FRACTIONAL DIVERGENCE-MEASURE FIELDS, LEIBNIZ RULE AND GAUSS–GREEN FORMULA

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ABSTRACT. Given $\alpha \in (0,1]$ and $p \in [1,+\infty]$, we define the space $\mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ of L^p vector fields whose α-divergence is a finite Radon measure, extending the theory of divergence-measure vector fields to the distributional fractional setting. Our main results concern the absolute continuity properties of the α-divergence-measure with respect to the Hausdorff measure and fractional analogues of the Leibniz rule and the Gauss–Green formula. The sharpness of our results is discussed via some explicit examples.

1. INTRODUCTION

1.1. **The classical framework.**

The theory of *divergence-measure fields* in the Euclidean space naturally emerged as the appropriate setting for the study of minimal regularity conditions allowing for integration-by-parts and Gauss–Green formulas. Since Anzellotti's seminal paper [3], several fundamental results have been established in the last 20 years, such as Leibniz rules for divergence-measure fields and suitably weakly differentiable scalar functions, well-posedness of generalized normal traces on rectifiable sets, and integration-by-parts formulas under minimal regularity assumptions, see [1, 7–9, 14, 15, 23–26, 33, 50–52]. Since its beginning, the theory of divergence-measure fields have found numerous applications in several areas, including Continuum Mechanics [30, 48, 49], hyperbolic systems of conservation laws [1, 8, 11, 29], gas dynamic [10] and Dirichlet problems for the 1-Laplacian operator and prescribed mean curvature-type equations [12, 27, 28, 37–40, 43, 46, 47], just to name a few. For recent extensions to non-Euclidean frameworks, we refer to [5, 6, 16].

The basic definition goes as follows (see Section 2.1 for the notation). Given $p \in [1,+\infty]$, we say that a vector field $F \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ has divergence-measure, and we write

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$F \in \mathcal{DM}^{1,p}(\mathbb{R}^n)$, if there exists a finite Radon measure $divF \in \mathcal{M}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} F \cdot \nabla\xi \,\mathrm{d}x = -\int_{\mathbb{R}^n} \xi \,\mathrm{d}divF \tag{1.1}$$

for all $\xi \in C_c^\infty(\mathbb{R}^n)$. The integration-by-parts formula (1.1) clearly generalizes the usual Divergence Theorem. In fact, if the vector field F is sufficiently regular, say $F \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, then $divF = \mathrm{div}F\,\mathscr{L}^n$ in (1.1), where $\mathscr{L}^n$ is the n-dimensional measure.

As for the analogous case of functions with bounded variation, the two principal questions regarding $\mathcal{DM}^{1,p}$ vector fields concern the absolute continuity properties of the divergence-measure with respect to the Hausdorff measure $\mathscr{H}^s$, for $s \in [0,n]$, and the well-posedness of a Leibniz rule with suitable scalar functions.

The absolute continuity properties of the divergence-measure of a $\mathcal{DM}^{1,p}$ vector field with respect to the Hausdorff measure hold as follows, see [49, Th. 3.2 and Exam. 3.3].

Theorem 1.1 (Absolute continuity properties of the divergence-measure)**.** *Assume that $F \in \mathcal{DM}^{1,p}(\mathbb{R}^n)$ with $p \in [1,+\infty]$. We have the following cases:*

(i) if $p \in \left[1, \frac{n}{n-1}\right)$, then $div\,F$ does not enjoy any absolute continuity property;

(ii) if $p \in \left[\frac{n}{n-1}, +\infty\right)$, then $|divF|(B) = 0$ on Borel sets B of σ-finite $\mathscr{H}^{n-\frac{p}{p-1}}$ measure;

(iii) if $p = +\infty$, then $|divF| \ll \mathscr{H}^{n-1}$.

The Leibniz rule involving $\mathcal{DM}^{1,p}$ vector fields and Sobolev functions is stated in Theorem 1.2 below, for which we refer to [7, Prop. 3.1], [8, Th. 3.1], [13, Th. 3.2.3] and [34, Th. 2.1]. Here and in the following, for $x \in \mathbb{R}^n$, we let

$$g^\star(x) = \begin{cases} \lim\limits_{r \to 0^+} \fint_{B_r(x)} g(y)\,\mathrm{d}y & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

be the *precise representative* of $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For the notion of *(Anzellotti's) pairing measure* briefly recalled in the statement, we refer the reader to [3, Def. 1.4], [8, Th. 3.2], or to [23, Sec. 2.5] for a more general formulation.

Theorem 1.2 (Leibniz rule for $\mathcal{DM}^{1,p}$ vector fields and weakly differentiable functions)**.** *Let $p, q \in [1,+\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in \mathcal{DM}^{1,p}(\mathbb{R}^n)$ and*

$$g \in \begin{cases} L^\infty(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n) & \text{for } p \in [1,+\infty), \\ L^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n) & \text{for } p = +\infty, \end{cases}$$

then $gF \in \mathcal{DM}^{1,r}(\mathbb{R}^n)$ for all $r \in [1,p]$, with

$$div(gF) = g^\star\, divF + (F, Dg)_q \quad \text{in } \mathcal{M}(\mathbb{R}^n). \tag{1.3}$$

Here

$$(F, Dg)_q = \begin{cases} F \cdot \nabla g\,\mathscr{L}^n & \text{if } q > 1, \text{ or } q = 1 \text{ and } g \in L^\infty(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n), \\ (F, Dg) & \text{if } q = 1 \text{ and } g \in L^\infty(\mathbb{R}^n) \cap (BV(\mathbb{R}^n) \setminus W^{1,1}(\mathbb{R}^n)), \end{cases}$$

is the pairing measure *between F and Dg, where (F, Dg) is the unique weak limit*

$$F \cdot \nabla(\varrho_\varepsilon * g)\,\mathscr{L}^n \rightharpoonup (F, Dg) \quad \text{in } \mathcal{M}(\mathbb{R}^n) \text{ as } \varepsilon \to 0^+,$$

being $\varrho_\varepsilon = \varepsilon^{-n}\varrho\left(\frac{\cdot}{\varepsilon}\right)$ *for* $\varepsilon > 0$, *with* $\varrho \in C_c^\infty(\mathbb{R}^n)$ *any non-negative radially symmetric function such that* $\operatorname{supp}\varrho \subset B_1$ *and* $\int_{B_1} \varrho \, \mathrm{d}x = 1$.

Remark 1.3 (Choice of $g^\star$ in (1.3) for $p < +\infty$)**.** For $p < +\infty$, the function $g^\star$ appearing in (1.3) can be defined in a more specific way. For $p \in \left[1, \frac{n}{n-1}\right)$, $g^\star$ can be taken as the continuous representative of g. Instead, for $p \in \left[\frac{n}{n-1}, +\infty\right)$, $g^\star$ can be taken as the q-*quasicontinuous representative* of g. See [13, Sec. 3.2] for a more detailed discussion.

1.2. **Fractional divergence-measure fields.** The aim of the present note is to introduce a fractional analogue of the theory of divergence-measure fields, following the distributional approach to fractional spaces recently introduced and studied by the authors and collaborators in the series of papers [4, 17–22]. For results close to the main topic of this paper, we also refer to [42, 53, 54], even though our point of view is different.

In the fractional setting, for $\alpha \in (0,1)$, one has the integration-by-parts formula

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \, \mathrm{d}x = -\int_{\mathbb{R}^n} \xi \operatorname{div}^\alpha F \, \mathrm{d}x \tag{1.4}$$

for all functions $\xi \in \operatorname{Lip}_c(\mathbb{R}^n)$ and vector fields $F \in \operatorname{Lip}_c(\mathbb{R}^n; \mathbb{R}^n)$, where

$$\nabla^\alpha \xi(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(\xi(y) - \xi(x))(y - x)}{|y - x|^{n+\alpha+1}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.5}$$

is the *fractional* α*-gradient*,

$$\operatorname{div}^\alpha F(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(F(y) - F(x)) \cdot (y - x)}{|y - x|^{n+\alpha+1}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.6}$$

is the *fractional* α*-divergence*, and

$$\mu_{n,\alpha} = 2^\alpha \, \pi^{-\frac{n}{2}} \, \frac{\Gamma\left(\frac{n+\alpha+1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}$$

is a renormalization constant, see [18, Sec. 2.2] for a detailed exposition. According to the main results of [4, 19], with a slight (but justified) abuse of notation, we may identify (1.5) with the usual gradient ∇ for $\alpha = 1$, and with the vector-valued Riesz transform $\nabla^0 = R$ for $\alpha = 0$ (see Section 2.1 for the definition).

As already done by the authors in the case of scalar functions, the basic idea is now to use formula (1.4) to define a fractional analogue of the divergence-measure (1.1).

Definition 1.4 ($\mathcal{DM}^{\alpha,p}$ vector fields)**.** Let $\alpha \in (0,1]$ and $p \in [1,+\infty]$. A vector field $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ has *fractional* α*-divergence-measure*, and we write $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, if

$$\sup\left\{ \int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \, \mathrm{d}x : \xi \in C_c^\infty(\mathbb{R}^n), \ \|\xi\|_{L^\infty(\mathbb{R}^n)} \le 1 \right\} < +\infty.$$

The case $\alpha = 1$ in Definition 1.4 corresponds to classical divergence-measure fields. Without loss of generality, we always assume $n \ge 2$, since for $n = 1$ one clearly identifies $\mathcal{DM}^{\alpha,p}(\mathbb{R}) = BV^{\alpha,p}(\mathbb{R})$, the space of L^p functions with finite totale fractional α-variation, see the aforementioned [17, 20, 21] for an extensive presentation of $BV^{\alpha,p}$ functions on $\mathbb{R}^n$. We also observe that $BV^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^n) \subset \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ for $n \ge 2$, with strict inclusion at least in the case $p \in \left[1, \frac{n}{n-\alpha}\right)$, due to the fact that the vector fields in Example 3.1 below cannot belong to $BV^{\alpha,p}(\mathbb{R}^n; \mathbb{R}^n)$, in the light of [17, Th. 1].

Similarly to the case of $BV^{\alpha,p}$ functions (see [18, Th. 3.2] and [4, Th. 5]), we can state the following structural result for $\mathcal{DM}^{\alpha,p}$ vector fields. The proof is very similar to the one of [18, Th. 3.2] and is therefore omitted.

Theorem 1.5 (Structure Theorem for $\mathcal{DM}^{\alpha,p}$ vector fields)**.** *Let $\alpha \in (0,1]$ and $p \in [1,+\infty]$. A vector field $F \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ belongs to $\mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ if and only if there exists a finite Radon measure $div^\alpha F \in \mathcal{M}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \,\mathrm{d}x = -\int_{\mathbb{R}^n} \xi \,\mathrm{d}div^\alpha F \tag{1.7}$$

for all $\xi \in C_c^\infty(\mathbb{R}^n)$. In addition, for any open set $U \subset \mathbb{R}^n$, it holds

$$|div^\alpha F|(U) = \sup\left\{\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \,\mathrm{d}x : \xi \in C_c^\infty(U),\ \|\xi\|_{L^\infty(U)} \le 1\right\}. \tag{1.8}$$

If the vector field is sufficiently regular, say $F \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$ for instance, then the fractional divergence-measure given by Theorem 1.5 is $div^\alpha F = \mathrm{div}^\alpha F \,\mathscr{L}^n$, where $\mathrm{div}^\alpha F$ is as in (1.6). Moreover, thanks to Theorem 1.5, the linear space

$$\mathcal{DM}^{\alpha,p}(\mathbb{R}^n) = \{F \in L^p(\mathbb{R}^n;\mathbb{R}^n) : |div^\alpha F|(\mathbb{R}^n) < +\infty\}$$

endowed with the norm

$$\|F\|_{\mathcal{DM}^{\alpha,p}(\mathbb{R}^n)} = \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} + |div^\alpha F|(\mathbb{R}^n), \quad F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n),$$

is a Banach space, and the fractional divergence-measure in (1.8) is lower semicontinuous with respect to the L^p convergence.

Remark 1.6 (On the space $\mathcal{DM}^{0,p}$)**.** Although not strictly necessary for the purposes of the present paper, let us briefly comment on the case $\alpha = 0$ in Definition 1.4. By exploiting [4, Lem. 26], if $F \in \mathcal{DM}^{0,p}(\mathbb{R}^n)$ for some $p \in (1,+\infty)$, then

$$div^0 F = \mathrm{div}^0 F \mathscr{L}^n = (R \cdot F)\,\mathscr{L}^n$$

with $R \cdot F \in L^p(\mathbb{R}^n)$, where $R = \nabla^0$ the vector-value Riesz transform (see Section 2.1 for the definition). Therefore, for $p \in (1,+\infty)$, we can write

$$\mathcal{DM}^{0,p}(\mathbb{R}^n) = \left\{F \in L^p(\mathbb{R}^n;\mathbb{R}^n) : \mathrm{div}^0 F \in L^1(\mathbb{R}^n)\right\}.$$

Hence, if $F \in \mathcal{DM}^{0,p}(\mathbb{R}^n)$ for some $p \in (1,+\infty)$, then $|div^0 F| \ll \mathscr{L}^n$. The limiting cases $p \in \{1,+\infty\}$ seem more intricate and we leave them for future investigations.

1.3. **Main results.** Our first main result deals with the absolute continuity properties of $\mathcal{DM}^{\alpha,p}$ vector fields with respect to the Hausdorff measure, extending Theorem 1.1.

Theorem 1.7 (Absolute continuity properties of the fractional divergence-measure)**.** *Let $\alpha \in (0,1)$, $p \in [1,+\infty]$ and assume that $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$. We have the following cases:*

(i) if $p \in \left[1, \frac{n}{n-\alpha}\right)$, then $div^\alpha F$ does not enjoy any absolute continuity property;

(ii) if $p \in \left[\frac{n}{n-\alpha}, \frac{n}{1-\alpha}\right)$, then $|div^\alpha F|(B) = 0$ on Borel sets $B \subset \mathbb{R}^n$ with σ-finite $\mathscr{H}^{n-\frac{p}{p-1+(1-\alpha)\frac{p}{n}}}$ measure;

(iii) if $p \in \left[\frac{n}{1-\alpha}, +\infty\right]$, then $|div^\alpha F| \ll \mathscr{H}^{n-\alpha-\frac{n}{p}}$.

In particular, Theorem 1.7 tells that, if $F \in \mathcal{DM}^{\alpha,\infty}(\mathbb{R}^n)$, then $|div^\alpha F| \ll \mathscr{H}^{n-\alpha}$, exactly as in Theorem 1.1 for $p = +\infty$. For $p < +\infty$, instead, the properties of the fractional divergence-measure are different from the corresponding ones in the classical setting. Indeed, as for the fractional variation of $BV^{\alpha,p}$ functions (see [17, Th. 1] for the corresponding result), the threshold $p = \frac{n}{1-\alpha}$ imposes an interesting change of dimension of the Hausdorff measure. This is quite customary in the distributional fractional framework, and is essentially due to the mapping properties of Riesz potential $I_{1-\alpha}$, see [18, Sec. 2.3].

Our second main result concerns Leibniz rules for $\mathcal{DM}^{\alpha,p}$-fields and Besov functions, see [20, Th 1.1] for the corresponding result for $BV^{\alpha,p}$ functions. We refer to Section 2.1 for the definitions of fractional Sobolev and Besov spaces.

Theorem 1.8 (Leibniz rule for $\mathcal{DM}^{\alpha,p}$ vector fields with Besov functions)**.** *Let* $\alpha \in (0,1)$ *and let* $p, q \in [1,+\infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ *and*

$$g \in \begin{cases} B^\alpha_{q,1}(\mathbb{R}^n) & \text{for } p \in \left[1, \frac{n}{n-\alpha}\right), \\ L^\infty(\mathbb{R}^n) \cap B^\gamma_{q,1}(\mathbb{R}^n) \text{ with } \gamma \in (\gamma_{n,q,\alpha}, 1) & \text{for } p \in \left[\frac{n}{n-\alpha}, \frac{n}{1-\alpha}\right), \\ L^\infty(\mathbb{R}^n) \cap B^\beta_{q,1}(\mathbb{R}^n) \text{ with } \beta \in (\beta_{n,q,\alpha}, 1) & \text{for } p \in \left[\frac{n}{1-\alpha}, +\infty\right), \\ L^\infty(\mathbb{R}^n) \cap W^{\alpha,1}(\mathbb{R}^n) & \text{for } p = +\infty, \end{cases} \tag{1.9}$$

where

$$\beta_{n,q,\alpha} = \frac{1}{q}\left(\alpha + n - \frac{n}{q}\right) \quad \text{and} \quad \gamma_{n,q,\alpha} = \frac{n}{n + (1-\alpha)q},$$

then $gF \in \mathcal{DM}^{\alpha,r}(\mathbb{R}^n)$ *for all* $r \in [1,p]$, *with*

$$div^\alpha(gF) = g^\star\, div^\alpha F + F \cdot \nabla^\alpha g\, \mathscr{L}^n + \mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\, \mathscr{L}^n \quad \text{in } \mathcal{M}(\mathbb{R}^n),$$

where

$$\mathrm{div}^\alpha_{\mathrm{NL}}(g,F) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(g(y)-g(x))(F(y)-F(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

is the non-local fractional divergence *of the couple* (g,F), *and satisfies*

$$\|\mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\|_{L^1} \le \mu_{n,\alpha}\, [g]_{B^\alpha_{q,1}(\mathbb{R}^n)}\, \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}.$$

In addition,

$$div^\alpha(gF)(\mathbb{R}^n) = \int_{\mathbb{R}^n} \mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\,\mathrm{d}x = 0, \tag{1.10}$$

and

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha g\,\mathrm{d}x = -\int_{\mathbb{R}^n} g^\star\,\mathrm{d}div^\alpha F. \tag{1.11}$$

Theorem 1.8, besides providing an extention of Theorem 1.2, provides a Gauss–Green formula for $\mathcal{DM}^{\alpha,\infty}$ vector fields on $W^{\alpha,1}$ sets. For the definitions of the *fractional reduced boundary* $\mathscr{F}^\alpha E$ and of the inner *fractional normal* $\nu^\alpha_E \colon \mathscr{F}^\alpha E \to \mathbb{S}^{n-1}$ of a set $E \subset \mathbb{R}^n$, we refer the reader to [18, Def. 4.7].

Corollary 1.9 (Generalized fractional Gauss–Green formula)**.** *Let* $\alpha \in (0,1)$. *If* $F \in \mathcal{DM}^{\alpha,\infty}(\mathbb{R}^n)$ *and* $\chi_E \in W^{\alpha,1}(\mathbb{R}^n)$, *then*

$$\int_{E^1} \mathrm{d}div^\alpha F = -\int_{\mathscr{F}^\alpha E} F \cdot \nu^\alpha_E\, |\nabla^\alpha \chi_E|\,\mathrm{d}x,$$

where

$$E^1 = \left\{ x \in \mathbb{R}^n : \exists \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = 1 \right\}.$$

Corollary 1.9 immediately follows from (1.11) with $g = \chi_E$, since $\chi_E^\star = \chi_{E^1}$ $\mathscr{H}^{n-\alpha}$-a.e. by [45, Prop. 3.1], and therefore $|div^\alpha F|$-a.e. thanks to point (iii) of Theorem 1.7.

Corollary 1.9 provides the most general version known so far of the fractional Gauss–Green formula proved in [18, Th. 4.2]. Unfortunately, we do not know if the assumption $\chi_E \in W^{\alpha,1}(\mathbb{R}^n)$ can be replaced with the weaker one $\chi_E \in BV^{\alpha,1}(\mathbb{R}^n)$ in Corollary 1.9. In fact, as observed in [17], we do not know whether the precise representative $g^\star$ defined in (1.2) of $g \in BV^{\alpha,\infty}(\mathbb{R}^n)$ is well defined up to $\mathscr{H}^{n-\alpha}$-negligible sets. We plan to tackle this and other strictly-connected challenging open questions in future works.

1.4. **Organization of the paper.** In Section 2, we collect all the needed intermediate results to prove our main theorems. In particular, Section 2.4 and Section 2.5 contain the proofs of points (ii) and (iii) of Theorem 1.7, respectively. The proof of Theorem 1.8, instead, can be found in Section 2.6. Section 3 collects several examples. In Section 3.1 we show point (i) of Theorem 1.7, while in Section 3.2 we discuss the sharpness of the other two points (ii) and (iii) of Theorem 1.7.

2. Proofs of the main results

In this section, we provide the proofs of our main results Theorem 1.7 and Theorem 1.8. The proof of Theorem 1.7 is split across Sections 3.1, 2.4 and 2.5, while the proof of Theorem 1.8 is given in Section 2.6.

2.1. **General notation.** We start with a brief description of the main notation used in this paper. In order to keep the exposition the most reader-friendly as possible, we retain the same notation adopted in our works [4, 17–22].

Lebesgue and Hausdorff measures. We let $\mathscr{L}^n$ and $\mathscr{H}^\alpha$ be the n-dimensional Lebesgue measure and the α-dimensional Hausdorff measure on $\mathbb{R}^n$, respectively, with $\alpha \in [0, n]$. We denote by $B_r(x)$ the standard open Euclidean ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. We let $B_r = B_r(0)$. Recall that $\omega_n = |B_1| = \pi^{\frac{n}{2}} / \Gamma\left(\frac{n+2}{2}\right)$ and $\mathscr{H}^{n-1}(\partial B_1) = n\omega_n$, where Γ is Euler's Gamma function.

Regular maps. Let $\Omega \subset \mathbb{R}^n$ be an open (non-empty) set. For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$, we let $C_c^k(\Omega; \mathbb{R}^m)$ and $\mathrm{Lip}_c(\Omega; \mathbb{R}^m)$ be the spaces of C^k-regular and, respectively, Lipschitz-regular, m-vector-valued functions defined on $\mathbb{R}^n$ with compact support in the open set $\Omega \subset \mathbb{R}^n$. Analogously, we let $C_b^k(\Omega; \mathbb{R}^m)$ and $\mathrm{Lip}_b(\Omega; \mathbb{R}^m)$ be the spaces of C^k-regular and, respectively, Lipschitz-regular, m-vector-valued bounded functions defined on the open set $\Omega \subset \mathbb{R}^n$. In the case $k = 0$, we drop the superscript and simply write $C_c(\Omega; \mathbb{R}^m)$ and $C_b(\Omega; \mathbb{R}^m)$.

Radon measures. For $m \in \mathbb{N}$, the total variation on Ω of the m-vector-valued Radon measure μ is defined as

$$|\mu|(\Omega) = \sup\left\{\int_\Omega \varphi \cdot d\mu : \varphi \in C_c^\infty(\Omega; \mathbb{R}^m),\ \|\varphi\|_{L^\infty(\Omega;\mathbb{R}^m)} \le 1\right\}.$$

We thus let $\mathcal{M}(\Omega; \mathbb{R}^m)$ be the space of m-vector-valued Radon measure with finite total variation on Ω. We say that $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ *weakly converges* to $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, and we write $\mu_k \rightharpoonup \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$ as $k \to +\infty$, if

$$\lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \tag{2.1}$$

for all $\varphi \in C_c(\Omega; \mathbb{R}^m)$. Note that we make a little abuse of terminology, since the limit in (2.1) actually defines the *weak*-convergence* in $\mathcal{M}(\Omega; \mathbb{R}^m)$.

Lebesgue, Sobolev and BV spaces. For any exponent $p \in [1, +\infty]$, we let $L^p(\Omega; \mathbb{R}^m)$ be the space of m-vector-valued Lebesgue p-integrable functions on Ω. We let

$$W^{1,p}(\Omega; \mathbb{R}^m) = \left\{u \in L^p(\Omega; \mathbb{R}^m) : [u]_{W^{1,p}(\Omega;\mathbb{R}^m)} = \|\nabla u\|_{L^p(\Omega;\mathbb{R}^{nm})} < +\infty\right\}$$

be the space of m-vector-valued Sobolev functions on Ω, see [41, Ch. 11], and

$$BV(\Omega; \mathbb{R}^m) = \left\{u \in L^1(\Omega; \mathbb{R}^m) : [u]_{BV(\Omega;\mathbb{R}^m)} = |Du|(\Omega) < +\infty\right\}$$

be the space of m-vector-valued functions of bounded variation on Ω, see [2, Ch. 3].

Fractional Sobolev spaces. For $\alpha \in (0, 1)$ and $p \in [1, +\infty)$, we let

$$W^{\alpha,p}(\Omega; \mathbb{R}^m) = \left\{u \in L^p(\Omega; \mathbb{R}^m) : [u]_{W^{\alpha,p}(\Omega;\mathbb{R}^m)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\alpha}}\, dx\, dy\right)^{\frac{1}{p}} < +\infty\right\}$$

be the space of m-vector-valued fractional Sobolev functions on Ω, see [31]. For $\alpha \in (0, 1)$ and $p = +\infty$, we simply let

$$W^{\alpha,\infty}(\Omega; \mathbb{R}^m) = \left\{u \in L^\infty(\Omega; \mathbb{R}^m) : \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < +\infty\right\},$$

so that $W^{\alpha,\infty}(\Omega; \mathbb{R}^m) = C_b^{0,\alpha}(\Omega; \mathbb{R}^m)$, the space of m-vector-valued bounded α-Hölder continuous functions on Ω.

Besov spaces. For $\alpha \in (0, 1)$ and $p, q \in [1, +\infty]$, we let

$$B^\alpha_{p,q}(\mathbb{R}^n; \mathbb{R}^m) = \left\{u \in L^p(\mathbb{R}^n; \mathbb{R}^m) : [u]_{B^\alpha_{p,q}(\mathbb{R}^n;\mathbb{R}^m)} < +\infty\right\}$$

be the space of m-vector-valued Besov functions on $\mathbb{R}^n$, see [41, Ch. 17], where

$$[u]_{B^\alpha_{p,q}(\mathbb{R}^n;\mathbb{R}^m)} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n} \frac{\|u(\cdot + h) - u\|^q_{L^p(\mathbb{R}^n;\mathbb{R}^m)}}{|h|^{n+q\alpha}}\, dh\right)^{\frac{1}{q}} & \text{if } q \in [1, +\infty), \\[2ex] \displaystyle\sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{\|u(\cdot + h) - u\|_{L^p(\mathbb{R}^n;\mathbb{R}^m)}}{|h|^\alpha} & \text{if } q = \infty. \end{cases}$$

Shorthand for scalar function spaces. In order to avoid heavy notation, if the elements of a function space $\mathcal{F}(\Omega;\mathbb{R}^m)$ are real-valued (i.e., $m=1$), then we will drop the target space and simply write $\mathcal{F}(\Omega)$.

Riesz potential. Given $\alpha\in(0,n)$, we let

$$I_\alpha f(x)=2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy,\quad x\in\mathbb{R}^n, \tag{2.2}$$

be the Riesz potential of order α of $f\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$. We recall that, if $\alpha,\beta\in(0,n)$ satisfy $\alpha+\beta<n$, then we have the following *semigroup property*

$$I_\alpha(I_\beta f)=I_{\alpha+\beta}f \tag{2.3}$$

for all $f\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^m)$. In addition, if $1<p<q<+\infty$ satisfy $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then there exists a constant $C_{n,\alpha,p}>0$ such that the operator in (2.2) satisfies

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^m)}\le C_{n,\alpha,p}\|f\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^m)} \tag{2.4}$$

for all $f\in C_c^\infty(\mathbb{R}^n;\,\mathbb{R}^m)$. As a consequence, the operator in (2.2) extends to a linear continuous operator from $L^p(\mathbb{R}^n;\mathbb{R}^m)$ to $L^q(\mathbb{R}^n;\mathbb{R}^m)$, for which we retain the same notation. For a proof of (2.3) and (2.4), see [55, Ch. V, Sec. 1] or [36, Sec. 1.2.1].

Riesz transform. We let

$$Rf(x)=\pi^{-\frac{n+1}{2}}\,\Gamma\left(\tfrac{n+1}{2}\right)\lim_{\varepsilon\to0^+}\int_{\{|y|>\varepsilon\}}\frac{y\,f(x+y)}{|y|^{n+1}}\,dy,\quad x\in\mathbb{R}^n, \tag{2.5}$$

be the (vector-valued) *Riesz transform* of a (sufficiently regular) function f. We refer the reader to [36, Sec. 2.1 and 2.4.4], [55, Ch. III, Sec. 1] and [56, Ch. III] for a more detailed exposition. We warn the reader that the definition in (2.5) agrees with the one in [56] and differs from the one in [36, 55] for a minus sign. The Riesz transform (2.5) is a singular integral of convolution type, thus in particular it defines a continuous operator $R\colon L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n;\mathbb{R}^n)$ for any given $p\in(1,+\infty)$, see [35, Cor. 5.2.8]. We also recall that its components R_i satisfy

$$\sum_{i=1}^n R_i^2=-\mathrm{Id}\quad\text{on } L^2(\mathbb{R}^n),$$

see [35, Prop. 5.1.16].

2.2. **Approximation by smooth vector fields.** Here and in the rest of the paper, we let $(\varrho_\varepsilon)\subset C_c^\infty(\mathbb{R}^n)$ be a family of standard mollifiers as in [18, Sec. 3.3]. The following approximation result is the natural generalization to $\mathcal{DM}^{\alpha,p}$ vector fields of [17, Th. 4]. We leave its proof to the reader.

Theorem 2.1 (Approximation by $C^\infty\cap\mathcal{DM}^{\alpha,p}$ fields)**.** *Let $\alpha\in(0,1]$ and $p\in[1,+\infty]$. Let $F\in\mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ and define $F_\varepsilon=F*\varrho_\varepsilon$ for all $\varepsilon>0$. Then $(F_\varepsilon)_{\varepsilon>0}\subset\mathcal{DM}^{\alpha,p}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $div^\alpha F_\varepsilon=(\varrho_\varepsilon*div^\alpha F)\mathscr{L}^n$ for all $\varepsilon>0$. Moreover, we have:*

(i) if $p<+\infty$, then $F_\varepsilon\to F$ in $L^p(\mathbb{R}^n;\mathbb{R}^n)$ as $\varepsilon\to0^+$; if $p=+\infty$, then $F_\varepsilon\to F$ in $L^q_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ as $\varepsilon\to0^+$ for all $q\in[1,+\infty)$;

(ii) $div^\alpha F_\varepsilon\rightharpoonup div^\alpha F$ in $\mathcal{M}(\mathbb{R}^n)$ and $|div^\alpha F_\varepsilon|(\mathbb{R}^n)\to|div^\alpha F|(\mathbb{R}^n)$ as $\varepsilon\to0^+$.

2.3. **Integration-by-parts with Sobolev tests.** For future convenience, we note that the integration-by-parts formula (1.7) actually holds for a wider class of test functions. To this aim, let us recall the notion of *non-local fractional gradient*

$$\nabla^\alpha_{\mathrm{NL}}(f,g)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(f(y)-f(x))(g(y)-g(x))(y-x)}{|y-x|^{n+\alpha+1}}\, dy, \quad x \in \mathbb{R}^n,$$

of a couple of functions $f, g \in \mathrm{Lip}_c(\mathbb{R}^n)$. The operator $\nabla^\alpha_{\mathrm{NL}}$ can be continuously extended to Lebesgue and Besov spaces, see [20, Cor. 2.7] for the precise statement.

Proposition 2.2 ($W^{1,q} \cap C_b$-regular test)**.** *Let $\alpha \in (0,1)$ and let $p, q \in [1,+\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \,\mathrm{d}x = -\int_{\mathbb{R}^n} \xi \,\mathrm{d}div^\alpha F \tag{2.6}$$

for all $\xi \in W^{1,q}(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$, and for all $\xi \in BV(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$ if $q = 1$.

Proof. The proof is analogous to the one of [17, Prop. 3], so we only sketch it for the reader's convenience. By a routine regularization-by-convolution argument, it is not restrictive to assume that $\xi \in W^{1,q}(\mathbb{R}^n) \cap \mathrm{Lip}_b(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Letting $(\eta_R)_{R>0} \subset C^\infty_c(\mathbb{R}^n)$ be a family of cut-off functions as in [18, Sec. 3.3], by [19, Lems. 2.3 and 2.4] we can write

$$\int_{\mathbb{R}^n} \eta_R\, F \cdot \nabla^\alpha \xi \,\mathrm{d}x = \int_{\mathbb{R}^n} F \cdot \nabla^\alpha(\eta_R \xi)\,\mathrm{d}x - \int_{\mathbb{R}^n} \xi\, F \cdot \nabla^\alpha \eta_R \,\mathrm{d}x - \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(\eta_R, \xi)\,\mathrm{d}x \tag{2.7}$$

for all $R > 0$. Moreover, since $\xi\eta_R \in C^\infty_c(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha(\eta_R \xi)\,\mathrm{d}x = -\int_{\mathbb{R}^n} \eta_R \xi \,\mathrm{d}div^\alpha F$$

for all $R > 0$. Since

$$\lim_{R\to+\infty} \int_{\mathbb{R}^n} \xi\, F \cdot \nabla^\alpha \eta_R \,\mathrm{d}x = \lim_{R\to+\infty} \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(\eta_R, \xi)\,\mathrm{d}x = 0,$$

the conclusion follows by passing to the limit as $R \to +\infty$ in (2.7). □

2.4. **Relation between $\mathcal{DM}^{\alpha,p}$ and $\mathcal{DM}^{1,p}$.** We now deal with point (ii) of Theorem 1.7. To this aim, we study the relationship between $\mathcal{DM}^{1,p}$ and $\mathcal{DM}^{\alpha,p}$ vector fields.

As one may expect, $\mathcal{DM}^{1,p}$ vector fields can be regarded as $\mathcal{DM}^{\alpha,p}$ vector fields, but only locally with respect to the divergence-measure. For $\alpha \in (0,1)$ and $p \in [1,+\infty]$, we write $F \in \mathcal{DM}^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n)$ if $F \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ and, for any $U \subset \mathbb{R}^n$ bounded open set,

$$\sup\left\{\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \,\mathrm{d}x : \xi \in C^\infty_c(\mathbb{R}^n),\ \|\xi\|_{L^\infty(\mathbb{R}^n)} \le 1,\ \mathrm{supp}\,\xi \subset U\right\} < +\infty.$$

Consequently, the Radon measure $div^\alpha F \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n)$ given by (1.7) may be such that $|div^\alpha F|(\mathbb{R}^n) = +\infty$. This issue is quite normal, and essentially due to the properties of Riesz potential, in view of the representation $\nabla^\alpha = \nabla I_{1-\alpha}$, see [18, Sec. 2.3].

Lemma 2.3 (Inclusion)**.** *If $\alpha \in (0,1)$ and $p \in [1,+\infty]$, then $\mathcal{DM}^{1,p}(\mathbb{R}^n) \subset \mathcal{DM}^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Let $F \in \mathcal{DM}^{1,p}(\mathbb{R}^n)$. Given $\xi \in C^\infty_c(\mathbb{R}^n)$, since $I_{1-\alpha}\xi \in C^\infty_b(\mathbb{R}^n)$ with $\nabla^\alpha \xi = \nabla I_{1-\alpha}\xi \in L^{p'}(\mathbb{R}^n)$, we can write

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \,\mathrm{d}x = \int_{\mathbb{R}^n} F \cdot \nabla I_{1-\alpha}\xi \,\mathrm{d}x = -\int_{\mathbb{R}^n} I_{1-\alpha}\xi \,\mathrm{d}divF.$$

Hence, for any bounded open set $U \supset \operatorname{supp}\xi$, by [18, Lem. 2.4] we can find a constant $C_{n,\alpha,U} > 0$, depending only on n, α and $\operatorname{diam}(U)$, such that

$$\left| \int_{\mathbb{R}^n} F \cdot \nabla^\alpha \xi \, \mathrm{d}x \right| \le C_{n,\alpha,U} |div F|(\mathbb{R}^n) \|\xi\|_{L^\infty(\mathbb{R}^n)}.$$

This implies that $F \in \mathcal{DM}^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n)$, as desired. □

The inclusion given by Lemma 2.3 can be somewhat reversed, as done in Lemma 2.4 below. Note that this result, besides providing analogues of [18, Lem. 3.28], [19, Lem. 3.7] and [17, Prop. 4], proves point (ii) of Theorem 1.7

Lemma 2.4 (Relation between $\mathcal{DM}^{\alpha,p}$ and $\mathcal{DM}^{1,p}$)**.** *Let $\alpha \in (0,1)$, $p \in \left(1, \frac{n}{1-\alpha}\right)$ and $q = \frac{np}{n-(1-\alpha)p}$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, then $G = I_{1-\alpha}F \in \mathcal{DM}^{1,q}(\mathbb{R}^n)$, with*

$$\|G\|_{L^q(\mathbb{R}^n;\,\mathbb{R}^n)} \le c_{n,\alpha,p} \|F\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \quad \text{and} \quad div\, G = div^\alpha F \text{ in } \mathcal{M}(\mathbb{R}^n).$$

As a consequence, the operator $I_{1-\alpha}\colon \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \to \mathcal{DM}^{1,q}(\mathbb{R}^n)$ is continuous. Moreover, for $p \in \left[\frac{n}{n-\alpha}, \frac{n}{1-\alpha}\right)$, if $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ then $|div^\alpha F|(B) = 0$ on Borel sets $B \subset \mathbb{R}^n$ of σ-finite $\mathscr{H}^{n-\frac{q}{q-1}}$ measure.

Proof. Let $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$ and note that $r = \frac{np'}{n+(1-\alpha)p'} \in \left(1, \frac{n}{1-\alpha}\right)$. By the Hardy–Littlewood–Sobolev inequality, we immediately get that $G = I_{1-\alpha}F \in L^q(\mathbb{R}^n;\mathbb{R}^n)$. Moreover, given $\xi \in C^\infty_c(\mathbb{R}^n)$, we clearly have $I_{1-\alpha}|\nabla\xi| \in L^{q'}(\mathbb{R}^n)$, because $|\nabla\xi| \in L^r(\mathbb{R}^n)$. Hence, by Fubini Theorem, we can write

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} F \cdot I_{1-\alpha}\nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^n} G \cdot \nabla\varphi \, \mathrm{d}x \tag{2.8}$$

for all $\xi \in C^\infty_c(\mathbb{R}^n)$, proving that $div^\alpha F = div\, G$ in $\mathcal{M}(\mathbb{R}^n)$. The remaining part of the statement easily follows from Theorem 1.1 (also see [49, Th. 3.2]). □

2.5. **Decay estimates.** We now deal with point (iii) of Theorem 1.7. To this aim, we prove some decay estimates of the fractional divergence-measure on balls.

Let us begin with the following result, which may be considered as a toy case for the more general result in Theorem 2.8 below.

Lemma 2.5 (Decay estimate for $div^\alpha F \ge 0$)**.** *Let $\alpha \in (0,1]$ and $p \in [1,+\infty]$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ satisfies $div^\alpha F \ge 0$ on some open set $A \subset \mathbb{R}^n$, then*

$$div^\alpha F(B_r(x)) \le C_{n,\alpha,p} \|F\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \, r^{n-\alpha-\frac{n}{p}}. \tag{2.9}$$

for all $x \in A$ and $r > 0$ such that $B_{2r}(x) \subset A$.

Proof. Let $\xi \in C^\infty_c(B_2)$ be such that $\xi \ge 0$ and $\xi \equiv 1$ on B_1. Then, for $x \in A$ and $r > 0$ such that $B_{2r}(x) \subset A$, we can estimate

$$div^\alpha F(B_r(x)) \le \int_{\mathbb{R}^n} \xi\left(\frac{y-x}{r}\right) \mathrm{d}div^\alpha F(y) = -\int_{\mathbb{R}^n} F(y) \cdot (\nabla^\alpha \xi)\left(\frac{y-x}{r}\right) r^{-\alpha} \, \mathrm{d}y.$$

Thus we easily get

$$\begin{aligned} div^\alpha F(B_r(x)) &\le \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \, r^{-\alpha} \left(\int_{\mathbb{R}^n} |\nabla^\alpha \xi(y)|^{p'} r^n \, \mathrm{d}y \right)^{\frac{1}{p'}} \\ &= \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \, \|\nabla^\alpha \xi\|_{L^{p'}(\mathbb{R}^n;\,\mathbb{R}^n)} \, r^{n-\alpha-\frac{n}{p}}, \end{aligned}$$

from which the conclusion immediately follows. □

Lemma 2.5, despite its simplicity, allows to recover the following rigidity result, which may be seen as the natural fractional analogue of [44, Th. 3.1].

Proposition 2.6 (Rigidity)**.** *Let $\alpha \in (0,1]$ and $p \in \left[1, \frac{n}{n-\alpha}\right]$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ satisfies $div^\alpha F \ge 0$, then $div^\alpha F = 0$.*

Proof. If $p < \frac{n}{n-\alpha}$, so that $n - \alpha - \frac{n}{p} < 0$, then

$$0 \le div^\alpha F(B_r) \le C_{n,\alpha,p} \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} r^{n-\alpha-\frac{n}{p}}$$

for all $r > 0$ by Lemma 2.5 in the case $x = 0$. Hence the conclusion follows by taking the limit as $r \to +\infty$. If instead $p = \frac{n}{n-\alpha}$, then $I_\alpha div^\alpha F = \operatorname{div}^0 F$ in $L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} I_\alpha \xi \, \mathrm{d}div^\alpha F = -\int_{\mathbb{R}^n} F \cdot \nabla^\alpha I_\alpha \xi \, \mathrm{d}x = -\int_{\mathbb{R}^n} F \cdot \nabla^0 \xi \, \mathrm{d}x = \int_{\mathbb{R}^n} \xi \operatorname{div}^0 F \, \mathrm{d}x$$

for all $\xi \in C_c^\infty(\mathbb{R}^n)$ by Proposition 2.2, Remark 1.6 and [4, Prop. 7 and Lem. 26]. However, for all $R > 0$ and $x \in \mathbb{R}^n$ we also have

$$I_\alpha div^\alpha F(x) \ge c_{n,\alpha} \int_{B_R} \frac{1}{|x-y|^{n-\alpha}} \, \mathrm{d}div^\alpha F(y) \ge \tilde{c}_{n,\alpha} \frac{div^\alpha F(B_R)}{(|x|+R)^{n-\alpha}},$$

and thus $I_\alpha div^\alpha F \notin L^{\frac{n}{n-\alpha}}$ unless $div^\alpha F = 0$. The proof is complete. □

To remove the non-negativity assumption $div^\alpha F \ge 0$ from the conclusion (2.9) in Lemma 2.5 we need to deal with integration-by-parts for $\mathcal{DM}^{\alpha,p}$ fields on balls. The following result is the analogue of [17, Th. 9].

Theorem 2.7 (Integration by parts on balls)**.** *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, $\xi \in \operatorname{Lip}_c(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, then*

$$\int_{B_r(x)} F \cdot \nabla^\alpha \xi \, \mathrm{d}y + \int_{\mathbb{R}^n} \xi F \cdot \nabla^\alpha \chi_{B_r(x)} \, \mathrm{d}y + \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \xi) \, \mathrm{d}y = -\int_{B_r(x)} \xi \, \mathrm{d}div^\alpha F \tag{2.10}$$

for $\mathscr{L}^1$-a.e. $r > 0$.

Proof. The proof is very similar to that of [17, Th. 9], so we only sketch it for the reader's convenience. Fix $x \in \mathbb{R}^n$ and $\xi \in \operatorname{Lip}_c(\mathbb{R}^n)$ be fixed.

In the case $p = +\infty$, we consider $h_{\varepsilon,r,x} \in \operatorname{Lip}_c(\mathbb{R}^n)$ for $\varepsilon > 0$ and $r > 0$ defined as

$$h_{\varepsilon,r,x}(y) = \begin{cases} 1 & \text{if } 0 \le |y-x| \le r, \\[1ex] \dfrac{r+\varepsilon-|y-x|}{\varepsilon} & \text{if } r < |y-x| < r+\varepsilon, \\[1ex] 0 & \text{if } |y-x| \ge r+\varepsilon, \end{cases}$$

for all $y \in \mathbb{R}^n$. By [18, Lem. 5.1], $\nabla^\alpha h_{\varepsilon,r,x} \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\nabla^\alpha h_{\varepsilon,r,x}(y) = \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)} \int_{B_{r+\varepsilon}(x)\setminus B_r(x)} \frac{x-z}{|x-z|} |z-y|^{1-n-\alpha} \, \mathrm{d}z \tag{2.11}$$

for $\mathscr{L}^n$-a.e. $y \in \mathbb{R}^n$.

Since $h_{\varepsilon,r,x}(y) \to \chi_{\overline{B_r(x)}}(y)$ as $\varepsilon \to 0^+$ for all $y \in \mathbb{R}^n$ and $|div^\alpha F|(\partial B_r(x)) = 0$ for $\mathscr{L}^1$-a.e. $r > 0$, we can use $h_{\varepsilon,r,x}$ to approximate $\chi_{B_r(x)}$ in (2.10). On the one hand, since $h_{\varepsilon,r,x}\,\varphi \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$, by Proposition 2.2 we have

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha(h_{\varepsilon,r,x}\,\varphi)\,\mathrm{d}y = -\int_{\mathbb{R}^n} h_{\varepsilon,r,x}\,\varphi\,\mathrm{d}div^\alpha F. \tag{2.12}$$

On the other hand, by [18, Lem. 2.6], we can compute

$$\nabla^\alpha(h_{\varepsilon,r,x}\,\varphi) = h_{\varepsilon,r,x}\,\nabla^\alpha\varphi + \varphi\,\nabla^\alpha h_{\varepsilon,r,x} + \nabla^\alpha_{\mathrm{NL}}(h_{\varepsilon,r,x},\varphi). \tag{2.13}$$

One then has to deal with each term of the right-hand side of (2.13) separately. The most difficult term is the second one, for which one has to observe that, by (2.11),

$$\begin{aligned}
&\int_{\mathbb{R}^n} \xi(y)\,F(y)\cdot\nabla^\alpha h_{\varepsilon,r,x}(y)\,\mathrm{d}y\\
&\qquad= \frac{\mu_{n,\alpha}}{\varepsilon(n+\alpha-1)}\int_{\mathbb{R}^n}\xi(y)\,F(y)\cdot\int_{B_{r+\varepsilon}(x)\setminus B_r(x)}\frac{x-z}{|x-z|}|z-y|^{1-n-\alpha}\,\mathrm{d}z\,\mathrm{d}y\\
&\qquad= \int_{B_{r+\varepsilon}(x)\setminus B_r(x)}\frac{x-z}{|x-z|}\cdot\int_{\mathbb{R}^n}F(y)\,\xi(y)\,|z-y|^{1-n-\alpha}\,\mathrm{d}y\,\mathrm{d}z\\
&\qquad= \int_r^{r+\varepsilon}\int_{\partial B_\varrho(x)}\frac{x-z}{|x-z|}\cdot\int_{\mathbb{R}^n}F(y)\,\xi(y)\,|z-y|^{1-n-\alpha}\,\mathrm{d}y\,\mathrm{d}\mathscr{H}^{n-1}(z)\,\mathrm{d}\varrho.
\end{aligned}$$

Hence, by Lebesgue's Differentiation Theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_{\mathbb{R}^n}\xi(y)\,F(y)\cdot\int_{B_{r+\varepsilon}(x)\setminus B_r(x)}\frac{x-z}{|x-z|}\,|z-y|^{1-n-\alpha}\,\mathrm{d}z\,\mathrm{d}y\\
&\qquad= \int_{\partial B_r(x)}\frac{x-z}{|x-z|}\cdot\int_{\mathbb{R}^n}F(y)\,\xi(y)\,|z-y|^{1-n-\alpha}\,\mathrm{d}y\,\mathrm{d}\mathscr{H}^{n-1}(z)\\
&\qquad= \int_{\mathbb{R}^n}\xi(y)\,F(y)\cdot\int_{\mathbb{R}^n}|z-y|^{1-n-\alpha}\,\mathrm{d}D\chi_{B_r(x)}(z)\,\mathrm{d}y
\end{aligned}$$

for $\mathscr{L}^1$-a.e. $r > 0$. Thus, by [18, Th. 3.18, Eq. (3.26)], we get that

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\xi\,F\cdot\nabla^\alpha h_{\varepsilon,r,x}\,\mathrm{d}y\\
&\qquad= \frac{\mu_{n,\alpha}}{n+\alpha-1}\int_{\mathbb{R}^n}\xi(y)\,F(y)\cdot\int_{\mathbb{R}^n}|z-y|^{1-n-\alpha}\,\mathrm{d}D\chi_{B_r(x)}(z)\,\mathrm{d}y\\
&\qquad= \int_{\mathbb{R}^n}\xi\,F\cdot\nabla^\alpha\chi_{B_r(x)}\,\mathrm{d}y
\end{aligned}\tag{2.14}$$

for $\mathscr{L}^1$-a.e. $r > 0$. The other terms are easier and hence left to the reader.

In the case $p \in \left(\frac{1}{1-\alpha}, +\infty\right)$, instead, one regularizes $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ to $(F_\varepsilon)_{\varepsilon>0} \subset \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n;\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ via convolution to reduce to the previous case $p = +\infty$. The conclusion then follows by exploiting the convergence properties given by Theorem 2.1 and recalling that, thanks to [17, Cor. 1], $\nabla^\alpha\chi_{B_r(x)} \in L^q(\mathbb{R}^n;\mathbb{R}^n)$ for any $p \in \left(\frac{1}{1-\alpha}, \infty\right)$, where $q = \frac{p}{p-1}$, and that $\nabla^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \xi) \in L^q(\mathbb{R}^n;\mathbb{R}^n)$ as well, thanks to [20, Cor. 2.7]. We leave the details to the reader. □

We are now ready to generalize Lemma 2.5 beyond the non-negativity assumption, as done in [17, Th. 10] for $BV^{\alpha,p}$ functions.

Theorem 2.8 (Decay estimates for $\mathcal{DM}^{\alpha,p}$ functions for $p > \frac{1}{1-\alpha}$)**.** *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. There exist two constants $A_{n,\alpha,p}, B_{n,\alpha,p} > 0$, depending on n, α and p only, with the following property. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ then, for $|div^\alpha F|$-a.e. $x \in \mathbb{R}^n$, there exists $r_x > 0$ such that*

$$|div^\alpha F|(B_r(x)) \le A_{n,\alpha,p} \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\, r^{\frac{n}{q}-\alpha} \tag{2.15}$$

and

$$|div^\alpha (\chi_{B_r(x)} F)|(\mathbb{R}^n) \le B_{n,\alpha,p} \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\, r^{\frac{n}{q}-\alpha} \tag{2.16}$$

for all $r \in (0, r_x)$, where $q \in [1,+\infty)$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof follows the same line of that of [17, Th. 10], so we only sketch it for the reader's ease. Since $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, by the Polar Decomposition Theorem for Radon measures there exists a Borel function $\sigma_F^\alpha \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$div^\alpha F = \sigma_F^\alpha\, |div^\alpha F| \quad \text{with} \quad |\sigma_F^\alpha(x)| = 1 \text{ for } |div^\alpha F|\text{-a.e. } x \in \mathbb{R}^n. \tag{2.17}$$

For $x \in \mathbb{R}^n$ such that $|\sigma_F^\alpha(x)| = 1$, given $r > 0$ we define $\xi_{x,r} \colon \mathbb{R}^n \to \mathbb{R}$ as

$$\xi_{x,r}(y) = \begin{cases} \sigma_F^\alpha(x) & \text{if } y \in B_r(x), \\ \sigma_F^\alpha(x)\left(2 - \frac{|y-x|}{r}\right) & \text{if } y \in B_{2r}(x) \setminus B_r(x), \\ 0 & \text{if } y \notin B_{2r}(x), \end{cases} \tag{2.18}$$

for all $y \in \mathbb{R}^n$. Since $\xi_{x,r} \in \mathrm{Lip}_c(\mathbb{R}^n)$ with $\|\varphi\|_{L^\infty(\mathbb{R}^n)} \le 1$, we can find $r_x \in (0,1)$ such that

$$\int_{B_r(x)} \xi_{x,r}(y)\, \mathrm{d}div^\alpha F(y) = \int_{B_r(x)} \sigma_F^\alpha(x)\, \sigma_F^\alpha(y)\, \mathrm{d}|div^\alpha F|(y) \ge \frac{1}{2}|div^\alpha F|(B_r(x)) \tag{2.19}$$

for all $r \in (0, r_x)$. Also, by (2.10), we can estimate

$$\begin{aligned} \int_{B_r(x)} \xi_{x,r}\, \mathrm{d}div^\alpha F \le & \left| \int_{B_r(x)} F \cdot \nabla^\alpha \xi_{x,r}\, \mathrm{d}y \right| + \left| \int_{\mathbb{R}^n} \xi_{x,r}\, F \cdot \nabla^\alpha \chi_{B_r(x)}\, \mathrm{d}x \right| \\ & + \left| \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \xi_{x,r})\, \mathrm{d}y \right| \end{aligned} \tag{2.20}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$. Hence the inequality in (2.15) follows by estimating the three terms in the right-hand side of (2.20), recalling the scaling property of ∇^α, [17, Cor. 1] and [20, Cor. 2.7]. For the inequality in (2.16), instead, one notes that, given any $\xi \in \mathrm{Lip}_c(\mathbb{R}^n)$ with $\|\xi\|_{L^\infty(\mathbb{R}^n)} \le 1$, from (2.10) it holds

$$\begin{aligned} \left| \int_{B_r(x)} F \cdot \nabla^\alpha \xi\, \mathrm{d}y \right| \le & |div^\alpha F|(B_r(x)) + \|F\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)}\, \|\nabla^\alpha \chi_{B_r(x)}\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} \\ & + \|F\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \|\nabla^\alpha_{\mathrm{NL}}(\chi_{B_r(x)}, \xi)\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} \end{aligned}$$

for $\mathscr{L}^1$-a.e. $r \in (0, r_x)$. The conclusion thus follows from (2.15) and again [17, Cor. 1] and [20, Cor. 2.7]. We leave the details to the reader. ☐

As a consequence of Theorem 2.8, we get the following result, in particular proving the validity of point (iii) in Theorem 1.7. Note that Corollary 2.9 below is actually relevant only in the case of point (iii) of Theorem 1.7, since $n - \frac{p}{p-1+(1-\alpha)\frac{p}{n}} \le n - \alpha - \frac{n}{p}$ if and only if $p \ge \frac{n}{1-\alpha}$ and $p \le \frac{n}{n-\alpha}$, but in this second case both exponents are negative.

Corollary 2.9 ($|div^\alpha F| \ll \mathscr{H}^{n-\alpha-\frac{n}{p}}$ for $p > \frac{1}{1-\alpha}$)**.** *Let $\alpha \in (0,1)$ and $p \in \left(\frac{1}{1-\alpha}, +\infty\right]$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, then there exists a $|div^\alpha F|$-negligible set $Z_F^{\alpha,p} \subset \mathbb{R}^n$ such that*

$$|div^\alpha F| \le 2^{\frac{n}{q}-\alpha} \frac{A_{n,\alpha,p}}{\omega_{\frac{n}{q}-\alpha}} \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)} \mathscr{H}^{\frac{n}{q}-\alpha} \llcorner \mathbb{R}^n \setminus Z_F^{\alpha,p}, \tag{2.21}$$

where $A_{n,\alpha,p}$ is as in (2.15) *and $q \in [1,+\infty)$ is such that $\frac{1}{p}+\frac{1}{q}=1$.*

Proof. By Theorem 2.8, there exists a set $Z_F^{\alpha,p} \subset \mathbb{R}^n$, which we can assume to be Borel without loss of generality, such that $|div^\alpha F|(Z_F^{\alpha,p}) = 0$ and (2.15) holds for any $x \notin Z_F^{\alpha,p}$. Hence, for all $x \in \mathbb{R}^n \setminus Z_F^{\alpha,p}$, we have

$$\Theta^*_{\frac{n}{q}-\alpha}(|div^\alpha F|, x) = \limsup_{r\to 0^+} \frac{|div^\alpha F|(B_r(x))}{\omega_{\frac{n}{q}-\alpha} r^{\frac{n}{q}-\alpha}} \le \frac{A_{n,\alpha,p}}{\omega_{\frac{n}{q}-\alpha}} \|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}.$$

Inequality (2.21) thus follows from [2, Th. 2.56]. □

Remark 2.10. Corollary 2.9 holds true also in the limit case as $\alpha \to 1^-$. Indeed, if $F \in \mathcal{DM}^{1,\infty}(\mathbb{R}^n)$, then [52, Prop. 1] implies that

$$|divF| \le c_n \|F\|_{L^\infty(\mathbb{R}^n;\,\mathbb{R}^n)} \mathscr{H}^{n-1} \llcorner (\mathbb{R}^n \setminus Z_F^{1,\infty}),$$

for some constant $c_n > 0$ and any $|divF|$-negligible set $Z_F^{1,\infty} \subset \mathbb{R}^n$.

2.6. **Proof of Theorem 1.8.** We begin with the following technical result.

Lemma 2.11 (Zero total divergence-measure)**.** *Let $\alpha \in (0,1]$ and $p \in \left[1, \frac{n}{n-\alpha}\right)$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, then $div^\alpha F(\mathbb{R}^n) = 0$.*

Proof. Let $\eta \in C_c^\infty(B_2)$ be such that $\eta \equiv 1$ on B_1 and set $\eta_k(x) = \eta\left(\frac{x}{k}\right)$ for $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. By (1.7) and the α-homogeneity of the fractional gradient, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \eta_k \,\mathrm{d}div^\alpha F\right| &= \left|\int_{\mathbb{R}^n} F \cdot \nabla^\alpha \eta_k \,\mathrm{d}x\right| \\ &\le k^{\frac{n}{q}-\alpha} \|F\|_{L^p(\mathbb{R}^n;\,\mathbb{R}^n)} \|\nabla^\alpha \eta\|_{L^q(\mathbb{R}^n;\mathbb{R}^n)} \to 0 \quad \text{as } k \to +\infty\end{aligned}$$

for $q > \frac{n}{\alpha}$, which means $p < \frac{n}{n-\alpha}$. Hence, by the Dominated Convergence Theorem with respect to the measure $|div^\alpha F|$, we get that

$$div^\alpha F(\mathbb{R}^n) = \int_{\mathbb{R}^n} \mathrm{d}div^\alpha F = \lim_{k\to+\infty} \int_{\mathbb{R}^n} \eta_k \,\mathrm{d}div^\alpha F = 0$$

concluding the proof. □

We can now deal with the Leibniz rule for $\mathcal{DM}^{\alpha,p}$ vector fields and bounded continuous Besov functions, in analogy with [20, Th. 3.1]. To this purpose, we need to recall the notion of *non-local fractional divergence*

$$\mathrm{div}^\alpha_{\mathrm{NL}}(g,F)(x) = \mu_{n,\alpha} \int_{\mathbb{R}^n} \frac{(g(y)-g(x))(F(y)-F(x))\cdot(y-x)}{|y-x|^{n+\alpha+1}}\,dy, \quad x \in \mathbb{R}^n,$$

of a couple (g,F), where $g \in \mathrm{Lip}_c(\mathbb{R}^n)$ and $F \in \mathrm{Lip}_c(\mathbb{R}^n;\mathbb{R}^n)$. The operator $\mathrm{div}^\alpha_{\mathrm{NL}}$ can be continuously extended to Lebesgue and Besov spaces, see [20, Cor. 2.7].

Theorem 2.12 (Leibniz rule for $\mathcal{DM}^{\alpha,p}$ with $C_b \cap B^\alpha_{q,1}$ for $\frac{1}{p}+\frac{1}{q}=1$)**.** *Let $\alpha \in (0,1)$ and let $p,q \in [1,+\infty]$ be such that $\frac{1}{p}+\frac{1}{q}=1$. If $F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ and $g \in C_b(\mathbb{R}^n) \cap B^\alpha_{q,1}(\mathbb{R}^n)$, then $gF \in \mathcal{DM}^{\alpha,r}(\mathbb{R}^n)$ for all $r \in [1,p]$, with $\operatorname{div}^\alpha_{\mathrm{NL}}(g,F) \in L^1(\mathbb{R}^n)$ and*

$$div^\alpha(gF) = g\, div^\alpha F + F \cdot \nabla^\alpha g\, \mathscr{L}^n + \operatorname{div}^\alpha_{\mathrm{NL}}(g,F)\, \mathscr{L}^n \quad in\ \mathcal{M}(\mathbb{R}^n). \tag{2.22}$$

In addition,

$$div^\alpha(gF)(\mathbb{R}^n) = 0, \qquad \int_{\mathbb{R}^n} \operatorname{div}^\alpha_{\mathrm{NL}}(g,F)\,\mathrm{d}x = 0, \tag{2.23}$$

and

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha g\,\mathrm{d}x = -\int_{\mathbb{R}^n} g\,\mathrm{d}div^\alpha F. \tag{2.24}$$

Proof. We mimic the proof of [20, Th. 3.1]. Since $g \in L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have $gF \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ by Hölder's inequality. In addition, [20, Cor. 2.7] implies that $\operatorname{div}^\alpha_{\mathrm{NL}}(g,F) \in L^1(\mathbb{R}^n)$. We now divide the proof in two steps.

Step 1: proof of (2.22). Let $\xi \in \mathrm{Lip}_c(\mathbb{R}^n)$ be given. By [20, Lem. 3.2(i)], we have

$$\nabla^\alpha(g\xi) = g\,\nabla^\alpha\xi + \xi\nabla^\alpha g + \nabla^\alpha_{\mathrm{NL}}(g,\xi) \quad \text{in } L^q(\mathbb{R}^n),$$

so that

$$\int_{\mathbb{R}^n} gF \cdot \nabla^\alpha\xi\,\mathrm{d}x = \int_{\mathbb{R}^n} F \cdot \nabla^\alpha(g\xi)\,\mathrm{d}x - \int_{\mathbb{R}^n} \xi F \cdot \nabla^\alpha g\,\mathrm{d}x - \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(g,\xi)\,\mathrm{d}x.$$

By [20, Lem. 2.9], we have that

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(g,\xi)\,dx = \int_{\mathbb{R}^n} \xi\,\operatorname{div}^\alpha_{\mathrm{NL}}(g,F)\,dx.$$

Now let $(F_\varepsilon)_{\varepsilon>0} \subset \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be given by $F_\varepsilon = \varrho_\varepsilon * F$ for all $\varepsilon > 0$. In particular, we have $F_\varepsilon \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$ for each $\varepsilon > 0$. Note that $W^{1,p}(\mathbb{R}^n;\mathbb{R}^n) \subset B^\alpha_{p,q}(\mathbb{R}^n;\mathbb{R}^n)$ for all $\alpha \in (0,1)$ and $p,q \in [1,+\infty]$, see [41, Th. 17.33]. As a consequence, $F_\varepsilon \in B^\alpha_{p,1}(\mathbb{R}^n;\mathbb{R}^n)$ for each $\varepsilon > 0$. Since $g\xi \in B^\alpha_{q,1}(\mathbb{R}^n)$, by [20, Lem. 2.6] we can write

$$\int_{\mathbb{R}^n} F_\varepsilon \cdot \nabla^\alpha(g\xi)\,\mathrm{d}x = -\int_{\mathbb{R}^n} g\xi\,\operatorname{div}^\alpha F_\varepsilon\,\mathrm{d}x$$

for all $\varepsilon > 0$. On the one side, we have

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} F_\varepsilon \cdot \nabla^\alpha(g\xi)\,\mathrm{d}x = \int_{\mathbb{R}^n} F \cdot \nabla^\alpha(g\xi)\,\mathrm{d}x$$

by Hölder's inequality in the case $p < +\infty$ and by the Dominated Convergence Theorem in the case $p = +\infty$. On the other side, since $g\xi \in C_c(\mathbb{R}^n)$, we also have

$$\lim_{\varepsilon\to0^+} \int_{\mathbb{R}^n} g\xi\,\operatorname{div}^\alpha F_\varepsilon\,\mathrm{d}x = \int_{\mathbb{R}^n} g\xi\,\mathrm{d}div^\alpha F,$$

thanks to Theorem 2.1. We thus conclude that

$$\int_{\mathbb{R}^n} F \cdot \nabla^\alpha(g\xi)\,\mathrm{d}x = -\int_{\mathbb{R}^n} g\xi\,\mathrm{d}div^\alpha F,$$

so that, for all $\xi \in \mathrm{Lip}_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} gF \cdot \nabla^\alpha\xi\,\mathrm{d}x = -\int_{\mathbb{R}^n} g\xi\,\mathrm{d}div^\alpha F - \int_{\mathbb{R}^n} \xi\,F \cdot \nabla^\alpha g\,\mathrm{d}x - \int_{\mathbb{R}^n} \xi\,\operatorname{div}^\alpha_{\mathrm{NL}}(g,F)\,\mathrm{d}x.$$

By a standard approximation argument for the test function, we get (2.22).

Step 2: proof of (2.23) *and* (2.24). Since $gF \in \mathcal{DM}^{\alpha,1}(\mathbb{R}^n)$ by Step 1, the first equation in (2.23) readily follows from Lemma 2.11. Moreover, since obviously $\nabla^\alpha_{\mathrm{NL}}(g,v) = 0$ for all $v \in \mathbb{R}$, by [20, Lem. 2.9] we get

$$v \int_{\mathbb{R}^n} \mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\,\mathrm{d}x = \int_{\mathbb{R}^n} v\,\mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\,\mathrm{d}x = \int_{\mathbb{R}^n} F \cdot \nabla^\alpha_{\mathrm{NL}}(g,v)\,\mathrm{d}x = 0$$

for all $v \in \mathbb{R}$ and also the second equation in (2.23) immediately follows. By combining (2.22) with (2.23), we get (2.24) and the proof is complete. □

We are now in position to prove our second main result Theorem 1.8.

Proof of Theorem 1.8. The proofs of the cases $p \in \left[1, \frac{n}{n-\alpha}\right)$, $p \in \left[\frac{n}{1-\alpha}, +\infty\right)$ and $p = +\infty$ are analogous to those of [20, Cors. 3.3, 3.6 and 3.7], respectively, and are hence omitted. We thus deal with the case $p \in \left[\frac{n}{n-\alpha}, \frac{n}{1-\alpha}\right)$. We start by noticing that $\gamma_{n,p,\alpha} \geq \alpha$ if and only if $p \geq \frac{n}{n-\alpha}$, so that $B^\gamma_{q,1}(\mathbb{R}^n) \subset B^\alpha_{q,1}(\mathbb{R}^n)$, thanks to [41, Th. 17.82]. Hence $g \in B^\alpha_{q,1}(\mathbb{R}^n)$ and so $\nabla^\alpha g \in L^q(\mathbb{R}^n;\mathbb{R}^n)$ by [20, Cor. 23 and Lem. 2.6]. Let $(\varrho_\varepsilon)_{\varepsilon>0}$ be as in Theorem 2.1 and set $g_\varepsilon = \varrho_\varepsilon * g$ for all $\varepsilon > 0$. Arguing as in the proof of [20, Cor. 3.5], we can exploit [17, Sec. 5.1 and Th. 11] to conclude that

$$\lim_{\varepsilon\to0^+} g_\varepsilon(x) = g^\star(x) \quad \text{for all } x \in \mathbb{R}^n \setminus D_g, \tag{2.25}$$

for some set $D_g \subset \mathbb{R}^n$ such that $\mathscr{H}^{n-\gamma q+\delta}(D_g) = 0$ for any $\delta > 0$ sufficiently small. Since

$$n - \frac{nq}{n+(1-\alpha)q} > n - \gamma q \iff \gamma > \frac{n}{n+(1-\alpha)q},$$

we conclude that $|div^\alpha F|(D_g) = 0$, by Theorem 1.7. Since $g_\varepsilon \in C_b(\mathbb{R}^n) \cap B^\alpha_{q,1}(\mathbb{R}^n)$ for all $\varepsilon > 0$ thanks to [41, Prop. 17.12], by Theorem 2.12 we get that $g_\varepsilon F \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$, with

$$div^\alpha(g_\varepsilon F) = g_\varepsilon\, div^\alpha F + F \cdot \nabla^\alpha g_\varepsilon\, \mathscr{L}^n + \mathrm{div}^\alpha_{\mathrm{NL}}(g_\varepsilon, F)\,\mathscr{L}^n \quad \text{in } \mathcal{M}(\mathbb{R}^n).$$

Now $\nabla^\alpha g_\varepsilon = \varrho_\varepsilon * \nabla^\alpha g$ in $L^q(\mathbb{R}^n;\mathbb{R}^n)$ (for example see [18, Lem. 3.5] and its proof), while [20, Cor. 2.7] implies that

$$\begin{aligned}\|\mathrm{div}^\alpha_{\mathrm{NL}}(g_\varepsilon,F) - \mathrm{div}^\alpha_{\mathrm{NL}}(g,F)\|_{L^1(\mathbb{R}^n)} &= \|\mathrm{div}^\alpha_{\mathrm{NL}}(g_\varepsilon - g, F)\|_{L^1(\mathbb{R}^n)} \\ &\leq 2\mu_{n,\alpha}\,\|F\|_{L^p(\mathbb{R}^n;\mathbb{R}^n)}\,[g - g_\varepsilon]_{B^\alpha_{q,1}(\mathbb{R}^n)}\end{aligned}$$

for all $\varepsilon > 0$. Therefore, since $\varrho_\varepsilon * \nabla^\alpha g \to \nabla^\alpha g$ in $L^q(\mathbb{R}^n;\mathbb{R}^n)$ and, by [41, Prop. 17.12], $[g-g_\varepsilon]_{B^\alpha_{q,1}(\mathbb{R}^n)} \to 0$, the conclusion follows by exploiting (2.25) and the Dominated Convergence Theorem with respect to the measure $|div^\alpha F|$. Finally, equations (1.10) and (1.11) can be proved as (2.23) and (2.24) in Theorem 2.12. □

3. Examples

In this section, we illustrate some examples concerning Theorem 1.7.

3.1. **Example for point** (i) **of Theorem 1.7.** Example 3.1 below shows that, if $p \in \left[1, \frac{n}{n-\alpha}\right)$, the fractional divergence-measure of $\mathcal{DM}^{\alpha,p}$ vector fields is not absolutely continuous with respect to $\mathscr{H}^{\varepsilon}$ for any $\varepsilon > 0$, in general, proving point (i) of Theorem 1.7.

Example 3.1. Let $\alpha \in (0,1)$, $y, z \in \mathbb{R}^n$, and define

$$F_{y,z,\alpha}(x) = \mu_{n,-\alpha}\left(\frac{(x-y)}{|x-y|^{n+1-\alpha}} - \frac{(x-z)}{|x-z|^{n+1-\alpha}}\right), \quad x \in \mathbb{R}^n \setminus \{y,z\}. \tag{3.1}$$

A plain computation yields $F_{y,z,\alpha} \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ for all $p \in \left[1, \frac{n}{n-\alpha}\right)$ (for example, see the proof of [18, Prop. 3.14]). Moreover, by [18, Prop. 3.13], we know that

$$div^{\alpha} F_{y,z,\alpha} = \delta_y - \delta_z.$$

Consequently, $F_{y,z,\alpha} \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ for all $p \in \left[1, \frac{n}{n-\alpha}\right)$.

Interestingly, the vector field (3.1) of Example 3.1 works also in the limit case $\alpha = 1$, proving point (i) of Theorem 1.1, see [49, Prop. 6.1].

Example 3.2. Let $y, z \in \mathbb{R}^n$ and define

$$F_{y,z,1}(x) = \mu_{n,-1}\left(\frac{(x-y)}{|x-y|^{n}} - \frac{(x-z)}{|x-z|^{n}}\right), \quad x \in \mathbb{R}^n \setminus \{y,z\}.$$

Computations as in Example 3.1 show that $F_{y,z,1} \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ for all $p \in \left(1, \frac{n}{n-1}\right)$, with

$$div F_{y,z,1} = \delta_y - \delta_z.$$

Hence $F_{y,z,1} \in \mathcal{DM}^{1,p}(\mathbb{R}^n)$ for all $p \in \left(1, \frac{n}{n-1}\right)$. Actually, we have $F_{y,z,1} \in \mathcal{DM}^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

3.2. **Partial sharpness of Theorem 1.7.** Arguing as in [49, Exam. 3.3 and Prop. 6.1], we can exploit the properties of the vector field (3.1) in Example 3.1 to construct additional examples proving a partial sharpness of Theorem 1.7.

The following result is the analogue of [17, Prop. 5].

Proposition 3.3 (The vector field $G_\alpha = F_\alpha * \nu$)**.** *Let $\alpha \in (0,1)$ and $F_\alpha = F_{0,\mathrm{e}_1,\alpha}$ be as in Example 3.1, and let $\nu \in \mathcal{M}(\mathbb{R}^n)$. Then we have*

$$G_\alpha = F_\alpha * \nu \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \quad \textit{for all } p \in \left[1, \frac{n}{n-\alpha}\right),$$

with

$$div^{\alpha} G_\alpha = \nu - (\tau_{\mathrm{e}_1})_{\#}\nu, \tag{3.2}$$

where $\tau_x(y) = y + x$ for all $x, y \in \mathbb{R}^n$. In addition, if there exist $C, \varepsilon > 0$ such that

$$|\nu|(B_r(x)) \le C r^{\varepsilon} \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } r > 0, \tag{3.3}$$

then

$$G_\alpha \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n) \quad \textit{for all } p \in \begin{cases} \left[1, \frac{n-\varepsilon}{n-\alpha-\varepsilon}\right) & \textit{if } \varepsilon \in (0, n-\alpha), \\ [1, +\infty) & \textit{if } \varepsilon = n-\alpha, \\ [1, +\infty] & \textit{if } \varepsilon \in (n-\alpha, n]. \end{cases} \tag{3.4}$$

Proof. We divide the proof into two steps.

Step 1. Let $\nu \in \mathscr{M}(\mathbb{R}^n)$. We claim that $G_\alpha \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ for all $p \in \left[1, \frac{n}{n-\alpha}\right)$ and that G_α satisfies (3.2). Indeed, by Young's inequality (for Radon measures), we can estimate

$$\|G_\alpha\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le \|F_\alpha\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} |\nu|(\mathbb{R}^n).$$

Moreover, thanks to the translation invariance of ∇^α and exploiting the explicit expression of F_α given in Example 3.1, we can write

$$\begin{aligned}\int_{\mathbb{R}^n} G_\alpha(x) \cdot \nabla^\alpha \xi(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F_\alpha(x-y) \cdot \nabla^\alpha \xi(x)\,\mathrm{d}\nu(y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F_\alpha(x-y) \cdot \nabla^\alpha \xi(x)\,\mathrm{d}x\,\mathrm{d}\nu(y) \\ &= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \xi(x+y)\,\mathrm{d}\left(\delta_0(x) - \delta_{\mathrm{e}_1}(x)\right)\,\mathrm{d}\nu(y) \\ &= -\int_{\mathbb{R}^n} \left(\xi(y) - \xi(y+\mathrm{e}_1)\right)\,\mathrm{d}\nu(y)\end{aligned}$$

for all $\xi \in C_c^\infty(\mathbb{R}^n)$. This proves $G_\alpha \in \mathcal{DM}^{\alpha,1}(\mathbb{R}^n)$ and (3.2). In addition, by Jensen's inequality and Tonelli's Theorem, we can estimate

$$\begin{aligned}\int_{\mathbb{R}^n} |G_\alpha(x)|^p\,\mathrm{d}x &\le \int_{\mathbb{R}^n} |\nu|(\mathbb{R}^n)^{p-1} \int_{\mathbb{R}^n} |F_\alpha(x-y)|^p\,\mathrm{d}|\nu|(y)\,\mathrm{d}x \\ &= |\nu|(\mathbb{R}^n)^p\,\|F_\alpha\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^n)} < +\infty\end{aligned}$$

for all $p \in \left[1, \frac{n}{n-\alpha}\right)$, thanks to the integrability properties of F_α given in Example 3.1.

Step 2. We prove that (3.3) implies (3.4). To this aim, let $q = \frac{p}{p-1}$ and $0 < \delta \le q$. Since $|F_\alpha| = |F_\alpha|^{\frac{\delta}{q}} |F_\alpha|^{1-\frac{\delta}{q}}$, by Hölder's inequality we get

$$\begin{aligned}|G_\alpha(x)|^p &\le \left(\int_{\mathbb{R}^n} |F_\alpha(x-y)|^{\frac{\delta}{q}}\,|F_\alpha(x-y)|^{1-\frac{\delta}{q}}\,\mathrm{d}|\nu|(y)\right)^p \\ &\le \left(\int_{\mathbb{R}^n} |F_\alpha(x-y)|^{\delta}\,\mathrm{d}|\nu|(y)\right)^{\frac{p}{q}} \left(\int_{\mathbb{R}^n} |F_\alpha(x-y)|^{p\left(1-\frac{\delta}{q}\right)}\,\mathrm{d}|\nu|(y)\right)\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$. We now recall the explicit expression of F_α in Example 3.1 and write

$$\begin{aligned}\int_{\mathbb{R}^n} |F_\alpha(x-y)|^\delta\,\mathrm{d}|\nu|(y) &= \int_{\mathbb{R}^n \setminus \left(B_{\frac{1}{2}}(x) \cup B_{\frac{1}{2}}(x-\mathrm{e}_1)\right)} |F_\alpha(x-y)|^\delta\,\mathrm{d}|\nu|(y) \\ &\quad + \sum_{j=1}^\infty \int_{C_j\left(x,\frac{1}{2}\right) \cup C_j\left(x-\mathrm{e}_1,\frac{1}{2}\right)} |F_\alpha(x-y)|^\delta\,\mathrm{d}|\nu|(y),\end{aligned} \tag{3.5}$$

where we have set

$$C_j(x,r) = B_{r^j}(x) \setminus B_{r^{j+1}}(x)$$

for all $x \in \mathbb{R}^n$, $r \in (0,1)$ and $j \in \mathbb{N}$, $j \ge 1$, for brevity. Now, on the one hand, if $y \in \mathbb{R}^n \setminus \left(B_{\frac{1}{2}}(x) \cup B_{\frac{1}{2}}(x-\mathrm{e}_1)\right)$, then $x - y \in \mathbb{R}^n \setminus \left(B_{\frac{1}{2}} \cup B_{\frac{1}{2}}(\mathrm{e}_1)\right)$, so that

$$|F_\alpha(x-y)| \le \mu_{n,-\alpha}\left(2^{n-\alpha} + 2^{n-\alpha}\right) = \mu_{n,-\alpha}\,2^{n+1-\alpha}$$

for all $y \in \mathbb{R}^n \setminus \left(B_{\frac{1}{2}}(x) \cup B_{\frac{1}{2}}(x - e_1)\right)$. Therefore, we can estimate

$$\int_{\mathbb{R}^n \setminus \left(B_{\frac{1}{2}}(x) \cup B_{\frac{1}{2}}(x-e_1)\right)} |F_\alpha(x-y)|^\delta \, d|\nu|(y) \le \left(\mu_{n,-\alpha} \, 2^{n+1-\alpha}\right)^\delta |\nu|(\mathbb{R}^n) \tag{3.6}$$

for all $x \in \mathbb{R}^n$. On the other hand, for all $x \in \mathbb{R}^n$ and $j \ge 1$, we have

$$\begin{aligned} \int_{C_j\left(x, \frac{1}{2}\right)} |F_\alpha(x-y)|^\delta \, d|\nu|(y) &\le \mu_{n,-\alpha}^\delta \int_{C_j\left(x,\frac{1}{2}\right)} \left(|x-y|^{\alpha-n} + |x-y-e_1|^{\alpha-n}\right)^\delta d|\nu|(y) \\ &\le \mu_{n,-\alpha}^\delta \left(2^{(j+1)(n-\alpha)} + \left(1 - 2^{-j}\right)^{\alpha-n}\right)^\delta |\nu|(B_{2^{-j}}(x)) \\ &\le \mu_{n,-\alpha}^\delta \left(2^{(j+1)(n-\alpha)} + 2^{n-\alpha}\right)^\delta C \, 2^{-j\varepsilon}. \end{aligned} \tag{3.7}$$

Reasoning analogously, we obtain

$$\int_{C_j\left(x-e_1, \frac{1}{2}\right)} |F_\alpha(x-y)|^\delta \, d|\nu|(y) \le C\mu_{n,-\alpha}^\delta \left(2^{(j+1)(n-\alpha)} + 2^{n-\alpha}\right)^\delta 2^{-j\varepsilon} \tag{3.8}$$

for all $x \in \mathbb{R}^n$ and $j \ge 1$. Therefore, inserting (3.6), (3.7) and (3.8) in (3.5), we get that

$$\int_{\mathbb{R}^n} |F_\alpha(x-y)|^\delta \, d|\nu|(y) \le C_{\alpha,\varepsilon,\delta} \tag{3.9}$$

for all $x \in \mathbb{R}^n$, where $C_{\alpha,\varepsilon,\delta} > 0$ is constant depending on α, ε, and δ which is finite provided that we choose $\delta < \frac{\varepsilon}{n-\alpha}$, as we are assuming from now on. We thus have

$$\begin{aligned} \int_{\mathbb{R}^n} |G_\alpha(x)|^p \, dx &\le C_{\alpha,\varepsilon,\delta}^{p-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F_\alpha(x-y)|^{p\left(1-\frac{\delta}{q}\right)} \, d|\nu|(y) \, dx \\ &= C_{\alpha,\varepsilon,\delta}^{p-1} \, |\nu|(\mathbb{R}^n) \int_{\mathbb{R}^n} |F_\alpha(x)|^{p\left(1-\frac{\delta}{q}\right)} \, dx. \end{aligned}$$

Now, recalling Example 3.1, we immediately see that

$$\int_{\mathbb{R}^n} |F_\alpha(x)|^{p\left(1-\frac{\delta}{q}\right)} \, dx < +\infty \iff p < \frac{n}{(n-\alpha)(1-\delta)} - \frac{\delta}{1-\delta} = \frac{n - \delta n + \alpha\delta}{(n-\alpha)(1-\delta)}.$$

Hence, since the function $\delta \mapsto \frac{n-\delta n+\alpha\delta}{(n-\alpha)(1-\delta)}$ is monotone increasing, we easily see that

$$\varepsilon \in (0, n-\alpha) \implies \delta < \frac{\varepsilon}{n-\alpha} < 1 \implies p \in \left[1, \frac{n-\varepsilon}{n-\alpha-\varepsilon}\right)$$

and, similarly,

$$\varepsilon \in [n-\alpha, n] \implies \delta(n-\alpha) < \varepsilon \text{ for all } \delta \in (0,1) \implies p \in [1, +\infty).$$

Finally, in the case $\varepsilon \in (n-\alpha, n]$, we exploit (3.9) for $\delta = 1$ in order to conclude that

$$|G_\alpha(x)| \le \int_{\mathbb{R}^n} |F_\alpha(x-y)| \, d|\nu|(y) = C_{\alpha,\varepsilon} < +\infty$$

for all $x \in \mathbb{R}^n$, which implies that $G_\alpha \in L^\infty(\mathbb{R}^n)$. The conclusion thus follows. □

Thanks to Proposition 3.3, we can now give the following example.

Example 3.4. Let $\alpha \in (0,1)$ and let ν and G_α be as in Proposition 3.3. By [32, Cor. 4.12], there exists a compact set $K \subset \mathbb{R}$ such that $\nu = \mathscr{H}^\varepsilon \llcorner K$, so that $|div^\alpha G_\alpha| \not\ll \mathscr{H}^s$ for all $s > \varepsilon$. Now we observe that, by (3.4), we have the following situations:

- in order to have $G_\alpha \in \mathcal{DM}^{\alpha,p}(\mathbb{R}^n)$ for some $p \in \left[\frac{n}{n-\alpha}, +\infty\right)$, we need $\varepsilon > n - \alpha q$, since, if $\varepsilon \in [n-\alpha, n]$, then $p \in [1, +\infty)$, while, for $\varepsilon \in (0, n-\alpha)$, we have $p < \frac{n-\varepsilon}{n-\alpha-\varepsilon}$, which implies $\varepsilon > n - \alpha q$;
- in order to have $G_\alpha \in \mathcal{DM}^{\alpha,\infty}(\mathbb{R}^n)$, we need $\varepsilon > n - \alpha$, since, if $\varepsilon \in (n-\alpha, n]$, then $p \in [1, +\infty]$.

Therefore, these lower bounds on ε imply that, for $p \in \left[\frac{n}{n-\alpha}, +\infty\right]$, we have

$$|div^\alpha G_\alpha| \not\ll \mathscr{H}^s \text{ for all } s > n - \alpha q.$$

Notice that

$$n - \alpha q \geq \max\left\{n - \frac{nq}{nq + (1-\alpha)q}, \ \frac{n}{q} - \alpha\right\}$$

for all $q \in \left[1, \frac{n}{\alpha}\right]$, which means $p \in \left[\frac{n}{n-\alpha}, +\infty\right]$, with equality only for $q = \frac{n}{\alpha}$ and $q = 1$. Consequently, Example 3.4 shows that points (ii) and (iii) in Theorem 1.7 cannot be improved beyond $|div^\alpha F| \ll \mathscr{H}^{n-\alpha q}$, which is actually sharp for $p = +\infty$.

Remark 3.5 (Correction to [17, Exam. 2])**.** For $n = 1$, Example 3.4 together with the above considerations corrects the conclusions of [17, Exam. 2].

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