Annales Fennici Mathematici
Volumen 47, 2022, 507–531

Restrictions of Sobolev $W_p^1(\mathbb{R}^2)$-spaces to planar rectifiable curves

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Abstract. We construct explicit examples of Frostman-type measures concentrated on arbitrary simple rectifiable curves $\Gamma \subset \mathbb{R}^2$ of positive length. Based on such constructions we obtain for each $p \in (1, \infty)$ an exact description of the trace space $W_p^1(\mathbb{R}^2)|_\Gamma$ of the first-order Sobolev space $W_p^1(\mathbb{R}^2)$ to an arbitrary simple rectifiable curve Γ of positive length.

Sobolevin avaruuden $W_p^1(\mathbb{R}^2)$ rajoittumat suoristuville tasokäyrille

Tiivistelmä. Rakennamme esimerkkejä mielivaltaiselle yksinkertaiselle positiivimittaiselle suoristuvalle käyrälle $\Gamma \subset \mathbb{R}^2$ keskittyneistä Frostmanin-tyyppisistä mitoista. Näiden avulla saamme jokaisella eksponentilla $p \in (1, \infty)$ tarkan kuvailun ensimmäisen kertaluvun Sobolevin avaruuden $W_p^1(\mathbb{R}^2)$ jäljestä $W_p^1(\mathbb{R}^2)|_\Gamma$ mielivaltaisella tällaisella käyrällä Γ.

1. Introduction

The problem of the exact description of restrictions of Sobolev spaces $W_p^1(\mathbb{R}^n)$, $p \in [1, \infty]$, to different subsets $S \subset \mathbb{R}^n$ has rich history. It takes the origin in the pioneering work of Gagliardo [6] where the case $S = \mathbb{R}^{n-1}$ was considered. In fact, the methods of [6] cover the case when S is a graph of a Lipschitz function $H \colon \mathbb{R}^{n-1} \to \mathbb{R}$. Note that this work extended the earlier results by Aronszajn [2] and Slobodetskii and Babich [18] concerning the case $p = 2$. It should be mentioned that the trace problem for higher order Sobolev spaces $W_p^m(\mathbb{R}^n)$, $p \in (1, \infty)$, $m \in \mathbb{N}$, in the case $S = \mathbb{R}^d$, $d \in [1, n-1] \cap \mathbb{N}$, was solved by Besov in the fundamental paper [3].

In the case $p = \infty$, the Sobolev space $W_\infty^1(\mathbb{R}^n)$ can be identified with the space $\mathrm{LIP}(\mathbb{R}^n)$ of Lipschitz functions on $\mathbb{R}^n$. It is known (see McShane–Whitney extension lemma in Section 4.1 of [7]) that for any closed set $S \subset \mathbb{R}^n$ the restriction $\mathrm{LIP}(\mathbb{R}^n)|_S$ coincides with the space $\mathrm{LIP}(S)$ of Lipschitz functions on S and that, furthermore, the classical Whitney extension operator linearly and continuously maps the space $\mathrm{LIP}(S)$ into the space $\mathrm{LIP}(\mathbb{R}^n)$ (see e.g., [19, Chapter 6]).

In the case $p = 1$ much less is known. Indeed, as it was shown in [6] for the case $S = \mathbb{R}^{n-1}$, the trace space on $\mathbb{R}^{n-1}$ of the Sobolev space $W_1^1(\mathbb{R}^n)$ can be identified with $L_1(\mathbb{R}^{n-1})$ as a linear space, the corresponding norms being equivalent. However, the extension operator constructed by Gagliardo is nonlinear. Furthermore, it was shown by Peetre [13] (see also Section 5 in [14]) that any bounded map from $L_1(\mathbb{R}^{n-1})$ to $W_1^1(\mathbb{R}^n)$ which is right inverse of the trace map is nonlinear.

https://doi.org/10.54330/afm.115393
2020 Mathematics Subject Classification: Primary 46E35, 28A78, 28A25, 28A12.
Key words: Traces, extensions, Sobolev spaces, Frostman measures, measures on curves.
The work of A. I. Tyulenev was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2019-1614).
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In the sequel we deal with the case $p \in (1, \infty)$ only. After [6] a substantial progress was made by several mathematicians [10, 11, 21, 15, 16, 17, 9] in the direction of relaxation of extra assumptions on the sets S. Note that in [17] the corresponding trace problem was solved without any assumption on S. However, in [17] only the case $p > n$ was considered. This case is special and exploits techniques different from that of used in other papers mentioned above. Unfortunately, such techniques do not allow to attack the case $p \in (1, n]$.

Recall that given $d \in (0, n]$, a closed set $S \subset \mathbb{R}^n$ is said to be *Ahlfors–David d-regular* provided that there exist constants $c_S^1, c_S^2 > 0$ such that

$$c_S^1 r^d \leq \mathcal{H}^d(Q(x,r) \cap S) \leq c_S^2 r^d \quad \text{for every } x \in S,\ r \in (0,1]. \tag{1.1}$$

In (1.1) we set $Q(x,r) := \prod_{i=1}^n [x_i - r, x_i + r]$ and by $\mathcal{H}^d$ we denote the d-Hausdorff measure. We will also call condition (1.1) the Ahlfors–David d-regularity condition.

Summarizing results and methods of papers [10, 16, 9] restricted to the case of the first-order Sobolev spaces $W_p^1(\mathbb{R}^n)$ one can obtain for any fixed $d \in (0, n]$ and $p \in (\max\{1, n-d\}, \infty)$ an exact description of the trace space of the space $W_p^1(\mathbb{R}^n)$ to any closed set $S \subset \mathbb{R}^n$ satisfying the Ahlfors–David d-regularity condition (1.1).

Rychkov introduced in [15] the concept of d-thick sets. Recall that given $d \in [0, n]$, a set $S \subset \mathbb{R}^n$ is said to be *d-thick* if there exists a constant $c_S^3 > 0$ such that

$$c_S^3 r^d \leq \mathcal{H}_\infty^d(Q(x,r) \cap S) \quad \text{for every } x \in S,\ r \in (0,1], \tag{1.2}$$

where by $\mathcal{H}_\infty^d$ we denote the so-called d-Hausdorff content. Note that condition (1.2) is much weaker than (1.1). It was noted in [15] and proved in [23] that every path-connected set $S \subset \mathbb{R}^n$ consisting of more than one point is 1-thick. It is clear that a generic path-connected set S fails to satisfy the Ahlfors–David 1-regularity condition. In [15] trace criteria for Besov $B_{p,q}^s(\mathbb{R}^n)$ and Lizorkin–Triebel $F_{p,q}^s(\mathbb{R}^n)$ spaces were obtained for d-thick sets S. However, that criteria were not fully intrinsic and were based on atomic-type characterizations of function spaces. Furthermore, in the case $s \in \mathbb{Z}$ the extra assumption $d > n - 1$ was required. In particular, with these restrictions one cannot hope to attack the trace problem for Sobolev spaces $W_p^1(\mathbb{R}^n)$ in the case of d-thick sets S with $d \in [0, n-1]$.

In [11] the trace problem for Sobolev spaces was considered in the case when $S \subset \mathbb{R}^2$ is a single cusp satisfying some extra regularity assumptions. In [21] the trace problem for Besov $B_{p,q}^s(\mathbb{R}^n)$ and Lizorkin–Triebel $F_{p,q}^s(\mathbb{R}^n)$ spaces was considered in the case when $S \subset \mathbb{R}^n$ is a domain satisfying the so-called internal and external regularity assumptions. However, the corresponding criteria were not fully intrinsic and involved atomic-type characterizations.

Very recently [23], given $d \in [0, n]$, $p \in (\max\{1, n-d\}, \infty)$, and a closed d-thick set $S \subset \mathbb{R}^n$, an exact description of traces of functions $F \in W_p^1(\mathbb{R}^n)$ to the set S was obtained. As far as we know, it was the first result concerning trace problems for the first-order Sobolev spaces $W_p^1(\mathbb{R}^n)$ obtained for the range $p \in (1, n]$ in such a high generality. Furthermore, these results were generalized to the case of weighted Sobolev spaces in [22].

Analysis of the results obtained in [10], [16], [9] shows that in the case of Ahlofrs-David d-regular sets S the only measure which played a crucial role in the solution of the trace problems is the d-Hausdorff measure $\mathcal{H}^d$. This is not the case for d-thick sets. Unfortunately, in this case one should work with a sequence of measures with prescribed growth conditions instead of the only "nice measure". Recall [23] that given a closed d-thick set $S \subset \mathbb{R}^n$, by a *d-regular on S sequence of measure* we will

mean any sequence of Radon measures $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k\in\mathbb{N}_0}$ satisfying the following properties:

(1) for every $k \in \mathbb{N}_0$

$$\operatorname{supp} \mathfrak{m}_k = S; \tag{1.3}$$

(2) there exists a constant $C^1 > 0$ such that for each $k \in \mathbb{N}_0$

$$\mathfrak{m}_k(Q(x,r)) \le C^1 r^d \quad \text{for every } x \in \mathbb{R}^n \text{ and every } r \in (0, 2^{-k}]; \tag{1.4}$$

(3) there exists a constant $C^2 > 0$ such that for each $k \in \mathbb{N}_0$

$$\mathfrak{m}_k(Q(x,2^{-k})) \ge C^2 2^{-dk} \quad \text{for every } x \in S; \tag{1.5}$$

(4) for each $k \in \mathbb{N}_0$ the measure $\mathfrak{m}_k = w_k \mathfrak{m}_0$ with $w_k \in L_\infty(\mathfrak{m}_0)$ and

$$2^{d-n} w_{k+1}(x) \le w_k(x) \le w_{k+1}(x) \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{1.6}$$

We denote by $C^1_{\{\mathfrak{m}_k\}}$ and $C^2_{\{\mathfrak{m}_k\}}$ the minimum among the constants C^1 in (1.4) and the maximum among the constants C^2 in (1.5) respectively. It was proved in [23] that for each closed d-thick set $S \subset \mathbb{R}^n$ there is a d-regular on S sequence of measures.

To describe the aim of this paper and for the sake of completeness of exposition we formulate one particular case of the main result from [23]. In doing so we firstly recall some concepts that were introduced in [23]. Given an arbitrary sequence $\{\mathfrak{m}_k\}$ of Radon measures on $\mathbb{R}^n$, we define for each $t \in (0,1]$ the *Calderon-type maximal function* of f with respect to $\{\mathfrak{m}_k\}$ as

$$f^\sharp_{\{\mathfrak{m}_k\}}(x,t) := \sup_{k\in\mathbb{N}_0, 2^{-k}\ge t} 2^k \inf_{c\in\mathbb{R}} \fint_{Q(x,2^{-k})} |f(y) - c| \, d\mathfrak{m}_k(y), \tag{1.7}$$

where the corresponding averaged integrals are assumed to be zero in the case $\mathfrak{m}_k(Q(x,2^{-k})) = 0$.

Finally, given a closed set $S \subset \mathbb{R}^n$ and a parameter $\lambda \in (0,1)$, we define for each $t \in (0,1]$ the *maximal λ-porous at the scale t subset of S* by

$$S_t(\lambda) := \{x \in S \colon \text{there exists } y \in Q(x,t) \text{ s.t. } Q(y,\lambda t) \subset \mathbb{R}^n \setminus S\}. \tag{1.8}$$

Given parameters $d \in (0,n]$, $p \in (\max\{1, n-d\}, \infty)$, and a closed set S with $\mathcal{H}^d(S) > 0$, we denote the trace space of the Sobolev space $W_p^1(\mathbb{R}^n)$ to the set S (see the next section for the precise definition) by $W_p^1(\mathbb{R}^n)|_S$.

Recall briefly the construction of the extension operator from [23]. Given $d \in (0,n]$ and a closed d-thick set $S \subset \mathbb{R}^n$, let $\{Q_\alpha\}_{\alpha\in I}$ be the Whitney decomposition of $\mathbb{R}^n \setminus S$ and let $\mathcal{I} \subset I$ be the index set corresponding to the Whitney cubes with side lengths ≤ 1. Let $\{\varphi_\alpha\}_{\alpha\in I}$ be the corresponding partition of unity (see [23] for details). For any cube $Q_\alpha = Q(x_\alpha, r_\alpha)$, $\alpha \in I$, we define the cube $\widetilde{Q}_\alpha := Q(\widetilde{x}_\alpha, r_\alpha)$, where $\widetilde{x}_\alpha$ is an arbitrary metric projection of x_α to the set S. Let $\{\mathfrak{m}_k\}$ be an arbitrary d-regular on S sequence of measures. Let $f \in L_1^{\mathrm{loc}}(\mathfrak{m}_k)$ for some (and hence every) $k \in \mathbb{N}_0$. We set $k(r) := [\log_2 r^{-1}]$ and define

$$F(x) = \operatorname{Ext}_{S,\{\mathfrak{m}_k\}}[f](x) := \sum_{\alpha\in\mathcal{I}} \varphi_\alpha(x) \fint_{\widetilde{Q}_\alpha \cap S} f(\widetilde{x}) \, d\mathfrak{m}_{k(r_\alpha)}(\widetilde{x}), \quad x \in \mathbb{R}^n. \tag{1.9}$$

Theorem A. *Let $d \in (0,n)$ and $p \in (\max\{1, n-d\}, \infty)$. Let $S \subset \mathbb{R}^n$ be a closed d-thick set with $\mathcal{H}^n(S) = 0$. Given a d-regular on S sequence of measures $\{\mathfrak{m}_k\} := \{\mathfrak{m}_k\}_{k\in\mathbb{N}_0}$, a Borel function $f \colon S \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^n)|_S$ if and only if the following conditions hold:*

(1) *for $\mathcal{H}^d$-almost every point $x \in S$ it holds,*

$$\lim_{k\to\infty} \fint_{Q(x,2^{-k})\cap S} |f(x) - f(z)|\, d\mathfrak{m}_k(z) = 0; \tag{1.10}$$

(2) *there exists a number $\lambda_0 \in (0,1)$ such that (we set $S_k(\lambda) := S_{2^{-k}}(\lambda)$)*

$$\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda_0}[f] := \|f|L_p(\mathfrak{m}_0)\| + \left(\sum_{k=1}^{\infty} 2^{k(d-n)} \int_{S_k(\lambda_0)} \big(f^{\sharp}_{\{\mathfrak{m}_k\}}(x,2^{-k})\big)^p\, d\mathfrak{m}_k(x)\right)^{\frac{1}{p}} < +\infty. \tag{1.11}$$

Furthermore, for every $\lambda \in (0,\lambda_0]$ there exists a constant $C > 0$ depending only on $p, n, d, C^1_{\{\mathfrak{m}_k\}}, C^2_{\{\mathfrak{m}_k\}}$, and λ such that

$$\frac{1}{C}\,\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}[f] \le \|f|W^1_p(\mathbb{R}^n)|_S\| \le C\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}[f]. \tag{1.12}$$

The extension operator $\mathrm{Ext}_{S,\{\mathfrak{m}_k\}}$ is a right inverse operator of the usual trace operator. It maps the trace space $W^1_p(\mathbb{R}^n)|_S$ to the space $W^1_p(\mathbb{R}^n)$ linearly and continuously.

In fact, Theorem A is an almost immediate consequence of Theorem 2.1 from [23]. The only delicate point which we should mention is that condition (1.10) is more rough than the corresponding condition in [23], because it is based on Hausdorff measures instead of capacities. On the other hand, our definition of the trace space given in this paper is based on Hausdorff measures rather than capacities (compare Definition 2.2 in this paper with Definitions 2.7, 2.8, 2.9 in [23]). Hence, a careful analysis of proofs of Lemma 4.3 and Theorem 4.2 in [23] in combination with very well known relations between capacities and Hausdorff measures shows that condition (1.10) is justified.

In practice the criterion given in Theorem A is not so easy to verify by the following reasons:

(1) it is difficult to check the delicate condition (1.10);

(2) in fact, the typical construction of a d-regular sequence of measures $\{\mathfrak{m}_k\}$ given in [23] is not fully explicit. Indeed, the classical Frostman-type arguments based on an inductive algorithm and on the weak limit procedure were used. Hence, the construction of explicit examples of d-regular sequences of measures $\{\mathfrak{m}_k\}$ on different closed d-thick sets $S \subset \mathbb{R}^n$ is of great importance.

The aim of this paper is to demonstrate that one can overcome the difficulties described above and simplify Theorem A in the case when S is *a planar simple (i.e. without self-intersections) rectifiable curve $\Gamma \subset \mathbb{R}^2$ of positive length.* In this case Γ is a 1-thick set. We construct *a special* 1*-regular on* Γ *sequence of measures* $\{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$. The advantage of the sequence $\{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$ is that the measures $\mu_k[\Gamma]$, $k \in \mathbb{N}_0$, have explicit expressions. Furthermore, having such measures at disposal one can get rid of the sophisticated condition (1.10) in Theorem A. Note that the results obtained in the present paper *are new and could not be obtained by previously known methods.* As far as we know, *explicit constructions of Frostman-type measures* on arbitrary simple planar rectifiable curves Γ of positive length were not considered in the literature before. Furthermore, even in the the case of a planar simple rectifiable curve $S = \Gamma \subset \mathbb{R}^2$ of positive length, the result formulated in Theorem A cannot be obtained by previously known methods. On the one hand, this result cannot be derived using the methods of [15] because in the case of Sobolev spaces the corresponding trace problem was considered only for d-thick sets $S \subset \mathbb{R}^n$ satisfying the additional requirement

$d > n-1$. On the other hand, there are planar rectifiable curves of positive length that fail to satisfy the Ahlfors–David 1-regularity condition (and hence, fail to satisfy the Ahlfors–David d-regularity conditions for all $d \in (0,2]$) and hence do not fall into the scope of [9].

Now we explain (informally) why in the present paper we restrict ourselves to simple rectifiable curves $\Gamma \subset \mathbb{R}^2$ only.

(A) In fact, the arguments of section 3 work for simple rectifiable curves of positive length in $\mathbb{R}^n$ for any $n \geq 2$. The only minor issue here is that in the case $n > 2$ the corresponding expressions for measures will be more technical. On the other hand, what is more important is the application of Theorem 3.1 in Section 4 where the restriction on the dimension $n \in \mathbb{N}$ of the ambient space becomes essential. More precisely, as we have already mentioned, any rectifiable curve of positive length is a 1-thick subset of $\mathbb{R}^n$. But Theorem 4.1 (which is a keystone for the main result of) works for d-thick sets with $d \in [n-1, n]$. This obstruction justifies our analysis in the 2-dimensional plane.

(B) The main reason why we restrict ourselves to the case of curves instead of general 1-thick sets in $\mathbb{R}^2$ is that the corresponding expressions for the Frostman-type measures concentrated on such sets will be much less transparent. Roughly speaking, the main technical advantage of our construction, which works for curves, is the reduction of 1-dimensional Frostman-type measures to the 0-dimensional Frostman-type measures. Indeed, it is well known that any simple planar rectifiable curve Γ has a finite number of intersections with "almost every" line parallel to any fixed coordinate axis. This allows to built the corresponding 0-dimensional Frostman-type measure with the help of elementary combinatorial arguments. Clearly, one cannot hope to make a similar trick for general 1-thick sets in $\mathbb{R}^2$ because intersections of such sets with lines parallel to coordinate axes can have much more complicated geometry.

Acknowledgements. I dedicate this paper to my "scientific grandfather" Academician of Russian Academy of Science, Professor S. M. Nikol'skii. The paper was written in 2020, the year of his 115-anniversary. I thank P. Shvartsman who read my joint paper with S. Vodop'yanov [23] and inspired me to built specific examples of Frostman-type measures. I am grateful to V. Bogachev and N. Gigli for fruitful discussions that helped to clarify some ideas of this paper. Furthermore, I am grateful to D. Stolyarov and A. Volberg for valuable remarks. I would like to thank the referees for very careful reading and numerous comments and suggestions, that led to improvements of the manuscript.

2. Preliminaries

Throughout the paper $C, C_1, C_2, \ldots$ will be generic positive constants. These constants can change even in a single line. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n,p,k)$. We write $A \approx B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$. Given a number $c \in \mathbb{R}$ we denote by $[c]$ the integer part of c.

2.1. Geometric measure theory background. We let $\mathbb{R}^n$, $n \geq 1$, denote the linear space of all strings $x = (x_1, \ldots, x_n)$ of real numbers *equipped with the uniform norm* $\|\cdot\|_\infty$ i.e., $\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}$. As usual, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Given a set $E \subset \mathbb{R}^n$, we denote by $\operatorname{int} E$, $\overline{E}$ and E^c the interior, the closure, and the complement (in $\mathbb{R}^n$) of E, respectively. Given a set $E \subset \mathbb{R}^n$, we will always denote

by χ_E the characteristic function of E. By *a cube* Q in $\mathbb{R}^n$ we mean a *closed* cube with sides parallel to the coordinate axes. We say that $E \subset \mathbb{R}^n$ is *a measurable set* if E belongs to the standard Lebesgue σ-algebra. Given a measurable set $E \subset \mathbb{R}^n$, we say that a function $f\colon E \to \mathbb{R}$ is *measurable* if $f^{-1}((c,+\infty])$ is measurable for any $c \in \mathbb{R}$.

In the sequel, given a metric space $\mathrm{X} = (\mathrm{X}, \mathrm{d})$, by a measure on X we mean only *a nonnegative Borel measure* on X. Given a measure $\mathfrak{m}$ on X and a nonempty Borel set $S \subset \mathrm{X}$, we define the *restriction* $\mathfrak{m}\lfloor_S$ *of* $\mathfrak{m}$ to S by

$$\mathfrak{m}\lfloor_S(E) := \mathfrak{m}(E \cap S) \quad \text{for any Borel set} \quad E \subset \mathrm{X}\,. \tag{2.1}$$

Given two metric spaces $(\mathrm{X}_1, \mathrm{d}_1)$, $(\mathrm{X}_2, \mathrm{d}_2)$, a Borel map $G\colon \mathrm{X}_1 \to \mathrm{X}_2$ and a Borel measure $\mathfrak{m}$ on X_1, we define the *push-forward measure* $G_\sharp\mathfrak{m}$ on X_2 by letting

$$G_\sharp\mathfrak{m}(E) := \mathfrak{m}(G^{-1}(E)) \quad \text{for any Borel set } E \subset \mathrm{X}_2\,. \tag{2.2}$$

Let $\mathfrak{m}$ be an arbitrary measure on $\mathbb{R}^n$. Given $f \in L_1^{\mathrm{loc}}(\mathbb{R}^2, \mathfrak{m})$ and a Borel set $G \subset \mathbb{R}^n$ with $\mathfrak{m}(G) < +\infty$, we put

$$\fint_G f(x)\,d\mathfrak{m}(x) := \begin{cases} \frac{1}{\mathfrak{m}(G)} \int_G f(x)\,d\mathfrak{m}(x), & \text{if } \mathfrak{m}(G) > 0; \\ 0, & \text{if } \mathfrak{m}(G) = 0. \end{cases} \tag{2.3}$$

Given a Radon measure $\mathfrak{m}$ on $\mathbb{R}^n$, we set for every cube $Q \subset \mathbb{R}^n$

$$\widetilde{\mathcal{E}}_{\mathfrak{m}}[f](Q) := \fint_Q \fint_Q |f(y) - f(z)|\,d\mathfrak{m}(y)\,d\mathfrak{m}(z). \tag{2.4}$$

Recall that Calderon-type maximal functions $f^\sharp_{\{\mathfrak{m}_k\}}$ were defined in the introduction.

Proposition 2.1. *Let $\{\mathfrak{m}_k\} = \{\mathfrak{m}_k\}_{k\in\mathbb{N}_0}$ be a sequence of Radon measures on $\mathbb{R}^n$. Then for each $t \in (0,1]$ and every $x \in \mathbb{R}^n$,*

$$\frac{1}{2} \sup_{k\in\mathbb{N}_0, 2^{-k}\geq t} 2^k \widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q(x,2^{-k})) \leq f^\sharp_{\{\mathfrak{m}_k\}}(x,t) \leq \sup_{k\in\mathbb{N}_0, 2^{-k}\geq t} 2^k \widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q(x,2^{-k})). \tag{2.5}$$

Proof. We fix arbitrary $x \in \mathbb{R}^n$,$r \in (0,1]$ and set $Q = Q(x,r)$ for brevity. The first inequality in (2.5) follows from the fact that for each $k \in \mathbb{N}_0$ and any constant $c \in \mathbb{R}$ we have

$$\begin{aligned} \widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q) &\leq \fint_Q |f(y) - c|\,d\mathfrak{m}_k(y) + \fint_Q |-f(z) + c|\,d\mathfrak{m}_k(z) \\ &\leq 2 \fint_Q |f(y) - c|\,d\mathfrak{m}_k(y). \end{aligned}$$

The second inequality in (2.5) follows from the estimate

$$\begin{aligned} \inf_{c\in\mathbb{R}} \fint_Q |f(y) - c|\,d\mathfrak{m}_k(y) &\leq \fint_Q \Big| f(y) - \fint_Q f(z)\,d\mathfrak{m}_k(z) \Big|\,d\mathfrak{m}_k(y) \\ &\leq \widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q), \quad k \in \mathbb{N}_0. \end{aligned}$$

□

It will be convenient to follow [5] (see Section 1.2 therein) and define for each $d \in [0,n]$ and $\delta \in (0,\infty]$

$$\mathcal{H}^d_\delta(E) := \inf \sum_{i=1}^\infty (\operatorname{diam} U_i)^d,$$

where the infimum is over all coverings $\{U_i\}_{i\in\mathbb{N}}$ of E with $\operatorname{diam} U_i < \delta$, $i \in \mathbb{N}$. Now, the *d-Hausdorff measure $\mathcal{H}^d$ of the set E* is defined as $\mathcal{H}^d(E) := \lim_{\delta\to 0} \mathcal{H}^d_\delta(E)$. By the *$d$-Hausdorff content of E* we mean $\mathcal{H}^d_\infty(E)$.

The following proposition is an immediate consequence of (1.4) and the definition of $\mathcal{H}^d$. We omit an elementary proof.

Proposition 2.2. *Let $d \in (0, n]$. Let $S \subset \mathbb{R}^n$ be a closed d-thick set. Let $\{\mathfrak{m}_k\}$ be a d-regular on S sequence of measures. Then, for every $k \in \mathbb{N}_0$ the measure $\mathfrak{m}_k$ is absolutely continuous with respect to $\mathcal{H}^d\lfloor_S$. Furthermore, for each $k \in \mathbb{N}_0$,*

$$\mathfrak{m}_k(E) \le C^1_{\{\mathfrak{m}_k\}}\mathcal{H}^d(E) \quad \text{for any Borel set } E \subset S. \tag{2.6}$$

Remark 2.1. Note that the right-hand side of (2.6) can be equal to $+\infty$ and hence the corresponding estimate is trivial.

Given a measurable set $E \subset \mathbb{R}^n$, recall that a map $f\colon E \to \mathbb{R}$, $E \subset \mathbb{R}$ is said to have *the Lusin property* if for any set $E_0 \subset E$ of Lebesgue measure zero the image $f(E_0)$ has Lebesgue measure zero. The following result is a particular case of Theorem 12 in [8] (see also Theorem 4.3.3 in [12] where minor modifications are required). In fact, this result will be a keystone for the whole Section 3 below.

Proposition 2.3. *Let a map $\Phi\colon \mathbb{R} \to \mathbb{R}$ be absolutely continuous. Let $g\colon \mathbb{R} \to [0, +\infty]$ be a measurable function. Then for any measurable set $E \subset \mathbb{R}$*

$$\begin{aligned}\int_E g(x)|\Phi'(x)|\,dx &= \int_{\Phi(E)} \sum_{x\in\Phi^{-1}(y)\cap E} g(x)\,d\mathcal{H}^1(y)\\ &= \int_{\Phi(E)} \left(\int_{\Phi^{-1}(y)\cap E} g(x)\,d\mathcal{H}^0(x)\right) d\mathcal{H}^1(y).\end{aligned} \tag{2.7}$$

Let $S \subset \mathbb{R}^n$ be a closed set. We recall that given $\lambda \in (0, 1)$ and $t \in (0, 1]$, the definition of the set $S_t(\lambda)$ was given in the introduction. In what follows for every $k \in \mathbb{N}$ we set $S_k(\lambda) := S_{2^{-k}}(\lambda)$. It is natural to ask whether there exists a parameter $\lambda \in (0, 1)$ such that the union $\bigcup_{k\in\mathbb{N}_0} S_k(\lambda)$ contains S or equivalently $\max_{k\in\mathbb{N}_0} \chi_{S_k(\lambda)}(x) = 1$ for every $x \in S$? Unfortunately, this is not the case in general. We introduce the following concept.

Definition 2.1. Let $d \in (0, n]$ and $\lambda \in (0, 1)$. Let $S \subset \mathbb{R}^n$ be a closed set with $\mathcal{H}^d(S) > 0$. We say that S is *(d, λ)-quasi-porous* if

$$\lim_{k\to\infty} \chi_{S_k(\lambda)}(x) = 1 \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in S. \tag{2.8}$$

Remark 2.2. Clearly, for each $k \in \mathbb{N}_0$ we have $S_k(\lambda) \subset S_k(\lambda')$ for all $\lambda' \in (0, \lambda]$. Hence, if a set S is (d, λ)-quasi-porous for some $\lambda \in (0, 1)$, then it is (d, λ')-quasi-porous for every $\lambda' \in (0, \lambda]$.

The following lemma gives a simple sufficient condition for a given closed set S to be (d, λ)-quasi-porous.

Lemma 2.1. *Let $d \in (0, n)$. Suppose that a closed set $S \subset \mathbb{R}^n$ is such that:*

(1) *$\mathcal{H}^d(S) \in (0, \infty)$;*
(2) *there is a constant $c_1 > 0$ such that*

$$\inf_{r\in(0,1]} \frac{\mathcal{H}^d(Q(x, r) \cap S)}{r^d} \ge c_1 \quad \text{for every} \quad x \in S. \tag{2.9}$$

Then, there exists $\lambda_0(S) \in (0,1)$ *such that* S *is* (d,λ)*-quasi-porous for every* $\lambda \in (0,\lambda_0]$.

Proof. The proof is very close in spirit to that of Proposition 9.18 in [20]. We present the details for completeness.

Since $\mathcal{H}^d(S) \in (0,\infty)$, it is well known (see, for example, Theorem 1.3.9 in [12]) that there exists a constant $c_2 > 0$ depending only on d and a set $S' \subset S$ with $\mathcal{H}^d(S \setminus S') = 0$ such that

$$\overline{\lim_{r\to 0}} \frac{\mathcal{H}^d(Q(x,r)\cap S)}{r^d} \le c_2 \quad \text{for every } x \in S'. \tag{2.10}$$

Now we fix an arbitrary point $x_0 \in S'$. By (2.9) and (2.10) there is a small $r_0 = r_0(x_0) > 0$ such that

$$c_1 \le \frac{\mathcal{H}^d(Q(x_0,r)\cap S)}{r^d} \le 2c_2 \quad \text{for all} \quad r \in (0,r_0). \tag{2.11}$$

Fix a sufficiently large number $N \in \mathbb{N}$ to be specified later. We fix $r \in (0, r_0/2)$, subdivide the cube $Q(x_0,r)$ into 2^{Nn} congruent cubes, and choose those of them that have nonempty intersections with S. Let $\{Q_i\}_{i=1}^M$ be the family of all such chosen cubes. For each $i \in \{1,\dots,M\}$ take an arbitrary point $x_i \in Q_i \cap S$. Clearly, we have $Q_i \subset Q(x_i, 2^{-N+1}r) := Q_i^*$ for every $i \in \{1,\dots,M\}$. Hence, $S \cap Q(x_0,r) \subset \bigcup_{i=1}^M Q_i^*$. The multiplicity of the covering of the set $Q(x_0,r)\cap S$ by the cubes Q_i^*, $i = 1,\dots,M$, is bounded from above by 5^n. As a result, using the subadditivity of $\mathcal{H}^d$, we get

$$2c_2 r^d \ge \mathcal{H}^d(Q(x_0,r)\cap S) \ge 5^{-n}\sum_{i=1}^M \mathcal{H}^d(Q_i^*\cap S) \ge 5^{-n} c_1 M 2^{d-Nd} r^d.$$

Hence, this gives

$$M \le \left(\frac{2c_2}{c_1} 5^n 2^{-d} 2^{(d-n)N}\right) 2^{nN}. \tag{2.12}$$

Since $d < n$ we can take $N = N(d,n,c_1,c_2) \in \mathbb{N}$ so large that $M < 2^{nN}$. This gives existence of at least one cube $Q_i \subset Q(x_0,r)\setminus S$. Hence, if we set $\lambda_0 = 1/N$, we get that $x_0 \in S_k(\lambda_0)$ for every $k > -\log_2 r + 1$. This observation together with Remark 2.2 completes the proof. □

Remark 2.3. Clearly, each Ahlfors–David d-regular set $S \subset \mathbb{R}^n$ (with $d \in (0,n)$) satisfies conditions (1) and (2) of Lemma 2.1. The converse is false. For example, each rectifiable curve $\Gamma \subset \mathbb{R}^n$ of positive length satisfies conditions (1) and (2) of Lemma 2.1 with $d = 1$. On the other hand, in section 5 we present simple examples of planar rectifiable curves that fail to satisfy Ahlfors–David 1-regularity condition.

2.2. Sobolev spaces. As usual, for each $p \in [1,\infty]$, we let $W_p^1(\mathbb{R}^n)$ denote the corresponding Sobolev space of all equivalence classes of real-valued functions $F \in L_p(\mathbb{R}^n)$ whose distributional partial derivatives $D^\beta F$ on $\mathbb{R}^n$ of order $|\beta| \le 1$ belong to $L_p(\mathbb{R}^n)$. This space is normed by

$$\|F\|_{W_p^1(\mathbb{R}^n)} := \sum_{|\beta|\le 1} \|D^\beta F\|_{L_p(\mathbb{R}^n)}.$$

Recall (see, e.g., [1, Section 6.2]) that given a parameter $p \in (1,n]$, for every element $F \in W_p^1(\mathbb{R}^n)$ there is *a representative* $\widehat{F}$ in the equivalence class of the

element F such that $\widehat{F}$ has Lebesgue points everywhere except a set E_F of zero $C_{1,p}$-capacity. Furthermore, according to the Sobolev embedding theorem (see, e.g., [1, Theorem 1.2.4]), given a parameter $p > n$, for every $F \in W_p^1(\mathbb{R}^n)$ there is a *continuous representative* $\widehat{F}$ of F. In the sequel we will call $\widehat{F}$ a *good representative of the element* F. Recall also (see [1, Theorem 5.1.13]) that if $p \in (1, n]$, $d \in (n-p, n]$, then for any given set $S \subset \mathbb{R}^n$ the condition $C_{1,p}(S) = 0$ implies $\mathcal{H}^d(S) = 0$. Since in this paper we mainly focus on traces of Sobolev functions to 1-dimensional path-connected sets in $\mathbb{R}^2$ we can use a little bit more rough definition of the trace of a given Sobolev function than the corresponding one used in [23]. More precisely, the later definition is based on $C_{1,p}$-capacities instead of the Hausdorff measures. However, taking into account well known relations between the d-Hausdorff measures and $C_{1,p}$-capacities, we believe that the use of $C_{1,p}$-capacities is not reasonable in the present framework. Informally speaking, our trace criterion in Theorem 4.2 is expressed in terms of Frostman-type measures $\mu_k[\Gamma]$ which "do not feel" changes of a trace function on a set of $\mathcal{H}^1$-measure zero. These remarks justify the following definition.

Definition 2.2. Let $d \in (0, n]$, $p \in (\max\{1, n-d\}, \infty)$, and $F \in W_p^1(\mathbb{R}^n)$. Let S be a Borel set with $\mathcal{H}^d(S) > 0$. We define the trace $F|_S$ of the element F to the set S as

$$F|_S := \{f\colon S \to \mathbb{R}\colon f(x) = \widehat{F}(x) \text{ for } \mathcal{H}^d\text{-a.e. } x \in S\}. \tag{2.13}$$

We define the trace space $W_p^1(\mathbb{R}^n)|_S$ of the space $W_p^1(\mathbb{R}^n)$ as

$$W_p^1(\mathbb{R}^n)|_S := \{f\colon S \to \mathbb{R}\colon f = F|_S \text{ for some } F \in W_p^1(\mathbb{R}^n)\} \tag{2.14}$$

and equip it with the usual quotient-space norm, i.e.

$$\|f|W_p^1(\mathbb{R}^n)|_S\| := \inf\{\|F|W_p^1(\mathbb{R}^n)\|\colon f = F|_S\}. \tag{2.15}$$

We denote by $\mathrm{Tr}\,|_S\colon W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)|_S$ the corresponding *trace operator.*

Remark 2.4. Since the trace $F|_S$ of a given Sobolev function is a class of equivalent (modulo coincidence on a set of $\mathcal{H}^d$-measure zero) functions $f\colon S \to \mathbb{R}$, the trace operator *is well defined and linear.*

2.3. Rectifiable curves in $\mathbb{R}^n$. By *a curve in* $\mathbb{R}^n$ we mean the image Γ of a continuous map $\gamma\colon [a, b] \to \mathbb{R}^n$, i.e. $\Gamma = \gamma([a, b])$. The map γ is *called a parametrization of the curve* Γ. We say that a curve Γ is *simple* if the map γ is injective. We say that Γ is *rectifiable* provided that

$$l(\Gamma) := \sup \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|_\infty < \infty, \tag{2.16}$$

where the supremum is taken over all tuples $\{t_i\}_{i=0}^n$ such that $a = t_0 < t_1 < \ldots < t_n = b$. The associated *length function* s_γ is defined as

$$s_\gamma(t) := l(\gamma([a, a+t])), \quad t \in [a, b].$$

The properties summarized in the next proposition are well known (see, e.g., [7, Chapter 5]).

Proposition 2.4. *Let $\Gamma \subset \mathbb{R}^n$ be a rectifiable curve and let $\gamma\colon [a, b] \to \mathbb{R}^n$ be its parametrization. Then the curve Γ admits the arc length parametrization. More precisely, there exists the 1-Lipschitz map $\gamma_s\colon [0, l(\gamma)] \to \mathbb{R}^n$ defined by*

$$\gamma_s(\tau) := \gamma(s_\gamma^{-1}(\tau)), \tag{2.17}$$

where

$$s_\gamma^{-1}(\tau) := \sup\{s\colon s_\gamma(s) = \tau\}. \tag{2.18}$$

If the curve Γ *is simple, then*

$$l(\Gamma) = \mathcal{H}^1(\Gamma). \tag{2.19}$$

It is also useful to recall the infinitesimal behavior of $\mathcal{H}^1\lfloor_\Gamma$. The following property is also well known (see, e.g., [5, Lemma 3.5])

Proposition 2.5. *Let* $\Gamma \subset \mathbb{R}^n$ *be a rectifiable curve. Then,*

$$\lim_{r\to 0} \frac{\mathcal{H}^1\lfloor_\Gamma(Q(x,r))}{2r} = 1 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{2.20}$$

Clearly, one could assume everywhere in the sequel that any given curve Γ has a 1-Lipschitz parametrization (for example, the arc-length parametrization). However, as we will see in Section 5, it is useful to have some sort of flexibility in the choice of parameterizations. This fact justifies the following definition.

Definition 2.3. Given a curve $\Gamma \subset \mathbb{R}^n$ we say that a map $\gamma\colon [a,b] \to \mathbb{R}^n$ *is an admissible parametrization of the curve* Γ if the following conditions hold:

(1) the map γ is absolutely continuous;
(2) $\|\dot\gamma(t)\| > 0$ for $\mathcal{H}^1$-a.e. $t \in [a,b]$.

Given $i = 1,\dots,n$, we denote by Π_i the *projection map along* the i-th coordinate axis, i.e. if $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, then $\Pi_i(x) := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) \in \mathbb{R}^{n-1}$. Sometimes we will use the shorthand $\widehat{x}^i := \Pi_i(x)$, $i = 1,\dots,n$. Given a parametrization $\gamma = (\gamma_1,\dots,\gamma_n)\colon [a,b] \to \mathbb{R}^n$ of the curve Γ, we let γ_i, $i = 1,\dots,n$ denote its i-th component. Similarly, if for some $t_0 \in [a,b]$ there exists the velocity vector

$$\dot\gamma(t_0) := \lim_{t\to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \in \mathbb{R}^n, \tag{2.21}$$

where we let $\dot\gamma_i(t_0)$, $i = 1,\dots,n$ denote the corresponding components of $\dot\gamma(t_0)$.

Definition 2.4. Let $\Gamma \subset \mathbb{R}^n$ be a simple rectifiable curve and let $\gamma\colon [a,b] \to \mathbb{R}^n$ be an admissible parametrization of Γ. Given $i \in \{1,\dots,n\}$, we identify $\mathbb{R}^{n-1}$ with a hyperplane orthogonal to the i-th coordinate axis and define

$$\mathcal{L}_i[\Gamma](x') := \Pi_i^{-1}(x') \cap \Gamma, \quad x' \in \mathbb{R}^{n-1}. \tag{2.22}$$

Informally speaking, $\mathcal{L}_i[\Gamma](x')$ is just the intersection of the line going through x' parallel to the i-th coordinate axis with Γ. We also define the Banach indicatrix functions of Γ by

$$N_i[\Gamma](x') := \operatorname{card} \mathcal{L}_i[\Gamma](x') = \operatorname{card}\{t \in [a,b] \mid t = (\Pi_i \circ \gamma)^{-1}(x')\}, \quad x' \in \mathbb{R}^{n-1}. \tag{2.23}$$

Remark 2.5. It follows from Theorem 4.3.2 in [12] that for each $i \in \{1,\dots,n\}$ the function $N_i[\Gamma] \in L_1^{\mathrm{loc}}(\mathbb{R}^{n-1})$. Hence, $N_i[\Gamma](x') \in \mathbb{N}_0$ for each $i \in \{1,\dots,n\}$ and almost all $x' \in \mathbb{R}^{n-1}$. Furthermore, the analysis of the proof of Lemma 4.1.4 in [12] allows to deduce that $N_i[\Gamma]$, $i = 1,...,n$ are Borel.

3. Construction of a special 1-regular sequence of measures.

As we mentioned in the introduction, every path-connected set in $\mathbb{R}^n$ containing at least two distinct points is 1-thick. In particular, every curve $\Gamma \subset \mathbb{R}^n$ with $l(\Gamma) > 0$ is 1-thick. Hence, there is a 1-regular sequence of measures on Γ. The

aim of this section is to construct, for *any planar simple rectifiable curve* $\Gamma \subset \mathbb{R}^2$ *of positive length, a special* 1*-regular sequence of measures* $\{\mu_k[\Gamma]\}$ concentrated on Γ. All constructions can be easily extended to simple rectifiable curves in $\mathbb{R}^n$. We consider the case $n = 2$ for simplicity.

Let us briefly describe the main idea of the construction. In this section we will assume without loss of generality that a rectifiable curve $\Gamma \subset [0,1) \times [0,1)$. By Remark 2.5 for each $i \in \{1, \dots, n\}$ intersections of Γ with almost all lines L^i parallel to the i-th coordinate axis consist of an at most finite number of distinct points. Hence, we can easily construct a special 0-regular sequence of measures concentrated on that finite sets. After that, taking an admissible parametrization $\gamma\colon [a,b] \to \mathbb{R}^2$ of Γ and applying Proposition 2.3 we easily obtain a 1-regular sequence of measures concentrated on Γ.

Let $I_0 := [0,1)$ be a half-open unit interval. Given $k \in \mathbb{N}_0$, let

$$\mathcal{D}_k := \left\{ I_{k,m} := \left[\frac{m}{2^k}, \frac{m+1}{2^k}\right),\ m = 0, \dots, 2^k - 1 \right\}$$

be the family of all dyadic half-open intervals of side length 2^{-k} contained in I_0.

Definition 3.1. Given a set of distinct points $\{x_i\}_{i=0}^N \subset I_0$, $N \in \mathbb{N}_0$, let $\mathrm{P}(\{x_i\}_{i=0}^N)$ be the set of all probability measures with the support $\{x_i\}_{i=0}^N$. In other words, $\nu \in \mathrm{P}(\{x_i\}_{i=0}^N)$ if and only if there exists a *density function* $\alpha\colon \{x_i\}_{i=0}^N \to [0, +\infty)$ with $\sum_{i=0}^N \alpha(x_i) = 1$ such that

$$\nu = \sum_{i=0}^N \alpha(x_i)\delta_{x_i}, \tag{3.1}$$

where $\delta_{x_i}, i = 0, \dots, N$ are the Dirac measures concentrated at the points x_i, $i = 0, \dots, N$.

Definition 3.2. Let $\{x_i\}_{i=0}^N \subset I_0$, $N \in \mathbb{N}$ be an arbitrary finite set of distinct points. Let $k^* := k^*(\{x_i\}_{i=0}^N)$ be the minimal among all $k \in \mathbb{N}_0$ for each of which the map sending every point x_i to the unique dyadic interval $I_{k,m}(x_i) \ni x_i$ is injective. For every $k \geq k^*$ we define the family

$$\mathcal{F}_k(\{x_i\}_{i=0}^N) := \left\{ I_{k,m}\colon I_{k,m} \cap \{x_i\}_{i=0}^N \neq \emptyset \right\}. \tag{3.2}$$

For each $i \in \{0, \dots, N\}$ and any $k \geq k^*(\{x_i\}_{i=0}^N)$ we denote by $I_{k,m}(x_i)$ the unique dyadic interval in $\mathcal{F}_k(\{x_i\}_{i=0}^N)$ containing x_i.

Given an arbitrary tree $\mathcal{T}$ with a root r, we introduce the intrinsic metric ρ on $\mathcal{T}$ and obtain the so-called metric tree $(\mathcal{T}, \rho)$. More precisely, given two vertices $\xi, \xi' \in \mathcal{V}(\mathcal{T})$ joined by some edge e we put $\rho(\xi, \xi') = 1$. For generic two vertices $\xi, \xi' \in \mathcal{V}(\mathcal{T})$ we define $\rho(\xi, \xi') = \inf \sum_i \rho(\xi_i, \xi_{i-1})$, where the infimum is taken over all paths $\xi =: \xi_0 \leftrightarrow \dots \leftrightarrow \xi_l := \xi'$ such that ξ_i and ξ_{i+1} are joined by the unique edge e_i for all $i \in \{0, ..., l-1\}$. Given $i \in \mathbb{N}_0$, let $\mathcal{V}^i(\mathcal{T}) := \{\xi \in \mathcal{T}\colon \rho(r, \xi) = i\}$. If $\mathcal{V}^i(\mathcal{T}) \neq \emptyset$ then for any $\xi \in \mathcal{V}^i(\mathcal{T})$ we denote by $n(\xi)$ the number of edges joining ξ with the corresponding vertices in $\mathcal{V}^{i+1}(\mathcal{T})$.

Definition 3.3. Let $k \in \mathbb{N}_0$. Given a nonempty family of dyadic intervals $\mathcal{F}_k \subset \mathcal{D}_k$, we define the tree $\mathcal{T} = \mathcal{T}(\mathcal{F}_k)$ as the metric tree whose vertices $\mathcal{V}^i(\mathcal{T})$, $i = 0, \dots, k$, correspond naturally to all those dyadic intervals in $\mathcal{D}_i$ each of which contains at least one interval from $\mathcal{F}_k$.

Definition 3.4. Let $\mathcal{F}_k \subset \mathcal{D}_k$, $k \in \mathbb{N}_0$ be an arbitrary nonempty family of dyadic intervals. For each $j \in \mathbb{N}_0$ and any $I \in \mathcal{F}_k$ we define *the Frostman-type weight* by letting

$$\alpha_F^j[\mathcal{F}_k](I) := \begin{cases} \prod_{l=j}^{k-1} \frac{1}{n(\xi_l(I))} & \text{if } j \in \{0,\dots,k-1\}, \\ 1 & \text{if } j > k-1, \end{cases} \tag{3.3}$$

where for each $j \in \{0,\dots,k\}$ we denote by $\xi_j(I) \in \mathcal{V}^j(\mathcal{T})$ the unique vertex corresponding to the unique dyadic interval in $\mathcal{D}_j$ containing the interval I and $r =: \xi_0(I) \leftrightarrow \ldots \leftrightarrow \xi_k(I) := \xi(I)$ is a unique path joining the root r with the vertex $\xi(I)$. Given a finite set $\{x_i\}_{i=0}^N$, $N \in \mathbb{N}$ of distinct points, we define *the Frostman-type weight* by letting

$$\alpha_F^j[\{x_i\}_{i=0}^N](x_i) := \alpha_F^j[\mathcal{F}_k(\{x_i\}_{i=0}^N)](I_{k,m}(x_i)), \quad k \geq k^*(\{x_i\}_{i=0}^N). \tag{3.4}$$

Definition 3.5. Given a finite set $\{x_i\}_{i=0}^N \subset I_0$, $N \in \mathbb{N}$ of distinct points, we define for each $j \in \mathbb{N}_0$ *the Frostman-type measure* $\nu_F^j[\{x_i\}_{i=0}^N]$ by letting

$$\nu_F^j[\{x_i\}_{i=0}^N] := \sum_{i=0}^N \alpha_F^j[\{x_i\}_{i=0}^N](x_i)\delta_{x_i}. \tag{3.5}$$

Remark 3.1. It follows immediately from the definition of k^* that Definition 3.4 is correct. Indeed, for each $k \geq k^*$ we have $\alpha_F^j[\mathcal{F}_k(\{x_i\}_{i=0}^N)](I) = \alpha_F^j[\mathcal{F}_{k^*}(\{x_i\}_{i=0}^N)](I)$ for all $I \in \mathcal{F}_k(\{x_i\}_{i=0}^N)$.

The following assertion exhibits basic properties of Frostman-type weights.

Lemma 3.1. *Let $\{x_i\}_{i=0}^N \subset I_0$, $N \in \mathbb{N}_0$, be a set of distinct points. Then, for each $j \in \mathbb{N}_0$ the following properties hold:*

(1) *for any dyadic interval $I_{j,m}$, $m \in \{0,\dots,2^j-1\}$*

$$\nu_F^j[\{x_i\}_{i=0}^N](I_{j,m}) = \begin{cases} 1, & \{x_i\}_{i=0}^N \cap I_{j,m} \neq \emptyset; \\ 0, & \{x_i\}_{i=0}^N \cap I_{j,m} = \emptyset; \end{cases} \tag{3.6}$$

(2) *for every $i \in \{0,\dots,N\}$*

$$\begin{aligned} \textit{either} \quad & \alpha_F^j[\{x_i\}_{i=0}^N](x_i) = \alpha_F^{j+1}[\{x_i\}_{i=0}^N](x_i) \\ \textit{or} \quad & \alpha_F^j[\{x_i\}_{i=0}^N](x_i) = 2^{-1}\alpha_F^{j+1}[\{x_i\}_{i=0}^N](x_i). \end{aligned} \tag{3.7}$$

Proof. For $j \geq k^*$ the corresponding result is obvious. Let $\mathcal{T} = \mathcal{T}(\mathcal{F}_{k^*}(\{x_i\}_{i=0}^N))$ be the corresponding metric tree. Note that in the case $j < k^*$ equalities (3.7) follow directly from (3.3) because for any $\xi \in \mathcal{V}(\mathcal{T})$ we have either $n(\xi) = 1$ or $n(\xi) = 2$.

To prove (3.6) we argue by induction. For $j \geq k^*$ this is obvious. Suppose that $k^* > 1$ and that (3.6) is proved for some $j_0 \in \{1,\dots,k^*\}$. Then from (3.7) it is easy to conclude the validity of (3.6) for $j_0 - 1$ using the same arguments as above. □

Remark 3.2. Note that given a set $\{x_i\}_{i=0}^N \subset I_0$, $N \in \mathbb{N}_0$, of distinct points and $I_{j,m}$ with $I_{j,m} \cap \{x_i\}_{i=0}^N \neq \emptyset$, the restriction $\nu_F^j\lfloor_{I_{j,m}} \in \mathrm{P}(I_{j,m} \cap \{x_i\}_{i=0}^N)$. Clearly, in general ν_F^j is not a probability measure on I_0.

We recall Definition 2.4.

Definition 3.6. Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be an admissible parametrization of Γ. Given $j \in \mathbb{N}_0$, we

define for each $t \in [a,b]$

$$\mathcal{W}_1^j[\gamma](t) := \begin{cases} \alpha_F^j[\mathcal{L}_1[\Gamma](\gamma_2(t))](\gamma_1(t)) & \text{if } N_1[\Gamma](\gamma_2(t)) < \infty; \\ 0 & \text{if } N_1[\Gamma](\gamma_2(t)) = \infty; \end{cases} \tag{3.8}$$

and similarly,

$$\mathcal{W}_2^j[\gamma](t) := \begin{cases} \alpha_F^j[\mathcal{L}_2[\Gamma](\gamma_1(t))](\gamma_2(t)) & \text{if } N_2[\Gamma](\gamma_1(t)) < \infty; \\ 0 & \text{if } N_2[\Gamma](\gamma_1(t)) = \infty. \end{cases} \tag{3.9}$$

Given $j \in \mathbb{N}_0$ we also define *the special density* as

$$\mathrm{D}^j[\gamma](t) := \max\{\mathcal{W}_2^j[\gamma](t)|\dot\gamma_1(t)|, \mathcal{W}_1^j[\gamma](t)|\dot\gamma_2(t)|\}, \quad t \in [a,b]. \tag{3.10}$$

Remark 3.3. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ and $\gamma'\colon [a',b'] \to \mathbb{R}^2$ be two admissible parameterizations of Γ. Suppose that $\gamma(t) = \gamma'(t'(t))$ for some strictly increasing absolutely continuous function $t'\colon [a,b] \to [a',b']$. Then it is easy to see that $\mathcal{W}_i^j[\gamma](t) = \mathcal{W}_i^j[\gamma'](t'(t))$ for each $i = 1,2$, every $j \in \mathbb{N}_0$ and all $t \in [a,b]$.

Proposition 3.1. *Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be an admissible parametrization of Γ. Then for every $j \in \mathbb{N}_0$ the functions $\mathcal{W}_i^j[\gamma]$, $i = 1,2$ and $\mathrm{D}^j[\gamma]$ are measurable.*

Proof. First of all we prove that the functions $\mathcal{W}_1^j[\gamma]$, $j \in \mathbb{N}_0$ are measurable. The proof of measurability of the functions $\mathcal{W}_2^j[\gamma]$, $j \in \mathbb{N}_0$ requires exactly the same arguments. By Lemma 3.1 it is sufficient to verify that $\mathcal{W}_1^0[\gamma]$ is measurable.

Fix an arbitrary number $k \in \mathbb{N}$. Let $\mathcal{F}_k \subset \mathcal{D}_k$ be an arbitrary nonempty family of dyadic intervals. We define

$$F_k := \bigcup_{I_{k,m} \in \mathcal{F}_k} I_{k,m}, \quad \widetilde{E}(F_k) := \gamma_2(\gamma_1^{-1}(F_k)) \setminus \gamma_2([a,b] \setminus \gamma_1^{-1}(F_k)).$$

In other words, $\widetilde{E}(F_k)$ is the set of all $x_2 \in \Pi_1(\Gamma)$ such that $\mathcal{L}_1[\Gamma](x_2) \subset F_k$. The set $\gamma_1^{-1}(F_k)$ is a Borel set. Since γ_2 is continuous, this implies that the set $\widetilde{E}(F_k)$ is a difference of two Souslin sets, and hence, is universally measurable. Now we define

$$E(F_k) := \widetilde{E}(F_k) \setminus \bigcup_{\substack{F_k' \subset F_k \\ F_k' \neq F_k}} \widetilde{E}(F_k'), \quad G(F_k) := \gamma_2^{-1}(E(F_k)).$$

Informally speaking, $E(F_k)$ is the set of all points on the second coordinate axis for each of which the line going through the point and parallel to the first coordinate axis the corresponding intersection with Γ consist of sets of points whose projections to the first coordinate axis meet every interval from the family $\mathcal{F}_k$ and do not meet the other dyadic intervals.

Clearly, $E(F_k)$ and $G(F_k)$ are universally measurable. We set

$$\mathcal{W}_{k,1}[\gamma](t) := \alpha_F[\mathcal{F}_k](I_{k,m}), \quad \text{if } t \in \gamma_1^{-1}(I_{k,m}) \cap G(F_k). \tag{3.11}$$

Since for different families $\mathcal{F}_k$ and $\mathcal{F}_k'$ the sets $G(F_k)$ and $G(F_k')$ are disjoint and since

$$[a,b] = \bigcup_{\mathcal{F}_k} G(F_k)$$

the function $\mathcal{W}_{k,1}[\gamma]$ is well defined everywhere on $[a,b]$ and is measurable.

From Remark 2.5 it follows that the set $\mathcal{G}$ where $N_1[\Gamma] = +\infty$ has measure $\mathcal{H}^1(\mathcal{G}) = 0$. Since $N_1[\Gamma]$ is a Borel function, the set $\mathcal{G}$ is Borel. Hence, the set $\gamma_2^{-1}(\mathcal{G})$ is a Borel set. As a result, by the very definition of the sets $E(F_k)$, $k \in \mathbb{N}_0$ and $\mathcal{G}$,

$$\lim_{k\to\infty} \mathcal{W}_{k,1}[\gamma](t) = \mathcal{W}_1^0[\gamma](t) \quad \text{for every } t \in [a,b] \setminus \gamma_2^{-1}(\mathcal{G}).$$

This implies that $\mathcal{W}_1^0[\gamma]$ is measurable. Note that the image of the function $\mathcal{W}_1^0[\gamma]$ is an at most countably set. Using this fact together with the measurability of $\mathcal{W}_1^0[\gamma]$ and (3.3), (3.4), (3.7), (3.8) it is easy to get measurability of $\mathcal{W}_1^j[\gamma]$ for all $j \in \mathbb{N}$.

Finally, to prove measurability of $\mathrm{D}^j[\gamma]$ it is sufficient to use measurability of $\mathcal{W}_i^j$ just established and take into account condition (1) of Definition 2.3. The proof is complete. □

Now we are ready to give the key definition.

Definition 3.7. Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be an admissible parametrization of Γ. We say that the sequence of measures $\{\mu_k[\Gamma]\} := \{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$ *is a special sequence of measures on* Γ if

$$\mu_k[\Gamma] := \gamma_\sharp(\mathrm{D}^k[\gamma]\mathcal{H}^1\lfloor_{[a,b]}) \quad \text{for every } k \in \mathbb{N}_0. \tag{3.12}$$

We are going to show that the measures $\mu_k[\Gamma]$, $k \in \mathbb{N}_0$ are well defined. For this purpose we recall the following property. Probably it looks like a folklore but we give a proof for the completeness.

Proposition 3.2. *Let $[a,b] \subset \mathbb{R}$, $[A,B] \subset \mathbb{R}$ and let $g\colon [a,b] \to [A,B]$ be absolutely continuous. Assume that*

$$\frac{dg}{dt}(t) > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [a,b]. \tag{3.13}$$

Then the inverse function $g^{-1}\colon [A,B] \to [a,b]$ is absolutely continuous.

Proof. From (3.13) it follows that $g\colon [a,b] \to [A,B]$ is strictly increasing. Therefore, there exists the inverse function $g^{-1}\colon [A,B] \to [a,b]$ which is continuous and strictly increasing. Hence, to prove that $g^{-1}\colon [A,B] \to [a,b]$ is absolutely continuous it is sufficient to check the Lusin property. Assume the contrary. Hence, there exists a set $E \subset [A,B]$ with $\mathcal{H}^1(E) = 0$ such that $\mathcal{H}^1(g^{-1}(E)) > 0$. Combing this with (3.13) we get

$$r_0 := \int_{g^{-1}(E)} \frac{dg}{dt}(t)\,dt > 0. \tag{3.14}$$

Let $\{(A_i,B_i)\}_{i=1}^\infty$ be an arbitrary sequence of nonempty intervals such that $E \subset \bigcup_{i=1}^\infty (A_i,B_i)$. Since g^{-1} is strictly increasing we have a sequence of nonempty intervals $\{(a_i,b_i)\}_{i=1}^\infty$ such that $(a_i,b_i) := g^{-1}((A_i,B_i))$, $i \in \mathbb{N}$. Clearly $g^{-1}(E) \subset \cup_{i=1}^\infty (a_i,b_i)$. Since g is absolutely continuous we can apply the Newton–Leibniz formula and take into account (3.14). As a result, we have

$$\sum_{i=1}^\infty |A_i - B_i| = \sum_{i=1}^\infty \int_{a_i}^{b_i} \frac{dg}{dt}(t)\,dt \ge r_0. \tag{3.15}$$

The sequence $\{(A_i,B_i)\}_{i=1}^\infty$ was chosen arbitrarily. Hence, by (3.15) and the definition of the measure $\mathcal{H}^1$ we conclude that $\mathcal{H}^1(E) \ge r_0 > 0$. This contradiction completes the proof. □

Proposition 3.3. *Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ and $\gamma'\colon [a',b'] \to \mathbb{R}^2$ be admissible parameterizations of Γ. Then*

$$\gamma_\sharp(\mathrm{D}^k[\gamma]\mathcal{H}^1\lfloor_{[a,b]}) = \gamma'_\sharp(\mathrm{D}^k[\gamma']\mathcal{H}^1\lfloor_{[a',b']}). \tag{3.16}$$

Proof. Since the parameterizations γ and γ' are admissible we see that the associated length functions s_γ and $s_{\gamma'}$ are absolutely continuous and strictly increasing. Furthermore,

$$\begin{aligned} \dot{s}_\gamma(t) &= \|\dot{\gamma}(t)\| > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [a,b]; \\ \dot{s}_{\gamma'}(t') &= \|\dot{\gamma'}(t')\| > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } t' \in [a',b']. \end{aligned} \tag{3.17}$$

Hence, there are continuous strictly increasing functions $t = t(s) = s_\gamma^{-1}(s)$ and $t' = t'(s) = s_{\gamma'}^{-1}(s)$. We set $t'(t) := s_{\gamma'}^{-1}(s_\gamma(t))$ for all $t \in [a,b]$. Note that

$$\frac{dt'}{dt}(t) > 0, \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [a,b].$$

By Proposition 3.2 not only the functions s_γ, $s_{\gamma'}$ but also the functions s_γ^{-1}, $s_{\gamma'}^{-1}$ have the Lusin property. Hence, the function $t'(\cdot)$ is absolutely continuous and strictly increasing. As a result, applying Proposition 2.3 with $\Phi(t) = t'(t)$, taking into account (3.10) and Remark 3.3, we establish for any measurable set E, the desirable equality

$$\int_E \mathrm{D}^k[\gamma](t)\,dt = \int_E \mathrm{D}^k[\gamma'](t'(t))\frac{dt'}{dt}(t)\,dt = \int_{t'(E)} \mathrm{D}^k[\gamma'](t')\,dt'.$$

The proof is complete. □

Recall Proposition 2.4. Now we can formulate *the main result of this section.*

Theorem 3.1. *Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Then the special sequence of measures $\{\mu_k[\Gamma]\} := \{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$ is 1-regular on Γ.*

Proof. Using Proposition 3.3 we may assume without loss of generality that the curve Γ is parameterized by the arc length γ_s. During the proof we use the shorthand $\gamma = \gamma_s$. By Proposition 2.4 the map $\gamma\colon [0, l(\Gamma)] \to \Gamma$ is 1-Lipschitz. Clearly the maps γ_i, $i = 1,2$ are 1-Lipschitz as compositions of γ with the corresponding projections Π_i. We set

$$l_i = \operatorname{diam} \gamma_i([a,b]) \quad \text{for } i = 1,2.$$

Since the curve Γ is rectifiable and $l(\Gamma) > 0$, we have

$$0 < l(\Gamma) \le l_1 + l_2 \le 2l(\Gamma) < +\infty. \tag{3.18}$$

Now we should verify that conditions (1)–(4) in the definition of a 1-regular on Γ sequence of measures (see the introduction) hold true for the sequence $\{\mu_k[\Gamma]\}$.

Step 1. Since γ is the arc length parametrization of Γ it follows directly from (3.10) that for each $k \in \mathbb{N}$

$$\mathrm{D}^k[\gamma](t) > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [a,b].$$

Hence, the construction (3.12) gives

$$\operatorname{supp} \mu_k[\Gamma] = \Gamma \quad \text{for every } k \in \mathbb{N}_0. \tag{3.19}$$

Step 2. Now it is convenient to verify condition (4). It follows immediately from (3.7), (3.8), (3.9) and (3.10) that, for every $k \in \mathbb{N}_0$,

$$2^{-1}\,\mathrm{D}^{k+1}[\gamma](t) \le \mathrm{D}^{k}[\gamma](t) \le \mathrm{D}^{k+1}[\gamma](t) \quad \text{for } \mathcal{H}^1\text{-a.e. } t \in [a,b]. \tag{3.20}$$

This fact together with (3.12) implies existence of a sequence of weights $\{w_k[\Gamma]\}$ such that for every $k \in \mathbb{N}$ we have $\mu_k[\Gamma] = w_k[\Gamma]\mu_0[\Gamma]$ and

$$2^{-1} w_{k+1}[\Gamma](x) \le w_k[\Gamma](x) \le w_{k+1}[\Gamma](x) \quad \text{for } \mu_0[\Gamma]\text{-a.e. } x \in \Gamma. \tag{3.21}$$

Step 3. Fix an arbitrary $j \in \mathbb{N}_0$. If $i = 1$ we set $i' = 2$, if $i = 2$ we set $i' = 1$. It is clear that for any (half-open) dyadic cube $Q_{j,m}$ with $j \ge k$, $m \in \mathbb{Z}^2$, we have by (3.10) and (3.12)

$$\begin{aligned} &\int_{\gamma^{-1}(Q_{j,m})} \mathcal{W}^k_{i'}[\gamma](s)|\dot\gamma_i(s)|\,ds \le \mu_k[\Gamma](Q_{j,m}) \\ &\le \int_{\gamma^{-1}(Q_{j,m})} \Big(\mathcal{W}^k_1[\gamma](s)|\dot\gamma_2(s)| + \mathcal{W}^k_2[\gamma](s)|\dot\gamma_1(s)|\Big)\,ds, \quad i = 1,2. \end{aligned} \tag{3.22}$$

Since the map γ is injective, we have, for $\mathcal{H}^1$-a.e. $x_i \in \Pi_{i'}(Q_{j,m})$,

$$\int_{\gamma^{-1}(Q_{j,m})\cap\gamma_i^{-1}(x_i)} \mathcal{W}^k_{i'}[\gamma](s)\,d\mathcal{H}^0(s) = \int_{\Pi_i(Q_{j,m})} d\nu^k_F[\mathcal{L}_{i'}[\Gamma](x_i)](x_{i'}). \tag{3.23}$$

Recall that γ_i, $i = 1,2$, are Lipschitz maps. Hence, we apply Proposition 2.3 and use (3.23) taking into account Definition 3.6 and Remark 3.2. We get for each $k \in \mathbb{N}_0$ and any $j \ge k$, $m \in \mathbb{Z}^2$,

$$\begin{aligned} &\int_{\gamma^{-1}(Q_{j,m})} \mathcal{W}^k_{i'}[\gamma](s)|\dot\gamma_i(s)|\,ds \\ &= \int_{\Pi_{i'}(Q_{j,m}\cap\Gamma)} \left(\int_{\Pi_i(Q_{j,m})} d\nu^k_F[\mathcal{L}_{i'}[\Gamma](x_i)](x_{i'})\right) d\mathcal{H}^1(x_i) \\ &\le \int_{\Pi_{i'}(Q_{j,m}\cap\Gamma)} d\mathcal{H}^1(y) = \mathcal{H}^1(\Pi_{i'}(Q_{j,m}\cap\Gamma)), \quad i = 1,2. \end{aligned} \tag{3.24}$$

On the other hand, similar arguments allow to deduce for every $k \in \mathbb{N}_0$, $m \in \mathbb{Z}^2$, the following equality

$$\int_{\gamma^{-1}(Q_{k,m})} \mathcal{W}^k_{i'}[\gamma](s)|\dot\gamma_i(s)|\,ds = \mathcal{H}^1(\Pi_{i'}(Q_{k,m}\cap\Gamma)), \quad i = 1,2. \tag{3.25}$$

Step 4. We verify condition (3) with $d = 1$ for the sequence $\{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$. Fix $k \in \mathbb{N}_0$ and $x \in \Gamma$. We set $Q_k(x) := Q(x, 2^{-k})$ for brevity. We use (3.19), (3.21), (3.25) and subadditivity of the measure $\mathcal{H}^1$. This gives

$$\begin{aligned} \mu_k[\Gamma](Q_k(x)\cap\Gamma) &= \mu_k[\Gamma](Q_k(x)) \ge \tfrac14\mu_{k+2}[\Gamma](Q_k(x)) \\ &\ge \frac14 \sum_{\substack{m\in\mathbb{Z}^2 \\ Q_{k+2,m}\cap\frac12 Q_k(x)\ne\emptyset}} \mu_{k+2}[\Gamma](Q_{k+2,m}) \\ &\ge \sum_{\substack{m\in\mathbb{Z}^2 \\ Q_{k+2,m}\cap\frac12 Q_k(x)\ne\emptyset}} \tfrac14\mathcal{H}^1(\Pi_i(Q_{k+2,m}\cap\Gamma)) \\ &\ge \tfrac14\mathcal{H}^1(\Pi_i(\frac12 Q_k(x)\cap\Gamma)), \quad i = 1,2. \end{aligned} \tag{3.26}$$

Since $Q_k(x) \cap \Gamma$ is path-connected we clearly have for $k \geq -\log_2(\max\{l_1, l_2\})$

$$\max\{\mathcal{H}^1(\Pi_1(\frac{1}{2}Q_k(x) \cap \Gamma)), \mathcal{H}^1(\Pi_2(\frac{1}{2}Q_k(x) \cap \Gamma))\} \geq 2^{-k-1}. \tag{3.27}$$

Finally, combining (3.26) and (3.27) we deduce

$$\begin{aligned} \mu_k[\Gamma](Q_k(x)) &\geq \frac{1}{8}\min\{1, 2^k \max\{l_1, l_2\}\}2^{-k} \\ &\geq \frac{1}{8}\min\{1, \max\{l_1, l_2\}\}2^{-k}, \quad k \in \mathbb{N}_0. \end{aligned} \tag{3.28}$$

Step 5. To verify condition (2) with $d = 1$ we fix $x \in \Gamma$ and $r \in (0, 2^{-k}]$. We set $k(r) := [\log_2 r^{-1}]$. Note that there are at most 25 dyadic cubes $Q_{k(r),m}$ whose intersections with $Q = Q(x, r)$ are nonempty. Hence, we apply the second inequality in (3.22) and then use (3.24) with $j = k(r)$ (it is possible because $k(r) \geq k$). We get

$$\begin{aligned} \mu_k[\Gamma](Q) &\leq \sum_{\substack{m \in \mathbb{Z}^2 \\ Q_{k(r),m} \cap Q \neq \emptyset}} \mu_k[\Gamma](Q_{k(r),m}) \\ &\leq \sum_{\substack{m \in \mathbb{Z}^n \\ Q_{k(r),m} \cap Q \neq \emptyset}} \sum_{i=1}^{2} \mathcal{H}^1(\Pi_i(Q_{k(r),m} \cap \Gamma)) \leq \frac{50}{2^{k(r)}} \leq 50r. \end{aligned} \tag{3.29}$$

The proof is complete. □

4. Main results

During the whole section we use the shorthand $Q_k(x) := Q(x, 2^{-k})$, $k \in \mathbb{N}_0$. The proof of the following lemma is based on standard arguments. Nevertheless, as far as we know, the assertion is new. We present the full proof for the completeness.

Lemma 4.1. *Let $\Gamma \subset [0, 1) \times [0, 1)$ be a simple rectifiable curve of positive length. Let $\{\mu_k[\Gamma]\}$ be a special 1-regular sequence of measures on Γ. Let $\mu_k[\Gamma] = w_k[\Gamma]\mu_0[\Gamma]$, $k \in \mathbb{N}$. Then,*

$$\overline{\lim_{k\to\infty}} \frac{1}{w_k[\Gamma](x)} \fint_{Q(x,2^{-k})} w_k[\Gamma](y)\, d\mu_0[\Gamma](y) < +\infty \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{4.1}$$

Proof. By Proposition 3.3 we may assume that Γ is parameterized by the arc length $\gamma_s \colon [0, l(\Gamma)] \to \mathbb{R}^2$. We split the proof into several steps.

Step 1. Since $\mathcal{H}^1\lfloor_\Gamma$ is a Radon measure, by Theorem 3.1 and Proposition 2.2 there exists a function $g[\Gamma] \in L_1(\mathcal{H}^1\lfloor_\Gamma)$ such that, for $\mathcal{H}^1$-a.e. point $x \in \Gamma$, (note that $w_0[\Gamma] \equiv 1$)

$$g[\Gamma](x) = \frac{d\mu_0[\Gamma]}{d\mathcal{H}^1\lfloor_\Gamma}(x) := \lim_{r\to 0} \frac{\mu_0[\Gamma](Q(x, r))}{\mathcal{H}^1(Q(x, r) \cap \Gamma)} \leq C^1_{\{\mu_k[\Gamma]\}}. \tag{4.2}$$

Step 2. At this step we are going to show that

$$g[\Gamma](x) > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{4.3}$$

By Proposition 2.5 it follows from (4.2) that in order to prove (4.3) it is sufficient to establish

$$\underline{\lim_{r\to 0}} \frac{\mu_0[\Gamma](Q(x, r))}{r} > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{4.4}$$

Since the set Γ is connected and the map γ_s is the arc-length parametrisation of Γ we get for each point $x \in \Gamma$ and any $r < \frac{\operatorname{diam}\Gamma}{3}$ that the preimage $\gamma_s^{-1}(Q(x,r))$ contains a closed interval $[t_x^1(r), t_x^2(r)] \ni \gamma_s^{-1}(x)$ such that:

(1) $\gamma_s(t_x^1(r)) = x$, $\gamma_s(t_x^2(r)) \in \partial Q(x,r)$;
(2) it holds

$$|t_x^1(r) - t_x^2(r)| \geq r. \tag{4.5}$$

Since γ_s is the arc-length parametrization, taking into account Proposition 2.5 we obtain

$$|t_x^1(r) - t_x^2(r)| \leq l(\Gamma \cap Q(x,r)) \to 0, \quad r \to 0. \tag{4.6}$$

Combining (3.12), (4.5) and (4.6) and taking into account that γ_s has the Lusin property (because γ_s is 1-Lipschitz) we get

$$\begin{aligned} \varliminf_{r\to 0} \frac{\mu_0[\Gamma](Q(x,r))}{r} &\geq \varliminf_{r\to 0} \fint_{t_x^1(r)}^{t_x^2(r)} \chi_{[0,l(\Gamma)]}(\tau)\, \mathrm{D}^0[\gamma_s](\tau)\, d\tau \\ &= \mathrm{D}^0[\gamma_s](\gamma_s^{-1}(x)) > 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \end{aligned} \tag{4.7}$$

Step 3. Using (4.3) and the definition of the measure $\mathcal{H}^1$ given in Section 2.1 it is easy to see that the measure $\mathcal{H}^1\lfloor_\Gamma$ is absolutely continuous with respect to $\mu_0[\Gamma]$. Furthermore, $w_0[\Gamma](x) := 1$ for all $x \in \Gamma$ by Theorem 3.1 and the definition of a d-regular sequence of measures given in the introduction. As a result, by (3.21) we have

$$w_k[\Gamma](x) \geq w_0[\Gamma](x) = 1 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \tag{4.8}$$

According to Theorem 3.1 the sequence of measures $\{\mu_k[\Gamma]\}$ is 1-regular on Γ. Hence, we can apply Proposition 2.2 with $d = 1$ and then use (4.3), (4.8). This gives

$$\begin{aligned} &\varlimsup_{k\to\infty} \frac{1}{w_k[\Gamma](x)} \fint_{Q_k(x)} w_k[\Gamma](y)\, d\mu_0[\Gamma](y) \\ &= \varlimsup_{k\to\infty} \frac{1}{w_k[\Gamma](x)} \frac{\mu_k[\Gamma](Q_k(x))}{\mathcal{H}^1\lfloor_\Gamma(Q_k(x))} \frac{\mathcal{H}^1\lfloor_\Gamma(Q_k(x))}{\mu_0[\Gamma](Q_k(x))} \\ &\leq \varlimsup_{k\to\infty} \Big(\frac{1}{w_k[\Gamma](x)} \frac{C^1_{\{\mu_k[\Gamma]\}}}{g(x)} \Big) \leq \frac{C^1_{\{\mu_k[\Gamma]\}}}{g(x)} < +\infty \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma. \end{aligned} \tag{4.9}$$

□

Recall that given a closed d-thick set S, the definition of a d-regular on S sequence of measures was given in the introduction. Recall also that the functional $\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}$ was defined in (1.11).

Theorem 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $d \in [n-1, n]$, $p \in (1,\infty)$ and $\lambda \in (0,1)$. Let $S \subset \mathbb{R}^n$ be a closed d-thick set. Let $\{\mathfrak{m}_k\}$ be a d-regular on S sequence of measures. Assume that the following conditions hold:*

(1) *the set S is (d,λ)-quasi-porous;*
(2) *for $\mathfrak{m}_0$-almost every $x \in S$*

$$\varlimsup_{k\to\infty} \frac{1}{w_k(x)} \fint_{Q(x,2^{-k})} w_k(y)\, d\mathfrak{m}_0(y) < +\infty. \tag{4.10}$$

Then, the condition $\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}[f] < +\infty$ implies

$$\lim_{k\to\infty} \fint_{Q(x,2^{-k})} |f(x) - f(y)|\, d\mathfrak{m}_k(y) = 0 \quad \textit{for } \mathfrak{m}_0\textit{-a.e. } x \in S. \tag{4.11}$$

Proof. We split the proof into several steps.

Step 1. Using Proposition 2.1 we get for each $k \in \mathbb{N}_0$

$$2^{k(p-(n-d))}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p \le 2^{k(d-n)}\Big(2f^{\sharp}_{\{\mathfrak{m}_k\}}(x,2^{-k})\Big)^p \quad \text{for every } x \in S. \tag{4.12}$$

Step 2. By B. Levi theorem we deduce from (4.12) and (1.11)

$$\begin{aligned}&\int_S \Big[\sum_{k=1}^{\infty} 2^{k(p-(n-d))}\chi_{S_k(\lambda)}(x)\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p w_k(x)\Big]\, d\mathfrak{m}_0(x)\\ &= \sum_{k=1}^{\infty} 2^{k(p-(n-d))}\int_{S_k(\lambda)}\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p w_k(x)\, d\mathfrak{m}_0(x) \le 2^p\Big(\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}[f]\Big)^p.\end{aligned} \tag{4.13}$$

Step 3. Since the set S is (d,λ)-quasi-porous and $\mathcal{BN}_{\{\mathfrak{m}_k\},p,\lambda}[f] < +\infty$ by (4.13) we get

$$\sum_{k=1}^{\infty} 2^{k(p-(n-d))} w_k(x)\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p < +\infty \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{4.14}$$

In particular, this gives

$$\lim_{k\to\infty} 2^{k(p-(n-d))} w_k(x)\Big(\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p = 0 \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{4.15}$$

Since $p > 1$, $d \in [n-1, n]$ and since $w_k(x) \le 2^{k(n-d)}$ for $\mathfrak{m}_0$-a.e. $x \in S$ we obtain

$$w_k^p(x) \le 2^{k(p-1)(n-d)} w_k(x) \le 2^{k(p-(n-d))} w_k(x) \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{4.16}$$

As a result, combining (4.15) and (4.16), we get

$$\lim_{k\to\infty}\Big(w_k(x)\widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x))\Big)^p = 0 \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S. \tag{4.17}$$

Step 4. Clearly, the condition $\mathcal{BN}_{\{\mathfrak{m}_k\}p,\lambda}[f] < +\infty$ implies $f \in L_1(\mathfrak{m}_0)$. Hence, $\mathfrak{m}_0$-almost every point $x_0 \in S$ is a Lebesgue point of the function f with respect to the measure $\mathfrak{m}_0$. This gives

$$\begin{aligned}&\overline{\lim_{k\to\infty}} \fint_{Q_k(x_0)} |f(x_0) - f(x)|\, d\mathfrak{m}_k(x)\\ &\le \overline{\lim_{k\to\infty}} \fint_{Q_k(x_0)} |f(x_0) - f(y)|\, d\mathfrak{m}_0(y)\\ &\quad + \overline{\lim_{k\to\infty}} \fint_{Q_k(x_0)} \fint_{Q_k(x_0)} |f(y) - f(x)|\, d\mathfrak{m}_0(y) d\mathfrak{m}_k(x)\\ &\le \overline{\lim_{k\to\infty}} \fint_{Q_k(x_0)} \fint_{Q_k(x_0)} |f(y) - f(x)|\, d\mathfrak{m}_0(y) d\mathfrak{m}_k(x) \quad \text{for } \mathfrak{m}_0\text{-a.e. } x \in S.\end{aligned} \tag{4.18}$$

Step 5. Since $w_k(x) \geq 1$ for $\mathfrak{m}_0$-a.e. $x \in S$ and since $\operatorname{supp} \mathfrak{m}_k = S$, $k \in \mathbb{N}_0$ we have by (4.10) and (4.17)

$$
\begin{aligned}
&\varlimsup_{k\to\infty} \fint_{Q_k(x_0)} \fint_{Q_k(x_0)} |f(y) - f(x)|\, d\mathfrak{m}_k(x)\, d\mathfrak{m}_0(y) \\
&\leq \varlimsup_{k\to\infty} \frac{\mathfrak{m}_k(Q_k(x_0))}{w_k(x_0)\mathfrak{m}_0(Q_k(x_0))} \frac{w_k(x_0)}{\mathfrak{m}_k(Q_k(x_0))} \\
&\quad \cdot \int_{Q_k(x_0)} \fint_{Q_k(x_0)} |f(y) - f(x)|\, d\mathfrak{m}_k(x) w_k(y)\, d\mathfrak{m}_0(y) \\
&\leq \varlimsup_{k\to\infty} \frac{\mathfrak{m}_k(Q_k(x_0))}{w_k(x_0)\mathfrak{m}_0(Q_k(x_0))} \varlimsup_{k\to\infty} w_k(x_0) \widetilde{\mathcal{E}}_{\mathfrak{m}_k}[f](Q_k(x_0)) = 0, \quad \mathfrak{m}_0\text{-a.e. } x_0 \in S.
\end{aligned} \tag{4.19}
$$

Combining (4.18) with (4.19) we get (4.11) and complete the proof. □

If $\Gamma \subset \mathbb{R}^2$ is a simple rectifiable curve of positive length then for any 1-regular on Γ sequence of measures $\{\mathfrak{m}_k\}_{k\in\mathbb{N}_0}$ every measure $\mathfrak{m}_k$, $k \in \mathbb{N}_0$ is finite on Γ. Hence, using (1.6) we have $f \in L_1(\mathfrak{m}_{k_0})$ for some $k_0 \in \mathbb{N}_0$ if and only if $f \in L_1(\mathfrak{m}_k)$ for all $k \in \mathbb{N}_0$. Furthermore, by Hölder inequality if $f \in L_p(\mathfrak{m}_{k_0})$ for some $k_0 \in \mathbb{N}_0$ and $p \in [1, \infty)$ then $f \in L_1(\mathfrak{m}_k)$ for all $k \in \mathbb{N}_0$. Recall the construction of the extension operator given in (1.9). Given a Borel function $f \in L_p(\mu_0[\Gamma])$, we define (we set $k(r) := [\log_2 r^{-1}]$)

$$
F(x) = \operatorname{Ext}_\Gamma[f](x) := \sum_{\alpha\in\mathcal{I}} \varphi_\alpha(x) \fint_{\widetilde{Q}_\alpha\cap\Gamma} f(\widetilde{x})\, d\mu_{k(r_\alpha)}[\Gamma](\widetilde{x}), \quad x \in \mathbb{R}^2. \tag{4.20}
$$

Now we can formulate the main result of this section.

Theorem 4.2. *Let $\Gamma \subset [0,1) \times [0,1)$ be a simple rectifiable curve of positive length. Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be an admissible parametrization of Γ. Let $\{\mu_k\} = \{\mu_k[\Gamma]\}_{k\in\mathbb{N}_0}$ be a special 1-regular on Γ sequence of measures. Given $p \in (1,\infty)$, a function $f\colon \Gamma \to \mathbb{R}$ belongs to the trace space $W^1_p(\mathbb{R}^2)|_\Gamma$ if and only if there exists $\lambda_0 := \lambda_0(\Gamma) \in (0,1]$ such that*

$$
\begin{aligned}
\mathcal{BN}_{\{\mu_k\},p,\lambda_0}[f] &:= \|f|L_p(\mu_0)\| + \left(\sum_{k=1}^{\infty} 2^{-k} \int_{\Gamma_k(\lambda_0)} \left(f^\sharp_{\{\mu_k\}}(x, 2^{-k})\right)^p d\mu_k(x)\right)^{\frac{1}{p}} \\
&< \infty.
\end{aligned} \tag{4.21}
$$

Furthermore, for every $\lambda \in (0, \lambda_0]$ there exists a constant $C > 0$ depending only on p, n, λ, $C^1_{\{\mu_k\}}$, $C^2_{\{\mu_k\}}$ such that

$$
C^{-1}\mathcal{BN}_{\{\mu_k\},p,\lambda}[f] \leq \|f|W^1_p(\mathbb{R}^2)|_\Gamma\| \leq C\mathcal{BN}_{\{\mu_k\},p,\lambda}[f] \tag{4.22}
$$

and the operator $\operatorname{Ext}_\Gamma\colon W^1_p(\mathbb{R}^2)|_\Gamma \to W^1_p(\mathbb{R}^2)$ defined in (4.20) is linear, bounded and $\operatorname{Tr}|_\Gamma \circ \operatorname{Ext}_\Gamma = \operatorname{Id}$ on the space $W^1_p(\mathbb{R}^2)|_\Gamma$.

Proof. Since Γ is a compact path-connected set and $l(\Gamma) > 0$ there exists a constant $c > 0$ such that $\mathcal{H}^1(Q(x,r) \cap \Gamma) \geq cr$ for all $x \in \Gamma$ and all $r \in (0,1]$. By Lemma 2.1, there is a number $\widetilde{\lambda}_0 := \widetilde{\lambda}_0(\Gamma) \in (0,1]$ such that the curve Γ is $(1,\lambda)$-quasi-porous for any $\lambda \in (0, \widetilde{\lambda}_0]$. Hence, combining Proposition 2.2, Theorem 3.1, Lemma 4.1 and Theorem 4.1 we deduce that for any $\lambda \in (0, \widetilde{\lambda}_0]$ the condition

$\mathcal{BN}_{\{\mu_k\},p,\lambda}[f] < +\infty$ implies that

$$\lim_{k\to\infty} \fint_{Q(x,2^{-k})} |f(x) - f(y)|\, d\mu_k(y) = 0 \quad \text{for } \mu_0[\Gamma]\text{-a.e. } x \in \Gamma.$$

On the other hand, it was mentioned in the proof of Lemma 4.1 that the measure $\mathcal{H}^1\lfloor_\Gamma$ is absolutely continuous with respect to $\mu_0[\Gamma]$. Hence, $\mathcal{BN}_{\{\mu_k\},p,\lambda}[f] < +\infty$ implies that

$$\lim_{k\to\infty} \fint_{Q(x,2^{-k})} |f(x) - f(y)|\, d\mu_k(y) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \Gamma.$$

Combining this observation with Theorem A and Theorem 3.1 we complete the proof. □

5. Example

Note that despite the fact that for a given simple rectifiable curve $\Gamma \subset \mathbb{R}^2$ of positive length the measures $\mu_k[\Gamma]$, $k \in \mathbb{N}_0$ constructed in Section 3 have explicit expressions, still without additional restrictions Γ can have extremely complicated geometry and functions $\mathrm{D}^k[\gamma]$ can oscillate wildly in general. As a result, given a function $f \in W_p^1(\mathbb{R}^2)|_\Gamma$, computations of the norm $\mathcal{BN}_{\{\mu_k[\Gamma]\},p,\lambda}[f]$ can be problematic in practice.

Below we present an illustrative example, which on the one hand is quite simple for computations, and on the other hand, it exhibits the typical effects when oscillations of a given curve Γ affect to the behavior of the densities $\mathrm{D}^k[\gamma]$ (for admissible parameterizations γ).

We restrict ourselves to the case when Γ is a graph of some locally Lipschitz nonnegative function. More precisely, we assume that $\gamma_1 = \mathrm{id}$ on $[0,1]$ and $\gamma_2\colon [0,1] \to \mathbb{R}_+$ is a locally Lipschitz (i.e. $\gamma_2|_{[a,b]}$ is Lipschitz for each $[a,b] \subset (0,1)$) function. We set $\Gamma := \{(x_1,x_2)\colon x_1 \in [0,1], x_2 = \gamma_2(x_1)\}$. Note that in order to make our example interesting our curve Γ should satisfy the following requirements:

a) the function γ_2 is not (globally) Lipschitz because otherwise we fall into the scope of [6];
b) the graph Γ fails to satisfy the Ahlfors–David 1-regularity condition because otherwise we fall into the scope of [9]. Hence, γ_2 should oscillate strongly.

First of all we define

$$\zeta(t) := \begin{cases} 2t, & t \in [0,2^{-1}], \\ 2-2t, & t \in (2^{-1},1], \\ 0, & t \notin [0,1]. \end{cases} \tag{5.1}$$

For each $k \in \mathbb{N}_0$ we set

$$\psi_k(t) := \sum_{m\in\mathbb{Z}} \zeta\left(2^k\left(t - \frac{m}{2^k}\right)\right), \quad t \in \mathbb{R}. \tag{5.2}$$

Let $\{c_k\} = \{c_k\}_{k\in\mathbb{N}} \subset [0,1)$ be a sequence of nonnegative numbers and let $\{n_k\} = \{n_k\}_{k\in\mathbb{N}}$ be a sequence of nonnegative integer numbers such that $n_k > k$ for every $k \in \mathbb{N}$. We define

$$\begin{cases} \gamma_1(t) = t, & t \in [0,1]; \\ \gamma_2(t) := \sum\limits_{k=1}^{\infty} c_k \chi_{[2^{-k},2^{-k+1})}(t)\psi_{n_k}(t), & t \in [0,1]. \end{cases} \tag{5.3}$$

In other words, the graph Γ looks like a sequence of triangles. The amount of congruent triangles on the interval $[2^{-k}, 2^{-k+1})$ equals 2^{n_k-k} and the height of every such triangle equals c_k.

Since γ_2 is a continuous function, the map $\gamma = (\gamma_1, \gamma_2)\colon [0,1] \to [0,1) \times [0,1)$ gives a parametrization of the simple planar curve. Clearly, γ_2 is *locally Lipschitz* (in fact γ_2 is Lipschitz on any closed interval $[a,1]$ with $a \in (0,1]$). It is easy to see that the sequences $\{c_k\}$ and $\{n_k\}$ can be chosen in such a way that:

$$\sum_{k=1}^{\infty} c_k 2^{n_k-k} < \infty; \tag{5.4}$$

$$\lim_{k\to\infty} c_k 2^{n_k} = +\infty. \tag{5.5}$$

As a typical example one can take $c_k := k^{-\alpha} 2^{k-n_k}$, $k \in \mathbb{N}_0$ with $\alpha > 1$.

From now we assume that our planar curve Γ satisfies (5.4)–(5.5). It is clear that condition (5.4) is necessary and sufficient for the *rectifiability* of Γ. This fact together with (5.1) and local Lipschitz property of γ_2 implies that the parametrization γ *is admissible*. From (5.5) it follows that

$$\lim_{j\to\infty} 2^j \sum_{k=j}^{\infty} c_k 2^{n_k-k} = +\infty. \tag{5.6}$$

Clearly, (5.6) leads to the distortion of the second inequality in (1.1) with $d = 1$ and $S = \Gamma$ (the first inequality in (1.1) always holds true with $S = \Gamma$ and $d = 1$). Hence, our curve is locally Lipschitz and *fails to satisfy the Ahlfors–David 1-regularity condition.* Finally, (5.5) implies that γ_2 is *not globally Lipschitz.* Hence, the exact description of the trace space $W_p^1(\mathbb{R}^2)|_\Gamma$ cannot be obtained by the earlier available methods of [6, 9].

It follows directly from the construction that for any $k \in \mathbb{N}_0$,

$$\mathcal{W}_2^k[\gamma](t)|\dot\gamma_1(t)| = 1 \quad \text{for a.e. } t \in [0,1]. \tag{5.7}$$

By (5.3) we obtain for every $k \in \mathbb{N}_0$

$$|\dot\gamma_2(t)| = c_k 2^{n_k+1} \quad \text{for a.e. } t \in [2^{-k}, 2^{-k+1}]. \tag{5.8}$$

Using (5.7) and (5.8) it is easy to deduce (with the help of elementary geometrical arguments) for every $k, l \in \mathbb{N}_0$

$$\chi_{[2^{-l},2^{-l+1})}(t)\, \mathrm{D}^k[\gamma](t) \approx \chi_{[2^{-l},2^{-l+1})}(t) \widetilde{\mathrm{D}}^k[\gamma](t), \tag{5.9}$$

where we set

$$\chi_{[2^{-l},2^{-l+1})}(t) \widetilde{\mathrm{D}}^k[\gamma](t) := \begin{cases} 1 & \text{if } 2^{-k} \in (0, 2^{-n_l}) \cup (c_l, 1] \text{ or } c_l \le 2^{-n_l}; \\ 2^k c_l & \text{if } 2^{-k} \in [2^{-n_l}, c_l]. \end{cases} \tag{5.10}$$

The corresponding constants in (5.9) do not depend on γ, k, l and t. We also define

$$\widetilde{\mu}_k[\Gamma] = \gamma_\sharp(\widetilde{\mathrm{D}}^k[\gamma]\mathcal{H}^1\lfloor_{[0,1]}), \quad k \in \mathbb{N}_0. \tag{5.11}$$

We would like to describe informally the main idea of (5.9). Firstly note that $2^{-n_k} < c_k$ for all sufficiently large $k \in \mathbb{N}$ because $c_k 2^{n_k} \to +\infty$, $k \to +\infty$. This implies that (5.10) is correct. In order to obtain a simplified version of Theorem A for this case we do not use Theorem 3.1. Indeed, in a trace criterion it is sufficient to obtain not precisely the special sequence of measures $\{\mu_k[\Gamma]\}$ but rather something comparable with it. Hence, we would like to guess in some sense how can we choose

the measures $\widetilde{\mu}_k[\Gamma]$, $k \in \mathbb{N}_0$ in order $\widetilde{\mu}_k[\Gamma]$ to satisfy conditions (1)–(4) in the corresponding definition of 1-regular on Γ sequence of measures. Fix a cube $Q = Q(x,r)$ with $x \in \Gamma$ and $r \approx 2^{-k}$. Roughly speaking, our weight function $\widetilde{\mathrm{D}}^k[\gamma] \approx 1$ in the case when either the side length r is comparable with the base of the corresponding triangle (where the center of the cube is located) or when the side length of the cube is comparable with the height of the corresponding triangle. In the case when the heights of the congruent triangles (located on the corresponding dyadic intervals) a much bigger than its bases and when the side length of the cube under consideration is somewhere between these two numbers we should focus on the intersections of lines parallel to the first coordinate axis with our curve Γ.

It remains to describe the porous subsets of Γ. First of all we fix $\lambda \in (0,1)$ and $k \in \mathbb{N}_0$. For each $l \in \mathbb{N}$ we set

$$\begin{aligned} E_k^l(\lambda) := {} & \Big(\gamma_2^{-1}((c_l - \frac{1-\lambda}{2^k}, c_l]) \cap [\frac{1}{2^l}, \frac{2}{2^l})\Big) \bigcup \Big(\gamma_2^{-1}([0, \frac{1-\lambda}{2^k})) \cap [\frac{1}{2^l}, \frac{2}{2^l})\Big) \\ & \bigcup \Big(\gamma_2^{-1}((c_{l+1} - \frac{1-\lambda}{2^k}, c_l]) \cap [\frac{1}{2^l}, \frac{1}{2^l} + \frac{1-\lambda}{2^k})\Big) \\ & \bigcup \Big(\gamma_2^{-1}((c_{l-1} - \frac{1-\lambda}{2^k}, c_l]) \cap (\frac{2}{2^l} - \frac{1-\lambda}{2^k}, \frac{2}{2^l}])\Big). \end{aligned} \tag{5.12}$$

Now for every $\lambda \in (0,1)$ and $k \in \mathbb{N}_0$ we set

$$E_k(\lambda) := \bigcup_{l=1}^{\infty} E_k^l(\lambda). \tag{5.13}$$

It is an elementary verification that for every $\lambda \in (0,1)$ and $k \in \mathbb{N}_0$

$$\Gamma_k(\lambda) = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \in E_k(\lambda),\ x_2 = \gamma_2(x_1)\}. \tag{5.14}$$

Let $\{\varphi_\alpha\}_{\alpha \in \mathcal{I}}$ be the same as in (4.20). Given a Borel function $f \in L_p(\widetilde{\mu}_0[\Gamma])$, we define

$$F(x) = \widetilde{\mathrm{Ext}}_\Gamma[f](x) := \sum_{\alpha \in \mathcal{I}} \varphi_\alpha(x) \fint_{\widetilde{Q}_\alpha \cap \Gamma} f(y)\, d\widetilde{\mu}_{k(r_\alpha)}[\Gamma](y), \quad x \in \mathbb{R}^2. \tag{5.15}$$

Combining Theorem 4.2 with (5.9), (5.11), (5.14), (5.15) and Proposition 2.1, we obtain the following criterion:

Given $p \in (1,\infty)$, a function $f \colon \Gamma \to \mathbb{R}$ belongs to the trace space $W_p^1(\mathbb{R}^2)|_\Gamma$ if and only if there is $\lambda_0 \in (0,1]$ such that

$$\begin{aligned} \mathrm{BN}_{\Gamma,p,\lambda_0}[f] := {} & \left(\int_0^1 |f(\gamma_2(t))|^p\, dt\right)^{\frac{1}{p}} \\ & + \left(\sum_{k=1}^{\infty} 2^{-k} \sum_{l=1}^{\infty} \int_{E_k^l(\lambda_0)} \left(\widetilde{f}_k^\sharp(t)\right)^p \widetilde{\mathrm{D}}_k[\gamma](t)\, dt\right)^{\frac{1}{p}} < \infty, \end{aligned} \tag{5.16}$$

where we set for each $k \in \mathbb{N}_0$ (we use the shorthand $U_k(t) := \gamma^{-1}(Q_k(\gamma(t)))$)

$$\widetilde{f}_k^\sharp(t) := \sup_{0 \le k' \le k} 2^{3k'} \int_{U_{k'}(t)} \int_{U_{k'}(t)} |f \circ \gamma(s) - f \circ \gamma(s')| \widetilde{\mathrm{D}}_k[\gamma](s) \widetilde{\mathrm{D}}_k[\gamma](s')\, ds\, ds', \tag{5.17}$$

$t \in [0,1)$. Furthermore, for each $\lambda \in (0,\lambda_0]$ there is a constant $C > 0$ depending only on p and λ such that

$$C^{-1}\, \mathrm{BN}_{\Gamma,p,\lambda}[f] \le \|f|W_p^1(\mathbb{R}^2)|_\Gamma\| \le C\, \mathrm{BN}_{\Gamma,p,\lambda}[f] \tag{5.18}$$

and the operator $\widetilde{\mathrm{Ext}}_\Gamma\colon W_p^1(\mathbb{R}^2)|_\Gamma \to W_p^1(\mathbb{R}^2)$ *defined in* (5.15) *is linear, bounded and* $\mathrm{Tr}\,|_\Gamma \circ \widetilde{\mathrm{Ext}}_\Gamma = \mathrm{Id}$ *on the space* $W_p^1(\mathbb{R}^2)|_\Gamma$.

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Received 17 December 2020 • Accepted 23 June 2021 • Published online 14 March 2022

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