A RELAXATION RESULT FOR A SECOND ORDER ENERGY OF MAPPINGS INTO THE SPHERE

DOMENICO MUCCI

Abstract. A relaxation problem for maps from 3-dimensional domains into the unit 2-sphere is analysed, the energy being given in the smooth case by the integral of the modulus of the Laplacean vector. For second order Sobolev maps, a complete explicit formula of the relaxed energy is obtained. Our proof is based on the following results: minimal energy computation of maps with fixed degree, Dipole-like problems, lower semicontinuity of the extended energy, and strong approximation properties on Cartesian currents.

Introduction

First order variational problems for maps taking values into isometrically embedded Riemannian manifolds $\mathcal{N}$ are widely studied, a relevant model being given by the Dirichlet integral

$$\mathbb{D}(u) := \frac{1}{2}\int_{B^n} |Du|^2\, dx \qquad (0.1)$$

of maps from the unit ball B^n into the $\mathfrak{p}$-dimensional unit sphere $\mathcal{N} = \mathbb{S}^{\mathfrak{p}}$.

When e.g. $n = 3$ and $\mathfrak{p} = 2$, unit vector fields minimizing the Dirichlet energy (under prescribed boundary conditions) represent a simplified model for the Ericksen-Leslie theory of liquid crystals, see [21] or [23, Sec. 5.1].

Harmonic maps u with values into the sphere $\mathbb{S}^{\mathfrak{p}}$ satisfy the Euler-Lagrange system $\tau(u) = 0$, where

$$\tau(u) := \Delta u + |Du|^2 u \qquad (0.2)$$

is the *intrinsic* Laplacean, or *tension field*, compare [23, Sec. 3.1.1]. More precisely, viewing the $\mathfrak{p}$-sphere as embedded into the Euclidean space $\mathbb{R}^{\mathfrak{p}+1}$, and working with maps $u : B^n \to \mathbb{R}^{\mathfrak{p}+1}$ such that $|u(x)| \equiv 1$, then Δu is the *Laplacean vector* in $\mathbb{R}^{\mathfrak{p}+1}$, and its normal component to $\mathbb{S}^{\mathfrak{p}}$ at $u(x)$ is $(\Delta u)^\perp = -|Du|^2\, u$, whence $\tau(u)$ is the tangential component of Δu, and

$$|\Delta u|^2 = |Du|^4 + |\tau(u)|^2\,. \qquad (0.3)$$

2010 *Mathematics Subject Classification.* 49Q15; 49J45; 28A75; 49Q45.
Key words and phrases. Laplacean, mapping into the sphere, relaxation, currents.
The research of D.M. was partially supported by the GNAMPA of INDAM.

In dimension $n = 2$, the Dirichlet integral is conformally invariant. Therefore, its second order analogous is probably given by the *bienergy* functional

$$\mathbb{H}(u) := \int_{B^n} |\Delta u|^2 \, dx \tag{0.4}$$

of maps u from B^n into $\mathbb{S}^{\mathfrak{p}}$. In dimension $n = 4$, in fact, the bienergy functional is conformally invariant. In addition, equation (0.3) implies the lower bound

$$\mathbb{H}(u) \geq \int_{B^n} |Du|^4 \, dx$$

where equality holds when $\tau(u) = 0$, i.e. for harmonic maps.

As a consequence, in any dimension $n \geq \mathfrak{p} \geq 2$, Sobolev maps u from B^n into $\mathbb{S}^{\mathfrak{p}}$ with finite bienergy belong to the Sobolev class $W^{1,4}(B^n, \mathbb{S}^{\mathfrak{p}})$. Moreover, when $n = \mathfrak{p}$, by the parallelogram inequality the Jacobian of a smooth map u from $\mathbb{R}^{\mathfrak{p}}$ into $\mathbb{S}^{\mathfrak{p}}$ satisfies the pointwise upper bound

$$J_{\mathfrak{p}} u \leq \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du|^{\mathfrak{p}} \tag{0.5}$$

where equality holds if and only if u is conformal.

Therefore, when in particular $\mathfrak{p} = 4$, it turns out that in any dimension n the "graph" in $B^n \times \mathbb{S}^4$ of a Sobolev map $u \in W^{2,1}(B^n, \mathbb{S}^4)$ with finite bienergy has finite "area" .

However, in view of analyzing the corresponding *relaxation problem*, a nontrivial open question comes into play: *finding a bienergy minimizer among smooth maps from* $\mathbb{R}^4$ *into* $\mathbb{S}^4$ *of* degree *one*, see Sec. 5.

One expects that it is given by the inverse σ_4^{-1} of the stereographic projection map from $\mathbb{S}^4$ to $\mathbb{R}^4$, compare (1.1). In fact, Angelsberg [5] showed that the energy minimum among degree one maps is attained and it is greater than $16\,\mathcal{H}^4(\mathbb{S}^4)$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. Moreover, recalling that $\int_{\mathbb{R}^4} |\Delta\sigma_4^{-1}|^2 \, dx = 24\,\mathcal{H}^4(\mathbb{S}^4)$, Cooper [15] proved that σ_4^{-1} minimizes the bienergy among degree one $O(4)$-equivariant maps from $\mathbb{R}^4$ into $\mathbb{S}^4$.

Laplacean energy. In this paper, we consider the functional

$$\mathbb{L}(u) := \int_{B^n} |\Delta u| \, dx$$

on maps $u : B^n \to \mathbb{S}^2$ taking values into the unit 2-sphere of $\mathbb{R}^3$. It will be called *Laplacean energy.* If u is sufficiently smooth, by (0.3) we get

$$|\Delta u| \geq |Du|^2 \tag{0.6}$$

where equality holds for harmonic maps. In addition, by (0.5), where $\mathfrak{p} = 2$, it turns out that the "graph" of u has finite "area" in $B^n \times \mathbb{S}^2$.

Minimal energy of degree one maps. Differently to the nontrivial case of the bienergy of maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, we now see that *the minimal*

Laplacean energy among degree one maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ is attained by the inverse σ_2^{-1} of the stereographic map, where

$$\int_{\mathbb{R}^2} |\Delta\sigma_2^{-1}|\, dx = \int_{\mathbb{R}^2} |D\sigma_2^{-1}|^2\, dx = 2\int_{\mathbb{R}^2} J_2\sigma_2^{-1}\, dx = 2\,\mathcal{H}^2(\mathbb{S}^2) = 8\pi\,. \tag{0.7}$$

More precisely, we denote

$$W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2) := \{u \in W^{2,1}_{\text{loc}}(\mathbb{R}^n,\mathbb{S}^2) \,:\, \Delta u \in L^1(\mathbb{R}^n,\mathbb{R}^3)\} \tag{0.8}$$

and also

$$\mathbb{L}(u,\mathbb{R}^n) := \int_{\mathbb{R}^n} |\Delta u|\, dx\,, \quad u \in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)\,. \tag{0.9}$$

If $u \in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)$, inequality (0.6) holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, where $\mathcal{L}^n$ is the Lebesgue measure, and hence $|Du| \in L^2(\mathbb{R}^n)$. In particular, in low dimension $n = 2$, by (0.5), with $\mathfrak{p} = 2$, any map $u \in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ satisfies the energy lower bound

$$\mathbb{L}(u,\mathbb{R}^2) \geq \int_{\mathbb{R}^2} |Du|^2\, dx \geq 2\int_{\mathbb{R}^2} J_2 u\, dx \tag{0.10}$$

where both inequalities are equalities if u is harmonic and conformal, and that is the case of the inverse σ_2^{-1} of the stereographic map $\sigma_2 : \mathbb{S}^2 \to \mathbb{R}^2$, compare (1.1). On the other hand, maps $u \in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ have a well-defined integer degree, and degree one maps as e.g. $u = \sigma_2^{-1}$ satisfy inequality $\int_{\mathbb{R}^2} J_2 u\, dx \geq 4\pi$, whence $\mathbb{L}(u,\mathbb{R}^2) \geq 8\pi = \mathbb{L}(\sigma_2^{-1},\mathbb{R}^2)$, by (0.10) and (0.7).

Relaxed energy. Following the classical Lebesgue-Serrin approach, we introduce in any dimension $n \geq 2$ the *relaxed energy*

$$\widetilde{\mathbb{L}}(u) := \inf\Big\{\liminf_{h\to\infty} \mathbb{L}(u_h) \mid \ \{u_h\} \subset C^\infty(B^n,\mathbb{S}^2),\ u_k \to u \text{ strongly in } L^1(B^n,\mathbb{R}^3)\Big\} \tag{0.11}$$

of maps u in $L^1(B^n,\mathbb{S}^2)$. Our first objective is to analyze the explicit formula of $\widetilde{\mathbb{L}}(u)$ on the class of maps with finite relaxed energy. We thus denote:

$$\mathbb{L}(B^n,\mathbb{S}^2) := \{\, u \in L^1(B^n,\mathbb{S}^2) \mid \widetilde{\mathbb{L}}(u) < \infty\,\} \tag{0.12}$$

and refer to Sec. 2 for details on the following preliminary discussion.

If $u \in \mathbb{L}(B^n,\mathbb{S}^2)$, inequality (0.6) implies that $u \in W^{1,2}(B^n,\mathbb{S}^2)$, whence the *distributional divergence* of the gradient Du is well defined by

$$\langle \text{Div} Du; \varphi\rangle := -\int_{B^n} \text{tr}\,[Du\,(D\varphi)^\top]\, dx\,, \quad \varphi \in C^\infty_c(B^n,\mathbb{R}^3) \tag{0.13}$$

where $A \mapsto A^\top$ is the transpose operator in $\mathbb{R}^{3\times n}$ and $B \mapsto \text{tr}\, B$ the trace operator in $\mathbb{R}^{3\times 3}$. By lower semicontinuity, we have:

$$\widetilde{\mathbb{L}}(u) \geq |\text{Div} Du|(B^n) \quad \forall\, u \in \mathbb{L}(B^n,\mathbb{S}^2) \tag{0.14}$$

and hence $\text{Div}Du$ is a *finite $\mathbb{R}^3$-valued regular measure*. Since moreover

$$\text{Div} Du = \Delta u\, \mathcal{L}^n \llcorner B^n \quad \forall\, u \in W^{2,1}(B^n,\mathbb{S}^2)$$

the measure $\mathrm{Div}Du$ may be called a *weak Laplacean.*

In the critical dimension $n = 2$, due to the continuous embedding of $W^{1,2}(B^2)$ in the class VMO of functions with *vanishing mean oscillation* it turns out that there is no gap:

$$\widetilde{\mathbb{L}}(u) = |\mathrm{Div}Du|(B^2) \quad \forall\, u \in \mathbb{L}(B^2, \mathbb{S}^2)\,. \tag{0.15}$$

In high dimension $n \geq 3$, the energy gap is positive, in general, i.e. strict inequality holds in (0.14). However, for a generic map $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, it is an open problem to find the explicit formula of the relaxed energy (0.11).

This is essentially due to a lack of sufficient information on the structure of the measure $\mathrm{Div}Du$. For that reason, in this paper we shall focus on the more regular subclass of second order Sobolev maps, since

$$|\mathrm{Div}Du|(B^n) = \int_{B^n} |\Delta u|\, dx =: \mathbb{L}(u) \quad \forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,. \tag{0.16}$$

Main Result. In case of dimension $n = 3$, and for maps u in $W^{2,1}(B^3, \mathbb{S}^2)$, we shall see that the energy gap only depends (up to the factor 8π) on the *mass* $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ of a *minimal connection* of the *current of the singularities* of u.

Referring to Sec. 1 for the precise notation, we only mention here that in any dimension $n \geq 3$, the *relevant singularities* of maps $u \in W^{2,1}(B^n, \mathbb{S}^2)$ are described by an *integral flat* $(n-3)$-chain $\mathbb{P}(u)$ in B^n. This means that the current $\mathbb{P}(u)$ is the boundary in B^n of an integer multiplicity (say i.m.) rectifiable $(n-2)$-current L, and the *integral mass* $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is the mass of a minimizer among $L \in \mathcal{R}_{n-2}(B^n)$ satisfying $(\partial L) \llcorner B^n = -\mathbb{P}(u)$.

If e.g. $n = 3$ and u is the harmonic map $u(x) = x/|x|$, then $|\Delta u| = |Du|^2 = 2/|x|^2$, and on account of (1.8) the current of the singularities is such that $-\mathbb{P}(u) = \delta_O$, the unit Dirac mass at the origin O, whence $\mathbf{m}_{i,B^3}(\mathbb{P}(u))$ is equal to the length of a segment connecting O to a point at the boundary of B^3, a so called *string* in the sense of Brezis–Coron–Lieb [13].

The Main Result of this paper is enclosed in the following theorem, where in the case of dimension $n = 3$ we are able to give an explicit formula for the relaxed energy (0.11) of maps in the Sobolev class $W^{2,1}(B^3, \mathbb{S}^2)$.

Theorem 0.1. *Let* $n = 3$ *and* $u \in W^{2,1}(B^3, \mathbb{S}^2)$. *Then*

$$\widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty\,.$$

In the proof of Theorem 0.1, we rely on the previous observation concerning the minimal Laplacean energy of degree one maps, and we follow a similar strategy to the one exploited in case of the Dirichlet energy (0.1). In particular, we make use of tools from the theory of *Cartesian currents* by Giaquinta-Modica-Souček [22, 23].

Finally, concerning the wider class

$$\mathbb{L}_{BV}(B^n, \mathbb{S}^2) := \big\{ u \in \mathbb{L}(B^n, \mathbb{S}^2) \mid Du \in \mathrm{BV}(B^n, \mathbb{R}^{3\times n}) \big\} \tag{0.17}$$

we prove in any dimension $n \geq 3$ the lower bound

$$\widetilde{\mathbb{L}}(u) \geq |\mathrm{Div} Du|(B^n) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \quad \forall\, u \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)\,. \tag{0.18}$$

Content of the paper. In Sec. 1, we collect some notation and background material, focusing in particular on the analogous relaxation problem for the Dirichlet integral (0.1) of mappings into the 2-sphere.

In Sec. 2, we preliminarily discuss some general properties of maps with finite relaxed energy, explaining the difficulties that one encounters in the general case when $u \notin W^{2,1}(B^n, \mathbb{S}^2)$ and $n \geq 3$. We also prove a lower semicontinuity result in dimension $n = 2$, Theorem 2.2. Finally, we report a cohomological criterion for strong density of smooth maps in $W^{2,1}(B^n, \mathbb{S}^2)$ recently obtained by Bousquet–Ponce–Van Schaftingen [12].

In Sec. 3, we introduce a suitable modification of the inverse to the stereographic map, Proposition 3.1. We then compute the *minimal Laplacean energy* among maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ with fixed degree, Theorem 3.2, and describe the related bubbling phenomenon. Finally, we extend to the Laplacean energy the classical *Dipole problem* of Brezis-Coron-Lieb [13] for the Dirichlet energy in 3D, Theorem 3.3.

In Sec. 4, we introduce the class $\mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ of Cartesian currents whose underlying functions belong to $W^{2,1}(B^n, \mathbb{S}^2)$. More precisely, see Definition 4.1, an element T in $\mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ is given by

$$T = G_u + L \times [\![\, \mathbb{S}^2 \,]\!]$$

where G_u is the graph current of a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ and L is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = -\mathbb{P}(u)$, if $n \geq 3$. We then extend the Laplacean energy to a functional $T \mapsto \mathbb{L}(T)$ on Cartesian currents, by letting

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi \cdot \mathbf{M}(L) \quad \text{if} \quad T = G_u + L \times [\![\, \mathbb{S}^2 \,]\!]$$

and prove a weak sequential *lower semicontinuity* property, Theorem 4.3. In dimension $n = 3$, we also obtain a *strong density result*. Namely, in Theorem 4.4 we show that every current in $\mathrm{cart}^{\mathbb{L}}(B^3 \times \mathbb{S}^2)$ can be approximated weakly and with energy convergence by a sequence of currents carried by graphs of smooth maps in $C^\infty(B^3, \mathbb{S}^2)$.

In Sec. 5, we deal with the explicit formula of the relaxed energy (0.11). The proof of Theorem 0.1 is based on the lower semicontinuity theorem 4.3 and on the density theorem 4.4. We then prove the energy lower bound (0.18). In the end, final remarks and open questions are reported.

1. Background material and preliminary results

In this section, we collect some well-known facts about stereographic maps, divergence-measure fields, and topics from Geometric Measure Theory, degree, Cartesian currents, singularities (for which we refer to the treatise [22, 23] or to [26]). We then describe the strong density and relaxation results for the Dirichlet energy of maps into the 2-sphere.

Let B^n be the open unit ball of dimension $n \geq 2$ centered at the origin, and $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. For $X = L^1$, $W^{k,p}$, or C^∞, we denote

$$X(B^n, \mathbb{S}^2) := \{u \in X(B^n, \mathbb{R}^3) \,:\, |u(x)| = 1 \text{ for } \mathcal{L}^n\text{-a.e. } x \in B^n\}\,.$$

Stereographic projection. For $\mathfrak{p} \geq 2$ integer, setting

$$\mathbb{S}^{\mathfrak{p}} := \{(y,z) \mid y \in \mathbb{R}^{\mathfrak{p}}\,,\ z \in \mathbb{R}\,,\ |(y,z)| = 1\} \subset \mathbb{R}^{\mathfrak{p}+1}$$

the stereographic projection from the "South Pole" $P_S := (0_{\mathbb{R}^{\mathfrak{p}}}, -1)$ is given by $\sigma_{\mathfrak{p}}(y,z) := \dfrac{y}{1+z}$. Its inverse $\sigma_{\mathfrak{p}}^{-1} : \mathbb{R}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ satisfies

$$\sigma_{\mathfrak{p}}^{-1}(x) = \Big(\frac{2x}{1+\rho^2}, \frac{1-\rho^2}{1+\rho^2}\Big)\,, \qquad x \in \mathbb{R}^{\mathfrak{p}}\,, \quad \rho := |x|\,. \tag{1.1}$$

The map $(-1)^{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is an orientation preserving conformal diffeomorphism from $\mathbb{R}^{\mathfrak{p}}$ onto $\mathbb{S}^{\mathfrak{p}} \setminus \{P_S\}$. In fact, denoting by $\bullet$ the scalar product in $\mathbb{R}^{\mathfrak{p}+1}$ and by δ^i_j the Kronecker symbol, the conformality relations

$$\partial_i \sigma_{\mathfrak{p}}^{-1} \bullet \partial_j \sigma_{\mathfrak{p}}^{-1} = \delta^i_j\, U^2 \quad \forall\, i,j = 1,\ldots,\mathfrak{p}$$

hold, with scaling factor $U(x) := \dfrac{2}{1+|x|^2}$, whence in (0.5) one has

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}} = J_{\mathfrak{p}} \sigma_{\mathfrak{p}}^{-1} = U^{\mathfrak{p}}$$

where $J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is the Jacobian of $\sigma_{\mathfrak{p}}^{-1}$. As a consequence, concerning the *conformal Dirichlet integral*, for any $\mathfrak{p} \geq 2$ integer one obtains:

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} \int_{\mathbb{R}^{\mathfrak{p}}} |D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}}\, dx = \int_{\mathbb{R}^{\mathfrak{p}}} J_{\mathfrak{p}} \sigma_{\mathfrak{p}}^{-1}\, dx = \mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \tag{1.2}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.

Most importantly, it turns out that the map $\sigma_{\mathfrak{p}}^{-1}$ is harmonic if and only if $\mathfrak{p} = 2$. Therefore, σ_2^{-1} satisfies the Euler-Lagrange system $\tau(u) = 0$, where $\tau(u)$ is the tension field (0.2). In conclusion, one readily obtains the energy computation (0.7).

Divergence-measure fields. Let $n \geq 2$. The *distributional divergence* of a vector field $F \in L^2(B^n, \mathbb{R}^n)$ is well defined by:

$$\langle \mathrm{Div} F; \phi \rangle := -\int_{B^n} F \cdot D\phi\, dx\,, \quad \phi \in C^\infty_c(B^n)\,.$$

Definition 1.1. We call $F \in L^2(B^n, \mathbb{R}^n)$ a *divergence-measure field*, say $F \in \mathcal{DM}^{1,2}(B^n)$, if $\mathrm{Div}F$ is a real finite Radon measure on B^n.

If $F \in \mathcal{DM}^{1,2}(B^n)$, a decomposition into mutually singular measures

$$\mathrm{Div}F = (\mathrm{Div}F)^a + (\mathrm{Div}F)^s\,, \quad (\mathrm{Div}F)^a = \widetilde{\mathrm{div}F}\, \mathcal{L}^n \llcorner B^n$$

holds, where $\widetilde{\mathrm{div}F} \in L^1(B^n)$ denotes the Radon-Nikodym derivative of $\mathrm{Div}F$ w.r.t. $\mathcal{L}^n$. Referring to [4] for further details on *functions of bounded variations*, we remark that if in addition $F \in \mathrm{BV}(B^n, \mathbb{R}^n)$, the density $\widetilde{\mathrm{div}F}$

agrees with the trace of the approximate gradient matrix ∇F, and that $(\mathrm{Div}F)^s = 0$ if in particular $F \in W^{1,1}(B^n, \mathbb{R}^n)$.

Šilhavý [30, Thm. 3.2] proved the following absolute continuity property:

Proposition 1.2. *If $F \in \mathcal{DM}^{1,2}(B^n)$, then $|\mathrm{Div}F|(B) = 0$ for each Borel set $B \subset B^n$ with σ-finite $\mathcal{H}^{n-2}$-measure. In particular, the measure $\mathrm{Div}F$ does not charge any atom.*

By the chain rule formula in BV, cf. [4, Thm. 3.96] and [22, p. 487], it turns out that if $v^1 \in W^{1,2}(B^n) \cap L^\infty(B^n)$ and $v^2 \in \mathrm{BV}(B^n) \cap L^2(B^n)$, then (denoting by D and ∇ the distributional derivative and the approximate gradient)

$$D(v^1 v^2) = v^1 Dv^2 + v^2 \nabla v^1 \, \mathcal{L}^n \llcorner B^n \,.$$

In this setting, the following version of the Leibnitz-rule is due to Comi [14].

Proposition 1.3. *Let $F \in \mathcal{DM}^{1,2}(B^n)$ and $g \in W^{1,2}(B^n) \cap L^\infty(B^n)$. Then, $gF \in \mathcal{DM}^{1,2}(B^n)$ and*

$$\mathrm{Div}(gF) = \widetilde{g}\, \mathrm{Div}F + F \cdot \nabla g \, \mathcal{L}^n \llcorner B^n$$

where $\widetilde{g}$ is the precise representative of g.

Integer rectifiable currents. For $U \subset \mathbb{R}^m$ an open set, and $k = 0, \dots, m$, we denote by $\mathcal{D}_k(U)$ the dual to the space $\mathcal{D}^k(U)$ of compactly supported smooth k-forms, whence $\mathcal{D}_0(U)$ is the class of distributions in U. For any $T \in \mathcal{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{\langle T; \omega\rangle \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \le 1\}$$

and (for $k \ge 1$) its *boundary* as the $(k-1)$-current ∂T defined by the relation

$$\langle \partial T; \eta\rangle := \langle T; d\eta\rangle \qquad \forall\, \eta \in \mathcal{D}^{k-1}(U)$$

where $d\eta$ is the differential of η. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathcal{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h; \omega\rangle = \langle T; \omega\rangle\,, \qquad \forall\, \omega \in \mathcal{D}^k(U)$$

and the mass is sequentially weakly lower semicontinuous, i.e.

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h) \quad \text{if} \quad T_h \rightharpoonup T\,.$$

For $k \ge 1$, a k-current T with finite mass is called *rectifiable* if

$$\langle T; \omega\rangle = \int_{\mathcal{M}} \theta\, \langle \omega; \xi\rangle \, d\mathcal{H}^k \qquad \forall\, \omega \in \mathcal{D}^k(U) \tag{1.3}$$

with $\mathcal{M}$ a k-rectifiable set in U, $\xi : \mathcal{M} \to \Lambda_k \mathbb{R}^n$ a $\mathcal{H}^k \llcorner \mathcal{M}$-measurable function such that $\xi(x)$ is a simple unit k-vector in $\Lambda_k\mathbb{R}^n$ orienting the approximate tangent space to $\mathcal{M}$ at $\mathcal{H}^k$-a.e. $x \in \mathcal{M}$, and $\theta : \mathcal{M} \to [0, +\infty)$ a $\mathcal{H}^k \llcorner \mathcal{M}$-summable non-negative function, so that $\mathbf{M}(T) = \int_{\mathcal{M}} \theta \, d\mathcal{H}^k < \infty$. In that case, the short-hand notation $T = [\![\mathcal{M}, \xi, \theta]\!]$ is commonly adopted, and $\mathrm{set}\,(T)$ denotes the set of points in $\mathcal{M}$ with positive multiplicity.

In addition, if θ is integer-valued, the current T is called *integer multiplicity* (in short *i.m.*) *rectifiable* and the corresponding class is denoted by $\mathcal{R}_k(U)$. The usefulness of i.m. rectifiable currents in the Calculus of Variations derives from Federer–Fleming's compactness theorem [17]. It states that if a sequence $\{T_h\} \subset \mathcal{R}_k(U)$ satisfies $\sup_h \big(\mathbf{M}(T_h) + \mathbf{M}((\partial T_h) \llcorner U)\big) < \infty$, where $\llcorner$ denotes restriction, then there exists $T \in \mathcal{R}_k(U)$ and a (not relabeled) subsequence of $\{T_h\}$ such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_k(U)$.

Example 1.4. If $\mathcal{M}$ is a smooth, oriented, compact k-submanifold of U, then $[\![\mathcal{M}]\!]$ is the current in $\mathcal{R}_k(U)$ given by integration of k-forms in the sense of differential geometry, i.e., $\langle [\![\mathcal{M}]\!]; \omega \rangle := \int_{\mathcal{M}} \omega$ for all $\omega \in \mathcal{D}^k(U)$.

Graph currents. If u is a map in $W^{1,1}(B^n, \mathbb{R}^N)$, where $n, N \geq 2$, then u has a Lusin representative on the subset $\widetilde{B}^n$ of Lebesgue points pertaining to both u and the gradient Du, where $\mathcal{L}^n(B^n \setminus \widetilde{B}^n) = 0$. Following [22], the *graph* of u is the countably n-rectifiable subset of $U = B^n \times \mathbb{R}^N$

$$\mathcal{G}_u := \{(x, y) \in B^n \times \mathbb{R}^N \mid x \in \widetilde{B}^n \,,\ y = \widetilde{u}(x)\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of u. By the area formula, one has $\mathcal{H}^n(\mathcal{G}_u) < \infty$ if in addition all the minors of Du are in $L^1(B^n)$. In that case, u is called a map in $\mathcal{A}^1(B^n, \mathbb{R}^N)$. More precisely, the approximate tangent n-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_i(x) = (\mathbf{e}_i, \partial_i u(x)) \in \mathbb{R}^{n+N}$, for $i = 1, \ldots, n$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the canonical basis in $\mathbb{R}^n$ and the partial derivative $\partial_i u(x)$ is the i-th column vector of the gradient matrix $Du(x)$ given by the Lebesgue value of Du at $x \in \widetilde{B}^n$. Therefore, the unit n-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)}{|\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)|} \in \Lambda_n \mathbb{R}^{n+N} \,, \quad x \in \widetilde{B}^n$$

provides an orientation to $\mathcal{G}_u$, and the *graph current* $G_u = [\![\mathcal{G}_u, \xi, 1]\!]$ is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$.

The action of G_u can be read (in an approximate $\mathcal{L}^n$-a.e. sense) through the pull-back of the *graph map* $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ by:

$$\langle G_u; \omega \rangle = \int_{B^n} (\mathrm{Id} \bowtie u)^{\#} \omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{R}^N) \,. \tag{1.4}$$

Therefore, the mass of G_u is equal to the *graph area* $\mathbb{A}(u)$, i.e.,

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = \mathbb{A}(u) := \int_{B^n} |M(Du)| \, dx < \infty \tag{1.5}$$

where $|M(Du)|$ is the Jacobian of $\mathrm{Id} \bowtie u$, so that $|M(Du)|^2$ is equal to 1 plus the sum of the square of all minors of the $N \times n$ matrix Du.

Let now $N = 3$ and $u \in W^{1,2}(B^n, \mathbb{S}^2)$. If $n \geq 3$, by the area formula all the 3×3 minors of Du are zero $\mathcal{L}^n$-a.e. in B^n. Therefore, for any $n \geq 2$ the Jacobian $|M(Du)|$ is $\mathcal{L}^n$-essentially bounded (up to an absolute constant factor c_n only depending on the dimension n) by $1+|Du|^2$, where $Du\in L^2(B^n,\mathbb{R}^{3\times n})$. Whence, $u\in\mathcal{A}^1(B^n,\mathbb{R}^3)$ and by (1.5) we get

$$\mathbf{M}(G_u)=\mathbb{A}(u)\le c_n\int_{B^n}(1+|Du|^2)\,dx<\infty\,. \tag{1.6}$$

In addition, by Federer's flatness theorem, the graph current G_u is an i.m. rectifiable current in $B^n\times\mathbb{S}^2$, say $G_u\in\mathcal{R}_n(B^n\times\mathbb{S}^2)$.

Now, if $u\in W^{1,2}(B^n,\mathbb{S}^2)$ is smooth, we have $G_u=[\![\,\mathcal{G}_u\,]\!]$, see (1.4), where the graph manifold $\mathcal{G}_u$ has no "fractures" or "holes". By Stokes' theorem, such a condition is read in terms of the graph current G_u by the property:

$$\langle\partial G_u;\eta\rangle:=\langle G_u;d\eta\rangle=0\qquad\forall\,\eta\in\mathcal{D}^{n-1}(B^n\times\mathbb{S}^2)\,. \tag{1.7}$$

Remark 1.5. Given $u\in W^{1,2}(B^n,\mathbb{S}^2)$, assume that there exists a sequence of smooth maps $\{u_h\}\subset C^\infty(B^n,\mathbb{S}^2)$ such that $u_h\to u$ strongly in $W^{1,2}(B^n,\mathbb{R}^3)$. We recall that this is always the case in the critical dimension $n=2$, by Schoen-Uhlenbeck density theorem [29]. Strong $W^{1,2}$ convergence implies that $G_{u_h}\rightharpoonup G_u$ weakly as currents in $\mathcal{D}_n(B^n\times\mathbb{S}^2)$, a convergence that preserves the homological property (1.7). Therefore, we conclude that the map u satisfies the *null-boundary condition* (1.7).

Remark 1.6. Condition (1.7) is violated in high dimension $n\ge 3$, in general. If e.g. $n=3$, the 0-homogeneous harmonic map $u(x)=x/|x|$ belongs to the class $W^{1,2}(B^3,\mathbb{S}^2)$, and one has (cf. [22, Sec. 3.2.2, Ex. 1])

$$(\partial G_u)\llcorner B^3\times\mathbb{S}^2=-\delta_O\times[\![\,\mathbb{S}^2\,]\!] \tag{1.8}$$

where δ_O is the unit Dirac mass at the origin O. Therefore, one cannot find a sequence $\{u_h\}\subset C^\infty(B^3,\mathbb{S}^2)$ strongly converging to u in $W^{1,2}(B^3,\mathbb{R}^3)$.

Remark 1.7. For maps $u\in W^{1,1}_{\rm loc}(\mathbb{R}^n,\mathbb{S}^2)$ with $|Du|\in L^2(\mathbb{R}^n)$, we denote

$$\mathbb{D}(u,\mathbb{R}^n):=\frac{1}{2}\int_{\mathbb{R}^n}|Du|^2\,dx \tag{1.9}$$

and notice that this time the graph current G_u is locally i.m. rectifiable in $\mathbb{R}^n\times\mathbb{S}^2$, i.e. $G_u\llcorner\Omega\times\mathbb{S}^2\in\mathcal{R}_n(\Omega\times\mathbb{S}^2)$ for each bounded open set $\Omega\subset\mathbb{R}^n$.

Degree. In dimension $n=2$, the degree of maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ is well defined provided that $u\in W^{1,1}_{\rm loc}(\mathbb{R}^2,\mathbb{S}^2)$ with $|Du|\in L^2(\mathbb{R}^2)$. In fact, by Remark 1.7 the graph current G_u is locally i.m. rectifiable. In addition, it satisfies the null-boundary condition

$$\langle\partial G_u;\omega\rangle=0\qquad\forall\,\omega\in\mathcal{D}^1(\mathbb{R}^2\times\mathbb{S}^2)\,.$$

Therefore, denoting by $\Pi_y(x,y):=y$ the orthogonal projection onto the target space $\mathbb{S}^2\subset\mathbb{R}^3$, the image current $\Pi_{y\#}G_u$ is an *integral* 2*-cycle* in $\mathbb{S}^2$, i.e., $\Pi_{y\#}G_u\in\mathcal{R}_2(\mathbb{S}^2)$ with $\partial(\Pi_{y\#}G_u)=0$. By the constancy theorem, compare [22, Sec. 4.3.1, Thm. 4], we thus have $\Pi_{y\#}G_u=\mathrm{d}\,[\![\,\mathbb{S}^2\,]\!]$ for some integer $\mathrm{d}\in\mathbb{Z}$. Moreover, if ω_2 denotes the volume 2-form on $\mathbb{S}^2$

$$\omega_2:=y^1dy^2\wedge dy^3+y^2dy^3\wedge dy^1+y^3dy^1\wedge dy^2 \tag{1.10}$$

by the action (1.4) we have

$$\int_{\mathbb{R}^2} u^{\#}\omega_2 = \langle \Pi_{y\#} G_u ; \omega_2\rangle = \langle \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!] ; \omega_2\rangle = \mathrm{d}\int_{\mathbb{S}^2}\omega_2 = \mathrm{d}\cdot 4\pi\,.$$

Definition 1.8. Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{S}^2)$ with $|Du| \in L^2(\mathbb{R}^2)$. We call *degree* $\deg u$ of u the integer $\mathrm{d}\in\mathbb{Z}$ given by formula

$$\deg u := \frac{1}{4\pi}\int_{\mathbb{R}^2} u^{\#}\omega_2 = \mathrm{d}\,.$$

Notice that the degree is *strongly continuous*: if $\{u_h\}\subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{S}^2)$ is a sequence converging to $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{S}^2)$ a.e. in $\mathbb{R}^2$, with $Du_h \to Du$ strongly in $L^2(\mathbb{R}^2,\mathbb{R}^{3\times 2})$, by dominated convergence we get

$$\lim_{h\to\infty}\frac{1}{4\pi}\cdot\Big|\int_{\mathbb{R}^2}(u_h^{\#}\omega_2 - u^{\#}\omega_2)\Big| = 0$$

and hence $\deg u_h = \deg u$, for h large enough.

Cartesian currents. Let $n\geq 2$ and $\{u_h\}\subset C^\infty(B^n,\mathbb{S}^2)$ be a sequence of smooth maps with equibounded Dirichlet energies, $\sup_h \mathbb{D}(u_h)<\infty$, see (0.1). The graph currents G_{u_h} belong to $\mathcal{R}_n(B^n\times\mathbb{S}^2)$ and satisfy condition (1.7) and $\sup_h \mathbf{M}(G_{u_h})<\infty$, by (1.6). Therefore, Federer-Fleming's theorem [17] yields that the currents G_{u_h} subconverge weakly in $\mathcal{D}_n(B^n\times\mathbb{S}^2)$ to a current $T\in\mathcal{R}_n(B^n\times\mathbb{S}^2)$ satisfying the *null-boundary condition*

$$(\partial T)\llcorner B^n\times\mathbb{S}^2 = 0\,. \tag{1.11}$$

In addition, compare [22, 26], there exists an i.m. rectifiable current $L\in\mathcal{R}_{n-2}(B^n)$ and a map $u_T\in W^{1,2}(B^n,\mathbb{S}^2)$ such that

$$T = G_{u_T} + L\times[\![\,\mathbb{S}^2\,]\!] \tag{1.12}$$

where the *underlying function* u_T is given by the weak $W^{1,2}$ limit of the u_h's. Finally, in low dimension $n=2$ we also have $(\partial G_{u_T})\llcorner B^2\times\mathbb{S}^2=0$. For that reason, Giaquinta-Modica-Souček [19] introduced the following

Definition 1.9. The class $\mathrm{cart}^{2,1}(B^n\times\mathbb{S}^2)$ is given by the i.m. rectifiable currents $T\in\mathcal{R}_n(B^n\times\mathbb{S}^2)$ satisfying the null-boundary condition (1.11) and the structure property (1.12) for some Sobolev map u_T in $W^{1,2}(B^n,\mathbb{S}^2)$ and some i.m. rectifiable current $L\in\mathcal{R}_{n-2}(B^n)$.

The Dirichlet energy of a current T in $\mathrm{cart}^{2,1}(B^n\times\mathbb{S}^2)$ is given by

$$\mathbb{D}(T) := \frac{1}{2}\int_{B^n}|Du_T|^2\,dx + 4\pi\cdot\mathbf{M}(L)\qquad \text{if (1.12) holds}\,.$$

Since the functional $T\mapsto\mathbb{D}(T)$ agrees with the *parametric polyconvex lower semicontinuous extension* of the Dirichlet integral, compare [23, Sec. 2.2.4], dealing with currents in $\mathrm{cart}^{2,1}\,(B^n\times\mathbb{S}^2)$ it turns out that if $T_h\rightharpoonup T$ weakly in $\mathcal{D}_n(B^n\times\mathbb{S}^2)$, then

$$\mathbb{D}(T)\leq\liminf_{h\to\infty}\mathbb{D}(T_h)\,. \tag{1.13}$$

Finally, a weak closure property holds: if a sequence $\{T_h\} \subset \text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ satisfies $\sup_h \mathbb{D}(T_h) < \infty$, then there exists a current T in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ and a (not relabeled) subsequence such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$.

Current of the singularities. Let $u \in W^{1,2}(B^n, \mathbb{S}^2)$, where $n \geq 3$. Following [23, Sec. 4.2.5], we denote by $\mathbb{P}(u)$ the $(n-3)$-current in $\mathcal{D}_{n-3}(B^n)$

$$\langle \mathbb{P}(u); \varphi \rangle := \frac{1}{4\pi} \int_{B^n} u^{\#}\omega_2 \wedge d\varphi \,, \qquad \varphi \in \mathcal{D}^{n-3}(B^n) \tag{1.14}$$

where ω_2 is the volume 2-form (1.10). It turns out that the boundary of the graph current G_u satisfies equation

$$(\partial G_u) \llcorner B^n \times \mathbb{S}^2 = \mathbb{P}(u) \times [\![\mathbb{S}^2]\!] \,. \tag{1.15}$$

Therefore, for a current $T \in \text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ as in (1.12), the null boundary condition (1.11) is equivalent to the following link between $\mathbb{P}(u_T)$ and L:

$$(\partial L) \llcorner B^n = -\mathbb{P}(u_T) \,. \tag{1.16}$$

Real and integral mass. The latter formula motivates the introduction of some more notation. Let again $n \geq 3$.

Definition 1.10. For any current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$, we denote by

$$\mathbf{m}_{r,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-2}(B^n) \,, \ (\partial D) \llcorner B^n = -\mathbb{P}\} \tag{1.17}$$

the *real mass of* $\mathbb{P}$ allowing connections to the boundary. We also define

$$\mathbf{m}_{i,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-2}(B^n) \,, \ (\partial L) \llcorner B^n = -\mathbb{P}\} \,. \tag{1.18}$$

Remark 1.11. By Federer-Fleming's theorem [17], if there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = -\mathbb{P}$, the minimum in (1.18) is attained. In that case, $\mathbf{m}_{i,B^n}(\mathbb{P})$ is called *integral mass*, and a minimizer L a *minimal integral connection of* $\mathbb{P}$ (allowing connections to the boundary).

Example 1.12. If e.g. $u(x) = x/|x|$, by (1.8) and (1.15) we get $\mathbb{P}(u) = -\delta_O$, and the integral mass $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is equal to the length of any line segment connecting a point at the boundary of B^3 to the origin O.

Remark 1.13. In dimension $n = 3$, Federer's theorem [16], compare [23, Sec. 3.1.4, Thm. 8], gives that if $\mathbf{m}_{i,B^3}(\mathbb{P}) < \infty$ for some $\mathbb{P} \in \mathcal{D}_0(B^3)$, then

$$\mathbf{m}_{i,B^3}(\mathbb{P}) = \mathbf{m}_{r,B^3}(\mathbb{P}) \,. \tag{1.19}$$

This is false in general when $n \geq 4$. More precisely, compare [28, 31], for a current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$ with $\mathbf{m}_{i,B^n}(\mathbb{P}) < \infty$, it may happen that

$$\mathbf{m}_{r,B^n}(\mathbb{P}) < \mathbf{m}_{i,B^n}(\mathbb{P}) \quad \text{if} \quad n \geq 4 \,.$$

Maps with small singular set. Due to the non-triviality of the second homotopy group $\pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$, in dimension $n \geq 3$ it is false that the class of smooth maps $C^\infty(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$. However, a wider class of maps with small singular set is dense.

Definition 1.14. For $n \geq 3$, we denote by $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ the class of maps $u : \bar{B}^n \to \mathbb{S}^2$ which are smooth on $\bar{B}^n \setminus S_u$, where S_u is a finite union of $(n-3)$-dimensional smooth sets with smooth boundary (a finite set of points when $n = 3$) and such that for every positive integer k there exists a positive real constant c, depending on u and k, such that the k-th order derivative

$$|D^k u(x)| \leq \frac{c}{(\mathrm{dist}\,(x, S_u))^k} \qquad \forall\, x \in \bar{B}^n \setminus S_u\,.$$

The following density property was proved in case $n = 3$ by Bethuel–Zheng [10], and extended to high dimension $n \geq 3$ by Bethuel [7].

Theorem 1.15. *The class $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$.*

Point singularities. Let $n = 3$ and assume that $u \in W^{1,2}(B^3, \mathbb{S}^2)$ is smooth outside a finite set S_u, compare Definition 1.14. For any singular point $a \in S_u$ and for $r > 0$ small, the restriction $u_{|\partial B^3_r(a)}$ of u to the boundary of the ball $B^3_r(a) := a + B^3_r$ is a smooth function. Therefore, arguing as before it turns out that the *degree of u at a* is well defined by the integer

$$\deg(u, a) := \frac{1}{4\pi} \int_{\partial B^3_r(a)} u^\# \omega_2 = \mathrm{d} \in \mathbb{Z}\,. \tag{1.20}$$

In fact, standard homotopy arguments imply that definition (1.20) does not depend on the choice of the (small) radius, whence it agrees with the classical Brouwer degree. Moreover, if $S_u = \{a_i\}_{i=1}^m$ and $\deg(u, a_i) = \mathrm{d}_i$, similarly to [23, Sec. 4.2.1, Prop. 1] we infer:

$$(\partial G_u) \llcorner B^3 \times \mathbb{S}^2 = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Therefore, formula (1.15) implies that the current of the singularities $\mathbb{P}(u)$ is i.m. rectifiable:

$$\mathbb{P}(u) = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \in \mathcal{R}_0(B^3)\,. \tag{1.21}$$

As e.g. to the 0-homogeneous map $u(x) = x/|x|$, one has $S_u = \{O\}$, the boundary condition (1.8) holds, $\mathbb{P}(u) = -\delta_O$, and $\deg(u, O) = 1$.

Relaxed Dirichlet energy. Similarly to (0.11), the relaxed Dirichlet energy of maps u in $L^1(B^n, \mathbb{S}^2)$ is defined by the functional

$$\widetilde{\mathbb{D}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{D}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^2),\ u_k \to u \text{ in } L^1(B^n, \mathbb{R}^3) \Big\}\,.$$

In dimension $n = 2$, we clearly have

$$\widetilde{\mathbb{D}}(u) = \begin{cases} \mathbb{D}(u) & \text{if } u \in W^{1,2}(B^n, \mathbb{S}^2) \\ +\infty & \text{if } u \in L^1(B^n, \mathbb{S}^2) \setminus W^{1,2}(B^n, \mathbb{S}^2)\,. \end{cases}$$

In dimension $n = 3$, following Brezis-Coron-Lieb [13], the *flat norm* $\mathbf{L}(u)$ of $u \in W^{1,2}(B^3, \mathbb{S}^2)$ (relative to the boundary) is given by

$$\mathbf{L}(u) := \frac{1}{4\pi} \cdot \sup_{\xi \in \mathcal{F}} \int_{B^3} D(u) \bullet D\xi \, dx \tag{1.22}$$

where $\bullet$ is the scalar product in $\mathbb{R}^3$. In the latter formula, $\mathcal{F}$ denotes the class of smooth test functions $\xi : B^3 \to \mathbb{R}$ such that $\|\xi\|_\infty \le 1$ and $\|D\xi\|_\infty \le 1$, and $D(u) : B^3 \to \mathbb{R}^3$ the *D-field*

$$D(u) := \big(u \bullet \partial_2 u \times \partial_3 u,\, u \bullet \partial_3 u \times \partial_1 u,\, u \bullet \partial_1 u \times \partial_2 u\big)\,.$$

Bethuel–Brezis–Coron [8] showed that for any $u \in W^{1,2}(B^3, \mathbb{S}^2)$ the relaxed Dirichlet energy is finite, and it satisfies the explicit formula

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{L}(u)\,.$$

Following Giaquinta–Modica–Souček [20], as distributions of $\mathcal{D}_0(B^3)$ one gets $\mathbb{P}(u) = \frac{1}{4\pi} \operatorname{Div} D(u)$, i.e.,

$$\langle \mathbb{P}(u); \varphi \rangle = -\frac{1}{4\pi} \int_{B^3} D(u) \bullet D\varphi \, dx \qquad \forall\, \varphi \in C_c^\infty(B^3)\,.$$

Equivalently, the current $\mathbf{D}(u) \in \mathcal{D}_1(B^3)$ given by

$$\langle \mathbf{D}(u); \eta \rangle := \frac{1}{4\pi} \int_{B^3} u^{\#}\omega_2 \wedge \eta\,, \qquad \eta \in \mathcal{D}^1(B^3)$$

is such that $(\partial \mathbf{D}(u)) \llcorner B^3 = \mathbb{P}(u)$ and $\mathbf{M}(\mathbf{D}(u)) < \infty$. Moreover, a duality argument yields that the minimal real connection $\mathbf{m}_{r,B^3}(\mathbb{P}(u))$ of the singularities agrees with the flat norm $\mathbf{L}(u)$, compare [23, Sec. 4.2.5].

Most importantly, in [20] the authors obtained that the flat norm agrees with the integral mass of the current of the singularities, i.e.,

$$\mathbf{L}(u) = \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty \qquad \forall\, u \in W^{1,2}(B^3, \mathbb{S}^2)\,.$$

Their argument relies on Theorem 1.15 and on the following result:

Proposition 1.16. *Let $u \in W^{1,2}(B^3, \mathbb{S}^2)$ and $\{u_k\} \subset W^{1,2}(B^3, \mathbb{S}^2) \cap R_0^\infty$ be such that $u_k \to u$ strongly in $W^{1,2}$. Then, for each k there exists an i.m. rectifiable current $L_k \in \mathcal{R}_1(B^3)$ with $(\partial L_k) \llcorner B^3 = \mathbb{P}(u) - \mathbb{P}(u_k)$ such that $\mathbf{M}(L_k) \to 0$ as $k \to \infty$.*

Remark 1.17. The proof of Proposition 1.16 makes use of the coarea formula by Almgren–Browder–Lieb [3] and of Federer's theorem [16], see (1.19). In high dimension $n \ge 4$, even if we knew a priori that $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$ for some $u \in W^{1,2}(B^n, \mathbb{S}^2)$, the cited Federer's theorem doesn't apply, see Remark 1.13. Therefore, Proposition 1.16 doesn't work anymore.

In [24], using a different approach we extended the explicit formula for the relaxed Dirichlet energy to any high dimension $n \ge 3$. Definitely, for any map $u \in W^{1,2}(B^n, \mathbb{S}^2)$, it turns out that the $(n-3)$-current $\mathbb{P}(u)$ of the singularities satisfies $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$, and we have:

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{1,2}(B^n, \mathbb{S}^2)\,. \tag{1.23}$$

2. Maps with finite relaxed Laplacean energy

In this section, we deal with some general properties of maps with finite relaxed energy (0.11). The case of low dimension $n = 2$ is then discussed, where a first lower semicontinuity property is also obtained. Finally, we report a recent cohomological criterion for strong density of smooth maps in $W^{2,1}(B^n, \mathbb{S}^2)$ obtained in [12].

Weak Laplacean. Let $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, see (0.12), and let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ be such that $u_h \to u$ in $L^1(B^n, \mathbb{R}^3)$ and $\sup_h \mathbb{L}(u_h) < \infty$. By inequality (0.6) we have $\sup_h \mathbb{D}(u_h) < \infty$, see (0.1), so that a (not relabeled) subsequence of $\{u_h\}$ converges to u weakly in $W^{1,2}(B^n, \mathbb{R}^3)$, and $u \in W^{1,2}(B^n, \mathbb{S}^2)$. Since $Du \in L^2(B^n, \mathbb{R}^{3\times n})$, the distributional divergence of the gradient is well defined by (0.13), and using that

$$\lim_{h\to\infty} \int_{B^n} \mathrm{tr}\,[Du_h\,(D\varphi)^\top]\,dx = \int_{B^n} \mathrm{tr}\,[Du\,(D\varphi)^\top]\,dx \quad \forall\,\varphi \in C_c^\infty(B^n, \mathbb{R}^3)$$

we infer that $\mathrm{Div}Du_h \rightharpoonup \mathrm{Div}Du$ weakly as $\mathbb{R}^3$-valued measures in B^n. Moreover, in case of Sobolev maps in $W^{2,1}(B^n, \mathbb{S}^2)$, integrating by parts we get:

$$\langle \mathrm{Div}Du; \varphi\rangle = \int_{B^n} \Delta u \bullet \varphi\,dx \quad \forall\,\varphi \in C_c^\infty(B^n, \mathbb{R}^3)$$

whence (0.16) holds true. Therefore, the lower semicontinuity property of the total variation gives

$$|\mathrm{Div}Du|(B^n) \le \liminf_{h\to\infty} |\mathrm{Div}Du_h|(B^n) \le \sup_h \mathbb{L}(u_h) < \infty \tag{2.1}$$

so that $\mathrm{Div}Du$ is a $\mathbb{R}^3$-valued finite Radon measure in B^n, and the lower bound (0.14) follows by lower semicontinuity.

Notice that if $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, denoting $u = (u^1, u^2, u^3)$, we have checked that Du^ℓ is a divergence-measure field in $\mathcal{DM}^{1,2}(B^n)$ for $\ell = 1, 2, 3$, see Definition 1.1. Moreover, the decomposition into mutually singular measures

$$\mathrm{Div}Du = (\mathrm{Div}Du)^a + (\mathrm{Div}Du)^s\,, \quad (\mathrm{Div}Du)^a = \widetilde{\Delta u}\,\mathcal{L}^n \llcorner B^n \tag{2.2}$$

holds, with density $\widetilde{\Delta u}$ in $L^1(B^n, \mathbb{R}^3)$.

The BV case. If u belongs to the class $\mathbb{L}_{BV}(B^n, \mathbb{S}^2)$ in (0.17), the weak hessian $\nabla(Du^\ell)$ of each component u^ℓ is a summable function in $L^1(B^n, \mathbb{R}^{n\times n})$, and the density $\widetilde{\Delta u}$ in (2.2) agrees with the approximate Laplacean $\Delta u = (\Delta u^1, \Delta u^2, \Delta u^3)$, where $\Delta u^\ell = \mathrm{tr}\,[\nabla(Du^\ell)]$, for $\ell = 1, 2, 3$. In addition, the singular part of the measure $(\mathrm{Div}Du)^s$ decomposes into a *Jump* and a *Cantor-type* component, the first one being concentrated on the countably $(n-1)$-rectifiable discontinuity set of the gradient Du, and the second one being equal to zero if Du is a *special function of bounded variation.*

As a consequence, we obtain a *tangential property* concerning the singular component of the weak Laplacean.

Proposition 2.1. *Let $n \geq 2$ and $u \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, see* (0.17). *Then*

$$u \bullet (\mathrm{Div} Du)^s = 0 \,. \tag{2.3}$$

Proof. By applying Proposition 1.3 with $F = Du^\ell$ and $g = u^\ell$, and by summating on $\ell = 1, 2, 3$, we obtain

$$\mathrm{Div}(u \bullet Du) = u \bullet \mathrm{Div} Du + |Du|^2 \mathcal{L}^n \llcorner B^n \,.$$

Since moreover $|u| \equiv 1$, we have $0 = \partial_i |u|^2 = 2\partial_i u \bullet u$ for $i = 1, \dots, n$, whence $\mathrm{Div}(u \bullet Du) = 0$ and therefore

$$u \bullet \mathrm{Div} Du = -|Du|^2 \mathcal{L}^n \llcorner B^n \,. \tag{2.4}$$

On the other hand, since Du is in BV, we have $u \bullet \Delta u = -|Du|^2$ for $\mathcal{L}^n$-a.e. in B^n, so that

$$u \bullet (\mathrm{Div} Du)^a = (u \bullet \Delta u) \, \mathcal{L}^n \llcorner B^n = -|Du|^2 \, \mathcal{L}^n \llcorner B^n \,. \tag{2.5}$$

Equation (2.3) follows from (2.4) and (2.5), on account of the decomposition formula (2.2), where $\widetilde{\Delta u} = \Delta u$. □

The low dimension case. In the critical dimension $n = 2$, formula (0.15) holds, and there is no gap. In fact, by the continuous embedding of $W^{1,2}(B^2)$ in VMO, convolutions $u_\varepsilon := \rho_\varepsilon * u$ with a smooth kernel $\rho_\varepsilon = \varepsilon^{-2} \rho(x/\varepsilon)$ of maps $u \in W^{1,2}(B^2, \mathbb{S}^2)$ have image whose distance to $\mathbb{S}^2$ goes to 0 with ε, and hence the projection argument from [29] applies, compare e.g. [22, Sec. 5.5.1, Thm. 3] or [26, Thm. 4.14]. More precisely, equation (0.15) holds true in the class $W^{2,1}(B^2, \mathbb{S}^2)$ by the strong $W^{2,1}$ density of maps in $C^\infty(B^2, \mathbb{S}^2)$. If $u \in \mathbb{L}(B^2, \mathbb{S}^2)$, arguing as e.g. in the proof of [4, Thm. 3.9], for each $\delta > 0$ we can find a smooth map $v_\delta \in C^\infty(B^2, \mathbb{R}^3)$ such that $\|u - v_\delta\|_{L^1(B^2)} < \delta$, $\mathrm{dist}\,(v_\delta(B^2), \mathbb{S}^2) < \delta$, and

$$|\mathrm{Div} D v_\delta|(B^2) \leq |\mathrm{Div} Du|(B^2) + \delta \,.$$

Setting then $u_h = \Pi(v_{\delta_h})$, where $\delta_h \searrow 0$ and $\Pi(y) = y/|y|$, for $y \in \mathbb{R}^3 \setminus \{0\}$, we have $u_h \to u$ in $L^1(B^2, \mathbb{R}^3)$ and

$$\limsup_{h \to \infty} \mathbb{L}(u_h) \leq \limsup_{h \to \infty} |\mathrm{Div} D v_{\delta_h}|(B^2) \,,$$

so that (2.1) gives the energy convergence $\mathbb{L}(u_h) \to |\mathrm{Div} Du|(B^2)$ as $h \to \infty$, whence (0.15) follows from the lower bound (0.14).

Using Proposition 1.2, we also obtain a lower semicontinuity property:

Theorem 2.2. *Let $\{u_k\} \subset C^\infty(B^2, \mathbb{S}^2)$ be such that the graph currents G_{u_k} weakly converge in $\mathcal{D}_2(B^2 \times \mathbb{S}^2)$ to the current $T = G_u + \mathrm{d}\, \delta_O \times [\![\mathbb{S}^2]\!]$, for some map $u \in \mathbb{L}(B^2, \mathbb{S}^2)$ and some integer $\mathrm{d} \in \mathbb{Z}$. Then*

$$\liminf_{k \to \infty} \mathbb{L}(u_k) \geq |\mathrm{Div} Du|(B^n) + 8\pi \, |\mathrm{d}| \,.$$

Proof. For any $v \in W^{2,1}(B^2, \mathbb{S}^2)$ and any Borel set $B \subset B^2$, we denote

$$\mathbb{L}(v, B) := |\mathrm{Div} Dv|(B) = \int_B |\Delta v| \, dx \,, \quad \mathbb{D}(v, B) := \frac{1}{2} \int_B |Dv|^2 \, dx \,.$$

Let $\varepsilon > 0$. Since by Proposition 1.2 the measure $|\mathrm{Div} Du|$ does not charge any atom, we can choose $r > 0$ small so that $|\mathrm{Div} Du|(\bar{B}^2_r) \leq \varepsilon$, and hence by lower semicontinuity and additivity

$$\liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) \geq |\mathrm{Div} Du|(B^2 \setminus \bar{B}^2_r) \geq |\mathrm{Div} Du|(B^2) - \varepsilon \,.$$

Moreover, by inequality (0.6) we get

$$\liminf_{k\to\infty} \mathbb{L}(u_k, \bar{B}^2_r) \geq 2 \cdot \liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r) \,.$$

On the other hand, by weak lower semicontinuity of the Dirichlet energy on Cartesian currents, see (1.13), using that

$$G_{u_k} \llcorner B^2_r \times \mathbb{S}^2 \rightharpoonup G_u \llcorner B^2_r \times \mathbb{S}^2 + \mathrm{d}\, \delta_O \times [\![\mathbb{S}^2]\!]$$

weakly in $\mathcal{D}_2(B^2_r \times \mathbb{S}^2)$ we obtain the energy lower bound

$$\liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r) \geq \mathbb{D}(u, B^2_r) + 4\pi \, |\mathrm{d}| \geq 4\pi \, |\mathrm{d}| \,.$$

Finally, putting the terms together we obtain

$$\begin{aligned} \liminf_{k\to\infty} \mathbb{L}(u_k, B^2) \geq & \liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) + \liminf_{k\to\infty} \mathbb{L}(u_k, B^2_r) \\ \geq & |\mathrm{Div} Du|(B^n) - \varepsilon + 8\pi \, |\mathrm{d}| \end{aligned}$$

for each $\varepsilon > 0$, as required. □

The high dimension case. If $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, to our knowledge there isn't enough information concerning both the density $\widetilde{\Delta u}$ and the singular part $(\mathrm{Div} Du)^s$ in formula (2.2). More precisely, Proposition 1.2 yields that $|(\mathrm{Div} Du)^s|(B) = 0$ for each Borel set $B \subset B^n$ with σ-finite $\mathcal{H}^{n-2}$-measure. However, it may happen that the gradient Du does not belong to the class $\mathrm{BV}(B^n, \mathbb{R}^{3\times n})$, so that we cannot conclude e.g. that for $\ell = 1, 2, 3$ the vector field $Du^\ell \in L^2(B^n, \mathbb{R}^n)$ is approximately differentiable $\mathcal{L}^n$-a.e. in B^n, and that the trace $\mathrm{tr}\, \nabla Du^\ell$ agrees $\mathcal{L}^n$-a.e. in B^n with the ℓ-th component of the Radon-Nikodym derivative $\widetilde{\Delta u}$ of the measure $\mathrm{Div} Du$.

A sufficient condition ensuring both enough regularity of the density $\widetilde{\Delta u}$ and property $(\mathrm{Div} Du)^s = 0$, is the membership of Du to the Sobolev class $W^{1,1}(B^n, \mathbb{R}^{3\times n})$, so that in particular equation (0.16) holds true. In fact, the computation of the energy gap for maps in $W^{2,1}(B^3, \mathbb{S}^2)$ is the content of our Main Result, Theorem 0.1.

A cohomological criterion. The following density result was proved in more generality by Gastel–Nerf [18] when $n = 3$, and then extended by Bousquet-Ponce-Van Schaftingen [11] in high dimension, see also [27, Lemma 4.5]. One of the main difficulties in the proof of Theorem 2.3 is to obtain Sobolev regularity of the derivatives of the functions involved.

Therefore, one cannot use the same construction as the one in the proof of Theorem 1.15.

Theorem 2.3. *The class* $R^\infty_{n-3}(B^n,\mathbb{S}^2)$ *is strongly dense in* $W^{2,1}(B^n,\mathbb{S}^2)$, *in any dimension* $n\geq 3$.

Concerning maps u in $W^{1,2}(B^n,\mathbb{S}^2)$, the only obstruction to strong density of smooth maps is encoded by the non-triviality of the current of the singularities $\mathbb{P}(u)$ in (1.14). This cohomological criterion was firstly proved by Bethuel [6] when $n=3$, and then extended in high dimension and for a wider class of target manifolds in [9]. For second order Sobolev maps, Bousquet–Ponce–Van Schaftingen [12] have recently proved the following

Theorem 2.4. ([12]) *Let* $u\in W^{2,1}(B^n,\mathbb{S}^2)$ *for some* $n\geq 3$. *If* $\mathbb{P}(u)=0$, *there exists a sequence* $\{u_h\}\subset C^\infty(B^n,\mathbb{S}^2)$ *such that* $u_h\to u$ *strongly in* $W^{2,1}(B^n,\mathbb{R}^3)$.

We briefly comment on the validity of the previous result. First, we observe that the converse implication in Theorem 2.4 is trivially checked. In fact, if $u\in W^{2,1}(B^n,\mathbb{S}^2)$ is the strong limit of a smooth sequence $\{u_h\}$ in $C^\infty(B^n,\mathbb{S}^2)$, by the $W^{1,2}$-convergence of u_h to u, in Remark 1.5 we have seen that the graph current G_u satisfies the null-boundary condition (1.7), which yields $\mathbb{P}(u)=0$, by (1.15).

As to the non-trivial implication in Theorem 2.4, in [12, Thm. 1.10] the authors show that for more general target Riemannian manifolds $\mathcal{N}$, a map $u\in W^{2,1}(B^n,\mathcal{N})$ is the strong $W^{2,1}$ limit of a smooth sequence in $C^\infty(B^n,\mathcal{N})$ if and only in u is 2-*extendable*. Referring to the cited paper for the precise notion of 2-extendability, in the special case in which $\mathcal{N}=\mathbb{S}^2$ we quote [12, Thm. 1.13], where it is shown that in any dimension $n\geq 3$ a map $u\in W^{2,1}(B^n,\mathbb{S}^2)$ is 2-extendable if and only if $(du^\#\omega_2)=0$ *in the sense of currents in* B^n. The latter property means that

$$\int_{B^n} u^\#\omega_2\wedge d\eta=0\quad\forall\,\eta\in\mathcal{D}^{n-3}(B^n)$$

and hence it is equivalent to property $\mathbb{P}(u)=0$.

3. Energy concentration and dipole problem

In this section, we define a suitable modification of the inverse to the stereographic map. We then compute the *minimal Laplacean energy* among maps $u:\mathbb{R}^2\to\mathbb{S}^2$ with fixed degree, Theorem 3.2, and describe the related bubbling phenomenon. Finally, we extend to the Laplacean energy the classical *Dipole problem* of Brezis–Coron–Lieb [13] for the Dirichlet energy in 3D, Theorem 3.3.

Modified stereographic projection. Consider the inverse of the stereographic map (1.1) in case $\mathfrak{p}=2$. Since ${\sigma_2^{-1}}_\#[\![\mathbb{R}^2]\!]=[\![\mathbb{S}^2]\!]$, one has $\deg\sigma_2^{-1}=1$, compare Definition 1.8.

Similarly to e.g. [23, Sec. 4.1.1], we now modify σ_2^{-1} in such a way that it is equal to the South Pole P_S outside some small disk, by paying a small amount of Laplacean energy.

Proposition 3.1. *For any $\varepsilon > 0$ and $\delta > 0$ sufficiently small, there exists a smooth and degree one map $u_{\varepsilon,\delta} \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, see* (0.8)*, such that:*

(1) $u_{\varepsilon,\delta}(x) = P_S$ *if* $|x| > \delta$ *;*
(2) $4\pi \le \mathbb{D}(u_{\varepsilon,\delta}, \mathbb{R}^2) \le 4\pi + O(\varepsilon)$*, see* (1.9) *;*
(3) $8\pi \le \mathbb{L}(u_{\varepsilon,\delta}, \mathbb{R}^2) \le 8\pi + O(\varepsilon)$*, see* (0.9)

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Proof. For $x \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$ and $\rho = |x|$ we can write

$$\sigma_2^{-1}(x) = \left(\frac{x}{\rho} \sin \widehat{\theta}(\rho), -\cos \widehat{\theta}(\rho) \right) , \quad \widehat{\theta}(\rho) = 2 \arctan(\rho^{-1})$$

in terms of the angular (geodesic) distance $\rho \mapsto \widehat{\theta}(\rho)$ in $\mathbb{S}^2$ of $\sigma_2^{-1}(\partial B_\rho^2)$ from the South Pole P_S.

For $\varepsilon > 0$ small, we modify the angular distance $\widehat{\theta}(\rho)$ and define

$$\widehat{\theta}_\varepsilon(\rho) = \begin{cases} 2\arctan(\rho^{-1}) & \text{if} \quad 0 < \rho < \varepsilon^{-1} \\ a_\varepsilon \rho^2 + b_\varepsilon \rho + c_\varepsilon & \text{if} \quad \varepsilon^{-1} \le \rho \le R_\varepsilon \\ 0 & \text{if} \quad \rho > R_\varepsilon \end{cases}$$

where the coefficients of the polynomial function in the second line and the radius $R_\varepsilon > \varepsilon^{-1}$ are given by

$$\begin{aligned} a_\varepsilon = & \quad \frac{\varepsilon^4}{(\varepsilon^2+1)^2} \cdot \frac{1}{2\arctan \varepsilon} , \\ b_\varepsilon = & \quad -\frac{\varepsilon^2}{(\varepsilon^2+1)^2} \left(\frac{\varepsilon}{\arctan \varepsilon} + 2(\varepsilon^2+1) \right) , \\ c_\varepsilon = & \quad 2\arctan \varepsilon + \frac{\varepsilon}{(\varepsilon^2+1)^2} \left(\frac{\varepsilon}{2\arctan \varepsilon} + 2(\varepsilon^2+1) \right) , \\ R_\varepsilon = & \quad \frac{1}{\varepsilon} + \frac{\varepsilon^2+1}{\varepsilon^2} \cdot 2\arctan \varepsilon . \end{aligned}$$

It can be checked that the function $\widehat{\theta}_\varepsilon$ is decreasing and of class C^1, and it is smooth outside the points ε^{-1} and R_ε. Therefore, letting

$$u_\varepsilon(x) = \left(\frac{x}{\rho} \sin \widehat{\theta}_\varepsilon(\rho), -\cos \widehat{\theta}_\varepsilon(\rho) \right) , \quad x \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\} ,$$

we have $\nabla u_\varepsilon \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^{3\times 2})$, and the Laplacean vector Δu_ε is a measurable function defined a.e. on $\mathbb{R}^2$. Moreover, we have:

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} |\Delta u_\varepsilon(x)| \, dx = 8\pi . \tag{3.1}$$

In fact, recalling that $\mathbb{L}(\sigma_2^{-1}, \mathbb{R}^2) = 8\pi$, for every $\varepsilon > 0$ small we obtain the estimate

$$\int_{\mathbb{R}^2} |\Delta u_\varepsilon(x)| \, dx \le 8\pi + \int_{\Omega_\varepsilon} |\Delta u_\varepsilon(x)| \, dx$$

where $\Omega_\varepsilon = \{x \in \mathbb{R}^2 \mid \varepsilon^{-1} < |x| < R_\varepsilon\}$. Now, for any sufficiently smooth function $\rho \mapsto \theta(\rho)$ we compute

$$\Delta\left(\frac{x_j}{\rho}\sin\theta(\rho)\right) = \left[\left(\theta''(\rho) + \frac{\theta'(\rho)}{\rho}\right)\cos\theta(\rho) - \left(\theta'(\rho)^2 + \frac{1}{\rho^2}\right)\sin\theta(\rho)\right]\frac{x_j}{\rho}$$

for $\rho = |x| > 0$, where $j = 1, 2$, and

$$\Delta\left(-\cos\theta(\rho)\right) = \theta'(\rho)^2\cos\theta(\rho) + \left(\theta''(\rho) + \frac{\theta'(\rho)}{\rho}\right)\sin\theta(\rho)\,.$$

Setting as before $u(x) = (\sin\theta(\rho)x/\rho, -\cos\theta(\rho))$, using that $|\sin\theta| \le |\theta|$ we obtain the pointwise estimate

$$|\Delta u(x)| \le \sqrt{2}\Big(|\theta''(\rho)| + \frac{|\theta'(\rho)|}{\rho} + \theta'(\rho)^2|\theta(\rho)| + \frac{|\theta(\rho)|}{\rho^2} + \theta'(\rho)^2 + |\theta''(\rho)\theta(\rho)| + \frac{|\theta(\rho)\theta'(\rho)|}{\rho}\Big)\,.$$

In particular, if $\varepsilon \in (0,1)$ is small enough in such a way that $\varepsilon/2 \le \arctan\varepsilon \le \varepsilon$, for $\theta(\rho) = \widehat{\theta}_\varepsilon(\rho)$ and $\rho \in (\varepsilon^{-1}, R_\varepsilon)$ we obtain the upper bounds

$$|\widehat{\theta}_\varepsilon(\rho)| \le 2\varepsilon\,, \quad |\widehat{\theta}'_\varepsilon(\rho)| \le 2\varepsilon^2\,, \quad |\widehat{\theta}''_\varepsilon(\rho)| \le 2\varepsilon^3$$

and hence for $u = u_\varepsilon$ we can find an absolute real constant $c > 0$ such that

$$|\Delta u_\varepsilon(x)| \le c\,\varepsilon^3 \quad \forall\, x \in \Omega_\varepsilon\,.$$

Using that the area of the set Ω_ε is lower than $24\pi/\varepsilon^2$, we thus obtain the upper bound

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon(x)|\,dx \le 24\pi\,c\,\varepsilon$$

and hence the energy limit (3.1) readily follows.

Now, in a similar way to (3.1) we can prove the energy convergence $\int_{\mathbb{R}^2} |\nabla u_\varepsilon(x)|^2\,dx \to 8\pi$ as $\varepsilon \to 0^+$. Therefore, the function $u_{\varepsilon,\delta}$ is readily obtained by means of a standard smoothing and rescaling argument applied to the angle function $\widehat{\theta}_\varepsilon$. □

Minimal energy of maps with fixed degree. According to (0.8), if $u \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, then $|Du| \in L^2(\mathbb{R}^2)$ and hence the degree of u is given by Definition 1.8. We now compute the minimal energy (0.9) in each class

(3.2) $$\mathcal{F}_{\mathrm{d}} := \{u \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2) \mid \deg u = \mathrm{d}\}\,, \qquad \mathrm{d} \in \mathbb{Z}\,.$$

We rely on the fact that the analogous problem for the Dirichlet energy is known. More precisely (cf. [23, Sec. 4.1]), denoting

(3.3) $$\mathcal{G}_{\mathrm{d}} := \{u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2) \mid Du \in L^2(\mathbb{R}^2, \mathbb{R}^{3\times 2}),\ \deg u = \mathrm{d}\}$$

and recalling (1.9), by the lower semicontinuity property (1.13) and Proposition 3.1 it turns out that

(3.4) $$\forall\,\mathrm{d} \in \mathbb{Z}\,, \quad \inf_{u \in \mathcal{G}_{\mathrm{d}}} \mathbb{D}(u, \mathbb{R}^2) = 4\pi\,|\mathrm{d}|\,.$$

Theorem 3.2. *For every integer* $\mathrm{d} \in \mathbb{Z}$ *we have:* $\inf_{u \in \mathcal{F}_{\mathrm{d}}} \mathbb{L}(u, \mathbb{R}^2) = 8\pi\, |\mathrm{d}|\,.$

Proof. When $\mathrm{d} = 0$, the claim is trivial, whereas the case $\mathrm{d} = 1$, and hence $\mathrm{d} = -1$, has been discussed in the introduction. In fact, by the lower bound (0.10), any map u in $\mathcal{F}_1$ has Laplacean energy at least 8π, and equality holds for harmonic and conformal maps of degree one, as e.g. $u = \sigma_2^{-1}$, compare (0.7). Therefore, it clearly suffices to consider the case $\mathrm{d} \geq 2$.

Since $\mathcal{F}_{\mathrm{d}} \subset \mathcal{G}_{\mathrm{d}}$, see (3.2) and (3.3), by inequality (0.10) and formula (3.4) we have

$$\inf_{u \in \mathcal{F}_{\mathrm{d}}} \mathbb{L}(u, \mathbb{R}^2) \geq 2 \cdot \inf_{u \in \mathcal{F}_{\mathrm{d}}} \mathbb{D}(u, \mathbb{R}^2) \geq 2 \cdot \inf_{u \in \mathcal{G}_{\mathrm{d}}} \mathbb{D}(u, \mathbb{R}^2) = 8\pi\, \mathrm{d}\,.$$

We now check the opposite inequality:

$$\inf_{u \in \mathcal{F}_{\mathrm{d}}} \mathbb{L}(u, \mathbb{R}^2) \leq 8\pi\, \mathrm{d}\,. \tag{3.5}$$

By Proposition 3.1, for each $\varepsilon > 0$ we find a degree one map $u_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, equal to P_S outside the unit disk B^2, and such that

$$\mathbb{L}(u_\varepsilon, \mathbb{R}^2) = \int_{B^2} |\Delta u_\varepsilon|\, dx \leq 8\pi + \frac{\varepsilon}{\mathrm{d}}\,.$$

Denoting $\mathbf{e}_1 := (1, 0)$, we define $w_\varepsilon(x) := u_\varepsilon(x - 3k\, \mathbf{e}_1)$ on the unit disk centered at $3k\, \mathbf{e}_1$, for $k = 0, 1, \ldots, \mathrm{d} - 1$, and $w_\varepsilon \equiv P_S$ outside the union of such d disks. The map w_ε satisfies $\mathbb{L}(w_\varepsilon, \mathbb{R}^2) \leq 8\pi\, \mathrm{d} + \varepsilon$ and belongs to the class $\mathcal{F}_{\mathrm{d}}$, whence (3.5) holds true, as required. □

Bubbling-off of spheres. We recall that the maps $u_{\varepsilon,\delta}$ from Proposition 3.1 have degree one. Therefore, letting e.g. $\varepsilon = \delta = 1/h$ we find a sequence $\{u_h\} \subset C^\infty(\mathbb{R}^2, \mathbb{S}^2)$ of smooth degree one maps weakly converging in $W^{2,1}$ to the constant map P_S, and such that

$$\lim_{h \to \infty} \mathbb{L}(u_h, \mathbb{R}^2) = \mathbb{L}(\sigma_2^{-1}, \mathbb{R}^2) = 8\pi\,.$$

Furthermore, it turns out that the above convergence is uniform far from the origin, and that the graph currents G_{u_h} weakly converge in $\mathcal{D}_2(\mathbb{R}^2 \times \mathbb{S}^2)$ to the Cartesian current

$$T = G_{P_S} + \delta_O \times [\![\, \mathbb{S}^2 \,]\!]$$

where G_{P_S} is the graph current of the constant map equal to P_S on $\mathbb{R}^2$.

A bubbling phenomenon occurs, and by Theorem 2.2 we infer that *the minimal Laplacean energy occurring for the formation of a* 2*-sphere is equal to* 8π. This *energy quantization* property is detected if one defines the energy

$$\mathbb{L}(T) := 8\pi\,, \qquad T = G_{P_S} + \delta_O \times [\![\, \mathbb{S}^2 \,]\!]\,. \tag{3.6}$$

We thus have a second order analogous to a similar feature concerning the conformal Dirichlet integral, where formula (1.2) yields that the minimum energy cost of a $\mathfrak{p}$-sphere is equal to $\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$, for any integer $\mathfrak{p} \geq 2$.

The Dipole problem. The classical Dipole problem by Brezis–Coron–Lieb [13] deals with Sobolev maps u in $W^{1,2}_{\rm loc}(\mathbb{R}^3,\mathbb{S}^2)$ which assume a given constant $P\in\mathbb{S}^2$ at infinity and which are smooth outside two singular points $a_\pm$, with

$$\deg(u,a_-)=-1\,,\qquad \deg(u,a_+)=+1$$

the degree being given by (1.20).

In [13], it is shown that the minimal Dirichlet energy $\mathbb{D}(u,\mathbb{R}^3)$ in such class, see (1.9), is equal to the distance $|a_+-a_-|$ between the singularities times the measure 4π of the unit sphere $\mathbb{S}^2$, compare [23, Sec. 4.2.3].

In this section, we discuss the Dipole problem for the Laplacean energy (0.9). We thus denote by $\mathcal{E}$ the subclass of maps u as above that in addition belong to the second order space $W_{\mathbb{L}}(\mathbb{R}^3,\mathbb{S}^2)$, see (0.8).

Theorem 3.3. *We have:* $\inf\{\mathbb{L}(u,\mathbb{R}^3)\mid u\in\mathcal{E}\}=|a_+-a_-|\cdot 8\pi\,.$

Proof. The lower bound inequality "$\geq$" readily follows, since by inequality (0.6) every map u in $\mathcal{E}$ belongs to $W^{1,2}_{\rm loc}(\mathbb{R}^3,\mathbb{S}^2)$ and hence by the Dipole problem for the Dirichlet energy we get

$$\mathbb{L}(u,\mathbb{R}^3)\geq 2\,\mathbb{D}(u,\mathbb{R}^3)\geq 2\,|a_+-a_-|\cdot 4\pi\,.$$

The energy upper bound

$$\inf\{\mathbb{L}(u,\mathbb{R}^3)\mid u\in\mathcal{E}\}\leq |a_+-a_-|\cdot 8\pi \tag{3.7}$$

will be obtained by means of a Dipole insertion argument which is re-adapted by [23, Sec. 4.2.3], see also [13]. Without loss of generality, we may and do assume $P=P_S$ and $a_+=(-r,0,0)$, $a_-=(r,0,0)$ for some $r>0$.

Firstly, by Proposition 3.1 we choose a smooth map $v_\varepsilon\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ with degree one, equal to the pole P_S outside B^2_r for some $r<1$, and such that with $\widehat{z}=(z_2,z_3)\in\mathbb{R}^2$

$$\mathbb{L}(v_\varepsilon,B^2):=\int_{B^2}|\Delta v_\varepsilon(\widehat{z})|\,d\widehat{z}\leq 8\pi+\varepsilon\,. \tag{3.8}$$

In formula (3.12), we wish to obtain a Sobolev map $u_\varepsilon\in W_{\mathbb{L}}(\mathbb{R}^3,\mathbb{S}^2)$, so that $\Delta u_\varepsilon\in L^1(\mathbb{R}^3,\mathbb{R}^3)$. Therefore, we have to replace the Lipschitz-continuous function $t\mapsto\min\{r+t,r-t,\delta\}$ on the interval $D^1_r:=(-r,r)$ with a function at least of class $C^1(D^1_r)$. With $\delta>0$ small, we can choose:

$$\varphi_\delta(t):=\begin{cases}\delta & \text{if}\quad |t|\leq r-\sqrt{2}\,\delta\\ \sqrt{\delta^2-\big(|t|-r+\sqrt{2}\,\delta\big)^2} & \text{if}\quad r-\sqrt{2}\,\delta\leq|t|\leq r-\dfrac{\delta}{\sqrt{2}}\\ r-|t| & \text{if}\quad r-\dfrac{\delta}{\sqrt{2}}\leq|t|<r\,.\end{cases} \tag{3.9}$$

For $x=(\widetilde{x},\widehat{x})\in\mathbb{R}\times\mathbb{R}^2\simeq\mathbb{R}^3$ and $z=(\widetilde{z},\widehat{z})\in D^1_r\times B^2$, we let

$$(\widetilde{x},\widehat{x})=\Phi_\delta(\widetilde{z},\widehat{z}):=(\widetilde{z},\varphi_\delta(\widetilde{x})\,\widehat{x}) \tag{3.10}$$

and define $u_{\varepsilon,\delta}(x) := \widehat{v}_\varepsilon(\Phi_\delta^{-1}(x))$ for $x \in \Phi_\delta(D_r^1 \times B^2)$, where $\widehat{v}_\varepsilon(z) := v_\varepsilon(\widehat{z})$, so that (3.8) holds. We then compute:

$$\begin{aligned}\Delta u_{\varepsilon,\delta}(x) = \quad & \frac{1}{\varphi_\delta(\widetilde{z})^2}\Delta\widehat{v}_\varepsilon(z) + \frac{{\varphi'_\delta}^2}{{\varphi_\delta}^4}(\widetilde{z}) \sum_{\alpha,\beta=2}^3 x_\alpha x_\beta \partial^2_{\alpha,\beta}\widehat{v}_\varepsilon(z) \\ & + \frac{2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta}{{\varphi_\delta}^3}(\widetilde{z}) \sum_{\alpha=2}^3 x_\alpha \partial_\alpha \widehat{v}_\varepsilon(z)\end{aligned}$$

where $z = \Phi_\delta^{-1}(x)$, so that $\widetilde{x} = \widetilde{z}$ and $\widehat{x} = \varphi_\delta(\widetilde{z})\,\widehat{z}$. Using that $\det D\Phi_\delta(z) = \varphi_\delta(\widetilde{z})^2$, we get:

$$\begin{aligned}\det D\Phi_\delta(z)\cdot \Delta u_{\varepsilon,\delta}(x) = \quad & \Delta v_\varepsilon(\widehat{z}) + \varphi'_\delta(\widetilde{z})^2 \sum_{\alpha,\beta=2}^3 z_\alpha z_\beta \partial^2_{\alpha,\alpha} v_\varepsilon(\widehat{z}) \\ & + (2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta)(\widetilde{z}) \sum_{\alpha=2}^3 z_\alpha \partial_\alpha v_\varepsilon(\widehat{z})\,.\end{aligned}$$

Therefore, since $\|\varphi'_\delta\|_{\infty,D_r^1} \le 1$ and $\|2{\varphi'_\delta}^2 - \varphi_\delta\varphi''_\delta\|_{\infty,D_r^1} \le 4$, by changing variables $z = \Phi_\delta^{-1}(x)$ we can estimate:

$$\begin{aligned}&\int_{\Phi_\delta(D_r^1\times B^2)} |\Delta u_{\varepsilon,\delta}|\,dx \le \int_{(-r,r)\times B^2} |\Delta v_\varepsilon(\widehat{z})|\,d\widetilde{z}\,d\widehat{z} \\ &\qquad + 8\int_{(r-\sqrt{2}\delta,r)\times B^2} \Big(\sum_{\alpha,\beta=2}^3 |z_\alpha z_\beta||\partial^2_{\alpha,\beta} v_\varepsilon(\widehat{z})| + |\widehat{z}||\nabla v_\varepsilon(\widehat{z})|\Big)\,d\widetilde{z}\,d\widehat{z}\,.\end{aligned}$$

Since moreover $v_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$ is smooth, the integral in the second line is small for $\delta > 0$ small, whence we can find $\delta(\varepsilon) \in (0, r/2)$ such that

$$\int_{\Phi_\delta(D_r^1\times B^2)} |\Delta u_{\varepsilon,\delta(\varepsilon)}|\,dx \le 2r\cdot \mathbb{L}(v_\varepsilon, B^2) + \varepsilon\,, \quad \text{see (3.8)}\,. \tag{3.11}$$

Recall that the map v_ε is equal to the pole P_S in a neighborhood of the boundary of B^2. Therefore, setting $\delta = \delta(\varepsilon)$ and

$$u_\varepsilon(x) := \begin{cases} u_{\varepsilon,\delta}(x) & \text{if} \quad x \in \Phi_\delta(D_r^1 \times B^2) \\ P_S & \text{if} \quad x \in \mathbb{R}^3 \setminus \big(\Phi_\delta(D_r^1 \times B^2) \cup \{(\pm r, 0, 0)\}\big)\end{cases} \tag{3.12}$$

it turns out that the map u_ε belongs to the class $\mathcal{E}$, whereas by (3.11)

$$\mathbb{L}(u_\varepsilon, \mathbb{R}^3) = \int_{\Phi_\delta(D_r^1\times B^2)} |\Delta u_{\varepsilon,\delta}|\,dx \le 2r\cdot 8\pi + (2r+1)\,\varepsilon\,.$$

The energy upper bound (3.7) follows by letting $\varepsilon \searrow 0$. □

A further dipole-like property. For future use, we state the following variant of Theorem 3.3. Recalling (3.9), for each $m > 0$ small we denote

$$\Phi_\delta^m(\widetilde{x}, \widehat{x}) := (\widetilde{x}, m\,\varphi_\delta(\widetilde{x})\,\widehat{x})\,, \quad \Omega_\delta^m := \Phi_\delta^m(D_r^1 \times B^2)\,.$$

Proposition 3.4. *Let U be a neighborhood of the segment joining a_- to a_+, and let $u : U \to \mathbb{S}^2$ be a $W^{2,1}$-map which is smooth in U outside the singular points $a_\pm$, where it has degree $\deg(u, a_\pm) = k_\pm$ for some $k_\pm \in \mathbb{Z}$, see (1.20). Let $\mathrm{d} \in \mathbb{Z}$. Then for all positive ε and for δ, $m > 0$ sufficiently small there exists a smooth function $u_\varepsilon : \mathbb{R}^3 \setminus \{a_-, a_+\} \to \mathbb{S}^2$ such that $u_\varepsilon \equiv u$ outside Ω^m_δ, $\deg(u, a_+) = k_+ - \mathrm{d}$, $\deg(u, a_-) = k_- + \mathrm{d}$, and*

$$\mathbb{L}(u_\varepsilon, \Omega^m_\delta) := \int_{\Omega^m_\delta} |\Delta u_\varepsilon|\, dx \le |a_+ - a_-| \cdot 8\pi\, |\mathrm{d}| + \varepsilon\,.$$

In addition, $u_\varepsilon \in W^{2,1}(U, \mathbb{S}^2)$ with

$$\int_{\Omega^m_\delta} |D^2 u_\varepsilon|\, dx \le C_2 \cdot |a_+ - a_-|\, |\mathrm{d}| + \varepsilon\,, \tag{3.13}$$

where the positive constant C_2 only depends on $\int_{\mathbb{R}^2} |D^2 {\sigma_2}^{-1}|\, dx$.

Proof. The first assertion is obtained by readapting an argument taken from [21], compare [23, Sec. 4.2.3], on account of Theorem 3.3. The second assertion follows from a similar estimate concerning the $W^{2,1}$ seminorm of u_ε on Ω^m_δ. Further details are omitted. □

4. The Laplacean energy on Cartesian currents

In this section, we define a Laplacean energy functional on a suitable class of Cartesian currents in such a way that a weak sequential lower semicontinuity property holds true, Theorem 4.3. In dimension $n = 3$, we also obtain a *strong density result*, Theorem 4.4.

Due to the embedding of $W^{2,1}(B^n, \mathbb{S}^2)$ into $W^{1,2}(B^n, \mathbb{S}^2)$, according to Definition 1.9 we give the following

Definition 4.1. We denote by $\mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with underlying function u_T in $W^{2,1}(B^n, \mathbb{S}^2)$.

Remark 4.2. For future use, given a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ we also denote

$$\mathcal{T}^{\mathbb{L}}_u := \{T \in \mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2) \text{ such that } u_T = u \text{ in (1.12)}\}\,. \tag{4.1}$$

By the explicit formula (1.23) for the relaxed Dirichlet energy, we infer that *the class $\mathcal{T}^{\mathbb{L}}_u$ is always non-empty.*

Theorems 2.2 and 3.2 suggest to introduce on the class $\mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the *Laplacean energy functional*

$$\mathbb{L}(T) := \int_{B^n} |\Delta u_T|\, dx + 8\pi \cdot \mathbf{M}(L) \qquad \text{if (1.12) holds} \tag{4.2}$$

so that we have:

(1) $\mathbb{L}(T) < \infty$ for every $T \in \mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$;
(2) $\mathbb{L}(G_u) = \mathbb{L}(u)$ if $T = G_u$ for some smooth map $u \in W^{2,1}(B^n, \mathbb{S}^2)$;
(3) when $n = 2$, formula (3.6) holds true.

Similarly to what happens for the Laplacean energy of σ_2^{-1} among degree one maps, see (0.7), the term $8\pi \cdot \mathbf{M}(L)$ in (4.2) is the optimal energy contribution of the vertical term $L \times [\![\mathbb{S}^2]\!]$ in (1.12). In fact, we are able to (partially) extend Theorem 2.2 as follows:

Theorem 4.3. *Let $n \geq 2$ and let $\{T_h\} \subset \mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ be such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to some $T \in \mathrm{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$. Then, we have:*

$$\mathbb{L}(T) \leq \liminf_{h\to\infty} \mathbb{L}(T_h), \qquad \textit{see } (4.2)\,.$$

Proof. According to (1.12), we let $u_{T_h} = u_h$, $u_T = u_\infty$ and $T = T_\infty$, so that for each $h \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ we have $T_h = G_{u_h} + L_h \times [\![\mathbb{S}^2]\!]$. We also denote

$$\mathbb{L}(u_h, B) := \int_B |\Delta u_h|\, dx\,, \quad \mathbb{D}(u_h, B) := \frac{1}{2}\int_B |Du_h|^2\, dx\,, \quad \forall\, B \in \mathcal{B}(B^n)$$

for every $h \in \overline{\mathbb{N}}$, where $\mathcal{B}(B^n)$ is the σ-algebra of Borel subsets of B^n.

Let L_h be the current in $\mathcal{R}_{n-2}(B^n)$ given by the decomposition formula (1.12) for T_h. According to the notation in (1.3), we can write $L_h = [\![\mathcal{L}_h, \xi_h, \theta_h]\!]$, and assume that the $(n-2)$-rectifiable set $\mathcal{L}_h \subset B^n$ agrees with set(L_h). We also denote by $\|L_h\|$ the Borel regular and finite measure $\|L_h\| := \theta_h\, \mathcal{H}^{n-2} \llcorner \mathcal{L}_h$, and consider the restriction $L_h \llcorner B$ given for every $B \in \mathcal{B}(B^n)$ by

$$\langle L_h \llcorner B; \omega \rangle := \int_B \theta\, \langle \omega; \xi \rangle\, d\|L_h\|\,, \qquad \omega \in \mathcal{D}^{n-2}(B^n)\,,$$

so that

$$\mathbf{M}(L_h \llcorner B) = \|L_h\|(B) = \int_{B \cap \mathcal{L}_h} \theta_h\, d\mathcal{H}^{n-2} < \infty \quad \forall\, B \in \mathcal{B}(B^n)\,, \quad h \in \overline{\mathbb{N}}\,.$$

We finally set for every $B \in \mathcal{B}(B^n)$ and $h \in \overline{\mathbb{N}}$

$$\mathbb{L}(T_h, B) := \mathbb{L}(u_h, B) + 8\pi\, \|L_h\|(B)\,, \quad \mathbb{D}(T_h, B) := \mathbb{D}(u_h, B) + 4\pi\, \|L_h\|(B)\,.$$

Since $\|L_\infty\|$ is a regular measure concentrated on a $(n-2)$-rectifiable set, we can find an open set of small measure (so that the energy $\mathbb{L}(u_\infty, \cdot)$ is small, by absolute continuity) that covers almost all the set where the measure $\|L_\infty\|$ is charged and in such a way that its boundary does not charge the measure $\mathbb{L}(T_\infty, \cdot)$. More precisely, for every $\varepsilon > 0$, we can find an open set $A_\varepsilon \subset B^n$ such that

(1) $\|L_\infty\|(B^n \setminus A_\varepsilon) \leq \varepsilon\,;$
(2) $\mathbb{L}(T_\infty, \partial A_\varepsilon \cap B^n) = 0\,;$
(3) $\mathbb{L}(u_\infty, A_\varepsilon) \leq \varepsilon\,.$

We clearly have

$$\liminf_{h\to\infty} \mathbb{L}(T_h) \geq \liminf_{h\to\infty} \mathbb{L}(T_h, B^n \setminus A_\varepsilon) + \liminf_{h\to\infty} \mathbb{L}(T_h, A_\varepsilon)$$

where by (4.2) and property (2)–(3) the first term in the right-hand side is bounded from below as follows:

$$\liminf_{h\to\infty}\mathbb{L}(T_h,B^n\setminus A_\varepsilon)\geq\liminf_{h\to\infty}\mathbb{L}(u_h,B^n\setminus A_\varepsilon)\geq\mathbb{L}(u_\infty,B^n\setminus A_\varepsilon)\geq\mathbb{L}(u_\infty)-\varepsilon\,.$$

As to the second term, using the lower bound (0.6) and the sequential weak lower semicontinuity of the Dirichlet integral $\mathbb{D}(T)$ in the class $\mathrm{cart}^{2,1}(A_\varepsilon\times\mathbb{S}^2)$, see (1.13), we have

$$\liminf_{h\to\infty}\mathbb{L}(T_h,A_\varepsilon)\geq 2\,\liminf_{h\to\infty}\mathbb{D}(T_h,A_\varepsilon)\geq 2\,\mathbb{D}(T_\infty,A_\varepsilon)$$

where by property (1), using that $\mathbf{M}(L_\infty)=\|L_\infty\|(B^n)=\|L_\infty\|(A_\varepsilon)+\|L_\infty\|(B^n\setminus A_\varepsilon)$, we get

$$\mathbb{D}(T_\infty,A_\varepsilon)=\mathbb{D}(u_\infty,A_\varepsilon)+4\pi\,\|L_\infty\|(A_\varepsilon)\geq 4\pi\,\left(\mathbf{M}(L_\infty)-\varepsilon\right)\,.$$

We thus obtain the estimate

$$\liminf_{h\to\infty}\mathbb{L}(T_h)\geq\mathbb{L}(u_\infty)+8\pi\,\mathbf{M}(L_\infty)-(1+8\pi)\,\varepsilon=\mathbb{L}(T_\infty)-(1+8\pi)\,\varepsilon$$

and hence the lower semicontinuity inequality is proved by letting $\varepsilon\searrow 0$. □

A strong density theorem. The following density result holds true in dimension $n=3$.

Theorem 4.4. *Let $n=3$ and $T\in\mathrm{cart}^{\mathbb{L}}(B^3\times\mathbb{S}^2)$, see Definition 4.1, so that (1.12) holds. Then, there exists a sequence $\{u_k\}\subset C^\infty(B^3,\mathbb{S}^2)$ such that $u_k\to u$ strongly in $L^1(B^3,\mathbb{R}^3)$, the currents G_{u_k} weakly converge to T in $\mathcal{D}_3(B^3\times\mathbb{S}^2)$, and $\mathbb{L}(u_k)\to\mathbb{L}(T)$ as $k\to\infty$.*

We adapt the argument used in the case of the Dirichlet energy analysed in [20]. Therefore, the proof of Theorem 4.4 relies on Proposition 1.16, on the density theorem 2.3, and on the cohomological criterion for strong density of smooth maps, Theorem 2.4. For these reasons, we refer to Sec. 4.2.5 and Sec. 4.2.6 in [23] for further details.

Proof of Theorem 4.4. Since $|\Delta u(x)|\leq 2|D^2u(x)|$ for a.e. $x\in B^n$ and every $u\in W^{2,1}(B^n,\mathbb{R}^3)$, by dominated convergence we infer that if $\{u_h\}\subset W^{2,1}(B^n,\mathbb{S}^2)$ is a sequence $W^{2,1}$-strongly converging to $u\in W^{2,1}(B^n,\mathbb{S}^2)$, then $\mathbb{L}(u_h)\to\mathbb{L}(u)$ as $h\to\infty$. Moreover, since the weak convergence of compactly supported i.m. rectifiable currents is metrizable, we can apply a diagonal procedure.

Step 1. By Theorem 2.3 and Proposition 1.16, we can find a sequence $\{u_k\}\subset R_0^\infty(B^3,\mathbb{S}^2)$ strongly converging to u in $W^{2,1}(B^3,\mathbb{R}^3)$, with $\mathbb{L}(u_k)\to\mathbb{L}(u)$ and $\mathbf{m}_{i,B^3}(\mathbb{P}(u_k)-\mathbb{P}(u))\to 0$. We thus reduce to the case in which $T=G_u+S_T$, where $u\in R_0^\infty(B^3,\mathbb{S}^2)$ and $S_T=L\times[\![\,\mathbb{S}^2\,]\!]$ for some $L\in\mathcal{R}_1(\mathbb{R}^3)$ with $(\partial L)\llcorner B^3=-\mathbb{P}(u)$, where $\mathbb{P}(u)\in\mathcal{R}_0(B^3)$ has finite mass. Therefore, formula (1.21) holds.

Step 2. Assume now in addition that $\mathbf{M}(\partial L) < \infty$, so that L is an integral current in $\mathbb{R}^3$. As in Steps 1–3 of the proof of Thm. 1 from [23, Sec. 4.2.5], we reduce to

$$S_T = \sum_{\mathrm{d}\in I} P^{\mathrm{d}} \times \mathrm{d}\, [\![\mathbb{S}^2]\!]$$

where I is a finite set of integer indices and the P^{d}'s are polyhedral lines in B^3 with pairwise disjoint supports. Notice in fact that at this step we apply Federer's strong polyhedral approximation theorem (cf. Thm. 4 in [22, Sec. 2.2.6]) and we deal with left compositions $u \circ \varphi$ with smooth diffeomorphisms φ of B^3 into itself, that preserve the membership of $u \circ \varphi$ to the class $R^\infty_0(B^3, \mathbb{S}^2)$.

Let now S_i be anyone of the segments of the P^{d}'s, and let $[\![S_i]\!] = [\![(n_i, p_i)]\!]$. By a suitable change of coordinates we can assume that $n_i = a_+$ and $p_i = a_-$, as in Sec. 3. We then apply Proposition 3.4. To this aim, notice that we can take m and δ sufficiently small so that the neighborhoods Ω^m_δ corresponding to different segments S_i are pairwise disjoint and contained in B^3. We then replace u in a small neighborhood of each S_i by a function $u_\varepsilon \in W^{2,1}(B^3, \mathbb{S}^2)$ satisfying

$$\mathbb{L}(u_\varepsilon) \le \mathbb{L}(u) + \sum_{d\in I} \mathbf{M}(P^{\mathrm{d}}) \cdot 8\pi\, |\mathrm{d}| + \varepsilon\, .$$

On account of the local upper bound (3.13), the function u_ε this way obtained satisfies

$$\int_{B^3} |D^2 u_\varepsilon|\, dx \le \int_{B^3} |D^2 u|\, dx + C_2 \cdot \mathbf{M}(P^{\mathrm{d}}) + \varepsilon$$

and hence $u_\varepsilon \in W^{2,1}(B^3, \mathbb{S}^2)$. Moreover, *it has degree zero around each end point of the segments* S_i which belongs to the open ball B^3, see (1.20), i.e. around each singular point of u_ε, whence $\mathbb{P}(u_\varepsilon) = 0$, see (1.14). Also, taking $\delta \searrow 0$ as $\varepsilon \searrow 0$, it turns out that $G_{u_\varepsilon} \rightharpoonup T$ weakly in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$ as $\varepsilon \to 0$ along a sequence.

Since $\mathbb{P}(u_\varepsilon) = 0$, by Theorem 2.4 we find a sequence $\{u^\varepsilon_k\} \subset C^\infty(B^3, \mathbb{S}^2)$ such that $u^\varepsilon_k \to u_\varepsilon$ strongly in $W^{2,1}(B^3, \mathbb{R}^3)$ and $\mathbb{L}(u^\varepsilon_k) \to \mathbb{L}(u)$, so that $G_{u^\varepsilon_k} \rightharpoonup G_{u_\varepsilon}$ weakly in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$. Therefore, a diagonal argument concludes the proof in case $\mathbf{M}(\partial L) < \infty$.

Step 3. After Step 1, in general we only have $\mathbf{M}((\partial L) \llcorner B^3) < \infty$ and therefore we cannot apply the strong polyhedral approximation theorem. We thus make use of a slicing argument similar to the one in the proof of Thm. 1 from [23, Sec. 4.2.6]. More precisely, denoting by B^3_r the ball of radius r centered at the origin, we may and do choose an increasing sequence $\{r_h\}$ such that $r_h \nearrow 1$ and $\mathbf{M}(\partial(L \llcorner B^3_{r_h})) < \infty$ for every h, i.e. the restriction $L \llcorner B^3_{r_h}$ is an integral current in $\mathbb{R}^3$.

Let $T_h := \psi_{h\#}(T \llcorner B^3_{r_h} \times \mathbb{S}^2)$, where $\psi_h(x, y) = (\varphi_h(x), y)$, with $\varphi_h(x) := x/r_h$, $x \in B^3_{r_h}$. It turns out that $\{T_h\}$ is a sequence of currents in $\mathrm{cart}^{\mathbb{L}}(B^3 \times \mathbb{S}^2)$ such that $T_h \rightharpoonup T$ and $\mathbb{L}(T_h) \to \mathbb{L}(T)$, where $T_h = G_{u_h} + L_h \times \mathbb{S}^2$ with

$u_h = u \circ \varphi_h$ and $L_h = \varphi_{h\#}(L \llcorner B^3_{r_h})$. Since L_h is an integral current in $\mathbb{R}^3$, we can apply Step 2 in order to approximate for each h the current T_h by a sequence $\{G_{u_{h_k}}\}_k$ of smooth graphs. A further diagonal argument concludes the proof. □

5. Main result, final remarks and open questions

In this section, we obtain the explicit formula of the relaxed energy (0.11) for Sobolev maps in $W^{2,1}(B^3, \mathbb{S}^2)$. We then prove the energy lower bound (0.18) and collect some final remarks and open questions.

With the notation from (1.14) and (1.18), Theorem 0.1 states:

$$\forall\, u \in W^{2,1}(B^3, \mathbb{S}^2)\,, \quad \widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty\,.$$

Notice that in terms of currents, and on account of definition (4.1) and Remark 4.2, the proof given below implies that the latter formula is equivalent to:

$$\widetilde{\mathbb{L}}(u) = \min\big\{\mathbb{L}(T) \mid T \in \mathcal{T}^{\mathbb{L}}_u \big\} \quad \forall\, u \in W^{2,1}(B^3, \mathbb{S}^2)\,. \tag{5.1}$$

Proof of Theorem 0.1. We first obtain in any dimension $n \geq 3$ the energy lower bound

$$\widetilde{\mathbb{L}}(u) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,. \tag{5.2}$$

Assume $\widetilde{\mathbb{L}}(u) < \infty$, and let $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ be any sequence converging to u in $L^1(B^n, \mathbb{R}^3)$ and such that $\sup_k \mathbb{L}(u_k) < \infty$. We thus have to show that

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,. \tag{5.3}$$

Possibly passing to a (not relabeled) subsequence, we can assume that the lim inf in the latter formula is a finite limit. The lower bound (0.10) yields $\sup_k \mathbb{D}(u_k) < \infty$. Therefore, compare Sec. 1, a (not relabeled) subsequence of G_{u_k} weakly converges in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to some current T in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$, see Definition 1.9. Moreover, the L^1-convergence $u_k \to u$ implies that the underlying function u_T agrees with u, whence $T \in \mathcal{T}^{\mathbb{L}}_u$, see (4.1). Therefore, by (1.16) and (1.18) we infer:

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi \cdot \mathbf{M}(L) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,.$$

Since moreover $\mathbb{L}(u_k) = \mathbb{L}(G_{u_k})$ for each k, the lower semicontinuity theorem 4.3, where $T_k = G_{u_k}$, gives

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(T)$$

and hence inequality (5.3) readily follows.

For any given $u \in W^{2,1}(B^n, \mathbb{S}^2)$, equality holds in (5.3) if we find a sequence $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ such that $u_k \to u$ strongly in $L^1(B^n, \mathbb{R}^3)$ and

$$\lim_{k\to\infty} \mathbb{L}(u_k) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u))\,.$$

To this aim, we first recall that the class $\mathcal{T}_u^{\mathbb{L}}$ is non-empty, see Remark 4.2, whence $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$. Setting then $T_u = G_u + L_u \times [\![\mathbb{S}^2]\!]$, where L_u is a minimal integral connection of $\mathbb{P}(u)$, see Remark 1.11, it turns out that T_u is an energy minimizing current in the class $\mathcal{T}_u^{\mathbb{L}}$, so that in particular

$$\mathbb{L}(T_u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty\,.$$

Therefore, in dimension $n = 3$ the requested strong approximation property is given by Theorem 4.4, when applied to $T = T_u$. In particular, equation (5.1) holds true. □

Extending the density theorem. In dimension $n = 3$, the relaxed Laplacean energy of $W^{2,1}$ maps satisfying a suitable Dirichlet-type boundary condition can be treated similarly to the case analysed here, and we refer to [24] for the corresponding results concerning the Dirichlet energy.

In high dimension $n \geq 4$, the expected equality in the energy lower bound (5.2) would be proved by extending the strong approximation property from Theorem 4.4. However, since we are not able to prove Proposition 1.16 in the case $n \geq 4$, see Remark 1.17, as for the Dirichlet energy we cannot argue as in the proof of Theorem 4.4. Moreover, the strategy from [24] doesn't work, since it is based on a partial regularity result for minimizers of the Dirichlet energy. On the other hand, the argument taken from [25] does not apply, too, since it strongly relies on slicing arguments that fail to hold for the Laplacean energy.

The energy lower bound. We now give the following

Proof of (0.18). If u is a map in $\mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, see (0.17), we have seen that the density $\widetilde{\Delta u}$ in (2.2) agrees with the approximate Laplacean Δu. As a consequence, the lower bound

$$|\mathrm{Div} Du|(B) \geq \int_B |\Delta u|\, dx \geq \int_B |Du|^2\, dx \tag{5.4}$$

holds true for each Borel set $B \in \mathcal{B}(B^n)$. Denoting by $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents $T = G_{u_T} + L \times [\![\mathbb{S}^2]\!]$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with $u_T \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, and setting correspondingly

$$\mathbb{L}(T) := |\mathrm{Div} Du_T|(B^n) + 8\pi \cdot \mathbf{M}(L)\,,$$

it turns out that the lower semicontinuity theorem 4.3 continues to hold in the wider class $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$. In fact, by using Proposition 1.2, when $n = 2$ it is obtained again as a consequence of Theorem 2.2, whereas in case $n \geq 3$ we can apply the same argument as in the proof Theorem 4.3, that essentially relies on the local inequality $|\mathrm{Div} Du|(B) \geq 2\,\mathbb{D}(u, B)$, which follows from (5.4). As a consequence, the lower bound (0.18) is obtained exactly as in the proof of (5.2). Further details are omitted. □

Open questions. As already mentioned in Sec. 2, even in dimension $n = 3$, we are not able to obtain an explicit formula of the relaxed energy in case of maps u in $L^1(B^3,\mathbb{S}^2)\setminus W^{2,1}(B^3,\mathbb{S}^2)$ satisfying $\widetilde{\mathbb{L}}(u)<\infty$. On account of (0.12), since we are dealing with maps in $W^{1,2}(B^n,\mathbb{S}^2)$, formula

$$\widetilde{\mathbb{L}}(u)=|\mathrm{Div}Du|(B^n)+8\pi\cdot\mathbf{m}_{i,B^n}(\mathbb{P}(u))\quad\forall\, u\in\mathbb{L}(B^n,\mathbb{S}^2)$$

is expected to hold true in any dimension $n\geq 3$, i.e. no extra terms in correspondence to the singular part of the measure $\mathrm{Div}Du$ should appear.

For example, in order to prove the validity of equality in (0.18) for maps in $\mathbb{L}_{BV}(B^3,\mathbb{S}^2)$, one should primarily extend Theorem 2.3, by showing for any $u\in\mathbb{L}_{BV}(B^3,\mathbb{S}^2)$ the existence of a sequence $\{u_h\}\subset R_0^\infty(B^3,\mathbb{S}^2)$ converging to u in $W^{1,2}(B^3,\mathbb{R}^3)$ and such that $\mathbb{L}(u_h)\to|\mathrm{Div}Du|(B^n)$ as $h\to\infty$.

Bienergy. Theorem 4.3 may be compared to the lower semicontinuity property (1.13) of the Dirichlet energy functional $\mathbb{D}(T)$ in $\mathrm{cart}^{2,1}(B^n\times\mathbb{S}^2)$, that holds true since $\mathbb{D}(T)$ agrees with the parametric polyconvex lower semicontinuous extension of the Dirichlet integrand.

Concerning *second order* functionals as e.g. the Laplacean energy $\mathbb{L}(u)$, a part from the easier case of 1-dimensional currents, compare [1, 2], to our knowledge it is not clear how to apply the approach from [23] in order to find the explicit formula of the parametric polyconvex lower semicontinuous extension. Therefore, we have followed a different strategy.

The same difficulty appears in case of the bienergy functional $\mathbb{H}(u)$ of maps $u:B^n\to\mathbb{S}^4$, see (0.4). In fact, finding the expression of the parametric polyconvex lower semicontinuous extension of the bienergy to Cartesian currents in $B^n\times\mathbb{S}^4$ would give us the explicit value of the minimal bienergy E_4 of degree one maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, a non-trivial open problem. We recall that in [5] it is proved that the bienergy minimum is attained, and that $E_4>16\cdot\mathcal{H}^4(\mathbb{S}^4)$, the expected weight being $E_4=24\cdot\mathcal{H}^4(\mathbb{S}^4)$.

Denote now on maps $u\in L^1(B^n,\mathbb{S}^4)$

$$\widetilde{\mathbb{H}}(u):=\inf\Big\{\liminf_{h\to\infty}\mathbb{H}(u_h)\mid\{u_h\}\subset C^\infty(B^n,\mathbb{S}^4),\ u_k\to u\text{ in }L^1(B^n,\mathbb{R}^5)\Big\}\,.$$

Under prescribed first order boundary conditions, by the Bochner inequality one infers that $\widetilde{\mathbb{H}}(u)<\infty$ if and only if $u\in W^{2,2}(B^n,\mathbb{S}^4)$. Since moreover by (0.6) we have $W^{2,2}(B^n,\mathbb{S}^4)\subset W^{1,4}(B^n,\mathbb{S}^4)$, due to the continuous embedding of $W^{1,4}(B^4)$ in VMO, we get $\widetilde{\mathbb{H}}(u)=\mathbb{H}(u)$ for every $u\in W^{2,2}(B^4,\mathbb{S}^4)$. Finally, in dimension $n=5$, with a similar strategy to the one adopted in this paper it could be shown that

$$\widetilde{\mathbb{H}}(u)=\mathbb{H}(u)+E_4\cdot\mathbf{m}_i(\mathbb{P}(u))<\infty\quad\forall\, u\in W^{2,2}(B^5,\mathbb{S}^4)$$

where $\mathbf{m}_i(\mathbb{P}(u))$ denotes the integral mass of the 0-current $\mathbb{P}(u)$ of the singularities, that is defined on maps $u\in W^{1,4}(B^5,\mathbb{S}^4)$ in a similar way to the case $n=3$ in (1.14), but in terms of a volume 4-form in $\mathbb{S}^4$.

References

[1] Acerbi E., Mucci D.: Curvature-dependent energies: a geometric and analytical approach. *Proc. Roy. Soc. Edinburgh, Sect. A*, **147A** (2017), 449–503.

[2] Acerbi E., Mucci D.: Curvature-dependent energies: the elastic case. *Nonl. Anal.* **153** (2017), 7–34.
[3] Almgren F.J., Browder W., Lieb E.H.: Co-area, liquid crystals, and minimal surfaces. In: *Partial Differential equations*, Springer Lecture Notes in Math. **1306** (1988), 1–22.
[4] Ambrosio L., Fusco N., Pallara D.: *Functions of bounded variation and free discontinuity problems.* Oxford Math. Monographs, Oxford, 2000.
[5] Angelsberg G.: Large solutions of biharmonic maps in four dimensions. *Calc. Var.* **30** (2007), 417–447.
[6] Bethuel F.: A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **7** (1990), 269–286.
[7] Bethuel F.: The approximation problem for Sobolev maps between manifolds. *Acta Math.* **167** (1992), 153–206.
[8] Bethuel F., Brezis H., Coron J. M.: Relaxed energies for harmonic maps. In: *Variational methods*, H. Berestycki, J.M. Coron, J. Ekeland (eds.), Birkauser, Basel (1990), 37–52.
[9] Bethuel F., Coron J.-M., Demengel F., Hélein F.: A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds. In: *Nematics* (Orsay, 1990), 15–23, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. **332**, 15–23. Kluwer Acad. Publ., Dordrecht, 1991.
[10] Bethuel F., Zheng X. M.: Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.* **80** (1988), 60–75.
[11] Bousquet P., Ponce A.C., Van Schaftingen J.: Strong density for higher order Sobolev spaces into compact manifolds. *J. Eur. Math. Soc.* **17** (2015), 763–817.
[12] Bousquet P., Ponce A.C., Van Schaftingen J.: Generic topological screening and approximation of Sobolev maps. *Preprint* (2025). See: https://arxiv.org/abs/2501.18149
[13] Brezis H., Coron J.M., Lieb E.H.: Harmonic maps with defects. *Commun. Math. Phys.* **107** (1986), 649–705.
[14] Comi G. E.: *Refined Gauss–Green formulas and evolution problems for Radon measures.* Scuola Normale Superiore, Pisa (2020). Ph.D. Thesis. Available at cvgmt.sns.it/paper/4579/ .
[15] Cooper M. K.: Critical $O(d)$-equivariant biharmonic maps. *Calc. Var.* **54** (2015), 2895–2919.
[16] Federer H.: Real flat chains, cochains and variational problems. *Indiana Univ. Math. J.* **24** (1974), 351–407.
[17] Federer H., Fleming W.H.: Normal and integral currents. *Ann. of Math.* **72** (1960), 458–520.
[18] Gastel A., Nerf J.: Density of smooth maps in $W^{k,p}(M, N)$ for a close to critical domain dimension. *Ann. Glob. Anal. Geom.* **39** (2011), 107–129.
[19] Giaquinta M., Modica G., Souček J.: Cartesian currents and variational problems for mappings into spheres. *Ann. Sc. Norm. Sup. Pisa* **16** (1989), 393–485.
[20] Giaquinta M., Modica G., Souček J.: The Dirichlet energy of mappings with values into the sphere. *Manuscr. Math.* **65** (1989), 489–507.
[21] Giaquinta M., Modica G., Souček J.: Liquid crystals: relaxed energies, dipoles, singular lines and singular points. *Ann. Sc. Norm. Sup. Pisa* **17** (1990), 415–437.
[22] Giaquinta M., Modica G., Souček J.: *Cartesian currents in the calculus of variations. I. Cartesian currents.* Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 37, Springer-Verlag, Berlin, 1998.
[23] Giaquinta M., Modica G., Souček J.: *Cartesian currents in the calculus of variations. II. Variational problems.* Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 38, Springer-Verlag, Berlin, 1998.
[24] Giaquinta M, Mucci D.: The relaxed Dirichlet energy of mappings into a manifold. *Calc. Var.* **24** (2005), 155–166.

[25] Giaquinta M, Mucci D.: The BV-energy of maps into a manifold: relaxation and density results. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **5** (2006), 483–548.
[26] Giaquinta M., Mucci D.: *Maps into manifolds and currents: area and* $W^{1,2}$*-,* $W^{1/2}$*-,* BV*-energies.* Edizioni della Normale, C.R.M. Series, Sc. Norm. Sup. Pisa, 2006.
[27] Hong M.-C., Wang C.: Regularity and relaxed problems of minimizing biharmonic maps into spheres. *Calc. Var.* **23** (2004), 425–450.
[28] Morgan F.: Area minimizing currents bounded by multiples of curves. *Rend. Circ. Mat. Palermo* **33** (1984), 37–46.
[29] Schoen R., Uhlenbeck K.: Boundary regularity and the Dirichlet problem for harmonic maps. *J. Diff. Geom.* **18** (1983), 253–268.
[30] Šilhavý M.: Divergence measure fields and Cauchy's stress theorem. *Rend. Sem. Mat. Univ. Padova* **113** (2005), 15–45.
[31] White B.: The least area bounded by multiples of a curve. *Proc. Am. Math. Soc.* **90** (1984), 230–232.

(D. Mucci) Dipartimento di Scienze Matematiche, Fisiche ed Informatiche, Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy
Email address: *domenico.mucci@unipr.it*