LONG-TIME CONVERGENCE OF A NONLOCAL BURGERS' EQUATION TOWARDS THE LOCAL N-WAVE

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Abstract. We study the long-time behavior of the unique weak solution of a nonlocal regularization of the (inviscid) Burgers' equation where the velocity is approximated by a one-sided convolution with an exponential kernel. The initial datum is assumed to be positive, bounded, and integrable. The asymptotic profile is given by the "N-wave" entropy solution of the Burgers' equation. The key ingredients of the proof are a suitable scaling argument and a nonlocal Oleinik-type estimate.

1. Introduction

Let us consider the following nonlocal regularization of the Burgers' equation:

$$\begin{cases} \partial_t \rho(t,x) + \partial_x\big(W[\rho](t,x)\rho(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

supplemented by the nonlocal term

$$W[\rho](t,x) := \int_{-\infty}^{x} \exp(y-x)\rho(t,y)\,\mathrm{d}y, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}, \tag{1.2}$$

which also satisfies the identity

$$\partial_x W[\rho](t,x) = \rho(t,x) - W[\rho](t,x), \qquad (t,x)\in(0,+\infty)\times\mathbb{R}. \tag{1.3}$$

In what follows, we assume that the initial data satisfies

$$\rho_0 \in L^1(\mathbb{R};\mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \tag{1.4}$$

and introduce the notation $M := \int_{\mathbb{R}} \rho_0(x)\,\mathrm{d}x$ for its L^1-mass.

Under these assumptions, the nonlocal conservation law (1.1) has a unique global non-negative weak solution ρ. In particular, in contrast to the case of the local Burgers' equation, no entropy condition is required to select a unique weak solution; moreover, the regularity and integrability of the initial datum is essentially preserved along the evolution (see [28] and Section 2 below).

The main aim of this paper is to study its asymptotic behavior when $t \to +\infty$. The main result asserts that, as $t\to+\infty$, the solution $\rho(t,\cdot)$ of (1.1) converges to the (unique) *N-wave solution* (or *source-type solution*) w of the local Burgers' equation (see [36, Eq. (2.1)]), i.e., the solution of the Burgers' equation with a Dirac delta as initial data,

$$\begin{cases} \partial_t w(t,x) + \partial_x(w^2(t,x)) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ w(0,x) = M\delta_{\{x=0\}}(x), & x\in\mathbb{R}, \end{cases} \tag{1.5}$$

which is given explicitly by

$$w(t,x) = \begin{cases} \frac{x}{2t} & \text{if } x \in (0,\sqrt{4Mt}), \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

We refer to [36] (and to Section 2 below) for the proof that (1.5) does indeed have a unique entropy solution (which is given by (1.6)) under suitable assumptions.

More precisely, our main theorem on the long-time behavior of the solution of (1.1) can be stated as follows.

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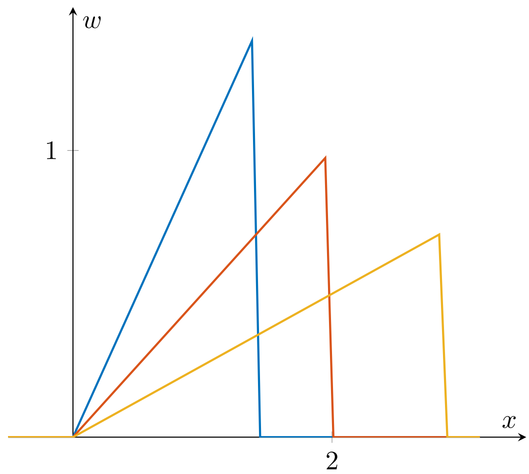

FIGURE 1. Plot of an N-wave solution (1.6) (with $M = 1$) for $t = 0.5$ (blue), $t = 1$ (red), and $t = 2$ (yellow).

Theorem 1.1 (Long-time asymptotics). *Let ρ_0 satisfy assumption* (1.4)*. Let ρ be the unique weak solution of the nonlocal Burgers equation* (1.1) *and let W be the corresponding nonlocal term. Then, for $p \in [1, +\infty)$, we have*

$$t^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|\rho(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} \to 0 \quad \text{and} \quad t^{\frac{1}{2}\left(1-\frac{1}{p}\right)}\|W(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} \to 0 \quad \text{as } t \to +\infty, \tag{1.7}$$

where w denotes the unique entropy solution (N-wave solution) of the Burgers equation (1.5) *defined in* (1.6)*.*

For local conservation laws with a general convex (or concave) flux function, the convergence of the entropy solution to the corresponding N-wave profile, as well as existence and uniqueness of entropy solutions to conservation laws with measure initial data, was first established rigorously in [36]. In [19, 18, 17, 16, 32, 31], the analysis was extended to classes of viscous conservation laws with flux $f(\xi) = \xi^{q-1}$ (for $\xi \in \mathbb{R}$). In case $1 < q < 2$, as $t \to +\infty$, the solutions converge to the N-wave profile of the inviscid conservation law; on the other hand, if $q = 2$, the limit profile is given by the fundamental solution of the viscous conservation law and, if $q > 2$, it is of Gaussian type (i.e., the fundamental solution of the heat equation with mass $\int_{\mathbb{R}} u_0 \,\mathrm{d}x$). Similar results have been also obtained for scalar conservation laws with fractional diffusion (see [3, 27, 15]). For the multi-dimensional setting, we also refer to the recent work [37], where is was proved that a multidimensional Burgers-type equation with a Dirac delta distribution as initial data is not well-posed (despite the L^1–L^∞ smoothing effect established in [38]).

For nonlocal conservation laws, this problem has not been considered in the literature. However, interestingly, it can be reduced to a type of nonlocal-to-local singular limit problem that has attracted much attention in recent years. Indeed, following [36], given $\lambda > 0$, we consider the rescaled function

$$\rho_\lambda(t, x) := \lambda\rho(\lambda^2 t, \lambda x), \tag{1.8}$$

which solves[1]

$$\begin{cases} \partial_t\rho_\lambda(t,x) + \partial_x\big(W_\lambda[\rho_\lambda](t,x)\rho_\lambda(t,x)\big) = 0, & (t,x) \in (0,+\infty)\times\mathbb{R},\\ \rho_\lambda(0,x) = \rho_{0,\lambda}(x) := \lambda\rho_0(\lambda x), & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

[1]Note that

$$W[\rho](\lambda^2 t, \lambda x) = \int_{-\infty}^{\lambda x} \exp(y - \lambda x)\rho(\lambda^2 t, y)\,\mathrm{d}y = \int_{-\infty}^{x} \exp(\lambda(z - x))\lambda\rho(\lambda^2 t, \lambda z)\,\mathrm{d}y = \frac{1}{\lambda}W_\lambda[\rho_\lambda](t, x).$$

Then

$$\begin{aligned} \partial_t\rho(t,x) + \partial_x\big(W[\rho](t,x)\rho(t,x)\big) = 0 &\iff \partial_t\rho(\lambda^2 t,\lambda x) + \lambda\partial_x\big(W[\rho](\lambda^2 t,\lambda x)\rho(\lambda^2 t,\lambda x)\big) = 0\\ &\iff \partial_t\lambda\rho(\lambda^2 t,\lambda x) + \lambda\partial_x\big(W_\lambda[\rho_\lambda](t,x)\rho(\lambda^2 t,\lambda x)\big) = 0\\ &\iff \partial_t\rho_\lambda(t,x) + \partial_x\big(W_\lambda[\rho_\lambda](t,x)\rho_\lambda(t,x)\big) = 0. \end{aligned}$$

with

$$W_\lambda[\rho_\lambda](t,x) := \lambda \int_{-\infty}^{x} \exp(\lambda(y-x))\rho_\lambda(t,y)\,\mathrm{d}y, \qquad (t,x) \in (0,+\infty)\times\mathbb{R}, \tag{1.10}$$

which satisfies

$$\lambda^{-1}\partial_x W_\lambda[\rho_\lambda](t,x) = \rho_\lambda(t,x) - W_\lambda[\rho_\lambda](t,x), \qquad (t,x)\in[0,+\infty)\times\mathbb{R}. \tag{1.11}$$

We shall prove that, for a fixed $t>0$, $\rho_\lambda(t,\cdot)\to w(t,\cdot)$ in $L^1(\mathbb{R})$ as $\lambda\to\infty$, which, in turn, will be shown to yield

$$\|\rho(t,\cdot)-w(t,\cdot)\|_{L^1(\mathbb{R})}\to 0 \quad \text{as } t\to+\infty$$

with w defined in (1.6) (and, by interpolation, the claim in Theorem 1.1).

This type of singular limit problem has been intensively studied in the case of initial data that are uniformly bounded with respect to the scaling parameter. First, in [2], it has been observed that, at least numerically, there is some hope that the solution of a nonlocal conservation law converges to the entropy solution of the corresponding local problem when the nonlocal term approaches a Dirac delta. Positive results in this direction were obtained in [29] for a large class of nonlocal conservation laws under the assumption of having monotone initial data; in [13] under the assumption that the initial datum has bounded total variation, is bounded away from zero and satisfies a one-sided Lipschitz condition. For the case of an exponential weight, in [7, 8], Bressan and Shen proved a convergence result under the assumption that the initial datum is bounded away from zero and has bounded total variation. The core of their argument is the observation that, under suitable changes of variables, the nonlocal problem can be rewritten as a hyperbolic system with relaxation terms. The assumption on the initial data being bounded away from zero played a key role in showing a uniform total variation bound for the solution of the nonlocal problem. Indeed, in [13], a counterexample shows that the total variation of the solution may blow up if the data is not bounded away from zero.

On the other hand, in [11], by arguing on the nonlocal term W rather than on the solution of the conservation law, it was possible to remove the additional assumption on the initial data – the key observation being that the nonlocal term W enjoys further regularity and, in particular, its total variation remains uniformly bounded. From the compactness of the sequence of nonlocal terms, it is then possible to deduce the convergence for the sequence of solutions as well. This approach was later adapted in [14] to classes of weights more general than the exponential one.

The difference and substantial added difficulty of the present contribution compared to the above-mentioned works is that, under the scaling transformation, we are considering initial data that concentrate to a Dirac delta distribution: i.e.,

$$\rho_\lambda(0,\cdot)\to M\delta_0 \quad \text{and} \quad W_\lambda(0,\cdot)\to M\delta_0 \tag{1.12}$$

in the sense of distributions as $\lambda\to+\infty$. That is, with respect to λ, the only uniform bound for the initial data ρ_0 is given in terms the L^1-mass.

To overcome this difficulty, we take advantage of an Oleinik-type inequality satisfied by the nonlocal term $W_\lambda[\rho_\lambda]$. Indeed, from [11], it is known that we can rewrite (1.9) as a conservation law with nonlocal source formulated purely in $W_\lambda[\rho_\lambda]$ (see (2.2) below). This motivates using the notation W_λ instead of $W_\lambda[\rho_\lambda]$ in what follows. From (2.2), arguing as in [10], we can deduce the Oleinik-type estimate (see Theorem 3.2):

$$\frac{W_\lambda(t,x)-W_\lambda(t,y)}{x-y}\le\frac{1}{t}, \qquad t>0, \quad x,y\in\mathbb{R},\ x\neq y.$$

Combining it with the uniform L^1-bound

$$\int_{\mathbb{R}} W_\lambda(t,x)\,\mathrm{d}x = M, \qquad t>0,$$

this inequality yields an L^∞-bound for $t>0$ (see Lemma 3.3):

$$0\le W_\lambda(t,x)\lesssim\sqrt{Mt^{-1}}, \qquad (t,x)\in(0,+\infty)\times\mathbb{R}.$$

With these ingredients, the approach of [16] leads to the claimed convergence of $\{W_\lambda\}_{\lambda>0}$ towards the N-wave solution of the (local) Burgers' equation and, thanks to (1.3), to the convergence of $\{\rho_\lambda\}_{\lambda>0}$ as well.

The paper is organized as follows. In Section 2, we recall the necessary preliminaries on the well-posedness of (1.9) (for fixed $\lambda > 0$). In Section 3, we prove the key and a priori estimates on W_λ sketched above. Then, in Section 4, we combine them and establish the convergence of $\{\rho_\lambda\}_{\lambda>0}$ and $\{W_\lambda\}_{\lambda>0}$ to the N-wave solution of the local Burgers' equation as $\lambda \to +\infty$; or, equivalently, of $\{\rho(t,\cdot)\}_{t>0}$ and $\{W(t,\cdot)\}_{t>0}$ as $t \to +\infty$. This convergence result is illustrated by several numerical simulations in Section 5 (together with some further conjectures). Finally, in Section 6, we conclude the paper by presenting some open problems.

2. Preliminaries

For the nonlocal conservation law in (1.9), we recall the following well-posedness result and some fundamental properties of the solution. We refer to [11, Theorem 2.1 & Lemma 3.1] (which, in turn, relies in part on [28, Theorem 2.20 & Theorem 3.2 & Corollary 4.3] or [12, Theorem 2.1 & Corollary 2.1]), [22, Theorem 2.1], [14, Proposition 2.1 & Corollary 2.2], or [10] for the proof of a similar statement.

Theorem 2.1 (Existence and uniqueness of weak solutions and maximum principle)**.** *Let assumptions* (1.4) *hold. Then, for every $\lambda > 0$, there exists a unique weak solution $\rho_\lambda \in C\big([0,+\infty); L^1(\mathbb{R})\big) \cap L^\infty((0,+\infty); L^\infty(\mathbb{R}))$ of the nonlocal Burgers' equation* (1.9) *and the following maximum principle holds*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_{0,\lambda}(x) \le \rho_\lambda(t,x) \le \|\rho_{0,\lambda}\|_{L^\infty(\mathbb{R})}, \qquad a.e.\ (t,x) \in (0,+\infty)\times\mathbb{R}. \tag{2.1}$$

Moreover, for the nonlocal term W_λ, the following properties hold:

1. $W_\lambda \in W^{1,\infty}((0,+\infty)\times\mathbb{R})$ *and* $\operatorname*{ess\,inf}_{x\in\mathbb{R}} \rho_{0,\lambda}(x) \le W_\lambda \le \|\rho_{0,\lambda}\|_{L^\infty(\mathbb{R})}$*;*
2. $W_\lambda \in C^0\big((0,+\infty); L^1(\mathbb{R})\big)$*; in particular, if* $\|\rho_\lambda(t,\cdot)\|_{L^1(\mathbb{R})} = M$*, then* $\|W_\lambda(t,\cdot)\|_{L^1(\mathbb{R})} = M$*;*
3. *if* $\rho_{0,\lambda} \in C^k(\mathbb{R})$*, then* $W_\lambda \in C^{k+1}((0,+\infty)\times\mathbb{R})$ *for* $k \ge 0$*.*

Furthermore, W_λ satisfies the following transport equation with nonlocal source in the strong sense:

$$\begin{cases} \partial_t W_\lambda(t,x) + \partial_x(W_\lambda^2(t,x)) \\ \quad = \lambda \displaystyle\int_{-\infty}^{x} \exp(\lambda(y-x))\Big(W_\lambda(t,x)\partial_x W_\lambda(t,x) - W_\lambda(t,y)\partial_y W_\lambda(t,y)\Big)\,\mathrm{d}y, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ W_\lambda(0,x) = \lambda \displaystyle\int_{-\infty}^{x} \exp(\lambda(y-x))\rho_{0,\lambda}(y)\,\mathrm{d}y, & x\in\mathbb{R}. \end{cases} \tag{2.2}$$

For the limit problem (1.5), we rely on a more general well-posedness result from [36, Theorem 1.1 & Remark 1.1].

Theorem 2.2 (Non-negative solutions with measure initial data)**.** *Let us consider the local conservation law*

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, & (t,x)\in(0,+\infty)\times\mathbb{R}, \\ u(0,x) = \mu, & x\in\mathbb{R}. \end{cases} \tag{2.3}$$

Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous with $f(0) = 0$ and $f([0,\infty)) \subset [0,\infty)$ and that μ is a non-negative finite measure on $\mathbb{R}$. Then there exists at most one non-negative solution $u \in C\big((0,+\infty); L^1(\mathbb{R})\big) \cap L^\infty((\tau,+\infty)\times\mathbb{R})$, for all $\tau \in (0,+\infty)$, which satisfies the Kružkov entropy condition, i.e.

$$\begin{gathered} \forall k \in \mathbb{R},\ \forall \psi \in C_c^\infty((0,+\infty)\times\mathbb{R}), \quad \psi \ge 0: \\ \int_0^{+\infty}\int_{\mathbb{R}} |u-k|\partial_t\psi + \operatorname{sign}(u-k)(\varphi(u)-\varphi(k))\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t \ge 0, \end{gathered}$$

and achieves the initial datum in the narrow (or weak) sense of measures[2],

$$\lim_{t\to0} u(t,\cdot) = \mu \qquad \textit{narrowly in } \mathbb{R}.$$

[2]A sequence of signed Radon measures $\{\mu_n\}_{n\in}$ on $\mathbb{R}$ *converges narrowly* (or *in the weak sense*) to μ if, for all bounded and continuous test functions $\varphi \in C_b(\mathbb{R})$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}} \varphi\,\mathrm{d}\mu_n = \int_{\mathbb{R}} \varphi\,\mathrm{d}\mu.$$

See [4, Chapter 8].

In particular, in our setting, Theorem 2.2 yields the uniqueness of the N-wave entropy solution (1.6) of (1.5).

Remark 2.3 (Non-negativity condition and uniqueness). As noted in [36, Remark 1.2], the uniqueness result in Theorem 2.2 fails without the assumption of non-negativity for the solutions. This hypothesis can, however, be replaced by taking $f(u) = \mathrm{sign}(u)|u|^q$ (with $q > 1$) or by $f(u) = |u|^q$ and assuming that the initial datum is achieved in a stronger sense (as shown in [36, Theorem 1.2] and [36, Theorem 1.3] respectively).

3. A priori estimates

Before presenting our key a priori estimates, let us recall the following stability result of the nonlocal conservation law (1.1) with respect to the initial datum (see [10]).

Lemma 3.1 (Stability of the nonlocal term with respect to the initial datum)**.** *Let $\rho_{0,1},\ \rho_{0,2} \in L^1(\mathbb{R})$ be given and denote by $W_1,\ W_2 \in L^\infty((0,T);W^{1,\infty}(\mathbb{R}))$ the nonlocal terms associated to the corresponding solutions of* (1.9)*. Then, the following stability result holds: for all $t \in [0,T]$,*

$$\|W_1(t,\cdot) - W_2(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C(\lambda, \|\rho_{0,1}\|_{L^\infty(\mathbb{R})}, \|\rho_{0,2}\|_{L^\infty(\mathbb{R})}, \|\rho_{0,1}\|_{L^1(\mathbb{R})}, \|\rho_{0,2}\|_{L^1(\mathbb{R})})\|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{R})},$$

where C is a suitable constant that depends only on the quantities mentioned above.

Proof. From the results in [28, 12], we know that the solution of (1.9) can be written as

$$\rho_1(t,x) = \rho_{0,1}\left(\xi_{W_1}(t,x;0)\right)\partial_2\xi_{W_1}(t,x;0) \quad \text{and} \quad \rho_2(t,x) = \rho_{0,2}\left(\xi_{W_2}(t,x;0)\right)\partial_2\xi_{W_2}(t,x;0),$$

where ξ_{W_1} and ξ_{W_2} solve the characteristic ODEs

$$\begin{aligned} \xi_{W_1}(t,x;\tau) &= x + \int_t^\tau W_1(s,\xi_{W_1}(t,x;s))\,\mathrm{d}s, \quad \tau \in [0,T], \\ \xi_{W_2}(t,x;\tau) &= x + \int_t^\tau W_2(s,\xi_{W_2}(t,x;s))\,\mathrm{d}s, \quad \tau \in [0,T]. \end{aligned} \tag{3.1}$$

In particular, we recall that the nonlocal terms corresponding to the initial data $\rho_{0,1}$ and $\rho_{0,2}$ satisfy the following fixed-point equations for $(t,x) \in (0,T)\times\mathbb{R}$:

$$\begin{aligned} W_1(t,x) &= \lambda\int_{-\infty}^x \exp(\lambda(y-x))\rho_1(t,y)\,\mathrm{d}y = \lambda\int_{-\infty}^x \exp(\lambda(y-x))\rho_{0,1}(\xi_{W_1}(t,y;0))\partial_2\xi_{W_1}(t,y;0)\,\mathrm{d}y \\ &= \lambda\int_{-\infty}^{\xi_{W_1}(t,x;0)} \exp\Big(\lambda(\xi_{W_1}(0,z;t)-x)\Big)\rho_{0,1}(z)\,\mathrm{d}z; \\ W_2(t,x) &= \lambda\int_{-\infty}^{\xi_{W_2}(t,x;0)} \exp\Big(\lambda(\xi_{W_2}(0,z;t)-x)\Big)\rho_{0,2}(z)\,\mathrm{d}z. \end{aligned}$$

Taking the absolute value of the difference, we have

$$\begin{aligned} &\lambda^{-1}\big|W_1(t,x) - W_2(t,x)\big| \\ &= \left|\int_{-\infty}^{\xi_{W_1}(t,x;0)} \exp\big(\lambda(\xi_{W_1}(0,z;t)-x)\big)\rho_0(z)\,\mathrm{d}z - \int_{-\infty}^{\xi_{W_2}(t,x;0)} \exp\big(\lambda(\xi_{W_2}(0,z;t)-x)\big)\rho_{0,2}(z)\,\mathrm{d}z\right| \\ &\leq \int_{\min\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}}^{\max\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}} \big(|\rho_{0,1}(y)| + |\rho_{0,2}(y)|\big)\,\mathrm{d}y \\ &\quad + \int_{-\infty}^{\min\{\xi_{W_1}(t,x;0),\xi_{W_2}(t,x;0)\}} \big(\exp\big(\lambda(\xi_{W_1}(0,z;t)-x)\big)\rho_{0,1}(z) - \exp\big(\lambda(\xi_{W_2}(0,z;t)-x)\big)\rho_{0,2}(z)\big)\,\mathrm{d}z \\ &\leq |\xi_{W_1}(t,x;0) - \xi_{W_2}(t,x;0)|\big(\|\rho_{0,1}\|_{L^\infty(\mathbb{R})} + \|\rho_{0,2}\|_{L^\infty(\mathbb{R})}\big) \\ &\quad + \lambda\|\xi_{W_1}(0,\cdot;t) - \xi_{W_2}(0,\cdot;t)\|_{L^\infty(\mathbb{R})}\big(\|\rho_{0,1}\|_{L^1(\mathbb{R})} + \|\rho_{0,2}\|_{L^1(\mathbb{R})}\big) + \|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.2}$$

To conclude, we need to study the stability of the characteristics with regard to W_1 and W_2. For $(t,x,\tau) \in (0,T)\times\mathbb{R}\times(0,T)$, we compute

$$\big|\xi_{W_1}(t,x;\tau) - \xi_{W_2}(t,x;\tau)\big| = \left|\int_t^\tau W_1(s,\xi_{W_1}(t,x;s)) - W_2(s,\xi_{W_2}(t,x;s))\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&= \left| \int_t^\tau W_1(s, \xi_{W_1}(t, x; s)) - W_2(s, \xi_{W_1}(t, x; s)) \, \mathrm{d}s \right| \\
&\quad + \left| \int_t^\tau W_2(s, \xi_{W_1}(t, x; s)) - W_2(s, \xi_{W_2}(t, x; s)) \, \mathrm{d}s \right| \\
&\leq \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s, \cdot) - W_2(s, \cdot)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}s \\
&\quad + \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|\xi_{W_1}(t, \cdot; s) - \xi_{W_2}(t, \cdot; s)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}s.
\end{aligned}
$$

Gronwall's inequality yields

$$
\begin{aligned}
&\left\|\xi_{W_1}(t, \cdot; \tau) - \xi_{W_2}(t, \cdot; \tau)\right\|_{L^\infty(\mathbb{R})} \\
&\leq \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s, \cdot) - W_2(s, \cdot)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}s \;\; \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right).
\end{aligned}
$$

Plugging this into (3.2), we get

$$
\begin{aligned}
&\|W_1(t, \cdot) - W_2(t, \cdot)\|_{L^\infty(\mathbb{R})} \\
&\leq \lambda\big(\|\rho_{0,1}\|_{L^\infty(\mathbb{R})} + \|\rho_{0,2}\|_{L^\infty(\mathbb{R})}\big) \int_{\min\{t,\tau\}}^{\max\{t,\tau\}} \|W_1(s, \cdot) - W_2(s, \cdot)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}s \\
&\qquad \times \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right) \\
&\quad + \lambda^2\big(\|\rho_{0,1}\|_{L^1(\mathbb{R})} + \|\rho_{0,2}\|_{L^1(\mathbb{R})}\big) \int_t^\tau \|W_1(s, \cdot) - W_2(s, \cdot)\|_{L^\infty(\mathbb{R})} \, \mathrm{d}s \\
&\qquad \times \exp\left(|t-\tau| \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}\right) \\
&\quad + \|\rho_{0,1} - \rho_{0,2}\|_{L^1(\mathbb{R})}.
\end{aligned}
$$

Applying again Gronwall's inequality on $W_1 - W_2$ and recalling that

$$
\partial_x W_2 = \lambda(\rho_2 - W_2) \implies \|\partial_x W_2\|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq 2\lambda \|\rho_{0,2}\|_{L^\infty(\mathbb{R})} \tag{3.3}
$$

(thanks to the maximum principle in Theorem 2.1), we conclude the proof. □

As a first step, we prove an Oleinik-type inequality on the nonlocal term W_λ. The result is essentially contained in [10] (in a more general form). We present the proof below for the sake of completeness.

Theorem 3.2 (Oleinik-type inequality for W_λ). *Given ρ_0 such that* (1.4) *holds, the solution W_λ of* (2.2) *satisfies*

$$
\frac{W_\lambda(t, x) - W_\lambda(t, y)}{x - y} \leq \frac{1}{t}, \qquad t > 0, \quad x, y \in \mathbb{R},\, x \neq y, \tag{3.4}
$$

for all $\lambda > 0$.

Proof of Theorem 3.2. We smooth the initial datum $\rho_{0,\lambda}$ by a $\rho_{0,\lambda}^\varepsilon$ for $\varepsilon > 0$ and call the corresponding smooth nonlocal term W_λ^ε. We then compute, differentiating the PDE in (2.2) with respect to x,

$$
\partial_{tx}^2 W_\lambda^\varepsilon = -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon + \lambda^2 \int_{-\infty}^x \exp(\lambda(y - x)) W_\lambda^\varepsilon(t, y) \partial_y W_\lambda^\varepsilon(t, y) \, \mathrm{d}y. \tag{3.5}
$$

For $t > 0$ fixed, considering $m(t) = \sup_{y \in \mathbb{R}} \partial_y W_\lambda^\varepsilon(t, y)$ and assuming – without loss of generality – that $m(t) \geq 0$, we estimate the right-hand side of (3.5) as follows:

$$
\begin{aligned}
\partial_{tx}^2 W_\lambda^\varepsilon &= -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon + \lambda^2 \int_{-\infty}^x \exp(\lambda(y - x)) W_\lambda^\varepsilon(t, y) \partial_y W_\lambda^\varepsilon(t, y) \, \mathrm{d}y \\
&= -W_\lambda^\varepsilon \partial_{xx}^2 W_\lambda^\varepsilon - (\partial_x W_\lambda^\varepsilon)^2 - \lambda W_\lambda^\varepsilon \partial_x W_\lambda^\varepsilon \\
&\quad + \lambda^2 \int_{-\infty}^x \exp(\lambda(y - x)) \Big(\rho_\lambda(t, y) - \lambda^{-1} \partial_y W_\lambda^\varepsilon(t, y)\Big) \partial_y W_\lambda^\varepsilon(t, y) \, \mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= -W^\varepsilon_\lambda \partial^2_{xx} W^\varepsilon_\lambda - (\partial_x W^\varepsilon_\lambda)^2 - \lambda W^\varepsilon_\lambda \partial_x W^\varepsilon_\lambda \\
&\quad + \lambda^2 \int_{-\infty}^x \exp(\lambda(y-x))\rho^\varepsilon_\lambda(t,y)\partial_y W^\varepsilon_\lambda(t,y)\,\mathrm{d}y \underbrace{-\lambda \int_{-\infty}^x \exp(\lambda(y-x))|\partial_y W^\varepsilon_\lambda(t,y)|^2\,\mathrm{d}y}_{\le 0} \\
&\le -W^\varepsilon_\lambda \partial^2_{xx} W^\varepsilon_\lambda - (\partial_x W^\varepsilon_\lambda)^2 - \lambda W^\varepsilon_\lambda \partial_x W^\varepsilon_\lambda + m(t)\lambda^2 \underbrace{\int_{-\infty}^x \exp(\lambda(y-x))\rho^\varepsilon_\lambda(t,y)\,\mathrm{d}y}_{=\lambda^{-1}W^\varepsilon_\lambda(t,x)}
\end{aligned}
$$

We have that, for every $t > 0$, there exists a maximum point of $\partial_y W^\varepsilon_\lambda(t,y)$ (by choosing, e.g., a compactly supported $\rho^\varepsilon_{0,\lambda}$ and relying on the regularity results of [28]). Let us consider $\bar{x}(t) \in \mathbb{R}$ such that $m(t) = \partial_x W^\varepsilon_\lambda(t,\bar{x}(t))$ and evaluate the previous expression at $x = \bar{x}$. Then

$$
\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \partial_t \partial_x W^\varepsilon_\lambda(t,\bar{x}(t)) + \partial^2_{xx} W^\varepsilon_\lambda(t,\bar{x}(t))\bar{x}'(t)
$$

and the second summand vanishes since $\bar{x}(t)$ is a critical point of $\partial_x W^\varepsilon_\lambda(t,\cdot)$. Then, using the computations carried out above, we deduce

$$
\frac{\mathrm{d}}{\mathrm{d}t} m(t) \le -m^2(t).
$$

Since $\widetilde{m}(t) = 1/t$ is a solution of the above Riccati-type differential inequality and $\widetilde{m}(0) = \infty$, we use the comparison principle for ODEs to conclude that $m(t) \le 1/t$ and thus

$$
\frac{W^\varepsilon_\lambda(t,x) - W^\varepsilon_\lambda(t,y)}{x-y} = \frac{1}{x-y}\int_y^x \partial_x W^\varepsilon_\lambda(t,\xi)\,\mathrm{d}\xi \le \frac{1}{t}, \qquad t > 0, \quad x,y \in \mathbb{R},\ x \ne y.
$$

Taking the limit $\varepsilon \to 0^+$, thanks to Lemma 3.1, we conclude the proof. □

As a by-product of (3.4), we prove (arguing as in [16, Lemma 1.3]) that a L^∞ bound holds for all $t > 0$ (which blows up as $t \to 0^+$).

Lemma 3.3 (L^∞-bound on W_λ)**.** *The following L^∞-bounds on W_λ and ρ_λ hold:*

$$
0 \le W_\lambda(t,x) \le \sqrt{\frac{2M}{t}}, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}, \tag{3.6}
$$

$$
0 \le \rho_\lambda(t,x) \le \sqrt{\frac{2M}{t}} + \frac{1}{\lambda t}, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{3.7}
$$

Proof. The fact that, for all $t > 0$, $W_\lambda(t,\cdot), \rho_\lambda(t,\cdot) \ge 0$ holds is contained in point (1) of Theorem 2.1. To prove the upper bound in (3.6), let us fix a time $t > 0$ and a point $\bar{x} \in \mathbb{R}$. By Lemma 3.2, we have

$$
W_\lambda(t,x) \ge W_\lambda(t,\bar{x}) - \frac{1}{t}(\bar{x} - x), \qquad \text{for all } x \le \bar{x},
$$

i.e.,

$$
W_\lambda(t,x) \ge \frac{1}{t}(x - (\bar{x} + W_\lambda(t,\bar{x})t)), \qquad \text{for all } 0 \le x - (\bar{x} + W_\lambda(t,\bar{x})t) \le W_\lambda(t,\bar{x})t.
$$

Integrating over $\mathbb{R}$, we deduce

$$
M = \int_{\mathbb{R}} W_\lambda(t,x)\,\mathrm{d}x \ge \int_{\mathbb{R}\cap\{x \ge \bar{x} + W_\lambda(t,\bar{x})t\}} \frac{x - (\bar{x} + W_\lambda(t,\bar{x})t)}{t}\,\mathrm{d}x \ge \int_0^{W_\lambda(t,\bar{x})t} \left(\frac{x}{t}\right)\mathrm{d}x = \frac{1}{2} W_\lambda(t,\bar{x})^2 t,
$$

which implies

$$
W_\lambda(t,\bar{x}) \le \sqrt{\frac{2M}{t}} \qquad \text{for all } t > 0,\ \bar{x} \in \mathbb{R}.
$$

The bound (3.7) follows from (3.6) and Theorem 3.2. Indeed, by (1.11), we have

$$
\begin{aligned}
0 \le \rho_\lambda(t,x) &= W_\lambda(t,x) + \frac{1}{\lambda}\partial_x W_\lambda(t,x) \\
&\le \sqrt{\frac{2M}{t}} + \frac{1}{\lambda t}.
\end{aligned}
$$

□

As a second corollary, from (3.4), we deduce the following $\mathrm{BV}_{\mathrm{loc}}$ regularization result (see [5, Eq. (4.3)] and [6, Lemma 2.2 (ii) & Remark 2.3]).

Corollary 3.4 (BV-regularization effect)**.** *The function $W_\lambda(t,\cdot)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for every $t>0$ and uniformly with respect to $\lambda>0$: namely, for every compact interval $K \Subset \mathbb{R}$,*

$$|W_\lambda(t,\cdot)|_{\mathrm{TV}(K)} \le 2\left(\frac{|K|}{t} + \|W_\lambda(t,\cdot)\|_{L^\infty(K)}\right), \quad t>0. \tag{3.8}$$

Proof. Let $K := [a,b] \Subset \mathbb{R}$ be a compact interval of $\mathbb{R}$ and fix $t>0$. Since $W_\lambda(t,\cdot) \in L^\infty(K) \subset L^1(K)$, we only need to prove, thanks to the characterization of BV functions in [40, Lemma 37.4] (see also [1, Remark 2.5 & Exercise 3.3] or [35, Corollary 2.17]), that there exists $C>0$ such that

$$\int_{K_h} \frac{|W_\lambda(x+h)-W_\lambda(x)|}{h}\,\mathrm{d}x \le C, \qquad \forall h>0, \quad \text{where } K_h := \{x\in K : x+h\in K\}.$$

Taking Lemma 3.2 into account, we note that

$$\frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h} = \frac{1}{t} - \underbrace{\left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right)}_{\ge 0}, \tag{3.9}$$

which implies

$$\frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h} \le \frac{1}{t} + \left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right).$$

Integrating over K_h and taking the absolute values on both sides yields

$$\begin{aligned}
\int_{K_h} \frac{|W_\lambda(t,x+h)-W_\lambda(t,x)|}{h}\,\mathrm{d}x &\le \int_{K_h} \left(\frac{1}{t} + \left(\frac{1}{t} - \frac{W_\lambda(t,x+h)-W_\lambda(t,x)}{h}\right)\right)\mathrm{d}x \\
&= 2\int_{K_h} \frac{1}{t}\,\mathrm{d}x + \int_{\mathbb{R}} W_\lambda(t,x)\left(\frac{\mathbb{1}_K(x+h)-\mathbb{1}_K(x)}{h}\right)\mathrm{d}x \\
&\le 2\int_{K_h} \frac{1}{t}\,\mathrm{d}x + \|W_\lambda(t,\cdot)\|_{L^\infty(K)} \underbrace{\int_{\mathbb{R}} \left(\frac{|\mathbb{1}_K(x+h)-\mathbb{1}_K(x)|}{h}\right)\mathrm{d}x}_{=\mathrm{TV}(\mathbb{1}_K)=2} \\
&= 2\left(\frac{|K|}{t} + \|W_\lambda(t,\cdot)\|_{L^\infty(K)}\right).
\end{aligned}$$

□

4. Long-time behavior

As a first step towards finishing the proof of Theorem 1.1, we shall show next that $\{W_\lambda\}_{\lambda>0}$ is compact in the canonical $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ topology. Note that the time-interval does not include $t=0$ because the L^∞ estimate from Lemma 3.3 blows up as $t\to 0^+$.

Lemma 4.1 (Compactness of $\{W_\lambda\}_{\lambda>0}$ in $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$)**.** *Let $t_0,\ T>0$ be fixed. The set $\{W_\lambda\}_{\lambda>0} \subseteq C\big([t_0,T];L^1_{loc}(\mathbb{R})\big)$ of solutions to* (1.1) *is compactly embedded into $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$, i.e.*

$$\Big\{W_\lambda \in C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big) :\ W_\lambda \textit{ satisfies } (1.10),\ \lambda>0\Big\} \overset{c}{\hookrightarrow} C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big).$$

Proof. Arguing as in [11, Theorem 4.1], we shall apply the compactness result in [39, Lemma 1]: given a Banach space B, a set $F\subset C([t_0,T];B)$ is relatively compact in $C([t_0,T];B)$ iff

- $F(t) := \big\{f(t)\in B : f\in F\big\}$ is relatively compact in B for all $t\in[t_0,T]$;
- F is uniformly equi-continuous, i.e.

$$\forall \sigma\in\mathbb{R}_{>0}\ \exists\delta\in\mathbb{R}_{>0} \text{ s. t. } \forall f\in F\ \forall (t_1,t_2)\in[t_0,T]^2 \text{ with } |t_1-t_2|\le\delta :\ \|f(t_1)-f(t_2)\|_B \le \sigma.$$

In our case, let us fix a compact interval $K\Subset\mathbb{R}$ and define $B=L^1(K)$ and $F(t) := \{W_\lambda(t,\cdot)\in L^1(K) : \lambda\in\mathbb{R}_{>0}\}$.

Thanks to Lemma 3.2, we know that $W_\lambda(t,\cdot)$ has a uniform total variation bound and by [34, Theorem 13.35], the set $F(t)$ is compact in $L^1(K)$, i.e.

$$F(t) \Subset L^1(K), \quad \forall t \in [t_0, T].$$

It remains to show the second point, the uniform equi-continuity. To this end, we again smooth the initial datum $\rho_{0,\lambda}$ by a $\rho^\varepsilon_{0,\lambda}$, with $\varepsilon > 0$, and call the corresponding smooth nonlocal term W^ε_λ. Then, we can estimate

$$\begin{aligned}
\big\|W^\varepsilon_\lambda(t_1,\cdot) - W^\varepsilon_\lambda(t_2,\cdot)\big\|_{L^1(\mathbb{R})} &= \left\|\int_{t_2}^{t_1} \partial_t W^\varepsilon_\lambda(s,\cdot)\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\le \left\|\int_{t_2}^{t_1} W^\varepsilon_\lambda(s,\cdot)\partial_2 W^\varepsilon_\lambda(s,\cdot)\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\quad + \left\|\int_{t_2}^{t_1} \lambda\int_*^\infty \exp(\lambda(*-y))\partial_y W^\varepsilon_\lambda(s,y)W^\varepsilon_\lambda(s,y)\,\mathrm{d}y\,\mathrm{d}s\right\|_{L^1(\mathbb{R})} \\
&\le \|W^\varepsilon_\lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}|W^\varepsilon_\lambda|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))}|t_1-t_2| \\
&\quad + \|W^\varepsilon_\lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}|W^\varepsilon_\lambda|_{L^\infty((0,T);\mathrm{TV}(\mathbb{R}))}|t_1-t_2|,
\end{aligned}$$

where we used Fubini-Tonelli's theorem to exchange the order of integration and estimate the last term. Thanks to Lemmas 3.2 and 3.3, we have that this is a uniform bound in $\lambda > 0$ and $\varepsilon > 0$. This yields the uniform equi-continuity so that we obtain indeed the claimed compactness.

□

We can now complete the proof of Theorem 1.1 arguing as in [16, Section 2].

Proof of Theorem 1.1. The core of the proof consists in showing that the family $\{\rho_\lambda\}_{\lambda>0}$ converges to the N-wave defined in (1.6). We shall divide the argument of this theorem in several steps.

Step 1. *Compactness of the family $\{W_\lambda\}_{\lambda>0}$ in $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$.* For any $0 < t_0 < T$, by Lemma 4.1, we have that W_λ converges (up to extracting a subsequence) to a limit point w^* strongly in $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$; hence, we also have $W_\lambda(t,\cdot) \to w^*(t,\cdot)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ for all $t \in [t_0,T]$ and $W_\lambda \to w^*$ pointwise (again up to subsequences) for all $t \in [t_0,T]$ and a.e. $x \in \mathbb{R}$.

Thanks to (1.11), we can deduce that ρ_λ also converges to w^* along the same subsequence. Indeed, first we observe that

$$\|W_\lambda(t,\cdot) - \rho_\lambda(t,\cdot)\|_{L^1(\mathbb{R})} = \lambda^{-1}|W_\lambda(t,\cdot)|_{\mathrm{TV}(\mathbb{R})}$$

and thus we also obtain

$$\lim_{\lambda\to+\infty} \|\rho_\lambda - w\|_{C([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R}))} = 0.$$

Step 2a. *Tail control and convergence of the family $\{\rho_\lambda\}_{\lambda>0}$ in $C([t_0,T],L^1(\mathbb{R}))$.* In order to pass from the convergence $\rho_\lambda \to w^*$ strongly in $C\big([t_0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$ to the convergence in $C\big([t_0,T];L^1(\mathbb{R})\big)$, we need a uniform bound on the "tail" of the functions $\{\rho_\lambda\}_{\lambda>1}$. We shall prove that there exists a constant $C = C(M)$ such that

$$\int_{\{|x|>2R\}} \rho_\lambda(t,x)\,\mathrm{d}x \le \int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R}t^{1/2}, \qquad t > 0. \tag{4.1}$$

Since $\rho_0 \in L^1(\mathbb{R})$, the right-hand side of (4.1) can be made arbitrarily small choosing R large enough. Then, from (4.1), the convergence

$$\rho_\lambda \to w^* \qquad \text{strongly in } C([t_0,T],L^1(\mathbb{R})) \quad \text{as } \lambda \to +\infty$$

follows by considering the splitting

$$\int_{\mathbb{R}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x = \int_{\{x<2R\}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x + \int_{\{x>2R\}} |\rho_\lambda(t,x) - w^*(t,x)|\,\mathrm{d}x.$$

In order to prove (4.1), let us consider a test function $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ for $|x| > 2$, and $\varphi \equiv 0$ for $|x| \le 1$; we consider the rescaling $\varphi_R := \varphi(\cdot/R)$ which satisfies $\|\partial_x\varphi_R\|_{L^\infty(\mathbb{R})} \le C/R$ for some $C > 0$. Let us multiply the PDE in (1.9) by φ_R, integrate in $(0,t)\times\mathbb{R}$ (for some $t > 0$), and perform an integration by parts (to rigorously justify this computation, we can use a smoothing argument based on Lemma 3.1):

$$\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi_R(x)\,\mathrm{d}x = \int_{\mathbb{R}} \rho_\lambda(0,x)\varphi_R(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \rho_\lambda(s,x)W_\lambda(s,x)\partial_x\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}s.$$

We remark that

$$\int_{\mathbb{R}} \rho_\lambda(0,x)\varphi_R(x)\,\mathrm{d}x = \int_{\{|x|\geq R\}} \rho_\lambda(0,x)\,\mathrm{d}x = \int_{\{|x|\geq \lambda R\}} \rho(0,x)\,\mathrm{d}x \leq \int_{\{|x|>R\}} \rho(0,x)\,\mathrm{d}x,$$

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \rho_\lambda(s,x)W_\lambda(s,x)\partial_x\varphi_R(x)\,\mathrm{d}x\,\mathrm{d}s &\leq \|\partial_x\varphi_R\|_{L^\infty(\mathbb{R})}\int_0^t \|\rho_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\ &\leq \frac{C}{R}\int_0^t M\sqrt{\frac{2M}{s}}\,\mathrm{d}s \leq \frac{C}{R}\frac{1}{\sqrt{2}}M^{3/2}t^{1/2},\end{aligned}$$

where we used Lemma 3.3 in the last line.

Step 2b. *Tail control and convergence of the family* $\{W_\lambda\}_{\lambda>0}$ *in* $C([t_0,T],L^1(\mathbb{R}))$. Since

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x = \lambda\int_{\{|x|>2R\}}\int_{-\infty}^x \exp(\lambda(y-x))\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x,$$

we use Fubini-Tonelli's Theorem to deduce

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x \leq \int_{\{|x|>2R\}} \rho_\lambda(t,x)\,\mathrm{d}x, \qquad t>0,$$

which yields, thanks to (4.1),

$$\int_{\{|x|>2R\}} W_\lambda(t,x)\,\mathrm{d}x \leq \int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R}t^{1/2}, \qquad t>0. \tag{4.2}$$

As a byproduct of Steps 1 and 2, we note that the limit point w^* satisfies

$$w^* \in C((0,+\infty);L^1(\mathbb{R};\mathbb{R}_{\geq 0}))\cap L^\infty((\tau,+\infty);L^\infty(\mathbb{R};\mathbb{R}_{\geq 0})) \text{ for all } \tau>0, \qquad \int_{\mathbb{R}} w^*(t,x)\,\mathrm{d}x = M.$$

Step 3. *Identification of the initial condition.* We now identify the initial datum taken by the limit point w^*, i.e., we verify that the initial condition $M\delta_0$ is achieved in the weak sense of non-negative measures on $\mathbb{R}$. We need to prove that, for all $\varphi\in C_b(\mathbb{R})$,

$$\lim_{t\to 0^+}\int_{\mathbb{R}} w^*(t,x)\varphi(x)\,\mathrm{d}x = M\varphi(0).$$

To this end, arguing as in [16, pp. 52-54], we shall split the argument into two steps. First, we consider a smaller class of test functions $\varphi\in C_c^\infty(\mathbb{R};[0,1])$ and secondly $\varphi\in C_b(\mathbb{R})$.

We start by estimating, for a test function $\varphi\in C_c^\infty(\mathbb{R};[0,1])$,

$$\begin{aligned}&\left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\lambda(0,x)\varphi(x)\,\mathrm{d}x\right| \\ &\leq \left|\int_0^t\int_{\mathbb{R}} \partial_x\varphi(x)W_\lambda(s,x)\rho_\lambda(s,x)\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\int_0^t \|\rho_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\ &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}C(M)\int_0^t s^{-1/2}\,\mathrm{d}s \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t}.\end{aligned}$$

Then, letting $\lambda\to+\infty$, we obtain

$$\left|\int_{\mathbb{R}} w^*(t,x) - M\varphi(0)\right| \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t},$$

which, in turn, goes to zero as $t\to 0^+$.

As a second step, let us consider the case of a bounded continuous function $\varphi\in C_b(\mathbb{R})$. We shall rely on an approximation argument and on the tail control of ρ_λ in (4.1). Let us consider a regularized test function φ_ε obtained as $\varphi_\varepsilon := \varphi*\eta_\varepsilon$ (where η_ε denotes a standard mollifier; see [20, Appendix C.4]), such that $\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|\varphi\|_{L^\infty(\mathbb{R})}$, $\varphi_\varepsilon \to \varphi$ uniformly on compact sets of $\mathbb{R}$ as $\varepsilon \to 0^+$, and $\|\varphi_\varepsilon\|_{W^{1,\infty}(\mathbb{R})} \leq C(\varepsilon)$. We then write

$$\left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi(x)\,\mathrm{d}x - M\varphi(0)\right| \leq \left|\int_{\mathbb{R}} \rho_\lambda(t,x)\varphi_\varepsilon(x)\,\mathrm{d}x - M\varphi(0)\right| + \left|\int_{\{|x|>2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\,\mathrm{d}x\right| + \left|\int_{\{|x|<2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\,\mathrm{d}x\right|.$$

The control of the first term follows by the same argument developed above. For the second and third term, we estimate

$$\left|\int_{\{|x|>2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\,\mathrm{d}x\right| \leq 2\|\varphi\|_{L^\infty(\mathbb{R})}\left(\int_{\{|x|>R\}} \rho_0(x)\,\mathrm{d}x + \frac{C(M)}{R}t^{1/2}\right),$$

$$\left|\int_{\{|x|<2R\}} \rho_\lambda(t,x)\left(\varphi(x)-\varphi_\varepsilon(x)\right)\,\mathrm{d}x\right| \leq \|\varphi-\varphi_\varepsilon\|_{L^\infty(\{|x|<2R\})}\int_{\mathbb{R}} \rho_\lambda(t,x)\,\mathrm{d}x = M\|\varphi-\varphi_\varepsilon\|_{L^\infty(\{|x|<2R\})},$$

which can both be made arbitrarily small provided that $\varepsilon > 0$ is small enough and $R > 0$ is large enough.

A similar argument can be used for $\{W_\lambda\}_{\lambda>0}$. Indeed, for $\varphi \in C_c^\infty(\mathbb{R};[0,1])$, we estimate

$$\begin{aligned}\left|\int_{\mathbb{R}} W_\lambda(t,x)\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} W_\lambda(0,x)\varphi(x)\,\mathrm{d}x\right| &= \left|\int_0^t\int_{\mathbb{R}} \partial_t W_\lambda(t,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}t\right| \\ \leq \underbrace{\int_0^t\int_{\mathbb{R}} \partial_x\varphi(x)\left|W_\lambda(s,x)\right|^2\,\mathrm{d}x\,\mathrm{d}s}_{=:I_1} &+ \lambda\underbrace{\left|\int_0^t\int_{\mathbb{R}}\varphi(x)\int_{-\infty}^x \exp(\lambda(y-x))W_\lambda(s,x)\partial_x W_\lambda(s,x) - W_\lambda(s,y)\partial_y W_\lambda(s,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s\right|}_{=:I_2}.\end{aligned}$$

For the term I_1, we compute

$$\begin{aligned}I_1 &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\int_0^t \|W_\lambda(s,\cdot)\|_{L^1(\mathbb{R})}\|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\mathrm{d}s \\ &\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{R})}C(M)\int_0^t s^{-1/2}\,\mathrm{d}s \leq C(M)\|\partial_x\varphi\|_{L^\infty(\mathbb{R})}\sqrt{t},\end{aligned}$$

where, in the last line, we used Lemma 3.3.

For I_2, using Fubini-Tonelli's theorem, we compute

$$\begin{aligned}I_2 &= \left|\int_0^t\int_{\mathbb{R}}\varphi(x)W_\lambda(s,x)\partial_x W_\lambda(s,x)\,\mathrm{d}x\,\mathrm{d}s \right.\\ &\quad\left. - \lambda\int_0^t\int_{\mathbb{R}}\varphi(x)\int_{-\infty}^x \exp(\lambda(y-x))W_\lambda(s,y)\partial_y W_\lambda(s,y)\,\mathrm{d}y\,\mathrm{d}x\right| \\ &= \left|\int_0^t\int_{\mathbb{R}}\varphi(x)W_\lambda(s,x)\partial_x W_\lambda(s,x)\,\mathrm{d}x\,\mathrm{d}s \right.\\ &\quad\left. - \lambda\int_0^t\int_{\mathbb{R}} W_\lambda(s,y)\partial_y W_\lambda(s,y)\int_y^\infty \varphi(x)\exp(\lambda(y-x))\,\mathrm{d}x\,\mathrm{d}y\right|;\end{aligned}$$

integrating by parts on the term $x \mapsto \exp(\lambda(y-x))$ yields

$$I_2 = \left|\int_0^t\int_{\mathbb{R}} W_\lambda(s,y)\partial_y W_\lambda(s,y)\int_y^\infty \varphi'(x)\exp(\lambda(y-x))\,\mathrm{d}x\,\mathrm{d}y\right|;$$

integrating by parts on the term $y \mapsto W_\lambda(s,y)\partial_y W_\lambda(s,y)$ and using the fact that $\lim_{x\to\pm\infty} W_\lambda(t,\cdot) = 0$ (which is a consequence of the fact that $W_\lambda(t,\cdot) \in L^1(\mathbb{R}) \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ for $t > 0$ and $\lambda > 0$), we then get

$$\begin{aligned} I_2 &= \left| -\lambda \frac{1}{2} \int_0^t \int_{\mathbb{R}} W_\lambda(s,y)^2 \int_y^\infty \partial_x \varphi(x) \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{2} \int_0^t \int_{\mathbb{R}} W_\lambda(s,y)^2 \partial_y \phi(y) \,\mathrm{d}y \right| \\ &\le \frac{\lambda}{2} \|\partial_x \phi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} W_\lambda(s,y)^2 \int_y^\infty \exp(\lambda(y-x)) \,\mathrm{d}x \,\mathrm{d}y + \frac{1}{2} \|\partial_x \phi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} W_\lambda(s,y)^2 \,\mathrm{d}y \\ &= \|\partial_x \phi\|_{L^\infty(\mathbb{R})} \int_0^t \int_{\mathbb{R}} W_\lambda(s,y)^2 \,\mathrm{d}y \\ &\le \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \int_0^t \|W_\lambda(s,\cdot)\|_{L^1(\mathbb{R})} \|W_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R})} \,\mathrm{d}s \\ &\le \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} C(M) \int_0^t s^{-1/2} \,\mathrm{d}s \le C(M) \|\partial_x \varphi\|_{L^\infty(\mathbb{R})} \sqrt{t}, \end{aligned}$$

where, in the last line, we used Lemma 3.3. Thus, for any $\varepsilon > 0$, we can choose $\tau > 0$ and $\lambda_0 > 0$ such that

$$\left| \int_{\mathbb{R}} W_\lambda(t,x)\varphi(x) \,\mathrm{d}x - M\varphi(0) \right| \le \varepsilon \qquad \text{for all } 0 < t < \tau,\ \lambda > \lambda_0.$$

The rest of the argument for $\varphi \in C_b(\mathbb{R})$ goes through as above.

Step 4. *Entropy admissibility of the limit point.* The limit point w^* is actually the unique entropy admissible N-wave solution w of the Burgers equation (1.5) defined in (1.6). This follows immediately from passing to the limit pointwise in the Oleinik inequality (3.4). Thanks to Tychonoff's theorem, from the uniqueness of the entropy solution of (1.5), we also deduce that the whole families $\{\rho_\lambda\}_{\lambda>0}$ and $\{W_\lambda\}_{\lambda>0}$ converge to w (not just up to extracting a subsequence).

Step 5. *Conclusion of the proof.* From the steps above, we have that

$$\|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } \lambda \to +\infty,$$

where w denotes the N-wave solution entropy of (1.5). For $p = 1$, (1.7) is a consequence of the fact that

$$\begin{aligned} \rho_\lambda(1,x) - w(1,x) &= \lambda\rho(\lambda^2, \lambda x) - w(1,x), \\ W_\lambda(1,x) - w(1,x) &= \lambda W(\lambda^2, \lambda x) - w(1,x), \end{aligned}$$

(and that the same would hold true replacing $t = 1$ by any fixed $\bar{t} > 0$), i.e., letting $\lambda \to +\infty$ for a fixed time $\bar{t} > 0$ is equivalent to fixing $\lambda = 1$ and letting $t \to +\infty$.

To prove the result also for $p \in (1, +\infty)$, we argue by interpolation. Indeed, we have that, for $t > 0$, $\{\rho_\lambda\}_{\lambda>0}$ and $\{W_\lambda\}_{\lambda>0}$ are also uniformly bounded in $L^q(\mathbb{R})$ (being in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $t > 0$) and $w \in L^q(\mathbb{R})$, with $q \in (1, +\infty)$. Then, we deduce

$$\begin{aligned} \|\rho_\lambda(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} &\le \|\rho_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})}^{\frac{1}{2p-1}} \Big(\|\rho_\lambda(t,\cdot)\|_{L^{2p}(\mathbb{R})} + \|w(t,\cdot)\|_{L^{2p}(\mathbb{R})} \Big)^{\frac{2p}{2p-1}}, \\ \|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^p(\mathbb{R})} &\le \|W_\lambda(t,\cdot) - w(t,\cdot)\|_{L^1(\mathbb{R})}^{\frac{1}{2p-1}} \Big(\|W_\lambda(t,\cdot)\|_{L^{2p}(\mathbb{R})} + \|w(t,\cdot)\|_{L^{2p}(\mathbb{R})} \Big)^{\frac{2p}{2p-1}}, \end{aligned}$$

from which the result follows. □

5. Numerical experiments

In this section, we showcase the result in Theorem 1.1 numerically. For the nonlocal problem, we rely on a non-dissipative solver based on characteristics (see [30] for further details). More precisely, the simulations illustrate the convergence

$$\bar{\rho}(t,\cdot) \to \bar{w} \quad \text{in } L^1(\mathbb{R}) \quad \text{as } t \to +\infty,$$

for the rescaled variables

$$\bar{\rho} = \sqrt{t}\rho, \quad \bar{w} = \sqrt{t}w, \quad y = x/\sqrt{t},$$

in which the N-wave is stationary (i.e., time-independent) and given by

$$\bar{w}(y) = \begin{cases} y & \text{if } y \in (0, \sqrt{4M}), \\ 0 & \text{otherwise.} \end{cases}$$

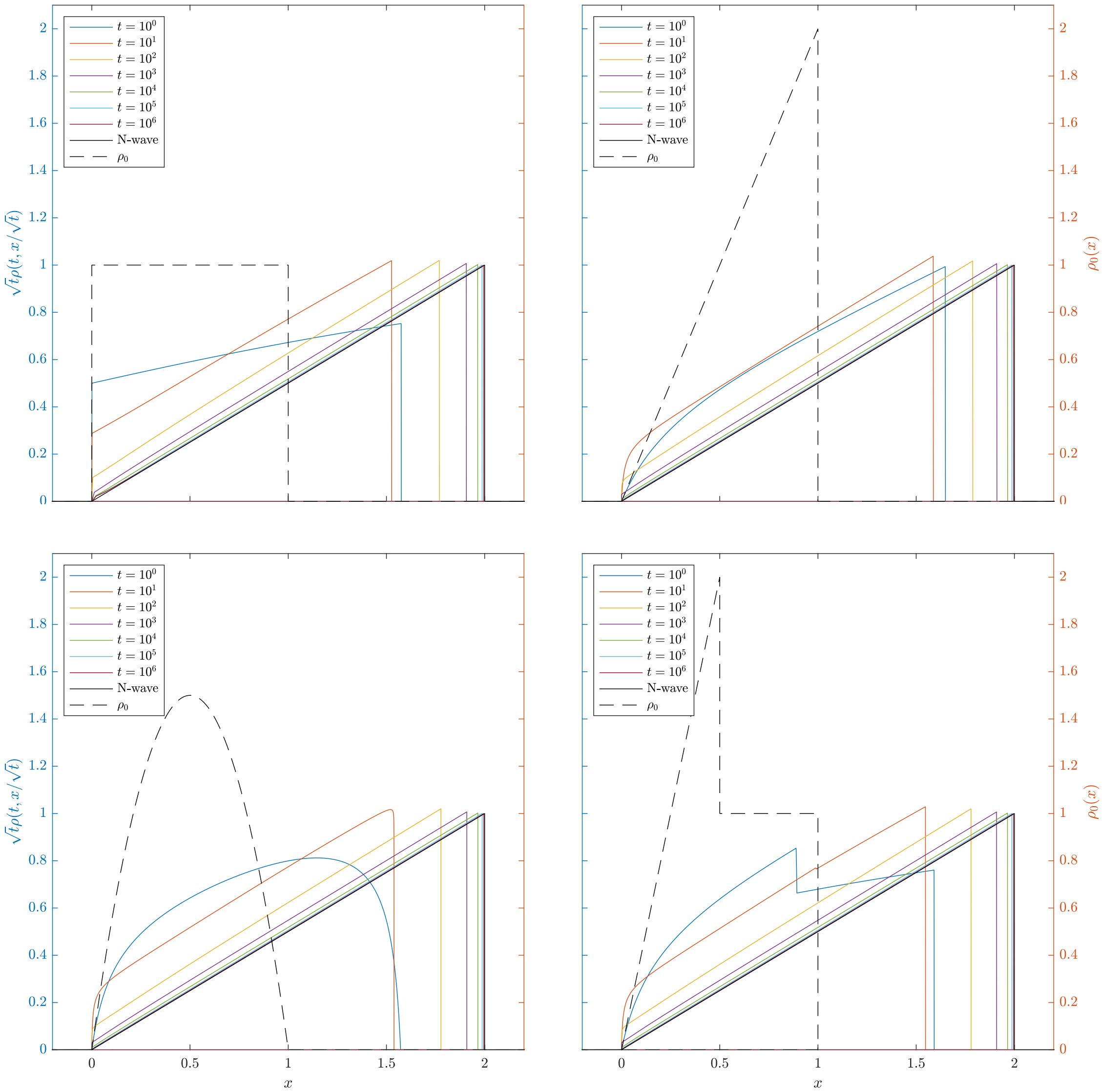

FIGURE 2. Convergence to the N-wave profile for the nonlocal regularization of the Burgers equation. TOP LEFT: $\rho_0(x) = \mathbb{1}_{[0,1]}(x)$. TOP RIGHT: $\rho_0(x) = 2x\,\mathbb{1}_{[0,1]}(x)$. BOTTOM LEFT: $\rho_0(x) = 6x(1-x)\mathbb{1}_{[0,1]}(x)$. BOTTOM RIGHT: $\rho_0(x) = 2x\,\mathbb{1}_{[0,0.5]}(x) + \mathbb{1}_{[0.5,1]}(x)$.

To start with, in Figure 2, we present the evolution of the solution of (1.1) on long time horizons for the following initial data:

(1) $\rho_0(x) = \mathbb{1}_{[0,1]}(x)$, **(2)** $\rho_0(x) = 2x\,\mathbb{1}_{[0,1]}(x)$,

(3) $\rho_0(x) = 6x(1-x)\mathbb{1}_{[0,1]}(x)$, **(4)** $\rho_0(x) = 2x\,\mathbb{1}_{[0,0.5]}(x) + \mathbb{1}_{[0.5,1]}(x)$,

for $x \in \mathbb{R}$. In all cases, we observe the convergence towards the N-wave profile of the (local) Burgers equation (1.5).

For (left) continuous initial data (as is the case in **(2)**,**(3)**, and **(4)**), the N-wave is also approximated by left-continuous functions. This is a well-known fact, as nonlocal conservation laws preserve regularity [28, Corollary 5.3] (see also Theorem 2.1). In particular, for the **(4)** case, there are two jumps downwards in the initial datum and the first one is damped out over time (still observable for $t = 10$ at $x \approx 1$). This can be understood when recalling that around $x \approx 1$ the velocity of the dynamics is smaller than for $x < 1$ so that the density increases between both points and the jump decreases (which is visible in particular for $t = 1$ and $t = 10$).

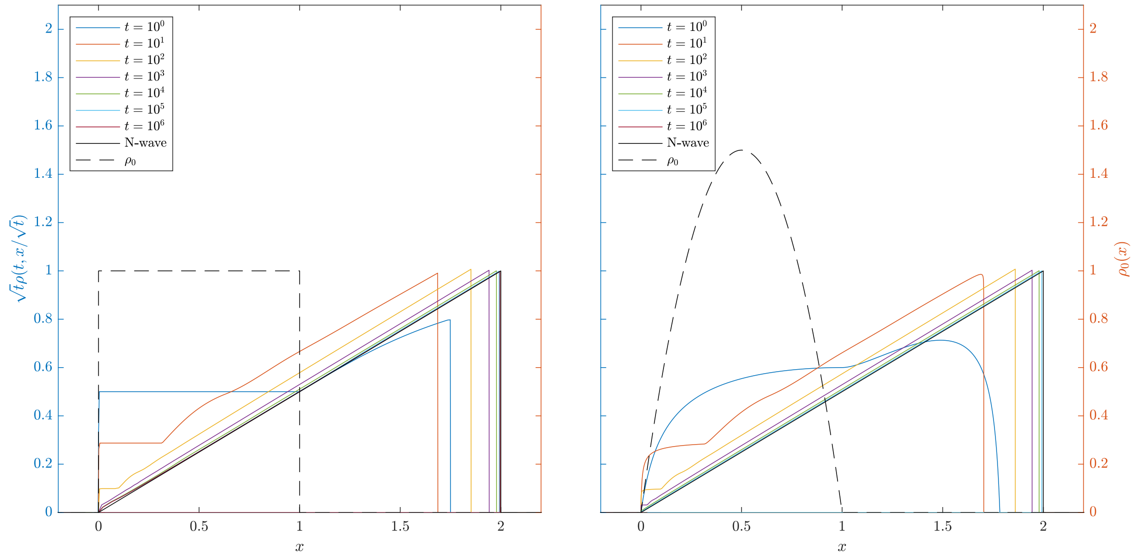

FIGURE 3. Convergence to the N-wave profile for the nonlocal regularization of Burgers equation with the constant kernel $\gamma := \mathbb{1}_{(-1,0)}$. LEFT: $\rho_0(x) = \mathbb{1}_{[0,1]}(x)$. RIGHT: $\rho_0(x) = 6x(1-x)\mathbb{1}_{[0,1]}(x)$.

Secondly, in Figure 3, we consider $\gamma(x) = \mathbb{1}_{(-1,0)}(x),\ x \in \mathbb{R}$, instead of an exponential weight in (1.2), i.e. we study

$$W[\rho](t,x) := \int_{x-1}^{x} \rho(t,y)\,\mathrm{d}y,\ \ (t,x) \in (0,T) \times \mathbb{R}.$$

The numerical simulation shows that, even in this case (which is not covered by the results of the present paper or by the ones on the singular limit problem contained in [11, 14]), a convergence result can be observed. However, the convergence seems to occur "less regularly" as the constant kernel generates more and more points where the solution is not differentiable. Indeed, in contrast to the exponential kernel case, the regularity of the solution for piece-wise constant kernels depends points-wise and locally (on the trace of backward characteristics) on initial data, kernel, and their interplay.

Finally, we present some simulations illustrating the case of a more general power-type velocity: namely,

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(W^{q-1}(t,x)\rho(t,x)) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.1}$$

for some for $q \geq 2$. In this case, the explicit N-wave solution of the corresponding local conservation law

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(\rho^q(t,x)) = 0, & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.2}$$

is given by

$$w_q(t,x) = \begin{cases} \operatorname{sign}(x)\,|x|^{\frac{1}{q-1}}(qt)^{\frac{1}{1-q}} & \text{if } x \in \Big(0,\ q\big(\frac{M}{q-1}\big)^{\frac{q-1}{q}} t^{\frac{1}{q}}\Big), \\ 0 & \text{otherwise.} \end{cases}$$

that is, in the rescaled variables

$$\bar{w}_q = t^{1/q} w_q, \quad y = xt^{-1/q},$$

$$\bar{w}_q(y) = \begin{cases} \operatorname{sign}(y)\,|y|^{\frac{1}{q-1}} q^{\frac{1}{1-q}} & \text{if } y \in \left(0,\ q(\frac{M}{q-1})^{\frac{q-1}{q}}\right), \\ 0 & \text{otherwise.} \end{cases}$$

(see [36, Eq. (2.1)]). In particular, in Figure 4 (for $q = 3$), the convergence result seems to hold.

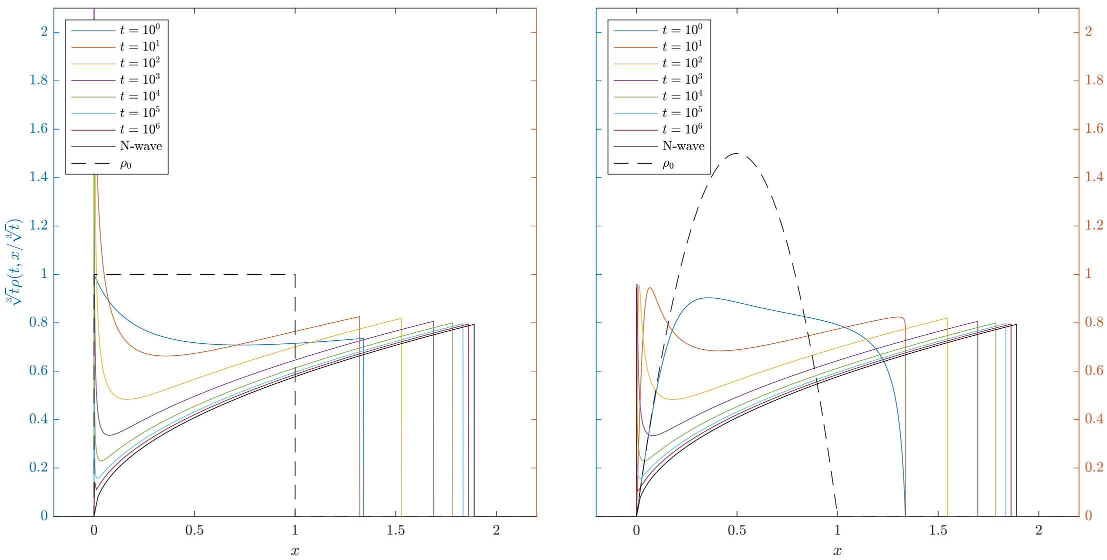

FIGURE 4. Convergence to the N-wave profile for the nonlocal regularization of $\partial_t \rho + \partial_x \rho^3 = 0$ with exponential kernel. LEFT: $\rho_0(x) = \mathbb{1}_{[0,1]}(x)$. RIGHT: $\rho_0(x) = 6x(1 - x)\mathbb{1}_{[0,1]}(x)$.

In this case, none of the previously established convergence results hold; however, the numerical experiments point to the fact that we may still observe the L^1 convergence to the N-wave profile. The behavior of the rescaled solution, which explodes at $x = 0$ is particularly noteworthy. It can be explained as follows. For the conservation law

$$\partial_t \rho(t, x) + \partial_x \big(W[\rho](t, x)^2 \rho(t, x)\big) = 0, \quad (t, x) \in (0, T) \times \mathbb{R},$$

we can compute, along characteristics (see [28, 12]),

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \rho(t, \xi(0, x; t)) &= \partial_t \rho(t, \xi(0, x; t)) + \partial_2 \rho(t, \xi(0, x; t)) \partial_3 \xi(0, x; t) \\ &= -\partial_2 \rho(t, \xi(0, x; t)) W[\rho](t, \xi(0, x; t))^2 \\ &\qquad - 2\rho(t, \xi(0, x; t)) W[\rho](t, \xi(0, x; t)) \partial_3 W[\rho](t, \xi(0, x; t)) \\ &\qquad + \partial_2 \rho(t, \xi(0, x; t)) W[\rho](t, \xi(0, x; t))^2 \\ &= -2\rho(t, \xi(0, x; t)) W[\rho](t, \xi(0, x; t)) \partial_3 W[\rho](t, \xi(0, x; t)). \end{aligned} \tag{5.3}$$

As W "looks" to the left and the solution vanishes on the left half space for all time $t > 0$, we have that $W[\rho](t, 0) = 0$ for all $t > 0$; thus, the value of the solution at $x = 0$ never changes, i.e. $\lim_{x \searrow 0} \rho(t, x) = 1$ for all $t > 0$.

On the other hand, in the case of the nonlocal Burgers' equation

$$\partial_t \rho(t, x) + \partial_x \big(W[\rho](t, x) \rho(t, x)\big) = 0,$$

(5.3) becomes

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \rho(t, \xi(0, x; t)) = -\rho(t, \xi(0, x; t)) \partial_3 W[\rho](t, \xi(0, x; t))$$

$$
\begin{aligned}
&= -\rho(t,\xi(0,x;t))\big(\rho(t,x) - W[\rho](t,\xi(0,x;t))\big) \\
&\longrightarrow -\rho(t,0)^2 \quad \text{as } x \searrow 0,
\end{aligned}
$$

which formally gets us back to the Riccati ordinary differential equation that results in the decay of Lemma 3.3 (as also observed in the numerics).

6. Conclusions

In this contribution, we have proved the convergence of the solution of the nonlocal conservation law (1.1) with bounded, integrable, and non-negative initial datum to the N-wave solution of the Burgers equation as $t \to +\infty$.

Several open problems and possible generalizations of this result could be of interest for future work. We mention a few below.

(1) *Sign-changing initial data.* The classical references that deal with the case of local conservation laws (starting from, e.g., [36]) study the case of sign-changing initial data as well; in the nonlocal setting, however, a non-negativity assumption seems to be needed in order to obtain a global existence result (see [28]).

(2) *Non-integrable initial data.* Considering initial data merely in L^∞ instead of $L^1 \cap L^\infty$ poses a significant issue: since solutions might then have infinite mass, the compactness arguments would need to be revised and the limit profile would no longer be governed by the initial mass, but rather be a non-integrable self-similar solution. We refer to [23] for the study of this problem for the heat equation. In the periodic setting, further information is available for the long-time behavior of (local) conservation laws (see, e.g., [33, 9]).

(3) *More general velocities.* While in the local case the available body of literature deals with general power nonlinearities (as mentioned in the introduction), in the nonlocal case a power-type velocity, as in $\partial_t\rho + \partial_x(W^{q-1}\rho) = 0$ (with $q > 2$), does not seem to generally allow for an Oleinik-type condition (see [10] for further discussion), which was pivotal in the approach used in the present paper.

(4) *General convolution kernels.* For more general convolution kernels, nonlocal-to-local convergence results have been obtained (see [14]); however, no Oleinik inequalities are currently available (whose regularization effect is needed to treat the case of initial data that are not uniformly bounded in L^∞ with respect to the scaling parameter). The numerical simulations from Section 5 still show that the convergence to the local N-wave should hold.

(5) *Nonlocality in the velocity.* A different type of nonlocal conservation laws presents nonlocal effect of the form $\partial_t\rho + \partial_x(\rho(\rho^{q-1} * \gamma)) = 0$, i.e. where the averaging is not done over the solution, but over the velocity. A recent contribution on the nonlocal-to-local singular limit problem for this kind of models is contained in [21].

(6) *Numerical schemes.* The development of numerical schemes that reproduce qualitatively the long-time behavior of the solution has been addressed in works dealing with diffusion operators and local velocities (see, e.g., [25, 24, 26]), but it appears to be fully unexplored in the nonlocal setting.

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