# A RELAXATION RESULT FOR A SECOND ORDER ENERGY OF MAPPINGS INTO THE SPHERE

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Abstract. A relaxation problem for maps from $n$-dimensional domains into the unit 2-sphere is analysed, the energy being given in the smooth case by the integral of the modulus of the Laplacean vector. For second order Sobolev maps, a complete explicit formula of the relaxed energy is obtained. Our proof is based on the following results: minimal energy computation of maps with fixed degree, Dipole-like problems, density of maps with small singular sets, lower semicontinuity of the extended energy, and strong approximation properties on Cartesian currents.

## 1. Introduction

First order variational problems for maps taking values into isometrically embedded Riemannian manifolds $\mathcal{N}$ are widely studied, a relevant model being given by the Dirichlet integral

$$\mathbb{D}(u) := \frac{1}{2}\int_{B^n} |Du|^2\, dx \tag{1.1}$$

of maps from the $n$-dimensional unit ball $B^n$ into the $\mathfrak{p}$-dimensional unit sphere $\mathcal{N} = \mathbb{S}^{\mathfrak{p}}$.

When e.g. $n = 3$ and $\mathfrak{p} = 2$, unit vector fields minimizing the Dirichlet energy (under prescribed boundary conditions) represent a simplified model for the Ericksen–Leslie theory of liquid crystals, see [21] or [23, Sec. 5.1].

Harmonic maps $u$ with values into the sphere $\mathbb{S}^{\mathfrak{p}}$ satisfy the Euler–Lagrange system $\tau(u) = 0$, where

$$\tau(u) := \Delta u + |Du|^2 u \tag{1.2}$$

is the *intrinsic* Laplacean, or *tension field*, compare [23, Sec. 3.1.1]. More precisely, viewing the $\mathfrak{p}$-sphere as embedded into the Euclidean space $\mathbb{R}^{\mathfrak{p}+1}$, and working with maps $u : B^n \to \mathbb{R}^{\mathfrak{p}+1}$ such that $|u(x)| \equiv 1$, then $\Delta u$ is the *Laplacean vector* in $\mathbb{R}^{\mathfrak{p}+1}$, and its normal component to $\mathbb{S}^{\mathfrak{p}}$ at $u(x)$ is $(\Delta u)^\perp = -|Du|^2\, u$, whence $\tau(u)$ is the tangential component of $\Delta u$, and

$$|\Delta u|^2 = |Du|^4 + |\tau(u)|^2\,. \tag{1.3}$$

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In dimension $n=2$, the Dirichlet integral is conformally invariant. Therefore, its second order analogous is probably given by the *Bienergy* functional

$$\mathbb{H}(u) := \int_{B^n} |\Delta u|^2 \, dx \tag{1.4}$$

of maps $u$ from $B^n$ into $\mathbb{S}^{\mathfrak{p}}$. In dimension $n=4$, in fact, the Bienergy functional is conformally invariant. In addition, equation (1.3) implies the lower bound

$$\mathbb{H}(u) \geq \int_{B^n} |Du|^4 \, dx$$

where equality holds when $\tau(u)=0$, i.e. for harmonic maps.

As a consequence, Sobolev maps $u$ from $B^n$ into $\mathbb{S}^{\mathfrak{p}}$ with finite Bienergy belong to the Sobolev class $W^{1,4}(B^n, \mathbb{S}^{\mathfrak{p}})$. Moreover, when $n=\mathfrak{p}$, by the parallelogram inequality the Jacobian of a smooth map $u$ from $\mathbb{R}^{\mathfrak{p}}$ into $\mathbb{S}^{\mathfrak{p}}$ satisfies the pointwise upper bound

$$J_{\mathfrak{p}} u \leq \frac{1}{\mathfrak{p}^{\mathfrak{p}/2}} |Du|^{\mathfrak{p}} \tag{1.5}$$

where equality holds if and only if $u$ is conformal.

Therefore, when in particular $\mathfrak{p}=4$, it turns out that in any dimension $n$ the "graph" in $B^n \times \mathbb{S}^4$ of a Sobolev map $u \in W^{2,1}(B^n, \mathbb{S}^4)$ with finite Bienergy has finite "area". This last property indicates the way to use tools from Geometric Measure Theory in order to analyze the corresponding *relaxation problem*.

However, in this framework a nontrivial open question comes into play: *finding a Bienergy minimizer among smooth maps from* $\mathbb{R}^4$ *into* $\mathbb{S}^4$ *of* degree *one*.

One expects that it is given by the inverse $\sigma_4^{-1}$ of the stereographic projection map from $\mathbb{S}^4$ to $\mathbb{R}^4$, compare (2.1). In fact, Angelsberg [5] showed that the energy minimum among degree one maps is attained and it is greater than $16\,\mathcal{H}^4(\mathbb{S}^4)$, where $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure. Moreover, recalling that $\int_{\mathbb{R}^4} |\Delta\sigma_4^{-1}|^2 \, dx = 24\,\mathcal{H}^4(\mathbb{S}^4)$, Cooper [15] proved that $\sigma_4^{-1}$ minimizes the Bienergy among degree one $O(4)$-equivariant maps from $\mathbb{R}^4$ into $\mathbb{S}^4$.

Comparing with the easier result in Proposition 1.1 below, the main difficulty here depends on the fact that the map $\sigma_4^{-1}$ fails to be harmonic. Further remarks on this subject are collected in Sec. 7.

### 1.1. **The Laplacean energy.**

In this paper, we consider in any dimension $n \geq 2$ the functional

$$\mathbb{L}(u) := \int_{B^n} |\Delta u| \, dx$$

on maps $u : B^n \to \mathbb{S}^2$ taking values into the unit 2-sphere of $\mathbb{R}^3$. It will be called *Laplacean energy*. We preliminarily notice that if $u$ is sufficiently smooth, by (1.3) we get

$$|\Delta u| \geq |Du|^2 \tag{1.6}$$

where equality holds for harmonic maps. In addition, by (1.5), where $\mathfrak{p} = 2$, it turns out that the "graph" of a map $u$ satisfying $\mathbb{L}(u) < \infty$ has finite "area" in $B^n \times \mathbb{S}^2$.

Differently to the nontrivial case of the Bienergy of maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, concerning the Laplacean energy, in the critical dimension $n = 2$ we have the following:

**Proposition 1.1.** *The minimal Laplacean energy among degree one smooth maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ is attained by the inverse $\sigma_2^{-1}$ of the stereographic map. Moreover, we have:*

$$\int_{\mathbb{R}^2} |\Delta\sigma_2^{-1}|\, dx = \int_{\mathbb{R}^2} |D\sigma_2^{-1}|^2\, dx = 2\int_{\mathbb{R}^2} J_2\sigma_2^{-1}\, dx = 2\,\mathcal{H}^2(\mathbb{S}^2) = 8\pi\,. \tag{1.7}$$

Proposition 1.1, whose proof is reported in Sec. 4, allows us to analyze (at least partially) the explicit formula of the *relaxed Laplacean energy.*

1.2. **The relaxed energy.** Following the classical Lebesgue-Serrin approach, we introduce in any dimension $n \geq 2$ the relaxed energy

$$\widetilde{\mathbb{L}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{L}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^2), \quad u_h \to u \text{ strongly in } L^1(B^n, \mathbb{R}^3) \Big\} \tag{1.8}$$

of maps $u$ in $L^1(B^n, \mathbb{S}^2)$. Our first objective is to analyze the explicit formula of $\widetilde{\mathbb{L}}(u)$ on the class of maps with finite relaxed energy. We thus denote:

$$\mathbb{L}(B^n, \mathbb{S}^2) := \{\, u \in L^1(B^n, \mathbb{S}^2) \mid \widetilde{\mathbb{L}}(u) < \infty \,\} \tag{1.9}$$

and refer to Sec. 3 for details on the following preliminary discussion.

If $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, inequality (1.6) implies that $u \in W^{1,2}(B^n, \mathbb{S}^2)$, whence the *distributional divergence* of the gradient $Du$ is well defined by

$$\langle \mathrm{Div} Du; \varphi\rangle := -\int_{B^n} \mathrm{tr}\,[Du\,(D\varphi)^\top]\, dx\,, \quad \varphi \in C_c^\infty(B^n, \mathbb{R}^3) \tag{1.10}$$

where $A \mapsto A^\top$ is the transpose operator in $\mathbb{R}^{3\times n}$ and $B \mapsto \mathrm{tr}\, B$ the trace operator in $\mathbb{R}^{3\times 3}$. By lower semicontinuity, we have:

$$\widetilde{\mathbb{L}}(u) \geq |\mathrm{Div} Du|(B^n) \quad \forall\, u \in \mathbb{L}(B^n, \mathbb{S}^2) \tag{1.11}$$

and hence $\mathrm{Div}Du$ is a *finite* $\mathbb{R}^3$*-valued regular measure*. Since moreover

$$\mathrm{Div} Du = \Delta u\, \mathcal{L}^n \llcorner B^n \quad \forall\, u \in W^{2,1}(B^n, \mathbb{S}^2)\,,$$

where $\llcorner$ denotes restriction, the measure $\mathrm{Div}Du$ may be called a *weak Laplacean.*

In the critical dimension $n = 2$, due to the continuous embedding of $W^{1,2}(B^2)$ in the class VMO of functions with *vanishing mean oscillation* it turns out that there is no gap:

$$\widetilde{\mathbb{L}}(u) = |\mathrm{Div} Du|(B^2) \quad \forall\, u \in \mathbb{L}(B^2, \mathbb{S}^2)\,. \tag{1.12}$$

In high dimension $n \geq 3$, the energy gap is positive, in general, i.e. strict inequality holds in (1.11). However, for a generic map $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, it is an open problem to find the explicit formula of the relaxed energy (1.8).

This is essentially due to a lack of sufficient information on the structure of the measure $\mathrm{Div}Du$. For that reason, in this paper we shall focus on the more regular subclass of second order Sobolev maps, since

$$(1.13) \qquad |\mathrm{Div}Du|(B^n) = \int_{B^n} |\Delta u| \, dx =: \mathbb{L}(u) \quad \forall \, u \in W^{2,1}(B^n, \mathbb{S}^2) \, .$$

Referring to Sec. 2 for the precise notation, we only mention here that in any dimension $n \geq 3$, the *relevant singularities* of maps $u \in W^{2,1}(B^n, \mathbb{S}^2)$ are described by an *integral flat* $(n-3)$-chain $\mathbb{P}(u)$ in $B^n$. This means that the current $\mathbb{P}(u)$ is the boundary in $B^n$ of an integer multiplicity (say i.m.) rectifiable $(n-2)$-current $L$, and the *integral mass* $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is the mass of a minimizer among $L \in \mathcal{R}_{n-2}(B^n)$ satisfying $(\partial L) \llcorner B^n = -\mathbb{P}(u)$.

If e.g. $n = 3$ and $u$ is the harmonic map $u(x) = x/|x|$, then $|\Delta u| = |Du|^2 = 2/|x|^2$, and on account of (2.8) the current of the singularities is such that $-\mathbb{P}(u) = \delta_O$, the unit Dirac mass at the origin $O$, whence $\mathbf{m}_{i,B^3}(\mathbb{P}(u))$ is equal to the length of a segment connecting $O$ to a point at the boundary of $B^3$, a so called *string* in the sense of Brezis–Coron–Lieb [13].

### 1.3. **The Main Result.**

The Main Result of this paper is enclosed in the following theorem, where in the case of dimension $n = 3$ we are able to give an explicit formula for the relaxed energy (1.8) of maps in the Sobolev class $W^{2,1}(B^3, \mathbb{S}^2)$.

**Theorem 1.2.** *Let* $n = 3$ *and* $u \in W^{2,1}(B^3, \mathbb{S}^2)$. *Then*

$$\widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty \, .$$

Therefore, the energy gap only depends (up to the factor $8\pi$) on the *mass* $\mathbf{m}_{i,B^3}(\mathbb{P}(u))$ of a *minimal integral connection* of the distribution $\mathbb{P}(u)$ of the singularities of $u$. In the proof of Theorem 1.2, we rely on Proposition 1.1, and we follow a similar strategy to the one exploited in case of the Dirichlet energy (1.1). In particular, we make use of tools from the theory of *Cartesian currents* by Giaquinta–Modica–Souček [22, 23].

To our knowledge, Theorem 1.2 is the first nontrivial result in high dimension concerning the explicit formula of a second order relaxed energy of Sobolev maps satisfying a constraint on the target space.

A brief discussion of the difficulties encountered in extending Theorem 1.2 is reported at the end of Sec. 7. In particular, concerning the class

$$(1.14) \qquad \mathbb{L}_{BV}(B^n, \mathbb{S}^2) := \left\{ u \in \mathbb{L}(B^n, \mathbb{S}^2) \mid Du \in \mathrm{BV}(B^n, \mathbb{R}^{3\times n}) \right\},$$

we are able to prove in any dimension $n \geq 3$ the energy lower bound

$$(1.15) \qquad \widetilde{\mathbb{L}}(u) \geq |\mathrm{Div}Du|(B^n) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \quad \forall \, u \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2) \, ,$$

the main open problem being to prove the validity of the equality in the latter formula.

1.4. **Content of the paper.** In Sec. 2, we collect some notation and background material, focusing in particular on the analogous relaxation problem for the Dirichlet integral (1.1) of mappings into the 2-sphere. We then report a cohomological criterion for strong density of smooth maps in $W^{2,1}(B^n, \mathbb{S}^2)$ recently obtained by Bousquet–Ponce–Van Schaftingen [12].

In Sec. 3, we preliminarily discuss some general properties of maps with finite relaxed energy, explaining the difficulties that one encounters in the general case when $u \notin W^{2,1}(B^n, \mathbb{S}^2)$ and $n \geq 3$. We also prove a lower semicontinuity result in dimension $n = 2$, Theorem 3.2.

In Sec. 4, we compute the *minimal Laplacean energy* among maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ with fixed degree, Theorem 4.1. To this purpose, we discuss a suitable modification of the inverse to the stereographic map, Proposition 4.2. Finally, we describe the related bubbling phenomenon.

In Sec. 5, we introduce the class $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ of Cartesian currents whose underlying functions belong to $W^{2,1}(B^n, \mathbb{S}^2)$. More precisely, see Definition 5.1, an element $T$ in $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ is given by

$$T = G_u + L \times [\![ \mathbb{S}^2 ]\!]$$

where $G_u$ is the graph current of a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ and $L$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = -\mathbb{P}(u)$, if $n \geq 3$. We then extend the Laplacean energy to a functional $T \mapsto \mathbb{L}(T)$ on Cartesian currents, by letting

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi \cdot \mathbf{M}(L) \quad \text{if} \quad T = G_u + L \times [\![ \mathbb{S}^2 ]\!]$$

and prove a weak sequential *lower semicontinuity* property, Theorem 5.3.

In Sec. 6, we prove a *strong density result* in dimension $n = 3$. Namely, in Theorem 6.1 we show that every current in $\text{cart}^{\mathbb{L}}(B^3 \times \mathbb{S}^2)$ can be approximated weakly and with energy convergence by a sequence of currents carried by graphs of smooth maps in $C^\infty(B^3, \mathbb{S}^2)$. For this purpose we extend to the Laplacean energy the classical *Dipole problem* of Brezis-Coron-Lieb [13] for the Dirichlet energy in 3D, Theorem 6.2.

In Sec. 7, we deal with the explicit formula of the relaxed energy (1.8). The proof of Theorem 1.2 is based on the lower semicontinuity theorem 5.3 and on the density theorem 6.1. We then prove the energy lower bound (1.15). In the end, final remarks and open questions are reported.

## 2. Background material and preliminary results

In this section, we collect some well-known facts about stereographic maps, divergence-measure fields, and topics from Geometric Measure Theory, degree, Cartesian currents, singularities (for which we refer to the treatise [22, 23] or to [26]). We then describe the strong density and relaxation results for the Dirichlet energy of maps into the 2-sphere. Finally, we report a recent cohomological criterion for strong density of smooth maps in $W^{2,1}(B^n,\mathbb{S}^2)$ obtained in [12].

Let $B^n$ be the open unit ball of dimension $n\geq 2$ centered at the origin, and $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. For $X=L^1$, $W^{k,p}$, or $C^\infty$, we denote

$$X(B^n,\mathbb{S}^2):=\{u\in X(B^n,\mathbb{R}^3)\,:\,|u(x)|=1 \text{ for } \mathcal{L}^n\text{-a.e. } x\in B^n\}\,.$$

2.1. **Stereographic projection.** For $\mathfrak{p}\geq 2$ integer, setting

$$\mathbb{S}^{\mathfrak{p}}:=\{(y,z)\mid y\in\mathbb{R}^{\mathfrak{p}}\,,\ z\in\mathbb{R}\,,\ |(y,z)|=1\}\subset\mathbb{R}^{\mathfrak{p}+1}$$

the stereographic projection from the "South Pole" $P_S:=(0_{\mathbb{R}^{\mathfrak{p}}},-1)$ is given by $\sigma_{\mathfrak{p}}(y,z):=\dfrac{y}{1+z}$. Its inverse $\sigma_{\mathfrak{p}}^{-1}:\mathbb{R}^{\mathfrak{p}}\to\mathbb{S}^{\mathfrak{p}}$ satisfies

$$\sigma_{\mathfrak{p}}^{-1}(x)=\Big(\frac{2x}{1+\rho^2},\frac{1-\rho^2}{1+\rho^2}\Big)\,,\qquad x\in\mathbb{R}^{\mathfrak{p}}\,,\quad \rho:=|x|\,. \tag{2.1}$$

The map $(-1)^{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is an orientation preserving conformal diffeomorphism from $\mathbb{R}^{\mathfrak{p}}$ onto $\mathbb{S}^{\mathfrak{p}}\setminus\{P_S\}$. In fact, denoting by $\bullet$ the scalar product in $\mathbb{R}^{\mathfrak{p}+1}$ and by $\delta^i_j$ the Kronecker symbol, the conformality relations

$$\partial_i\sigma_{\mathfrak{p}}^{-1}\bullet\partial_j\sigma_{\mathfrak{p}}^{-1}=\delta^i_j\,U^2\quad\forall\, i,j=1,\dots,\mathfrak{p}$$

hold, with scaling factor $U(x):=\dfrac{2}{1+|x|^2}$, whence in (1.5) one has

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}|D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}}=J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}=U^{\mathfrak{p}}$$

where $J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}$ is the Jacobian of $\sigma_{\mathfrak{p}}^{-1}$. As a consequence, concerning the *conformal Dirichlet integral*, for any $\mathfrak{p}\geq 2$ integer one obtains:

$$\frac{1}{\mathfrak{p}^{\mathfrak{p}/2}}\int_{\mathbb{R}^{\mathfrak{p}}}|D\sigma_{\mathfrak{p}}^{-1}|^{\mathfrak{p}}\,dx=\int_{\mathbb{R}^{\mathfrak{p}}}J_{\mathfrak{p}}\sigma_{\mathfrak{p}}^{-1}\,dx=\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}) \tag{2.2}$$

where $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure.

Most importantly, it turns out that the map $\sigma_{\mathfrak{p}}^{-1}$ is harmonic if and only if $\mathfrak{p}=2$. Therefore, $\sigma_2^{-1}$ satisfies the Euler-Lagrange system $\tau(u)=0$, where $\tau(u)$ is the tension field (1.2). In conclusion, one readily obtains the energy computation (1.7).

2.2. **Divergence-measure fields.** Let $n\geq 2$. The *distributional divergence* of a vector field $F\in L^2(B^n,\mathbb{R}^n)$ is well defined by:

$$\langle \mathrm{Div}F;\phi\rangle:=-\int_{B^n}F\cdot D\phi\,dx\,,\quad \phi\in C^\infty_c(B^n)\,.$$

**Definition 2.1.** We call $F\in L^2(B^n,\mathbb{R}^n)$ a *divergence-measure field*, say $F\in\mathcal{DM}^{1,2}(B^n)$, if $\mathrm{Div}F$ is a real finite Radon measure on $B^n$.

If $F \in \mathcal{DM}^{1,2}(B^n)$, a decomposition into mutually singular measures

$$\mathrm{Div}F = (\mathrm{Div}F)^a + (\mathrm{Div}F)^s\,, \quad (\mathrm{Div}F)^a = \widetilde{\mathrm{div}F}\,\mathcal{L}^n \llcorner B^n$$

holds, where $\widetilde{\mathrm{div}F} \in L^1(B^n)$ denotes the Radon–Nikodym derivative of $\mathrm{Div}F$ w.r.t. $\mathcal{L}^n$. Referring to [4] for further details on *functions of bounded variations*, we remark that if in addition $F \in \mathrm{BV}(B^n, \mathbb{R}^n)$, the density $\widetilde{\mathrm{div}F}$ agrees with the trace of the approximate gradient matrix $\nabla F$, and that $(\mathrm{Div}F)^s = 0$ if in particular $F \in W^{1,1}(B^n, \mathbb{R}^n)$.

Šilhavý [30, Thm. 3.2] proved the following absolute continuity property:

**Proposition 2.2.** *If $F \in \mathcal{DM}^{1,2}(B^n)$, then $|\mathrm{Div}F|(B) = 0$ for each Borel set $B \subset B^n$ with $\sigma$-finite $\mathcal{H}^{n-2}$-measure. In particular, the measure* $\mathrm{Div}F$ *does not charge any atom.*

By the chain rule formula in BV, cf. [4, Thm. 3.96] and [22, p. 487], it turns out that if $v^1 \in W^{1,2}(B^n) \cap L^\infty(B^n)$ and $v^2 \in \mathrm{BV}(B^n) \cap L^2(B^n)$, then (denoting by $D$ and $\nabla$ the distributional derivative and the approximate gradient)

$$D(v^1 v^2) = v^1 D v^2 + v^2 \nabla v^1\, \mathcal{L}^n \llcorner B^n\,.$$

In this setting, the following version of the Leibnitz-rule is due to Comi [14].

**Proposition 2.3.** *Let $F \in \mathcal{DM}^{1,2}(B^n)$ and $g \in W^{1,2}(B^n) \cap L^\infty(B^n)$. Then, $gF \in \mathcal{DM}^{1,2}(B^n)$ and*

$$\mathrm{Div}(gF) = \widetilde{g}\,\mathrm{Div}F + F \cdot \nabla g\, \mathcal{L}^n \llcorner B^n$$

*where $\widetilde{g}$ is the precise representative of $g$.*

2.3. **Integer rectifiable currents.** For $U \subset \mathbb{R}^m$ an open set, and $k = 0, \dots, m$, we denote by $\mathcal{D}_k(U)$ the dual to the space $\mathcal{D}^k(U)$ of compactly supported smooth $k$-forms, whence $\mathcal{D}_0(U)$ is the class of distributions in $U$. For any $T \in \mathcal{D}_k(U)$, we define its *mass* $\mathbf{M}(T)$ as

$$\mathbf{M}(T) := \sup\{\langle T; \omega\rangle \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \le 1\}$$

and (for $k \ge 1$) its *boundary* as the $(k-1)$-current $\partial T$ defined by the relation

$$\langle \partial T; \eta\rangle := \langle T; d\eta\rangle \qquad \forall\, \eta \in \mathcal{D}^{k-1}(U)$$

where $d\eta$ is the differential of $\eta$. The *weak convergence* $T_h \rightharpoonup T$ in the sense of currents in $\mathcal{D}_k(U)$ is defined through the formula

$$\lim_{h\to\infty} \langle T_h; \omega\rangle = \langle T; \omega\rangle\,, \qquad \forall\, \omega \in \mathcal{D}^k(U)$$

and the mass is sequentially weakly lower semicontinuous, i.e.

$$\mathbf{M}(T) \le \liminf_{h\to\infty} \mathbf{M}(T_h) \quad \text{if} \quad T_h \rightharpoonup T\,.$$

For $k \ge 1$, a $k$-current $T$ with finite mass is called *rectifiable* if

$$\langle T; \omega\rangle = \int_{\mathcal{M}} \theta\, \langle \omega; \xi\rangle\, d\mathcal{H}^k \qquad \forall\, \omega \in \mathcal{D}^k(U) \tag{2.3}$$

with $\mathcal{M}$ a $k$-rectifiable set in $U$, $\xi : \mathcal{M} \to \Lambda_k \mathbb{R}^n$ a $\mathcal{H}^k \llcorner \mathcal{M}$-measurable function such that $\xi(x)$ is a simple unit $k$-vector in $\Lambda_k \mathbb{R}^n$ orienting the approximate tangent space to $\mathcal{M}$ at $\mathcal{H}^k$-a.e. $x \in \mathcal{M}$, and $\theta : \mathcal{M} \to [0, +\infty)$ a $\mathcal{H}^k \llcorner \mathcal{M}$-summable non-negative function, so that $\mathbf{M}(T) = \int_{\mathcal{M}} \theta \, d\mathcal{H}^k < \infty$. In that case, the short-hand notation $T = [\![ \mathcal{M}, \xi, \theta ]\!]$ is commonly adopted, and set $(T)$ denotes the set of points in $\mathcal{M}$ with positive multiplicity.

In addition, if $\theta$ is integer-valued, the current $T$ is called *integer multiplicity* (in short *i.m.*) *rectifiable* and the corresponding class is denoted by $\mathcal{R}_k(U)$. The usefulness of i.m. rectifiable currents in the Calculus of Variations derives from Federer–Fleming's compactness theorem [17]. It states that if a sequence $\{T_h\} \subset \mathcal{R}_k(U)$ satisfies $\sup_h \big( \mathbf{M}(T_h) + \mathbf{M}((\partial T_h) \llcorner U) \big) < \infty$, where $\llcorner$ denotes restriction, then there exists $T \in \mathcal{R}_k(U)$ and a (not relabeled) subsequence of $\{T_h\}$ such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_k(U)$.

**Example 2.4.** If $\mathcal{M}$ is a smooth, oriented, compact $k$-submanifold of $U$, then $[\![ \mathcal{M} ]\!]$ is the current in $\mathcal{R}_k(U)$ given by integration of $k$-forms in the sense of differential geometry, i.e., $\langle [\![ \mathcal{M} ]\!]; \omega \rangle := \int_{\mathcal{M}} \omega$ for all $\omega \in \mathcal{D}^k(U)$.

2.4. **Graph currents.** If $u$ is a map in $W^{1,1}(B^n, \mathbb{R}^N)$, where $n, N \geq 2$, then $u$ has a Lusin representative on the subset $\widetilde{B}^n$ of Lebesgue points pertaining to both $u$ and the gradient $Du$, where $\mathcal{L}^n(B^n \setminus \widetilde{B}^n) = 0$. Following [22], the *graph* of $u$ is the countably $n$-rectifiable subset of $U = B^n \times \mathbb{R}^N$

$$\mathcal{G}_u := \{(x, y) \in B^n \times \mathbb{R}^N \mid x \in \widetilde{B}^n \,,\ y = \widetilde{u}(x)\},$$

where $\widetilde{u}(x)$ is the Lebesgue value of $u$. By the area formula, one has $\mathcal{H}^n(\mathcal{G}_u) < \infty$ if in addition all the minors of $Du$ are in $L^1(B^n)$. In that case, $u$ is called a map in $\mathcal{A}^1(B^n, \mathbb{R}^N)$. More precisely, the approximate tangent $n$-plane at $(x, \widetilde{u}(x))$ is generated by the vectors $\mathbf{t}_i(x) = (\mathbf{e}_i, \partial_i u(x)) \in \mathbb{R}^{n+N}$, for $i = 1, \ldots, n$, where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is the canonical basis in $\mathbb{R}^n$ and the partial derivative $\partial_i u(x)$ is the $i$-th column vector of the gradient matrix $Du(x)$ given by the Lebesgue value of $Du$ at $x \in \widetilde{B}^n$. Therefore, the unit $n$-vector

$$\xi(x) := \frac{\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)}{|\mathbf{t}_1(x) \wedge \cdots \wedge \mathbf{t}_n(x)|} \in \Lambda_n \mathbb{R}^{n+N} \,, \quad x \in \widetilde{B}^n$$

provides an orientation to $\mathcal{G}_u$, and the *graph current* $G_u = [\![ \mathcal{G}_u, \xi, 1 ]\!]$ is i.m. rectifiable in $\mathcal{R}_n(B^n \times \mathbb{R}^N)$.

The action of $G_u$ can be read (in an approximate $\mathcal{L}^n$-a.e. sense) through the pull-back of the *graph map* $(\mathrm{Id} \bowtie u)(x) := (x, u(x))$ by:

$$\langle G_u; \omega \rangle = \int_{B^n} (\mathrm{Id} \bowtie u)^{\#} \omega \qquad \forall\, \omega \in \mathcal{D}^n(B^n \times \mathbb{R}^N) \,. \tag{2.4}$$

Therefore, the mass of $G_u$ is equal to the *graph area* $\mathbb{A}(u)$, i.e.,

$$\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_u) = \mathbb{A}(u) := \int_{B^n} |M(Du)| \, dx < \infty \tag{2.5}$$

where $|M(Du)|$ is the Jacobian of $\mathrm{Id} \bowtie u$, so that $|M(Du)|^2$ is equal to 1 plus the sum of the square of all minors of the $N \times n$ matrix $Du$.

Let now $N = 3$ and $u \in W^{1,2}(B^n, \mathbb{S}^2)$. If $n \geq 3$, by the area formula all the $3 \times 3$ minors of $Du$ are zero $\mathcal{L}^n$-a.e. in $B^n$. Therefore, for any $n \geq 2$ the Jacobian $|M(Du)|$ is $\mathcal{L}^n$-essentially bounded (up to an absolute constant factor $c_n$ only depending on the dimension $n$) by $1 + |Du|^2$, where $Du \in L^2(B^n, \mathbb{R}^{3\times n})$. Whence, $u \in \mathcal{A}^1(B^n, \mathbb{R}^3)$ and by (2.5) we get

$$\mathbf{M}(G_u) = \mathbb{A}(u) \leq c_n \int_{B^n} (1 + |Du|^2)\, dx < \infty\,. \tag{2.6}$$

In addition, by Federer's flatness theorem, the graph current $G_u$ is an i.m. rectifiable current in $B^n \times \mathbb{S}^2$, say $G_u \in \mathcal{R}_n(B^n \times \mathbb{S}^2)$.

Now, if $u \in W^{1,2}(B^n, \mathbb{S}^2)$ is smooth, we have $G_u = [\![ \mathcal{G}_u ]\!]$, see (2.4), where the graph manifold $\mathcal{G}_u$ has no "fractures" or "holes". By Stokes' theorem, such a condition is read in terms of the graph current $G_u$ by the property:

$$\langle \partial G_u; \eta\rangle := \langle G_u; d\eta\rangle = 0 \qquad \forall\, \eta \in \mathcal{D}^{n-1}(B^n \times \mathbb{S}^2)\,. \tag{2.7}$$

**Remark 2.5.** Given $u \in W^{1,2}(B^n, \mathbb{S}^2)$, assume that there exists a sequence of smooth maps $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ such that $u_h \to u$ strongly in $W^{1,2}(B^n, \mathbb{R}^3)$. We recall that this is always the case in the critical dimension $n = 2$, by Schoen–Uhlenbeck density theorem [29]. The strong $W^{1,2}$ convergence implies that $G_{u_h} \rightharpoonup G_u$ weakly as currents in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$, a convergence that preserves the homological property (2.7). Therefore, we conclude that the map $u$ satisfies the *null-boundary condition* (2.7).

**Remark 2.6.** Condition (2.7) is violated in high dimension $n \geq 3$, in general. If e.g. $n = 3$, the 0-homogeneous harmonic map $u(x) = x/|x|$ belongs to the class $W^{1,2}(B^3, \mathbb{S}^2)$, and one has (cf. [22, Sec. 3.2.2, Ex. 1])

$$(\partial G_u) \llcorner B^3 \times \mathbb{S}^2 = -\delta_O \times [\![ \mathbb{S}^2 ]\!] \tag{2.8}$$

where $\delta_O$ is the unit Dirac mass at the origin $O$. Therefore, one cannot find a sequence $\{u_h\} \subset C^\infty(B^3, \mathbb{S}^2)$ strongly converging to $u$ in $W^{1,2}(B^3, \mathbb{R}^3)$.

**Remark 2.7.** For maps $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{S}^2)$ with $|Du| \in L^2(\mathbb{R}^n)$, we denote

$$\mathbb{D}(u, \mathbb{R}^n) := \frac{1}{2} \int_{\mathbb{R}^n} |Du|^2\, dx \tag{2.9}$$

and notice that this time the graph current $G_u$ is locally i.m. rectifiable in $\mathbb{R}^n \times \mathbb{S}^2$, i.e. $G_u \llcorner \Omega \times \mathbb{S}^2 \in \mathcal{R}_n(\Omega \times \mathbb{S}^2)$ for each bounded open set $\Omega \subset \mathbb{R}^n$.

2.5. **Degree.** In dimension $n = 2$, the degree of maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ is well defined provided that $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ with $|Du| \in L^2(\mathbb{R}^2)$. In fact, by Remark 2.7 the graph current $G_u$ is locally i.m. rectifiable. In addition, it satisfies the null-boundary condition

$$\langle \partial G_u; \eta\rangle = 0 \qquad \forall\, \eta \in \mathcal{D}^1(\mathbb{R}^2 \times \mathbb{S}^2)\,.$$

Therefore, denoting by $\Pi_y(x, y) := y$ the orthogonal projection onto the target space $\mathbb{S}^2 \subset \mathbb{R}^3$, the image current $\Pi_{y\#} G_u$ is an *integral* 2-*cycle* in $\mathbb{S}^2$, i.e., $\Pi_{y\#} G_u \in \mathcal{R}_2(\mathbb{S}^2)$ with $\partial(\Pi_{y\#} G_u) = 0$. By the constancy theorem, compare [22, Sec. 4.3.1, Thm. 4], we thus have $\Pi_{y\#}G_u = \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!]$ for some integer $\mathrm{d} \in \mathbb{Z}$. Moreover, if $\omega_2$ denotes the volume 2-form on $\mathbb{S}^2$

$$\omega_2 := y^1 dy^2 \wedge dy^3 + y^2 dy^3 \wedge dy^1 + y^3 dy^1 \wedge dy^2 \tag{2.10}$$

by the action (2.4) we have

$$\int_{\mathbb{R}^2} u^{\#}\omega_2 = \langle \Pi_{y\#}G_u; \omega_2 \rangle = \langle \mathrm{d}\,[\![\,\mathbb{S}^2\,]\!]; \omega_2 \rangle = \mathrm{d} \int_{\mathbb{S}^2} \omega_2 = \mathrm{d} \cdot 4\pi\,.$$

**Definition 2.8.** Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ with $|Du| \in L^2(\mathbb{R}^2)$. We call *degree* $\deg u$ of $u$ the integer $\mathrm{d} \in \mathbb{Z}$ given by formula

$$\deg u := \frac{1}{4\pi} \int_{\mathbb{R}^2} u^{\#}\omega_2 = \mathrm{d}\,.$$

Notice that the degree is *strongly continuous*: if $\{u_h\} \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ is a sequence converging to $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2)$ a.e. in $\mathbb{R}^2$, with $Du_h \to Du$ strongly in $L^2(\mathbb{R}^2, \mathbb{R}^{3\times 2})$, by dominated convergence we get

$$\lim_{h\to\infty} \frac{1}{4\pi} \cdot \Big| \int_{\mathbb{R}^2} (u_h^{\#}\omega_2 - u^{\#}\omega_2) \Big| = 0$$

and hence $\deg u_h = \deg u$, for $h$ large enough.

2.6. **Cartesian currents.** Let $n \geq 2$ and $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ be a sequence of smooth maps with equibounded Dirichlet energies, $\sup_h \mathbb{D}(u_h) < \infty$, see (1.1). The graph currents $G_{u_h}$ belong to $\mathcal{R}_n(B^n \times \mathbb{S}^2)$ and satisfy condition (2.7) and $\sup_h \mathbf{M}(G_{u_h}) < \infty$, by (2.6). Therefore, Federer–Fleming's theorem [17] yields that the currents $G_{u_h}$ subconverge weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to a current $T \in \mathcal{R}_n(B^n \times \mathbb{S}^2)$ satisfying the *null-boundary condition*

$$(\partial T) \llcorner B^n \times \mathbb{S}^2 = 0\,. \tag{2.11}$$

In addition, compare [22, 26], there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ and a map $u_T \in W^{1,2}(B^n, \mathbb{S}^2)$ such that

$$T = G_{u_T} + L \times [\![\,\mathbb{S}^2\,]\!] \tag{2.12}$$

where the *underlying function* $u_T$ is given by the weak $W^{1,2}$ limit of the $u_h$'s. Finally, in low dimension $n = 2$ we also have $(\partial G_{u_T}) \llcorner B^2 \times \mathbb{S}^2 = 0$. For that reason, Giaquinta–Modica–Souček [19] introduced the following

**Definition 2.9.** The class $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ is given by the i.m. rectifiable currents $T \in \mathcal{R}_n(B^n \times \mathbb{S}^2)$ satisfying the null-boundary condition (2.11) and the structure property (2.12) for some Sobolev map $u_T$ in $W^{1,2}(B^n, \mathbb{S}^2)$ and some i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$.

The Dirichlet energy of a current $T$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ is given by

$$\mathbb{D}(T) := \frac{1}{2} \int_{B^n} |Du_T|^2\, dx + 4\pi \cdot \mathbf{M}(L) \qquad \text{if (2.12) holds}\,.$$

Since the functional $T \mapsto \mathbb{D}(T)$ agrees with the *parametric polyconvex lower semicontinuous extension* of the Dirichlet integral, compare [23, Sec. 2.2.4], dealing with currents in $\operatorname{cart}^{2,1}(B^n \times \mathbb{S}^2)$ it turns out that if $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$, then

$$\mathbb{D}(T) \leq \liminf_{h\to\infty} \mathbb{D}(T_h) . \tag{2.13}$$

Finally, a weak closure property holds: if a sequence $\{T_h\} \subset \operatorname{cart}^{2,1}(B^n \times \mathbb{S}^2)$ satisfies $\sup_h \mathbb{D}(T_h) < \infty$, then there exists a current $T$ in $\operatorname{cart}^{2,1}(B^n \times \mathbb{S}^2)$ and a (not relabeled) subsequence such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$.

2.7. **Current of the singularities.** Let $u \in W^{1,2}(B^n, \mathbb{S}^2)$, where $n \geq 3$. Following [23, Sec. 4.2.5], we denote by $\mathbb{P}(u)$ the $(n-3)$-current in $\mathcal{D}_{n-3}(B^n)$ given by

$$\langle \mathbb{P}(u); \varphi \rangle := \frac{1}{4\pi} \int_{B^n} u^{\#}\omega_2 \wedge d\varphi , \qquad \varphi \in \mathcal{D}^{n-3}(B^n) \tag{2.14}$$

where $\omega_2$ is the volume 2-form (2.10). It turns out that the boundary of the graph current $G_u$ satisfies equation

$$(\partial G_u) \llcorner B^n \times \mathbb{S}^2 = \mathbb{P}(u) \times [\![ \mathbb{S}^2 ]\!] . \tag{2.15}$$

Therefore, for a current $T \in \operatorname{cart}^{2,1}(B^n \times \mathbb{S}^2)$ as in (2.12), the null boundary condition (2.11) is equivalent to the following link between $\mathbb{P}(u_T)$ and $L$:

$$(\partial L) \llcorner B^n = -\mathbb{P}(u_T) . \tag{2.16}$$

2.8. **Real and integral mass.** The latter formula motivates the introduction of some more notation. Let again $n \geq 3$.

**Definition 2.10.** For any current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$, we denote by

$$\mathbf{m}_{r,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{n-2}(B^n) , \ (\partial D) \llcorner B^n = -\mathbb{P}\} \tag{2.17}$$

the *real mass of* $\mathbb{P}$ allowing connections to the boundary. We also define

$$\mathbf{m}_{i,B^n}(\mathbb{P}) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-2}(B^n) , \ (\partial L) \llcorner B^n = -\mathbb{P}\} . \tag{2.18}$$

**Remark 2.11.** By Federer-Fleming's theorem [17], if there exists an i.m. rectifiable current $L \in \mathcal{R}_{n-2}(B^n)$ such that $(\partial L) \llcorner B^n = -\mathbb{P}$, the minimum in (2.18) is attained. In that case, $\mathbf{m}_{i,B^n}(\mathbb{P})$ is called *integral mass*, and a minimizer $L$ a *minimal integral connection of* $\mathbb{P}$ (allowing connections to the boundary).

**Example 2.12.** If e.g. $u(x) = x/|x|$, by (2.8) and (2.15) we get $\mathbb{P}(u) = -\delta_O$, and the integral mass $\mathbf{m}_{i,B^n}(\mathbb{P}(u))$ is equal to the length of any line segment connecting a point at the boundary of $B^3$ to the origin $O$.

**Remark 2.13.** In dimension $n = 3$, Federer's theorem [16], compare [23, Sec. 3.1.4, Thm. 8], gives that if $\mathbf{m}_{i,B^3}(\mathbb{P}) < \infty$ for some $\mathbb{P} \in \mathcal{D}_0(B^3)$, then

$$\mathbf{m}_{i,B^3}(\mathbb{P}) = \mathbf{m}_{r,B^3}(\mathbb{P}) . \tag{2.19}$$

This is false in general when $n \geq 4$. More precisely, compare [28, 31], for a current $\mathbb{P} \in \mathcal{D}_{n-3}(B^n)$ with $\mathbf{m}_{i,B^n}(\mathbb{P}) < \infty$, it may happen that

$$\mathbf{m}_{r,B^n}(\mathbb{P}) < \mathbf{m}_{i,B^n}(\mathbb{P}) \quad \text{if} \quad n \geq 4 .$$

2.9. **Maps with a small singular set.** Due to the non-triviality of the second homotopy group $\pi_2(\mathbb{S}^2) \simeq \mathbb{Z}$, in dimension $n \geq 3$ it is false that the class of smooth maps $C^\infty(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$. However, a wider class of maps with small singular set is dense.

**Definition 2.14.** For $n \geq 3$, we denote by $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ the class of maps $u : \bar{B}^n \to \mathbb{S}^2$ which are smooth on $\bar{B}^n \setminus S_u$, where $S_u$ is a finite union of $(n-3)$-dimensional smooth sets with smooth boundary (a finite set of points when $n = 3$) and such that for every positive integer $k$ there exists a positive real constant $c$, depending on $u$ and $k$, such that the $k$-th order derivative satisfies

$$|D^k u(x)| \leq \frac{c}{(\mathrm{dist}\,(x, S_u))^k} \qquad \forall\, x \in \bar{B}^n \setminus S_u\,.$$

The following density property was proved in case $n = 3$ by Bethuel–Zheng [10], and extended to high dimension $n \geq 3$ by Bethuel [7].

**Theorem 2.15.** *The class $R^\infty_{n-3}(B^n, \mathbb{S}^2)$ is strongly dense in $W^{1,2}(B^n, \mathbb{S}^2)$.*

2.10. **Point singularities.** Let $n = 3$ and assume that $u \in W^{1,2}(B^3, \mathbb{S}^2)$ is smooth outside a finite set $S_u$, compare Definition 2.14. For any singular point $a \in S_u$ and for $r > 0$ small, the restriction $u_{|\partial B^3_r(a)}$ of $u$ to the boundary of the ball $B^3_r(a) := a + B^3_r$ is a smooth function. Therefore, arguing as before it turns out that the *degree of $u$ at $a$* is well defined by the integer

$$\deg(u, a) := \frac{1}{4\pi} \int_{\partial B^3_r(a)} u^\# \omega_2 = \mathrm{d} \in \mathbb{Z}\,. \tag{2.20}$$

In fact, standard homotopy arguments imply that definition (2.20) does not depend on the choice of the (small) radius, whence it agrees with the classical Brouwer degree. Moreover, if $S_u = \{a_i\}_{i=1}^m$ and $\deg(u, a_i) = \mathrm{d}_i$, similarly to [23, Sec. 4.2.1, Prop. 1] we infer:

$$(\partial G_u) \llcorner B^3 \times \mathbb{S}^2 = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \times [\![\, \mathbb{S}^2 \,]\!]\,.$$

Therefore, formula (2.15) implies that the current of the singularities $\mathbb{P}(u)$ is i.m. rectifiable:

$$\mathbb{P}(u) = -\sum_{i=1}^m \mathrm{d}_i\, \delta_{a_i} \in \mathcal{R}_0(B^3)\,. \tag{2.21}$$

As e.g. to the 0-homogeneous map $u(x) = x/|x|$, one has $S_u = \{O\}$, the boundary condition (2.8) holds, $\mathbb{P}(u) = -\delta_O$, and $\deg(u, O) = 1$.

2.11. **The relaxed Dirichlet energy.** Similarly to (1.8), the relaxed Dirichlet energy of maps $u$ in $L^1(B^n, \mathbb{S}^2)$ is defined by the functional

$$\widetilde{\mathbb{D}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{D}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^2),\ u_h \to u \text{ in } L^1(B^n, \mathbb{R}^3) \Big\}\,.$$

In dimension $n = 2$, we clearly have

$$\widetilde{\mathbb{D}}(u) = \begin{cases} \mathbb{D}(u) & \text{if } u \in W^{1,2}(B^n, \mathbb{S}^2) \\ +\infty & \text{if } u \in L^1(B^n, \mathbb{S}^2) \setminus W^{1,2}(B^n, \mathbb{S}^2) . \end{cases}$$

In dimension $n = 3$, following Brezis-Coron-Lieb [13], the *flat norm* $\mathbf{L}(u)$ of $u \in W^{1,2}(B^3, \mathbb{S}^2)$ (relative to the boundary) is given by

$$\mathbf{L}(u) := \frac{1}{4\pi} \cdot \sup_{\xi \in \mathcal{F}} \int_{B^3} D(u) \bullet D\xi \, dx \tag{2.22}$$

where $\bullet$ is the scalar product in $\mathbb{R}^3$. In the latter formula, $\mathcal{F}$ denotes the class of smooth test functions $\xi : B^3 \to \mathbb{R}$ such that $\|\xi\|_\infty \leq 1$ and $\|D\xi\|_\infty \leq 1$, and $D(u) : B^3 \to \mathbb{R}^3$ the *D-field*

$$D(u) := \big( u \bullet \partial_2 u \times \partial_3 u,\, u \bullet \partial_3 u \times \partial_1 u,\, u \bullet \partial_1 u \times \partial_2 u \big) .$$

Bethuel–Brezis–Coron [8] showed that for any $u \in W^{1,2}(B^3, \mathbb{S}^2)$ the relaxed Dirichlet energy is finite, and it satisfies the explicit formula

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{L}(u) .$$

Following Giaquinta–Modica–Souček [20], as distributions of $\mathcal{D}_0(B^3)$ one gets $\mathbb{P}(u) = \frac{1}{4\pi} \operatorname{Div} D(u)$, i.e.,

$$\langle \mathbb{P}(u); \varphi \rangle = -\frac{1}{4\pi} \int_{B^3} D(u) \bullet D\varphi \, dx \qquad \forall\, \varphi \in C_c^\infty(B^3) .$$

Equivalently, the current $\mathbf{D}(u) \in \mathcal{D}_1(B^3)$ given by

$$\langle \mathbf{D}(u); \eta \rangle := \frac{1}{4\pi} \int_{B^3} u^\# \omega_2 \wedge \eta , \qquad \eta \in \mathcal{D}^1(B^3)$$

is such that $(\partial \mathbf{D}(u)) \llcorner B^3 = \mathbb{P}(u)$ and $\mathbf{M}(\mathbf{D}(u)) < \infty$. Moreover, a duality argument yields that the minimal real connection $\mathbf{m}_{r,B^3}(\mathbb{P}(u))$ of the singularities agrees with the flat norm $\mathbf{L}(u)$, compare [23, Sec. 4.2.5].

Most importantly, in [20] the authors obtained that the flat norm agrees with the integral mass of the current of the singularities, i.e.,

$$\mathbf{L}(u) = \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty \qquad \forall\, u \in W^{1,2}(B^3, \mathbb{S}^2) .$$

Their argument relies on Theorem 2.15 and on the following result:

**Proposition 2.16.** *Let $u \in W^{1,2}(B^3, \mathbb{S}^2)$ and $\{u_k\} \subset W^{1,2}(B^3, \mathbb{S}^2) \cap R_0^\infty$ be such that $u_k \to u$ strongly in $W^{1,2}$. Then, for each $k$ there exists an i.m. rectifiable current $L_k \in \mathcal{R}_1(B^3)$ with $(\partial L_k) \llcorner B^3 = \mathbb{P}(u) - \mathbb{P}(u_k)$ such that $\mathbf{M}(L_k) \to 0$ as $k \to \infty$.*

**Remark 2.17.** The proof of Proposition 2.16 makes use of the coarea formula by Almgren–Browder–Lieb [3] and of Federer's theorem [16], see (2.19). In high dimension $n \geq 4$, even if we knew a priori that $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$ for some $u \in W^{1,2}(B^n, \mathbb{S}^2)$, the cited Federer's theorem doesn't apply, see Remark 2.13. Therefore, Proposition 2.16 doesn't work anymore.

In [24], using a different approach we extended the explicit formula for the relaxed Dirichlet energy to any high dimension $n \geq 3$. Definitely, for any map $u \in W^{1,2}(B^n, \mathbb{S}^2)$, it turns out that the $(n-3)$-current $\mathbb{P}(u)$ of the singularities satisfies $\mathbf{m}_{i,B^n}(\mathbb{P}(u)) < \infty$, and we have:

$$\widetilde{\mathbb{D}}(u) = \mathbb{D}(u) + 4\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{1,2}(B^n, \mathbb{S}^2)\,. \tag{2.23}$$

2.12. **A cohomological criterion.** The following density result was proved in more generality by Gastel–Nerf [18] when $n = 3$, and then extended by Bousquet–Ponce–Van Schaftingen [11] in high dimension, see also [27, Lemma 4.5].

**Theorem 2.18.** *The class $R^{\infty}_{n-3}(B^n, \mathbb{S}^2)$ is strongly dense in $W^{2,1}(B^n, \mathbb{S}^2)$, in any dimension $n \geq 3$.*

One of the main difficulties in the proof of Theorem 2.18 is to obtain Sobolev regularity of the derivatives of the functions involved. Therefore, one cannot apply the same construction as the one in the proof of Theorem 2.15.

Concerning maps $u$ in $W^{1,2}(B^n, \mathbb{S}^2)$, the only obstruction to strong density of smooth maps is encoded by the non-triviality of the current of the singularities $\mathbb{P}(u)$ in (2.14). This cohomological criterion was firstly proved by Bethuel [6] when $n = 3$, and then extended in high dimension and for a wider class of target manifolds in [9]. For second order Sobolev maps, Bousquet–Ponce–Van Schaftingen [12] have recently proved the following

**Theorem 2.19.** ([12]) *Let $u \in W^{2,1}(B^n, \mathbb{S}^2)$ for some $n \geq 3$. If $\mathbb{P}(u) = 0$, there exists a sequence $\{u_h\} \subset C^{\infty}(B^n, \mathbb{S}^2)$ such that $u_h \to u$ strongly in $W^{2,1}(B^n, \mathbb{R}^3)$.*

We briefly comment on the validity of the previous result. First, we observe that the converse implication in Theorem 2.19 is trivially checked. In fact, if $u \in W^{2,1}(B^n, \mathbb{S}^2)$ is the strong limit of a smooth sequence $\{u_h\}$ in $C^{\infty}(B^n, \mathbb{S}^2)$, by the $W^{1,2}$-convergence of $u_h$ to $u$, in Remark 2.5 we have seen that the graph current $G_u$ satisfies the null-boundary condition (2.7), which yields $\mathbb{P}(u) = 0$, by (2.15).

As to the non-trivial implication of Theorem 2.19, in [12, Thm. 1.10] the authors show that for more general target Riemannian manifolds $\mathcal{N}$, a map $u \in W^{2,1}(B^n, \mathcal{N})$ is the strong $W^{2,1}$ limit of a smooth sequence in $C^{\infty}(B^n, \mathcal{N})$ if and only in $u$ is 2-*extendable*. Referring to the cited paper for the precise notion of 2-extendability, in the special case in which $\mathcal{N} = \mathbb{S}^2$ we quote [12, Thm. 1.13], where it is shown that in any dimension $n \geq 3$ a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ is 2-extendable if and only if $(du^{\#}\omega_2) = 0$ *in the sense of currents in $B^n$*. The latter property means that

$$\int_{B^n} u^{\#}\omega_2 \wedge d\varphi = 0 \quad \forall\, \varphi \in \mathcal{D}^{n-3}(B^n)$$

and hence it is equivalent to property $\mathbb{P}(u) = 0$.

## 3. Maps with finite relaxed Laplacean energy

In this section, we discuss in any dimension $n \geq 2$ some general properties of maps with finite relaxed energy (1.8). The case of low dimension $n = 2$ is then analyzed, where a first lower semicontinuity property is also obtained.

3.1. **Weak Laplacean.** Let $n \geq 2$ and $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, see (1.9), and let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^2)$ be a smooth sequence such that $u_h \to u$ in $L^1(B^n, \mathbb{R}^3)$ and $\sup_h \mathbb{L}(u_h) < \infty$. By the inequality (1.6) we have $\sup_h \mathbb{D}(u_h) < \infty$, see (1.1), so that a (not relabeled) subsequence of $\{u_h\}$ converges to $u$ weakly in $W^{1,2}(B^n, \mathbb{R}^3)$, and $u \in W^{1,2}(B^n, \mathbb{S}^2)$. Since $Du \in L^2(B^n, \mathbb{R}^{3\times n})$, the distributional divergence of the gradient is well defined by (1.10), and using that

$$\lim_{h\to\infty} \int_{B^n} \operatorname{tr}\,[Du_h\,(D\varphi)^\top]\,dx = \int_{B^n} \operatorname{tr}\,[Du\,(D\varphi)^\top]\,dx \quad \forall\,\varphi \in C^\infty_c(B^n, \mathbb{R}^3)$$

we infer that $\text{Div}Du_h \rightharpoonup \text{Div}Du$ weakly as $\mathbb{R}^3$-valued measures in $B^n$. In case of Sobolev maps in $W^{2,1}(B^n, \mathbb{S}^2)$, integrating by parts we get:

$$\langle \text{Div}Du; \varphi\rangle = \int_{B^n} \Delta u \bullet \varphi\,dx \quad \forall\,\varphi \in C^\infty_c(B^n, \mathbb{R}^3)$$

whence (1.13) holds true. Moreover, the lower semicontinuity property of the total variation gives

$$|\text{Div}Du|(B^n) \leq \liminf_{h\to\infty} |\text{Div}Du_h|(B^n) \leq \sup_h \mathbb{L}(u_h) < \infty \tag{3.1}$$

so that $\text{Div}Du$ is a $\mathbb{R}^3$-valued finite Radon measure in $B^n$, and the lower bound (1.11) follows by lower semicontinuity.

Notice that if $u \in \mathbb{L}(B^n, \mathbb{S}^2)$, denoting $u = (u^1, u^2, u^3)$, we have checked that $Du^\ell$ is a divergence-measure field in $\mathcal{DM}^{1,2}(B^n)$ for $\ell = 1, 2, 3$, see Definition 2.1. Moreover, the decomposition into mutually singular measures

$$\text{Div}Du = (\text{Div}Du)^a + (\text{Div}Du)^s\,, \quad (\text{Div}Du)^a = \widetilde{\Delta u}\,\mathcal{L}^n \llcorner B^n \tag{3.2}$$

holds, with a density function $\widetilde{\Delta u}$ in $L^1(B^n, \mathbb{R}^3)$.

To our knowledge, there isn't enough information concerning both the density $\widetilde{\Delta u}$ and the singular part $(\text{Div}Du)^s$ in formula (3.2) in order to analyze the relaxed energy (1.8) of maps $u$ in $\mathbb{L}(B^n, \mathbb{S}^2)$.

More precisely, Proposition 2.2 yields that $|(\text{Div}Du)^s|(B) = 0$ for each Borel set $B \subset B^n$ with $\sigma$-finite $\mathcal{H}^{n-2}$-measure. However, it may happen that the gradient $Du$ does not belong to the class $\text{BV}(B^n, \mathbb{R}^{3\times n})$, so that we cannot conclude e.g. that for $\ell = 1, 2, 3$ the vector field $Du^\ell \in L^2(B^n, \mathbb{R}^n)$ is approximately differentiable $\mathcal{L}^n$-a.e. in $B^n$, and that the trace $\operatorname{tr}\nabla Du^\ell$ agrees $\mathcal{L}^n$-a.e. in $B^n$ with the $\ell$-th component of the Radon-Nikodym derivative $\widetilde{\Delta u}$ of the measure $\text{Div}Du$.

3.2. **The BV case.** Assume now that $u$ belongs to the class $\mathbb{L}_{BV}(B^n,\mathbb{S}^2)$ in (1.14). Then the weak hessian $\nabla(Du^\ell)$ of each component $u^\ell$ is a summable function in $L^1(B^n,\mathbb{R}^{n\times n})$, and the density $\widetilde{\Delta u}$ in (3.2) agrees with the approximate Laplacean $\Delta u=(\Delta u^1,\Delta u^2,\Delta u^3)$, where $\Delta u^\ell=\mathrm{tr}\,[\nabla(Du^\ell)]$, for $\ell=1,2,3$. In addition, the singular part of the measure $(\mathrm{Div}Du)^s$ decomposes into a *Jump* and a *Cantor-type* component, the first one being concentrated on the countably $(n-1)$-rectifiable discontinuity set of the gradient $Du$, and the second one being equal to zero if $Du$ is a *special function of bounded variation*.

As a consequence, we obtain a *tangential property* concerning the singular component of the weak Laplacean.

**Proposition 3.1.** *Let $n\geq 2$ and $u\in\mathbb{L}_{BV}(B^n,\mathbb{S}^2)$, see* (1.14)*. Then*

$$u\bullet(\mathrm{Div}Du)^s=0\,. \tag{3.3}$$

*Proof.* By applying Proposition 2.3 with $F=Du^\ell$ and $g=u^\ell$, and by summating on $\ell=1,2,3$, we obtain

$$\mathrm{Div}(u\bullet Du)=u\bullet\mathrm{Div}Du+|Du|^2\mathcal{L}^n\llcorner B^n\,.$$

Since moreover $|u|\equiv 1$, we have $0=\partial_i|u|^2=2\partial_i u\bullet u$ for $i=1,\dots,n$, whence $\mathrm{Div}(u\bullet Du)=0$ and therefore

$$u\bullet\mathrm{Div}Du=-|Du|^2\mathcal{L}^n\llcorner B^n\,. \tag{3.4}$$

On the other hand, since $Du$ is in BV, we have $u\bullet\Delta u=-|Du|^2$ for $\mathcal{L}^n$-a.e. in $B^n$, so that

$$u\bullet(\mathrm{Div}Du)^a=(u\bullet\Delta u)\,\mathcal{L}^n\llcorner B^n=-|Du|^2\,\mathcal{L}^n\llcorner B^n\,. \tag{3.5}$$

Equation (3.3) follows from (3.4) and (3.5), on account of the decomposition formula (3.2), where $\widetilde{\Delta u}=\Delta u$. □

A sufficient condition ensuring both enough regularity of the density $\widetilde{\Delta u}$ and property $(\mathrm{Div}Du)^s=0$, is the membership of $Du$ to the Sobolev class $W^{1,1}(B^n,\mathbb{R}^{3\times n})$, so that in particular equation (1.13) holds true. In fact, the computation of the energy gap for maps in $W^{2,1}(B^3,\mathbb{S}^2)$ is the content of our Main Result, Theorem 1.2.

3.3. **The low dimension case.** We now see that in dimension $n=2$, formula (1.12) holds and hence there is no energy gap.

*Proof of* (1.12). By the continuous embedding of $W^{1,2}(B^2)$ in VMO, convolutions $u_\varepsilon:=\rho_\varepsilon*u$ with a smooth kernel $\rho_\varepsilon=\varepsilon^{-2}\rho(x/\varepsilon)$ of maps $u\in W^{1,2}(B^2,\mathbb{S}^2)$ have image whose distance to $\mathbb{S}^2$ goes to 0 with $\varepsilon$, and hence the projection argument from [29] applies, compare e.g. [22, Sec. 5.5.1, Thm. 3] or [26, Thm. 4.14]. More precisely, equation (1.12) holds true in the class $W^{2,1}(B^2,\mathbb{S}^2)$ by the strong $W^{2,1}$ density of maps in $C^\infty(B^2,\mathbb{S}^2)$. If $u\in\mathbb{L}(B^2,\mathbb{S}^2)$, arguing as e.g. in the proof of [4, Thm. 3.9], for each $\delta>0$

we can find a smooth map $v_\delta \in C^\infty(B^2, \mathbb{R}^3)$ such that $\|u - v_\delta\|_{L^1(B^2)} < \delta$, $\operatorname{dist}(v_\delta(B^2), \mathbb{S}^2) < \delta$, and

$$|\mathrm{Div} D v_\delta|(B^2) \le |\mathrm{Div} D u|(B^2) + \delta\,.$$

Setting then $u_h = \Pi(v_{\delta_h})$, where $\delta_h \searrow 0$ and $\Pi(y) = y/|y|$, for $y \in \mathbb{R}^3 \setminus \{0\}$, we have $u_h \to u$ in $L^1(B^2, \mathbb{R}^3)$ and

$$\limsup_{h\to\infty} \mathbb{L}(u_h) \le \limsup_{h\to\infty} |\mathrm{Div} D v_{\delta_h}|(B^2)\,,$$

so that (3.1) gives the energy convergence $\mathbb{L}(u_h) \to |\mathrm{Div} D u|(B^2)$ as $h \to \infty$, whence (1.12) follows from the lower bound (1.11). □

Using Proposition 2.2, we finally obtain a lower semicontinuity property:

**Theorem 3.2.** *Let $n = 2$ and let $\{u_k\} \subset C^\infty(B^2, \mathbb{S}^2)$ be such that the graph currents $G_{u_k}$ weakly converge in $\mathcal{D}_2(B^2 \times \mathbb{S}^2)$ to the current $T = G_u + \mathrm{d}\,\delta_O \times [\![\, \mathbb{S}^2 \,]\!]$, for some map $u \in \mathbb{L}(B^2, \mathbb{S}^2)$ and some integer $\mathrm{d} \in \mathbb{Z}$. Then*

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \ge |\mathrm{Div} D u|(B^n) + 8\pi\, |\mathrm{d}|\,.$$

*Proof.* For any $v \in W^{2,1}(B^2, \mathbb{S}^2)$ and any Borel set $B \subset B^2$, we denote

$$\mathbb{L}(v, B) := |\mathrm{Div} D v|(B) = \int_B |\Delta v|\, dx\,, \quad \mathbb{D}(v, B) := \frac{1}{2} \int_B |Dv|^2\, dx\,.$$

Let $\varepsilon > 0$. Since by Proposition 2.2 the measure $|\mathrm{Div} D u|$ does not charge any atom, we can choose $r > 0$ small so that $|\mathrm{Div} D u|(\bar{B}^2_r) \le \varepsilon$, and hence by lower semicontinuity and additivity

$$\liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) \ge |\mathrm{Div} D u|(B^2 \setminus \bar{B}^2_r) \ge |\mathrm{Div} D u|(B^2) - \varepsilon\,.$$

Moreover, by the inequality (1.6) we get

$$\liminf_{k\to\infty} \mathbb{L}(u_k, \bar{B}^2_r) \ge 2 \cdot \liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r)\,.$$

On the other hand, by the weak lower semicontinuity of the Dirichlet energy on Cartesian currents, see (2.13), using that

$$G_{u_k} \llcorner B^2_r \times \mathbb{S}^2 \rightharpoonup G_u \llcorner B^2_r \times \mathbb{S}^2 + \mathrm{d}\,\delta_O \times [\![\, \mathbb{S}^2 \,]\!]$$

weakly in $\mathcal{D}_2(B^2_r \times \mathbb{S}^2)$ we obtain the energy lower bound

$$\liminf_{k\to\infty} \mathbb{D}(u_k, B^2_r) \ge \mathbb{D}(u, B^2_r) + 4\pi\, |\mathrm{d}| \ge 4\pi\, |\mathrm{d}|\,.$$

Finally, putting the terms together we obtain

$$\begin{aligned} \liminf_{k\to\infty} \mathbb{L}(u_k, B^2) \ge\ & \liminf_{k\to\infty} \mathbb{L}(u_k, B^2 \setminus \bar{B}^2_r) + \liminf_{k\to\infty} \mathbb{L}(u_k, B^2_r) \\ \ge\ & |\mathrm{Div} D u|(B^n) - \varepsilon + 8\pi\, |\mathrm{d}| \end{aligned}$$

for each $\varepsilon > 0$, as required. □

## 4. Energy concentration

In this section, we compute the *minimal Laplacean energy* among maps $u:\mathbb{R}^2\to\mathbb{S}^2$ with fixed degree, the degree one case being the content of Proposition 1.1. In order to prove the general case, Theorem 4.1, we define a suitable modification of the inverse to the stereographic map, Proposition 4.2. Finally, the related bubbling phenomenon is briefly discussed.

We first introduce in any dimension $n\geq 2$ the class

$$W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2):=\{u\in W^{2,1}_{\rm loc}(\mathbb{R}^n,\mathbb{S}^2)\,:\,\Delta u\in L^1(\mathbb{R}^n,\mathbb{R}^3)\}\tag{4.1}$$

and the corresponding energy

$$\mathbb{L}(u,\mathbb{R}^n):=\int_{\mathbb{R}^n}|\Delta u|\,dx\,,\quad u\in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)\,.\tag{4.2}$$

Notice that if $u\in W_{\mathbb{L}}(\mathbb{R}^n,\mathbb{S}^2)$, inequality (1.6) holds $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, and hence $|Du|\in L^2(\mathbb{R}^n)$.

Assume now $n=2$. According to (4.1), the degree of a map $u$ in $W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ is given by Definition 2.8. We thus introduce the classes

$$\mathcal{F}_{\rm d}:=\{u\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)\mid \deg u={\rm d}\}\,,\qquad {\rm d}\in\mathbb{Z}\,.\tag{4.3}$$

The main result of this section is contained in the following

**Theorem 4.1.** *For every integer* $\rm d\in\mathbb{Z}$ *we have:*

$$\inf_{u\in\mathcal{F}_{\rm d}}\mathbb{L}(u,\mathbb{R}^2)=8\pi\,|{\rm d}|\,.$$

4.1. **The degree one case.** The case $\rm d=1$ of Theorem 4.1 is the content of Proposition 1.1. We recall that $\sigma_2^{-1}$ denotes the inverse of the stereographic map $\sigma_2:\mathbb{S}^2\to\mathbb{R}^2$, compare (2.1). Since ${\sigma_2^{-1}}_{\#}[\![\mathbb{R}^2]\!]=[\![\mathbb{S}^2]\!]$, one has $\deg\sigma_2^{-1}=1$.

*Proof of Proposition 1.1.* By (1.5), where we take $\mathfrak{p}=2$, any map $u\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ satisfies the lower bound

$$\mathbb{L}(u,\mathbb{R}^2)\geq\int_{\mathbb{R}^2}|Du|^2\,dx\geq 2\int_{\mathbb{R}^2}J_2u\,dx\,.\tag{4.4}$$

On the other hand, any degree one map $u$ in $W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ satisfies the inequality

$$\int_{\mathbb{R}^2}J_2u\,dx\geq 4\pi\,.\tag{4.5}$$

In addition, both inequalities in formula (4.4) are equalities if $u$ is harmonic and conformal, and that is the case of the degree one map $\sigma_2^{-1}$, compare (2.1). More precisely, formula (1.7) holds, so that by (4.4) and (4.5) we infer that

$$\mathbb{L}(u,\mathbb{R}^2)\geq 8\pi=\mathbb{L}(\sigma_2^{-1},\mathbb{R}^2)$$

for every degree one map $u$ in $W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$. □

4.2. **Modified stereographic projection.** Similarly to e.g. [23, Sec. 4.1.1], we modify the map $\sigma_2^{-1}$ in such a way that it is equal to the South Pole $P_S$ outside some small disk, by paying a small amount of Laplacean energy.

**Proposition 4.2.** *For any $\varepsilon > 0$ and $\delta > 0$ sufficiently small, there exists a smooth and degree one map $u_{\varepsilon,\delta} \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$, see (4.1), such that:*

1. $u_{\varepsilon,\delta}(x) = P_S$ *if* $|x| > \delta$ *;*
2. $4\pi \le \mathbb{D}(u_{\varepsilon,\delta}, \mathbb{R}^2) \le 4\pi + O(\varepsilon)$, *see* (2.9) *;*
3. $8\pi \le \mathbb{L}(u_{\varepsilon,\delta}, \mathbb{R}^2) \le 8\pi + O(\varepsilon)$, *see* (4.2)

*where $O(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.*

*Proof.* For $x \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$ and $\rho = |x|$ we can write

$$\sigma_2^{-1}(x) = \left( \frac{x}{\rho} \sin \widehat{\theta}(\rho), -\cos \widehat{\theta}(\rho) \right) , \quad \widehat{\theta}(\rho) = 2 \arctan(\rho^{-1})$$

in terms of the angular (geodesic) distance $\rho \mapsto \widehat{\theta}(\rho)$ in $\mathbb{S}^2$ of $\sigma_2^{-1}(\partial B^2_\rho)$ from the South Pole $P_S$.

For $\varepsilon > 0$ small, we modify the angular distance $\widehat{\theta}(\rho)$ and define

$$\widehat{\theta}_\varepsilon(\rho) = \begin{cases} 2\arctan(\rho^{-1}) & \text{if} \quad 0 < \rho < \varepsilon^{-1} \\ a_\varepsilon \rho^2 + b_\varepsilon \rho + c_\varepsilon & \text{if} \quad \varepsilon^{-1} \le \rho \le R_\varepsilon \\ 0 & \text{if} \quad \rho > R_\varepsilon \end{cases}$$

where the coefficients of the polynomial function in the second line and the radius $R_\varepsilon > \varepsilon^{-1}$ are given by

$$\begin{aligned} a_\varepsilon &= \frac{\varepsilon^4}{(\varepsilon^2+1)^2} \cdot \frac{1}{2\arctan\varepsilon} , \\ b_\varepsilon &= -\frac{\varepsilon^2}{(\varepsilon^2+1)^2} \left( \frac{\varepsilon}{\arctan\varepsilon} + 2(\varepsilon^2+1) \right) , \\ c_\varepsilon &= 2\arctan\varepsilon + \frac{\varepsilon}{(\varepsilon^2+1)^2} \left( \frac{\varepsilon}{2\arctan\varepsilon} + 2(\varepsilon^2+1) \right) , \\ R_\varepsilon &= \frac{1}{\varepsilon} + \frac{\varepsilon^2+1}{\varepsilon^2} \cdot 2\arctan\varepsilon . \end{aligned}$$

It can be checked that the function $\widehat{\theta}_\varepsilon$ is decreasing and of class $C^1$, and it is smooth outside the points $\varepsilon^{-1}$ and $R_\varepsilon$. Therefore, letting

$$u_\varepsilon(x) = \left( \frac{x}{\rho} \sin \widehat{\theta}_\varepsilon(\rho), -\cos \widehat{\theta}_\varepsilon(\rho) \right) , \quad x \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\} ,$$

we have $\nabla u_\varepsilon \in W^{1,1}(\mathbb{R}^2, \mathbb{R}^{3\times 2})$, and the Laplacean vector $\Delta u_\varepsilon$ is a measurable function defined a.e. on $\mathbb{R}^2$. Moreover, we have:

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2} |\Delta u_\varepsilon(x)| \, dx = 8\pi . \tag{4.6}$$

In fact, recalling that $\mathbb{L}(\sigma_2^{-1}, \mathbb{R}^2) = 8\pi$, for every $\varepsilon > 0$ small we obtain the estimate

$$\int_{\mathbb{R}^2} |\Delta u_\varepsilon(x)| \, dx \le 8\pi + \int_{\Omega_\varepsilon} |\Delta u_\varepsilon(x)| \, dx$$

where $\Omega_\varepsilon = \{x \in \mathbb{R}^2 \mid \varepsilon^{-1} < |x| < R_\varepsilon\}$. Now, for any sufficiently smooth function $\rho \mapsto \theta(\rho)$ we compute

$$\Delta\left(\frac{x_j}{\rho}\sin\theta(\rho)\right) = \left[\left(\theta''(\rho) + \frac{\theta'(\rho)}{\rho}\right)\cos\theta(\rho) - \left(\theta'(\rho)^2 + \frac{1}{\rho^2}\right)\sin\theta(\rho)\right]\frac{x_j}{\rho}$$

for $\rho = |x| > 0$, where $j = 1, 2$, and

$$\Delta\left(-\cos\theta(\rho)\right) = \theta'(\rho)^2\cos\theta(\rho) + \left(\theta''(\rho) + \frac{\theta'(\rho)}{\rho}\right)\sin\theta(\rho)\,.$$

Setting as before $u(x) = (\sin\theta(\rho)x/\rho, -\cos\theta(\rho))$, using that $|\sin\theta| \le |\theta|$ we obtain the pointwise estimate

$$\begin{aligned} |\Delta u(x)| \le \quad & \sqrt{2}\Bigg(|\theta''(\rho)| + \frac{|\theta'(\rho)|}{\rho} + \theta'(\rho)^2|\theta(\rho)| \\ & + \frac{|\theta(\rho)|}{\rho^2} + \theta'(\rho)^2 + |\theta''(\rho)\theta(\rho)| + \frac{|\theta(\rho)\theta'(\rho)|}{\rho}\Bigg)\,. \end{aligned}$$

In particular, if $\varepsilon \in (0,1)$ is small enough in such a way that $\varepsilon/2 \le \arctan\varepsilon \le \varepsilon$, for $\theta(\rho) = \widehat{\theta}_\varepsilon(\rho)$ and $\rho \in (\varepsilon^{-1}, R_\varepsilon)$ we obtain the upper bounds

$$|\widehat{\theta}_\varepsilon(\rho)| \le 2\varepsilon\,, \quad |\widehat{\theta}'_\varepsilon(\rho)| \le 2\varepsilon^2\,, \quad |\widehat{\theta}''_\varepsilon(\rho)| \le 2\varepsilon^3$$

and hence for $u = u_\varepsilon$ we can find an absolute real constant $c > 0$ such that

$$|\Delta u_\varepsilon(x)| \le c\,\varepsilon^3 \quad \forall\, x \in \Omega_\varepsilon\,.$$

Using that the area of the set $\Omega_\varepsilon$ is lower than $24\pi/\varepsilon^2$, we thus obtain the upper bound

$$\int_{\Omega_\varepsilon} |\Delta u_\varepsilon(x)|\,dx \le 24\pi\,c\,\varepsilon$$

and hence the energy limit (4.6) readily follows.

Now, in a similar way to (4.6) we prove that $\int_{\mathbb{R}^2} |\nabla u_\varepsilon(x)|^2\,dx \to 8\pi$ as $\varepsilon \to 0^+$. Therefore, the function $u_{\varepsilon,\delta}$ is readily obtained by means of a standard smoothing and rescaling argument applied to the angle function $\widehat{\theta}_\varepsilon$. □

4.3. **Minimal energy of maps with fixed degree.** In the proof of Theorem 4.1, we apply Proposition 4.2 and rely on the fact that the analogous problem for the Dirichlet energy is known. More precisely (cf. [23, Sec. 4.1]), denoting

$$\mathcal{G}_{\mathrm{d}} := \{u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{S}^2) \mid Du \in L^2(\mathbb{R}^2, \mathbb{R}^{3\times 2}),\ \deg u = \mathrm{d}\} \tag{4.7}$$

and recalling formula (2.9), by the lower semicontinuity property (2.13) and Proposition 4.2 it turns out that

$$\forall\,\mathrm{d} \in \mathbb{Z}\,, \quad \inf_{u \in \mathcal{G}_{\mathrm{d}}} \mathbb{D}(u, \mathbb{R}^2) = 4\pi\,|\mathrm{d}|\,. \tag{4.8}$$

*Proof of Theorem 4.1.* For $\mathrm{d} = 0$ the claim is trivial, whereas the cases $\mathrm{d} = \pm 1$ follow from Proposition 1.1. Therefore, it clearly suffices to consider the case $\mathrm{d} \geq 2$.

Since $\mathcal{F}_{\mathrm{d}} \subset \mathcal{G}_{\mathrm{d}}$, see (4.3) and (4.7), by inequality (4.4) and formula (4.8) we have

$$\inf_{u\in\mathcal{F}_{\mathrm{d}}} \mathbb{L}(u,\mathbb{R}^2) \geq 2 \cdot \inf_{u\in\mathcal{F}_{\mathrm{d}}} \mathbb{D}(u,\mathbb{R}^2) \geq 2 \cdot \inf_{u\in\mathcal{G}_{\mathrm{d}}} \mathbb{D}(u,\mathbb{R}^2) = 8\pi\,\mathrm{d}\,.$$

We now check the opposite inequality:

$$\inf_{u\in\mathcal{F}_{\mathrm{d}}} \mathbb{L}(u,\mathbb{R}^2) \leq 8\pi\,\mathrm{d}\,. \tag{4.9}$$

By Proposition 4.2, for each $\varepsilon > 0$ small enough we find a degree one map $u_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$, equal to $P_S$ outside the unit disk $B^2$, and such that

$$\mathbb{L}(u_\varepsilon,\mathbb{R}^2) = \int_{B^2} |\Delta u_\varepsilon|\,dx \leq 8\pi + \frac{\varepsilon}{\mathrm{d}}\,.$$

Denoting $\mathbf{e}_1 := (1,0)$, we define $w_\varepsilon(x) := u_\varepsilon(x - 3k\,\mathbf{e}_1)$ on the unit disk centered at $3k\,\mathbf{e}_1$, for $k = 0,1,\dots,\mathrm{d}-1$, and $w_\varepsilon \equiv P_S$ outside the union of such d disks. The map $w_\varepsilon$ satisfies $\mathbb{L}(w_\varepsilon,\mathbb{R}^2) \leq 8\pi\,\mathrm{d} + \varepsilon$ and belongs to the class $\mathcal{F}_{\mathrm{d}}$, whence (4.9) holds true, as required. □

4.4. **Bubbling-off of spheres.** We recall that the maps $u_{\varepsilon,\delta}$ from Proposition 4.2 have degree one. Therefore, letting e.g. $\varepsilon = \delta = 1/h$ we find a sequence $\{u_h\} \subset C^\infty(\mathbb{R}^2,\mathbb{S}^2)$ of degree one maps weakly converging in $W^{2,1}$ to the constant map $P_S$, and such that

$$\lim_{h\to\infty} \mathbb{L}(u_h,\mathbb{R}^2) = \mathbb{L}(\sigma_2^{-1},\mathbb{R}^2) = 8\pi\ .$$

Furthermore, it turns out that the above convergence is uniform far from the origin, and that the graph currents $G_{u_h}$ weakly converge in $\mathcal{D}_2(\mathbb{R}^2\times\mathbb{S}^2)$ to the Cartesian current

$$T = G_{P_S} + \delta_O \times [\![\,\mathbb{S}^2\,]\!]$$

where $G_{P_S}$ is the graph current of the constant map equal to $P_S$ on $\mathbb{R}^2$.

A bubbling phenomenon occurs, and by Theorem 3.2 we infer that *the minimal Laplacean energy occurring for the formation of a* 2*-sphere is equal to* $8\pi$. This *energy quantization* property is detected if one defines the energy

$$\mathbb{L}(T) := 8\pi\,, \qquad T = G_{P_S} + \delta_O \times [\![\,\mathbb{S}^2\,]\!]\,. \tag{4.10}$$

We thus have a second order analogous to a similar feature concerning the conformal Dirichlet integral, where formula (2.2) yields that the minimum energy cost of a $\mathfrak{p}$-sphere is equal to $\mathcal{H}^{\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}})$, for any integer $\mathfrak{p} \geq 2$.

## 5. The Laplacean energy on Cartesian currents

In this section, we define in any dimension $n \geq 2$ a Laplacean energy functional on a suitable class of Cartesian currents in such a way that a weak sequential lower semicontinuity property holds true, Theorem 5.3.

Due to the embedding of $W^{2,1}(B^n, \mathbb{S}^2)$ into $W^{1,2}(B^n, \mathbb{S}^2)$, according to Definition 2.9 we give the following

**Definition 5.1.** We denote by $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents in $\text{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with underlying function $u_T$ in $W^{2,1}(B^n, \mathbb{S}^2)$.

**Remark 5.2.** For future use, given a map $u \in W^{2,1}(B^n, \mathbb{S}^2)$ we also denote

$$\mathcal{T}_u^{\mathbb{L}} := \{T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2) \text{ such that } u_T = u \text{ in (2.12)}\}\,. \tag{5.1}$$

By the explicit formula (2.23) for the relaxed Dirichlet energy, we infer that *the class $\mathcal{T}_u^{\mathbb{L}}$ is always non-empty.*

Theorems 3.2 and 4.1 suggest to introduce on the class $\text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ the *Laplacean energy functional*

$$\mathbb{L}(T) := \int_{B^n} |\Delta u_T|\,dx + 8\pi \cdot \mathbf{M}(L) \qquad \text{if (2.12) holds} \tag{5.2}$$

so that we have:

1. (1) $\mathbb{L}(T) < \infty$ for every $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$;
2. (2) $\mathbb{L}(G_u) = \mathbb{L}(u)$ if $T = G_u$ for some smooth map $u \in W^{2,1}(B^n, \mathbb{S}^2)$;
3. (3) in low dimension $n = 2$, formula (4.10) holds true.

Similarly to what happens for the Laplacean energy of $\sigma_2^{-1}$ among degree one maps, see (1.7), the term $8\pi \cdot \mathbf{M}(L)$ in (5.2) is the optimal energy contribution of the vertical term $L \times [\![ \mathbb{S}^2 ]\!]$ in (2.12). In fact, we are able to (partially) extend Theorem 3.2 as follows:

**Theorem 5.3.** *Let $n \geq 2$ and let $\{T_h\} \subset \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$ be such that $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ for some $T \in \text{cart}^{\mathbb{L}}(B^n \times \mathbb{S}^2)$. Then, we have:*

$$\mathbb{L}(T) \leq \liminf_{h\to\infty} \mathbb{L}(T_h), \qquad \textit{see } (5.2)\,.$$

*Proof.* According to the decomposition formula (2.12), we let $u_{T_h} = u_h$, $u_T = u_\infty$ and $T = T_\infty$, so that we have

$$T_h = G_{u_h} + L_h \times [\![ \mathbb{S}^2 ]\!] \tag{5.3}$$

for each $h \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. We also denote for every $h \in \overline{\mathbb{N}}$

$$\mathbb{L}(u_h, B) := \int_B |\Delta u_h|\,dx\,, \quad \mathbb{D}(u_h, B) := \frac{1}{2}\int_B |Du_h|^2\,dx\,, \quad \forall\, B \in \mathcal{B}(B^n)\,,$$

where $\mathcal{B}(B^n)$ is the $\sigma$-algebra of Borel subsets of $B^n$.

Let $L_h$ be the current in the decomposition formula (5.3) for $T_h$. Since $L_h \in \mathcal{R}_{n-2}(B^n)$, according to the notation in (2.3) we can write $L_h = [\![ \mathcal{L}_h, \xi_h, \theta_h ]\!]$, and assume that the $(n-2)$-rectifiable set $\mathcal{L}_h \subset B^n$ agrees with set $(L_h)$. We also denote by $\|L_h\|$ the Borel regular and finite measure $\|L_h\| := \theta_h\,\mathcal{H}^{n-2} \llcorner \mathcal{L}_h$, and consider the restriction $L_h \llcorner B$ given for every $B \in \mathcal{B}(B^n)$ by

$$\langle L_h \llcorner B; \omega\rangle := \int_B \theta\,\langle \omega; \xi\rangle\, d\|L_h\|\,, \qquad \omega \in \mathcal{D}^{n-2}(B^n)\,,$$

so that

$$\mathbf{M}(L_h \llcorner B) = \|L_h\|(B) = \int_{B\cap\mathcal{L}_h} \theta_h\, d\mathcal{H}^{n-2} < \infty \quad \forall\, B \in \mathcal{B}(B^n)\,, \quad h \in \overline{\mathbb{N}}\,.$$

We finally set for every $B \in \mathcal{B}(B^n)$ and $h \in \overline{\mathbb{N}}$

$$\mathbb{L}(T_h, B) := \mathbb{L}(u_h, B) + 8\pi\,\|L_h\|(B)\,, \quad \mathbb{D}(T_h, B) := \mathbb{D}(u_h, B) + 4\pi\,\|L_h\|(B)\,.$$

Since $\|L_\infty\|$ is a regular measure concentrated on a $(n-2)$-rectifiable set, we can find an open set of small measure (so that the energy $\mathbb{L}(u_\infty, \cdot)$ is small, by absolute continuity) that covers almost all the set where the measure $\|L_\infty\|$ is charged and in such a way that its boundary does not charge the measure $\mathbb{L}(T_\infty, \cdot)$. More precisely, for every $\varepsilon > 0$, we can find an open set $A_\varepsilon \subset B^n$ such that

(1) $\|L_\infty\|(B^n \setminus A_\varepsilon) \le \varepsilon$ ;
(2) $\mathbb{L}(T_\infty, \partial A_\varepsilon \cap B^n) = 0$ ;
(3) $\mathbb{L}(u_\infty, A_\varepsilon) \le \varepsilon$ .

We clearly have

$$\liminf_{h\to\infty} \mathbb{L}(T_h) \ge \liminf_{h\to\infty} \mathbb{L}(T_h, B^n \setminus A_\varepsilon) + \liminf_{h\to\infty} \mathbb{L}(T_h, A_\varepsilon)$$

where by (5.2) and properties (2)–(3) the first term in the right-hand side is bounded from below as follows:

$$\liminf_{h\to\infty} \mathbb{L}(T_h, B^n\setminus A_\varepsilon) \ge \liminf_{h\to\infty} \mathbb{L}(u_h, B^n\setminus A_\varepsilon) \ge \mathbb{L}(u_\infty, B^n\setminus A_\varepsilon) \ge \mathbb{L}(u_\infty) - \varepsilon\,.$$

As to the second term, using the lower bound (1.6) and the sequential weak lower semicontinuity of the Dirichlet integral $\mathbb{D}(T)$ in the class $\mathrm{cart}^{2,1}(A_\varepsilon \times \mathbb{S}^2)$, see (2.13), we have

$$\liminf_{h\to\infty} \mathbb{L}(T_h, A_\varepsilon) \ge 2\,\liminf_{h\to\infty} \mathbb{D}(T_h, A_\varepsilon) \ge 2\,\mathbb{D}(T_\infty, A_\varepsilon)$$

where by property (1), using that $\mathbf{M}(L_\infty) = \|L_\infty\|(B^n) = \|L_\infty\|(A_\varepsilon) + \|L_\infty\|(B^n \setminus A_\varepsilon)$, we get

$$\mathbb{D}(T_\infty, A_\varepsilon) = \mathbb{D}(u_\infty, A_\varepsilon) + 4\pi\,\|L_\infty\|(A_\varepsilon) \ge 4\pi\,\left(\mathbf{M}(L_\infty) - \varepsilon\right)\,.$$

We thus obtain the estimate

$$\liminf_{h\to\infty} \mathbb{L}(T_h) \ge \mathbb{L}(u_\infty) + 8\pi\,\mathbf{M}(L_\infty) - (1+8\pi)\,\varepsilon = \mathbb{L}(T_\infty) - (1+8\pi)\,\varepsilon$$

and hence the lower semicontinuity inequality is proved by letting $\varepsilon \searrow 0$. $\square$

## 6. A strong density result

Let $n = 3$, and recall that a current $T$ in the class $\text{cart}^{\mathbb{L}}(B^3 \times \mathbb{S}^2)$ is given by $T = G_{u_T} + L \times [\![ \mathbb{S}^2 ]\!]$ for some $u_T \in W^{2,1}(B^3, \mathbb{S}^2)$ and some $L \in \mathcal{R}_1(B^3)$ such that $(\partial L) \llcorner B^3 = -\mathbb{P}(u_T)$, where the current of the singularities $\mathbb{P}(u_T) \in \mathcal{D}_0(B^3)$ is given by (2.14). In this section we prove the following:

**Theorem 6.1.** *Let $n = 3$ and $T \in \text{cart}^{\mathbb{L}}(B^3 \times \mathbb{S}^2)$. Then, there exists a sequence $\{u_k\} \subset C^\infty(B^3, \mathbb{S}^2)$ such that $u_k \to u_T$ strongly in $L^1(B^3, \mathbb{R}^3)$, the currents $G_{u_k}$ weakly converge to $T$ in $\mathcal{D}_3(B^3 \times \mathbb{S}^2)$, and $\mathbb{L}(u_k) \to \mathbb{L}(T)$ as $k \to \infty$.*

The proof of Theorem 6.1 is based on previous results and on the dipole-like construction contained in Proposition 6.3. We thus preliminary discuss the Dipole problem for the Laplacean energy (4.2).

### 6.1. The Dipole problem.

The classical Dipole problem by Brezis–Coron–Lieb [13] deals with Sobolev maps $u$ in $W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathbb{S}^2)$ which assume a given constant $P \in \mathbb{S}^2$ at infinity and which are smooth outside two singular points $a_\pm$, with

$$\deg(u, a_-) = -1\,, \qquad \deg(u, a_+) = +1$$

the degree being given by (2.20). In [13], it is shown that the minimal Dirichlet energy $\mathbb{D}(u, \mathbb{R}^3)$ in such class, see (2.9), is equal to the distance $|a_+ - a_-|$ between the singularities times the measure $4\pi$ of the unit sphere $\mathbb{S}^2$, compare [23, Sec. 4.2.3].

In this section, we discuss the Dipole problem for the Laplacean energy (4.2). We thus denote by $\mathcal{E}$ the subclass of maps $u$ as above that in addition belong to the second order space $W_{\mathbb{L}}(\mathbb{R}^3, \mathbb{S}^2)$, see (4.1).

**Theorem 6.2.** *We have:*

$$\inf\{\mathbb{L}(u, \mathbb{R}^3) \mid u \in \mathcal{E}\} = |a_+ - a_-| \cdot 8\pi\,.$$

*Proof.* The lower bound "$\geq$" readily follows, since by inequality (1.6) every map $u$ in $\mathcal{E}$ belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^3, \mathbb{S}^2)$ and hence by the Dipole problem for the Dirichlet energy we get

$$\mathbb{L}(u, \mathbb{R}^3) \geq 2\,\mathbb{D}(u, \mathbb{R}^3) \geq 2\,|a_+ - a_-| \cdot 4\pi\,.$$

We now prove the energy upper bound

$$\inf\{\mathbb{L}(u, \mathbb{R}^3) \mid u \in \mathcal{E}\} \leq |a_+ - a_-| \cdot 8\pi \tag{6.1}$$

by means of a Dipole insertion argument which is re-adapted by [23, Sec. 4.2.3], see also [13]. Without loss of generality, we may and do assume $P = P_S$ and $a_+ = (-r, 0, 0)$, $a_- = (r, 0, 0)$ for some $r > 0$.

Firstly, by Proposition 4.2 we choose a smooth map $v_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^2, \mathbb{S}^2)$ with degree one, equal to the South pole $P_S$ outside $B^2_r$ for some $r < 1$, and such that with $\widehat{z} = (z_2, z_3) \in \mathbb{R}^2$

$$\mathbb{L}(v_\varepsilon, B^2) := \int_{B^2} |\Delta v_\varepsilon(\widehat{z})|\, d\widehat{z} \leq 8\pi + \varepsilon\,. \tag{6.2}$$

In formula (6.6), we wish to obtain a Sobolev map $u_\varepsilon \in W_{\mathbb{L}}(\mathbb{R}^3,\mathbb{S}^2)$, so that $\Delta u_\varepsilon \in L^1(\mathbb{R}^3,\mathbb{R}^3)$. Therefore, we have to replace the Lipschitz-continuous function $t\mapsto \min\{r+t,r-t,\delta\}$ on the interval $D^1_r:=(-r,r)$ with a function at least of class $C^1(D^1_r)$. With $\delta>0$ small, we can choose:

$$\varphi_\delta(t):=\begin{cases}\delta & \text{if} \quad |t|\le r-\sqrt{2}\,\delta\\ \sqrt{\delta^2-\big(|t|-r+\sqrt{2}\,\delta\big)^2} & \text{if} \quad r-\sqrt{2}\,\delta\le |t|\le r-\dfrac{\delta}{\sqrt{2}}\\ r-|t| & \text{if} \quad r-\dfrac{\delta}{\sqrt{2}}\le |t|<r\,.\end{cases} \tag{6.3}$$

For $x=(\widetilde{x},\widehat{x})\in\mathbb{R}\times\mathbb{R}^2\simeq\mathbb{R}^3$ and $z=(\widetilde{z},\widehat{z})\in D^1_r\times B^2$, we let

$$(\widetilde{x},\widehat{x})=\Phi_\delta(\widetilde{z},\widehat{z}):=(\widetilde{z},\varphi_\delta(\widetilde{z})\,\widehat{z}) \tag{6.4}$$

and define $u_{\varepsilon,\delta}(x):=\widehat{v}_\varepsilon(\Phi_\delta^{-1}(x))$ for $x\in\Phi_\delta(D^1_r\times B^2)$, where $\widehat{v}_\varepsilon(z):=v_\varepsilon(\widehat{z})$, so that (6.2) holds. We then compute:

$$\begin{aligned}\Delta u_{\varepsilon,\delta}(x)= \quad & \frac{1}{\varphi_\delta(\widetilde{z})^2}\Delta\widehat{v}_\varepsilon(z)+\frac{{\varphi'_\delta}^2}{{\varphi_\delta}^4}(\widetilde{z})\sum_{\alpha,\beta=2}^3 x_\alpha x_\beta\partial^2_{\alpha,\beta}\widehat{v}_\varepsilon(z)\\ &+\frac{2{\varphi'_\delta}^2-\varphi_\delta\varphi''_\delta}{{\varphi_\delta}^3}(\widetilde{z})\sum_{\alpha=2}^3 x_\alpha\partial_\alpha\widehat{v}_\varepsilon(z)\end{aligned}$$

where $z=\Phi_\delta^{-1}(x)$, so that $\widetilde{x}=\widetilde{z}$ and $\widehat{x}=\varphi_\delta(\widetilde{z})\,\widehat{z}$. Using that $\det D\Phi_\delta(z)=\varphi_\delta(\widetilde{z})^2$, we get:

$$\begin{aligned}\det D\Phi_\delta(z)\cdot\Delta u_{\varepsilon,\delta}(x)= \quad & \Delta v_\varepsilon(\widehat{z})+\varphi'_\delta(\widetilde{z})^2\sum_{\alpha,\beta=2}^3 z_\alpha z_\beta\partial^2_{\alpha,\beta}v_\varepsilon(\widehat{z})\\ &+(2{\varphi'_\delta}^2-\varphi_\delta\varphi''_\delta)(\widetilde{z})\sum_{\alpha=2}^3 z_\alpha\partial_\alpha v_\varepsilon(\widehat{z})\,.\end{aligned}$$

Therefore, since $\|\varphi'_\delta\|_{\infty,D^1_r}\le 1$ and $\|2{\varphi'_\delta}^2-\varphi_\delta\varphi''_\delta\|_{\infty,D^1_r}\le 4$, by changing variables $z=\Phi_\delta^{-1}(x)$ we can estimate:

$$\begin{aligned}\int_{\Phi_\delta(D^1_r\times B^2)}|\Delta u_{\varepsilon,\delta}|\,dx\le &\int_{(-r,r)\times B^2}|\Delta v_\varepsilon(\widehat{z})|\,d\widetilde{z}\,d\widehat{z}\\ &+16\int_{(r-\sqrt{2}\delta,r)\times B^2}\Big(\sum_{\alpha,\beta=2}^3|z_\alpha z_\beta||\partial^2_{\alpha,\beta}v_\varepsilon(\widehat{z})|+|\widehat{z}||\nabla v_\varepsilon(\widehat{z})|\Big)\,d\widetilde{z}\,d\widehat{z}\,.\end{aligned}$$

Since moreover $v_\varepsilon\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ is smooth, the integral in the second line is small for $\delta>0$ small, whence we can find $\delta(\varepsilon)\in(0,r/2)$ such that

$$\int_{\Phi_\delta(D^1_r\times B^2)}|\Delta u_{\varepsilon,\delta(\varepsilon)}|\,dx\le 2r\cdot\mathbb{L}(v_\varepsilon,B^2)+\varepsilon\,,\quad \text{see (6.2)}\,. \tag{6.5}$$

Recall that the map $v_\varepsilon$ is equal to the pole $P_S$ in a neighborhood of the boundary of $B^2$. Therefore, setting $\delta=\delta(\varepsilon)$ and

$$u_\varepsilon(x):=\begin{cases} u_{\varepsilon,\delta}(x) & \text{if} \quad x\in\Phi_\delta(D^1_r\times B^2) \\ P_S & \text{if} \quad x\in\mathbb{R}^3\setminus\big(\Phi_\delta(D^1_r\times B^2)\cup\{(\pm r,0,0)\}\big)\end{cases} \tag{6.6}$$

it turns out that the map $u_\varepsilon$ belongs to the class $\mathcal{E}$, whereas by (6.5)

$$\mathbb{L}(u_\varepsilon,\mathbb{R}^3)=\int_{\Phi_\delta(D^1_r\times B^2)}|\Delta u_{\varepsilon,\delta}|\,dx\le 2r\cdot 8\pi+(2r+1)\,\varepsilon\,.$$

The energy upper bound (6.1) follows by letting $\varepsilon\searrow 0$. ☐

6.2. **A further dipole-like property.** We now consider the following variant of Theorem 6.2. Recalling (6.3), for each $m>0$ small we denote

$$\Phi^m_\delta(\widetilde{x},\widehat{x}):=(\widetilde{x},m\,\varphi_\delta(\widetilde{x})\,\widehat{x})\,,\quad \Omega^m_\delta:=\Phi^m_\delta(D^1_r\times B^2)\,.$$

**Proposition 6.3.** *Let $U\subset\mathbb{R}^3$ be a neighborhood of the segment joining $a_-$ to $a_+$, and let $u:U\to\mathbb{S}^2$ be a $W^{2,1}$-map which is smooth in $U$ outside the singular points $a_\pm$, where it has degree $\deg(u,a_\pm)=k_\pm$ for some $k_\pm\in\mathbb{Z}$, see (2.20). Let $\mathrm{d}\in\mathbb{Z}$. Then for all positive $\varepsilon$ and for $\delta$, $m>0$ sufficiently small there exists a smooth function $u_\varepsilon:\mathbb{R}^3\setminus\{a_-,a_+\}\to\mathbb{S}^2$ such that $u_\varepsilon\equiv u$ on $U\setminus\Omega^m_\delta$, $\deg(u,a_+)=k_+-\mathrm{d}$, $\deg(u,a_-)=k_-+\mathrm{d}$, and*

$$\mathbb{L}(u_\varepsilon,\Omega^m_\delta):=\int_{\Omega^m_\delta}|\Delta u_\varepsilon|\,dx\le|a_+-a_-|\cdot 8\pi\,|\mathrm{d}|+\varepsilon\,.$$

*In addition, $u_\varepsilon\in W^{2,1}(U,\mathbb{S}^2)$ with*

$$\int_{\Omega^m_\delta}|D^2u_\varepsilon|\,dx\le C_2\cdot|a_+-a_-|\,|\mathrm{d}|+\varepsilon\,, \tag{6.7}$$

*where the constant $C_2$ only depends on the $W^{2,1}$ seminorm $\int_{\mathbb{R}^2}|D^2{\sigma_2}^{-1}|\,dx$ of ${\sigma_2}^{-1}$.*

*Proof.* The first assertion is obtained by readapting an argument taken from [21], compare [23, Sec. 4.2.3], on account of Theorem 6.2. In particular, we have to replace the map $v_\varepsilon$ satisfying (6.2) with a smooth degree d map $v_\varepsilon\in W_{\mathbb{L}}(\mathbb{R}^2,\mathbb{S}^2)$ that is equal to the South pole $P_S$ outside $B^2_r$, as in the proof of Theorem 4.1, in such a way that this time

$$\mathbb{L}(v_\varepsilon,B^2):=\int_{B^2}|\Delta v_\varepsilon(\widehat{x})|\,d\widehat{x}\le 8\pi\,|d|+\varepsilon\,.$$

The second assertion follows from a similar estimate concerning the $W^{2,1}$ seminorm of $u_\varepsilon$ on $\Omega^m_\delta$. Further details are omitted. ☐

6.3. **The strong density theorem.** We are now in position to prove Theorem 6.1, where we adapt the argument used in the case of the Dirichlet energy analyzed in [20]. Therefore, we rely on Proposition 2.16, on the density theorem 2.18, on the dipole-like construction in Proposition 6.3, and on the cohomological criterion for strong density of smooth maps, Theorem 2.19. For these reasons, we refer to Sec. 4.2.5 and Sec. 4.2.6 in [23] for further details.

*Proof of Theorem 6.1.* Since $|\Delta u(x)| \leq c\,|D^2 u(x)|$ for a.e. $x \in B^3$ and every $u \in W^{2,1}(B^3, \mathbb{R}^3)$, where $c > 0$ is an absolute real constant, by dominated convergence we infer that if $\{u_h\} \subset W^{2,1}(B^3, \mathbb{S}^2)$ is a sequence $W^{2,1}$-strongly converging to $u \in W^{2,1}(B^3, \mathbb{S}^2)$, then $\mathbb{L}(u_h) \to \mathbb{L}(u)$ as $h \to \infty$. Moreover, since the weak convergence of Cartesian currents is metrizable (cf. [23, Sec. 5.3.2]), we can apply a diagonal procedure.

*Step 1.* By Theorem 2.18 and Proposition 2.16, we can find a sequence $\{u_k\} \subset R_0^\infty(B^3, \mathbb{S}^2)$ strongly converging to $u$ in $W^{2,1}(B^3, \mathbb{R}^3)$, with $\mathbb{L}(u_k) \to \mathbb{L}(u)$ and $\mathbf{m}_{i,B^3}(\mathbb{P}(u_k) - \mathbb{P}(u)) \to 0$. We thus reduce to the case in which $T = G_u + S_T$, where $u \in R_0^\infty(B^3, \mathbb{S}^2)$ and $S_T = L \times [\![ \mathbb{S}^2 ]\!]$ for some i.m. rectifiable current $L \in \mathcal{R}_1(\mathbb{R}^3)$ with support contained in the closure $\bar{B}^3$ and with boundary satisfying $(\partial L) \llcorner B^3 = -\mathbb{P}(u)$. Moreover, $\mathbb{P}(u)$ is an i.m. rectifiable current in $\mathcal{R}_0(B^3)$ with finite mass, so that formula (2.21) holds.

*Step 2.* Assume now in addition that $\mathbf{M}(\partial L) < \infty$, so that $L$ is an integral current in $\mathbb{R}^3$. As in Steps 1–3 of the proof of Thm. 1 from [23, Sec. 4.2.5], we reduce to the case in which

$$S_T = \sum_{\mathrm{d} \in I} P^{\mathrm{d}} \times \mathrm{d}\, [\![ \mathbb{S}^2 ]\!]$$

where $I$ is a finite set of integer indices and the $P^{\mathrm{d}}$'s are polyhedral lines in $B^3$ with pairwise disjoint supports. Notice in fact that at this step we apply Federer's strong polyhedral approximation theorem (cf. Thm. 4 in [22, Sec. 2.2.6]) and we deal with left compositions $u \circ \varphi$ with smooth diffeomorphisms $\varphi$ of $B^3$ into itself, that preserve the membership of $u \circ \varphi$ to the class $R_0^\infty(B^3, \mathbb{S}^2)$.

Let now $S_i$ be anyone of the segments of the $P^{\mathrm{d}}$'s, and let $[\![ S_i ]\!] = [\![ (n_i, p_i) ]\!]$. By a suitable change of coordinates we can assume that $n_i = a_+$ and $p_i = a_-$, as in Sec. 4. We then apply the dipole-like construction in Proposition 6.3. To this aim, notice that we can take $m$ and $\delta$ sufficiently small so that the neighborhoods $\Omega_\delta^m$ corresponding to different segments $S_i$ are pairwise disjoint and contained in $B^3$. We then replace $u$ in a small neighborhood of each $S_i$ by a function $u_\varepsilon \in W^{2,1}(B^3, \mathbb{S}^2)$ satisfying

$$\mathbb{L}(u_\varepsilon) \leq \mathbb{L}(u) + \sum_{d \in I} \mathbf{M}(P^{\mathrm{d}}) \cdot 8\pi\, |\mathrm{d}| + \varepsilon\,.$$

On account of the local upper bound (6.7), the function $u_\varepsilon$ this way obtained satisfies

$$\int_{B^3} |D^2 u_\varepsilon|\,dx \le \int_{B^3} |D^2 u|\,dx + C_2 \cdot \mathbf{M}(P^{\mathrm{d}}) + \varepsilon$$

and hence $u_\varepsilon \in W^{2,1}(B^3,\mathbb{S}^2)$. Moreover, *it has degree zero around each end point of the segments* $S_i$ which belongs to the open ball $B^3$, see (2.20), i.e. around each singular point of $u_\varepsilon$, whence $\mathbb{P}(u_\varepsilon)=0$, see (2.14). Also, taking $\delta \searrow 0$ as $\varepsilon \searrow 0$, it turns out that $G_{u_\varepsilon} \rightharpoonup T$ weakly in $\mathcal{D}_3(B^3\times\mathbb{S}^2)$ as $\varepsilon \to 0$ along a sequence.

Since $\mathbb{P}(u_\varepsilon)=0$, by Theorem 2.19 we find a sequence $\{u_k^\varepsilon\} \subset C^\infty(B^3,\mathbb{S}^2)$ such that $u_k^\varepsilon \to u_\varepsilon$ strongly in $W^{2,1}(B^3,\mathbb{R}^3)$ and $\mathbb{L}(u_k^\varepsilon) \to \mathbb{L}(u)$, so that $G_{u_k^\varepsilon} \rightharpoonup G_{u_\varepsilon}$ weakly in $\mathcal{D}_3(B^3\times\mathbb{S}^2)$. Therefore, a diagonal argument concludes the proof in case $\mathbf{M}(\partial L)<\infty$.

*Step 3.* After Step 1, in general we only have $\mathbf{M}((\partial L) \llcorner B^3) < \infty$ and therefore we cannot apply the strong polyhedral approximation theorem. We thus make use of a slicing argument similar to the one in the proof of Thm. 1 from [23, Sec. 4.2.6]. More precisely, denoting by $B^3_r$ the ball of radius $r$ centered at the origin, we may and do choose an increasing sequence $\{r_h\}$ such that $r_h \nearrow 1$ and $\mathbf{M}(\partial(L \llcorner B^3_{r_h})) < \infty$ for every $h$, i.e. the restriction $L \llcorner B^3_{r_h}$ is an integral current in $\mathbb{R}^3$.

Let $T_h := \psi_{h\#}(T \llcorner B^3_{r_h} \times \mathbb{S}^2)$, where $\psi_h(x,y)=(\varphi_h(x),y)$, with $\varphi_h(x) := x/r_h$, $x \in B^3_{r_h}$. It turns out that $\{T_h\}$ is a sequence of currents in $\mathrm{cart}^{\mathbb{L}}(B^3\times\mathbb{S}^2)$ such that $T_h \rightharpoonup T$ and $\mathbb{L}(T_h)\to\mathbb{L}(T)$, where $T_h = G_{u_h} + L_h \times \mathbb{S}^2$ with $u_h = u\circ\varphi_h$ and $L_h = \varphi_{h\#}(L \llcorner B^3_{r_h})$. Since $L_h$ is an integral current in $\mathbb{R}^3$, we can apply Step 2 in order to approximate for each $h$ the current $T_h$ by a sequence $\{G_{u_{h_k}}\}_k$ of smooth graphs. A further diagonal argument concludes the proof. □

## 7. Main result, final remarks and open questions

In this section, we obtain the explicit formula of the relaxed energy (1.8) for Sobolev maps in $W^{2,1}(B^3,\mathbb{S}^2)$. We then prove the energy lower bound (1.15) and collect some final remarks and open questions.

With the notation from (2.14) and (2.18), Theorem 1.2 states:

$$\forall\, u \in W^{2,1}(B^3,\mathbb{S}^2)\,, \quad \widetilde{\mathbb{L}}(u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty\,.$$

Notice that in terms of currents, and on account of definition (5.1) and Remark 5.2, the proof given below implies that the latter formula is equivalent to:

$$\widetilde{\mathbb{L}}(u) = \min\big\{\mathbb{L}(T) \mid T \in \mathcal{T}_u^{\mathbb{L}}\big\} \quad \forall\, u \in W^{2,1}(B^3,\mathbb{S}^2)\,. \tag{7.1}$$

*Proof of Theorem 1.2.* We first obtain in any dimension $n \ge 3$ the energy lower bound

$$\widetilde{\mathbb{L}}(u) \ge \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in W^{2,1}(B^n,\mathbb{S}^2)\,. \tag{7.2}$$

Assume $\widetilde{\mathbb{L}}(u) < \infty$, and let $\{u_k\} \subset C^\infty(B^n, \mathbb{S}^2)$ be any sequence converging to $u$ in $L^1(B^n, \mathbb{R}^3)$ and such that $\sup_k \mathbb{L}(u_k) < \infty$. We thus have to show that

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \,. \tag{7.3}$$

Possibly passing to a (not relabeled) subsequence, we can assume that the lim inf in the latter formula is a finite limit. The lower bound (4.4) yields $\sup_k \mathbb{D}(u_k) < \infty$. Therefore, compare Sec. 2, a (not relabeled) subsequence of $G_{u_k}$ weakly converges in $\mathcal{D}_n(B^n \times \mathbb{S}^2)$ to some current $T$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$, see Definition 2.9. Moreover, the $L^1$-convergence $u_k \to u$ implies that the underlying function $u_T$ agrees with $u$, whence $T \in \mathcal{T}_u^{\mathbb{L}}$, see (5.1). Therefore, by (2.16) and (2.18) we infer:

$$\mathbb{L}(T) := \mathbb{L}(u) + 8\pi \cdot \mathbf{M}(L) \geq \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \,.$$

Since moreover $\mathbb{L}(u_k) = \mathbb{L}(G_{u_k})$ for each $k$, the lower semicontinuity theorem 5.3, where $T_k = G_{u_k}$, gives

$$\liminf_{k\to\infty} \mathbb{L}(u_k) \geq \mathbb{L}(T)$$

and hence inequality (7.3) readily follows.

Assume now $n = 3$. For any given $u \in W^{2,1}(B^3, \mathbb{S}^2)$, equality holds in (7.3) if we find a sequence $\{u_k\} \subset C^\infty(B^3, \mathbb{S}^2)$ such that $u_k \to u$ strongly in $L^1(B^3, \mathbb{R}^3)$ and

$$\lim_{k\to\infty} \mathbb{L}(u_k) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) \,.$$

To this aim, we first recall that the class $\mathcal{T}_u^{\mathbb{L}}$ is non-empty, see Remark 5.2, whence $\mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty$. Setting then $T_u = G_u + L_u \times [\![\, \mathbb{S}^2 \,]\!]$, where $L_u$ is a minimal integral connection of $\mathbb{P}(u)$, see Remark 2.11, it turns out that $T_u$ is an energy minimizing current in the class $\mathcal{T}_u^{\mathbb{L}}$, so that in particular

$$\mathbb{L}(T_u) = \mathbb{L}(u) + 8\pi \cdot \mathbf{m}_{i,B^3}(\mathbb{P}(u)) < \infty \,.$$

Therefore, the requested strong density property is given by Theorem 6.1, when applied to $T = T_u$. In particular, equation (7.1) holds true. □

7.1. **Extending the density theorem.** In dimension $n = 3$, the relaxed Laplacean energy of $W^{2,1}$ maps satisfying a suitable Dirichlet-type boundary condition can be treated similarly to the case analyzed here. For the sake of brevity we do not discuss this case, and we refer to [24] for the corresponding result concerning the Dirichlet energy.

In high dimension $n \geq 4$, the expected equality in the energy lower bound (7.2) would be proved by extending the strong approximation property from Theorem 6.1. However, since we are not able to prove Proposition 2.16 in the case $n \geq 4$, see Remark 2.17, we cannot argue as in the proof of Theorem 6.1. Moreover, the strategy from [24] doesn't work, since it is based on a partial regularity result for minimizers of the Dirichlet energy. On the other hand, the argument taken from [25] does not apply, too, since it strongly relies on slicing arguments that fail to hold for the Laplacean energy.

7.2. **The energy lower bound.** We now give a sketch of the proof of the energy lower bound (1.15) in any dimension $n \geq 3$.

*Proof of* (1.15). If $u$ is a map in $\mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, see (1.14), we have seen that the density $\widetilde{\Delta u}$ in (3.2) agrees with the approximate Laplacean $\Delta u$. As a consequence, the lower bound

$$|\mathrm{Div} Du|(B) \geq \int_B |\Delta u|\, dx \geq \int_B |Du|^2\, dx \tag{7.4}$$

holds true for each Borel set $B \in \mathcal{B}(B^n)$. As a consequence, denoting by $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$ the class of Cartesian currents $T = G_{u_T} + L \times [\![ \mathbb{S}^2 ]\!]$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$ with $u_T \in \mathbb{L}_{BV}(B^n, \mathbb{S}^2)$, and setting correspondingly

$$\mathbb{L}(T) := |\mathrm{Div} Du_T|(B^n) + 8\pi \cdot \mathbf{M}(L)\,,$$

it turns out that the lower semicontinuity theorem 5.3 continues to hold in the wider class $\mathrm{cart}^{\mathbb{L}_{BV}}(B^n \times \mathbb{S}^2)$. In fact, by using Proposition 2.2, when $n = 2$ it is obtained again as a consequence of Theorem 3.2, whereas in case $n \geq 3$ we can apply the same argument as in the proof Theorem 5.3, that essentially relies on the local inequality $|\mathrm{Div} Du|(B) \geq 2\, \mathbb{D}(u, B)$, which follows from (7.4). As a consequence, the lower bound (1.15) is obtained exactly as in the proof of (7.2). Further details are omitted. □

7.3. **Open questions.** As already mentioned in Sec. 3, even in dimension $n = 3$, we are not able to obtain an explicit formula of the relaxed energy in case of maps $u$ in $L^1(B^3, \mathbb{S}^2) \setminus W^{2,1}(B^3, \mathbb{S}^2)$ satisfying $\widetilde{\mathbb{L}}(u) < \infty$. On account of (1.9), since we are dealing with maps in $W^{1,2}(B^n, \mathbb{S}^2)$, formula

$$\widetilde{\mathbb{L}}(u) = |\mathrm{Div} Du|(B^n) + 8\pi \cdot \mathbf{m}_{i,B^n}(\mathbb{P}(u)) \quad \forall\, u \in \mathbb{L}(B^n, \mathbb{S}^2)$$

is expected to hold true in any dimension $n \geq 3$, i.e. no extra terms in correspondence to the singular part of the measure $\mathrm{Div} Du$ should appear.

For example, in order to prove the validity of the equality sign in (1.15) for maps in $\mathbb{L}_{BV}(B^3, \mathbb{S}^2)$, one should primarily extend Theorem 2.18, by showing for any $u \in \mathbb{L}_{BV}(B^3, \mathbb{S}^2)$ the existence of a sequence $\{u_h\} \subset R_0^\infty(B^3, \mathbb{S}^2)$ converging to $u$ in $W^{1,2}(B^3, \mathbb{R}^3)$ and such that $\mathbb{L}(u_h) \to |\mathrm{Div} Du|(B^3)$ as $h \to \infty$.

We recall that the optimal lower bound of the relaxed Laplacean energy follows from Theorem 5.3. This result may be compared to the lower semicontinuity property (2.13) of the Dirichlet energy functional $\mathbb{D}(T)$ in $\mathrm{cart}^{2,1}(B^n \times \mathbb{S}^2)$, that holds true since $\mathbb{D}(T)$ agrees with the parametric polyconvex lower semicontinuous extension of the Dirichlet integrand.

Concerning *second order* functionals as e.g. the Laplacean energy $\mathbb{L}(u)$, a part from the easier case of 1-dimensional currents, compare [1, 2], to our knowledge it is not clear how to apply the approach from [23] in order to find the explicit formula of the parametric polyconvex lower semicontinuous extension. Therefore, in the proof of Theorem 5.3 we have followed a different strategy, that is based on Proposition 1.1.

7.4. **On the Bienergy functional.** Similar difficulties appears in the case of the Bienergy functional $\mathbb{H}(u)$ of maps $u : B^n \to \mathbb{S}^4$, see (1.4). In fact, finding the expression of the parametric polyconvex lower semicontinuous extension of the Bienergy to Cartesian currents in $B^n \times \mathbb{S}^4$ would give us (in the critical dimension $n = 4$) the explicit value of the minimal Bienergy $E_4$ of degree one maps from $\mathbb{R}^4$ into $\mathbb{S}^4$, a non-trivial open problem. We recall that in [5] it is proved that such a minimum $E_4$ is attained, and that $E_4 > 16 \cdot \mathcal{H}^4(\mathbb{S}^4)$, the expected weight being $E_4 = 24 \cdot \mathcal{H}^4(\mathbb{S}^4)$.

We finally give some hints concerning a possible approach to the analysis of the explicit formula of the *relaxed Bienergy functional*. Similarly to (1.8), it is defined on maps $u \in L^1(B^n, \mathbb{S}^4)$ by the formula

$$\widetilde{\mathbb{H}}(u) := \inf\Big\{ \liminf_{h\to\infty} \mathbb{H}(u_h) \mid \{u_h\} \subset C^\infty(B^n, \mathbb{S}^4),\ u_h \to u \text{ in } L^1(B^n, \mathbb{R}^5) \Big\}\,,$$

where we assume $n \geq 4$, the case of low dimension $n \leq 3$ being trivially solved.

Under prescribed first order boundary conditions, by the Bochner inequality one infers that $\widetilde{\mathbb{H}}(u) < \infty$ if and only if $u \in W^{2,2}(B^n, \mathbb{S}^4)$. Furthermore, by (1.6) we have $W^{2,2}(B^n, \mathbb{S}^4) \subset W^{1,4}(B^n, \mathbb{S}^4)$. Therefore, in the critical dimension $n = 4$, due to the continuous embedding of $W^{1,4}(B^4)$ in VMO we get $\widetilde{\mathbb{H}}(u) = \mathbb{H}(u)$ for every $u \in W^{2,2}(B^4, \mathbb{S}^4)$. In addition, in dimension $n = 5$, by using a similar strategy to the one adopted in this paper it can be shown that

$$\widetilde{\mathbb{H}}(u) = \mathbb{H}(u) + E_4 \cdot \mathbf{m}_i(\mathbb{P}(u)) < \infty \quad \forall\, u \in W^{2,2}(B^5, \mathbb{S}^4)$$

where $\mathbf{m}_i(\mathbb{P}(u))$ denotes the integral mass of the 0-current $\mathbb{P}(u)$ of the singularities, that is defined on maps $u \in W^{1,4}(B^5, \mathbb{S}^4)$ in a similar way to the case $n = 3$ in (2.14), but in terms of a volume 4-form in $\mathbb{S}^4$. In high dimension $n \geq 6$, similarly to the inequality (1.15) concerning the Laplacean energy, we can only prove the energy lower bound

$$\widetilde{\mathbb{H}}(u) \geq \mathbb{H}(u) + E_4 \cdot \mathbf{m}_i(\mathbb{P}(u)) < \infty \quad \forall\, u \in W^{2,2}(B^n, \mathbb{S}^4)\,,$$

where the current $\mathbb{P}(u)$ of the singularities is an $(n-5)$-dimensional integral flat chain in $B^n$. However, as we mentioned before, the main open problem concerning the analysis of the relaxed Bienergy functional $\widetilde{\mathbb{H}}(u)$ is to find the explicit value of the energy quantization constant $E_4$.

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