u_h}) < \infty$, by (1.5). Therefore, Federer-Fleming's theorem [18] yields that the currents G_{u_h} subconverge weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to a current $T \in \mathcal{R}_n(B^n \times \mathbb{S}^2)$ satisfying the *null-boundary condition*

$$(\partial T) \llcorner B^n \times \mathbb{S}^2 = 0\,. \tag{1.10}$$

In addition, compare [24, 28], there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ and a map $u_T \in W^{1,2}(B^n, \mathbb{S}^2)$ such that

$$T = G_{u_T} + L \times [\![\,\mathbb{S}^2\,]\!] \tag{1.11}$$

where the *underlying function* u_T is given by the weak $W^{1,2}$ limit of the u_h's. Finally, in low dimension $n = 2$ we also have $(\partial G_{u_T}) \llcorner B^2 \times \mathbb{S}^2 = 0$. For that reason, Giaquinta-Modica-Souček [21] introduced the following

Definition 1.9. The class $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ is given by the i.m. rectifiable currents $T \in \mathcal{R}_n(B^n \times \mathbb{S}^2)$ satisfying the null-boundary condition (1.10) and the structure property (1.11) for some Sobolev map u_T in $W^{1,2}(B^n, \mathbb{S}^2)$ and some i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$.

The Dirichlet energy of a current T in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ is given by

$$\mathbb{D}(T) := \frac{1}{2} \int_{B^n} |Du_T|^2\, dx + 4\pi \cdot \mathbf{M}(L) \qquad \text{if (1.11) holds}\,.$$

Since the functional $T \mapsto \mathbb{D}(T)$ agrees with the *parametric polyconvex lower semicontinuous extension* of the Dirichlet integral, compare [25, Sec. 2.2.4], dealing with currents in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ it turns out that if $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$, then

$$\mathbb{D}(T) \le \liminf_{h\to\infty} \mathbb{D}(T_h)\,. \tag{1.12}$$

Finally, a sequential weak closure property holds: if a sequence $\{T_h\} \subset \text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ satisfies $\sup_h \mathbb{D}(T_h) < \infty$, then there exists a current T in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ and a (not relabeled) subsequence such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$.

Current of the singularities. Let $u \in W^{1,2}(B^n, \mathbb{S}^2)$, where $n \ge 3$. Following [25, Sec. 4.2.5], we denote by $\mathbb{P}(u)$ the $(n-3)$-current in $\mathcal{D}_{n-3}(B^n)$

$$\langle \mathbb{P}(u); \varphi\rangle := \frac{1}{4\pi}\int_{B^n} u^{\#}\omega_2 \wedge d\varphi\,, \qquad \varphi \in \mathcal{D}^{n-3}(B^n) \tag{1.13}$$

where ω_2 is the volume 2-form (1.9). It turns out that the boundary of the graph current G_u satisfies equation

$$(\partial G_u) \llcorner B^n \times \mathbb{S}^2 = \mathbb{P}(u) \times [\![\, \mathbb{S}^2 \,]\!]\,. \tag{1.14}$$

Therefore, for a current $T \in \text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ as in (1.11), the null boundary condition (1.10) is equivalent to the following link between $\mathbb{P}(u_T)$ and L:

$$(\partial L) \llcorner B^n = -\mathbb{P}(u_T)\,. \tag{1.15}$$

Real and integral mass. The latter formula motivates the introduction of some more notation. Let again $n \ge 3$.

Definition 1.10. For any current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$, we denote by

$$\mathbf{m}_{r,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-2}(B^n)\,,\ (\partial D)\llcorner B^n = -\mathbb{P}\} \tag{1.16}$$

the *real mass of* $\mathbb{P}$ allowing connections to the boundary. We also define

$$\mathbf{m}_{i,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-2}(B^n)\,,\ (\partial L)\llcorner B^n = -\mathbb{P}\}\,. \tag{1.17}$$

Remark 1.11. By Federer-Fleming's theorem [18], if there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ such that $(\partial L)\llcorner B^n = -\mathbb{P}$, the minimum in (1.17) is attained. In that case, $\mathbf{m}_{i,B^n}(\mathbb{P})$ is called *integral mass*, and a minimizer L a *minimal integral connection of* $\mathbb{P}$ (allowing connections to the boundary).

Example 1.12. If e.g. $u(x) = x/|x|$, by (1.7) and (1.14) we get $\mathbb{P}(u) = -\delta_O$, and hence the integral mass $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is equal to the length of any line segment connecting a point at the boundary of B^3 to the origin O.

Remark 1.13. In dimension $n = 3$, Federer's theorem [17], compare [25, Sec. 3.1.4, Thm. 8], gives that if $\mathbf{m}_{i,B^3}(\mathbb{P}) < \infty$ for some $\mathbb{P} \in \mathcal{D}_0(B^3)$, then

$$\mathbf{m}_{i,B^3}(\mathbb{P}) = \mathbf{m}_{r,B^3}(\mathbb{P})\,. \tag{1.18}$$

This is false in general when $n \ge 4$. More precisely, compare [31, 35], for a current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$ with $\mathbf{m}_{i,B^n}(\mathbb{P}) < \infty$, it may happen that

$$\mathbf{m}_{r,B^n}(\mathbb{P}) < \mathbf{m}_{i,B^n}(\mathbb{P}) \quad \text{if} \quad n \ge 4\,.$$

Maps with small singular set. Due to the non-triviality of the second homotopy group $\pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$, in dimension $n \geq 3$ it is false that the class of smooth maps $C^\infty(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$. However, a wider class of maps with small singular set is dense.

Definition 1.14. For $n \geq 3$, we denote by $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ the class of maps $u : \bar{B}^n \to \mathbb{S}^2$ which are smooth on $\bar{B}^n \setminus S_u$, where S_u is a finite union of $(n-3)$-dimensional smooth sets with smooth boundary (a finite set of points when $n = 3$) and such that for every positive integer k there exists a positive real constant c, depending on u and k, such that the k-th order derivative

$$|D^k u(x)| \leq \frac{c}{(\text{dist}\,(x, S_u))^k} \qquad \forall\, x \in \bar{B}^n \setminus S_u\,.$$

The following density property was proved in case $n = 3$ by Bethuel-Zheng [10], and extended to high dimension $n \geq 3$ by Bethuel [7].

Theorem 1.15. *The class $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$.*

Point singularities. Let $n = 3$ and assume that $u \in W^{1,2}(B^3, \mathbb{S}^2)$ is smooth outside a finite set S_u, compare Definition 1.14. For any singular point $a \in S_u$ and for $r > 0$ small, the restriction $u_{|\partial B^3_r(a)}$ of u to the boundary of the ball $B^3_r(a) := a + B^3_r$ is a smooth function. Therefore, arguing as before it turns out that the *degree of u at a* is well defined by the integer

$$\deg(u, a) := \frac{1}{4\pi} \int_{\partial B^3_r(a)} u^\# \omega_2 = \mathrm{d} \in \mathbb{Z}\,. \tag{1.19}$$

In fact, standard homotopy arguments imply that definition (1.19) does not depend on the choice of the (small) radius, whence it agrees with the classical Brouwer degree. Moreover, if $S_u = \{a_i\}_{i=1}^m$ and $\deg(u, a_i) = \mathrm{d}_i$, similarly to [25, Sec. 4.2.1, Prop. 1] we infer:

$$(\partial G_u) \llcorner B^3 \times \mathbb{S}^2 = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Therefore, formula (1.14) implies that the current of the singularities $\mathbb{P}(u)$ is i.m. rectifiable:

$$\mathbb{P}(u) = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \in \mathcal{R}_0(B^3)\,.$$

As e.g. to the 0-homogeneous map $u(x) = x/|x|$, one has $S_u = \{O\}$, the boundary condition (1.7) holds, $\mathbb{P}(u) = -\delta_O$, and $\deg(u, O) = 1$.

Relaxed Dirichlet energy. Similarly to (0.11), the relaxed Dirichlet energy of maps u in $L^1(B^n, \mathbb{S}^2)$ is defined by

$$\widetilde{\mathbb{D}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{D}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^2),\ u_k \to u \text{ in } L^1(B^n, \mathbb{R}^3) \Big\}\,.$$

In dimension $n = 2$, we clearly have

$$\widetilde{\mathbb{D}}(u) = \begin{cases} \mathbb{D}(u) & \text{if } u \in W^{1,2}(B^n, \mathbb{S}^2) \\ +\infty & \text{if } u \in L^1(B^n, \mathbb{S}^2) \setminus W^{1,2}(B^n, \mathbb{S}^2) \,. \end{cases}$$

In dimension $n = 3$, following Brezis-Coron-Lieb [13], the *flat norm* $\mathbf{L}(u)$ of $u \in W^{1,2}(B^3, \mathbb{S}^2)$ (relative to the boundary) is given by

$$\mathbf{L}(u) := \frac{1}{4\pi} \cdot \sup_{\xi \in \mathcal{F}} \int_{B^3} D(u) \bullet D\xi \, dx \tag{1.20}$$

where $\bullet$ is the scalar product in $\mathbb{R}^3$. In the latter formula, $\mathcal{F}$ denotes the class of smooth test functions $\xi : B^3 \to \mathbb{R}$ such that $\|\xi\|_\infty \leq 1$ and $\|D\xi\|_\infty \leq 1$, and $D(u) : B^3 \to \mathbb{R}^3$ the *D-field*

$$D(u) := \big(u \bullet \partial_2 u \times \partial_3 u,\, u \bullet \partial_3 u \times \partial_1 u,\, u \bullet \partial_1 u \times \partial_2 u \big) \,.$$

Bethuel-Brezis-Coron [8] showed that for any $u \in W^{1,2}(B^3, \mathbb{S}^2)$ the relaxed Dirichlet energy is finite, and it satisfies the explicit formula

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{L}(u) \,.$$

Following Giaquinta-Modica-Souček [22], as distributions of $\mathcal{D}_0(B^3)$ one gets $\mathbb{P}(u) = \frac{1}{4\pi} \operatorname{Div} D(u)$, i.e.,

$$\langle \mathbb{P}(u); \varphi \rangle = -\frac{1}{4\pi} \int_{B^3} D(u) \bullet D\varphi \, dx \qquad \forall\, \varphi \in C_c^\infty(B^3) \,.$$

Equivalently, the current $\mathbf{D}(u) \in \mathcal{D}_1(B^3)$ given by

$$\langle \mathbf{D}(u); \eta \rangle := \frac{1}{4\pi} \int_{B^3} u^\# \omega_2 \wedge \eta \,, \qquad \eta \in \mathcal{D}^1(B^3)$$

is such that $(\partial \mathbf{D}(u)) \llcorner B^3 = \mathbb{P}(u)$ and $\mathbf{M}(\mathbf{D}(u)) < \infty$. Moreover, a duality argument yields that the minimal real connection $\mathbf{m}_{r,B^3}(\mathbb{P}(u))$ of the singularities agrees with the flat norm $\mathbf{L}(u)$, compare [25, Sec. 4.2.5].

Most importantly, in [22] the authors obtained that the flat norm agrees with the integral mass of the current of the singularities, i.e.,

$$\mathbf{L}(u) = \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty \qquad \forall\, u \in W^{1,2}(B^3, \mathbb{S}^2) \,.$$

Their argument relies on Theorem 1.15 and on the following result:

Proposition 1.16. *Let $u \in W^{1,2}(B^3, \mathbb{S}^2)$ and $\{u_k\} \subset W^{1,2}(B^3, \mathbb{S}^2) \cap R_0^\infty$ be such that $u_k \to u$ strongly in $W^{1,2}$. Then, for each k there exists an i.m. rectifiable current $L_k \in \mathcal{R}_1(B^3)$ with $(\partial L_k) \llcorner B^3 = \mathbb{P}(u) - \mathbb{P}(u_k)$ such that $\mathbf{M}(L_k) \to 0$ as $k \to \infty$.*

Remark 1.17. The proof of Proposition 1.16 makes use of the coarea formula by Almgren-Browder-Lieb [3] and of Federer's theorem [17], see (1.18). In high dimension $n \geq 4$, even if we knew a priori that $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$ for some $u \in W^{1,2}(B^n, \mathbb{S}^2)$, the cited Federer's theorem doesn't apply, see Remark 1.13. Therefore, Proposition 1.16 doesn't work anymore.

In [26], using a different approach we extended the explicit formula for the relaxed Dirichlet energy to any high dimension $n \geq 3$. Definitely, for any map $u \in W^{1,2}(B^n, \mathbb{S}^2)$, it turns out that the $(n-3)$-current $\mathbb{P}(u)$ of the singularities satisfies $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$, and we have:

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{1,2}(B^n, \mathbb{S}^2)\,. \tag{1.21}$$

2. Maps with finite relaxed Laplacean energy

In this section, we deal with some general properties of maps with finite relaxed energy (0.11). The case of low dimension $n = 2$ is discussed, where a first lower semicontinuity property is also obtained.

Weak Laplacean. Let $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, see (0.12), and let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ be such that $u_h \to u$ in $L^1(B^n, \mathbb{R}^3)$ and $\sup_h \mathbb{L}(u_h) < \infty$. By inequality (0.6) we have $\sup_h \mathbb{D}(u_h) < \infty$, see (0.1), so that a (not relabeled) subsequence of $\{u_h\}$ converges to u weakly in $W^{1,2}(B^n, \mathbb{R}^3)$, and $u \in W^{1,2}(B^n, \mathbb{S}^2)$. Since $Du \in L^2(B^n, \mathbb{R}^{3\times n})$, the distributional divergence of the gradient is well defined by (0.13), and using that

$$\lim_{h\to\infty} \int_{B^n} \mathrm{tr}\,[Du_h\,(D\varphi)^t]\,dx = \int_{B^n} \mathrm{tr}\,[Du\,(D\varphi)^t]\,dx \quad \forall\, \varphi \in C_c^\infty(B^n, \mathbb{R}^3)$$

we infer that $\mathrm{Div}Du_h \rightharpoonup \mathrm{Div}Du$ weakly as $\mathbb{R}^3$-valued measures in B^n. More precisely, for Sobolev maps in $W^{2,1}(B^n, \mathbb{S}^2)$, integrating by parts we get:

$$\langle \mathrm{Div}Du; \varphi\rangle = \int_{B^n} \Delta u \bullet \varphi\,dx \quad \forall\, \varphi \in C_c^\infty(B^n, \mathbb{R}^3)$$

whence (0.16) holds true. Therefore, the lower semicontinuity property of the total variation gives

$$|\mathrm{Div}Du|(B^n) \leq \liminf_{h\to\infty} |\mathrm{Div}Du_h|(B^n) \leq \sup_h \mathbb{L}(u_h) < \infty \tag{2.1}$$

so that $\mathrm{Div}Du$ is a $\mathbb{R}^3$-valued finite Radon measure in B^n, and the lower bound (0.14) follows by lower semicontinuity.

Notice that if $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, denoting $u = (u^1, u^2, u^3)$, we have checked that Du^ℓ is a divergence-measure field in $\mathcal{DM}^{1,2}(B^n)$ for $\ell = 1, 2, 3$, see Definition 1.1. Moreover, the decomposition into mutually singular measures

$$\mathrm{Div}Du = (\mathrm{Div}Du)^a + (\mathrm{Div}Du)^s\,, \quad (\mathrm{Div}Du)^a = \widetilde{\Delta u}\,\mathcal{L}^n \llcorner B^n \tag{2.2}$$

holds, with density $\widetilde{\Delta u}$ in $L^1(B^n, \mathbb{R}^3)$.

The BV case. Assume now that $u \in \mathbb{L}(B^n, \mathbb{S}^2)$ satisfies condition $Du \in \mathrm{BV}(B^n, \mathbb{R}^{3\times n})$, see (0.17). Then, the weak hessian $\nabla^2 u^\ell$ of each component u^ℓ is a summable function in $L^1(B^n, \mathbb{R}^{n\times n})$, and the density $\widetilde{\Delta u}$ in (2.2) agrees with the approximate Laplacean $\Delta u = (\Delta u^1, \Delta u^2, \Delta u^3)$, where $\Delta u^\ell = \mathrm{tr}\,\nabla^2 u^\ell$, for $\ell = 1, 2, 3$. In addition, the singular part of the measure $(\mathrm{Div}Du)^s$ decomposes into a *Jump* and a *Cantor-type* component, the first one being concentrated on the countably $(n-1)$-rectifiable discontinuity set of the gradient Du, and the second one being equal to zero if Du is a *special function of bounded variation.*

As a consequence, in the BV-case we obtain a *tangential property* concerning the singular component of the distributional divergence.

Proposition 2.1. *Let $n \geq 2$ and $u \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, see* (0.17)*. Then*

$$u \bullet (\mathrm{Div} Du)^s = 0 \,. \tag{2.3}$$

Proof. By applying Proposition 1.3 with $F = Du^\ell$ and $g = u^\ell$, and by summating on $\ell = 1, 2, 3$, we obtain

$$\mathrm{Div}(u \bullet Du) = u \bullet \mathrm{Div} Du + |Du|^2 \mathcal{L}^n \llcorner B^n \,.$$

Since moreover $|u| \equiv 1$, we have $0 = \partial_i |u|^2 = 2 \partial_i u \bullet u$ for $i = 1, \ldots, n$, whence $\mathrm{Div}(u \bullet Du) = 0$ and therefore

$$u \bullet \mathrm{Div} Du = -|Du|^2 \mathcal{L}^n \llcorner B^n \,. \tag{2.4}$$

On the other hand, since Du is in BV, we have $u \bullet \Delta u = -|Du|^2$ for $\mathcal{L}^n$-a.e. in B^n, so that

$$u \bullet (\mathrm{Div} Du)^a = (u \bullet \Delta u) \, \mathcal{L}^n \llcorner B^n = -|Du|^2 \, \mathcal{L}^n \llcorner B^n \,. \tag{2.5}$$

Equation (2.3) follows from (2.4) and (2.5), on account of the decomposition formula (2.2), where $\widetilde{\Delta u} = \Delta u$. □

The low dimension case. In the critical dimension $n = 2$, due to the continuous embedding of $W^{1,2}(B^2)$ in VMO, Schoen-Uhlenbeck density theorem [33] applies. More precisely, by a convolution and projection argument, we can find a sequence $\{u_h\} \subset C^\infty(B^2, \mathbb{S}^2)$ such that $u_h \to u$ strongly in $W^{1,2}(B^2, \mathbb{R}^3)$, compare e.g. [24, Sec. 5.5.1, Thm. 3] or [28, Thm. 4.14]. Since moreover

$$\mathbb{L}(u_h) = |\mathrm{Div} Du_h|(B^2) \leq |\mathrm{Div} Du|(B^2) + \varepsilon_h$$

where $\varepsilon_h \to 0^+$ as $h \to \infty$, by the lower semicontinuity inequality (2.1) we infer that $\mathbb{L}(u_h) \to |\mathrm{Div} Du|(B^2)$ as $h \to \infty$, whence formula (0.15) holds, and there is no gap.

Using Proposition 1.2, we also obtain a lower semicontinuity property:

Theorem 2.2. *Let $\{u_k\} \subset C^\infty(B^2, \mathbb{S}^2)$ be such that the graph currents G_{u_k} weakly converge in $\mathcal{D}_2(B^2 \times \mathbb{S}^2)$ to the current $T = G_u + \mathrm{d}\, \delta_O \times [\![\mathbb{S}^2]\!]$, for some map $u \in \mathbb{L}(B^2, \mathbb{S}^2)$ and some integer $\mathrm{d} \in \mathbb{Z}$. Then*

$$\liminf_{k \to \infty} \mathbb{L}(u_k) \geq |\mathrm{Div} Du|(B^n) + 8\pi \, |\mathrm{d}| \,.$$

Proof. For any $v \in W^{2,1}(B^2, \mathbb{S}^2)$ and any Borel set $B \subset B^2$, we denote

$$\mathbb{L}(v, B) := |\mathrm{Div} Dv|(B) = \int_B |\Delta v| \, dx \,, \quad \mathbb{D}(v, B) := \frac{1}{2} \int_B |Dv|^2 \, dx \,.$$

Let $\varepsilon > 0$. Since by Proposition 1.2 the measure $|\mathrm{Div}Du|$ does not charge any atom, we can choose $r > 0$ small so that $|\mathrm{Div}Du|(\bar{B}^2_r) \leq \varepsilon$, so that by lower semicontinuity and additivity

$$\liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) \geq |\mathrm{Div}Du|(B^2 \setminus \bar{B}^2_r) \geq |\mathrm{Div}Du|(B^2) - \varepsilon\,.$$

Moreover, by inequality (0.6) we get

$$\liminf_{k\to\infty} \mathbb{L}(u_k, \bar{B}^2_r) \geq 2 \cdot \liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r)\,.$$

On the other hand, by weak lower semicontinuity of the Dirichlet energy on Cartesian currents, see (1.12), using that

$$G_{u_k} \llcorner B^2_r \times \mathbb{S}^2 \rightharpoonup G_u \llcorner B^2_r \times \mathbb{S}^2 + \mathrm{d}\,\delta_O \times [\![\, \mathbb{S}^2 \,]\!]$$

weakly in $\mathcal{D}_2(B^2_r \times \mathbb{S}^2)$ we obtain the energy lower bound

$$\liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r) \geq \mathbb{D}(u, B^2_r) + 4\pi\,|\mathrm{d}| \geq 4\pi\,|\mathrm{d}|\,.$$

Finally, putting the terms together we obtain

$$\begin{aligned} \liminf_{k\to\infty} \mathbb{L}(u_k, B^2) \geq &\ \liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) + \liminf_{k\to\infty} \mathbb{L}(u_k, B^2_r) \\ \geq &\ |\mathrm{Div}Du|(B^n) - \varepsilon + 8\pi\,|\mathrm{d}| \end{aligned}$$

for each $\varepsilon > 0$, as required. □

THE HIGH DIMENSION CASE. If $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, in general we do not have enough information concerning both the density $\widetilde{\Delta u}$ and the singular part $(\mathrm{Div}Du)^s$ in formula (2.2). More precisely, Proposition 1.2 yields that $|(\mathrm{Div}Du)^s|(B) = 0$ for each Borel set $B \subset B^n$ with σ-finite $\mathcal{H}^{n-2}$-measure. However, it mat happen that the gradient Du does not belong to the class $\mathrm{BV}(B^n, \mathbb{R}^{3\times n})$, so that we cannot conclude e.g. that for $\ell = 1, 2, 3$ the vector field $Du^\ell \in L^2(B^n, \mathbb{R}^n)$ is approximately differentiable $\mathcal{L}^n$-a.e. in B^n, and that the trace $\mathrm{tr}\,\nabla Du^\ell$ agrees $\mathcal{L}^n$-a.e. in B^n with the ℓ-th component of the Radon-Nikodym derivative $\widetilde{\Delta u}$ of the measure $\mathrm{Div}Du$.

A sufficient condition ensuring both enough regularity of the density $\widetilde{\Delta u}$ and property $(\mathrm{Div}Du)^s = 0$, is the membership of Du to the Sobolev class $W^{1,1}(B^n, \mathbb{R}^{3\times n})$, so that in particular equation (0.16) holds true. In fact, the computation of the energy gap for maps in $W^{2,1}(B^n, \mathbb{S}^2)$ and in any dimension $n \geq 3$ is the content of our Main Result, Theorem 0.1.

3. ENERGY CONCENTRATION

In this section, we define a suitable modification of the inverse to the stereographic map. We then compute the *minimal Laplacean energy* among maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ with fixed degree, and describe the bubbling phenomenon.

MODIFIED STEREOGRAPHIC PROJECTION. Consider the inverse of the stereographic map (1.1) in case $\mathfrak{p} = 2$. Since ${\sigma_2^{-1}}_{\#}[\![\, \mathbb{R}^2 \,]\!] = [\![\, \mathbb{S}^2 \,]\!]$, one has $\deg \sigma_2^{-1} = 1$, compare Definition 1.8.

Similarly to e.g. [25, Sec. 4.1.1], we now modify σ_2^{-1} in such a way that it is equal to the South Pole P_S outside some small disk, by paying a small amount of Laplacean energy.

Proposition 3.1. *For any $\varepsilon > 0$ and $\delta > 0$ sufficiently small, there exists a smooth and degree one map $u_{\varepsilon,\delta} \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, see* (0.8)*, such that:*

(1) $u_{\varepsilon,\delta}(x) = P_S$ *if* $|x| > \delta$ *;*
(2) $4\pi \leq \mathbb{D}(u_{\varepsilon,\delta}, \mathbb{R}^2) \leq 4\pi + O(\varepsilon)$*, see* (1.8) *;*
(3) $8\pi \leq \mathbb{L}(u_{\varepsilon,\delta}, \mathbb{R}^2) \leq 8\pi + O(\varepsilon)$*, see* (0.9)

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Proof. Choose a smooth decreasing function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(\rho) = 1$ if $\rho \leq -1$ and $\Phi(\rho) = 0$ if $\rho \geq 0$, and define for $k \in \mathbb{N}$ large the smooth map $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\varphi_k(x) := \Phi(|x| - k)\, \sigma_2^{-1}(x) + \big(1 - \Phi(|x| - k)\big)\, P_S$$

so that $\varphi_k(x) = \sigma_2^{-1}(x)$ if $|x| \leq k - 1$ and $\varphi_k(x) = P_S$ if $|x| \geq k$. For $\sigma > 0$ small, we also have $|\varphi_k(x) - P_S| < \sigma$ if $|x| \geq k - 1$ and k is sufficiently large. Therefore, letting

$$\widetilde{\varphi}_k(x) := \Pi \circ \varphi_k(x)\,, \qquad \Pi(y) := \frac{y}{|y|}, \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R}^3 \setminus \{0\}$$

it turns out that $\mathbb{D}(\widetilde{\varphi}_k, \mathbb{R}^2) \to \mathbb{D}(\sigma_2^{-1}, \mathbb{R}^2)$ and $\mathbb{L}(\widetilde{\varphi}_k, \mathbb{R}^2) \to \mathbb{L}(\sigma_2^{-1}, \mathbb{R}^2)$ as $k \to \infty$. In addition, by scale invariance of the energy, and setting for $\delta < 1$

$$\widetilde{\varphi}_{k,\delta}(x) := \widetilde{\varphi}_k\Big(\frac{x}{\delta}\Big)\,, \quad x \in \mathbb{R}^2$$

for k large we have $\mathbb{D}(\widetilde{\varphi}_{k,\delta}, \mathbb{R}^2) = \mathbb{D}(\widetilde{\varphi}_k, \mathbb{R}^2)$ and $\mathbb{L}(\widetilde{\varphi}_{k,\delta}, \mathbb{R}^2) = \mathbb{D}(\widetilde{\varphi}_k, \mathbb{R}^2)$, with $\widetilde{\varphi}_{k,\delta}(x) = P_S$ if $|x| > \delta\, k$. Finally, we have $\widetilde{\varphi}_{k,\delta\,\#}[\![\, \mathbb{R}^2 \,]\!] = [\![\, \mathbb{S}^2 \,]\!]$, whence $\deg \widetilde{\varphi}_{k,\delta} = 1$ for k large and δ small. The claim readily follows. □

Minimal energy of maps with fixed degree. According to (0.8), if $u \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, then $|Du| \in L^2(\mathbb{R}^2)$ and hence the degree of u is given by Definition 1.8. We now compute the minimal energy (0.9) in each class

$$\mathcal{F}_{\rm d} := \{u \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2) \mid \deg u = {\rm d}\}\,, \qquad {\rm d} \in \mathbb{Z}\,. \tag{3.1}$$

We rely on the fact that the analogous problem for the Dirichlet energy is known. More precisely, denoting

$$\mathcal{G}_{\rm d} := \{u \in W^{1,1}_{\rm loc}(\mathbb{R}^2, \mathbb{S}^2) \mid Du \in L^2(\mathbb{R}^2, \mathbb{R}^{3\times 2}),\ \deg u = {\rm d}\} \tag{3.2}$$

and recalling (1.8), by the lower semicontinuity property (1.12) and Proposition 3.1, it turns out that

$$\forall\, {\rm d} \in \mathbb{Z}\,, \quad \inf_{u \in \mathcal{G}_{\rm d}} \mathbb{D}(u, \mathbb{R}^2) = 4\pi\, |{\rm d}|\,. \tag{3.3}$$

Theorem 3.2. *For every integer* ${\rm d} \in \mathbb{Z}$ *we have:* $\displaystyle\inf_{u \in \mathcal{F}_{\rm d}} \mathbb{L}(u, \mathbb{R}^2) = 8\pi\, |{\rm d}|\,.$

Proof. When $\mathrm{d}=0$, the claim is trivial, whereas the case $\mathrm{d}=1$, and hence $\mathrm{d}=-1$, has been discussed in the introduction. In fact, by the lower bound (0.10), any map u in $\mathcal{F}_1$ has Laplacean energy at least 8π, and equality holds for harmonic and conformal maps of degree one, as e.g. $u=\sigma_2^{-1}$, compare (0.7). Therefore, it clearly suffices to consider the case $\mathrm{d}\geq 2$.

Since $\mathcal{F}_{\mathrm{d}}\subset\mathcal{G}_{\mathrm{d}}$, see (3.1) and (3.2), by inequality (0.10) and formula (3.3)

$$\inf_{u\in\mathcal{F}_{\mathrm{d}}}\mathbb{L}(u,\mathbb{R}^2)\geq 2\cdot\inf_{u\in\mathcal{F}_{\mathrm{d}}}\mathbb{D}(u,\mathbb{R}^2)\geq 2\cdot\inf_{u\in\mathcal{G}_{\mathrm{d}}}\mathbb{D}(u,\mathbb{R}^2)=8\pi\,\mathrm{d}\,.$$

We now check the opposite inequality:

$$\inf_{u\in\mathcal{F}_{\mathrm{d}}}\mathbb{L}(u,\mathbb{R}^2)\leq 8\pi\,\mathrm{d}\,. \tag{3.4}$$

By Proposition 3.1, for each $\varepsilon>0$ we find a degree one map $u_\varepsilon\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$, equal to P_S outside the unit disk B^2, and such that

$$\mathbb{L}(u_\varepsilon,\mathbb{R}^2)=\int_{B^2}|\Delta u_\varepsilon|\,dx\leq 8\pi+\frac{\varepsilon}{\mathrm{d}}\,.$$

Denoting $\mathbf{e}_1:=(1,0)$, we define $w_\varepsilon(x):=u_\varepsilon(x-3k\,\mathbf{e}_1)$ on the unit disk centered at $3k\,\mathbf{e}_1$, for $k=0,1,\ldots,\mathrm{d}-1$, and $w_\varepsilon\equiv P_S$ outside the union of such d disks. The map w_ε satisfies $\mathbb{L}(w_\varepsilon,\mathbb{R}^2)\leq 8\pi\,\mathrm{d}+\varepsilon$ and belongs to the class $\mathcal{F}_{\mathrm{d}}$, whence (3.4) holds true, as required. □

Bubbling-off of spheres. We recall that the maps $u_{\varepsilon,\delta}$ from Proposition 3.1 have degree one. Therefore, letting e.g. $\varepsilon=\delta=1/h$ we find a sequence $\{u_h\}\subset C^\infty(\mathbb{R}^2,\mathbb{S}^2)$ of smooth degree one maps weakly converging in $W^{1,2}$ to the constant map P_S, and such that

$$\lim_{h\to\infty}\mathbb{L}(u_h,\mathbb{R}^2)=\mathbb{L}(\sigma_2^{-1},\mathbb{R}^2)=8\pi\ .$$

Furthermore, it turns out that the above convergence is uniform far from the origin, and that the graph currents G_{u_h} weakly converge in $\mathcal{D}_2(\mathbb{R}^2\times\mathbb{S}^2)$ to the Cartesian current

$$T=G_{P_S}+\delta_O\times[\![\,\mathbb{S}^2\,]\!]$$

where G_{P_S} is the graph current of the constant map equal to P_S on $\mathbb{R}^2$.

A bubbling phenomenon occurs, and by Theorem 2.2 we infer that *the minimal Laplacean energy occurring for the formation of a* 2*-sphere is equal to* 8π. This *energy quantization* property is detected if one defines the energy

$$\mathbb{L}(T):=8\pi\,,\qquad T=G_{P_S}+\delta_O\times[\![\,\mathbb{S}^2\,]\!]\,. \tag{3.5}$$

We thus have a second order analogous to a similar feature concerning the conformal Dirichlet integral, where formula (1.2) yields that the minimum energy cost of a $\mathfrak{p}$-sphere is equal to $\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$, for any integer $\mathfrak{p}\geq 2$.

4. The Dipole problem

The classical Dipole problem by Brezis-Coron-Lieb [13] deals with Sobolev maps u in $W^{1,2}(\mathbb{R}^3, \mathbb{S}^2)$ which assume a given constant $P \in \mathbb{S}^2$ at infinity and which are smooth outside two singular point $a_\pm$, with

$$\deg(u, a_-) = -1\,, \qquad \deg(u, a_+) = +1$$

the degree being given by (1.19).

In [13], it is shown that the minimal Dirichlet energy $\mathbb{D}(u, \mathbb{R}^3)$ in such class, see (1.8), is equal to the distance $|a_+ - a_-|$ between the singularities times the measure 4π of the unit sphere $\mathbb{S}^2$, compare [25, Sec. 4.2.3].

In this section, we discuss the Dipole problem for the Laplacean energy (0.9). We thus denote by $\mathcal{E}$ the subclass of maps u as above that in addition belong to the second order space $W_{\mathbb{L}}(\mathbb{R}^3, \mathbb{S}^2)$, see (0.8).

Theorem 4.1. *We have:* $\inf\{\mathbb{L}(u, \mathbb{R}^3) \mid u \in \mathcal{E}\} = |a_+ - a_-| \cdot 8\pi\,.$

Proof. Without loss of generality, we may and do assume $P = P_S$ and $a_+ = (-r, 0, 0)$, $a_- = (r, 0, 0)$ for some $r > 0$. The energy upper bound

$$\inf\{\mathbb{L}(u, \mathbb{R}^3) \mid u \in \mathcal{E}\} \leq |a_+ - a_-| \cdot 8\pi \tag{4.1}$$

is obtained by means of a Dipole insertion argument which is re-adapted by [25, Sec. 4.2.3], see also [13]. Firstly, by Proposition 3.1 we choose a smooth map $v_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$ with degree one, equal to the pole P_S outside B^2_r for some $r < 1$, and such that with $\widehat{z} = (z_2, z_3) \in \mathbb{R}^2$

$$\mathbb{L}(v_\varepsilon, B^2) := \int_{B^2} |\Delta v_\varepsilon(\widehat{z})|\, d\widehat{z} \leq 8\pi + \varepsilon\,. \tag{4.2}$$

In formula (4.6), we wish to obtain a Sobolev map $u_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^3, \mathbb{S}^2)$, so that $\Delta u_\varepsilon \in L^1(\mathbb{R}^3, \mathbb{R}^3)$. Therefore, we have to replace the Lipschitz-continuous function $t \mapsto \min\{r + t, r - t, \delta\}$ on the interval $D^1_r := (-r, r)$ with a function at least of class $C^1(D^1_r)$. With $\delta > 0$ small, we can choose:

$$\varphi_\delta(t) := \begin{cases} \delta & \text{if} \quad |t| \leq r - \sqrt{2}\,\delta \\ \sqrt{\delta^2 - \big(|t| - r + \sqrt{2}\,\delta\big)^2} & \text{if} \quad r - \sqrt{2}\,\delta \leq |t| \leq r - \dfrac{\delta}{\sqrt{2}} \\ r - |t| & \text{if} \quad r - \dfrac{\delta}{\sqrt{2}} \leq |t| < r\,. \end{cases} \tag{4.3}$$

For $x = (\widetilde{x}, \widehat{x}) \in \mathbb{R} \times \mathbb{R}^2 \simeq \mathbb{R}^3$ and $z = (\widetilde{z}, \widehat{z}) \in D^1_r \times B^2$, we let

$$(\widetilde{x}, \widehat{x}) = \Phi_\delta(\widetilde{z}, \widehat{z}) := (\widetilde{z}, \varphi_\delta(\widetilde{x})\, \widehat{x}) \tag{4.4}$$

and define $u_{\varepsilon,\delta}(x) := \widehat{v}_\varepsilon(\Phi_\delta^{-1}(x))$ for $x \in \Phi_\delta(D^1_r \times B^2)$, where $\widehat{v}_\varepsilon(z) := v_\varepsilon(\widehat{z})$, so that (4.2) holds. We then compute:

$$\begin{aligned} \Delta u_{\varepsilon,\delta}(x) = \quad & \frac{1}{\varphi_\delta(\widetilde{z})^2}\Delta\widehat{v}_\varepsilon(z) + \frac{{\varphi'_\delta}^2}{{\varphi_\delta}^4}(\widetilde{z}) \sum_{\alpha,\beta=2}^3 x_\alpha x_\beta \partial^2_{\alpha,\beta}\widehat{v}_\varepsilon(z) \\ & + \frac{2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta}{{\varphi_\delta}^3}(\widetilde{z}) \sum_{\alpha=2}^3 x_\alpha \partial_\alpha \widehat{v}_\varepsilon(z) \end{aligned}$$

where $z = \Phi_\delta^{-1}(x)$, so that $\widetilde{x} = \widetilde{z}$, and $\widehat{x} = \varphi_\delta(\widetilde{z})\,\widehat{z}$. Using that $\det D\Phi_\delta(z) = \varphi_\delta(\widetilde{z})^2$, we get:

$$\begin{aligned} \det D\Phi_\delta(z)\cdot\Delta u_{\varepsilon,\delta}(x) = \quad & \Delta v_\varepsilon(\widehat{z}) + \varphi'_\delta(\widetilde{z})^2 \sum_{\alpha,\beta=2}^3 z_\alpha z_\beta \partial^2_{\alpha,\alpha} v_\varepsilon(\widehat{z}) \\ & + (2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta)(\widetilde{z}) \sum_{\alpha=2}^3 z_\alpha \partial_\alpha v_\varepsilon(\widehat{z})\,. \end{aligned}$$

Therefore, since $\|\varphi'_\delta\|_{\infty,D^1_r} \le 1$ and $\|2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta\|_{\infty,D^1_r} \le 4$, by changing variables $z = \Phi_\delta^{-1}(x)$ we can estimate:

$$\begin{aligned} & \int_{\Phi_\delta(D^1_r\times B^2)} |\Delta u_{\varepsilon,\delta}|\,dx \le \int_{(-r,r)\times B^2} |\Delta v_\varepsilon(\widehat{z})|\,d\widetilde{z}\,d\widehat{z} \\ & \qquad + 8\int_{(r-\sqrt{2}\delta,r)\times B^2} \Big(\sum_{\alpha,\beta=2}^3 |z_\alpha z_\beta||\partial^2_{\alpha,\beta} v_\varepsilon(\widehat{z})| + |\widehat{z}||\nabla v_\varepsilon(\widehat{z})|\Big)\,d\widetilde{z}\,d\widehat{z}\,. \end{aligned}$$

Since moreover $v_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$ is smooth, the integral in the second line is small for $\delta > 0$ small, whence we can find $\delta(\varepsilon) \in (0, r/2)$ such that

$$\int_{\Phi_\delta(D^1_r\times B^2)} |\Delta u_{\varepsilon,\delta(\varepsilon)}|\,dx \le 2r\cdot\mathbb{L}(v_\varepsilon, B^2) + \varepsilon\,, \quad \text{see (4.2)}\,. \tag{4.5}$$

Recall that the map v_ε is equal to the pole P_S in a neighborhood of the boundary of B^2. Therefore, setting $\delta = \delta(\varepsilon)$ and

$$u_\varepsilon(x) := \begin{cases} u_{\varepsilon,\delta}(x) & \text{if} \quad x \in \Phi_\delta(D^1_r \times B^2) \\ P_S & \text{if} \quad x \in \mathbb{R}^3 \setminus \big(\Phi_\delta(D^1_r\times B^2) \cup \{(\pm r, 0, 0)\}\big) \end{cases} \tag{4.6}$$

it turns out that the map u_ε belongs to the class $\mathcal{E}$, whereas by (4.5)

$$\mathbb{L}(u_\varepsilon, \mathbb{R}^3) = \int_{\Phi_\delta(D^1_r\times B^2)} |\Delta u_{\varepsilon,\delta}|\,dx \le 2r\cdot 8\pi + (2r+1)\,\varepsilon\,.$$

The energy upper bound (4.1) follows by letting $\varepsilon \searrow 0$.

We now prove the energy lower bound

$$\inf\{\mathbb{L}(u, \mathbb{R}^3) \mid u \in \mathcal{E}\} \ge |a_+ - a_-|\cdot 8\pi$$

by means of a slicing argument and of Theorem 3.2, case $d = 1$. For this purpose, recalling the notation $x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}\times\mathbb{R}^2 \simeq \mathbb{R}^3$, and choosing $u \in \mathcal{E}$, for a.e. $t \in (-r, r)$ the restriction u_t of u to the 2-dimensional space

$\mathbb{R}^2_t := \{x \in \mathbb{R}^3 \mid \widetilde{x} = t\}$ is a function in $W_{\mathbb{L}}(\mathbb{R}^2_t, \mathbb{S}^2)$ with degree one. This can be seen by closing the 2-space $\mathbb{R}^2_t$ "near infinity" around the degree one singularity a_+, by means of a standard homotopy argument. On account of Theorem 3.2, we thus have $\mathbb{L}(u_t, \mathbb{R}^2_t) \geq 8\pi$ for a.e. $t \in (-r, r)$, whence

$$\mathbb{L}(u, \mathbb{R}^3) \geq \int_{-r}^{r} \mathbb{L}(u_t, \mathbb{R}^2_t)\, dt \geq 2r \cdot 8\pi\,, \quad 2r = |a_+ - a_-|$$

and the proof is complete □

5. Strong density results

In this section, we prove in any dimension $n \geq 3$ a strong density property in $W^{2,1}(B^n, \mathbb{S}^2)$ of the class $R^\infty_{n-3}(B^n, \mathbb{S}^3)$ from Definition 1.14. We then prove a cohomological criterion for strong density of smooth maps.

Second order density results. The following density result was proved in more generality in [19] when $n = 3$, and then extended by Bousquet-Ponce-Van Schaftingen [12] in high dimension, see also [30, Lemma 4.5].

Theorem 5.1. *The class $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ is strongly dense in $W^{2,1}(B^n, \mathbb{S}^2)$, in any dimension $n \geq 3$.*

As in the Dipole problem previously discussed, one of the main difficulties in the proof of Theorem 5.1 is to obtain Sobolev regularity of the derivatives of the functions involved. Therefore, one cannot use the same construction as the one in the proof of Theorem 1.15. We also have:

Theorem 5.2. *Let $n \geq 3$ and $u \in W^{2,1}(B^n, \mathbb{S}^2)$. There exists a sequence $\{u_h\} \subset R^\infty_{n-3}(B^n, \mathbb{S}^2)$ such that $u_h \to u$ strongly in $W^{1,2}(B^n, \mathbb{R}^3)$ and $\mathbb{L}(u_h) \to \mathbb{L}(u)$.*

Proof. We argue as in the proof of [12, Thm. 3], with the following modifications. Firstly, we let $m = n$ and consider maps defined on the open unit cube Q^n of $\mathbb{R}^n$ and taking values in $\mathcal{N} = \mathbb{S}^2$. The ambient functional space is $X := W^{1,2} \cap W^{2,1}(Q^n, \mathbb{R}^3)$, equipped with the norm $\|u\|_X := \|u\|_{W^{1,2}} + \|D^2 u\|_{L^1}$. Then, $(X, \|\cdot\|_X)$ is a Banach space, and we work with the induced distance $d_X(u, v) = \|u - v\|_X$. When treating first order derivatives $D^1 u$, i.e., for $j = 1$, we choose the exponent $p = 2$, whereas for second order derivatives, i.e., for $j = 2$, we choose $p = 1$. Therefore, we have $kp = 2$. Condition $D^1 u \in L^2$, that is obtained in [12, pp. 795, 798] by means of the Gagliardo-Nirenberg interpolation inequality, in our hypotheses follows from the pointwise inequality $|\Delta u| \geq |Du|^2$, that holds true for maps $u \in X$ taking values in $\mathcal{N} = \mathbb{S}^2$. Moreover, the Poincaré-Wirtinger inequality applied in [12, p. 795] continues to hold, with $kp = 2$, since it deals with first order derivatives. For the sake of brevity, further details are omitted. □

Remark 5.3. We now point out that the opening, smoothening, and thickening arguments from [12] make use of left or right compositions with smooth maps. Therefore, if $u \in W^{2,1}(B^n, \mathbb{S}^2)$ satisfies condition $\mathbb{P}(u) = 0$, in Theorem 5.2 we can approximate u by a sequence $\{u_h\} \subset R^\infty_{n-3}(B^n, \mathbb{S}^2)$ satisfying $\mathbb{P}(u_h) = 0$ for every h.

A COHOMOLOGICAL CRITERION. Obstructions to strong density of smooth maps are encoded by the non-triviality of the current of the singularities $\mathbb{P}(u)$ in (1.13). Concerning the Dirichlet energy, this cohomological argument was firstly proved by Bethuel [6] when $n = 3$, and then extended in high dimension and for a wider class of target manifolds in [9].

Theorem 5.4. *Let $u \in W^{2,1}(B^n, \mathbb{S}^2)$ for some $n \geq 3$. If $\mathbb{P}(u) = 0$, there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ such that $u_h \to u$ strongly in $W^{1,2}(B^n, \mathbb{R}^3)$ and $\mathbb{L}(u_h) \to \mathbb{L}(u)$ as $h \to \infty$. Moreover, the converse property holds, too.*

Proof. If $u \in W^{2,1}(B^n, \mathbb{S}^2)$ is the strong limit of a smooth sequence $\{u_h\}$ in $C^\infty(B^n, \mathbb{S}^2)$, by Remark 1.5 we know that the graph current G_u satisfies the null-boundary condition (1.6), which yields $\mathbb{P}(u) = 0$, by (1.14).

Assume now that $u \in W^{2,1}(B^n, \mathbb{S}^2)$ satisfies condition $\mathbb{P}(u) = 0$. We follow the proof of Thm. 1 from [12, Sec. 9]. Their claim on page 812 concerning a removable singularity property for $W^{2,1}$-maps is of topological nature, as it holds true under the assumption $\pi_2(\mathcal{N}) = 0$ on the target manifold $\mathcal{N}$. When $\mathcal{N} = \mathbb{S}^2$, such a topological condition is not satisfied. However, taking into account Remark 5.3, when approximating a map u in $W^{2,1}(B^n, \mathbb{S}^2)$, property $\mathbb{P}(u) = 0$ yields a zero degree condition that allows to conclude that the cited removable singularity argument continues to hold. For that reason, the proof follows by using similar arguments to the ones of [12, Thm. 1], with the modification that we made in the proof of Theorem 5.2. Further details are omitted. □

6. The Laplacean energy on Cartesian currents

In this section, we define a Laplacean energy functional on a suitable class of Cartesian currents in such a way that a weak sequential lower semicontinuity property holds true.

Due to the embedding of $W^{2,1}(B^n, \mathbb{S}^2)$ into $W^{1,2}(B^n, \mathbb{S}^2)$, according to Definition 1.9 we introduce the following

Definition 6.1. We denote by $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with underlying function u_T in $W^{2,1}(B^n, \mathbb{S}^2)$.

Remark 6.2. For future use, given a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ we also denote

$$\mathcal{T}^{\mathbb{L}}_u := \{T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2) \text{ such that } u_T = u \text{ in (1.11)}\}. \tag{6.1}$$

By the explicit formula (1.21) for the relaxed Dirichlet energy, we infer that *the class $\mathcal{T}^{\mathbb{L}}_u$ is always non-empty.*

Laplacean energy on currents. Theorems 2.2 and 3.2 suggest to introduce on the class $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the *Laplacean energy functional*

$$\mathbb{L}(T) := \int_{B^n} |\Delta u_T|\, dx + 8\pi \cdot \mathbf{M}(L) \qquad \text{if (1.11) holds} \tag{6.2}$$

so that we have:

(1) $\mathbb{L}(T) < \infty$ for every $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$;
(2) $\mathbb{L}(G_u) = \mathbb{L}(u)$ if $T = G_u$ for some smooth map $u \in W^{2,1}(B^n, \mathbb{S}^2)$;
(3) when $n = 2$, formula (3.5) holds true.

Lower semicontinuity. Similarly to what happens for the Laplacean energy of σ_2^{-1} among degree one maps, see (0.7), the term $8\pi \cdot \mathbf{M}(L)$ in (6.2) is the optimal energy contribution of the vertical term $L \times [\![\mathbb{S}^2]\!]$ in(1.11). In fact, making use of Theorem 2.2 we obtain:

Theorem 6.3. *Let $n \geq 2$ and let $\{T_k\} \subset \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ be such that $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to some $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$. Then, we have:*

$$\mathbb{L}(T) \leq \liminf_{k \to \infty} \mathbb{L}(T_k), \qquad \textit{see } (6.2)\,.$$

Proof. As in [27, Thm. 2.12] and [32, Thm. 5.1], we make use of a dimension reduction argument that goes back to [11], see also [4, Thm. 5.4]. Firstly, we consider the case $n = 2$, where we rely on Theorem 2.2, obtaining inequality (6.3). We then deal with the case $n = 3$, where we apply the dimension reduction argument and inequality (6.3) for $n = 2$. A similar reduction to the case $n - 1$ and an induction argument gives the proof in any high dimension $n \geq 4$.

In (1.11), we denote $u_{T_k} = u_k$, $u_T = u_\infty$ and $T = T_\infty$, so that for each $k \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ we have $T_k = G_{u_k} + L_k \times [\![\mathbb{S}^2]\!]$ and, in case $n \geq 3$, also $(\partial L_k) \llcorner B^n = -\mathbb{P}(u_k)$. We define the localized energy on open sets $A \subset B^n$

$$\mathbb{L}(T_k, A) := \mathbb{L}(u_k, A) + 8\pi \cdot \mathbf{M}(L_k \llcorner A)\,, \qquad k \in \overline{\mathbb{N}}\,.$$

The case $n = 2$: For each $k \in \overline{\mathbb{N}}$, the current L_{T_k} is in $\mathcal{R}_0(B^2)$, whence it is a finite sum of unit Dirac masses with integer coefficients. Therefore, by a localization and elimination argument, we may and do assume that $L_{T_\infty} = \mathrm{d}\,\delta_O$ for some $\mathrm{d} \in \mathbb{Z}$, and that for each $k \in \mathbb{N}$ we have $T_k = G_{u_k}$ for some $u_k \in W^{2,1}(B^2, \mathbb{S}^2)$, with $\sup_k \mathbb{L}(u_k) < \infty$. Moreover, for each $k \in \mathbb{N}$ we can find a sequence $\{u_{k,h}\}_h \subset C^\infty(B^2, \mathbb{S}^2)$ strongly converging to u_k in $W^{1,2}$ and such that $\mathbb{L}(u_{k,h}) \to \mathbb{L}(u_k)$ as $h \to \infty$. Using that $G_{u_{k,h}} \rightharpoonup G_{u_k}$ weakly in $\mathcal{D}_2(B^2 \times \mathbb{S}^2)$, by a diagonal argument (which holds true since the weak convergence is metrizable, see Remark 7.2) we deduce that we can also assume that $\{u_k\} \subset C^\infty(B^2, \mathbb{S}^2)$. In this case, the lower semicontinuity inequality follows from Theorem 2.2. Finally, in a similar way we obtain for any open set $A \subset B^2$

$$\mathbb{L}(T_\infty, A) \leq \liminf_{k \to \infty} \mathbb{L}(T_k, A)\,. \tag{6.3}$$

The case $n = 3$: Let $S^2_+ := \{x \in \mathbb{R}^3 \,:\, |x| = 1,\ x_1 \geq 0\}$. For a given direction $\nu \in S^2_+$, denote by π_ν the 1D vector space spanned by ν, and by A_ν the orthogonal projection of an open set $A \subset B^3$ onto π_ν. We also fix an orthonormal basis $\tau(\nu) = (\tau_1, \tau_2)$ of the 2D vector space orthogonal to π_ν and for $z = (z_1, z_2) \in \mathbb{R}^2$ we let $z \bullet \tau(\nu) := \sum_{i=1}^2 z_i \tau_i$. For any $y \in A_\nu$ we also denote by

$$A^\nu_y := \{z \in \mathbb{R}^2 \mid y\,\nu + z \bullet \tau(\nu) \in A\}$$

the (non-empty) section of A corresponding to y, and for any $y \in A_\nu$ and any $u_k : A \subset B^3 \to \mathbb{S}^2$ we define the sliced function $(u_k)^\nu_y : A^\nu_y \to \mathbb{S}^2$

$$(u_k)^\nu_y(z) := u_k(y\,\nu + z \bullet \tau(\nu))\,.$$

Taking $A = B^3$, for any $k \in \overline{\mathbb{N}}$ the 2-dimensional slice (cf. [24, Sec. 2.2.5])

$$(T_k)^\nu_y := T_k \llcorner (B^3)^\nu_y \times \mathbb{S}^2$$

defines a Cartesian current in $\mathrm{cart}^{\mathbb{L}}((B^3)^\nu_y \times \mathbb{S}^2)$ for $\mathcal{L}^1$-a.e. $y \in (B^3)_\nu$, where

$$(T_k)^\nu_y = G_{(u_k)^\nu_y} + (L_k \llcorner (B^3)^\nu_y) \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Also, the energy of the sliced current $(T_k)^\nu_y$ is given for $\mathcal{L}^1$-a.e. $y \in (B^3)_\nu$ by

$$\mathbb{L}((T_k)^\nu_y, A^\nu_y) = \int_{A^\nu_y} |\Delta (u_k)^\nu_y(z)|\, d\mathcal{L}^2(z) + 8\pi \cdot \mathbf{M}(L_k \llcorner A^\nu_y)$$

for any open set $A \subset B^3$. Therefore, setting

$$\mathbb{L}(T_k, A; \nu) := \int_{A_\nu} \mathbb{L}((T_k)^\nu_y, A^\nu_y)\, d\mathcal{L}^1(y)\,, \qquad k \in \overline{\mathbb{N}}$$

by the inequalities

$$\int_A |\Delta u_k|\, dx \geq \int_{A_\nu} \int_{A^\nu_y} |\Delta (u_k)^\nu_y(z)|\, d\mathcal{L}^2(z)\, d\mathcal{L}^1(y)\,,$$
$$\mathbf{M}(L_k \llcorner A) \geq \int_{A_\nu} \mathbf{M}(L_k \llcorner A^n_y)\, d\mathcal{L}^1(y)$$

we infer that

$$\mathbb{L}(T_k, A) \geq \mathbb{L}(T_k, A; \nu) \qquad \forall\, k \in \overline{\mathbb{N}}\,. \tag{6.4}$$

Moreover, for any open set $A \subset B^3$, using that

$$\lim_{k\to\infty} \int_{\pi_\nu} \int_{A^\nu_y} |(u_k)^\nu_y - (u_\infty)^\nu_y|\, d\mathcal{L}^2(z)\, d\mathcal{L}^1(y) = \lim_{k\to\infty} \int_A |u_k - u_\infty|\, dx = 0$$

we can find a strictly increasing sequence $\{k(h)\} \subset \mathbb{N}$ such that

$$\liminf_{k\to\infty} \mathbb{L}(T_k, A; \nu) = \lim_{h\to\infty} \mathbb{L}(T_{k(h)}, A; \nu)$$

and the sliced currents $(T_{k(h)})^\nu_y$ converge to $(T_\infty)^\nu_y$ weakly in $\mathcal{D}_2(A^\nu_y \times \mathbb{S}^2)$ as $h \to \infty$ for $\mathcal{L}^1$-a.e. $y \in A_\nu$. By (6.3), where $n = 2$, we thus have

$$\liminf_{h\to\infty} \mathbb{L}(T_{k(h)}, A^\nu_y) \geq \mathbb{L}(T_\infty, A^\nu_y) \tag{6.5}$$

for any such y. Integrating both sides of (6.5) on A_ν, using Fatou's lemma and (6.4) we thus get for any $\nu \in S^2_+$ and $A \subset B^3$ open

$$\liminf_{k\to\infty} \mathbb{L}(T_k, A) \geq \liminf_{h\to\infty} \mathbb{L}(T_{k(h)}, A;\nu) \geq \mathbb{L}(T_\infty, A;\nu)\,. \tag{6.6}$$

Consider now the Radon measure $\lambda := \mathcal{L}^3 + \theta_\infty\, \mathcal{H}^1 \llcorner \operatorname{set}(L_\infty)$, where θ_∞ is the integer-valued density of the current $L_\infty = L_{T_\infty} \in \mathcal{R}_1(B^3)$ corresponding to T_∞, and set (L_∞) the 1-rectifiable set of points with positivity density θ_∞.

Let $\{\nu^{(i)}\}_i \subset S^2_+$ be a countable dense sequence. Setting

$$\varphi_i(x) := \begin{cases} |\Delta(u_\infty)^{\nu^{(i)}}_y(z)| & \text{if } x \in B^3 \setminus \operatorname{set}(L_\infty)\,,\ x = y\,\nu^{(i)} + z \bullet \tau(\nu^{(i)}) \\ 8\pi & \text{if } x \in \operatorname{set}(L_\infty) \end{cases}$$

we obtain for every i and for each open set $A \subset B^3$:

$$\mathbb{L}(T_\infty, A;\nu^{(i)}) = \int_A \varphi_i\, d\lambda\,.$$

By the superadditivity of the lim inf operator, using (6.6) we thus get

$$\liminf_{k\to\infty} \mathbb{L}(T_k) \geq \sum_i \int_{A_i} \varphi_i\, d\lambda \tag{6.7}$$

for any finite family of pairwise disjoint open sets $A_i \subset B^3$. We now recall that by [4, Lemma 2.35]

$$\int_{B^3} \sup_{i\in\mathbb{N}} \varphi_i\, d\lambda = \sup\Big\{ \sum_{i\in I} \int_{A_i} \varphi_i\, d\lambda \Big\}$$

where the supremum ranges over all finite sets of indices $I \subset \mathbb{N}$ and all families $\{A_i\}_{i\in I}$ of pairwise disjoint open sets with compact closure in B^3. By (6.7), we then conclude with

$$\liminf_{k\to\infty} \mathbb{L}(T_k) \geq \int_{B^3} \sup_{i\in\mathbb{N}} \varphi_i\, d\lambda = \mathbb{L}(T_\infty)\,.$$

Finally, inequality (6.3) is similarly obtained for any open set $A \subset B^3$, and hence the dimension reduction argument from [11] applies. □

7. The explicit formula

In this section, we obtain in any dimension $n \geq 3$ the explicit formula for the relaxed energy (0.11) of Sobolev maps in $W^{2,1}(B^n, \mathbb{S}^2)$.

With the notation from (1.13) and (1.17), Theorem 0.1 states:

$$\forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,, \quad \widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty\,.$$

In terms of currents, and on account of definition (6.1) and Remark 6.2, the proof given below implies that the latter formula is equivalent to:

$$\widetilde{\mathbb{L}}(u) = \min\big\{ \mathbb{L}(T) \mid T \in \mathcal{T}^{\mathbb{L}}_u \big\} \quad \forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,. \tag{7.1}$$

Proof of Theorem 0.1. We first prove the lower bound

$$\widetilde{\mathbb{L}}(u) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,.$$

Assume $\widetilde{\mathbb{L}}(u) < \infty$, and let $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ be any sequence converging to u in $L^1(B^n, \mathbb{R}^3)$ and such that $\sup_k \mathbb{L}(u_k) < \infty$. We thus have to show that

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,. \tag{7.2}$$

Possibly passing to a (not relabeled) subsequence, we can assume that the lim inf in the latter formula is a limit. The lower bound (0.10) yields $\sup_k \mathbb{D}(u_k) < \infty$. Therefore, compare Sec. 1, a (not relabeled) subsequence of G_{u_k} weakly converges in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to some current T in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$, see Definition 1.9. Moreover, the L^1-convergence $u_k \to u$ implies that the underlying function u_T agrees with u, whence $T \in \mathcal{T}_u^{\mathbb{L}}$, see (6.1). Therefore, by (1.15) and (1.17) we infer:

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi \cdot \mathbf{M}(L) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,.$$

Since moreover $\mathbb{L}(u_k) = \mathbb{L}(G_{u_k})$ for each k, the lower semicontinuity theorem 6.3, where $T_k = G_{u_k}$, gives

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(T)$$

and hence inequality (7.2) readily follows, as required.

Thanks to the energy lower bound, for any given $u \in W^{2,1}(B^n, \mathbb{S}^2)$ we now have to find a sequence $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ such that $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^3)$ and

$$\lim_{k\to\infty} \mathbb{L}(u_k) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,.$$

To this aim, we first recall that the class $\mathcal{T}_u^{\mathbb{L}}$ is non-empty, see Remark 6.2, whence $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$. Setting then $T_u = G_u + L_u \times [\![\, \mathbb{S}^2 \,]\!]$, where L_u is a minimal integral connection of $\mathbb{P}(u)$, see Remark 1.11, it turns out that T_u is an energy minimizing current T_u in the class $\mathcal{T}_u^{\mathbb{L}}$, so that in particular

$$\mathbb{L}(T_u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty\,.$$

Therefore, the requested strong approximation property is given by Theorem 7.1 below, when applied to $T = T_u$. In particular, equation (7.1) holds true. $\square$

The density theorem. The following density result concludes the proof of Theorem 0.1.

Theorem 7.1. *Let $n \geq 3$ and $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$, see Definition 6.1, so that (1.11) holds. Then, there exists a sequence $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ such that $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^3)$, the currents G_{u_k} weakly converge to T in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$, and $\mathbb{L}(u_k) \to \mathbb{L}(T)$ as $k \to \infty$.*

In Appendix A, we give a shorter proof of Theorem 7.1 in low dimension $n = 3$. Following arguments from the case of the Dirichlet energy analysed in [22], we make use of Proposition 1.16.

In dimension $n \geq 4$, we are not able to extend Proposition 1.16, see Remark 1.17, whereas the strategy from [26] doesn't apply, since it is based on a partial regularity result for the Dirichlet energy. Therefore, making use of an idea by M. Giaquinta from [27], we have to proceed by means of Theorem 7.3 below. Some further notation is in order.

For currents $T \in \mathcal{D}_n(B^n \times \mathbb{S}^2)$, we denote by $\mathbf{F}(T)$ the *flat norm*

$$\mathbf{F}(T) := \sup\{\langle T; \omega\rangle \mid \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^2)\,,\ \mathbf{F}(\omega) \leq 1\}$$

where for every $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^2)$

$$\mathbf{F}(\omega) := \max\left\{ \sup_{z \in B^n \times \mathbb{S}^2} \|\omega(z)\|\,,\ \sup_{z \in B^n \times \mathbb{S}^2} \|d\omega(z)\| \right\}.$$

Remark 7.2. As $|T(\omega)| \leq \mathbf{F}(T)\,\mathbf{F}(\omega)$, we infer that $T_k \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ provided that $\mathbf{F}(T_k - T) \to 0$. Notice that the converse implication holds true on the class $\mathcal{R}_n(B^n \times \mathbb{S}^2)$, as we deal with compactly supported currents, compare e.g. [24, Sec. 5.1.3, Thm. 2].

If $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$, so that (1.11) holds, we denote by μ_T the finite Radon measure on B^n given for every Borel set $B \subset B^n$ by

$$\mu_T(B) := 8\pi \cdot \mathbf{M}(L \llcorner B)\,. \tag{7.3}$$

Theorem 7.3. *Let $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$, let $\varepsilon \in (0, 1/2)$ and $k \in \mathbb{N}$. We can find a current $\widetilde{T} \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ such that*

$$\mathbb{L}(\widetilde{T}) \leq \mathbb{L}(T) + \varepsilon^k\,, \quad \mathbf{F}(\widetilde{T} - T) \leq \varepsilon^k\,, \quad \textit{and} \quad \mu_{\widetilde{T}}(B^n) \leq \frac{1}{2} \cdot \mu_T(B^n)\,.$$

The proof of the *approximation theorem* 7.3 is rather technical. Therefore, it is postponed to the Appendices B and C.

Proof of Theorem 7.1. By Theorem 7.3 and Remark 7.2, using a diagonal argument we find a sequence $\{T_k\} \subset \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ that weakly converges to T in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ with $\mathbb{L}(T_k) \to \mathbb{L}(T)$ as $k \to \infty$, and such that $\mu_{T_k}(B^n) = 0$ for each k. Therefore, $T_k = G_{u_k}$ for some $u_k \in W^{2,1}(B^n, \mathbb{S}^2)$, and hence $\mathbb{L}(T_k) = \mathbb{L}(u_k)$. Since

$$(\partial T_k) \llcorner B^n \times \mathbb{S}^2 = 0 \quad \forall\, k$$

by (1.14) we have $\mathbb{P}(u_k) = 0$. Therefore, by Theorem 5.4 each u_k is the strong limit of a sequence $\{u_{k,h}\}_h$ in $C^\infty(B^n, \mathbb{S}^2)$, with $G_{u_{k,h}} \rightharpoonup G_{u_k}$ in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ and also $\mathbb{L}(u_{k,h}) \to \mathbb{L}(u_k)$ as $h \to \infty$. Using Remark 7.2, a further diagonal argument yields the smooth approximating sequence. □

8. Final remarks and open questions

The relaxed Laplacean energy of maps satisfying a suitable Dirichlet-type boundary condition can be treated similarly to the case analysed here, and we refer to [26] for the corresponding results concerning the Dirichlet energy.

As already mentioned in Sec. 2, a part from the case of low dimension $n = 2$, we are not able to give an explicit formula of the relaxed energy in case of maps u in $L^1(B^n, \mathbb{S}^2) \setminus W^{2,1}(B^n, \mathbb{S}^2)$ satisfying $\widetilde{\mathbb{L}}(u) < \infty$.

To this purpose, we have seen in Proposition 2.1 that in the particular case when the gradient Du belongs to $\mathrm{BV}(B^n, \mathbb{R}^{3\times n})$, the singular component $(\mathrm{Div}Du)^s$ is "tangential" to $\mathbb{S}^2$. On the other hand, in the $W^{2,1}$-case we have proved that the energy gap $8\pi \cdot \mathbf{m}_{i,B^n}(B^n)$ does not depend on the tangential component $\tau(u)$ of the Laplacean vector, see (0.2).

Therefore, since we are dealing with maps in $W^{1,2}(B^n, \mathbb{S}^2)$, formula

$$\widetilde{\mathbb{L}}(u) = |\mathrm{Div}Du|(B^n) + 8\pi \cdot \mathbf{m}_{i,B^n}(B^n) < \infty \quad \forall\, u \in \mathbb{L}(B^n, \mathbb{S}^2)$$

is expected to hold in any dimension $n \geq 3$, i.e., no extra terms in correspondence to the singular part of the measure $\mathrm{Div}Du$ should appear.

The BV case. On account of (0.17), we now show in any dimension $n \geq 3$ the lower bound (0.18). To this purpose, denoting by $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents $T = G_{u_T} + L \times [\![\mathbb{S}^2]\!]$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with $u_T \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, and defining

$$\mathbb{L}(T) := |\mathrm{Div}Du_T|(B^n) + 8\pi \cdot \mathbf{M}(L)$$

it turns out that the lower semicontinuity theorem 6.3 continues to hold in $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$.

In fact, when $n = 2$ it is obtained again as a consequence of Theorem 2.2, whereas in case $n \geq 3$ we can apply the same dimension reduction procedure, due to the structure of the measure $\mathrm{Div}Du$ inherited by the membership of the gradient function Du to the class BV. As a consequence, the lower bound (0.18) is obtained exactly as in the proof of Theorem 0.1 from Sec. 7.

On the other hand, in order to obtain the equality sign in (0.18), one should extend Theorem 5.2, by proving for any $u \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, where $n \geq 3$, the existence of a sequence $\{u_h\} \subset R^\infty_{n-3}(B^n, \mathbb{S}^2)$ converging to u in $W^{1,2}(B^n, \mathbb{R}^3)$ and such that $\mathbb{L}(u_h) \to |\mathrm{Div}Du|(B^n)$ as $h \to \infty$. Therefore, it is not clear how to extend the density theorem 7.1 to the larger class $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$, that would give the equality sign in (0.18).

Remark 8.1. Extending the validity of the lower semicontinuity theorem 6.3 to the case of general maps $u \in \mathbb{L}(B^2, \mathbb{S}^2)$ is another open question. In fact, when $Du \notin \mathrm{BV}(B^2, \mathbb{R}^{3\times 2})$ we do not have enough information concerning the singular part of the measure $\mathrm{Div}Du$ in order to apply the same dimension reduction procedure. Therefore, we are not able to extend the lower bound (0.18) to the whole class of maps with finite relaxed energy.

Bienergy. Theorem 6.3 may be compared to the lower semicontinuity property (1.12) of the Dirichlet energy functional $\mathbb{D}(T)$ in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$, that holds true since $\mathbb{D}(T)$ agrees with the parametric polyconvex lower semicontinuous extension of the Dirichlet integrand.

Concerning *second order* functionals as e.g. the Laplacean energy $\mathbb{L}(u)$, a part from the easier case of 1-dimensional currents, compare [1, 2], to our knowledge it is not clear how to apply the approach from [25] in order to find the explicit formula of the parametric polyconvex lower semicontinuous extension. Therefore, we have followed a different strategy.

The same difficulty appears in case of the bienergy functional $\mathbb{H}(u)$ of maps $u : B^n \to \mathbb{S}^4$, see (0.4). In fact, finding the expression of the parametric polyconvex lower semicontinuous extension of the bienergy to Cartesian currents in $B^n \times \mathbb{S}^4$ would give us the explicit value of the minimal bienergy E_4 of degree one maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, a non-trivial open problem.

We recall that in [5] it is proved that the bienergy minimum is attained, and that $E_4 > 16 \cdot \mathcal{H}^4(\mathbb{S}^4)$, the expected weight being $E_4 = 24 \cdot \mathcal{H}^4(\mathbb{S}^4)$.

Denote now on maps $u \in L^1(B^n, \mathbb{S}^4)$

$$\widetilde{\mathbb{H}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{H}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^4),\ u_k \to u \text{ in } L^1(B^n, \mathbb{R}^5) \Big\} \,.$$

Under prescribed first order boundary conditions, by the Bochner inequality one infers that $\widetilde{\mathbb{H}}(u) < \infty$ if and only if $u \in W^{2,2}(B^n, \mathbb{S}^4)$. Since moreover by (0.6) we have $W^{2,2}(B^n, \mathbb{S}^4) \subset W^{1,2}(B^n, \mathbb{S}^4)$, due to the continuous embedding of $W^{1,4}(B^n)$ in VMO, in low dimension $n \leq 4$ we get $\widetilde{\mathbb{H}}(u) = \mathbb{H}(u)$ for every $u \in W^{2,2}(B^n, \mathbb{S}^4)$. Finally, in high dimension $n \geq 5$, with a similar strategy to the one adopted in this paper it could be shown that

$$\widetilde{\mathbb{H}}(u) = \mathbb{H}(u) + E_4 \cdot \mathbf{m}_i(\mathbb{P}(u)) < \infty \quad \forall\, u \in W^{2,2}(B^n, \mathbb{S}^4)$$

where $\mathbf{m}_i(\mathbb{P}(u))$ denotes the integral mass of the $(n-5)$-current $\mathbb{P}(u)$ of the singularities, that is defined on maps $u \in W^{1,4}(B^n, \mathbb{S}^4)$ in a similar way to (1.13), but in terms of a volume 4-form in $\mathbb{S}^4$.

Appendix A. The density theorem in 3D

We give a sketch of a proof of Theorem 7.1 in low dimension $n = 3$, by following the lines of Thm. 1 from [25, Sec. 4.2.6], to which we refer for further details.

We make use of the following variant of Theorem 4.1. Recalling (4.3), for each $m > 0$ small we denote

$$\Phi^m_\delta(\widetilde{x}, \widehat{x}) := (\widetilde{x}, m\, \varphi_\delta(\widetilde{x})\, \widehat{x})\,, \quad \Omega^m_\delta := \Phi^m_\delta(D^1_r \times B^2)\,.$$

Proposition A.1 is obtained by readapting an argument taken from [23], compare [25, Sec. 4.2.3]. Therefore, its proof is omitted.

Proposition A.1. *Let U be a neighborhood of the segment joining a_- to a_+, and let $u : U \to \mathbb{S}^2$ be a $W^{1,2}$-map with finite Laplacean energy which is smooth in U outside the singular points $a_\pm$, where it has degree*

$\deg(u, a_\pm) = k_\pm$ *for some* $k_\pm \in \mathbb{Z}$, *see* (1.19). *Let* $\mathrm{d} \in \mathbb{Z}$. *Then for all positive* ε *and for* δ, $m > 0$ *sufficiently small there exists a smooth function* $u_\varepsilon : \mathbb{R}^3 \setminus \{a_-, a_+\} \to \mathbb{S}^2$ *such that* $u_\varepsilon \equiv u$ *outside* Ω^m_δ, $\deg(u, a_+) = k_+ - \mathrm{d}$, $\deg(u, a_-) = k_- + \mathrm{d}$, *and finally*

$$\mathbb{L}(u_\varepsilon, \Omega^m_\delta) := \int_{\Omega^m_\delta} |\Delta u_\varepsilon|\, dx \le |a_+ - a_-| \cdot 8\pi\, |\mathrm{d}| + \varepsilon\,.$$

Proof of Theorem 7.1, case $n = 3$. By Theorem 5.2 and Proposition 1.16, we find a sequence $\{u_k\} \subset W^{2,1}(B^3, \mathbb{S}^2) \cap R^\infty_0$ such that $u_k \to u$ strongly in $W^{1,2}(B^3, \mathbb{R}^3)$, with $\mathbb{L}(u_k) \to \mathbb{L}(u)$ and $\mathbf{m}_{i,B^3}(\mathbb{P}(u_k) - \mathbb{P}(u)) \to 0$. We thus reduce to the case when $T = G_u + S_T$, where $u \in W^{2,1}(B^3, \mathbb{S}^2) \cap R^\infty_0$ and $S_T = L \times [\![\, \mathbb{S}^2 \,]\!]$ for some $L \in \mathcal{R}_1(\mathbb{R}^3)$ with $(\partial L) \llcorner B^3 = -\mathbb{P}(u)$, where $\mathbb{P}(u) \in \mathcal{R}_0(B^3)$ with $\mathbf{M}(\mathbb{P}(u)) < \infty$.

If in addition we have $\mathbf{M}(\partial L) < \infty$, i.e., L is an integral current in $\mathbb{R}^3$, as in Steps 1–3 of the proof of Thm. 1 from [25, Sec. 4.2.5], we reduce to

$$S_T = \sum_{\mathrm{d} \in I} P^{\mathrm{d}} \times \mathrm{d}\, [\![\, \mathbb{S}^2 \,]\!]$$

where I is a finite set of integer indices and the P^{d}'s are polyhedral lines in B^3 with pairwise disjoint supports. Let now S_i be anyone of the segments of the P^{d}'s, and let $[\![\, S_i \,]\!] = [\![\, (n_i, p_i) \,]\!]$. By a suitable change of coordinates we can assume that $n_i = a_+$ and $p_i = a_-$, as in Sec. 4. We then apply Proposition A.1. To this aim, notice that we can take m and δ sufficiently small so that the neighborhoods Ω^m_δ corresponding to different segments S_i are pairwise disjoint and contained in B^3. We then replace u in a small neighborhood of each S_i by a function $u_\varepsilon \in W^{2,1}(B^3, \mathbb{S}^2)$ satisfying

$$\mathbb{L}(u_\varepsilon) \le \mathbb{L}(u) + \sum_{\mathrm{d} \in I} \mathbf{M}(P^{\mathrm{d}}) \cdot 8\pi\, |\mathrm{d}| + \varepsilon\,.$$

The function u_ε this way obtained belongs to $W^{2,1}(B^3, \mathbb{S}^2)$ and *it has degree zero around each end point of the segments* S_i which belongs to the open ball B^3, see (1.19), i.e., around each singular point of u_ε, whence $\mathbb{P}(u_\varepsilon) = 0$, see (1.13). Moreover, taking $\delta \searrow 0$ as $\varepsilon \searrow 0$, it turns out that $G_{u_\varepsilon} \rightharpoonup T$ weakly in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$ as $\varepsilon \to 0$ along a sequence. Since moreover $\mathbb{P}(u_\varepsilon) = 0$, by Theorem 5.4 we find a sequence $\{u^\varepsilon_k\} \subset C^\infty(B^3, \mathbb{S}^2)$ such that $u^\varepsilon_k \to u_\varepsilon$ strongly in $W^{1,2}(B^3, \mathbb{R}^3)$ and $\mathbb{L}(u^\varepsilon_k) \to \mathbb{L}(u)$, so that $G_{u^\varepsilon_k} \rightharpoonup G_{u_\varepsilon}$ weakly in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$. Therefore, a diagonal argument concludes the proof in case $\mathbf{M}(\partial L) < \infty$.

Finally, in general we only have $\mathbf{M}((\partial L) \llcorner B^3) < \infty$. In that case, in order to apply the strong polyhedral approximation theorem we make use of a slicing argument as in the proof of Thm. 1 from [25, Sec. 4.2.6]. □

Appendix B. The approximation theorem. I

The proof of Theorem 7.3 is given in Appendix C. It is based on the following local arguments. Firstly, Proposition B.1, we show how to "deform" a current T, satisfying suitable energy estimates on the boundary of a ball, into a current satisfying a *bound on the oscillation*. Secondly, Propositions B.2 and B.3, we make use of a local approximation argument that relies on the Dipole construction from Theorem 4.1. Since we follow arguments from [27, 28, 29, 20], some details will be omitted.

Notation. If $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$, in formula (1.11) we can write

$$T = G_u + S_T \,, \quad S_T := \sum_{\mathrm{d}\in\mathbb{Z}} L_{\mathrm{d}} \times \mathrm{d}\,[\![\, \mathbb{S}^2 \,]\!] \tag{B.1}$$

where $u = u_T \in W^{2,1}(B^n, \mathbb{S}^2)$, every L_{d} is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ with multiplicity one, and the $(n-2)$-rectifiable sets $\mathcal{L}_{\mathrm{d}} := \text{set}\,(L_{\mathrm{d}})$ are pairwise disjoint. As a consequence, by definition (7.3) we have

$$\mu_T(B) = 8\pi \cdot \sum_{\mathrm{d}\in\mathbb{Z}} |\mathrm{d}| \cdot \mathbf{M}(L_d \llcorner B) \tag{B.2}$$

and the *rectifiable measure* μ_T satisfies $\mu_T = \theta_T\, \mathcal{H}^{n-2} \llcorner \mathcal{L}_T$, where $\mathcal{L}_T := \bigcup_{\mathrm{d}\in\mathbb{Z}} \mathcal{L}_{\mathrm{d}}$ is an $(n-2)$-rectifiable set, with $\mathcal{H}^{n-2}(\mathcal{L}_T) < \infty$, and the density $\theta_T : \mathcal{L}_T \to [0, +\infty)$ is $\mathcal{H}^{n-2} \llcorner \mathcal{L}_T$-summable, with $\theta_T(x) = |\mathrm{d}|$ if $x \in \mathcal{L}_{\mathrm{d}}$. Therefore, there exists $\overline{\mathrm{d}} \in \mathbb{N}^+$, only depending on T, such that

$$\mu_T(B^n \setminus \mathcal{L}_T(\overline{\mathrm{d}})) < \frac{1}{4}\, \mu_T(B^n) \,, \quad \text{where} \quad \mathcal{L}_T(\overline{\mathrm{d}}) := \bigcup_{|\mathrm{d}|\le\overline{\mathrm{d}}} \mathcal{L}_{\mathrm{d}} \,. \tag{B.3}$$

Finally, for each Borel set $B \subset B^n$ we let

$$\mathbb{L}(T, B) := \mathbb{L}(u, B) + \mu_T(B) \,, \quad \mathbb{L}(u, B) := \int_B |\Delta u| \, dx \,.$$

For $\varepsilon > 0$ small, we denote by $\mathbb{S}^2_\varepsilon := \{y \in \mathbb{R}^3 \mid \text{dist}\,(y, \mathbb{S}^2) \le \varepsilon\}$ the ε-neighborhood of $\mathbb{S}^2$ and by Π_ε the nearest point projection of $\mathbb{S}^2_\varepsilon$ onto $\mathbb{S}^2$. Notice that the Lipschitz constant L_ε of the smooth map Π_ε goes to 1 as $\varepsilon \to 0^+$. For $y \in \mathbb{S}^2$ we also denote by

$$B_{\mathbb{S}^2}(y, \varepsilon) := \bar{B}^3_\varepsilon(y) \cap \mathbb{S}^2$$

the intersection of $\mathbb{S}^2$ with the closed 3-ball $\bar{B}^3_\varepsilon(y)$ in $\mathbb{R}^3$ of radius ε centered at y, so that we have $\Pi_\varepsilon(\bar{B}^3_\varepsilon(y)) = B_{\mathbb{S}^2}(y, \varepsilon)$. Moreover, we let $\Psi_{(y,\varepsilon)} : \mathbb{R}^3 \to B_{\mathbb{S}^2}(y, \varepsilon)$ be the retraction map $\Psi_{(y,\varepsilon)}(z) := \Pi_\varepsilon \circ \xi_{(y,\varepsilon)}$, where

$$\xi_{(y,\varepsilon)}(z) := \begin{cases} z & \text{if} \quad z \in \bar{B}^3_\varepsilon(y) \\ \varepsilon\, \dfrac{z - y}{|z - y|} & \text{if} \quad z \in \mathbb{R}^3 \setminus \bar{B}^3_\varepsilon(y) \,. \end{cases} \tag{B.4}$$

Then, $\Psi_{(y,\varepsilon)}$ is Lipschitz continuous and $\text{Lip}\, \Psi_{(y,\varepsilon)} = \text{Lip}\, \Pi_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$.

Finally, in the sequel we denote by $c > 0$ an absolute real constant, possibly varying from line to line.

Slicing properties. For every point $x_0 \in B^n$ we consider on $(x, y) \in B^n \times \mathbb{R}^3$ the distance functions $\widehat{d}_{x_0}(x, y) = \widetilde{d}_{x_0}(x) := |x - x_0|$. Since T is an i.m. rectifiable current satisfying the null-boundary condition (1.10), for a.e. radius $r \in (0, r_0)$, where $r_0 := \mathrm{dist}\,(x_0, \partial B^n) > 0$, the *sliced current*

$$\langle T, \widehat{d}_{x_0}, r\rangle = \langle G_{u_T}, \widehat{d}_{x_0}, r\rangle + \langle S_T, \widehat{d}_{x_0}, r\rangle$$

(cf. [24, Sec. 2.2.5]) is i.m. rectifiable in $\mathcal{R}_{n-1}(\partial B^n_r(x_0) \times \mathbb{S}^2)$, with no boundary, $\partial\langle T, \widehat{d}_{x_0}, r\rangle = 0$ on $\mathcal{D}^{n-2}(\partial B^n_r(x_0) \times \mathbb{S}^2)$.

Denoting by $u_{r,x_0} := u_{|\partial B^n_r(x_0)}$ the restriction of u to $\partial B^n_r(x_0)$, then u_{r,x_0} is a Sobolev map in $W^{2,1}(\partial B^n_r(x_0), \mathbb{S}^2)$, with $\int_{\partial B^n_r(x_0)} |\Delta u_{r,x_0}|\, d\mathcal{H}^{n-1} < \infty$, where $\Delta u_{r,x_0}$ denotes the Laplacean of the sliced map u_{r,x_0} w.r.t. a tangential frame to $\partial B^n_r(x_0)$, and on account of (1.3)

$$\langle\langle G_u, \widehat{d}_{x_0}, r\rangle; \omega\rangle = \int_{\partial B^n_r(x_0)} (\mathrm{Id} \bowtie u_{r,x_0})^{\#}\omega\,, \qquad \omega \in \mathcal{D}^{n-1}(\partial B^n_r(x_0) \times \mathbb{S}^2)$$

whereas by (B.1)

$$\langle S_T, \widehat{d}_{x_0}, r\rangle = \sum_{\mathrm{d}\in\mathbb{Z}} \langle L_{\mathrm{d}}, \widetilde{d}_{x_0}, r\rangle \times \mathrm{d}\,[\![\, \mathbb{S}^2 \,]\!] \qquad \text{on} \quad \mathcal{D}^{n-1}(\partial B^n_r(x_0) \times \mathbb{S}^2)\,.$$

As in Definition 6.1, the sliced current $\langle T, \widehat{d}_{x_0}, r\rangle$ is said to be a Cartesian current in $\mathrm{cart}^{\mathbb{L}}(\partial B^n_r(x_0) \times \mathbb{S}^2)$. The Laplacean energy of $\langle T, \widehat{d}_{x_0}, r\rangle$ is

$$\mathbb{L}(\langle T, \widehat{d}_{x_0}, r\rangle) := \mathbb{L}(\langle T, \widehat{d}_{x_0}, r\rangle, B^n)$$

where for each Borel set $B \subset B^n$ we let

$$\mathbb{L}(\langle T, \widehat{d}_{x_0}, r\rangle, B) := \mathbb{L}(u_{r,x_0}, B) + 8\pi \cdot \sum_{\mathrm{d}\in\mathbb{Z}} |\mathrm{d}| \cdot \mathbf{M}(\langle L_{\mathrm{d}}, \widetilde{\delta}_{x_0}, r\rangle \llcorner B) \tag{B.5}$$

and

$$\mathbb{L}(u_{r,x_0}, B) := \int_{\partial B^n_r(x_0)\cap B} |\Delta u_{r,x_0}|\, d\mathcal{H}^{n-1}\,.$$

Projecting the image of a current. Let $D_r := B^{n-2}(0_{\mathbb{R}^{n-2}}, r)$, and denote $O := 0_{\mathbb{R}^n}$. We have:

Proposition B.1. *Let $0 < R < R_0 < 1$ and $T \in \mathrm{cart}^{\mathbb{L}}(B^n_{R_0} \times \mathbb{S}^2)$ such that $\theta_T(O) \leq \overline{\mathrm{d}}$, see (B.3), and*

$$\begin{aligned} \mathbb{L}(\langle T, \widehat{d}_O, R\rangle, \partial B^n_R \setminus (\bar{D}_R \times \{0_{\mathbb{R}^2}\})) &\leq \ c\,\sigma\,\theta_T(O) R^{n-3} \\ \mathbb{L}(\langle T, \widehat{d}_O, R\rangle) &\leq \ c\,\theta_T(O)\, R^{n-3} \\ \int_{\partial B^n_R} |u(x) - y|^2\, d\mathcal{H}^{n-1} &\leq \ c\,\sigma\, R^{n-1} \end{aligned} \tag{B.6}$$

for some $y \in \mathbb{S}^2$ and for $\sigma > 0$ small enough, in such a way that $\sigma^{1/3}\,\overline{\mathrm{d}} \leq 1$.

Then, there exists an absolute constant $c > 0$ such that, if $q \in \mathbb{N}^+$ is the integer part of $\sigma^{\alpha(n)}$, where $\alpha(n) := 1/(6(2 - n)) < 0$, and $r = R(1 - 1/q)$, *we can find a Cartesian current* $\widetilde{T}\in \mathrm{cart}^{\mathbb{L}}((B^n_R\setminus \bar B^n_r)\times\mathbb{S}^2)$ *for which the following facts hold:*

(*a*) $\langle \widetilde T,\widehat d_O,R\rangle=\langle T,\widehat d_O,R\rangle$ *and* $\langle \widetilde T,\widehat d_O,r\rangle=(\psi_{R,r}\bowtie\Psi_{(y,\varepsilon_\sigma)})_\#\langle T,\widehat d_O,R\rangle$, *see* (B.4), *where* $\varepsilon_\sigma:=c\,\sigma^{1/3}$ *and* $\psi_{R,r}(x):=rx/R$, *so that the support* $\mathrm{spt}\,\langle \widetilde T,\widehat d_O,r\rangle\subset\partial B^n_r\times B_{\mathbb{S}^2}(y,\varepsilon_\sigma)$*;*

(*b*) $\widetilde T$ *has small energy on* $B^n_R\setminus\bar B^n_r$, *i.e.,*

$$\mathbb{L}(\widetilde T,B^n_R\setminus\bar B^n_r)\le c\,\frac{R}{q}\,\mathbb{L}(\langle T,\widehat d_O,R\rangle)\,; \tag{B.7}$$

(*c*) *the flat distance to the graph* G_y *of the constant map* y *is small:*

$$\mathbf{F}((\widetilde T-G_y)\llcorner(B^n_R\setminus\bar B^n_r)\times\mathbb{S}^2)\le c\,\frac{\sigma}{q}\,R^n\le c\,\sigma\,R_0^{n-1}\,. \tag{B.8}$$

Proof. We find a suitable subdivision of ∂B^n_R in a grid made of small $(n-1)$-dimensional "cubes" of side R/q. Denoting by Σ^k_R the union of the k-faces of the grid that do not intersect $\bar D_R\times\{0_{\mathbb{R}^2}\}$, using the first inequality in (B.6) we may and do estimate the energy of the restriction of $\langle T,\widehat d_O,R\rangle$ to $\Sigma^k_R\times\mathbb{S}^2$ by $c\,\sigma\,\theta_T(O)\,q^{n-1-k}\,R^{k-2}$ for every $k=1,\dots,n-2$.

In particular, since $\sigma^{1/3}\,\Theta_T(O)\le\sigma^{1/3}\,\overline{\mathrm{d}}\le 1$, then the energy of the restriction of $\langle T,\widehat d_O,R\rangle$ to $\Sigma^2_R\times\mathbb{S}^2$ is smaller than the quantity $E(\sigma):=c\,\sigma^{2/3}\,q^{n-3}$. Therefore, since $E(\sigma)\to 0$ as $\sigma\to 0$, provided that $q\in\mathbb{N}^+$ is chosen as in the hypothesis, taking $\sigma>0$ small it turns out that the restriction of $\langle T,\widehat d_O,R\rangle$ to $\Sigma^2_R\times\mathbb{S}^2$ has no vertical part, hence it agrees with the graph current of a $W^{2,1}$ map w with values into $\mathbb{S}^2$. In addition,

$$\int_{\Sigma^1_R}|Dw_{|\Sigma^1_R}|^2\,d\mathcal{H}^1\le\int_{\Sigma^1_R}|\Delta w_{|\Sigma^1_R}|\,d\mathcal{H}^1\le c\,\sigma^{2/3}\,q^{n-2}\,\frac{1}{R}\,.$$

The grid of ∂B^n_R being made of approximately q^{n-1} cubes of side R/q, we have $\mathcal{H}^1(\Sigma^1_R)\le c\,R\,q^{n-2}$ and hence, by Hölder inequality

$$\int_{\Sigma^1_R}|Dw_{|\Sigma^1_R}|\,d\mathcal{H}^1\le c\;\sigma^{1/3}q^{n-2}\le c\,\sigma^{1/3}=:\varepsilon_\sigma\,.$$

By using the third inequality in (B.6) and the above formula, we then infer that we may and do assume that the image $w(\Sigma^1_R)$ is contained in the geodesic ball $B_{\mathbb{S}^2}(y,\varepsilon_\sigma)$.

Therefore, using the argument of Step 3 of [29], we define the current $\widetilde T$ on the cubes of $B^n_R\setminus B^n_r$ that do not intersect $\bar D_R\times\{0_{\mathbb{R}^2}\}$. In fact, since it is obtained by means of extensions of the restriction of w to the 1-skeleton Σ^1_R, no boundary is produced in the construction, as $w(\Sigma^1_R)\subset B_{\mathbb{S}^2}(y,\varepsilon_\sigma)$.

This way we obtain a bound of the energy of $\widetilde T$ in terms of

$$c\,\frac{R}{q}\,\mathbb{L}(\langle T,\widehat d_O,R\rangle,\partial B^n_R\setminus(\bar D_R\times\{0_{\mathbb{R}^2}\}))$$

and hence in terms of the right-hand side of (B.7). Using a slightly different argument when defining $\widetilde{T}$ on the cubes of $B^n_R \setminus B^n_r$ that intersect the $(n-2)$-disk $\bar{D}_R \times \{0_{\mathbb{R}^2}\}$, by the second inequality in (B.6) we obtain an extra term in the energy estimate of $\widetilde{T}$ given again by the right-hand side of (B.7).

By the third inequality in (B.6), by the construction of $\widetilde{T}$, and since $0 < R < R_0 < 1$, we obtain the bound (B.8) of the flat distance, whereas property (a) follows by using the argument of [29, Step 3]. □

Approximation on a ball. Let $\varphi_\delta : (-r, r) \to [0, \delta]$ be the smooth function defined in (4.3) for the Dipole problem, and let Φ_δ be given by (4.4), where this time $z = (\widetilde{z}, \widehat{z}) \in D_r \times \bar{B}^2 \subset \mathbb{R}^{n-2} \times \mathbb{R}^2$, so that $\Omega_\delta := \Phi_\delta(D_r \times \bar{B}^2)$ is a small neighborhood of the interior of the $(n-2)$-disk $D_r \times \{0_{\mathbb{R}^2}\}$ in B^n_R. Moreover, let

$$\widetilde{\Omega}_\delta := \Phi_\delta(D_r \times \bar{B}^2_{1/2}) = \{(\widetilde{x}, \widehat{x}) \mid \widetilde{x} \in D_r\,,\ |\widehat{x}| \le \varphi_\delta(|\widetilde{x}|)/2\}\,, \tag{B.9}$$

where $|\widetilde{x}| := |\widetilde{x}|_{\mathbb{R}^{n-2}}$, $|\widehat{x}| := |\widehat{x}|_{\mathbb{R}^2}$, and

$$\Omega_{(r,\delta)} := \Omega_\delta \setminus (D_r \times \{0_{\mathbb{R}^2}\})\,.$$

In the proof of Theorem 7.3, we make use of the following

Proposition B.2. *Let $T \in \mathrm{cart}^{\mathbb{L}}(B^n_r \times \mathbb{S}^2)$ be such that* (B.1) *holds. Assume that $\mathrm{spt}\, T \subset \bar{B}^n_r \times B_{\mathbb{S}^2}(y, \varepsilon_\sigma)$, where $y \in \mathbb{S}^2$ and $\varepsilon_\sigma = c\,\sigma^{1/3}$, with $\sigma > 0$ small, and that $D_r \times \{0_{\mathbb{R}^2}\} \subset \mathrm{set}\,(L_{\mathrm{d}})$ for some $\mathrm{d} \in \mathbb{Z}$. For $\delta > 0$ small enough, we can find a current $\widetilde{T} \in \mathrm{cart}^{\mathbb{L}}((B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2)$ satisfying:*

i)
$$\begin{aligned}\partial(\widetilde{T} \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2) = \;& \partial(T \llcorner B^n_r \times \mathbb{S}^2) - \partial[\![\,\widetilde{\Omega}_\delta\,]\!] \times \delta_y \\ & - [\![\,\partial D_r \times \{0_{\mathbb{R}^2}\}\,]\!] \times \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!]\,;\end{aligned}$$

ii) $\mathbb{L}(\widetilde{T}, (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2) \le \mathbb{L}(u, (B^n_r \setminus \Omega_\delta)) + c\,\sigma\, r^{n-2} + c\,\mu_T(\Omega_{(r,\delta)})$;

iii) $\mathbf{F}((\widetilde{T} - T) \llcorner (B^n_r \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2) \le c\,\sigma\, r^{n-2}$.

Proof. Let $\psi_\delta : \Omega_\delta \setminus \widetilde{\Omega}_\delta \to \Omega_{(r,\delta)}$ be the bijective map

$$\psi_\delta(\widetilde{x}, \widehat{x}) := \Big(\widetilde{x}, \Big(2 - \frac{\varphi_\delta(|\widetilde{x}|)}{|\widehat{x}|}\Big)\,\widehat{x}\Big)\,.$$

Similarly to [28, Sec. 5.5], we infer that the current

$$\overline{T} := ((\psi_\delta)^{-1} \bowtie \mathrm{Id}_{\mathbb{R}^3})_\#(T \llcorner (\mathrm{int}\,(\Omega_{(r,\delta)}) \times \mathbb{S}^2))$$

belongs to $\mathrm{cart}^{\mathbb{L}}(\mathrm{int}\,(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2)$, its underlying $W^{2,1}$ function is $v := u \circ \psi_\delta : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to B_{\mathbb{S}^2}(y, \varepsilon_\sigma)$, where $u : B^n_r \to B_{\mathbb{S}^2}(y, \varepsilon_\sigma)$ is the $W^{2,1}$ function corresponding to T in (B.1), and

$$\mu_{\overline{T}}(\mathrm{int}\,(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \le \mu_T(\mathrm{int}\,(\Omega_{(r,\delta)}))\,.$$

Setting then $w : (\Omega_\delta \setminus \widetilde{\Omega}_\delta) \to \mathbb{R}^3$ by

$$w(x) := \Big(\frac{2|\widehat{x}|}{\varphi_\delta(|\widetilde{x}|)} - 1\Big)\, v(x) + \Big(2 - \frac{2|\widehat{x}|}{\varphi_\delta(|\widetilde{x}|)}\Big)\, y$$

using that the oscillation of v is small with $\sigma > 0$, we infer that the energy $\mathbb{L}(w, \Omega_\delta \setminus \widetilde{\Omega}_\delta)$ is small if δ and σ are small. Moreover, by projecting w into $\mathbb{S}^2$, we may and do assume that w belongs to $W^{2,1}(\Omega_\delta \setminus \widetilde{\Omega}_\delta, \mathbb{S}^2)$.

Therefore, we can define a current $\widehat{T} \in \mathrm{cart}^{\mathbb{L}}(\mathrm{int}\,(\Omega_\delta \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2)$, with underlying $W^{2,1}$ function equal to w, that satisfies the boundary condition

$$\partial \widehat{T} = \partial (T \llcorner \Omega_\delta \times \mathbb{S}^2) - \partial [\![\widetilde{\Omega}_\delta]\!] \times \delta_y - [\![\partial D_r \times \{0_{\mathbb{R}^2}\}]\!] \times \mathrm{d}\, [\![\mathbb{S}^2]\!]$$

and (taking δ small) also the energy estimate

$$\mathbb{L}(\widehat{T}, \mathrm{int}\,(\Omega_\delta \setminus \widetilde{\Omega}_\delta)) \leq c\,\sigma\, r^{n-2} + c\,\mu_T(\Omega_{(r,\delta)})\,.$$

We finally set

$$\widetilde{T} := T \llcorner (B^n_r \setminus \mathrm{int}\,(\Omega_\delta)) \times \mathbb{S}^2 + \widehat{T} \llcorner (\mathrm{int}\,(\Omega_\delta) \setminus \widetilde{\Omega}_\delta) \times \mathbb{S}^2\,.$$

Properties i)–iii) readily follows, for $\delta > 0$ small. □

THE DIPOLE CONSTRUCTION. We finally make use of the following:

Proposition B.3. *Let* $\mathrm{d} \in \mathbb{Z}$ *and* $y \in \mathbb{S}^2$. *For every* $\sigma > 0$ *there exists a function* $v_\sigma \in W^{2,1}(\widetilde{\Omega}_\delta, \mathbb{S}^2)$, *with* $\delta > 0$ *sufficiently small, such that* $G_{v_\sigma} \in \mathrm{cart}^{\mathbb{L}}(\mathrm{int}\,(\widetilde{\Omega}_\delta) \times \mathbb{S}^2)$, $v_{\sigma\#}[\![\widetilde{\Omega}_\delta]\!] = \mathrm{d}\, [\![\mathbb{S}^2]\!]$,

$$\mathbb{L}(v_\sigma, \widetilde{\Omega}_\delta) \leq \sigma\, r^{n-2} + \mathcal{H}^{n-2}(D_r) \cdot 8\pi\, |\mathrm{d}| \tag{B.10}$$

and

$$\partial G_{v_\sigma} = \partial [\![\widetilde{\Omega}_\delta]\!] \times \delta_y + [\![\partial D_r \times \{0_{\mathbb{R}^2}\}]\!] \times \mathrm{d}\, [\![\mathbb{S}^2]\!]\,. \tag{B.11}$$

Proof. By Theorem 3.2, we find $u \in \mathcal{F}_{\mathrm{d}}$ with energy $\mathbb{L}(u, \mathbb{R}^2) \leq 8\pi\, |\mathrm{d}| + \varepsilon$. Arguing as in Proposition 3.1 (where we choose $\delta = 1/2$), for each $\varepsilon > 0$ we then find a smooth map $v_\varepsilon \in C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ such that:

(1) v_ε is equal to y outside the disk $B^2_{1/2}$;
(2) $\deg v_\varepsilon = \deg u = \mathrm{d}$;
(3) $\mathbb{L}(v_\varepsilon, \mathbb{R}^2) \leq \mathbb{L}(u, \mathbb{R}^2) + \varepsilon$.

Setting $u_\varepsilon(x) := v_\varepsilon(\widehat{x})$, where $x = (\widetilde{x}, \widehat{x}) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$, we obtain $u_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^n, \mathbb{S}^2)$ such that for every $\rho \in (0, r]$

$$\mathbb{L}(u_\varepsilon, D_\rho \times B^2_{1/2}) \leq \mathcal{H}^{n-2}(D_\rho) \cdot \mathbb{L}(v_\varepsilon, \mathbb{R}^2) \leq \mathcal{H}^{n-2}(D_\rho) \cdot (8\pi\, |\mathrm{d}| + 2\varepsilon)\,.$$

Let now $0 < \delta < 1$ and $u^\varepsilon_\delta := u_\varepsilon \circ \Phi_\delta^{-1} : \widetilde{\Omega}_\delta \to \mathbb{S}^2$, where Φ_δ is given by (4.4). Arguing as e.g. in [28, Sec. 5.5], we estimate

$$\mathbb{L}(u^\varepsilon_\delta, \widetilde{\Omega}_\delta) \leq \mathbb{L}(u_\varepsilon, D_r \times B^2_{1/2}) + c\, \mathbb{L}(u_\varepsilon, (D_r \setminus D_{r-\delta}) \times B^2_{1/2})$$

where $c > 0$ is an absolute constant. Therefore, setting $v_\sigma := u^\varepsilon_\delta$ for $\varepsilon > 0$ sufficiently small, and for δ sufficiently small in dependence of ε, we get (B.10), whereas (B.11) readily follows. □

Appendix C. The approximation theorem. II

Proof of Theorem 7.3. On account of (B.3), we let

$$\widetilde{\mu}_T := \theta_T\, \mathcal{H}^{n-2} \llcorner \mathcal{L}_T(\overline{\mathrm{d}})\,. \tag{C.1}$$

By applying arguments as for instance in the proof of [16, 4.2.19] to the rectifiable measure $\widetilde{\mu}_T$, by [16, 3.2.29] we find a countable family $\mathcal{G}$ of $(n-2)$-dimensional C^1-submanifolds $\mathcal{M}_j$ of B^n such that $\widetilde{\mu}_T$-almost all of B^n is covered by $\mathcal{G}$.

Let $\sigma \in (0,1)$ to be fixed. By Vitali-Besicovitch theorem, and by the properties of the class $\mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$, we can find a number $t = t_\sigma \in (1/2, 1)$, a countable disjoint family of closed balls $B_j := \bar{B}^n(x_j, r_j)$, contained in B^n and centered at points x_j in $\mathcal{L}_T(\overline{\mathrm{d}})$, satisfying the properties listed below. In the sequel we denote by $c > 0$ an absolute constant, possibly varying from line to line, which is independent of σ and of the radii r_j of the balls B_j.

i) $\widetilde{\mu}_T(B^n \setminus \bigcup_j B_j) = 0$.

ii) For every j there is a manifold $\mathcal{M}_j$ of $\mathcal{G}$ such that the center x_j of B_j belongs to $\mathcal{M}_j$.

iii) Since $\mathcal{H}^{n-2}(\mathcal{L}_T(\overline{\mathrm{d}})) < \infty$, then

$$\sum_{j=1}^{\infty} {r_j}^{n-2} \leq c\, \mathcal{H}^{n-2}(\mathcal{L}_T(\overline{\mathrm{d}})) < \infty\,. \tag{C.2}$$

iv) We have

$$\widetilde{\mu}_T(B(x_j, r_j) \setminus (B(x_j, tr_j) \cap \mathcal{M}_j)) \leq \sigma\, \widetilde{\mu}_T(B(x_j, r_j)) \qquad \forall\, j\,. \tag{C.3}$$

v) Recalling (B.2) and(B.3), we have $\mathcal{M}_j \subset \mathcal{L}_{\mathrm{d}_j} = \mathrm{set}\,(L_{\mathrm{d}_j})$ for some $\mathrm{d}_j \in \mathbb{Z}$ with $|d_j| \leq \overline{\mathrm{d}}$, so that in particular $\Theta_T(x_j) \leq \overline{\mathrm{d}}$.

vi) All the x_j's are Lebesgue points of u, Du, and Δu, with Lebesgue values $u(x_j) = z_j$, and by a slicing argument

$$\int_{\partial B(x_j, tr_j)} |u(x) - z_j|^2\, d\mathcal{H}^{n-1} \leq c\, \sigma\, {r_j}^{n-1}\,. \tag{C.4}$$

vii) Using a blow-up argument at x_j in the x-variables, we may and do assume that the current $S_j := [\![\, B_j \,]\!] \times \delta_{z_j} + [\![\, \mathcal{M}_j \,]\!] \times \mathrm{d}_j\, [\![\, \mathbb{S}^2 \,]\!]$ has small flat distance from T on $B_j \times \mathbb{S}^2$, i.e.,

$$\mathbf{F}((S_j - T) \llcorner B_j \times \mathbb{S}^2) \leq c\, \sigma\, {r_j}^{n-2}\,. \tag{C.5}$$

viii) By a slicing argument, we may and do assume that for some $R \in (tr_j, 2tr_j)$ the $(n-1)$-dimensional current $\langle T, \widehat{d}_{x_j}, tr_j \rangle$ belongs to $\mathrm{cart}^{\mathbb{L}}$ and satisfies

$$\mathbb{L}(\langle T, \widehat{d}_{x_j}, tr_j \rangle, \partial B(x_j, tr_j) \setminus \mathcal{M}_j) \leq \frac{c}{r_j} \cdot \mathbb{L}(T, B(x_j, R) \setminus \mathcal{M}_j)\,.$$

Also, by the construction we may assume that both (C.3) and

$$\widetilde{\mu}_T(B(x_j, \rho)) \leq c\, \theta_T(x_j)\, \rho^{n-2}\,, \qquad \mathbb{L}(u, B(x_j, \rho)) \leq c\, |\Delta u(x_j)|\, \rho^n \tag{C.6}$$

hold true for any $0 < \rho < 2r_j$. Therefore, taking r_j small so that $|\Delta u(x_j)|\, {r_j}^2 \leq \sigma\, \theta_T(x_j)$, we readily obtain

$$\mathbb{L}(\langle T, \widehat{d}_{x_j}, tr_j\rangle, \partial B(x_j, tr_j) \setminus \mathcal{M}_j) \leq c\,\sigma\,\theta_T(x_j)\,{r_j}^{n-3}\,. \tag{C.7}$$

ix) Using a similar slicing argument and (C.6), we also may and do assume that

$$\mathbb{L}(\langle T, \widehat{d}_{x_j}, tr_j\rangle) \leq c\,\theta_T(x_j)\,{r_j}^{n-3}\,. \tag{C.8}$$

x) Since $\theta_T(x_j)$ is the $(n-2)$-dimensional density of $\widetilde{\mu}_T$ at x_j, and $x_j \in \text{set}\,(L_{\mathrm{d}_j})$, denoting by α_{n-2} the measure of the unit ball of dimension $n-2$, we also may and do assume that

$$|\widetilde{\mu}_T(B_j) - 8\pi\,|\mathrm{d}_j| \cdot \alpha_{n-2}\,{r_j}^{n-2}| \leq \sigma\,\alpha_{n-2}\,{r_j}^{n-2}\,. \tag{C.9}$$

xi) There exists a suitable bilipschitz homeomorphism ψ_σ from B^n onto itself, with $\text{Lip}\,\psi_\sigma \leq 2$ and $\text{Lip}\,\psi_\sigma^{-1} \leq 2$, such that ψ_σ maps bijectively B_j onto B_j with $\psi_{\sigma|\partial B_j} = \text{Id}_{|\partial B_j}$ for all j, and ψ_σ is equal to the identity outside the union of the balls B_j. Moreover:

xii) For every j we have

$$\psi_\sigma(B(x_j, t_\sigma r_j) \cap \mathcal{M}_j) = B(x_j, \rho_j) \cap (x_j + \text{Tan}(\mathcal{M}_j, x_j))$$

where $\rho_j \in (r_j/2, r_j)$ and $\text{Tan}(\mathcal{M}_j, x_j)$ is the $(n-2)$-dimensional tangent space to $\mathcal{M}_j$ at x_j, and also

$$\psi_\sigma(\partial B(x_j, t_\sigma r_j)) = \partial B(x_j, \rho_j)\,.$$

Setting now for any j

$$T_j^\sigma := (\psi_\sigma \bowtie \text{Id}_{\mathbb{R}^3})_\# T \llcorner \text{int}\,(B_j) \times \mathbb{S}^2$$

then T_j^σ belongs to $\text{cart}^{\mathbb{L}}(\text{int}\,(B_j) \times \mathbb{S}^2)$, with underlying function $u_j^\sigma := (u \circ \psi_\sigma^{-1})_{|\text{int}\,(B_j)}$ in $W^{2,1}(\text{int}\,(B_j), \mathbb{S}^2)$, and $\mu_{T_j^\sigma} = \psi_{\sigma\#}(\widetilde{\mu}_T \llcorner \text{int}\,(B_j))$. Moreover, by (C.7), (C.8), and (C.4) we infer that T_j^σ satisfies (B.6), where $y = z_j \in \mathbb{S}^2$ is the Lebesgue value of u at x_j, with $x_j = O$, $R_0 = r_j$, and $R = \rho_j$, i.e.,

$$B_j = \bar{B}^n_{R_0}\,, \quad B(x_j, \rho_j) = B^n_R\,,$$
$$B(x_j, \rho_j) \cap (x_j + \text{Tan}(\mathcal{M}_j, x_j)) = D_R \times \{0_{\mathbb{R}^2}\} \subset \mathbb{R}^{n-2} \times \mathbb{R}^2\,.$$

Since $\Theta_T(x_j) \leq \overline{\mathrm{d}}$, we can apply Proposition B.1 in order to obtain a current $\widetilde{T}_j \in \text{cart}^{\mathbb{L}}((B(x_j, \rho_j) \setminus \bar{B}(x_j, \delta_j)) \times \mathbb{S}^2)$, where $\delta_j := \rho_j(1 - 1/q)$. Set now $\beta(n) := 1/(12(n-2)) > 0$. Since $1/q \leq c\,\sigma^{1/(6(n-2))}$, by (B.7), (B.6), and (C.8), taking $\sigma > 0$ small so that $\sigma^{\beta(n)} < 1/\overline{\mathrm{d}}$, see (B.3), we obtain

$$\mathbb{L}(\widetilde{T}_j, B(x_j, \rho_j) \setminus \bar{B}(x_j, \delta_j)) \leq c\,\sigma^{\beta(n)}\,{\rho_j}^{n-2} \tag{C.10}$$

whereas by (B.8)

$$\mathbf{F}((\widetilde{T}_j - G_{z_j}) \llcorner (B(x_j, \rho_j) \setminus \bar{B}(x_j, \delta_j)) \times \mathbb{S}^2) \leq c\,\sigma\,{r_j}^{n-1}\,. \tag{C.11}$$

Setting now

$$\breve{T}_j^\sigma := (\psi_j \bowtie \Psi_{(z_j, \varepsilon_\sigma)})_\# (T_j^\sigma \llcorner \bar{B}(x_j, \rho_j) \times \mathbb{S}^2)$$

where $\psi_j(x) := x_j + \dfrac{\delta_j}{\rho_j}\,(x - x_j)$, we have $\operatorname{spt} \breve T^\sigma_j \subset \bar B(x_j,\delta_j) \times B_{\mathbb{S}^2}(z_j,\varepsilon_\sigma)$, whence $\breve T^\sigma_j$ satisfies the hypotheses of Proposition B.2, with $B(x_j,\delta_j)$ instead of B^n_r, $y = z_j$, and $\mathrm{d} = \mathrm{d}_j$. By defining $\widetilde\Omega^j_\delta$ similarly to (B.9), but in correspondence to $B(x_j,\delta_j)$, the cited proposition yields a current $\widehat T^\sigma_j \in \mathrm{cart}^{\mathbb{L}}((B(x_j,\delta_j)\setminus \widetilde\Omega^j_\delta)\times\mathbb{S}^2)$.

Moreover, by applying Proposition B.3, with $B(x_j,\delta_j)$ instead of B^n_r and $\mathrm{d} = \mathrm{d}_j$, we find a suitable function $v^\sigma_j \in W^{2,1}(\widetilde\Omega^j_\delta, \mathbb{S}^2)$. Setting then

$$\overline T^\sigma_j := \widehat T^\sigma_j + G_{v^\sigma_j}$$

(B.11) and i) in Proposition B.2 yield that $\overline T^\sigma_j \in \mathrm{cart}^{\mathbb{L}}(B(x_j,\delta_j)\times\mathbb{S}^2)$ and

$$\partial(\overline T^\sigma_j \llcorner B(x_j,\delta_j)\times\mathbb{S}^2) = \partial(\breve T^\sigma_j \llcorner B(x_j,\delta_j)\times\mathbb{S}^2)\,. \tag{C.12}$$

Also, by (B.10) we have

$$\mathbb{L}(v^\sigma_j, \widetilde\Omega_\delta) \le \sigma\,{\delta_j}^{n-2} + \mathcal{H}^{n-2}(D_{r_j})\cdot 8\pi\,|\mathrm{d}_j|\,.$$

Therefore, since $\delta_j \in (r_j/2, r_j)$, by (C.9) we get:

$$\mathbb{L}(v^\sigma_j, \widetilde\Omega_\delta) \le c\,\sigma\,{r_j}^{n-2} + \widetilde\mu_T(B_j)\,.$$

Finally, using (C.3) in order to estimate the last term in the right-hand side of ii) in Proposition B.2, we obtain

$$\begin{array}{rcl}\mathbb{L}(\overline T^\sigma_j, B(x_j,\delta_j)) & \le & \mathbb{L}(u^\sigma_j, B(x_j,\delta_j)) \\ & + & c\,\sigma\,{r_j}^{n-2} + (1+c\,\sigma)\,\widetilde\mu_T(B(x_j,\delta_j))\,. \end{array} \tag{C.13}$$

We now set

$$\widetilde T^\sigma_j := \overline T^\sigma_j + \widetilde T_j + T^\sigma_j \llcorner (B(x_j,r_j)\setminus B(x_j,\rho_j))\times\mathbb{S}^2\,.$$

Property (a) in Proposition B.1, the definition of $\breve T^\sigma_j$, and (C.12), yield that $\widetilde T^\sigma_j$ belongs to the class $\mathrm{cart}^{\mathbb{L}}(\mathrm{int}\,(B_j)\times\mathbb{S}^2)$. Moreover, by (C.10) and (C.13)

$$\mathbb{L}(\widetilde T^\sigma_j, B_j) \le \mathbb{L}(T^\sigma_j, B_j) + c\,\sigma^{\beta(n)}\,{r_j}^{n-2} + c\,\sigma\widetilde\mu_T(B_j)\,. \tag{C.14}$$

Finally, arguing as in [28, Sec. 5.5, Step 3], by (B.8), property iii) in Proposition B.2, and the dipole construction from Proposition B.3, for ε, δ small we obtain

$$\mathbf{F}((\widetilde T^\sigma_j - T^\sigma_j)\llcorner B_j\times\mathbb{S}^2) \le c\,\sigma\,{r_j}^{n-2}\,.$$

Setting now

$$T^{(\sigma)}_j := (\psi^{-1}_\sigma \bowtie \mathrm{Id}_{\mathbb{R}^3})_\# \widetilde T^{(\sigma)}_j \llcorner \mathrm{int}\,(B_j)\times\mathbb{S}^2\,,$$

by (C.14) we infer that for every j

$$\mathbb{L}(T^{(\sigma)}_j, \mathrm{int}\,(B_j)) \le \mathbb{L}(u, B_j) + (1+c\,\sigma)\,\widetilde\mu_T(B_j) + c\,\sigma^{\beta(n)}\,{r_j}^{n-2} \tag{C.15}$$

whereas

$$\mathbf{F}((T^{(\sigma)}_j - T)\llcorner B_j\times\mathbb{S}^2) \le c\,\sigma\,{r_j}^{n-2}\,. \tag{C.16}$$

In conclusion, setting $T^\sigma \in \mathcal{D}_n(B^n \times \mathbb{S}^2)$ by

$$T^\sigma := \sum_{j=1}^\infty T_j^{(\sigma)} + T \llcorner \big(B^n \setminus \bigcup_{j=1}^\infty \mathrm{int}\,(B_j)\big) \times \mathbb{S}^2$$

we have $T^\sigma \in \mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$. By (B.3), (C.2), and (C.15), we thus obtain

$$\mathbb{L}(T^\sigma) \le \mathbb{L}(u) + \mu_T(B^n \setminus \mathcal{L}_T(\overline{\mathrm{d}})) + (1 + c\,\sigma)\,\widetilde{\mu}_T(B^n) + c\,\sigma^{\beta(n)}\,\mathcal{H}^{n-2}(\mathcal{L}_T)$$

so that if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu_T) > 0$ is small, by (C.1) we have

$$\mathbb{L}(T^\sigma) \le \mathbb{L}(T) + \varepsilon^k\,.$$

Moreover, by (C.3), taking σ small, the above construction yields that

$$\begin{aligned}\mu_{T^\sigma}(B^n) \quad &\le \quad c\sum_{j=1}^\infty \widetilde{\mu}_T(B(x_j, r_j) \setminus (B(x_j, tr_j) \cap \mathcal{M}_j)) + \mu_T(B^n \setminus \mathcal{L}_T(\overline{\mathrm{d}})) \\ &\le \quad c\,\sigma\,\mu_T(B^n) + \frac{1}{4}\,\mu_T(B^n) < \frac{1}{2}\,\mu_T(B^n)\,.\end{aligned}$$

Finally, by (C.16) and (C.2) we have

$$\mathbf{F}(T^\sigma - T) \le \sum_{j=1}^\infty \mathbf{F}((T_j^{(\sigma)} - T) \llcorner B_j \times \mathbb{S}^2) \le c\,\sigma \sum_{j=1}^\infty {r_j}^{n-2} < \varepsilon^k$$

if $\sigma = \sigma(\varepsilon, k, \mathcal{L}_T, \mu_T) > 0$ is small. Taking $\widetilde{T} = T^\sigma$ for $\sigma > 0$ small, the proof is complete. □

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