ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2022, Vol. 319, pp. 283–306. © Pleiades Publishing, Ltd., 2022.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2022, Vol. 319, pp. 298–323.

Some Porosity-Type Properties of Sets Related to the d-Hausdorff Content

A. I. Tyulenev [a]

Received December 9, 2021; revised June 6, 2022; accepted June 22, 2022

Abstract—Let $S \subset \mathbb{R}^n$ be a nonempty set. Given $d \in [0, n)$ and a cube $\overline{Q} \subset \mathbb{R}^n$ with side length $l = l(\overline{Q}) \in (0, 1]$, we show that if the d-Hausdorff content $\mathcal{H}^d_\infty(\overline{Q} \cap S)$ of the set $\overline{Q} \cap S$ satisfies the inequality $\mathcal{H}^d_\infty(\overline{Q} \cap S) < \overline{\lambda} l^d$ for some $\overline{\lambda} \in (0, 1)$, then the set $\overline{Q} \setminus S$ contains a specific cavity. More precisely, we prove the existence of a pseudometric $\rho = \rho_{S,d}$ such that for every sufficiently small $\delta > 0$ the δ-neighborhood $U^\rho_\delta(S)$ of S in the pseudometric ρ does not cover $\overline{Q}$. Moreover, we establish the existence of constants $\overline{\delta} = \overline{\delta}(n, d, \overline{\lambda}) > 0$ and $\underline{\gamma} = \underline{\gamma}(n, d, \overline{\lambda}) > 0$ such that $\mathcal{L}^n(\overline{Q} \setminus U^\rho_{\delta l}(S)) \ge \underline{\gamma} l^n$ for all $\delta \in (0, \overline{\delta})$, where $\mathcal{L}^n$ is the Lebesgue measure. If in addition the set S is lower content d-regular, we prove the existence of a constant $\underline{\tau} = \underline{\tau}(n, d, \overline{\lambda}) > 0$ such that the cube $\overline{Q}$ is $\underline{\tau}$-porous. The sharpness of the results is illustrated by several examples.

DOI: 10.1134/S0081543822050194

1. INTRODUCTION

In many areas of analysis the so-called porous sets play a significant role. The corresponding literature is so huge that we mention only the groundbreaking papers [4, 7, 13, 21], the beautiful survey [14], and the monograph [9]. Roughly speaking, S is porous if, for any point $x \in S$, there are cubic holes located in $\mathbb{R}^n \setminus S$ arbitrary close to x whose diameter is comparable to the distance to the point x. Under a different nomenclature, porosity was used as early as 1920 by A. Denjoy [3]. As far as we know, E. P. Dolzhenko was the first to use the term “porous set” [4].

There are intimate connections between porosity properties of sets and their dimensions. It was proved in [8] that a set $S \subset \mathbb{R}^n$ is porous if and only if its Assouad dimension is strictly less than n. The situation becomes more complicated in the context of the Hausdorff dimension. The papers [7, 11, 13] contain results stating that a “sufficiently strong porosity” of a set $S \subset \mathbb{R}^n$ implies the existence of an appropriate upper bound on its Hausdorff dimension. However (in contrast to the Assouad dimension), one can construct a nonporous set $S \subset \mathbb{R}^n$ whose Hausdorff dimension is strictly less than n. The main goal of the present paper is to understand how the behavior of the Hausdorff contents of the intersections of cubes $Q \cap S$ with a given set $S \subset \mathbb{R}^n$ affects porosity-type properties of S. Such kind of questions arose naturally in the study of trace problems for Sobolev spaces [17, 18]. This was a motivation for the present paper.

In order to briefly describe the main results and ideas of the present paper, we fix some notation and introduce basic concepts. As usual, given $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the linear space of all rows $x = (x_1, \dots, x_n)$ of real numbers. It will be convenient to equip this space with the uniform norm, i.e., $\mathbb{R}^n := (\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$. Furthermore, as usual, by $\mathcal{L}^n$ we denote the classical Lebesgue measure on $\mathbb{R}^n$. In what follows, given a number $d \in [0, n]$ and a set $E \subset \mathbb{R}^n$, by $\mathcal{H}^d(E)$ and $\mathcal{H}^d_\infty(E)$ we will denote the d-Hausdorff measure and the d-Hausdorff content of E, respectively (see the next section for the precise definitions). Throughout the paper, by a cube we will mean a *closed cube in* $\mathbb{R}^n$ *with sides parallel to the coordinate axes.* Given $x \in \mathbb{R}^n$ and $l \ge 0$,

[a] Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.
E-mail addresses: tyulenev-math@yandex.ru, tyulenev@mi-ras.ru

we set $Q_l(x) := \prod_{i=1}^n [x_i - l, x_i + l]$. In other words, $Q_l(x)$ is the ball centered at x of radius l in the space $(\mathbb{R}^n, \|\cdot\|_\infty)$. Given a cube Q, we will denote by $l(Q)$ its side length. Given a $c > 0$ and a cube $Q = Q_l(x)$, we let cQ denote the dilation of Q with respect to its center by a factor of c, i.e., $cQ_l(x) := Q_{cl}(x)$.

We recall a slightly modified modern definition of a porous set that is commonly used in the literature [14]. First of all, given a nonempty set $S \subset \mathbb{R}^n$ and a parameter $\tau \in (0, 1]$, we say that a cube $Q_l(x)$ is *(S, τ)-porous* if there is a cube $Q_{l'}(x') \subset Q_l(x) \setminus S$ with $l' \geq \tau l$. A cube $Q_l(x)$ is said to be *S-porous* if it is (S, τ)-porous for some $\tau \in (0, 1]$. The family of all (S, τ)-porous cubes will be denoted by $\mathcal{POR}_S(\tau)$. Finally, we say that a set S is *τ-porous* if $Q_l(x) \in \mathcal{POR}_S(\tau)$ for all $x \in S$ and all $l \in (0, 1]$.

Recall that for $d \in [0, n]$ a closed set $S \subset \mathbb{R}^n$ is said to be *Ahlfors–David d-regular* if there are constants $\mathrm{c}^1_S, \mathrm{c}^2_S > 0$ such that

$$\mathrm{c}^1_S l^d \leq \mathcal{H}^d(Q_l(x) \cap S) \leq \mathrm{c}^2_S l^d \qquad \forall\, x \in S, \quad \forall\, l \in (0, 1]. \tag{1.1}$$

In what follows, given $d \in [0, n]$, by $\mathcal{ADR}(d)$ we denote the class of all closed Ahlfors–David d-regular sets.

The starting point of our investigation is the following elementary but beautiful observation made by A. Jonsson [6] (see also [16, Proposition 9.18]). Note that this result was an important tool in [5, 6], where traces of Besov and Lizorkin–Triebel spaces on Ahlfors–David d-regular sets were studied.

Theorem A. *Let $d \in [0, n)$ and $S \in \mathcal{ADR}(d)$. Then there exists a constant $\tau \in (0, 1/2)$ depending only on d, n, c^1_S, and c^2_S such that S is τ-porous.*

We should make several remarks concerning Theorem A.

Remark 1.1. The requirement $d < n$ is essential. Indeed, in the case $d = n$ it is obvious that $S = \mathbb{R}^n \in \mathcal{ADR}(n)$ but the set S fails to satisfy any porosity-type properties.

Remark 1.2. Example 6.1 below shows that an analog of Theorem A fails for sets satisfying only the right inequality in (1.1). One can easily show that an analog of Theorem A also fails for sets satisfying only the left inequality in (1.1).

Remark 1.3. The Ahlfors–David d-regularity is only a sufficient condition for the porosity of S but it is far from being necessary.

Remark 1.4. Theorem A has an essential drawback. Indeed, only d-dimensional sets fall into the scope of the theorem.

Recent investigations related to trace problems for Sobolev-type spaces [12, 15, 18–20, 22] called for the study of porosity-type properties of more complicated (in comparison with Ahlfors–David regular sets) sets that can be composed of pieces of different dimensions. This gives a motivation for finding some less restrictive conditions on a set S that are still sufficient for some porosity-type properties of S.

Given $d \in [0, n]$, a set $S \subset \mathbb{R}^n$ is said to be *lower content d-regular* (or equivalently, *d-thick*) if there exists a constant $\lambda \in (0, 1]$ such that

$$\mathcal{H}^d_\infty(Q_l(x) \cap S) \geq \lambda l^d \qquad \forall\, x \in S, \quad \forall\, l \in (0, 1]. \tag{1.2}$$

Since the parameter λ will play some role below, we introduce the following notation. Given $d \in [0, n]$ and $\lambda \in (0, 1]$, we denote by $\mathcal{LCR}(d, \lambda)$ the class of all sets $S \subset \mathbb{R}^n$ for which (1.2) holds. Furthermore, we set $\mathcal{LCR}(d) := \bigcup_{\lambda \in (0,1]} \mathcal{LCR}(d, \lambda)$.

As far as we know, d-thick sets were introduced by V. Rychkov in [12]. Recently, d-thick sets were thoroughly studied in [1, 2], where they were called lower content d-regular sets. The class

$\mathcal{LCR}(d)$ is a natural and far reaching generalization of the class $\mathcal{ADR}(d)$. Indeed,

$$\mathcal{ADR}(d) \subset \mathcal{LCR}(d) \qquad \forall\, d \in (0, n]. \tag{1.3}$$

It is clear that $\mathcal{ADR}(n) = \mathcal{LCR}(n)$. If $d \in (0, n)$, the inclusion (1.3) is strict. Indeed, it was noticed in [12] and showed in [22] that any path-connected set containing at least two distinct points is 1-thick. On the other hand, it is easy to built planar rectifiable curves that fail to satisfy the Ahlfors–David 1-regularity condition (see [19]) and hence fail to satisfy the Ahlfors–David d-regularity conditions for all $d \in (0, 2]$. In the recent papers [12, 19, 20, 22] it was discovered that d-thick sets can be effectively used in the theory of traces of function spaces.

Problem A. Suppose we are given parameters $d \in (0, n)$ and $\lambda \in (0, 1)$. Does there exist a constant $\tau \in (0, 1/2)$ such that $\mathcal{LCR}(d, \lambda) \subset \mathcal{POR}(\tau)$?

Unfortunately, the answer to Problem A is negative. For example, in the case $S = \mathbb{R}^n$ we have $S \in \mathcal{LCR}(d, \lambda)$ for all $d \in (0, n)$ and all $\lambda \in (0, 1]$ but $S \notin \mathcal{POR}(\tau)$ for any $\tau \in (0, 1/2)$. The reason for that is clear. In contrast to condition (1.1), condition (1.2) contains a nontrivial lower bound for the content but does not give any nontrivial estimate from above. The estimate $\mathcal{H}^d_\infty(Q_l(x) \cap S) \le l^d$ holds automatically and does not give any information.

Our first main result looks like a natural generalization of Theorem A.

Theorem 1.1. *Let $d \in [0, n)$ and $\lambda \in (0, 1)$. For every $\overline{\lambda} \in (0, 1)$ there exists a constant $\tau = \underline{\tau}(n, d, \lambda, \overline{\lambda}) \in (0, 1)$ such that, for every set $S \in \mathcal{LCR}(d, \lambda)$, every cube $\overline{Q}$ with $l(\overline{Q}) \in (0, 1]$ and $\mathcal{H}^d_\infty(\overline{Q} \cap S) < \overline{\lambda}(l(\overline{Q}))^d$ is $(S, \underline{\tau})$-porous.*

In fact we will show that Theorem 1.1 is a simple corollary to a much deeper and more complicated result. In order to formulate it, we need some notation.

Recall that a pseudometric on $\mathbb{R}^n$ is a symmetric nonnegative function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ satisfying the triangle inequality. In what follows, given a pseudometric ρ on $\mathbb{R}^n$, we will denote the pseudometric space $(\mathbb{R}^n, \rho)$ by $\mathbb{R}^n_\rho$. By $B^{n,\rho}_r(x)$ we will denote the closed ball centered at $x \in \mathbb{R}^n$ with radius r (in the pseudometric ρ), i.e., $B^{n,\rho}_r(x) := \{y \in \mathbb{R}^n \colon \rho(x, y) \le r\}$. Given a pseudometric ρ on $\mathbb{R}^n$, a nonempty set $S \subset \mathbb{R}^n$, and a parameter $\tau \in (0, 1]$, we say that the ball $B^{n,\rho}_r(x)$ is (S, ρ, τ)-*porous* if there is a ball $B^{n,\rho}_{r'}(x') \subset B^{n,\rho}_r(x) \setminus S$ with $r' \ge \tau r$. A ball B is said to be (S, ρ)-porous if it is (S, ρ, τ)-porous for some $\tau \in (0, 1]$. Given $\tau \in (0, 1/2)$, a set $S \subset \mathbb{R}^n$ is said to be (ρ, τ)-porous if, for every $x \in S$ and every $r \in (0, 1]$, the ball $B^{n,\rho}_r(x)$ is (S, ρ, τ)-porous.

Given a nonempty set $S \subset \mathbb{R}^n$ and parameters $d \in [0, n]$ and $\lambda \in (0, 1]$, we put

$$\mathcal{F}_S(d, \lambda) := \bigl\{Q\colon\ \mathcal{H}^d_\infty(Q \cap S) \ge \lambda(l(Q))^d\bigr\}. \tag{1.4}$$

Since all the cubes Q are assumed to be closed, we obviously have $\{x\} \in \mathcal{F}_S(d, \lambda)$ for all $x \in S$.

Given parameters $d \in [0, n]$, $\lambda \in (0, 1]$ and a nonempty set S, we define the (d, λ)-*thick distance with respect to* S from an arbitrary point $y \in \mathbb{R}^n \setminus S$ to S by the formula

$$\mathrm{D}_{S,d,\lambda}(y, S) := \begin{cases} \inf\{l(Q)\colon Q \in \mathcal{F}_S(d, \lambda),\ y \in Q\}, & \{Q \in \mathcal{F}_S(d, \lambda)\colon y \in Q\} \ne \varnothing, \\ +\infty, & \{Q \in \mathcal{F}_S(d, \lambda)\colon y \in Q\} = \varnothing. \end{cases}$$

Given $\delta \ge 0$, we also define the (d, λ)-*thick* δ-*neighborhood* of S by the formula

$$S_\delta(d, \lambda) := \bigl\{y \in \mathbb{R}^n\colon\ \mathrm{D}_{S,d,\lambda}(y, S) \le \delta\bigr\}. \tag{1.5}$$

In Section 5 we introduce a pseudometric $\rho := \rho_{S,d,\lambda}$ on $\mathbb{R}^n$ and show that the set $S_\delta(d, \lambda)$ is a δ-neighborhood of S in the pseudometric ρ.

If we disregard the particular form of holes in a cube, we can obtain a natural generalization of the concept of porous cubes. Given a nonempty set $S \subset \mathbb{R}^n$ and a number $\gamma \in (0, 1]$, we say that a cube $Q = Q_l(x)$ with $x \in \mathbb{R}^n$ and $l > 0$ is (S, γ)-*hollow* if there is a Borel set $\Omega \subset Q \setminus S$ (called an (S, γ)-*cavity of* Q) such that $\mathcal{L}^n(\Omega) \ge \gamma l^n$. Now we are ready to formulate our second main result.

Theorem 1.2. *Let $d \in [0,n)$, $\overline{\lambda} \in (0,1)$, and $\lambda \in (0,1)$. Then there exist constants $\underline{\gamma} = \underline{\gamma}(n,d,\overline{\lambda}) \in (0,1]$ and $\overline{\delta} = \overline{\delta}(n,d,\lambda,\overline{\lambda}) \in (0,1)$ such that, for every set $S \subset \mathbb{R}^n$, every cube $\overline{Q}$ with $l = l(\overline{Q}) \in (0,1]$ and $\mathcal{H}^d_\infty(\overline{Q} \cap S) < \overline{\lambda} l^d$, and any $\delta \in (0,\overline{\delta}]$, the set*

$$W_{\delta l}(\overline{Q}) := \overline{Q} \setminus S_{\delta l}(d,\lambda) \tag{1.6}$$

is an $(S,\underline{\gamma})$-cavity of Q.

We should make several remarks clarifying Theorem 1.2.

Remark 1.5. It will follow from the proof of Theorem 1.2 that

$$\lim_{\overline{\lambda}\to 1} \underline{\gamma}(n,d,\overline{\lambda}) = 0, \qquad \lim_{\overline{\lambda}\to 1} \overline{\delta}(n,d,\lambda,\overline{\lambda}) = 0, \qquad \lim_{\lambda\to 0} \overline{\delta}(n,d,\lambda,\overline{\lambda}) = 0.$$

Remark 1.6. Note that if $\overline{Q} \setminus S_{\delta l}(d,\lambda) \neq \varnothing$, then there is a ball in the pseudometric $\rho_{S,d,\lambda}$ inside $\mathbb{R}^n \setminus S$. In Section 5 we will show that $\rho_{S,0,\lambda}(x,y) = \|x-y\|_\infty$. Hence, in the case $d = 0$ the condition $\overline{Q} \setminus S_{\delta l}(0,\lambda) \neq \varnothing$ is equivalent to the $(S, \|\cdot\|_\infty)$-porosity of the cube $2\overline{Q}$.

Remark 1.7. We show in Example 6.2 that condition $d < n$ is essential and cannot be dropped.

One can show that if S is (d,λ)-thick, then, for all points $x \in \mathbb{R}^n \setminus S$ close enough to S, the (d,λ)-thick (with respect to S) distance from x to S is comparable to the usual distance from x to S. This is a key observation underlying the derivation of Theorem 1.1 from Theorem 1.2.

Structure of the paper. The paper is organized as follows.

Section 2 contains elementary background. Section 3, which is a technical core of the paper, is based on beautiful combinatorial ideas of Yu. Netrusov [10]. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we introduce a new pseudometric and establish its basic properties. Finally, Section 6 contains elementary examples demonstrating the sharpness of the main theorems.

2. PRELIMINARIES

Throughout the paper, $C, C_1, C_2, \dots$ will be generic positive constants. These constants can vary even in a single chain of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n,p,k)$. We write $A \approx B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$. Given a number $c \in \mathbb{R}$, we denote by $[c]$ the integer part of c.

Recall that we consider the linear space $\mathbb{R}^n$ of row vectors with the uniform norm $\|x\| := \|x\|_\infty := \max\{|x_1|,\dots,|x_n|\}$. Given a set $E \subset \mathbb{R}^n$, we will denote by $\operatorname{int} E$ the interior of E. By $\#E$ we will denote the cardinality of a (finite) set E. Recall that by a cube $Q \subset \mathbb{R}^n$ we mean a closed ball in the space $(\mathbb{R}^n, \|\cdot\|_\infty)$. By a *dyadic cube* we mean an arbitrary closed cube $Q_{k,m} := \prod_{i=1}^n [m_i/2^k, (m_i+1)/2^k]$ with $k \in \mathbb{Z}$ and $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$. For each $k \in \mathbb{Z}$ by $\mathcal{D}_k$ we denote the family of all closed dyadic cubes with side lengths 2^{-k}. We set

$$\mathcal{D} := \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k, \qquad \mathcal{D}_+ := \bigcup_{k\in\mathbb{N}_0} \mathcal{D}_k.$$

Given a family of cubes $\mathcal{Q}$ in $\mathbb{R}^n$ and a number $c > 1$, we set

$$c\mathcal{Q} := \{cQ \colon Q \in \mathcal{Q}\}.$$

Given a family $\mathcal{G}$ of subsets of $\mathbb{R}^n$, by $M(\mathcal{G})$ we denote its covering multiplicity, i.e., the minimal $M' \in \mathbb{N}$ such that every point $x \in \mathbb{R}^n$ belongs to at most M' sets from $\mathcal{G}$.

We need the following elementary assertion (for details, see [18]).

Proposition 2.1. *Let $c \geq 1$ and $k \in \mathbb{N}_0$. Then*

$$M(c\mathcal{D}_k) \leq ([c]+2)^n. \tag{2.1}$$

Definition 2.1. Let $\mathcal{G} = \{G_\alpha\}_{\alpha\in\mathcal{I}}$ be a family of sets and let $U \subset \mathbb{R}^n$ be a set. We define the *restriction of the family* $\mathcal{G}$ to U by the formula

$$\mathcal{G}|_U := \{G\colon\ G \subset U\}.$$

A family $\mathcal{G}$ of sets is said to be *nonoverlapping* if

$$\operatorname{int} G \cap \operatorname{int} G' = \varnothing \qquad \forall\, G, G' \in \mathcal{G}, \quad G \neq G'.$$

In what follows, by a *measure* we only mean a *nonnegative Borel* measure on $\mathbb{R}^n$. By $\mathcal{L}^n$ we denote the classical n-dimensional Lebesgue measure on $\mathbb{R}^n$. We say that a set $E \subset \mathbb{R}^n$ is *measurable* if it belongs to the standard Lebesgue σ-algebra in $\mathbb{R}^n$.

In what follows we will commonly use the following *partial order* on the set of all nonoverlapping families of dyadic cubes. Given two nonoverlapping families $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{D}$, we write $\mathcal{Q} \succeq \mathcal{Q}'$ provided that, for every $Q' \in \mathcal{Q}'$, there exists a unique cube $Q \in \mathcal{Q}$ such that $Q \supset Q'$. If, in addition, $l(Q) > l(Q')$ for all such Q and Q', we write $\mathcal{Q} \succ \mathcal{Q}'$. We say that two nonoverlapping families of dyadic cubes $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{D}$ are *comparable* if either $\mathcal{Q} \succeq \mathcal{Q}'$ or $\mathcal{Q}' \succeq \mathcal{Q}$. Otherwise we call the families *incomparable*.

Given a set $E \subset \mathbb{R}^n$, by a *covering of* E we mean a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ such that $E \subset \bigcup_{F\in\mathcal{F}} F$. Given a set $E \subset \mathbb{R}^n$, by a *dyadic nonoverlapping covering of* E we mean a nonoverlapping family $\mathcal{Q} \subset \mathcal{D}$ such that $\mathcal{Q}$ is a covering of E.

Given an at most countable family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ and a number $d \in [0, n]$, we set

$$\mathrm{H}^d(\mathcal{F}) := \sum_{F\in\mathcal{F}} (\operatorname{diam} F)^d. \tag{2.2}$$

We also define the *metric floor* and the *metric ceiling* of $\mathcal{F}$ by letting

$$\underline{\mu}(\mathcal{F}) := \inf\{\operatorname{diam} F\colon\ F \in \mathcal{F}\}, \qquad \overline{\mu}(\mathcal{F}) := \sup\{\operatorname{diam} F\colon\ F \in \mathcal{F}\}. \tag{2.3}$$

In this paper we will deal not only with the classical Hausdorff measures and contents but also with their corresponding dyadic analogs.

Definition 2.2. Let $E \subset \mathbb{R}^n$ be a nonempty set and $d \in [0, n]$. For any $\delta \in (0, \infty]$, we set

$$\mathcal{H}^d_\delta(E) := \inf \mathrm{H}^d(\mathcal{F}), \qquad \mathcal{DH}^d_\delta(E) := \inf \mathrm{H}^d(\mathcal{Q}), \tag{2.4}$$

where the first infimum is taken over all at most countable coverings $\mathcal{F}$ of E such that $\overline{\mu}(\mathcal{F}) < \delta$ and the second infimum is taken over all dyadic nonoverlapping coverings $\mathcal{Q}$ of E with $\overline{\mu}(\mathcal{Q}) < \delta$. The quantity $\mathcal{H}^d_\infty(E)$ is called the *d-Hausdorff content* of E. The quantity $\mathcal{DH}^d_\infty(E)$ is called the *dyadic d-Hausdorff content* of E. We define the *d-Hausdorff measure* and the *dyadic d-Hausdorff measure* of the set E by letting, respectively,

$$\mathcal{H}^d(E) := \lim_{\delta\to 0} \mathcal{H}^d_\delta(E), \qquad \mathcal{DH}^d(E) := \lim_{\delta\to 0} \mathcal{DH}^d_\delta(E). \tag{2.5}$$

Remark 2.1. Given a set $E \subset \mathbb{R}^n$ and a parameter $\delta \in (0, +\infty]$, it is easy to show that

$$\mathcal{H}^d_\delta(E) \le \mathcal{DH}^d_\delta(E) \le 2^n \mathcal{H}^d_\delta(E). \tag{2.6}$$

Remark 2.2. Let $d \in [0, n]$ and $\delta \in (0, \infty]$. Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then by [9, Lemma 4.6] and Remark 2.1 we get

$$\mathcal{H}^d(E) = 0 \quad\Leftrightarrow\quad \mathcal{DH}^d(E) = 0 \quad\Leftrightarrow\quad \mathcal{H}^d_\delta(E) = 0 \quad\Leftrightarrow\quad \mathcal{DH}^d_\delta(E) = 0.$$

Definition 2.3. Let $d \in [0,n]$ and $E \subset \mathbb{R}^n$ be an arbitrary set with $\mathcal{H}^d_\infty(E) > 0$. We say that a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ is a *d-almost covering of E* if there exists a set $E' \subset E$ such that $\mathcal{H}^d_\infty(E') = 0$ and $\mathcal{F}$ is a covering of $E \setminus E'$.

Definition 2.4. Let $d \in (0,n]$ and $S \subset \mathbb{R}^n$ be a set with $\mathcal{H}^d_\infty(S) \in (0,+\infty)$. Given $\varepsilon > 0$, we say that a d-almost covering $\mathcal{F}$ of S is *ε-optimal* if

$$\mathrm{H}^d(\mathcal{F}) \le (1+\varepsilon)\mathcal{H}^d_\infty(S).$$

Similarly, a dyadic nonoverlapping d-almost covering $\mathcal{Q}$ of S is *ε-optimal* if

$$\mathrm{H}^d(\mathcal{Q}) \le (1+\varepsilon)\mathcal{DH}^d_\infty(S).$$

We say that a dyadic ε-optimal nonoverlapping d-almost covering $\mathcal{Q}$ of S is *maximal* if $\mathcal{Q} \succeq \mathcal{Q}'$ for any ε-optimal dyadic nonoverlapping d-almost covering $\mathcal{Q}'$ of S comparable to $\mathcal{Q}$.

Remark 2.3. Let $d \in (0,n]$ and $S \subset \mathbb{R}^n$ be a set with $\mathcal{H}^d_\infty(S) \in (0,+\infty)$. It is easy to see that a maximal ε-optimal dyadic nonoverlapping d-almost covering of S exists but is not unique in general.

Definition 2.5. Let $d \in [0,n]$ and $\lambda \in (0,1]$. Let $S \subset \mathbb{R}^n$ be a set with $\mathcal{H}^d_\infty(S) > 0$. We say that a cube $Q = Q_l(x)$ with $x \in \mathbb{R}^n$ and $l \in [0,\infty)$ is *(d,λ)-thick with respect to S* if

$$\mathcal{H}^d_\infty(Q \cap S) \ge \lambda l^d. \tag{2.7}$$

Similarly, a cube $Q \subset \mathbb{R}^n$ is said to be *(d,λ)-dyadically thick with respect to S* if

$$\mathcal{DH}^d_\infty(Q \cap S) \ge \lambda l^d. \tag{2.8}$$

Recall that, given parameters $d \in [0,n]$, $\lambda \in (0,1]$ and a set $S \subset \mathbb{R}^n$, we denote the family of all (d,λ)-thick cubes by $\mathcal{F}_S(d,\lambda)$ (see (1.4)).

Remark 2.4. Note that $Q_0(x) = x$ for any point $x \in \mathbb{R}^n$. Hence, inequality (2.7) holds trivially with $Q = \{x\}$ for any $d \in [0,n]$, $\lambda \in (0,1]$, and $x \in S$. Sometimes it will be convenient for us to consider a point $x \in S$ of a given set $S \subset \mathbb{R}^n$ as a (d,λ)-thick cube with respect to S (whose side length is zero) for some $d \in [0,n]$ and $\lambda \in (0,1]$.

In Section 3 we will constantly deal with a special family of dyadic cubes. This family provides a sort of building blocks for the proof of the main results of the present paper.

Definition 2.6. Let $S \subset \mathbb{R}^n$ be a nonempty set. Given $d \in [0,n]$ and $\lambda \in (0,1]$, we define the *(d,λ)-keystone* family of cubes for S by the formula

$$\mathcal{DF}_S(d,\lambda) := \bigl\{Q \in \mathcal{D}_+ \colon \mathcal{DH}^d_\infty(Q \cap S) \ge \lambda (l(Q))^d\bigr\}.$$

Remark 2.5. Let $S \subset \mathbb{R}^n$ be an arbitrary nonempty set. If $Q = Q_l(x) \in \mathcal{F}_S(d,\lambda)$ for some $d \in (0,n]$ and $\lambda \in (0,1]$, then $Q_{cl}(x) \in \mathcal{F}_S(d,\lambda/c^d)$. Indeed, using the monotonicity property of the d-Hausdorff content, we get

$$\mathcal{H}^d_\infty(Q_{cl}(x) \cap S) \ge \mathcal{H}^d_\infty(Q_l(x) \cap S) \ge \lambda l^d = \frac{\lambda}{c^d}(cl)^d.$$

Proposition 2.2. *Let $S \subset \mathbb{R}^n$ be a set. Let $d \in (0,n]$ and $\lambda \in (0,1]$. Then there exists an $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0,\varepsilon_0)$, any cube $\overline{Q} = Q_l(x)$ with $\mathcal{DH}^d(\overline{Q} \cap S) < \lambda l^d$, and any ε-optimal dyadic nonoverlapping d-almost covering $\mathcal{Q}$ of the set $\overline{Q} \cap S$, we have*

$$l(Q) \le 2^{k(l)} \qquad \forall\, Q \in \mathcal{Q}, \tag{2.9}$$

where $k(l)$ is a unique integer for which $l \in (2^{k(l)}, 2^{k(l)+1}]$.

Proof. We set $\overline{\lambda} := \mathcal{DH}^d_\infty(\overline{Q} \cap S)$. By the assumptions, $\overline{\lambda} < \lambda l^d$. Choose $\varepsilon_0 > 0$ so small that $(1+\varepsilon_0)\overline{\lambda} < \lambda l^d$. Hence, taking $\varepsilon \in (0, \varepsilon_0)$ and taking an arbitrary ε-optimal dyadic nonoverlapping d-almost covering $\mathcal{Q}$ of the set $Q \cap S$, we clearly get

$$(l(Q))^d \le \mathrm{H}^d(\mathcal{Q}) \le (1+\varepsilon)\overline{\lambda} < \lambda l^d \qquad \forall\, Q \in \mathcal{Q}.$$

Since $\mathcal{Q} \subset \mathcal{D}$, we obtain (2.9). $\square$

The following elementary proposition will be quite useful in the sequel. It clarifies relations between (d,λ)-thick cubes and (d,λ)-dyadically thick cubes.

Proposition 2.3. *Let $d \in (0,n)$ and $\lambda \in (0,1)$, and let $S \subset \mathbb{R}^n$ be a Borel set with $\mathcal{H}^d_\infty(S) > 0$. Let $Q = Q_l(x)$ be a cube with $l \in (0,1]$ and let $k := [-\log_2 l] \in \mathbb{N}_0$. Then,*

- (i) *if $cQ \in \mathcal{F}_S(d,\lambda)$ ($cQ \in \mathcal{DF}_S(d,\lambda)$) for some $c \ge 1$, then there exists a dyadic cube $Q_{k,m} \in \mathcal{F}_S(d, \lambda/([2c]+1)^n)$ ($Q_{k,m} \in \mathcal{DF}_S(d, \lambda/([2c]+1)^n)$) such that $Q_{k,m} \cap Q \ne \varnothing$;*
- (ii) *if $Q \notin \mathcal{F}_S(d, \lambda/2^{dj})$ ($Q \in \mathcal{D}_+ \setminus \mathcal{DF}_S(d, \lambda/2^{dj})$) for some $j \in \mathbb{N}_0$, then no dyadic cube $Q_{k+j,m} \subset Q$ belongs to $\mathcal{F}_S(d,\lambda)$ (to $\mathcal{DF}_S(d,\lambda)$).*

Proof. It is clear that $l \in (2^{-k-1}, 2^{-k}]$. Since Q is closed, there are at most $([2c]+1)^n$ dyadic cubes $Q_{k,m}$ such that $Q_{k,m} \cap cQ \ne \varnothing$.

To prove assertion (i), we assume the contrary. Using the subadditivity property of $\mathcal{H}^d_\infty$, we get

$$\mathcal{H}^d_\infty(cQ \cap S) \le \sum_{Q_{k,m} \cap cQ \ne \varnothing} \mathcal{H}^d_\infty(Q_{k,m} \cap S) < \frac{\lambda([2c]+1)^n}{2^{kd}([2c]+1)^n} \le \lambda(l(Q))^d.$$

This contradicts the assumption that $cQ \in \mathcal{F}_S(d,\lambda)$.

To prove assertion (ii), assume contrarily that there is a dyadic cube $Q_{k+j,m} \subset Q$ such that $Q_{k+j,m} \in \mathcal{F}_S(d,\lambda)$. Due to the monotonicity of $\mathcal{H}^d_\infty$, by the definition of k we get

$$\mathcal{H}^d_\infty(Q \cap S) \ge \mathcal{H}^d_\infty(Q_{k+j,m} \cap S) \ge \frac{\lambda}{2^{(k+j)d}} \ge \frac{\lambda l^d}{2^{dj}}.$$

However, this contradicts the assumption that $Q \notin \mathcal{F}_S(d, \lambda/2^{dj})$.

The corresponding dyadic analogs of the claims can be proved similarly. $\square$

3. KEYSTONE FAMILIES OF CUBES

The following data are *assumed to be fixed* throughout this section:

(D1) arbitrary numbers $n \in \mathbb{N}$ and $d \in (0,n]$;

(D2) a set $S \subset \mathbb{R}^n$ with $\mathcal{H}^d_\infty(S) > 0$.

Recall Definition 2.6. Given $\lambda \in (0,1]$, in this section we set $\mathcal{DF}(\lambda) := \mathcal{DF}_S(d,\lambda)$ for brevity. In the sequel we will deal with special subfamilies of $\mathcal{DF}(\lambda)$.

Definition 3.1. Given $\lambda \in (0,1]$, we say that a family $\mathcal{Q}$ of cubes is (d,λ)-*nice for the set S* if the following conditions hold:

(1) the family $\mathcal{Q}$ is a dyadic nonoverlapping d-almost covering of S;

(2) $\mathcal{Q} \subset \mathcal{DF}(\lambda)$.

The following result is a modification of Netrusov's Lemma 2.1 from [10] adapted to our framework. We present a full proof to make our paper self-contained. Furthermore, we hope that the proof will clarify the underlying ideas of this section.

Lemma 3.1. *Let a cube $\overline{Q} \in \mathcal{D}_+$ be such that*

$$0 < \mathcal{DH}^d_\infty(\overline{Q} \cap S) < 1. \tag{3.1}$$

Then, for every $\lambda \in (0,1)$*, there exists a family of cubes* $\widehat{\mathcal{Q}}(\lambda) := \widehat{\mathcal{Q}}(\overline{Q},\lambda) \subset \mathcal{D}$ *such that*

(1) $\widehat{\mathcal{Q}}(\lambda)$ *is* (d,λ)*-nice for* $Q \cap S$*;*

(2) *for every cube* $Q \in \widehat{\mathcal{Q}}(\lambda)$*,*

$$l(Q) \le \frac{l(\overline{Q})}{2}; \tag{3.2}$$

(3) *the Carleson-type packing condition*

$$\mathrm{H}^d(\widehat{\mathcal{Q}}(\lambda)|_Q) \le (l(Q))^d \tag{3.3}$$

holds for every dyadic cube $Q \subset \overline{Q}$*.*

Proof. Given $\lambda \in (0,1)$, we fix $\varepsilon > 0$ so small that

$$0 < \tau := \frac{\varepsilon}{1-\lambda} < 1. \tag{3.4}$$

We split the proof into several steps.

Step 1. Recall Definition 2.4. Given a dyadic cube $K \subset \overline{Q}$ with

$$0 < \mathcal{H}^d_\infty(K \cap S) < (l(K))^d, \tag{3.5}$$

let $\mathcal{Q}(K)$ be a maximal ε-optimal dyadic nonoverlapping d-almost covering of the set $K \cap S$. By (3.5) and Proposition 2.2 (decreasing $\varepsilon > 0$ if necessary) we have

$$l(Q) \le \frac{l(K)}{2} \qquad \forall\, Q \in \mathcal{Q}(K). \tag{3.6}$$

The key property of the family $\mathcal{Q}(K)$ is that the Carleson-type packing condition holds true. More precisely, by the construction and Definition 2.4 we have, for every dyadic cube $Q \subset K$,

$$\mathrm{H}^d(\mathcal{Q}(K)|_Q) \le (l(Q))^d. \tag{3.7}$$

Indeed, otherwise, if inequality (3.7) fails for some dyadic cube $Q \subset K$, then we modify the family $\mathcal{Q}(K)$ by including Q and excluding all cubes $Q' \subset Q$, $Q' \in \mathcal{Q}(K)$. This gives an ε-optimal dyadic nonoverlapping d-almost covering of $K \cap S$. But this contradicts the maximality of $\mathcal{Q}(K)$.

Step 2. Given a dyadic cube $K \subset \overline{Q}$, we set

$$\mathcal{Q}^1(K) := \mathcal{Q}(K) \cap \mathcal{DF}(\lambda) \qquad \text{and} \qquad \widetilde{\mathcal{Q}}^1(K) := \mathcal{Q}(K) \setminus \mathcal{Q}^1(K).$$

Using Definition 2.4 and the subadditivity of $\mathcal{DH}^d_\infty$, we have

$$\begin{aligned} \frac{1}{1+\varepsilon}\,\mathrm{H}^d(\mathcal{Q}(K)) &= \frac{1}{1+\varepsilon}\,\mathrm{H}^d(\mathcal{Q}^1(K)) + \frac{1}{1+\varepsilon}\,\mathrm{H}^d(\widetilde{\mathcal{Q}}^1(K)) \le \mathcal{DH}^d_\infty(K \cap S) \\ &\le \sum_{Q \in \mathcal{Q}^1(K)} \mathcal{DH}^d_\infty(Q \cap S) + \sum_{Q \in \widetilde{\mathcal{Q}}^1(K)} \mathcal{DH}^d_\infty(Q \cap S) \le \mathrm{H}^d(\mathcal{Q}^1(K)) + \lambda\,\mathrm{H}^d(\widetilde{\mathcal{Q}}^1(K)). \end{aligned}$$

This clearly gives

$$\varepsilon\,\mathrm{H}^d(\mathcal{Q}^1(K)) \ge \bigl(1 - \lambda(1+\varepsilon)\bigr)\,\mathrm{H}^d(\widetilde{\mathcal{Q}}^1(K)). \tag{3.8}$$

Hence, using (3.8), Definition 2.4, and (3.4), we get

$$\mathrm{H}^d(\widetilde{\mathcal{Q}}^1(K)) \le \frac{\varepsilon}{(1-\lambda)(1+\varepsilon)}\,\mathrm{H}^d(\mathcal{Q}(K)) \le \tau (l(K))^d. \tag{3.9}$$

Step 3. Suppose that we have already built, for some $k_0 \in \mathbb{N}$ and for every $j \in \{1,\dots,k_0\}$, families of cubes $\mathcal{Q}^j = \mathcal{Q}^j(\overline{Q})$ and $\widetilde{\mathcal{Q}}^j = \widetilde{\mathcal{Q}}^j(\overline{Q})$ such that

(i) $\mathcal{Q}^1 \subset \ldots \subset \mathcal{Q}^{k_0}$ and $\widetilde{\mathcal{Q}}^1 \succ \ldots \succ \widetilde{\mathcal{Q}}^{k_0}$;

(ii) $\mathcal{Q}^{k_0} \subset \mathcal{DF}(\lambda)|_{\overline{Q}}$;

(iii) $\widetilde{\mathcal{Q}}^{k_0} \subset \mathcal{D}_+|_{\overline{Q}} \setminus \mathcal{DF}(\lambda)|_{\overline{Q}}$;

(iv) the inequality

$$\mathrm{H}^d\big(\mathcal{Q}^{k_0}|_Q \cup \widetilde{\mathcal{Q}}^{k_0}|_Q\big) \le (l(Q))^d \tag{3.10}$$

holds for every dyadic cube $Q \subset \overline{Q}$;

(v) it holds that

$$\mathrm{H}^d(\widetilde{\mathcal{Q}}^{k_0}) \le \tau^{k_0}(l(\overline{Q}))^d. \tag{3.11}$$

We recall the notation and constructions of steps 1 and 2. We put

$$\mathcal{Q}^{k_0+1} := \bigcup_{Q\in\widetilde{\mathcal{Q}}^{k_0}} \mathcal{Q}^1(Q) \cup \mathcal{Q}^{k_0} \qquad \text{and} \qquad \widetilde{\mathcal{Q}}^{k_0+1} := \bigcup_{Q\in\widetilde{\mathcal{Q}}^{k_0}} \widetilde{\mathcal{Q}}^1(Q). \tag{3.12}$$

It is clear that conditions (i)–(iii) are satisfied with k_0 replaced by k_0+1. It remains to verify that (3.10) and (3.11) hold with k_0+1 instead of k_0. Indeed, an application of (3.7) with K replaced by Q' gives, for any $Q \subset \overline{Q}$,

$$\sum_{Q''\colon\ \exists Q'\in\widetilde{\mathcal{Q}}^{k_0}|_Q,\ Q''\in\mathcal{Q}^1(Q')\cup\widetilde{\mathcal{Q}}^1(Q')} (l(Q''))^d \le \sum_{Q'\in\widetilde{\mathcal{Q}}^{k_0}|_Q} (l(Q'))^d = \mathrm{H}^d(\widetilde{\mathcal{Q}}^{k_0}|_Q). \tag{3.13}$$

By the construction it is clear that $\widetilde{\mathcal{Q}}^{k_0} \cap \mathcal{Q}^{k_0} = \varnothing$. Hence, combining (3.10), (3.12), and (3.13), we get

$$\begin{aligned}\mathrm{H}^d(\mathcal{Q}^{k_0+1}|_Q \cup \widetilde{\mathcal{Q}}^{k_0+1}|_Q) &= \sum_{Q''\colon\ \exists Q'\in\widetilde{\mathcal{Q}}^{k_0}|_Q,\ Q''\in\mathcal{Q}^1(Q')\cup\widetilde{\mathcal{Q}}^1(Q')} (l(Q''))^d + \mathrm{H}^d(\mathcal{Q}^{k_0}|_Q)\\ &\le \mathrm{H}^d(\mathcal{Q}^{k_0}|_Q) + \mathrm{H}^d(\widetilde{\mathcal{Q}}^{k_0}|_Q) = \mathrm{H}^d(\mathcal{Q}^{k_0}|_Q \cup \widetilde{\mathcal{Q}}^{k_0}|_Q) \le (l(Q))^d\end{aligned}$$

for any dyadic cube $Q \subset K$. Hence, (3.10) holds with k_0+1 instead of k_0.

Combining (3.9), (3.11), and (3.12), we obtain

$$\mathrm{H}^d(\widetilde{\mathcal{Q}}^{k_0+1}) = \sum_{Q\in\widetilde{\mathcal{Q}}^{k_0}} \mathrm{H}^d(\mathcal{Q}^1(Q)) \le \tau\,\mathrm{H}^d(\widetilde{\mathcal{Q}}^{k_0}) \le \tau^{k_0+1}(l(\overline{Q}))^d. \tag{3.14}$$

Step 4. As a result, by induction we built sequences $\{\mathcal{Q}^k\}_{k\in\mathbb{N}} := \{\mathcal{Q}^k(\overline{Q})\}_{k\in\mathbb{N}}$ and $\{\widetilde{\mathcal{Q}}^k\}_{k\in\mathbb{N}} := \{\widetilde{\mathcal{Q}}^k(\overline{Q})\}_{k\in\mathbb{N}}$ such that conditions (i)–(v) are satisfied for any $k \in \mathbb{N}$ instead of a fixed $k_0 \in \mathbb{N}$. We set

$$\widehat{\mathcal{Q}}(\lambda) := \bigcup_{k\in\mathbb{N}} \mathcal{Q}^k \subset \mathcal{DF}(\lambda)|_{\overline{Q}}. \tag{3.15}$$

Note also that according to our construction, estimate (3.9) implies

$$\widetilde{\mathcal{Q}}^k \succ \widetilde{\mathcal{Q}}^{k+1} \qquad \text{and} \qquad \mathrm{H}^d(\widetilde{\mathcal{Q}}^k) < \tau^k(l(\overline{Q}))^d \qquad \forall\, k \in \mathbb{N}. \tag{3.16}$$

Furthermore,

$$\overline{Q} \cap S \setminus \bigcup_{Q\in\widehat{\mathcal{Q}}(\lambda)} Q \subset \widetilde{\mathcal{Q}}^k \qquad \forall\, k \in \mathbb{N}.$$

Since $\tau \in (0,1)$, this leads to

$$\mathcal{H}^d_\infty\Biggl(\overline{Q} \cap S \setminus \bigcup_{Q \in \widehat{\mathcal{Q}}(\lambda)} Q\Biggr) \le \varlimsup_{k\to\infty} \mathrm{H}^d(\widetilde{\mathcal{Q}}^k) = 0.$$

Hence, by (3.15) the family $\widehat{\mathcal{Q}}(\lambda)$ is a dyadic nonoverlapping (d,λ)-thick d-almost covering of the set $S \cap \overline{Q}$. This proves assertion (1) of the lemma.

By our construction, assertion (2) follows easily from (3.6).

Finally, it is clear from our construction that $\mathcal{Q}^k \subset \mathcal{Q}^{k+1}$ for all $k \in \mathbb{N}$. Combining this fact with (3.16) and using inequality (3.10) with k_0 replaced by $k \in \mathbb{N}$, we get

$$\mathrm{H}^d(\widehat{\mathcal{Q}}(\lambda)|_Q) = \lim_{k\to\infty} \mathrm{H}^d\bigl(\mathcal{Q}^k|_Q \cup \widetilde{\mathcal{Q}}^k|_Q\bigr) \le (l(Q))^d \tag{3.17}$$

for every dyadic cube $Q \subset \overline{Q}$. This shows assertion (3) of the lemma. $\square$

The following concept will play a crucial role in what follows.

Definition 3.2. Given $\lambda \in (0,1]$, we say that a sequence $\{\widehat{\mathcal{Q}}^s(\lambda)\}_{s\in\mathbb{N}_0} := \{\widehat{\mathcal{Q}}^s_S(d,\lambda)\}_{s\in\mathbb{N}_0}$ of families of cubes is a (d,λ)-*nice sequence for* S if the following conditions hold:

(1) $\widehat{\mathcal{Q}}^0 := \{Q \in \mathcal{D}_0 \colon \mathcal{DH}^d_\infty(S \cap Q) > 0\}$;

(2) for every $s \in \mathbb{N}$ the family $\widehat{\mathcal{Q}}^s(\lambda)$ is (d,λ)-nice for S;

(3) $\widehat{\mathcal{Q}}^{s+1}(\lambda) \prec \widehat{\mathcal{Q}}^s(\lambda)$ for all $s \in \mathbb{N}_0$;

(4) for each $s \in \mathbb{N}_0$, every $\overline{Q} \in \widehat{\mathcal{Q}}^s(\lambda)$, and every dyadic cube $Q \subset \overline{Q}$,

$$\sum_{Q' \in \widehat{\mathcal{Q}}^{s+1}(\lambda)|_Q} (l(Q'))^d \le \begin{cases} 2^{n-d}(l(Q))^d & \text{if } Q = \overline{Q} \text{ and } Q \in \mathcal{DF}(1), \\ (l(Q))^d & \text{in the other cases.} \end{cases} \tag{3.18}$$

Theorem 3.1. *Given* $\lambda \in (0,1)$, *there exists a sequence* $\{\widehat{\mathcal{Q}}^s(\lambda)\}_{s\in\mathbb{N}_0}$ *of families of cubes that is* (d,λ)-*nice for* S.

Proof. We split the proof into two steps.

Step 1. We fix an arbitrary cube $\overline{Q} \in \mathcal{D}_+$ such that

$$\mathcal{DH}^d_\infty(\overline{Q} \cap S) > 0$$

and consider two cases.

In the first case,

$$\mathcal{DH}^d_\infty(\overline{Q} \cap S) < (l(\overline{Q}))^d.$$

We apply Lemma 3.1 to the cube $\overline{Q}$ and obtain a (d,λ)-nice family $\widehat{\mathcal{Q}}(\overline{Q},\lambda)$ for $\overline{Q} \cap S$ satisfying (3.2) and (3.3).

In the second case,

$$\mathcal{DH}^d_\infty(\overline{Q} \cap S) = (l(\overline{Q}))^d,$$

i.e., $\overline{Q} \in \mathcal{DF}(1)$. We divide $\overline{Q}$ into 2^n congruent dyadic cubes. Let $\mathcal{K}_{\overline{Q}}$ be those of them whose intersection with S has positive $\mathcal{H}^d_\infty$-content. We put $\mathcal{K}^{\mathrm{g}}_{\overline{Q}} := \mathcal{K}_{\overline{Q}} \cap \mathcal{DF}(\lambda)$ and $\mathcal{K}^{\mathrm{b}}_{\overline{Q}} := \mathcal{K}_{\overline{Q}} \setminus \mathcal{K}^{\mathrm{g}}_{\overline{Q}}$. For every $\overline{Q}' \in \mathcal{K}^{\mathrm{b}}_{\overline{Q}}$ we apply Lemma 3.1. This gives families $\widehat{\mathcal{Q}}(\overline{Q}',\lambda)$, $\overline{Q}' \in \mathcal{K}^{\mathrm{b}}_{\overline{Q}}$, satisfying conditions (1) and (2) of Lemma 3.1 with $\widehat{\mathcal{Q}}(\lambda)$ replaced by $\widehat{\mathcal{Q}}(\overline{Q}',\lambda)$. We set

$$\widehat{\mathcal{Q}}(\overline{Q},\lambda) := \mathcal{K}^{\mathrm{g}}_{\overline{Q}} \cup \bigcup_{\overline{Q}' \in \mathcal{K}^{\mathrm{b}}_{\overline{Q}}} \widehat{\mathcal{Q}}(\overline{Q}',\lambda).$$

It is clear by the construction that

$$\mathcal{H}^d_\infty\Bigg(\overline{Q}\cap S\setminus\bigcup_{Q\in\widehat{\mathcal{Q}}(\overline{Q},\lambda)}Q\Bigg)=0. \tag{3.19}$$

Furthermore, by the construction,

$$l(Q)=\frac{l(\overline{Q})}{2}\qquad\forall\, Q\in\mathcal{K}_{\overline{Q}}. \tag{3.20}$$

Using (3.3) with $\widehat{\mathcal{Q}}(\overline{Q},\lambda)$ replaced by $\widehat{\mathcal{Q}}(\overline{Q}',\lambda)$, $\overline{Q}'\in\mathcal{K}^{\mathrm{b}}_{\overline{Q}}$, taking into account that $\#\big(\mathcal{K}^{\mathrm{g}}_{\overline{Q}}\cup\mathcal{K}^{\mathrm{b}}_{\overline{Q}}\big)\le 2^n$, and finally using (3.20), we obtain

$$\begin{aligned}\mathrm{H}^d(\widehat{\mathcal{Q}}(\overline{Q},\lambda))&\le\sum_{\overline{Q}'\in\mathcal{K}^{\mathrm{b}}_{\overline{Q}}}\mathrm{H}^d(\widehat{\mathcal{Q}}(\overline{Q}',\lambda))+\mathrm{H}^d(\mathcal{K}^{\mathrm{g}}_{\overline{Q}})\le\mathrm{H}^d(\mathcal{K}^{\mathrm{b}}_{\overline{Q}})+\mathrm{H}^d(\mathcal{K}^{\mathrm{g}}_{\overline{Q}})\\&=\#\big(\mathcal{K}^{\mathrm{g}}_{\overline{Q}}\cup\mathcal{K}^{\mathrm{b}}_{\overline{Q}}\big)\bigg(\frac{l(\overline{Q})}{2}\bigg)^d\le 2^{n-d}(l(\overline{Q}))^d.\end{aligned} \tag{3.21}$$

On the other hand, it is easy to see by construction that $\mathrm{H}^d(\widehat{\mathcal{Q}}(\overline{Q},\lambda)|_Q)\le(l(Q))^d$ for any cube $Q\subset\overline{Q}$ with side length $l(Q)<l(\overline{Q})$. Thus, an analog of formula (3.18) is valid with $\widehat{\mathcal{Q}}^{s+1}(\lambda)|_Q$ replaced by $\widehat{\mathcal{Q}}(\overline{Q},\lambda)$.

Step 2. We built the desirable sequence by induction. Clearly, by condition (D2) the family $\widehat{\mathcal{Q}}^0(\lambda)$, which consists of all dyadic cubes $Q\in\mathcal{D}_0$ with $\mathcal{DH}^d_\infty(Q\cap S)>0$, is nonempty. We define

$$\widehat{\mathcal{Q}}^1(\lambda):=\bigcup_{\overline{Q}\in\widehat{\mathcal{Q}}^0(\lambda)}\widehat{\mathcal{Q}}(\overline{Q},\lambda)\subset\mathcal{DF}(\lambda).$$

Suppose that we have already built, for some $j_0\in\mathbb{N}$, families $\widehat{\mathcal{Q}}^0(\lambda),\dots,\widehat{\mathcal{Q}}^{j_0}(\lambda)$ such that conditions (1)–(4) of Definition 3.2 are satisfied for any $s\in\{0,\dots,j_0-1\}$. Then we define

$$\widehat{\mathcal{Q}}^{j_0+1}(\lambda):=\bigcup_{\overline{Q}\in\widehat{\mathcal{Q}}^{j_0}(\lambda)}\widehat{\mathcal{Q}}(\overline{Q},\lambda)\subset\mathcal{DF}(\lambda). \tag{3.22}$$

By (3.19), (3.21), and (3.22) conditions (1)–(4) of Definition 3.2 are satisfied for any $s\in\{0,\dots,j_0\}$.

As a result, by induction we get the required sequence $\{\widehat{\mathcal{Q}}^s(\lambda)\}_{s\in\mathbb{N}_0}$. $\square$

Although the proof of the following result is quite elementary, as far as we know it has never been formulated in the literature in the present form. For $\lambda\in(0,1)$, we get a canonical decomposition of the family $\mathcal{DF}(\lambda)$. Informally speaking, this result can be thought of as a natural generalization of the decomposition of the family of all dyadic cubes $\mathcal{D}_+$ into the subfamilies $\mathcal{D}_k$, $k\in\mathbb{N}_0$.

Theorem 3.2. *For every $\lambda\in(0,1)$, there exists a unique sequence $\{\mathcal{Q}^s(\lambda)\}_{s\in\mathbb{N}}$ such that*

(1) $\mathcal{DF}(\lambda)=\bigcup_{s\in\mathbb{N}}\mathcal{Q}^s(\lambda)$;

(2) *for every $s\in\mathbb{N}$ the family $\mathcal{Q}^s(\lambda)$ is (d,λ)-nice for S*;

(3) $\mathcal{Q}^s(\lambda)\succ\mathcal{Q}^{s+1}(\lambda)$ *for every $s\in\mathbb{N}$*;

(4) *if, for some cubes $\overline{Q}\in\mathcal{Q}^s(\lambda)$ and $\underline{Q}\in\mathcal{Q}^{s+1}(\lambda)$, there is a cube $Q\in\mathcal{D}_+$ such that*

$$\underline{Q}\subset Q\subset\overline{Q}\qquad\text{and}\qquad l(Q)\in(l(\underline{Q}),l(\overline{Q})),$$

then the cube Q does not belong to the family $\mathcal{DF}(\lambda)$, i.e., $\mathcal{H}^d_\infty(Q\cap S)<\lambda(l(Q))^d$.

Proof. We split the proof into several steps.

Step 1. First of all, we fix $\lambda \in (0,1)$ and for every $Q \in \mathcal{D}_0$ with $\mathcal{DH}^d_\infty(Q \cap S) > 0$ denote by $\mathcal{Q}(Q,\lambda)$ the family of all maximal dyadic cubes $Q' \in \mathcal{DF}(\lambda)$ whose side lengths are strictly less than $l(Q)$. Then, for any $Q \in \mathcal{D}_0$ with $\mathcal{DH}^d_\infty(Q \cap S) > 0$ we have the following properties:

(A) $\mathcal{Q}(Q,\lambda) \subset \mathcal{DF}(\lambda)$;

(B) $\{Q\} \succ \mathcal{Q}(Q,\lambda)$;

(C) the family $\mathcal{Q}(Q,\lambda)$ is (d,λ)-nice for $Q \cap S$.

Properties (A) and (B) are clear by the construction. To establish (C), we apply Theorem 3.1 and fix a (d,λ)-nice sequence $\{\widehat{\mathcal{Q}}^s(\lambda)\}_{s\in\mathbb{N}_0}$ for S. Let $j_0 \in \mathbb{N}_0$ be the least number among all $j \in \mathbb{N}_0$ satisfying $\{Q\} \succ \widehat{\mathcal{Q}}^j(\lambda)$. It is clear by the construction that $\mathcal{Q}(\lambda) \succeq \widehat{\mathcal{Q}}^{j_0}(\lambda)$. Property (C) follows from the fact that for every $Q \in \mathcal{D}_0$ with $\mathcal{DH}^d_\infty(Q \cap S) > 0$ the family $\widehat{\mathcal{Q}}^{j_0}(\lambda)|_Q$ is (d,λ)-nice for the set $S \cap Q$.

Step 2. We build the desirable sequence $\{\mathcal{Q}^s(\lambda)\}_{s\in\mathbb{N}_0}$ by induction.

The base of induction. We set

$$\mathcal{K}^0_{\mathrm{g}} := \mathcal{D}_0 \cap \mathcal{DF}(\lambda), \qquad \mathcal{K}^0_{\mathrm{b}} := \bigl\{Q \in \mathcal{D}_0 \colon 0 < \mathcal{DH}^d_\infty(Q \cap S) < \lambda\bigr\} \tag{3.23}$$

and define

$$\mathcal{Q}^1(\lambda) := \bigcup_{Q\in\mathcal{K}^0_{\mathrm{b}}} \mathcal{Q}(Q,\lambda) \cup \mathcal{K}^0_{\mathrm{g}}. \tag{3.24}$$

It follows immediately from the construction that the family $\mathcal{Q}^1(\lambda)$ is (d,λ)-nice for S.

The induction step. Suppose that, for some $j_0 \in \mathbb{N}$, we have already built the families $\mathcal{Q}^s(\lambda)$, $s \in \{1,\dots,j_0\}$. We put

$$\mathcal{Q}^{j_0+1}(\lambda) := \bigcup_{Q\in\mathcal{Q}^{j_0}(\lambda)} \mathcal{Q}(Q,\lambda). \tag{3.25}$$

Hence, by induction we obtain the families $\mathcal{Q}^s(\lambda)$ for all $s \in \mathbb{N}$.

Step 3. It is clear that

$$\mathcal{Q}^s(\lambda) \subset \mathcal{DF}(\lambda) \qquad \text{and} \qquad \mathcal{Q}^s(\lambda) \succ \mathcal{Q}^{s+1}(\lambda) \qquad \forall\, s \in \mathbb{N}.$$

Furthermore, for each $s \in \mathbb{N}$ the family $\mathcal{Q}^s(\lambda)$ is (d,λ)-nice for S. This proves properties (2) and (3) from the statement of the theorem.

Suppose now that there exist a number $j \in \mathbb{N}$ and cubes $\overline{Q} \in \mathcal{Q}^j(\lambda)$, $\underline{Q} \in \mathcal{Q}^{j+1}(\lambda)$, and $Q \in \mathcal{D}$ such that

$$\underline{Q} \subset Q \subset \overline{Q} \qquad \text{and} \qquad l(Q) \in (l(\underline{Q}), l(\overline{Q})).$$

Note that $Q \notin \mathcal{DF}(\lambda)$, since otherwise we get a contradiction with the maximality of $\underline{Q} \in \mathcal{Q}(\overline{Q},\lambda)$. To complete the proof, it is sufficient to note that the already established property (4) from the statement of the theorem, combined with (3.24) and (3.25), gives property (1). $\square$

Definition 3.3. For $\lambda \in (0,1)$ the sequence $\{\mathcal{Q}^s(\lambda)\}_{s\in\mathbb{N}}$ will be called the *canonical decomposition of the family* $\mathcal{DF}(\lambda)$.

The following result will be important in the proof of Theorem 4.2; however, we believe that it can be interesting in itself. It describes some combinatorial properties of the canonical decomposition $\{\mathcal{Q}^s(\lambda)\}_{s\in\mathbb{N}}$ of the family $\mathcal{DF}(\lambda)$.

Theorem 3.3. *Let* $\lambda_1, \lambda_2 \in (0,1)$. *Let* $\{\widehat{\mathcal{Q}}^s(\lambda_1)\}_{s\in\mathbb{N}}$ *be a* (d,λ_1)*-nice sequence for* S. *Let* $Q \in \mathcal{D}_+$ *and let*

$$j_0 := \min\bigl\{j \in \mathbb{N}_0 \colon \widehat{\mathcal{Q}}^j(\lambda_1)|_Q \neq \varnothing\bigr\}.$$

Then

$$\mathrm{H}^d(\mathcal{C}) \le \begin{cases} 2^{n-d}\dfrac{(l(Q))^d}{\lambda_2}, & Q \in \mathcal{DF}(1), \\ \dfrac{(l(Q))^d}{\lambda_2}, & Q \notin \mathcal{DF}(1), \end{cases} \tag{3.26}$$

for any family $\mathcal{C} \subset \mathcal{DF}(\lambda_2)$ satisfying the following conditions:

(1) $\operatorname{int} Q \cap \operatorname{int} Q' = \varnothing$ *for any* $Q, Q' \in \mathcal{C}$ *such that* $Q \ne Q'$;

(2) $\{Q\} \succeq \mathcal{C} \succeq \widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_Q$.

Proof. In the case $\{Q\} = \widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_Q$ we clearly get $\mathcal{C} = \{Q\}$, and hence (3.26) holds trivially.

Suppose now that $\{Q\} \succ \widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_Q$. Note that for every $Q' \in \mathcal{C}$ the family $\widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_{Q'}$ is (d,λ_1)-nice for $S \cap Q'$. Hence, taking into account the inclusion $\mathcal{C} \subset \mathcal{DF}(\lambda_2)$ and using (3.18), we obtain

$$\begin{aligned} \mathrm{H}^d(\mathcal{C}) &\le \frac{1}{\lambda_2}\sum_{Q'\in\mathcal{C}} \mathcal{DH}^d_\infty(Q' \cap S) \le \frac{1}{\lambda_2}\sum_{Q'\in\mathcal{C}} \mathrm{H}^d\big(\widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_{Q'}\big) \\ &\le \frac{1}{\lambda_2}\,\mathrm{H}^d\big(\widehat{\mathcal{Q}}^{j_0}(\lambda_1)|_Q\big) \le \frac{c}{\lambda_2}(l(Q))^d, \end{aligned} \tag{3.27}$$

where $c = 1$ in the case when $Q \notin \mathcal{DF}(1)$ and $c = 2^{n-d}$ in the case when $Q \in \mathcal{DF}(1)$. $\square$

4. MAIN RESULTS

Recall that given a number $\delta > 0$ and a set $S \subset \mathbb{R}^n$, the *δ-neighborhood of S* is defined by the formula

$$U_\delta(S) := \big\{y \in \mathbb{R}^n\colon\ \|y - x\|_\infty < \delta \text{ for some } x \in S\big\}. \tag{4.1}$$

We start with the following elementary observation. Recall that the metric floor and metric ceiling of a given family of sets were defined in (2.3).

Proposition 4.1. *Let $\overline{Q}$ be an arbitrary cube in $\mathbb{R}^n$. Let $\mathcal{F}$ be a family of subsets in $\mathbb{R}^n$ such that $\overline{\mu} := \overline{\mu}(\mathcal{F}) < l(\overline{Q})/2$. Then*

$$\mathcal{L}^n\Bigg(\bigcup_{F\in\mathcal{F}\colon\, F\cap\partial\overline{Q}\ne\varnothing} F\Bigg) \le 8n(l(\overline{Q}))^{n-1}\overline{\mu}. \tag{4.2}$$

Proof. Every n-dimensional cube has $2n$ facets. Given $i \in \{1,\dots,2^n\}$, we denote by $\partial^i\overline{Q}$ the ith facet of $\overline{Q}$. It is clear that for any $\delta > 0$ we have

$$U_\delta(\partial\overline{Q}) \subset \bigcup_{i=1}^{2n} U_\delta(\partial^i\overline{Q}). \tag{4.3}$$

Elementary geometrical considerations give

$$\mathcal{L}^n(U_\delta(\partial^i\overline{Q})) \le 2(2\delta + l(\overline{Q}))^{n-1}\delta \qquad \forall\, i \in \{1,\dots,2^n\}. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\mathcal{L}^n(U_\delta(\partial\overline{Q})) \le \sum_{i=1}^{2n}\mathcal{L}^n(U_\delta(\partial^i\overline{Q})) \le 4n(2\delta + l(\overline{Q}))^{n-1}\delta \qquad \forall\, \delta > 0. \tag{4.5}$$

By (2.3) it is clear that if $F \cap \partial\overline{Q} \neq \varnothing$ for some $F \in \mathcal{F}$, then $F \subset U_{(1+\epsilon)\overline{\mu}}(\partial Q)$ for a sufficiently small $\epsilon > 0$. Hence, using (4.5) with $\delta = \overline{\mu}$ and taking into account that $(1+\epsilon)\overline{\mu} \leq 2^{-1}l(\overline{Q})$, we obtain the desirable estimate

$$\mathcal{L}^n\Bigg(\bigcup_{F\in\mathcal{F}\colon F\cap\partial Q\neq\varnothing} F\Bigg) \leq \mathcal{L}^n(U_{\overline{\mu}}(\partial\overline{Q})) \leq 4n(2\overline{\mu} + l(\overline{Q}))^{n-1}\overline{\mu} \leq 8n(l(Q))^{n-1}\overline{\mu}. \qquad \square$$

Below we will need the following auxiliary result, which can be of independent interest.

Lemma 4.1. *Let $d \in (0,n)$, $\overline{c} > 1$, and $r > 1$. Then there exists a number $\overline{\delta} = \overline{\delta}(n,\overline{c},r) > 0$ such that, for any $\tau > 0$, $\delta \in (0,\overline{\delta}]$ and any at most countable family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ with the properties*

(1) $\mathrm{H}^d(\mathcal{F}) < +\infty$ *and*

(2) $\delta\tau \leq \underline{\mu}(\mathcal{F}) \leq \overline{\mu}(\mathcal{F}) \leq \tau < +\infty$,

the inequality

$$\mathcal{L}^n(U_{r\delta\tau}(\mathrm{F})) \leq \overline{c}\tau^{n-d}\,\mathrm{H}^d(\mathcal{F}) \tag{4.6}$$

holds with $\mathrm{F} := \bigcup_{F\in\mathcal{F}} F$.

Proof. We fix $\theta > 1$ so close to 1 and choose $k^* \in \mathbb{N}$ so large that

$$\theta^n + \frac{([2r]+1)^n}{2^{k^*(n-d)}} < \overline{c}. \tag{4.7}$$

Now we set

$$\overline{\delta} := \frac{\theta - 1}{[2r]+1}\cdot 2^{-k^*}. \tag{4.8}$$

We fix $\tau > 0$, $\delta \in (0,\overline{\delta}]$ and an arbitrary family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ with properties (1) and (2). We set $\underline{\mu} := \underline{\mu}(\mathcal{F})$ and $\overline{\mu} := \overline{\mu}(\mathcal{F})$ for brevity. It is clear that for every set $F \in \mathcal{F}$ there is a cube $Q(F) \supset F$ with $l(Q(F)) = \operatorname{diam} F$. Such a cube is not unique in general. We fix some choice of cubes $Q(F)$, $F \in \mathcal{F}$, and define the family

$$\mathcal{Q} := \mathcal{Q}(\mathcal{F}) := \big\{Q\colon Q = Q(F) \text{ for some } F \in \mathcal{F}\big\}.$$

By our construction it is clear that

$$\mathrm{H}^d(\mathcal{Q}) = \mathrm{H}^d(\mathcal{F}), \qquad \underline{\mu}(\mathcal{Q}) = \underline{\mu}, \qquad \overline{\mu}(\mathcal{Q}) = \overline{\mu}. \tag{4.9}$$

Since $l(Q) \geq \delta\tau$, it is easy to see that for any cube $Q \in \mathcal{Q}$ there is a constant $c(Q) \in (1,[2r]+1]$ such that

$$U_{r\delta\tau}(\mathrm{F}) \subset \bigcup_{Q\in\mathcal{Q}} U_{r\delta\tau}(Q) \subset \bigcup_{Q\in\mathcal{Q}} c(Q)Q. \tag{4.10}$$

Our goal is to make a smart choice of the constants $c(Q)$, $Q \in \mathcal{Q}$. For this purpose we split the family $\mathcal{Q}$ into two disjoint subfamilies. Namely, we set

$$\mathcal{Q}^1 := \big\{Q \in \mathcal{Q}\colon l(Q) > 2^{-k^*}\tau\big\}, \qquad \mathcal{Q}^2 := \mathcal{Q}\setminus\mathcal{Q}^1. \tag{4.11}$$

Since $\delta \in (0,\overline{\delta}]$, by (4.8) we have $U_{r\delta\tau}(Q) \subset \theta Q$ for all $Q \in \mathcal{Q}^1$. On the other hand, since $l(Q) \geq \delta\tau$ for all $Q \in \mathcal{Q}$, it is clear that $U_{r\delta\tau}(Q) \subset ([2r]+1)Q$ for all $Q \in \mathcal{Q}^2$. Hence, inclusion (4.10) holds with

$$c(Q) = \begin{cases} \theta & \text{if } Q \in \mathcal{Q}^1, \\ [2r]+1 & \text{if } Q \in \mathcal{Q}^2. \end{cases} \tag{4.12}$$

By (4.10)–(4.12) we have

$$\mathcal{L}^n(U_{r\delta\tau}(\mathrm{F})) \le \theta^n\,\mathrm{H}^n(\mathcal{Q}^1) + ([2r]+1)^n\,\mathrm{H}^n(\mathcal{Q}^2). \tag{4.13}$$

Since $d \in (0,n)$ by (2.3) and (4.11) we obtain the first key estimate

$$\mathrm{H}^n(\mathcal{Q}^2) \le \left(\frac{\tau}{2^{k^*}}\right)^{n-d}\mathrm{H}^d(\mathcal{Q}^2) \le \left(\frac{\tau}{2^{k^*}}\right)^{n-d}\mathrm{H}^d(\mathcal{Q}) = \left(\frac{\tau}{2^{k^*}}\right)^{n-d}\mathrm{H}^d(\mathcal{F}). \tag{4.14}$$

Similarly, we have the second key estimate

$$\mathrm{H}^n(\mathcal{Q}^1) \le \tau^{n-d}\,\mathrm{H}^d(\mathcal{Q}^1) = \tau^{n-d}\,\mathrm{H}^d(\mathcal{F}). \tag{4.15}$$

Combining (4.13)–(4.15) and taking into account (4.7), we obtain (4.6). $\square$

Now we are ready to prove a relatively simple result, which however will play an important role in the proof of the main results of the present paper. We believe that it can be interesting in itself. Roughly speaking, we show that if a cube $\overline{Q}$ is not $(d,\overline{\lambda})$-thick with respect to a given set S, then one can find a cube $\underline{Q} \subset \overline{Q}$ such that $\mathcal{H}^d_\infty(\underline{Q}\cap S)/(l(\underline{Q}))^d$ is much smaller than $\overline{\lambda}$ but the side length $l(\underline{Q})$ is controlled from below in a reasonable way.

Since the proof below will be quite technical, for the reader's convenience we give some informal explanations of the underlying ideas. Roughly speaking, if $d \in (0,n)$ and a cube $\overline{Q}$ is $(d,\overline{\lambda})$-thin, then, for sufficiently large $k = k(d,\overline{\lambda}) \in \mathbb{N}$, one can construct a family $\mathcal{F} \subset \mathcal{D}_k|_{\overline{Q}}$ of cardinality $\approx 2^{kn}$ such that

$$\sum_{Q\in\mathcal{F}} \mathcal{H}^d_\infty(Q\cap S) \le C(n)\mathcal{H}^d_\infty(\overline{Q}\cap S)$$

for some universal constant $C(n) \ge 1$. Hence, if we assume that $\mathcal{H}^d_\infty(Q\cap S)$ is not small for any cube $Q \in \mathcal{F}$ and if k is large enough, then, taking into account that $d<n$, by elementary cardinality arguments we get a contradiction with the smallness of $\mathcal{H}^d_\infty(\overline{Q}\cap S)$. In order to construct $\mathcal{F}$, one should fix a small enough $\varepsilon > 0$ and fix an ε-optimal covering $\mathcal{Q}$ of $\overline{Q}\cap S$. The main technical difficulty is to split the family $\mathcal{Q}$ into a "large part" $\mathcal{Q}^{\mathrm{l}}$ and a "small part" $\mathcal{Q}^{\mathrm{s}}$. The cubes $Q \in \mathcal{Q}^{\mathrm{l}}$ have relatively large side lengths but the Lebesgue measure $\mathcal{L}^n$ of the union of such cubes is relatively small. Hence, one should fix $k \in \mathbb{N}$ such that 2^{-k} is approximately equal to the minimal side length of cubes from $\mathcal{Q}^{\mathrm{l}}$, and then select cubes from $\mathcal{D}_k|_{\overline{Q}}$ that do not meet cubes $Q \in \mathcal{Q}^{\mathrm{l}}$.

Theorem 4.1. *Let $d \in (0,n)$ and $\overline{\lambda} \in (0,1)$. For every $c > 1$ there exists a constant $\underline{\kappa} = \underline{\kappa}(\overline{\lambda},n,d,c) > 0$ such that, for every set $S \subset Q_{0,0}$ and any cube $\overline{Q}$ satisfying*

$$l(\overline{Q}) < 1 \qquad \text{and} \qquad \mathcal{H}^d_\infty(S\cap\overline{Q}) < \overline{\lambda}(l(\overline{Q}))^d, \tag{4.16}$$

there exists a cube $\underline{Q} \subset \overline{Q}$ with the following properties:

(i) $\underline{Q} \in \mathcal{D}_+$ *and* $\mathcal{H}^d(\underline{Q}\cap S) < (\overline{\lambda}/c)(l(\underline{Q}))^d$;

(ii) $l(\underline{Q}) \ge \underline{\kappa}\, l(\overline{Q})$.

Proof. We fix an arbitrary set $S \subset Q_{0,0}$ and a cube $\overline{Q}$ satisfying (4.16). Without loss of generality we may assume that $\mathcal{H}^d_\infty(S) > 0$, because otherwise the assertion is trivial. Since $\overline{\lambda} \in (0,1)$, we fix a sufficiently small $\varepsilon \in (0,1/2)$ and $\overline{c} > 1$ sufficiently close to 1 in such a way that

$$(1+\varepsilon)\overline{\lambda} < 1 \qquad \text{and} \qquad 1 - \overline{c}((1+\varepsilon)\overline{\lambda})^{n/d} > \frac{3}{4}(1-\overline{\lambda}^{n/d}). \tag{4.17}$$

We split the proof into several steps.

Step 1. Since $\overline{Q} \notin \mathcal{F}_S(d,\overline{\lambda})$, there is an ε-optimal covering $\mathcal{Q} := \mathcal{Q}_\varepsilon$ of the set $\overline{Q}\cap S$ such that

$$\mathrm{H}^d(\mathcal{Q}) \le (1+\varepsilon)\overline{\lambda}(l(\overline{Q}))^d. \tag{4.18}$$

Hence, by (4.16)–(4.18) we get

$$\overline{\mu} := \overline{\mu}(\mathcal{Q}) \le \big(\mathrm{H}^d(\mathcal{Q})\big)^{1/d} \le (\overline{\lambda}(1+\varepsilon))^{1/d}\, l(\overline{Q}) =: \tau < 1. \tag{4.19}$$

Step 2. Let $\overline{\delta} = \overline{\delta}(n, \overline{c}, 3) > 0$ be the same number as in Lemma 4.1. We set

$$\underline{\kappa} := \min\biggl\{\frac{\overline{\delta}}{3}, \frac{\overline{\lambda}^{n/d}}{64n}, \biggl(\frac{1-\overline{\lambda}^{n/d}}{3^{n+1}c}\biggr)^{1/(n-d)}\biggr\}. \tag{4.20}$$

We split the family $\mathcal{Q}$ into two disjoint subfamilies. Namely, we define subfamilies of "large" and "small" cubes of $\mathcal{Q}$ by letting, respectively,

$$\mathcal{Q}^{\mathrm{l}}_{\underline{\kappa}} := \bigl\{Q \in \mathcal{Q} \colon\ l(Q) \ge \underline{\kappa}\tau\bigr\} \qquad \text{and} \qquad \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}} := \mathcal{Q} \setminus \mathcal{Q}^{\mathrm{l}}_{\underline{\kappa}}. \tag{4.21}$$

We define

$$F := \bigcup_{Q \in \mathcal{Q}^{\mathrm{l}}_{\underline{\kappa}}} Q.$$

The main idea is to show that (4.20) guaranties that for

$$k := [-\log_2(\underline{\kappa}\tau)] \tag{4.22}$$

there are a lot of cubes in $\mathcal{D}_k$ inside $\overline{Q}$ that meet neither F nor ∂Q. We set

$$\mathcal{F}^1_k := \bigl\{Q \in \mathcal{D}_k \colon\ Q \cap F \ne \varnothing\bigr\}, \qquad \mathcal{F}^2_k := \bigl\{Q \in \mathcal{D}_k \colon\ Q \cap \partial\overline{Q} \ne \varnothing\bigr\},$$
$$\mathcal{F}^3_k := \bigl\{Q \in \mathcal{D}_k \colon\ Q \subset \overline{Q},\ Q \notin (\mathcal{F}^1_k \cup \mathcal{F}^2_k)\bigr\}.$$

Step 3. By (4.22) we have $2^{-k} \le 2\underline{\kappa}\tau$. Hence, $Q \subset U_{3\underline{\kappa}\tau}(F)$ for every $Q \in \mathcal{F}^1_k$. We apply Lemma 4.1 with $\mathcal{F} = \mathcal{Q}^{\mathrm{l}}_{\underline{\kappa}}$, $\delta = \underline{\kappa}$, $r = 3$ and take into account (4.18). This gives

$$V^1_k := \mathcal{L}^n\biggl(\bigcup_{Q \in \mathcal{F}^1_k} Q\biggr) \le \mathcal{L}^n(U_{3\underline{\kappa}\tau}(F)) \le \overline{c}\tau^{n-d}\,\mathrm{H}^d(\mathcal{Q}) \le \overline{c}((1+\varepsilon)\overline{\lambda})^{n/d}(l(\overline{Q}))^n. \tag{4.23}$$

On the other hand, using Proposition 4.1 and taking into account the first inequality in (4.17) we obtain

$$V^2_k := \mathcal{L}^n\biggl(\bigcup_{Q \in \mathcal{F}^2_k} Q\biggr) \le 16n(l(\overline{Q}))^{n-1}\underline{\kappa}\tau \le 16n\underline{\kappa}(l(\overline{Q}))^n. \tag{4.24}$$

Using (4.20) we continue (4.24) and get

$$V^2_k \le \frac{(\overline{\lambda})^{n/d}}{4}(l(\overline{Q}))^n. \tag{4.25}$$

Step 4. By the very definition of $\mathcal{F}^3_k$ we obviously have the following fact. If $Q \in \mathcal{F}^3_k$ and $Q' \in \mathcal{Q}$ is such that $Q' \cap Q \cap S \ne \varnothing$, then $Q' \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}$ and

$$\sum_{Q' \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}} \colon Q' \cap Q \cap S \ne \varnothing} (l(Q'))^d = \sum_{Q' \in \mathcal{Q} \colon Q' \cap Q \cap S \ne \varnothing} (l(Q'))^d.$$

We use this observation and take into account that $\mathcal{Q}$ is a covering of the set $\overline{Q} \cap S$. Hence, by the definition of the Hausdorff content it is clear that for every cube $Q \in \mathcal{F}^3_k$ we have

$$\mathcal{H}^d_\infty(Q \cap S) \le \sum_{Q' \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}} \colon Q' \cap Q \cap S \ne \varnothing} (l(Q'))^d. \tag{4.26}$$

By (4.21) and (4.22) it follows that

$$2^{-k} \geq l(Q) \qquad \forall\, Q \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}.$$

Hence, every cube $Q' \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}$ meets at most 3^n cubes from the family $\mathcal{F}^3_k$. A combination of this observation with (4.26) gives

$$\sum_{Q \in \mathcal{F}^3_k} \mathcal{H}^d_\infty(Q \cap S) \leq \sum_{Q \in \mathcal{F}^3_k} \; \sum_{Q' \in \mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}:\, Q' \cap Q \cap S \neq \varnothing} (l(Q'))^d \leq 3^n \,\mathrm{H}^d(\mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}). \tag{4.27}$$

On the other hand, since the family $\mathcal{F}^3_k$ consists of dyadic nonoverlapping cubes with the side length 2^{-k}, it is clear that the number of cubes in $\mathcal{F}^3_k$ can be calculated by the formula

$$\#\mathcal{F}^3_k = 2^{kn}\,\mathrm{H}^n(\mathcal{F}^3_k) = 2^{kn}\mathcal{L}^n\Biggl(\bigcup_{Q \in \mathcal{F}^3_k} Q\Biggr).$$

From (4.17), (4.23), and (4.25) it follows that

$$\mathcal{L}^n\Biggl(\bigcup_{Q \in \mathcal{F}^3_k} Q\Biggr) \geq (l(\overline{Q}))^n - V^1_k - V^2_k \geq \frac{1-\overline{\lambda}^{n/d}}{2}(l(\overline{Q}))^n.$$

As a result, we obtain

$$\#\mathcal{F}^3_k \geq 2^{kn}\frac{1-\overline{\lambda}^{n/d}}{2}(l(\overline{Q}))^n. \tag{4.28}$$

Step 5. If we assume that $\mathcal{F}^3_k \subset \mathcal{F}_S(d, \overline{\lambda}/c)$, then a combination of (4.18), (4.21), (4.22), (4.27), and (4.28) gives

$$\begin{aligned}(1+\varepsilon)\overline{\lambda}(l(\overline{Q}))^d &\geq \mathrm{H}^d(\mathcal{Q}^{\mathrm{s}}_{\underline{\kappa}}) \geq \frac{1}{3^n}\frac{\overline{\lambda}}{c}\cdot 2^{-kd}\,\#\mathcal{F}^3_k > \frac{\overline{\lambda}}{2c}2^{k(n-d)}\frac{1-\overline{\lambda}^{n/d}}{3^n}(l(\overline{Q}))^n \\ &\geq \frac{\overline{\lambda}(1-\overline{\lambda}^{n/d})}{2c\cdot 3^n}\left(\frac{l(\overline{Q})}{2\underline{\kappa}\tau}\right)^{n-d}(l(\overline{Q}))^d.\end{aligned} \tag{4.29}$$

Using (4.29) and taking into account the definition of τ given in (4.19), we get

$$(1+\varepsilon)((1+\varepsilon)\overline{\lambda})^{(n-d)/d}\,\underline{\kappa}^{n-d} \geq \frac{1-\overline{\lambda}^{n/d}}{c\cdot 2^{n-d+1}\cdot 3^n}.$$

Hence, using the first inequality in (4.17), we get (recall that $\varepsilon \in (0, 1/2)$)

$$\underline{\kappa}^{n-d} > \frac{1-\overline{\lambda}^{n/d}}{c\cdot 3^{n+1}}.$$

This inequality is in contradiction with (4.20). $\square$

The following concept, which was already mentioned in the Introduction, gives a natural generalization of the concept of porous cubes.

Definition 4.1. Given a set $S \subset \mathbb{R}^n$, a cube Q, and a parameter $\gamma \in (0, 1]$, we say that a set $U \subset Q$ is an (S, γ)-*cavity* of Q if

$$U \subset Q \setminus S \qquad \text{and} \qquad \mathcal{L}^n(U \setminus S) \geq \gamma(l(Q))^n.$$

We say that Q is (S, γ)-*hollow* if there exists an (S, γ)-cavity U of Q.

We need some notation. Given numbers $d \in (0,n)$, $\lambda \in (0,1)$, $r \geq 1$ and a set $S \subset \mathbb{R}^n$ with $\mathcal{H}^d_\infty(S) > 0$, we define, for every $\varkappa \in (0,1]$ and any cube $Q \subset \mathbb{R}^n$, the set

$$U_\varkappa(Q,r) := U_\varkappa(Q,d,\lambda,r) := Q \setminus \bigcup_{Q' \in \mathcal{DF}_S(d,\lambda)\colon l(Q') \leq \varkappa l(Q)} rQ'.$$

Theorem 4.2. *Let $d \in (0,n)$, $\lambda \in (0,1)$, and $r \geq 1$. Then, for every $\gamma \in (0, 1-2^{d-n})$, there exists a number $\overline{\varkappa} = \overline{\varkappa}(\gamma,n,d,\lambda,r) \in (0,1)$ such that, for every set $S \subset \mathbb{R}^n$ and every cube $\overline{Q} = Q_{k,m} \in \mathcal{D}_+$ with*

$$\mathcal{DH}^d_\infty(\overline{Q} \cap S) < (l(\overline{Q}))^d, \tag{4.30}$$

the sets $U_\varkappa(\overline{Q},d,\lambda,r)$ are (S,γ)-cavities of the cube $\overline{Q}$ for all $\varkappa \in (0,\overline{\varkappa})$.

Proof. We fix a set $S \subset \mathbb{R}^n$ and a cube $\overline{Q} = Q_{k,m} \in \mathcal{D}_+ \notin \mathcal{DF}_S(d,1)$. Without loss of generality we assume that $\mathcal{H}^d_\infty(S) > 0$, because otherwise the assertion is trivial. We also fix a parameter $\gamma \in (0, 1-2^{d-n})$. During the proof we write for brevity $\mathcal{DF} := \mathcal{DF}_S(d,\lambda)$. Let $\{\widehat{\mathcal{Q}}^s\}_{s\in\mathbb{N}}$ be an arbitrary (d,λ)-nice sequence for S. We split the proof into several steps.

Step 1. We fix a number $\overline{c} > 1$ close to 1 so that

$$\frac{\overline{c}}{2^{n-d}} < \frac{1}{2}(1-\gamma+2^{d-n}). \tag{4.31}$$

Let $\overline{\delta} = \overline{\delta}(\overline{c},n,r)$ be the same as in Lemma 4.1. Now we fix the minimal $k \in \mathbb{N}$ for which

$$2^{-k} < \overline{\delta} \qquad \text{and} \qquad \frac{r^d}{2^{k(n-d)}\lambda} + \frac{8n}{2^k} < \frac{1}{2}(1-\gamma-2^{d-n}). \tag{4.32}$$

Step 2. We define

$$s_0 := \min\bigl\{s \in \mathbb{N}_0\colon \{Q\} \succ \widehat{\mathcal{Q}}^{s+1}|_{\overline{Q}}\bigr\}. \tag{4.33}$$

Since $Q \in \mathcal{D}_+ \notin \mathcal{DF}_S(d,1)$, by (3.18) we have

$$\mathrm{H}^d\bigl(\widehat{\mathcal{Q}}^{s_0+1}|_{\overline{Q}}\bigr) \leq (l(\overline{Q}))^d. \tag{4.34}$$

Step 3. We introduce the family

$$\mathcal{K} := \bigl\{Q \in \mathcal{DF}\colon\ rQ \cap \overline{Q} \neq \varnothing,\ l(rQ) \leq 2^{-k-1} l(\overline{Q})\bigr\}.$$

We split $\mathcal{K}$ into three subfamilies. More precisely, we set

$$\begin{aligned} \mathcal{K}^1 &:= \bigl\{Q \in \mathcal{K}\colon\ \operatorname{int} Q \subset \mathbb{R}^n \setminus \overline{Q}\bigr\}, \\ \mathcal{K}^2 &:= \bigl\{Q \in \mathcal{K} \setminus \mathcal{K}^1\colon\ \exists \widehat{Q} \in \widehat{\mathcal{Q}}^{s_0+1},\ \widehat{Q} \subset Q\bigr\}, \\ \mathcal{K}^3 &:= \bigl\{Q \in \mathcal{K} \setminus \mathcal{K}^1\colon\ \exists \widehat{Q} \in \widehat{\mathcal{Q}}^{s_0+1},\ Q \subset \widehat{Q},\ l(Q) < l(\widehat{Q})\bigr\}. \end{aligned} \tag{4.35}$$

It follows directly from the construction that

$$\mathcal{K} \subset \mathcal{K}^1 \cup \mathcal{K}^2 \cup \mathcal{K}^3. \tag{4.36}$$

Step 4. Using Proposition 4.1, we get

$$\mathcal{L}^n\Biggl(\bigcup_{Q\in\mathcal{K}^1} (rQ \cap \overline{Q})\Biggr) \leq 8n(l(\overline{Q}))^n \frac{1}{2^k}. \tag{4.37}$$

Step 5. Let $\overline{\mathcal{K}}^2 \subset \mathcal{K}^2$ be the family of all maximal (with respect to inclusion) dyadic cubes from the family $\mathcal{K}^2$. We obviously get

$$\bigcup_{Q\in\mathcal{K}^2} Q \subset \bigcup_{Q\in\overline{\mathcal{K}}^2} Q. \tag{4.38}$$

Consider also the family

$$\mathcal{C} := \overline{\mathcal{K}}^2 \cup \{Q \in \widehat{Q}^{s_0+1}|_{\overline{Q}} \colon \operatorname{int} Q \cap \operatorname{int} Q' = \varnothing \ \forall Q' \in \overline{\mathcal{K}}^2\}.$$

We use (4.38), then take into account that $l(Q) \leq 2^{-k} l(\overline{Q})$ for all $Q \in \overline{\mathcal{K}}^2$ and finally apply Theorem 3.3 for $\lambda = \lambda_1 = \lambda_2$. As a result, we obtain

$$\mathcal{L}^n\Bigg(\bigcup_{Q\in\mathcal{K}^2} rQ\Bigg) \leq \mathcal{L}^n\Bigg(\bigcup_{Q\in\overline{\mathcal{K}}^2} rQ\Bigg) \leq r^n\, \mathrm{H}^n(\overline{\mathcal{K}}^2) \leq r^d \bigg(\frac{l(\overline{Q})}{2^k}\bigg)^{n-d} \mathrm{H}^d(\overline{\mathcal{K}}^2)$$

$$\leq r^d \bigg(\frac{l(\overline{Q})}{2^k}\bigg)^{n-d} \mathrm{H}^d(\mathcal{C}) \leq \frac{r^d}{2^{k(n-d)}\lambda} (l(\overline{Q}))^n. \tag{4.39}$$

Step 6. Let $\widehat{\mathcal{Q}} \subset \widehat{\mathcal{Q}}^{s_0+1}|_{\overline{Q}}$ be the family consisting of all cubes $\widehat{Q}$ for each of which there exists a cube $Q \in \mathcal{K}^3$ such that $Q \subset \widehat{Q}$ and $l(Q) < l(\widehat{Q})$. Letting $\tau := 2^{-1} l(\overline{Q})$, by (4.33) we get

$$\overline{\mu}(\widehat{\mathcal{Q}}) \leq \overline{\mu}(\widehat{\mathcal{Q}}^{s_0+1}|_{\overline{Q}}) \leq \tau. \tag{4.40}$$

Using the first inequality in (4.32), we have

$$\bigcup_{Q\in\mathcal{K}^3} rQ \subset U_{r\overline{\delta}\tau}\Bigg(\bigcup_{\widehat{Q}\in\widehat{\mathcal{Q}}} \widehat{Q}\Bigg). \tag{4.41}$$

We use (4.41), then apply Lemma 4.1, and finally use (4.34). This yields

$$\mathcal{L}^n\Bigg(\bigcup_{Q\in\mathcal{K}^3} rQ\Bigg) \leq \mathcal{L}^n\Bigg(U_{r\overline{\delta}\tau}\Bigg(\bigcup_{\widehat{Q}\in\widehat{\mathcal{Q}}} \widehat{Q}\Bigg)\Bigg) \leq \overline{c}\bigg(\frac{l(\overline{Q})}{2}\bigg)^{n-d} \mathrm{H}^d(\widehat{\mathcal{Q}})$$

$$\leq \overline{c}\bigg(\frac{l(\overline{Q})}{2}\bigg)^{n-d} \mathrm{H}^d(\widehat{\mathcal{Q}}^{s_0+1}|_{\overline{Q}}) \leq \frac{\overline{c}}{2^{n-d}} (l(\overline{Q}))^n. \tag{4.42}$$

Step 7. We set $\overline{\varkappa} := 2^{-k-1} r^{-1}$. Collecting (4.31), (4.32), (4.37), (4.39), and (4.42), we obtain

$$\mathcal{L}^n(\overline{Q} \setminus U_{\overline{\varkappa}}(\overline{Q}, d, \lambda, r)) \leq (1-\gamma)(l(\overline{Q}))^n. \tag{4.43}$$

Taking into account that $U_{\overline{\varkappa}}(\overline{Q}, d, \lambda, r) \subset U_{\varkappa}(\overline{Q}, d, \lambda, r)$ for all $\varkappa \in (0, \overline{\varkappa})$ we complete the proof. □

Now we are ready to prove the second main result of the present paper. We recall that the (d,λ)-thick δ-neighborhood of a set $S \subset \mathbb{R}^n$ was defined in (1.5).

Proof of Theorem 1.2. The case $d = 0$ is obvious, so we further assume that $d \in (0, n)$. Fix an arbitrary set $S \subset \mathbb{R}^n$ and an arbitrary cube $\overline{Q} = Q_l(x)$ satisfying the assumptions of the theorem. An application of Theorem 4.1 with $c = 2^n$ gives the existence of a constant $\underline{\kappa} := \underline{\kappa}(\overline{\lambda}, n, d, 2^n)$ and a cube $\underline{Q} \in \mathcal{D}_+$ with side length

$$l(\underline{Q}) \geq \underline{\kappa} l(\overline{Q}) \tag{4.44}$$

such that $\mathcal{H}^d_\infty(\underline{Q} \cap S) < \overline{\lambda}/2^n$. By Remark 2.1,

$$\mathcal{D}\mathcal{H}^d_\infty(\underline{Q} \cap S) < \overline{\lambda} l(\underline{Q}).$$

Hence, by Theorem 4.2 there exists a constant $\overline{\varkappa} := \overline{\varkappa}((1-2^{d-n})/2, n, d, \lambda/3^n, 3)$ such that

$$\mathcal{L}^n\bigg(U_{\overline{\varkappa}}\bigg(\underline{Q}, d, \frac{\lambda}{3^n}, 3\bigg)\bigg) \geq \frac{1-2^{d-n}}{2} (l(\underline{Q}))^n. \tag{4.45}$$

By Proposition 2.3 for any cube $Q \in \mathcal{F}_S(d,\lambda)$ there is a cube $Q' \in \mathcal{D}_j$ with $j = [-\log_2 l(Q)]$ such that $Q' \in \mathcal{DF}_S(d,\lambda/3^n)$. Clearly, $Q \subset 3Q'$. Hence, if we set

$$\overline{\delta} := \overline{\delta}(n,d,\overline{\lambda},\lambda) := \underline{\kappa}(\overline{\lambda},n,d,2^n)\,\overline{\varkappa}\biggl(\frac{1-2^{d-n}}{2},n,d,\frac{\lambda}{3^n},3\biggr) = \underline{\kappa}\overline{\varkappa},$$

then for any $\delta \in (0,\overline{\delta})$ we have

$$W_{\delta l}(\overline{Q},d,\lambda) \supset U_{\overline{\varkappa}}\biggl(\underline{Q},d,\frac{\lambda}{3^n},3\biggr). \tag{4.46}$$

Now we set

$$\underline{\gamma}(\overline{\lambda},n,d) = \frac{1-2^{d-n}}{2}\,(\underline{\kappa}(\overline{\lambda},n,d,2^n))^n.$$

As a result, by (4.44)–(4.46) we deduce

$$\mathcal{L}^n(W_{\delta l}(\overline{Q},d,\lambda)) \ge \mathcal{L}^n\biggl(U_{\overline{\varkappa}}\biggl(\underline{Q},d,\frac{\lambda}{3^n},3\biggr)\biggr) \ge \underline{\gamma}(\overline{\lambda},n,d)(l(\overline{Q}))^n. \tag{4.47}$$

This completes the proof. $\square$

Remark 4.1. It is easy to show that if $d \in (0,n]$, $\overline{\lambda} \in (0,1)$, $S \subset \mathbb{R}^n$ is a nonempty set, and a cube $\overline{Q} = Q_l(x)$ with $l \in (0,1]$ is such that $\overline{Q} \notin \mathcal{F}_S(d,\overline{\lambda})$, then the cube $\overline{Q}$ is $(S,1-\overline{\lambda}^{n/d})$-hollow.

Indeed, by Definition 2.2 there is an at most countable covering $\mathcal{U}$ of the set $\overline{Q} \cap S$ such that

$$\mathrm{H}^d(\mathcal{U}) < \overline{\lambda} l^d.$$

For every set $U \in \mathcal{U}$ there is a cube $Q(U) \supset U$ with $l(Q) = \operatorname{diam} U$. It is clear that

$$l(Q(U)) < \overline{\lambda}^{1/d} l \qquad \forall\, U \in \mathcal{U}.$$

This gives

$$\mathrm{H}^n(\mathcal{U}) < \overline{\lambda}^{n/d-1} l^{n-d}\, \mathrm{H}^d(\mathcal{U}) \le \overline{\lambda}^{n/d} l^n. \tag{4.48}$$

Since $\overline{\lambda} < 1$, the required result follows from (4.48) and the subadditivity property of the Lebesgue measure $\mathcal{L}^n$.

It is clear that there is a huge difference between the elementary observation given above and Theorem 1.2. The former observation does not give any information about the structure of cavities in cubes whose intersections with S have relatively small d-Hausdorff content. On the other hand, informally speaking, Theorem 1.2 claims that the corresponding cavities in cubes are located at some "nonzero depth" in $\mathbb{R}^n \setminus S$ with respect to the special distance.

Now we show that Theorem 1.2 admits a significant refinement in the context of d-thick sets.

Proof of Theorem 1.1. The case $d = 0$ is obvious, so we further assume that $d \in (0,n)$. Since S is (d,λ)-thick, we have (recall the notation (1.5))

$$S_{2\varepsilon}(d,\lambda) \supset U_\varepsilon(S) \qquad \forall\, \varepsilon \in (0,1].$$

By Theorem 1.2 this implies that

$$\overline{Q} \setminus U_{\delta l}(S) \ne \varnothing \qquad \forall\, \delta \in (0,\overline{\delta}].$$

Hence, letting $\underline{\tau} := \underline{\tau}(n,d,\overline{\lambda},\lambda) = \overline{\delta}(n,d,\overline{\lambda},\lambda)/2$ and taking an arbitrary point $x^* \in \overline{Q} \setminus U_{\delta l}(S)$, we obtain

$$Q_{\underline{\tau}}(x^*) \subset \overline{Q} \setminus S,$$

which completes the proof. $\square$

5. APPLICATIONS

In this section we introduce some new concepts, which may be of independent interest.

The following data are assumed to be fixed throughout this section:

(D1′) arbitrary numbers $n \in \mathbb{N}$ and $d \in (0, n)$;

(D2′) a set $S \subset Q_{0,0}$ with $\lambda_S := \mathcal{H}^d_\infty(S) > 0$.

Recall (1.4) and Definition 2.6. Given $\lambda \in (0, 1]$, we write $\mathcal{F}(\lambda)$ and $\mathcal{DF}(\lambda)$ instead of $\mathcal{F}_S(d, \lambda)$ and $\mathcal{DF}_S(d, \lambda)$, respectively. Furthermore, given $\lambda \in (0, 1]$, for any $x, y \in \mathbb{R}^n$ we define the family

$$\mathcal{Q}_{x,y}(\lambda) := \bigl\{Q \ni x, y \colon Q \in \mathcal{F}(\lambda)\bigr\}.$$

Now, for every $\lambda \in (0, 1]$ and any $x, y \in \mathbb{R}^n$, we set

$$\widetilde{\rho}_{S,d,\lambda}(x, y) := \widetilde{\rho}_\lambda(x, y) := \begin{cases} \inf\{l(Q) \colon Q \in \mathcal{Q}_{x,y}(\lambda)\}, & x \neq y,\ \mathcal{Q}_{x,y}(\lambda) \neq \varnothing, \\ +\infty, & \mathcal{Q}_{x,y}(\lambda) = \varnothing, \\ 0, & x = y. \end{cases} \tag{5.1}$$

For any two points $x, y \in \mathbb{R}^n$ such that at least one of them belongs to the set S, we put

$$\rho_\lambda(x, y) := \rho_{S,d,\lambda}(x, y) := \inf \sum_{i=0}^{N-1} \widetilde{\rho}_\lambda(x^i, x^{i+1}), \tag{5.2}$$

where the infimum is taken over all finite sets $\{x^i\}_{i=0}^N \subset \mathbb{R}^n$ such that $x^0 = x$ and $x^N = y$. Finally, in the case when $x, y \in \mathbb{R}^n \setminus S$, we define

$$\rho_\lambda(x, y) := \rho_{S,d,\lambda}(x, y) := \max\biggl\{\|x - y\|_\infty, \sup_{\xi \in S} |\rho_\lambda(x, \xi) - \rho_\lambda(y, \xi)|\biggr\}, \tag{5.3}$$

where we set $|\rho_\lambda(x, \xi) - \rho_\lambda(y, \xi)| := +\infty$ if $\max\{\rho_\lambda(x, \xi), \rho_\lambda(y, \xi)\} = +\infty$.

Recall that a pseudometric on $\mathbb{R}^n$ is a symmetric nonnegative function $\rho \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ satisfying the triangle inequality.

Proposition 5.1. *For every $\lambda \in (0, 1]$ the function $\rho_\lambda \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is a pseudometric on $\mathbb{R}^n$.*

Proof. The symmetry is obvious by (5.2) and (5.3). Furthermore, note that

$$\rho_\lambda(x, y) \geq \|x - y\|_\infty \qquad \forall\, x, y \in \mathbb{R}^n.$$

Hence, $\rho_\lambda(x, y) = 0$ implies $x = y$.

It remains to verify the triangle inequality. We fix an arbitrary triple of points $x, y, z \in \mathbb{R}^n$. In the case when $\rho_\lambda(x, y) = +\infty$ or $\rho_\lambda(y, z) = +\infty$, the triangle inequality is obvious. Consider the case when $\rho_\lambda(x, y) < +\infty$ and $\rho_\lambda(y, z) < +\infty$. We should consider two subcases. In the first subcase at least one of the three points x, y, z (suppose that $y \in S$, as the case of $z \in S$ and $x \notin S$ immediately follows from (5.3)) belongs to the set S. Given $\delta > 0$, let $\{x^i\}_{i=0}^N, \{x^i\}_{i=N+1}^L \subset \mathbb{R}^n$ be finite sets of points such that $x^0 = x$, $x^N = y$, $x^L = z$ and

$$\sum_{i=0}^{N-1} \widetilde{\rho}_\lambda(x^i, x^{i+1}) \leq \rho_\lambda(x, y) + \frac{\delta}{2}, \qquad \sum_{i=N}^{L-1} \widetilde{\rho}_\lambda(x^i, x^{i+1}) \leq \rho_\lambda(y, z) + \frac{\delta}{2}.$$

Adding the two inequalities and using (5.2), we obtain

$$\rho_\lambda(x, z) \leq \sum_{i=0}^{L-1} \widetilde{\rho}_\lambda(x^i, x^{i+1}) \leq \rho_\lambda(x, y) + \rho_\lambda(y, z) + \delta.$$

Since $\delta > 0$ was chosen arbitrarily, we deduce the triangle inequality for this subcase,

$$\rho_\lambda(x,z) \le \rho_\lambda(x,y) + \rho_\lambda(y,z). \tag{5.4}$$

Finally, consider the subcase when $x, y, z \in \mathbb{R}^n \setminus S$. If $\|x - z\| \ge \sup_{\xi\in S} |\rho_\lambda(x,\xi) - \rho_\lambda(z,\xi)|$, then by (5.3) we get

$$\rho_\lambda(x,z) = \|x - z\| \le \|x - y\| + \|y - z\| \le \rho_\lambda(x,y) + \rho_\lambda(y,z). \tag{5.5}$$

If $\|x - z\| < \sup_{\xi\in S} |\rho_{\lambda,\varepsilon}(x,\xi) - \rho_{\lambda,\varepsilon}(z,\xi)|$, then by (5.3) we have, for any $\xi \in S$,

$$|\rho_\lambda(x,\xi) - \rho_\lambda(z,\xi)| \le |\rho_\lambda(x,\xi) - \rho_\lambda(y,\xi)| + |\rho_\lambda(y,\xi) - \rho_\lambda(z,\xi)| \le \rho_\lambda(x,y) + \rho_\lambda(y,z). \tag{5.6}$$

Taking the supremum in (5.6) over all $\xi \in S$, we also obtain (5.4). Together with (5.5) this yields the triangle inequality for the subcase when $x, y, z \in \mathbb{R}^n \setminus S$.

Thus, we have proved that the triangle inequality holds for any triple of points $x, y, z \in \mathbb{R}^n$. $\square$

Remark 5.1. Note that if $\lambda \in (0, \lambda_S]$, then $Q_{0,0} \in \mathcal{DF}(\lambda)$ (by assumption (D2$'$) and Remark 2.1). Hence, by (5.2) and (5.3) it is easy to see that $\rho_\lambda(x,y) < +\infty$ for all $x, y \in Q_{0,0}$.

Recall that the (d,λ)-*thick distance* between a nonempty set $E \subset \mathbb{R}^n \setminus S$ and S is defined by the formula

$$\mathrm{D}_\lambda(E,S) := \mathrm{D}_{S,d,\lambda}(E,S) := \inf\bigl\{\rho_\lambda(x,\xi)\colon\ x \in E,\ \xi \in S\bigr\}. \tag{5.7}$$

The following proposition gives a simpler way to compute the (d,λ)-thick distance from a given point $x \in \mathbb{R}^n \setminus S$ to the set S.

Proposition 5.2. *Let $\lambda \in (0, \lambda_S]$. Then the equality*

$$\mathrm{D}_\lambda(x,S) = \inf\bigl\{l(Q)\colon\ Q \ni x,\ Q \in \mathcal{F}(\lambda)\bigr\} \tag{5.8}$$

holds for any $x \in (\lambda_S/\lambda)^{1/d} Q_{0,0} \setminus S$.

Proof. By Remark 2.5 and assumption (D2$'$) it follows that $(\lambda_S/\lambda)^{1/d} Q_{0,0} \in \mathcal{F}(\lambda)$. Hence, we have $\mathcal{Q}_{x,y}(\lambda) \ne \varnothing$ for every $x \in (\lambda_S/\lambda)^{1/d} Q_{0,0} \setminus S$ and $y \in S$. As a result, $0 < \rho_\lambda(x,y) < +\infty$ for such x and y.

Now we fix an arbitrary $x \in (\lambda_S/\lambda)^{1/d} Q_{0,0} \setminus S$. We denote the right-hand side of (5.8) by $\widetilde{\mathrm{D}}_\lambda(x,S)$. Our aim is to show that $\mathrm{D}_\lambda(x,S) = \widetilde{\mathrm{D}}_\lambda(x,S)$. Given $\varepsilon > 0$, using (5.2) and (5.7), we choose $x_\varepsilon \in S$ and points $\{x_i\}_{i=0}^{N_\varepsilon}$ in such a way that $x_0 = x$, $x_{N_\varepsilon} = x_\varepsilon$ and

$$\sum_{i=0}^{N_\varepsilon - 1} \widetilde{\rho}_\lambda(x_i, x_{i+1}) < \mathrm{D}_\lambda(x,S) + \varepsilon.$$

Hence, we obtain

$$\mathrm{D}_\lambda(x,S) \le \widetilde{\mathrm{D}}_\lambda(x,S) \le \widetilde{\rho}_\lambda(x_0,x_1) < \mathrm{D}_\lambda(x,S) + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrary small, we get the required equality. $\square$

Remark 5.2. The pseudometric $\rho_\lambda := \rho_{S,d,\lambda}$ introduced above is a natural generalization of the metric induced by the $\|\cdot\|_\infty$-norm. Indeed, for $\lambda \in (0,1]$ we have $\rho_{S,0,\lambda}(x,y) = \|x - y\|_\infty$ for any $x, y \in \mathbb{R}^n$.

Given $\lambda \in (0, \lambda_S]$, recall the concept of (d,λ)-thick sets in $\mathbb{R}^n$ formulated in the Introduction. Recall also the notion of δ-neighborhood of S. If S is (d,λ)-thick, then using Proposition 5.2 it is easy to see that

$$\operatorname{dist}(x,S) \le \mathrm{D}_\lambda(x,S) \le 2\operatorname{dist}(x,S) \qquad \forall\, x \in U_{1/2}(S).$$

Given $\lambda \in (0,1]$, for any $\varepsilon > 0$ we introduce the ε-neighborhood of the set S with respect to the metric ρ_λ by the formula

$$U^{\rho_\lambda}_{\varepsilon}(S) := \{x \in \mathbb{R}^n\colon\ \mathrm{D}_\lambda(x,S) < \varepsilon\}.$$

By Remark 2.4 we have

$$U^{\rho_\lambda}_{\varepsilon}(S) \supset S \qquad \forall\, \varepsilon \geq 0, \quad \forall\, \lambda \in (0,1].$$

Remark 5.3. For every $\lambda \in (0,\lambda_S]$ and $\varepsilon > 0$ small enough, we have (recall (1.5))

$$U^{\rho_\lambda}_{\varepsilon}(S) = S_\varepsilon(\lambda) := \bigcup_{Q\in\mathcal{F}(\lambda)\colon\, 0\leq l(Q)<\varepsilon} Q. \tag{5.9}$$

Indeed, by Proposition 5.2 for a given point $x \in (\lambda_S/\lambda)^d Q_{0,0} \setminus S$ we have $\mathrm{D}_\lambda(x,S) < \varepsilon$ if and only if there is a cube $Q \in \mathcal{F}(\lambda)$ with $0 \leq l(Q) < \varepsilon$. This proves the claim.

6. EXAMPLES

In this concluding section we present elementary examples demonstrating the sharpness of the main results.

The following example shows that the d-thickness condition in Theorem 1.1 cannot be dropped.

Example 6.1. Let $K \subset [0,1]$ be the standard middle-third Cantor set. For each $j \in \mathbb{N}$ we define $K_j := \{3^{-j}x\colon x \in K\}$. We set

$$S_j := \bigcup_{i=0}^{2^j-1}\biggl(\frac{i}{2^j} + K_{2j}\biggr).$$

Obviously, given $j \in \mathbb{N}$, the maximal size of closed one-dimensional cubes $Q \subset [0,1] \setminus S_j$ is at most 2^{-j}. If $d = \ln 2/\ln 3$, we clearly have

$$\mathcal{H}^d(S_j) \leq \frac{2^j}{2^{2j}} \to 0, \qquad j \to \infty. \tag{6.1}$$

Finally, we define the set

$$S := \bigcup_{j=0}^{\infty}\biggl(\biggl(1-\frac{1}{2^j}\biggr) + \frac{1}{2^{j+1}}S_{j+1}\biggr),$$

where $cS_{j+1} = \{cx\colon x \in S_{j+1}\}$ for $c > 0$. It follows from (6.1) that the set S is not (d,λ)-thick for any $\lambda \in (0,1)$. On the other hand, any dyadic interval $[1-2/2^j, 1-1/2^j]$, $j \in \mathbb{N}$, can be (S,τ)-porous only with $\tau < 2^{-s}$.

Now we show that the restriction $d < n$ in Theorem 1.2 cannot be dropped.

Example 6.2. For each $j \in \mathbb{N}$ we set

$$S_j := \bigcup_{i=0}^{2^j-1}\biggl[\frac{i}{2^j}, \frac{i}{2^j} + \frac{1}{10}\cdot\frac{1}{2^j}\biggr]. \tag{6.2}$$

Define the set

$$S := \bigcup_{j=0}^{\infty}\biggl(\biggl(1-\frac{1}{2^j}\biggr) + \frac{1}{2^{j+1}}S_{j+1}\biggr).$$

It is easy to see that

$$\mathcal{H}^1_\infty\biggl(\biggl[1-\frac{2}{2^j}, 1-\frac{1}{2^j}\biggr] \cap S\biggr) < \frac{1}{8}\cdot\frac{1}{2^j} \qquad \forall\, j \in \mathbb{N}. \tag{6.3}$$

On the other hand, for any $j \in \mathbb{N}$,

$$\left[1 - \frac{2}{2^j}, 1 - \frac{1}{2^j}\right] \setminus \bigcup_{Q' : \, l(Q') \le 2^{-2j}, \; Q' \in \mathcal{F}_S(1,1/10)} Q' = \varnothing. \tag{6.4}$$

This shows that the conclusion of Theorem 1.2 cannot hold in the case $d = n = 1$ and $\lambda = 1/10$.

ACKNOWLEDGMENTS

The author would like to thank Alexey Alimov and Roman Karasev, who have read the first versions of this paper and made valuable remarks. The author is grateful to the anonymous referee, who find several misprints.

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This article was submitted by the author simultaneously in Russian and English