Isoperimetric conditions, lower semicontinuity, and existence results for perimeter functionals with measure data

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Abstract

We establish lower semicontinuity results for perimeter functionals with measure data on $\mathbb{R}^n$ and deduce the existence of minimizers to these functionals with Dirichlet boundary conditions, obstacles, or volume-constraints. In other words, we lay foundations of a perimeter-based variational approach to mean curvature measures on $\mathbb{R}^n$ capable of proving existence in various prescribed-mean-curvature problems with measure data. As crucial and essentially optimal assumption on the measure data we identify a new condition, called small-volume isoperimetric condition, which sharply captures cancellation effects and comes with surprisingly many properties and reformulations in itself. In particular, we show that the small-volume isoperimetric condition is satisfied for a wide class of $(n-1)$-dimensional measures, which are thus admissible in our theory. Our analysis includes infinite measures and semicontinuity results on very general domains.

Contents

1 Introduction

Prescribed mean curvature hypersurfaces and Massari's functional. This paper contributes to the theory of (generalized) hypersurfaces of prescribed mean curvature in $\mathbb{R}^n$, approached from a parametric

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calculus-of-variations side. Given a function $H \in \mathrm{L}^1(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$, this amounts to the study of functionals of the type

$$\mathscr{P}_H[A;\Omega] := \mathrm{P}(A,\Omega) - \int_{A\cap\Omega} H\,\mathrm{d}x \qquad \text{on measurable sets } A \subset \mathbb{R}^n\,, \tag{1.1}$$

where the perimeter $\mathrm{P}(A,\Omega)$ of A in Ω gives, in sufficiently regular cases, the $(n-1)$-dimensional Hausdorff measure of $\Omega \cap \partial A$. In order to obtain prescribed mean curvature hypersurfaces one seeks to minimize the functional $\mathscr{P}_H[\,\cdot\,;\Omega]$ among sets A of finite perimeter in Ω, which are usually required to satisfy boundary conditions at $\partial\Omega$ and possibly further constraints. If a minimizer A with sufficiently smooth boundary $\Omega\cap\partial A$ exists, at least in cases with constraints *only* at $\partial\Omega$, it should solve the parametric prescribed mean curvature equation

$$\operatorname{div}\nu_A = H \qquad \text{on } \Omega\cap\partial A\,, \tag{1.2}$$

where ν_A denotes the outward unit normal to A at points of $\Omega\cap\partial A$ and the divergence can be taken either as the tangential divergence of ν_A along ∂A or equivalently as the standard divergence of any smooth continuation of ν_A to Ω as a (sub-)unit vector field. The equation (1.2), if valid in a suitably strong sense, does express that the mean curvature of ∂A is indeed the prescribed function $\frac{-1}{n-1}H$ — or more precisely that, for every $x \in \Omega\cap\partial A$, the number $\frac{-1}{n-1}H(x)$ is the mean curvature at x of the hypersurface $\Omega\cap\partial A$ oriented by ν_A.

A major step in the program described has been achieved by Massari [27, 28] who introduced the approach via the functional $\mathscr{P}_H[\,\cdot\,;\Omega]$ and extended De Giorgi's pioneering work [11] from the minimal surface case $H \equiv 0$ to general prescribed functions H. In fact, the papers [27, 28] establish an existence result for minimizers of $\mathscr{P}_H[\,\cdot\,;\Omega]$ in case $H \in \mathrm{L}^1(\Omega)$ and also a minimal-surface-type[1] partial $\mathrm{C}^{1,\alpha}$ regularity result under the optimal assumption that $H \in \mathrm{L}^p_{(\mathrm{loc})}(\Omega)$ holds for some $p > n$. If H is additionally continuous, it follows in a standard way (e.g. by locally computing the non-parametric first variation) that minimizers A of $\mathscr{P}_H[\,\cdot\,;\Omega]$ satisfy (1.2) on the regular portions of $\Omega\cap\partial A$ and that $\frac{-1}{n-1}H$ is the mean curvature of $\Omega\cap\partial A$. For discontinuous H, in contrast, the geometric significance of H is far less clear, and in general it seems to be a widely open problem if and in which precise sense one can still restrict H to $\Omega\cap\partial A$ and make any sense of equation (1.2).

Lower semicontinuity for a Massari-type functional with measures. In the present paper, though we take the geometric situation as a background motivation and in fact have some hope for a connection with the open problem just mentioned, we deal with the minimization of prescribed-mean-curvature functionals mostly in its own right. In fact, we replace the prescribed function $H \in \mathrm{L}^1(\Omega)$ with prescribed non-negative Radon measures μ_+ and μ_- concentrated on Ω and possibly of dimension lower than n, and correspondingly we replace Massari's functional (1.1) with its natural generalization

$$\mathscr{P}_{\mu_+,\mu_-}[A;\Omega] := \mathrm{P}(A,\Omega) + \mu_+(A^1) - \mu_-(A^+) \qquad \text{on measurable sets } A \subset \mathbb{R}^n\,, \tag{1.3}$$

where A^+ denotes the measure-theoretic closure and A^1 the measure-theoretic interior of A (see Section 2 for the definitions). Our central results on the functional $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega]$ are semicontinuity results, which apply under sharp hypotheses on the measures $\mu_\pm$ and are suitable to prove the existence of minimizers of $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega]$ in several cases with standard boundary conditions or constraints. In fact, our semicontinuity statements take slightly different forms in the full-space case $\Omega = \mathbb{R}^n$ (see Section 4), in versions adapted to Dirichlet problems on domains $\Omega \subset \mathbb{R}^n$ (see Section 6), and generally on domains $\Omega \subset \mathbb{R}^n$ (see Section 9). For the purposes of this introduction, we restrict the detailed discussion to the full-space case and the functional

$$\mathscr{P}_{\mu_+,\mu_-} := \mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\mathbb{R}^n]\,,$$

for which we introduce the crucial hypothesis on $\mu_\pm$ and state a prototypical case of our results as follows:

Definition 1.1 (small-volume isoperimetric condition)**.** *We say that a non-negative Radon measure μ on $\mathbb{R}^n$ satisfies the small-volume isoperimetric condition* (*briefly*: *the small-volume IC*) *in $\mathbb{R}^n$ with constant* 1 *if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\mu(A^+) \le \mathrm{P}(A,\mathbb{R}^n) + \varepsilon \qquad \textit{for all measurable } A \subset \mathbb{R}^n \textit{ with } |A| < \delta\,. \tag{1.4}$$

[1]By minimal-surface-type partial regularity we mean regularity up to an exceptional set of Hausdorff dimension at most $n-8$.

Theorem 1.2 (lower semicontinuity on full space; proptotypical case)**.** *Consider non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ which both satisfy the small-volume IC in $\mathbb{R}^n$ with constant* 1*. Then the full-space functional $\mathscr{P}_{\mu_+,\mu_-}$ introduced above is finite and lower semicontinuous with respect to convergence in measure on $\mathcal{BV}(\mathbb{R}^n) := \{A \subset \mathbb{R}^n : A \text{ measurable}, |A| + \mathrm{P}(A,\mathbb{R}^n) < \infty\}$.*

We emphasize that, for this and similar semicontinuity results, we necessarily need to use some closed representative of A in the μ_--volume term of (1.3), since measurable sets A are considered in an $\mathcal{L}^n$-a.e. sense, and other choices of representative would not ensure lower semicontinuity of $\mathscr{P}_{\mu_+,\mu_-}$ along basic strictly decreasing sequences $A_k \searrow A_\infty$ with $\mathrm{P}(A_k,\mathbb{R}^n) \to \mathrm{P}(A_\infty,\mathbb{R}^n)$, as soon as μ_- assigns mass to the boundary of A_∞. Indeed, the usage of A^+ as a precise $\mathcal{H}^{n-1}$-a.e. defined representative of A is perfectly suited for our purposes and is inspired by related developments in the theory of one-sided obstacle problems; cf. [6, 37, 4, 38, 39, 45]. In a very similar way, the choice of A^1 in the μ_+-volume term allows to cope with basic increasing sequences $A_k \nearrow A_\infty$.

Lower semicontinuity also on general domains. Our semicontinuity results for functionals of type (1.3) on general domains $\Omega \subset \mathbb{R}^n$ rely on closely related (small-volume) ICs, which partially can be understood as relative ICs adapted to the domain at hand. However, at this introductory stage we will only briefly touch upon some aspects of the results, while postponing the discussion of the adapted ICs entirely to the later sections. We mention that basically all results on general domains will be deduced from the ones on full space by extension/restriction to/from all of $\mathbb{R}^n$. For cases with a generalized Dirichlet boundary condition on a bounded domain Ω, this deduction is essentially standard. However, as a technical addition, when working out the details, we also include a careful treatment of (strongly) unbounded domains Ω and infinite measures $\mu_\pm$; see Section 6 for the details. Furthermore, in the final Section 9, we also obtain two semicontinuity results on general domains independent of any boundary condition. The first result is somewhat different from the usual semicontinuity on open sets and yields lower semicontinuity of a functional $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega^1]$ *on the measure-theoretic interior* Ω^1 of a set Ω of locally finite perimeter in $\mathbb{R}^n$. This type of semicontinuity on Ω^1 does not seem to be standard even in case of the relative perimeter $\mathscr{P}_{0,0}[\,\cdot\,;\Omega^1] = \mathrm{P}(\,\cdot\,,\Omega^1)$ alone, but in the perimeter case is in fact not entirely new and can also be deduced from a recent result of Lahti [24]. Anyway, our theory allows for a new and very natural proof by incorporating the perimeter measure $\mathrm{P}(\Omega,\,\cdot\,)$ (and potentially even some other measures on the reduced boundary $\partial^*\Omega$) into the measures $\mu_\pm$ of the full-space functional $\mathscr{P}_{\mu_+,\mu_-}$. As a complement, the second result gives lower semicontinuity of $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega]$ also on an arbitrary open set $\Omega \subset \mathbb{R}^n$ and thus can dispense with any regularity of Ω at the price of requiring openness even in the standard topological sense. Finally, we will also further underpin the results with several examples of admissible domains and measures and with a detailed discussion of the relevant (relative) ICs and their optimality.

The small-volume IC as decisive assumption for semicontinuity. For now, we return to the full space-setting of Theorem 1.2 and discuss its crucial assumption, the small-volume IC, in some more detail. We first highlight that this condition is not only sufficient for the lower semicontinuity conclusion, but in itself expresses lower semicontinuity of the functional $\mathscr{P}_{0,\mu}$ *at the empty set* and thus in most cases is also necessary for lower semicontinuity. Indeed, if $\mu = \mu_-$ violates the small-volume IC in $\mathbb{R}^n$ with constant 1, for some $\varepsilon > 0$ there exists a sequence of counterexamples in form of measurable sets $A_k \subset \mathbb{R}^n$ with $\lim_{k\to\infty} |A_k| = 0$ and $\mu(A_k^+) > \mathrm{P}(A_k,\mathbb{R}^n) + \varepsilon$. This, however, means that A_k converge in measure to the empty set $\emptyset$ with $\limsup_{k\to\infty} \mathscr{P}_{0,\mu}[A_k] \le -\varepsilon < 0 = \mathscr{P}_{0,\mu}[\emptyset]$, and lower semicontinuity fails as well. Therefore, the small-volume IC with constant 1 is in fact the optimal assumption on μ_- in Theorem 1.2. Moreover, if $\mu = \mu_+$ is supported in a ball B and A_k are as before, then $B \setminus A_k$ converge in measure to B, and one finds $\limsup_{k\to\infty} \mathscr{P}_{\mu,0}[B \setminus A_k] \le \mathscr{P}_{\mu,0}[B] - \varepsilon$. Therefore, at least case of bounded support, the small-volume IC with constant 1 is the optimal assumption on μ_+ as well.

In the proof of Theorem 1.2, the small-volume IC is decisive in coping with cases in which (the singular part of) $\mu = \mu_-$ has mass on an $(n-1)$-dimensional surface S and, for a decreasing sequence $A_k \searrow A_\infty$, the sets A_k include thinner and thinner neighborhoods of S, while A_∞^+ does not intersect S anymore; see Figure 1 below for an illustration in case $n = 2$. In such situations, with $-\mu(A_\infty^+) > \liminf_{k\to\infty}[-\mu(A_k^+)]$ the μ-volume term in $\mathscr{P}_{0,\mu}$ is *not* lower semicontinuous, but it holds the *strict* inequality $\mathrm{P}(A_\infty,\mathbb{R}^n) < \liminf_{k\to\infty} \mathrm{P}(A_k,\mathbb{R}^n)$. Under the small-volume IC from (1.4) we will show that it is possible to quantitatively relate these opposite effects, to compensate for the increase of the μ-volume with the decrease of the perimeter and thus to admit a certain cancellation effect while still preserving lower semicontinuity of the functional $\mathscr{P}_{0,\mu}$. The functional $\mathscr{P}_{\mu,0}$ with $\mu = \mu_+$ can be handled in a dual manner (where the decisive sequences are the increasing ones), and the results can be combined in order to reach functionals of the general type $\mathscr{P}_{\mu_+,\mu_-}$.

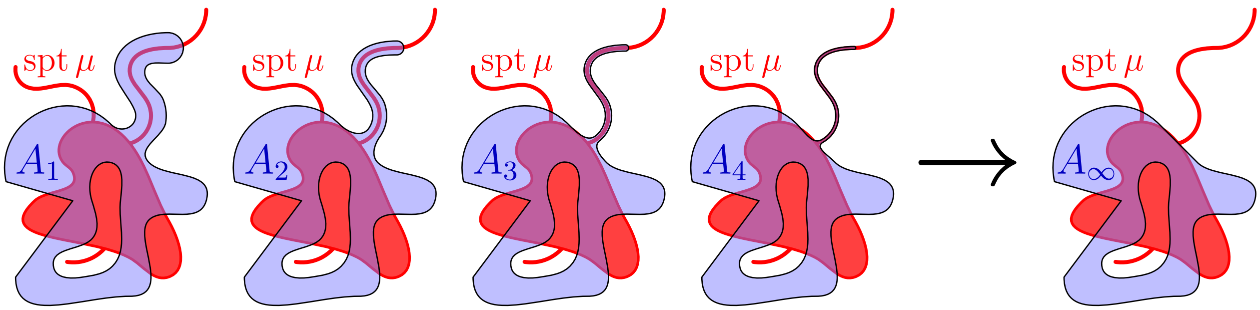

Figure 1: An illustration of the decisive cancellation effect in $\mathbb{R}^2$: A sequence $(A_k)_{k\in\mathbb{N}}$ forms thinner and thinner tentacles around a 1d portion of spt μ, but in the limit A_∞^+ does not cover this portion anymore.

Beside the decisive effect just described, the small-volume IC also has a role in preventing a breakdown of lower semicontinuity at infinity, which in general can occur already in the function case $\mu_\pm = H_\pm\mathcal{L}^n$. Indeed, for each $H \in \mathrm{L}^1(\mathbb{R}^n)$, continuity of the H-volume term and thus lower semicontinuity of $\mathscr{P}_H$ are immediate. However, this does not extend to $H \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, where for similar reasons as above one needs to prevent that A_k move away to infinity with $\lim_{k\to\infty} |A_k| = 0$, $\limsup_{k\to\infty} \mathrm{P}(A_k, \mathbb{R}^n) < \infty$, but $\limsup_{k\to\infty} \int_{A_k} H\,\mathrm{d}x = \infty$. As our result is formulated for *locally* finite measures $\mu_\pm$, it also singles out functions $H \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \setminus \mathrm{L}^1(\mathbb{R}^n)$ such that $\mathscr{P}_H$ is lower semicontinuous. We are aware of previous results in this direction only on specific unbounded domains in the different setting of [13, 14] (compare also below), but still consider this aspect mostly as a side benefit of our treatment of possibly singular measure data.

Existence results. As standard consequences of semicontinuity we derive existence results for minimizers of $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega]$ with obstacles, prescribed volume, or a Dirichlet boundary condition as side conditions. Since the obstacle and prescribed-volume constraints fit into the full-space setting described so far, we exemplarily state our corresponding existence results at least for the case of finite μ_-, while the somewhat more technical treatment of Dirichlet problems is postponed to the later Section 6. In all cases, we impose the small-volume IC as the decisive assumption on $\mu_\pm$.

Theorem 1.3 (existence in obstacle and prescribed-volume problems)**.** *Consider non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ such that both μ_+ and μ_- satisfy the small-volume IC with constant* 1 *on $\mathbb{R}^n$ and such that μ_- is finite. Then, with $\mathcal{BV}(\mathbb{R}^n)$ as in Theorem 1.2, we have:*

Obstacle problem: *Whenever, for given measurable sets $I, O \subset \mathbb{R}^n$, the admissible class $\{A \in \mathcal{BV}(\mathbb{R}^n) : I \subset A \subset O$ up to negligible sets$\}$ is non-empty, then there exists a minimizer of $\mathscr{P}_{\mu_+,\mu_-}$ in this class.*

Prescribed-volume problem with $\mu_+ \equiv 0$: *For every $v \in (0,\infty)$, there exists a minimizer of $\mathscr{P}_{0,\mu_-}$ in $\{A \in \mathcal{BV}(\mathbb{R}^n) : |A| = v\}$.*

Theorem 1.3 will be established in Section 5, where existence in the obstacle problem will also be extended to some infinite measures μ_-, while in the prescribed-volume problem we will not go beyond the statement given above. The proof uses the direct method in the calculus of variations and at least in the obstacle case is standard once suitable semicontinuity is at hand. However, since in the full-space situation out of a minimizing sequence we can only extract a subsequence which converges *locally* in measure on all of $\mathbb{R}^n$, we in fact need a semicontinuity statement adapted to *local* convergence in measure. As we will see in Section 4, such a variant can be deduced from the above statement of Theorem 1.2 by cut-off arguments. In case of the prescribed-volume problem, the local-convergence issue additionally brings up the more severe difficulty that a limit in the sense of local convergence may exhibit a “volume drop” at infinity and thus may fall out of the admissible class. The strategy for preventing this is technically more involved and consists in constructing an improved minimizing sequence by “shifting volume” into a bounded region; see Section 5 for detailed discussion and implementation.

More on the small-volume IC: criteria and exemplary cases. We further support the semicontinuity and existence results by identifying wide classes of measures for which the small-volume IC holds. First let us remark that related ICs without the additive ε-term have been considered in classical literature (compare also below for related discussion) with the typical background idea that such conditions can be deduced for $\mu_\pm = H_\pm \mathcal{L}^n$, $H \in \mathrm{L}^p(\mathbb{R}^n)$, $p > n$, by the classical estimate via the Hölder and isoperimetric inequalities $\int_A H_\pm \,\mathrm{d}x \le C_n \|H\|_{\mathrm{L}^p(\mathbb{R}^n)} |A|^{\frac{1}{n}-\frac{1}{p}} \,\mathrm{P}(A,\mathbb{R}^n)$, where C_n is a dimensional constant. As a first indication that our small-volume IC is substantially different, we record that it is in fact trivially satisfied, beyond the previous L^p cases and due to the ε-term alone, for all finite absolutely continuous measures $\mu_\pm = H_\pm \mathcal{L}^n$ with $H \in \mathrm{L}^1(\mathbb{R}^n)$. Hence, our semicontinuity results include Massari's standard case of the functional $\mathscr{P}_H$. In addition, however, our results do admit singular measures, as will become clear from the following abstract criterion:

Theorem 1.4 (divergence criterion for the small-volume IC)**.** *If a non-negative Radon measure μ on $\mathbb{R}^n$ can be expressed as $\mu = H\mathcal{L}^n + \operatorname{div}\sigma$ with $H \in \mathrm{L}^1(\mathbb{R}^n)$ and a divergence-measure field $\sigma \in \mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ such that $\|\sigma\|_{\mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^n)} \le 1$, then μ satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1.*

Theorem 1.4 and its proof are not very surprising. For instance, one may read off the result from a divergence theorem for L^∞ divergence-measure fields on sets of finite perimeter (similar to the later formula (2.13)). Alternatively, one can also argue by approximation, and this is the route we take when picking up the result in the somewhat wider context of the later Section 7.

For the moment, we mainly record that the condition of Theorem 1.4 holds for infinite measures $\mu = \theta\mathcal{H}^{n-1} \llcorner S$ with $\theta \in [0,2]$ and with a hyperplane $S \subset \mathbb{R}^n$ or a union S of finitely many parallel hyperplanes in $\mathbb{R}^n$. Thus, we obtain basic examples of singular measures with small-volume IC. However, the condition remains valid for a much broader class of $(n-1)$-dimensional measures, as in fact we have:

Theorem 1.5 (small-volume IC for rectifiable $\mathcal{H}^{n-1}$-measures)**.** *Whenever, for a non-negative Radon measure μ on $\mathbb{R}^n$, we have $\mu \le 2\mathcal{H}^{n-1} \llcorner S$ with some $\mathcal{H}^{n-1}$-finite and countably $\mathcal{H}^{n-1}$-rectifiable Borel set $S \subset \mathbb{R}^n$, then μ satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1.*

Theorem 1.5 will be established in Section 8, where the case $\mu = 2\mathcal{H}^{n-1} \llcorner \partial^* E$ with the reduced boundary $\partial^* E$ of a set E of finite perimeter will be vital and will be resolved by a reasoning interesting in its own right: The argument is based on the construction of a sub-unit extension $\sigma_E \in \mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ of a unit normal vector field to $\partial^* E$ with $\operatorname{div}\sigma_E \in \mathrm{L}^1(\mathbb{R}^n)$ and then reads off the condition of Theorem 1.4 for $\mu = 2\mathcal{H}^{n-1} \llcorner \partial^* E$ from Gauss-Green formulas which involve weak normal traces of σ_E. While, for smooth ∂E, the field σ_E can be obtained by more elementary means, in the general case we rely on the theory and construction of an optimal variational mean curvature $H_E \in \mathrm{L}^1(\mathbb{R}^n)$ of E due to Barozzi & Gonzalez & Tamanini [3] and Barozzi [2], read off a certain auxiliary IC for H_E, and only then deduce the existence of σ_E with $\operatorname{div}\sigma_E = H_E$.

We will postpone most of the more detailed discussion on reformulations and further properties of ICs to the latter sections. However, already at this stage we wish to mention one more specific property of the small-volume IC, since it came quite unexpected and allows to obtain further examples of measures admissible in our theory from those already discussed:

Proposition 1.6 (small-volume IC for the sum of singular measures)**.** *Consider non-negative Radon measures μ_1 and μ_2 on $\mathbb{R}^n$ such that μ_1 and μ_2 are singular to each other and least one of μ_1 and μ_2 is finite. If μ_1 and μ_2 both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1, then $\mu_1{+}\mu_2$ satisfies the small-volume IC in $\mathbb{R}^n$ still with the same constant 1 (and not merely in the evident way with an additional multiplicative factor 2 in front of the perimeter).*

The proof of Proposition 1.6 will be given in Section 7 and is based on a certain relative-perimeter characterization of the small-volume IC and an elementary separation argument.

On the usage of ICs and related results in the literature. To the state of our knowledge, the precise form of our small-volume IC and its flexibility, as underlined by Theorem 1.5, are new. Nevertheless, related linear ICs have been around in the theory of prescribed mean curvature surfaces for a long time, and thus we now comment on the previous literature in some more detail.

In fact, ICs have been prominently used in the theory of *non-parametric* prescribed-mean-curvature functionals, which correspond to $\mathscr{P}_H[A;\Omega]$ from (1.1) for subgraphs A and $\Omega = D \times \mathbb{R}$ with a bounded

Lipschitz domain $D \subset \mathbb{R}^{n-1}$. However, the considerations on such functionals in [32, 19, 18, 17, 21] differ from ours, since e.g. the assumptions in [17] imply (in the terminology of our setting) $\partial_n H \le 0$, $H(\,\cdot\,,0) \in \mathrm{L}^{n-1}(D)$ and the settings of the other papers tend in similar, but rather more restrictive directions. In any case, these works exclude cancellation in the previously described sense, and thus the perimeter and the H-volume are even separately lower semicontinuous for basic reasons and without need for imposing an IC. In fact, in these non-parametric cases it is not semicontinuity but rather coercivity of the problem which is obtained from stronger ICs of type

$$\left| \int_A H(\overline{x},0)\,\mathrm{d}\overline{x} \right| \le C\mathrm{P}(A,\mathbb{R}^{n-1}) \qquad \text{for all measurable } A \subset D\,, \qquad \text{with fixed } C \in [0,1)\,. \tag{1.5}$$

When comparing with our results, the need for assuming (1.5) may be viewed as a result of considering on the unbounded cylinder $D \times \mathbb{R}$ an infinite measure $H\mathcal{L}^n$, and analogous conditions occur also in our theory when later addressing the existence issue with infinite measures in Theorems 5.1 and 6.4. Moreover, in case $H(\overline{x},x_n) = H_0(\overline{x})$, having (1.5) with $C = 1$ is also necessary for classical solvability of the prescribed mean curvature equation $-\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big) = H_0$ (compare with [21] for finer related discussion). It is not clear to us if there is an effective necessary condition of a similar type also for general H with x_n-dependence.

Still in the non-parametric framework, a direction partially analogous to ours has been pursued in [46, 7, 8]: Indeed, Ziemer [46] gives an existence result for non-parametric functionals which involve a finite non-negative measure datum μ_0 with compact support in a bounded Lipschitz domain $D \subset \mathbb{R}^{n-1}$. However, his central assumption

$$\mu_0(\mathrm{B}_r(x)) \le Cr^\kappa \qquad \text{for all balls } \mathrm{B}_r(x) \subset D\,, \qquad \text{with fixed } C \in [0,\infty) \text{ and } \kappa \in (n{-}2,n{-}1) \tag{1.6}$$

is considerably stronger than a linear IC and in particular excludes the interesting borderline case of $(n{-}2)$-dimensional measures μ_0. Moreover, Dai & Trudinger & Wang [7] and Dai & Wang & Zhou [8] introduce an approximation-based notion of a mean curvature measure and establish a corresponding existence result for generalized solutions to the prescribed mean curvature equation on a smooth bounded domain $D \subset \mathbb{R}^{n-1}$ with a finite signed measure μ_0 on D as right-hand side. They require that the singular part of μ_0 has compact support in D and in analogy with (1.5) impose on μ_0 an IC of type

$$|\mu_0(A^1)| \le C\mathrm{P}(A,\mathbb{R}^{n-1}) \qquad \text{for all measurable } A \subset D\,, \qquad \text{with fixed } C \in [0,1)\,. \tag{1.7}$$

Since the settings differ, a comparison of the preceding results with ours is necessarily incomplete, but one may say that the results in [46, 7, 8] work for product measures $\mu = \mu_0 \otimes \mathcal{L}^1$ on $D \times \mathbb{R}$, while we admit general measures μ on $\Omega \subset \mathbb{R}^n$. Alternatively, from a more PDE-based viewpoint, one may put it the way that [46, 7, 8] treat right-hand sides of type $H_0(x)$ with $H_0 \in \mathrm{L}^1(D)$ replaced by a measure μ_0 on D, while for the non-parametric equations corresponding to our functionals one expects right-hand sides of type $H(x,u(x))$ (with dependence on the unknown u) with $H \in \mathrm{L}^1(D \times \mathbb{R})$ replaced by a measure μ on $D \times \mathbb{R}$. Beyond this partial comparison we stress that the approaches taken are technically very different from ours and that the works [46, 7, 8] do *not* involve any semicontinuity by cancellation. In fact, the more restrictive assumption (1.6) of [46] still ensures separate semicontinuity of the μ_0-volume, and the approach of [7, 8] works much more on the PDE side rather than the variational side of the field and does not involve semicontinuity of a functional with measure datum at all.

Finally, when a first version of this article was already finalized, an independent preprint of Leonardi & Comi [25] on non-parametric functionals closely analogous to the parametric ones in (1.3) became available. In this interesting work the authors obtain (among other results) lower semicontinuity and existence results over a bounded Lipschitz domain $D \subset \mathbb{R}^{n-1}$ in case of specific measures $\mu_0 = h\mathcal{L}^{n-1} + \gamma\mathcal{H}^{n-2} \llcorner \Gamma$ with $h \in \mathrm{L}^q(D)$, $q > n{-}1$, an $(n{-}2)$-dimensional set $\Gamma \subset D$ with bounded $(n{-}2)$-dimensional density ratio, and $\gamma \in \mathrm{L}^\infty(\Gamma;\mathcal{H}^{n-2})$ such that moreover the IC (1.7) holds. Though also these results concern the non-parametric setting and differ considerably from ours in the framework and the technical approach, we put on record that at its heart the work [25] brings up a semicontinuity-by-cancellation effect analogous to ours.

Returning to the parametric case, we point out that ICs have been introduced into the classical 2-dimensional Douglas-Radó theory of prescribed mean curvature surfaces by Steffen [40, 41]. Among the ICs considered in his work, a central type for functions $H\colon S \to \mathbb{R}$ on $S \subset \mathbb{R}^n$ reads in our terminology

$$\left| \int_A H(x)\,\mathrm{d}x \right| \le C\mathrm{P}(A,\mathbb{R}^n) \qquad \text{for all measurable } A \subset S \text{ with } H \in \mathrm{L}^1(A)\,,\ \mathrm{P}(A,\mathbb{R}^n) \le R\,, \tag{1.8}$$

where $C \in [0,1]$ and $R \in [0,\infty]$ are fixed. In the classical case with $n = 3$ such ICs are then exploited in [40, 41] in establishing lower semicontinuity of prescribed mean curvature functionals and in case $C < 1$ also existence results, where in a spirit similar to ours the ICs compensate for a lack of separate lower semicontinuity of a certain H-volume term. However, while in our theory the main issue originates from passing from functions H to measures $\mu_\pm$ and from a possible loss of a hypersurface portion in the limit, in [40, 41] an analogous issue occurs already for functions H and is connected with a typical phenomenon of the parametric theory, namely the possible bubbling-off of regions of positive volume in the limit. In addition, Duzaar [13] and Duzaar & Steffen [14] have established existence results based on ICs of type (1.8) with $C < 1$ also in Euclidean space $\mathbb{R}^n$ and in Riemannian manifolds of arbitrary dimension n by working in a general GMT framework with codimension-1 currents. However, also the results in [13, 14] are limited to functions H and not measures $\mu_\pm$ in the volume term. Yet again, since bubbling off is not an issue in the framework of currents, the role of the ICs is once more a bit different and consists mostly in preventing a breakdown of semicontinuity at ∞, as it has already been discussed and needs to be excluded in our theory as well.

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2 Preliminaries

We work in Euclidean space $\mathbb{R}^n$ of arbitrary dimension $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ (unless indicated otherwise).

Basic notation for sets and balls

Our basic notation for sets is widely standard. However, we mention that we use A^{c} for the complement of a set A (in $\mathbb{R}^n$ or in some other base set clear from the context), $A\Delta B := (A\setminus B) \cup (B\setminus A)$ for the symmetric difference of sets A and B, and $\mathbb{1}_A$ for the characteristic function of a set A with $\mathbb{1}_A \equiv 1$ on A and $\mathbb{1}_A \equiv 0$ on A^{c}. By $\overline{A}$ and $\mathrm{int}(A)$ we denote the closure and the interior, respectively, of a set A (taken once more in $\mathbb{R}^n$ or another base space). We write $A \Subset B$ if $\overline{A}$ is compact and satisfies $\overline{A} \subset B$. Moreover, we use $\mathrm{B}_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}$ for balls in $\mathbb{R}^n$, we abbreviate $\mathrm{B}_r := \mathrm{B}_r(0)$, and we denote by $\alpha_n = |\mathrm{B}_1|$ the volume of the unit ball B_1 in $\mathbb{R}^n$. Finally, for $a \in \mathbb{R}^n$, $A, B \subset \mathbb{R}^n$ we use $\mathrm{dist}(a, B) := \inf_{b\in B} |a-b|$ and $\mathrm{dist}(A, B) := \inf_{a\in A} \mathrm{dist}(a, B)$ for Euclidean distances.

Measures and convergence in measure

We write $\mathcal{B}(\mathbb{R}^n)$ for the Borel σ-algebra on the full space $\mathbb{R}^n$ and $\mathcal{B}(\Omega) = \{A \in \mathcal{B}(\mathbb{R}^n) : A \subset \Omega\}$ for the Borel σ-algebra on a Borel subset $\Omega \in \mathcal{B}(\mathbb{R}^n)$. By a non-negative Borel measure μ on a set $\Omega \in \mathcal{B}(\mathbb{R}^n)$ we mean a σ-additive set function on $\mathcal{B}(\Omega)$ with values in $[0,\infty]$. The support $\mathrm{spt}\,\mu$ of such a measure μ is the smallest closed set $S \subset \Omega$ with $\mu(S^{\mathrm{c}}) = 0$, and μ is called finite if $\mu(\Omega) < \infty$ holds. A non-negative Radon measure on an open set $\Omega \subset \mathbb{R}^n$ is a non-negative Borel measure on Ω with finite value on all compacts subsets of Ω.

Specifically, we work with the n-dimensional Lebesgue measure $\mathcal{L}^n$, which is a non-negative Radon measure on $\mathbb{R}^n$, and with the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, which is at least a non-negative Borel measure on $\mathbb{R}^n$. In case of $\mathcal{L}^n$ we also consider its extension from $\mathcal{B}(\mathbb{R}^n)$ to the completed σ-algebra $\mathcal{M}(\mathbb{R}^n)$ of Lebesgue measurable subsets of $\mathbb{R}^n$. We write $|A| := \mathcal{L}^n(A)$ for the volume of $A \in \mathcal{M}(\mathbb{R}^n)$ and generally adopt the convention that *measure-theoretic notions are taken with respect to the Lebesgue measure unless indicated otherwise*. Specifically, this applies for a.e. properties and the following convergences. For $\Omega, A_k, A \in \mathcal{M}(\mathbb{R}^n)$ we define

$$A_k \text{ converge } (\textit{globally}) \textit{ in measure} \text{ on } \Omega \text{ to } A_\infty :\Longleftrightarrow \lim_{k\to\infty} |(A_k\Delta A_\infty)\cap\Omega| = 0\,, \tag{2.1}$$

$$A_k \text{ converge } \textit{locally in measure} \text{ on } \Omega \text{ to } A_\infty :\Longleftrightarrow \lim_{k\to\infty} |(A_k\Delta A_\infty)\cap K| = 0 \text{ for all compact } K \subset \Omega\,. \tag{2.2}$$

We remark that in most of the following we will apply (2.1) and (2.2) in the standard case of open Ω only, but in fact we have intentionally given the definitions for arbitrary measurable Ω, since this more general viewpoint will become relevant for Theorem 9.1 and Corollary 9.2 in the final section of this paper. Indeed, the reasonableness of this framework is supported by the fact that just as the convergence in (2.1) also the convergence in (2.2) depends on Ω only up to negligible sets, as one can verify in case of (2.2) by a short reasoning with the inner regularity of the Lebesgue measure. Moreover, the same reasoning shows that equivalent with (2.2) is having $\lim_{k\to\infty}|(A_k\Delta A_\infty)\cap S|=0$ even for all $S\in\mathcal{M}(\mathbb{R}^n)$ with $|S\setminus\Omega|=0$ and $|S|<\infty$. Finally, we briefly remark that local convergence in measure is closely tied to almost everywhere convergence in the sense of $\lim_{k\to\infty}\mathbb{1}_{A_k}=\mathbb{1}_{A_\infty}$ a.e. on Ω: In fact, almost everywhere convergence implies local convergence in measure, and local convergence in measure implies almost everywhere convergence of a subsequence.

In connection with signed measures and vector measures we adopt mostly the conventions of [1, Sections 1.1, 1.3]. Specifically, as a signed Radon measure ν on open $\Omega\subset\mathbb{R}^n$ we consider any set function which is defined and σ-additive with finite real values (at least) on the relatively compact Borel subsets of Ω, and an $\mathbb{R}^m$-valued Radon measure is defined analogously with values in $\mathbb{R}^m$. A signed or $\mathbb{R}^m$-valued Radon measure ν on Ω is called finite if it extends to a finite-valued σ-additive set function on the full Borel σ-algebra $\mathcal{B}(\Omega)$. With these conventions the (total) variation measure $|\nu|$ of a signed or $\mathbb{R}^m$-valued Radon measure ν on Ω can always be regarded as a non-negative Radon measure on Ω (where $|\nu|$ is finite if and only if ν is finite). Moreover, every signed Radon measure ν on Ω admits a unique decomposition $\nu=\nu_+-\nu_-$ into mutually singular non-negative Radon measures ν_+ and ν_- on Ω, which also satisfy $|\nu|=\nu_++\nu_-$.

Finally, for any measure ν on a measurable space $(\Omega,\mathcal{A})$, the weighted measure $f\nu$ on $(\Omega,\mathcal{A})$ with $f\in\mathrm{L}^1(\Omega\,;\nu)$ is defined by setting $(f\nu)(A):=\int_A f\,\mathrm{d}\nu$ for all $A\in\mathcal{A}$. Specifically, the restriction measure $\nu\llcorner S$ on $(\Omega,\mathcal{A})$ with $S\in\mathcal{A}$ is obtained through $(\nu\llcorner S)(A):=(\mathbb{1}_S\nu)(A)=\nu(S\cap A)$ for all $A\in\mathcal{A}$.

Coarea formula for Lipschitz functions

For a (locally) Lipschitz function $\Omega\to\mathbb{R}$ on open $\Omega\subset\mathbb{R}^n$, Rademacher's theorem guarantees the existence of the derivative $\nabla u(x)\in\mathbb{R}^n$ at a.e. $x\in\Omega$; compare e.g. with [1, Section 2.3], [15, Section 3.1], [26, Section 7.3], or [29, Theorem 7.3]. With the derivative at hand the coarea formula for Lipschitz functions can then be stated as follows.

Theorem 2.1 (coarea formula for Lipschitz functions)**.** *Consider a Lipschitz function $u\colon\Omega\to\mathbb{R}$ on open $\Omega\subset\mathbb{R}^n$. Then we have*

$$\int_A|\nabla u|\,\mathrm{d}x=\int_{-\infty}^{\infty}\mathcal{H}^{n-1}(A\cap\{u=t\})\,\mathrm{d}t\qquad\text{for all }A\in\mathcal{B}(\Omega)\,.$$

For the proof (of actually more general statements) we refer to [1, Section 2.12], [15, Section 3.4], or [26, Section 18.1], for instance.

Sets of finite perimeter (and BV functions)

In working with spaces of integrable and weakly differentiable functions such as $\mathrm{L}^p_{(\mathrm{loc})}(\Omega)$, $\mathrm{W}^{1,p}_{(\mathrm{loc})}(\Omega)$, $\mathrm{BV}_{(\mathrm{loc})}(\Omega)$ we follow once more the terminology of [1]. In particular, for a real-valued BV function $u\in\mathrm{BV}_{\mathrm{loc}}(\Omega)$ on open $\Omega\subset\mathbb{R}^n$, we write $\mathrm{D}u$ for the $\mathbb{R}^n$-valued Radon measure which represents the distributional gradient of u on Ω. Moreover, we generally use $u_\pm:=\max\{\pm u,0\}$ for the positive and negative part of functions, but we directly warn the reader that in addition to this convention with *lower* indices $_\pm$ we will soon introduce *upper* indices $^\pm$ for certain approximate limits as well.

We introduce the perimeter $\mathrm{P}(A,\Omega)$ of a measurable set $A\in\mathcal{M}(\mathbb{R}^n)$ in an arbitrary Borel set $\Omega\in\mathcal{B}(\mathbb{R}^n)$ by setting $\mathrm{P}(A,\Omega):=|\mathrm{D}\mathbb{1}_A|(\Omega)$ whenever there exists an open neighborhood U of Ω in $\mathbb{R}^n$ such that $\mathbb{1}_A\in\mathrm{BV}_{\mathrm{loc}}(U)$ and by trivially setting $\mathrm{P}(A,\Omega):=\infty$ otherwise. For open Ω this coincides with more standard distributional definitions, while in general we have $\mathrm{P}(A,\Omega)=\inf\{\mathrm{P}(A,U)\,:\,U\text{ open neighborhood of }\Omega\text{ in }\mathbb{R}^n\}$. As usual we abbreviate $\mathrm{P}(A):=\mathrm{P}(A,\mathbb{R}^n)$.

We next record two standard results, where the former can be inferred from [1, Theorem 3.39] or [26, Corollary 12.27], and the later from [1, Proposition 3.38(b)], [15, Theorem 5.2], or [26, Proposition 12.15].

Lemma 2.2 (compactness from perimeter bounds)**.** *Consider an open set* $\Omega \subset \mathbb{R}^n$*. If* $(A_k)_{k\in\mathbb{N}}$ *is a sequence in* $\mathcal{M}(\mathbb{R}^n)$ *with* $\sup_{k\in\mathbb{N}} \mathrm{P}(A_k,\Omega) < \infty$*, then a subsequence of* $(A_k)_{k\in\mathbb{N}}$ *converges locally in measure on* Ω *to some limit* $A_\infty \in \mathcal{M}(\mathbb{R}^n)$*.*

Lemma 2.3 (lower semicontinuity of the perimeter)**.** *Consider an open set* $\Omega \subset \mathbb{R}^n$*. If a sequence* $(A_k)_{k\in\mathbb{N}}$ *in* $\mathcal{M}(\mathbb{R}^n)$ *converges locally in measure on* Ω *to* $A_\infty \in \mathcal{M}(\mathbb{R}^n)$*, then we have*

$$\liminf_{k\to\infty} \mathrm{P}(A_k,\Omega) \geq \mathrm{P}(A_\infty,\Omega)\,.$$

Whenever we have $\mathrm{P}(A,\Omega) < \infty$ for $A \in \mathcal{M}(\mathbb{R}^n)$ and $\Omega \in \mathcal{B}(\mathbb{R}^n)$, we call A a set of finite perimeter in Ω, and we write the class of sets of finite measure and finite perimeter in Ω as

$$\mathcal{BV}(\Omega) := \{A \in \mathcal{M}(\mathbb{R}^n) \,:\, |A\cap\Omega| + \mathrm{P}(A,\Omega) < \infty\} = \bigcup_{U \text{ open},\, \Omega\subset U} \{A \in \mathcal{M}(\mathbb{R}^n) \,:\, \mathbb{1}_A \in \mathrm{BV}(U)\}\,.$$

Moreover, we call $A \in \mathcal{M}(\mathbb{R}^n)$ a set of locally finite perimeter in open $\Omega \subset \mathbb{R}^n$ if $\mathrm{P}(A,K) < \infty$ holds for all compact $K \subset \Omega$. The corresponding class is written, still for open Ω, as

$$\mathcal{BV}_{\mathrm{loc}}(\Omega) := \{A \in \mathcal{M}(\mathbb{R}^n) \,:\, \mathrm{P}(A,K) < \infty \text{ for all compact } K \subset \Omega\} = \{A \in \mathcal{M}(\mathbb{R}^n) \,:\, \mathbb{1}_A \in \mathrm{BV}_{\mathrm{loc}}(\Omega)\}\,,$$

The reduced boundary of $A \in \mathcal{BV}(\Omega)$ in $\Omega \in \mathcal{B}(\mathbb{R}^n)$ in the sense of [1, Definition 3.54], [15, Definition 5.4], [26, Section 15] is denoted by ∂^*A or by $\Omega\cap\partial^*A$. Its significance is partially highlighted by the following result, which can be read off from [1, Theorem 3.59], [15, Theorem 5.15], or [26, Theorem 15.9].

Theorem 2.4 (De Giorgi's structure theorem; partial statement)**.** *For* $A \in \mathcal{M}(\mathbb{R}^n)$ *and* $\Omega \in \mathcal{B}(\mathbb{R}^n)$ *with* $\mathrm{P}(A,\Omega) < \infty$*, it holds*

$$\mathrm{P}(A,\,\cdot\,) = |\mathrm{D}\mathbb{1}_A| = \mathcal{H}^{n-1} \llcorner \partial^*A \qquad \textit{as measures on } \Omega\,.$$

With this result in mind, from here on we mostly use $\mathrm{P}(A,\,\cdot\,)$ as the preferred notation for the perimeter measure of a set A of (locally) finite perimeter.

In view of the conventions for BV functions and $\mathcal{BV}$ sets we can also state a variant of the coarea formula, which is contained in e.g. [1, Theorem 3.40] or [15, Theorem 5.9].

Theorem 2.5 (Fleming-Rishel coarea formula)**.** *Consider an open set* $\Omega \subset \mathbb{R}^n$ *and* $u \in \mathrm{BV}(\Omega)$*. Then, for* $\mathcal{L}^1$*-a.e.* $t \in \mathbb{R}$*, we have* $\{u > t\} \in \mathcal{BV}(\Omega)$*, and it holds*

$$|\mathrm{D}u|(A) = \int_{-\infty}^{\infty} \mathrm{P}(\{u>t\},A)\,\mathrm{d}t \qquad \textit{for all } A \in \mathcal{B}(\Omega)\,.$$

Finally, we use the following result, which in this form is provided by [1, Theorem 3.46], for instance.

Theorem 2.6 (isoperimetric estimate)**.** *For* $n \geq 2$ *and* $A \in \mathcal{M}(\mathbb{R}^n)$*, we have*

$$\min\{|A|,|A^{\mathrm{c}}|\} \leq \Gamma_n \mathrm{P}(A)^{\frac{n}{n-1}}$$

with a constant $\Gamma_n > 0$ *which depends only on* n*. Evidently, in case* $|A| < \infty$ *this reduces to* $|A| \leq \Gamma_n \mathrm{P}(A)^{\frac{n}{n-1}}$*.*

With the determination of the optimal constant $\Gamma_n = \mathrm{P}(\mathrm{B}_1)^{-\frac{n}{n-1}}|\mathrm{B}_1| = \mathrm{P}(\mathrm{B}_r)^{-\frac{n}{n-1}}|\mathrm{B}_r|$, the preceding statement turns into the isoperimetric inequality

$$\mathrm{P}(\mathrm{B}_r) \leq \mathrm{P}(A) \qquad \text{for } r \in (0,\infty) \text{ and all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } |A| = |\mathrm{B}_r|\,; \tag{2.3}$$

for a proof see [26, Chapter 14], for instance. For the purposes of this paper we need (2.3) only at a single point in the proof of Theorem 5.2, while otherwise the estimate of Theorem 2.6 with any constant Γ_n suffices.

Finally, we record the following basic estimate (which has also variants for sets with *finite* $\mathcal{H}^{n-1}$-measure):

Lemma 2.7. *For every* $\mathcal{H}^{n-1}$*-negligible* $N \in \mathcal{B}(\mathbb{R}^n)$ *and every* $\varepsilon > 0$*, there exists an open set* A *such that*

$$N \subset A \subset \mathcal{N}_\varepsilon(N)\,, \qquad |A| < \varepsilon\,, \qquad \textit{and} \qquad \mathrm{P}(A) < \varepsilon$$

(with the ε*-neighborhood* $\mathcal{N}_\varepsilon(N) := \{x \in \mathbb{R}^n : \mathrm{dist}(x,N) < \varepsilon\}$ *of* N*).*

Proof. By definition of $\mathcal{H}^{n-1}$, there exist open balls $B_i \subset \mathcal{N}_\varepsilon(N)$ with corresponding radii $r_i \in (0,n]$ such that $N \subset \bigcup_{i=1}^\infty B_i$ and $n\alpha_n \sum_{i=1}^\infty r_i^{n-1} < \varepsilon$ hold. For the open set $A := \bigcup_{i=1}^\infty B_i$ with $N \subset A \subset \mathcal{N}_\varepsilon(N)$, we get

$$|A| \le \sum_{i=1}^\infty |B_i| = \alpha_n \sum_{i=1}^\infty r_i^n \le n\alpha_n \sum_{i=1}^\infty r_i^{n-1} < \varepsilon \qquad \text{and} \qquad \mathrm{P}(A) \le \sum_{i=1}^\infty \mathrm{P}(B_i) = n\alpha_n \sum_{i=1}^\infty r_i^{n-1} < \varepsilon\,.$$

This completes the proof. □

$\mathcal{H}^{n-1}$-a.e. representatives and set operations for sets of finite perimeter

For $A \in \mathcal{M}(\mathbb{R}^n)$, $\vartheta \in [0,1]$ we introduce the Borel sets

$$A^\vartheta := \left\{ x \in \mathbb{R}^n \,:\, \lim_{\varrho \searrow 0} \frac{|\mathrm{B}_\varrho(x) \cap A|}{|\mathrm{B}_\varrho|} = \vartheta \right\} \qquad \text{and} \qquad A^+ := \big(A^0\big)^{\mathrm{c}} = \left\{ x \in \mathbb{R}^n \,:\, \limsup_{\varrho \searrow 0} \frac{|\mathrm{B}_\varrho(x) \cap A|}{|\mathrm{B}_\varrho|} > 0 \right\}$$

of density-ϑ points and positive-upper-density points of A, and we record that $A^1 = A^+ = A$ holds up to negligible sets (see e.g. by [15, Theorem 1.35], [26, eq. (5.19)], or [29, Corollary 2.14(1)]). More can be said in case A has finite perimeter: Then the A^ϑ are significant only for $\vartheta \in \{0, \frac{1}{2}, 1\}$, and the essential boundary

$$\partial^{\mathrm{e}} A := A^+ \setminus A^1$$

is not only negligible, but in fact coincides with the reduced boundary $\partial^* A$ up to an $\mathcal{H}^{n-1}$-negligible sets. In fact, this is made precise in the next result, for which we refer to [1, Theorem 3.61] or [26, Theorem 16.2].

Theorem 2.8 (Federer's structure theorem)**.** *For $A \in \mathcal{M}(\mathbb{R}^n)$, $\Omega \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A,\Omega) < \infty$, there hold $\Omega \cap \partial^* A \subset A^{\frac{1}{2}}$ and*

$$\mathcal{H}^{n-1}((\partial^{\mathrm{e}} A \setminus \partial^* A) \cap \Omega) = 0$$

In particular, in the situation of the theorem we infer $\mathcal{H}^{n-1}(A^\vartheta \cap \Omega) = 0$ for all $\vartheta \in [0,1] \setminus \{0, \frac{1}{2}, 1\}$, and the equalities $\partial^* A \cap \Omega = A^{\frac{1}{2}} \cap \Omega = \partial^{\mathrm{e}} A \cap \Omega$ and $A^+ \cap \Omega = (A^1 \cup \partial^* A) \cap \Omega$ hold up to $\mathcal{H}^{n-1}$-negligible sets. Altogether this supports viewing A^+ as measure-theoretic closure and A^1 as measure-theoretic interior of A.

Next we discuss basic set operations and corresponding estimates for sets of finite perimeter.

Lemma 2.9. *For $A, B \in \mathcal{M}(\mathbb{R}^n)$, $\Omega \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A,\Omega) + \mathrm{P}(B,\Omega) < \infty$, there holds*

$$\mathrm{P}(A \cap B, G) \le \mathrm{P}(A, B^1 \cap G) + \mathrm{P}(B, A^+ \cap G) \qquad \textit{for all } G \in \mathcal{B}(\Omega) \tag{2.4}$$

and in particular $\mathrm{P}(A \cap B, \Omega) < \infty$. If either $|(A \setminus B) \cap G| = 0$ or $|(B \setminus A) \cap G| = 0$ or $\mathcal{H}^{n-1}(\partial^* A \cap \partial^* B \cap G) = 0$ holds, then we have equality in (2.4).

Similarly, for $A, S \in \mathcal{M}(\mathbb{R}^n)$, $\Omega \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A,\Omega) + \mathrm{P}(S,\Omega) < \infty$, there holds

$$\mathrm{P}(A \setminus S, G) \le \mathrm{P}(A, S^0 \cap G) + \mathrm{P}(S, A^+ \cap G) \qquad \textit{for all } G \in \mathcal{B}(\Omega) \tag{2.5}$$

and in particular $\mathrm{P}(A \setminus S, \Omega) < \infty$. If either $|A \cap S \cap G| = 0$ or $|G \setminus (A \cup S)| = 0$ or $\mathcal{H}^{n-1}(\partial^* A \cap \partial^* S \cap G) = 0$ holds, then we have equality in (2.5).

Proof. We observe that $\mathrm{P}(A \cap B, \Omega) < \infty$ is ensured, for instance, by applying the basic product rule estimate [1, eq. (3.10)] for the derivative of $\mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B$. Now we consider $x \in (A^1 \cup A^{\frac{1}{2}} \cup A^0) \cap (B^1 \cup B^{\frac{1}{2}} \cup B^0)$. Then $x \in (A \cap B)^{\frac{1}{2}}$ necessarily implies that either $x \in A^{\frac{1}{2}} \cap B^1$ or $x \in B^{\frac{1}{2}} \cap A^+$ holds. In view of Theorem 2.8 this means $\partial^*(A \cap B) \subset (\partial^* A \cap B^1) \cup (\partial^* B \cap A^+)$ up to $\mathcal{H}^{n-1}$-negligible sets, and via Theorem 2.4 we arrive at (2.4). In order to discuss equality, one can use the full statement of De Giorgi's theorem as provided in [1, Theorem 3.59] to verify more precisely $\partial^*(A \cap B) = (\partial^* A \cap B^1) \cup (\partial^* B \cap A^1) \cup (\partial^* A \cap \partial^* B \cap \{\nu_A = \nu_B\})$ up to $\mathcal{H}^{n-1}$-negligible sets, where ν_A and ν_B denote the generalized outward unit normals of A and B. Then one reads off that equality occurs in (2.4) if and only if $\nu_A = \nu_B$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial^* A \cap \partial^* B \cap G$, and the latter can be checked to follow from each of the conditions claimed to be sufficient for equality.

We find worth recording also the following alternative derivation of (2.4). From the rule for the derivative of composite functions in [1, Theorem 3.84] we get

$$\mathrm{P}(A\cap B,G)=|\mathrm{D}(\mathbb{1}_A\mathbb{1}_B)|(G)=|\mathrm{D}\mathbb{1}_A|(B^1\cap G)+\big(|(\mathbb{1}_A)^{\mathrm{int}}_{\partial^*B}|\mathcal{H}^{n-1}\big)(\partial^*B\cap G)\qquad\text{for } G\in\mathcal{B}(\Omega)$$

and specifically $\mathrm{P}(A\cap B,\Omega)<\infty$, where $(\mathbb{1}_A)^{\mathrm{int}}_{\partial^*B}$ stands for the interior trace of $\mathbb{1}_A$ on ∂^*B. Since the trace is $\{0,1\}$-valued with value 1 on $A^1\cap\partial^*B$ and value 0 on $A^0\cap\partial^*B=(A^+)^{\mathrm{c}}\cap\partial^*B$, with the help of Theorem 2.4 we obtain

$$\mathrm{P}(A\cap B,G)\le|\mathrm{D}\mathbb{1}_A|(B^1\cap G)+\mathcal{H}^{n-1}(\partial^*B\cap A^+\cap G)=\mathrm{P}(A,B^1\cap G)+\mathrm{P}(B,A^+\cap G)\qquad\text{for } G\in\mathcal{B}(\Omega)$$

and arrive once more at (2.4). From these arguments one reads off that equality occurs in (2.4) if and only if $(\mathbb{1}_A)^{\mathrm{int}}_{\partial^*B}=1$ holds $\mathcal{H}^{n-1}$-a.e. on $(A^+\setminus A^1)\cap\partial^*B\cap G$. In view of Theorem 2.8 it is equivalent that $(\mathbb{1}_A)^{\mathrm{int}}_{\partial^*B}=1$ holds $\mathcal{H}^{n-1}$-a.e. on $\partial^*A\cap\partial^*B\cap G$, and once more this can be checked to follow from each of the conditions in the statement.

Finally, the inequality (2.5) is nothing but the inequality (2.4) for $B=S^{\mathrm{c}}$. □

Also the following combined estimate for the perimeters of union and intersection is well known.

Lemma 2.10. *For $A,B\in\mathcal{M}(\mathbb{R}^n)$, $\Omega\in\mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A,\Omega)+\mathrm{P}(B,\Omega)<\infty$, we have*

$$\mathrm{P}(A\cup B,G)+\mathrm{P}(A\cap B,G)\le\mathrm{P}(A,G)+\mathrm{P}(B,G)\qquad\text{for all } G\in\mathcal{B}(\Omega)\tag{2.6}$$

and thus in particular $\mathrm{P}(A\cup B,\Omega)+\mathrm{P}(A\cap B,\Omega)<\infty$.

Proofs. A basic approach is given in the proofs of [1, Proposition 3.38(d)] and [26, Lemma 12.22], where the claim is shown for open G by approximating $\mathbb{1}_A$ and $\mathbb{1}_B$ with smooth functions. Our claim for arbitrary $G\in\mathcal{B}(\Omega)$ then follows by regularity of the perimeter measures.

Alternatively, one may obtain the lemma from the equality $|\mathrm{D}u_+|+|\mathrm{D}u_-|=|\mathrm{D}u|$ for $u\in\mathrm{BV}_{\mathrm{loc}}(U)$ on open $U\subset\mathbb{R}^n$ (which in turn results from an approximation argument somewhat similar to the previously mentioned one). In fact, using the equality for $u:=\mathbb{1}_A+\mathbb{1}_B-1$ with $u_+=\mathbb{1}_{A\cap B}$ and $u_-=1-\mathbb{1}_{A\cup B}$ we directly obtain $\mathrm{P}(A\cap B,G)+\mathrm{P}(A\cup B,G)=|\mathrm{D}u_+|(G)+|\mathrm{D}u_-|(G)=|\mathrm{D}u|(G)\le\mathrm{P}(A,G)+\mathrm{P}(B,G)$.

Finally, we find worth recording that the claim can also be derived from the preceding Lemma 2.9. Indeed, elementary rules for complements and (2.4) with B^{c} in place of A and A^{c} in place of B yield

$$\mathrm{P}(A\cup B,G)=\mathrm{P}(B^{\mathrm{c}}\cap A^{\mathrm{c}},G)\le\mathrm{P}(B^{\mathrm{c}},(A^{\mathrm{c}})^1\cap G)+\mathrm{P}(A^{\mathrm{c}},(B^{\mathrm{c}})^+\cap G)=\mathrm{P}(B,(A^+)^{\mathrm{c}}\cap G)+\mathrm{P}(A,(B^1)^{\mathrm{c}}\cap G)\,.$$

Summing up the original version of (2.4) and the variant just derived, we arrive at (2.6) once more. □

Pseudoconvexity

Pseudoconvexity, a weak version of mean-convexity, has been introduced by Miranda [31] and will eventually be relevant for us in connection with the discussion of a basic example. We restate the definition and a first lemma in versions adapted to our framework.

Definition 2.11 (pseudoconvexity)**.** *We say that $K\in\mathcal{BV}(\mathbb{R}^n)$ is pseudoconvex if it satisfies*

$$\mathrm{P}(K)\le\mathrm{P}(B)\qquad\text{whenever } K\subset B\in\mathcal{M}(\mathbb{R}^n)\text{ with }|B|<\infty\,.\tag{2.7}$$

Lemma 2.12. *For every pseudoconvex set $K\in\mathcal{BV}(\mathbb{R}^n)$, we have*

$$\mathrm{P}(A\cap K)\le\mathrm{P}(A)\qquad\text{for all } A\in\mathcal{M}(\mathbb{R}^n)\text{ with }|A|<\infty\,.$$

Proof. From (2.6) and the definition of pseudoconvexity, applied with $B=A\cup K$, we get

$$\mathrm{P}(A\cap K)\le\mathrm{P}(A)+\mathrm{P}(K)-\mathrm{P}(A\cup K)\le\mathrm{P}(A)\,.$$ □

Clearly, a basic feature of pseudoconvexity is that convex sets are pseudoconvex. Though this may be considered as geometrically quite obvious, we prefer to sketch at least one possible precise proof.

Lemma 2.13 (convexity implies pseudoconvexity)**.** *Every bounded, convex set* $K \in \mathcal{M}(\mathbb{R}^n)$ *with* $\mathrm{int}(K) \neq \emptyset$ *satisfies* $K \in \mathcal{BV}(\mathbb{R}^n)$ *with* $\mathcal{H}^{n-1}(\partial K \setminus \partial^* K) = 0$ *and is actually pseudoconvex.*

Sketch of proof. The claims $K \in \mathcal{BV}(\mathbb{R}^n)$ and $\mathcal{H}^{n-1}(\partial K \setminus \partial^* K) = 0$ follow from [1, Proposition 3.62]. We now establish the inequality (2.7) for the convex set K, at first only with the extra assumption that B is a bounded C^1 domain. Indeed, for every $x \in \partial K$, moving from x in an outward normal direction, we find some $y \in \partial B = \partial^* B$ with $p_{\overline{K}}(y) = x$ for the nearest-point projection $p_{\overline{K}} \colon \mathbb{R}^n \to \overline{K}$ onto $\overline{K}$. This shows $\partial K \subset p_{\overline{K}}(\partial^* B)$. Then, since $p_{\overline{K}}$ is a contraction, we get $\mathrm{P}(K) = \mathcal{H}^{n-1}(\partial K) \leq \mathcal{H}^{n-1}(\partial^* B) = \mathrm{P}(B)$ as claimed. In a next step, we weaken the extra assumption to merely $B \in \mathcal{BV}(\mathbb{R}^n)$ and show that (2.7) still applies. To this end we approximate B with bounded C^1 domains B_ℓ such that $\lim_{\ell \to \infty} \mathrm{P}(B_\ell) = \mathrm{P}(B)$ as in [1, Theorem 3.42], where we can additionally arrange for $K_\ell \subset B_\ell$ with the bounded, convex sets $K_\ell := \{x \in \mathbb{R}^n : \mathrm{dist}(x, K^{\mathrm{c}}) > \varepsilon_\ell\}$, suitable $\varepsilon_\ell > 0$, and $\lim_{\ell \to \infty} \varepsilon_\ell = 0$. As we infer $\liminf_{\ell \to \infty} \mathrm{P}(K_\ell) \geq \mathrm{P}(K)$ by Lemma 2.3, we can then carry over (2.7) from K_ℓ and B_ℓ to K and B as claimed. Finally, we deduce (2.7) in full generality by approximating B with $B \cap \mathrm{B}_R$ and exploiting the convergence $\liminf_{R \to \infty} \mathrm{P}(B \cap \mathrm{B}_R) = \mathrm{P}(B)$ (which in turn results from Lemma 2.3, the estimate $\mathrm{P}(B \cap \mathrm{B}_R) \leq \mathrm{P}(B) + \mathcal{H}^{n-1}(B^1 \cap \partial \mathrm{B}_R)$, and $\int_0^\infty \mathcal{H}^{n-1}(B^1 \cap \partial \mathrm{B}_R)\,\mathrm{d}R = |B| < \infty$). ☐

$\mathcal{H}^{n-1}$-a.e. representatives of BV functions

For measurable $u \colon \Omega \to \mathbb{R}$ on open $\Omega \subset \mathbb{R}^n$, by taking the approximate upper and lower limits in the sense of

$$u^+(x) := \sup\{t \in \mathbb{R} : x \in \{u > t\}^+\} \qquad \text{and} \qquad u^-(x) := \sup\{t \in \mathbb{R} : x \in \{u > t\}^1\} \qquad \text{for } x \in \Omega$$

(where as usual $\sup \emptyset := -\infty$) we obtain two extended-real-valued Borel functions $u^+ \geq u^-$ on Ω. Occasionally we also work with their arithmetic mean $u^* := \frac{1}{2}(u^+ + u^-)$. We record that, whenever u has value $y_0 \in \mathbb{R}$ at a Lebesgue point $x_0 \in \Omega$ (in the sense that $\lim_{r \searrow 0} |\mathrm{B}_r|^{-1} \int_{\mathrm{B}_r(x_0)} |u - y_0|\,\mathrm{d}x = 0$), then $u^*(x_0) = u^+(x_0) = u^-(x_0) = y_0$ holds. Hence, it follows from [1, Corollary 2.23] that in case of $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ the representatives u^+, u^-, u^* of u coincide a.e. on Ω. Moreover, as a consequence of the Federer-Volpert theorem (see e.g. [1, Theorem 3.78]), for $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$ the coincidence $u^* = u^+ = u^-$ stays valid even $\mathcal{H}^{n-1}$-a.e. on Ω, and for $u \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ one has $u^* = u^+ = u^-$ at least $\mathcal{H}^{n-1}$-a.e. on $\Omega \setminus \mathrm{J}_u$, while on the approximate jump set J_u the values u^+ and u^- correspond $\mathcal{H}^{n-1}$-a.e. to the two jump values in the sense of [1, Definition 3.67]. In particular, for $A \in \mathcal{M}(\mathbb{R}^n)$, $\Omega \in \mathcal{B}(\mathbb{R}^n)$ with $\mathrm{P}(A, \Omega) < \infty$, we observe that $(\mathbb{1}_A)^+ = \mathbb{1}_{A^+}$ holds $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$.

1-capacity

A decisive role in at least one central proof of this paper is taken by 1-capacity, also known as BV-capacity, in the sense of the next definition.

Definition 2.14 (1-capacity)**.** *For an arbitrary set* $E \subset \mathbb{R}^n$*, we define*

$$\mathrm{Cap}_1(E) := \inf\left\{ \int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x \,:\, u \in \mathrm{W}^{1,1}(\mathbb{R}^n)\,,\, u \geq 1 \textit{ a.e. on an open neighborhood of } E \right\} \in [0, \infty]$$

(*with the usual understanding that* $\mathrm{Cap}_1(E) = \infty$ *if no such* u *exists, as, for instance, in case* $|E| = \infty$).

The geometric meaning of 1-capacity is captured by the following result.

Proposition 2.15 (perimeter characterization of 1-capacity)**.** *For every set* $E \subset \mathbb{R}^n$*, we have*

$$\mathrm{Cap}_1(E) = \inf\{\mathrm{P}(H) : H \in \mathcal{BV}(\mathbb{R}^n)\,,\, E \subset H^+\}\,.$$

Proof. By [6, Theorem 2.1], the claim holds with the inclusion $E \subset H^+$ replaced either by $E \subset \mathrm{int}(H)$ (for any pointwise representative of H) or by $\mathcal{H}^{n-1}(E \setminus H^+) = 0$. Since we trivially have $E \subset \mathrm{int}(H) \implies E \subset H^+ \implies \mathcal{H}^{n-1}(E \setminus H^+) = 0$, the claimed intermediate version of the formula follows. (In fact, taking into account Lemma 2.7, the claimed version can alternatively be deduced from the version with $\mathcal{H}^{n-1}(E \setminus H^+) = 0$ only.) ☐

The following result from [16, Section 4] can also be found in [6, Proposition 2.2(f)] and [15, Theorem 5.12], for instance (where the latter statement is made for $n \geq 2$ and compact sets, but easily extends to the remaining cases).

Proposition 2.16. *For $S \in \mathcal{B}(\mathbb{R}^n)$, we have*

$$\mathrm{Cap}_1(S) = 0 \iff \mathcal{H}^{n-1}(S) = 0\,.$$

Finally, we record a continuity property of weakly differentiable functions, where our localized claim easily follows from the original statements established in [16, Section 9, 10] for full-space case. Alternatively, our statement may be viewed as a consequence of the *semi*continuity property provided by [6, Theorem 2.5] and the $\mathcal{H}^{n-1}$-a.e. coincide $u^* = u^+ = u^-$ for $\mathrm{W}^{1,1}$ functions u.

Lemma 2.17 (quasi continuity of a $\mathrm{W}^{1,1}$ function)**.** *For open $\Omega \subset \mathbb{R}^n$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega)$, the representative u^* of u is Cap_1-quasi continuous, that is, for every $\varepsilon > 0$, there exists an open set $E \subset \Omega$ with $\mathrm{Cap}_1(E) < \varepsilon$ such that u^* is defined and continuous on E^{c}.*

Strict and $\mathcal{H}^{n-1}$-a.e. convergence and approximation

Lemma 2.18 (one-sided $\mathcal{H}^{n-1}$-a.e. approximation of a BV function)**.** *For every $u \in \mathrm{BV}(\mathbb{R}^n)$, there exists a sequence of functions $v_\ell \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $v_{\ell+1} \leq v_\ell$ holds a.e. on $\mathbb{R}^n$ for all $\ell \in \mathbb{N}$ and v_ℓ^* converge $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$ to u^+. If u is bounded from above, one can additionally achieve $\sup_{\mathbb{R}^n} v_1 \leq \sup_{\mathbb{R}^n} u$.*

The main claim of Lemma 2.18 follows, for instance, by combining [6, Theorem 2.5] and [9, Lemma 1.5, Section 6]; compare also [16, Section 4, Section 10]. The additional boundedness assertion can be obtained by passage to the pointwise minimum of v_ℓ and $\sup_{\mathbb{R}^n} u$.

Lemma 2.19 (strong convergence in $\mathrm{W}^{1,1}$ implies $\mathcal{H}^{n-1}$-a.e. convergence)**.** *If v_ℓ converge to v in $\mathrm{W}^{1,1}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$, then v_ℓ^* converge $\mathcal{H}^{n-1}$-a.e. on Ω to v^*.*

The case $\Omega = \mathbb{R}^n$ of Lemma 2.19 is contained in [16, Section 10] (where in view of Theorem 2.16 we may use $\mathcal{H}^{n-1}$ instead of Cap_1). Since the claim can be localized, one may pass to general domains Ω by simple cut-off arguments.

Definition 2.20 (strict convergence in BV)**.** *We say that a sequence of functions $u_\ell \in \mathrm{BV}(\Omega)$ converges strictly in $\mathrm{BV}(\Omega)$ to $u \in \mathrm{BV}(\Omega)$ if u_ℓ converge to u in $\mathrm{L}^1(\Omega)$ with $\lim_{\ell\to\infty} |\mathrm{D}u_\ell|(\Omega) = |\mathrm{D}u|(\Omega)$.*

The following statement slightly adapts the one-sided approximation result of [6, Theorem 3.3] in order to additionally preserve boundedness of the support and possibly the function itself.

Lemma 2.21 (one-sided strict approximation of a BV function)**.** *Consider an open set $\Omega \subset \mathbb{R}^n$ and $u \in \mathrm{BV}(\Omega)$ with $\mathrm{spt}\, u \Subset \Omega$. Then there exists a sequence of functions $v_k \in \mathrm{W}^{1,1}(\Omega)$ such that v_k converge strictly in $\mathrm{BV}(\Omega)$ to u with $\mathrm{spt}\, v_k \Subset \Omega$ and $v_k \geq u$ a.e. on Ω for all $k \in \mathbb{N}$. If u is bounded from above, one can additionally achieve $\sup_\Omega v_k \leq \max\{0, \sup_\Omega u\}$ for all $k \in \mathbb{N}$.*

Proof. Since $\mathrm{spt}\, u$ is compact in Ω, there is no loss of generality in assuming boundedness of Ω. Then, by [6, Theorem 3.3], there exist $w_k \in \mathrm{W}^{1,1}(\Omega)$ such that w_k converge strictly in $\mathrm{BV}(\Omega)$ to u with $w_k \geq u$ a.e. on Ω for all $k \in \mathbb{N}$ (where in fact the convergence in area guaranteed by [6, Theorem 3.3] is even stronger than the strict convergence of Definition 2.20). We now fix a cut-off function $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\mathbb{1}_{\mathrm{spt}\, u} \leq \eta \leq 1$ on Ω. Then, for $v_k := \eta w_k \in \mathrm{W}^{1,1}(\Omega)$ with $\mathrm{spt}\, v_k \Subset \Omega$, it is standard to verify that v_k still converge strictly in $\mathrm{BV}(\Omega)$ to u with $v_k \geq u$ a.e. on Ω for all $k \in \mathbb{N}$. This establishes the main claim.

If u is additionally bounded, we replace v_k already constructed with $\min\{v_k, L\}$ for $L := \max\{0, \sup_\Omega u\}$. Taking into account the lower semicontinuity of the total variation, this preserves all previous properties and additionally ensures boundedness from above by L. □

We conclude this subsection with one more lemma which is tailored out for constructing approximations with suitable smallness conditions on the support in the proof of the later Theorem 7.6.

Lemma 2.22 (control on the support of strict approximations)**.** *Consider an open set* $\Omega \subset \mathbb{R}^n$*. If* $v_k \in \mathrm{W}_0^{1,1}(\Omega)$ *converge to* $u \in \mathrm{BV}(\Omega)$ *strictly in* $\mathrm{BV}(\Omega)$ *with* $u \ge 0$ *a.e. on* Ω *and* $|\{u > 0\}| < M < \infty$*, then there also exists a modified sequence of functions* $w_\ell \in \mathrm{W}_0^{1,1}(\Omega)$ *such that* w_ℓ *still converge to* u *strictly in* $\mathrm{BV}(\Omega)$ *with* $w_\ell \ge 0$ *a.e. on* Ω *and* $|\{w_\ell > 0\}| < M$ *for all* $\ell \in \mathbb{N}$*. Moreover, if all* v_k *are even in* $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$*, all* w_ℓ *can be taken in* $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ *as well, and in this case* $|\{w_\ell > 0\}| < M$ *can be strengthened to* $|\operatorname{spt} w_\ell| < M$*. Finally, if* v_k *converge even in* $\mathrm{W}^{1,1}(\Omega)$ *(and thus to* $u \in \mathrm{W}_0^{1,1}(\Omega)$*), also* w_ℓ *can be taken to converge in* $\mathrm{W}^{1,1}(\Omega)$*.*

Proof. We first establish the original claim. For fixed $\ell \in \mathbb{N}$ we observe $|\{v_k > \frac{2}{\ell}\} \setminus \{u > \frac{1}{\ell}\}| \le \ell \|v_k - u\|_{\mathrm{L}^1(\Omega)}$ and deduce $\limsup_{k\to\infty} |\{v_k > \frac{2}{\ell}\}| \le |\{u > \frac{1}{\ell}\}| < M$. Hence, for each $\ell \in \mathbb{N}$, we can choose $k_\ell \in \mathbb{N}$ such that in addition to $\|v_{k_\ell} - u\|_{\mathrm{L}^1(\Omega)} < \frac{1}{\ell}$ and $\|\nabla v_{k_\ell}\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \le |\mathrm{D}u|(\Omega) + \frac{1}{\ell}$ we have $|\{v_{k_\ell} > \frac{2}{\ell}\}| < M$. For the non-negative functions $w_\ell := \big(v_{k_\ell} - \frac{2}{\ell}\big)_+ \in \mathrm{W}_0^{1,1}(\Omega)$, the previous properties and the non-negativity of u imply via $\|w_\ell - u\|_{\mathrm{L}^1(\Omega)} \le \frac{3}{\ell}$ and $\|\nabla w_\ell\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \le |\mathrm{D}u|(\Omega) + \frac{1}{\ell}$ the claimed strict convergence of w_ℓ, and in view of $\{w_\ell > 0\} = \{v_{k_\ell} > \frac{2}{\ell}\}$ we additionally get $|\{w_\ell > 0\}| < M$. This completes the main part of the reasoning.

If all v_k are even in $\mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, in order to preserve smoothness and control the support we slightly modify the choice of w_ℓ. In fact, since in this situation $\{v_{k_\ell} \ge \frac{3}{\ell}\}$ is compact in the open set $\{v_{k_\ell} > \frac{2}{\ell}\}$, we even get $\operatorname{spt} w_\ell \subset \{v_{k_\ell} > \frac{2}{\ell}\}$ *for a suitable mollification* $w_\ell \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ of $(v_{k_\ell} - \frac{3}{\ell})_+$. Then, also exploiting standard estimates for mollifications, we conclude the reasoning by a straightforward adaptation of the preceding arguments.

Finally, if the convergence is even in $\mathrm{W}^{1,1}(\Omega)$, we still argue in the same way, where the gradients can even be kept L^1-close in the sense of $\|\nabla v_{k_\ell} - \nabla u\|_{\mathrm{L}^1(\Omega,\mathbb{R}^n)} \le \frac{1}{\ell}$. □

We remark that essentially the same proof yields versions of Lemma 2.22 for sequences in other spaces, e.g. in $\mathrm{W}^{1,1}(\Omega)$ or $\mathrm{BV}(\Omega)$ instead of $\mathrm{W}_0^{1,1}(\Omega)$. However, since the above version suffices for our later purposes, we do not discuss this any further.

Normal traces of L^∞ vector fields with L^1 divergence

We next discuss, for vector fields σ with L^1 distributional divergence, a notion of normal trace on the reduced boundary of a set of finite perimeter. The considerations are given for the case of a base domain $\Omega \subset \mathbb{R}^n$ which need not necessarily be bounded, and in fact we are mostly interested in the full-space situation $\Omega = \mathbb{R}^n$.

Definition 2.23 (distributional normal traces)**.** *Consider an open set* Ω *in* $\mathbb{R}^n$*, a set* $E \in \mathcal{M}(\mathbb{R}^n)$ *with* $\mathrm{P}(E,\Omega) < \infty$*, and a vector field* $\sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ *with distributional divergence* $\operatorname{div}\sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$*. Then we call the distribution*

$$\mathrm{Tr}_E(\sigma) := \mathbb{1}_E(\operatorname{div}\sigma) - \operatorname{div}(\mathbb{1}_E\sigma)$$

on Ω *the distributional normal trace (with respect to the outward normal) of* σ *on* $\Omega \cap \partial^* E$*.*

We remark that, spelling out the definition of $\mathrm{Tr}_E(\sigma)$, we have

$$\langle \mathrm{Tr}_E(\sigma); \varphi \rangle = \int_E (\operatorname{div}\sigma)\varphi \,\mathrm{d}x + \int_E \sigma \cdot \nabla\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,. \tag{2.8}$$

Taking into account the definition of the distributional divergence (or merely its linearity), we also infer $\mathrm{Tr}_E(\sigma) = -\mathrm{Tr}_{E^{\mathrm{c}}}(\sigma) = -\mathbb{1}_{E^{\mathrm{c}}} \operatorname{div}\sigma + \operatorname{div}(\mathbb{1}_{E^{\mathrm{c}}}\sigma)$ in the sense of distributions on Ω, that is,

$$\langle \mathrm{Tr}_E(\sigma); \varphi \rangle = -\int_{E^{\mathrm{c}}} (\operatorname{div}\sigma)\varphi \,\mathrm{d}x - \int_{E^{\mathrm{c}}} \sigma \cdot \nabla\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)\,. \tag{2.9}$$

For *bounded* σ, the distributional normal trace actually admits a more concrete representation:

Lemma 2.24 (measure representation of the distributional normal trace)**.** *Consider an open set* Ω *in* $\mathbb{R}^n$*, a set* E *of finite perimeter in* Ω*, and a bounded vector field* $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ *with distributional divergence* $\operatorname{div}\sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$*. Then* $\mathrm{Tr}_E(\sigma)$ *is a finite signed Radon measure on* Ω *and satisfies*

$$|\mathrm{Tr}_E(\sigma)| \le \|\sigma\|_{\mathrm{L}^\infty;\Omega} \mathcal{H}^{n-1} \llcorner (\Omega \cap \partial^* E) \qquad \text{as measures on } \Omega\,.$$

Proof. We fix $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ and consider standard mollifications σ_ε of σ, which are defined on all of $\operatorname{spt}\varphi$ at least for $0 < \varepsilon \ll 1$. Then from (2.8) and standard properties of mollifications we deduce

$$\begin{aligned}\big|\langle \mathrm{Tr}_E(\sigma);\varphi\rangle\big| &= \lim_{\varepsilon\searrow 0}\bigg|\int_E (\operatorname{div}\sigma_\varepsilon)\varphi\,\mathrm{d}x + \int_E \sigma_\varepsilon\cdot\nabla\varphi\,\mathrm{d}x\bigg| = \lim_{\varepsilon\searrow 0}\bigg|\int_\Omega \mathbb{1}_E \operatorname{div}(\varphi\sigma_\varepsilon)\,\mathrm{d}x\bigg| \\ &= \lim_{\varepsilon\searrow 0}\bigg|\int_\Omega \varphi\sigma_\varepsilon\cdot\mathrm{dD}\mathbb{1}_E\bigg| \le \|\sigma\|_{\mathrm{L}^\infty;\Omega}\int_\Omega |\varphi|\,\mathrm{d}|\mathrm{D}\mathbb{1}_E|\,,\end{aligned}$$

where specifically in the last step we used the bound $\|\sigma_\varepsilon\|_{\mathrm{L}^\infty;\operatorname{spt}\varphi} \le \|\sigma\|_{\mathrm{L}^\infty;\Omega}$. This implies that $\mathrm{Tr}_E(\sigma)$ extends to a continuous linear functional on $\mathrm{C}^0_0(\Omega)$, which satisfies the resulting estimate $|\langle \mathrm{Tr}_E(\sigma);\varphi\rangle| \le \|\sigma\|_{\mathrm{L}^\infty;\Omega}\int_\Omega |\varphi|\,\mathrm{d}|\mathrm{D}\mathbb{1}_E|$ for arbitrary $\varphi \in \mathrm{C}^0_{\mathrm{cpt}}(\Omega)$. An application of the Riesz representation theorem now identifies $\mathrm{Tr}_E(\sigma)$ as finite signed Radon measure with $|\mathrm{Tr}_E(\sigma)| \le \|\sigma\|_{\mathrm{L}^\infty;\Omega}|\mathrm{D}\mathbb{1}_E|$ as measures on Ω. Since we have $|\mathrm{D}\mathbb{1}_E| = \mathcal{H}^{n-1}\llcorner(\Omega\cap\partial^*E)$ from Theorem 2.4, the claimed estimate follows. ☐

Lemma 2.24 and the Radon-Nikodým theorem yield the representation

$$\mathrm{Tr}_E(\sigma) = (\sigma\cdot\nu_E)\mathcal{H}^{n-1}\llcorner(\Omega\cap\partial^*E) \tag{2.10}$$

with a density $\sigma\cdot\nu_E \in \mathrm{L}^\infty(\Omega\cap\partial^*E;\mathcal{H}^{n-1})$ such that $|\sigma\cdot\nu_E| \le \|\sigma\|_{\mathrm{L}^\infty;\Omega}$ holds $\mathcal{H}^{n-1}$-a.e. on $\Omega\cap\partial^*E$.

Definition 2.25 (generalized normal traces)**.** *Consider an open set Ω in $\mathbb{R}^n$, a set E of finite perimeter in Ω, and a bounded vector field $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with distributional divergence $\operatorname{div}\sigma \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Then we call the density $\sigma\cdot\nu_E$ from (2.10) the generalized normal trace of σ on $\Omega\cap\partial^*E$.*

In the setting of Definition 2.25, the formulas (2.8), (2.9) can be recast in form of the Gauss-Green formulas

$$\int_E \varphi(\operatorname{div}\sigma)\,\mathrm{d}x + \int_E \sigma\cdot\nabla\varphi\,\mathrm{d}x = \int_{\partial^*E}\varphi\,\sigma\cdot\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.11}$$

$$-\int_{E^{\mathrm{c}}} \varphi(\operatorname{div}\sigma)\,\mathrm{d}x - \int_{E^{\mathrm{c}}} \sigma\cdot\nabla\varphi\,\mathrm{d}x = \int_{\partial^*E}\varphi\,\sigma\cdot\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.12}$$

valid for all $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$. If we additionally assume $\operatorname{div}\sigma \in \mathrm{L}^1(\Omega\cap E)$ and $|\Omega\cap E| < \infty$, then (2.11) stays valid for bounded functions $\varphi \in \mathrm{C}^\infty(\Omega)$ with bounded gradient $\nabla\varphi$ and possibly unbounded support $\operatorname{spt}\varphi \subset \Omega$. This is straightforwardly verified by approximating φ with $\eta_k\varphi$, where $\eta_k \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ are cut-off functions with $0 \le \eta_k \nearrow 1$ and $|\nabla\eta_k| \le 1/k$ on $\mathbb{R}^n$. Specifically, we record for later application that in case $\Omega = \mathbb{R}^n$, we can use $\varphi \equiv 1$ to obtain

$$\int_E \operatorname{div}\sigma\,\mathrm{d}x = \int_{\partial^*E}\sigma\cdot\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{2.13}$$

for all $E \in \mathcal{BV}(\mathbb{R}^n)$ and all $\sigma \in \mathrm{L}^\infty(\mathbb{R}^n,\mathbb{R}^n)$ with $\operatorname{div}\sigma \in \mathrm{L}^1(\mathbb{R}^n)$.

3 Isoperimetric conditions

In order to conveniently specify assumptions on the measure data we introduce the following terminology (which for our main results will mostly be needed in the small-volume version with the optimal constant 1):

Definition 3.1 (isoperimetric conditions)**.** *Consider a non-negative Radon measure μ on an open set $\Omega \subset \mathbb{R}^n$ and $C \in [0,\infty)$. We say that μ satisfies the* strong *isoperimetric condition (strong IC) in Ω with constant C if we have*

$$\mu(A^+) \le C\,\mathrm{P}(A) \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } \overline{A} \subset \Omega \text{ and } |A| < \infty\,. \tag{3.1}$$

We say that μ satisfies the small-volume *isoperimetric condition (small-volume IC) in Ω with constant C if, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that we have*

$$\mu(A^+) \le C\,\mathrm{P}(A) + \varepsilon \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } \overline{A} \subset \Omega \text{ and } |A| < \delta\,. \tag{3.2}$$

We briefly point out two equivalent reformulations of ICs in Ω, which will be treated in detail only in Section 7. First, it is equivalent to require the ICs merely for $A \Subset \Omega$ or to admit even for $A^+ \subset \Omega$ instead of $\overline{A} \subset \Omega$. Second, it is equivalent to replace $\mu(A^+)$ in the ICs with $\mu(A^1)$ (or to use any other precise representative between A^1 and A^+ at this point). The latter possibility is in sharp contrast, however, with the necessity of sticking to A^+ in the μ_--term and to A^1 in the μ_+-term of the functional $\mathscr{P}_{\mu_+,\mu_-}$, as explained in the introduction.

We next record some basic properties which are somewhat reminiscent of the theory of charges discussed e.g. in [33, 5]. However, as we are not aware of a precise link between our ICs with fixed constant C and that theory, we work out the details in our framework. We first recall that, if a finite measure μ is absolutely continuous with respect to the Lebesgue measure, then the absolute continuity of the integral gives, for every $\varepsilon > 0$ some $\delta > 0$ such that we have even $\mu(A^+) = \mu(A) < \varepsilon$ whenever $|A| < \delta$ holds. Therefore, for this type of n-dimensional measures, we trivially have the small-volume IC condition even with constant 0. Back to the general case we now show by a basic covering argument that a measure with IC cannot have any part of dimension smaller than $n{-}1$:

Lemma 3.2. *If a Radon measure μ on open $\Omega \subset \mathbb{R}^n$ satisfies, for $C \in [0,\infty)$, the small-volume IC in Ω with constant C, then, for every $\mathcal{H}^{n-1}$-negligible set $N \in \mathcal{B}(\Omega)$, we have $\mu(N) = 0$.*

Proof. By inner regularity of μ it suffices to treat an $\mathcal{H}^{n-1}$-negligible Borel set $N \Subset \Omega$. Consider an arbitrary $\varepsilon > 0$ with corresponding $\delta > 0$. By Lemma 2.7, there exists an open set A (in particular $A \subset A^+$) such that $N \subset A \Subset \Omega$, $|A| < \delta$, $\mathrm{P}(A) < \varepsilon$. Bringing in the IC, we get $\mu(N) \le \mu(A^+) \le C\mathrm{P}(A){+}\varepsilon < (C{+}1)\varepsilon$. As $\varepsilon > 0$ is arbitrary, this means $\mu(N) = 0$. $\square$

In other words, measures with IC can only have parts of dimension in $[n{-}1, n]$, and for the limit case of $(n{-}1)$-dimensional measures we will actually show in Section 8 that $\mathcal{H}^{n-1}$-rectifiable measures satisfy the small-volume IC with constant C if and only if the $(n{-}1)$-dimensional density of μ does not exceed $2C$. Moreover, examples with fractional dimension κ between $n{-}1$ and n can be obtained from the basic observation that a Radon measure μ on $\mathbb{R}$ satisfies the strong IC in $\mathbb{R}$ with constant C if and only if $\mu(\mathbb{R}) \le 2C$ holds. In particular, for every fractal $F \in \mathcal{B}(\mathbb{R})$ with $0 < \mathcal{H}^\kappa(F) \le 2C$, the measure $\mathcal{H}^\kappa \llcorner F$ satisfies even the strong IC in $\mathbb{R}$ with constant C. With the help of a slicing theory similar to [26, Theorem 18.11] it follows successively for arbitrary $n \in \mathbb{N}$ that the product measure $(\mathcal{H}^\kappa \llcorner F) \otimes (\mathcal{L}^{n-1} \llcorner [0,1])$ satisfies the strong IC in $\mathbb{R}^n$ with constant C. However, since we do not work with such fractional examples or with slicing elsewhere in this paper, we refrain from going into details on these issues.

Next, as a technical preparation, which in the sequel ensures finiteness of our functionals even on unbounded sets A, we record:

Lemma 3.3. *Consider a Radon measure μ on open $\Omega \subset \mathbb{R}^n$, which satisfies, for $C \in [0,\infty)$, the small-volume IC in Ω with constant C or at least satisfies the defining condition (3.2) for one fixed choice of $\varepsilon > 0$ and $\delta > 0$. Then, for every $A \in \mathcal{BV}(\mathbb{R}^n)$ with $\overline{A} \subset \Omega$, we have $\mu(A^+) < \infty$.*

Proof. We fix ε and δ such that (3.2) applies. Since we have $|A| < \infty$ and since $t \mapsto |A \cap ((t_0,t) {\times} \mathbb{R}^{n-1})|$ is continuous, we can divide $\mathbb{R}^n$ into finitely many parallel strips $S_i := (t_{i-1}, t_i) {\times} \mathbb{R}^{n-1}$ with $-\infty = t_0 < t_1 < t_2 < \ldots t_{k-1} < t_k = \infty$ such that $|A \cap S_i| < \delta$ holds for $i = 1, 2, \ldots, k$. Since we assumed in fact $A \in \mathcal{BV}(\mathbb{R}^n)$, we have $\mathrm{P}(A \cap S_i) \le \mathrm{P}(A) < \infty$, and via the IC we get $\mu((A \cap S_i)^+) < \infty$ for $i = 1, 2, \ldots, k$. Taking into account $A^+ \subset \bigcup_{i=1}^k (A \cap S_i)^+$, we conclude $\mu(A^+) < \infty$. $\square$

At the end of this section we wish to underline that the small-volume requirement $|A| < \delta$ in (3.2) is absolutely decisive for our purposes. As a first indication in this direction, we record that an analogous small-*diameter* IC, in which the condition $\operatorname{diam}(A) < \delta$ substitutes for $|A| < \delta$, does not share the same desirable features. Indeed, a compacteness argument shows that the small-diameter IC with any constant $C \in [0,\infty)$ for a non-negative finite Radon measure μ on open $\Omega \subset \mathbb{R}^n$, $n \ge 2$, reduces to the simple requirement that μ is non-atomic (i.e. $\mu(\{x\}) = 0$ for all $x \in \Omega$). Hence, in case[2] $n \ge 2$, the small-diameter IC admits many measures of dimension strictly smaller than $n{-}1$ and cannot yield any semicontinuity results for the functionals $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega]$ considered here.

[2]For $n = 1$, in contrast, the small-volume and small-diameter ICs with constant $C \in (0,\infty)$ are in fact equivalent, since small-length sets of finite perimeter can always be decomposed into short intervals with disjoint closures.

4 Lower semicontinuity on full space

After the preparations of Section 3 we are ready to state, in extension of Theorem 1.2, our main semicontinuity result for the full-space case. The result applies under ICs on given non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ and yields lower semicontinuity of a functional $\mathscr{P}_{\mu_+,\mu_-}$, in which μ_+ and μ_- are each evaluated on a suitable representative. In fact, this functional is defined by

$$\mathscr{P}_{\mu_+,\mu_-}[E] \coloneqq \mathrm{P}(E) + \mu_+(E^1) - \mu_-(E^+) \tag{4.1}$$

for $E \in \mathcal{M}(\mathbb{R}^n)$ with at least one of $\mathrm{P}(E){+}\mu_+(E^1)$ and $\mu_-(E^+)$ finite. In the sequel we keep $\mathscr{P}_{\mu_+,\mu_-}[E]$ well-defined either by generally requiring finiteness of μ_- (in which case $\mathrm{P}(E)$ and $\mu_+(E^1)$ may be finite or infinite) or by drawing on the ICs and Lemma 3.3 to ensure finiteness of all three terms in (4.1) at least for the restricted class of sets $E \in \mathcal{BV}(\mathbb{R}^n)$. We find it worth pointing out that, whenever the measures μ_+ and μ_- are singular to each other, they may be viewed as positive and negative part of a signed Radon measure $\mu_+{-}\mu_-$, and we presently consider this the most relevant case. However, our actual semicontinuity result does not depend on any relation between μ_+ and μ_-.

Theorem 4.1 (lower semicontinuity on full space)**.** *Consider non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$, which both satisfy the* small-volume *IC in $\mathbb{R}^n$ with constant* 1*. For a set $A_\infty \in \mathcal{M}(\mathbb{R}^n)$, and a sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^n)$, assume that* one of the following sets of additional assumptions *is valid*:

(a) *The measure μ_- is* finite*, and A_k converge to A_∞* locally *in measure on $\mathbb{R}^n$.*

(b) *The measure μ_- additionally satisfies an* almost-strong IC with constant 1 near ∞ *in the sense that, for every $\varepsilon > 0$, there exists some $R_\varepsilon \in (0,\infty)$ such that*

$$\mu_-(A^+) \le \mathrm{P}(A) + \varepsilon \qquad \textit{for all } A \in \mathcal{M}(\mathbb{R}^n) \textit{ with } |A \cap \mathrm{B}_{R_\varepsilon}| = 0 \textit{ and } |A| < \infty\,, \tag{4.2}$$

and $A_k \in \mathcal{BV}(\mathbb{R}^n)$ converge to $A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ locally *in measure on $\mathbb{R}^n$.*

(c) *The sets $A_k \in \mathcal{BV}(\mathbb{R}^n)$ converge to $A_\infty \in \mathcal{BV}(\mathbb{R}^n)$* globally *in measure on $\mathbb{R}^n$.*

Then we have

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k] \ge \mathscr{P}_{\mu_+,\mu_-}[A_\infty] \tag{4.3}$$

We emphasize that the μ_+- and μ_--terms in Theorem 4.1 behave fully dual to each other only for finite measures $\mu_\pm$. In contrast, in case of infinite measures, the μ_--term features a more subtle interplay with the perimeter term due to the opposite signs and the resulting well-definedness and cancellation issues whenever both these terms are infinite or approach infinity. This is in fact the reason why the settings (a), (b), (c) in the theorem differ in the assumptions only on μ_- and not on μ_+. In brief, the actual differences are that in (a) we assume *finiteness* of μ_-, that in (b) we impose the *almost-strong* IC near ∞ on μ_-, and that finally in (c) we have neither finiteness nor any strong IC for μ_-, but in exchange require the convergence of A_k to A_∞ in a more restrictive *global* L^1 sense. We point out that a finite measure μ_- generally fulfills $\lim_{R\to\infty} \mu_-((\mathrm{B}_R)^{\mathrm{c}}) = 0$ and thus satisfies (4.2). Thus, the result under (a) is a special case of the one under (b) when disregarding the marginal point that in (a) we can formally allow infinite perimeters of A_k and A_∞. Nevertheless, we believe that also the much simpler setting (a) deserves its explicit recording in the above statement (and in similar ones to follow later on).

Interestingly, having at least one of the extra features from the settings (a), (b), (c) is necessary for having (4.3), as shown by the following examples with sequences $(A_k)_{k\in\mathbb{N}}$ which „loose mass at infinity“.

Example 4.2 (for the failure of lower semicontinuity)**.** *For $n \ge 2$, we consider the infinite Radon measure*

$$\mu_- \coloneqq 2\mathcal{H}^{n-1}\llcorner(\mathbb{R}^{n-1}{\times}\{0,1\})$$

(twice the area measure on two parallel hyperplanes). Then μ_- satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1 *by Proposition A.3 in the appendix, while it satisfies the strong IC in $\mathbb{R}^n$ and its variant of type (4.2) only with constant* 2*, but not with constant* 1*. Furthermore, for fixed $B \in \mathcal{BV}(\mathbb{R}^{n-1})$ with $\mathrm{P}(B) < 2|B| < \infty$ (a large ball in $\mathbb{R}^{n-1}$, for instance) and a fixed direction $0 \ne v \in \mathbb{R}^{n-1}$, we consider the shifted*

cylinders $A_k := (B+kv)\times[0,1] \in \mathcal{BV}(\mathbb{R}^n)$*; see Figure 2 for a basic illustration. Then* A_k *converge only locally, but not globally in measure on* $\mathbb{R}^n$ *to* $\emptyset$*, and from* $\mathrm{P}(A_k) = 2|B|+\mathrm{P}(B)$ *and* $\mu_-(A_k^+) = \mu_-(A_k) = 4|B|$ *we deduce*

$$\lim_{k\to\infty} \mathscr{P}_{0,\mu_-}[A_k] = \mathrm{P}(B)-2|B| < 0 = \mathscr{P}_{0,\mu_-}[\emptyset]\,.$$

Thus, lower semicontinuity of $\mathscr{P}_{0,\mu_-}$ *fails along this sequence.*

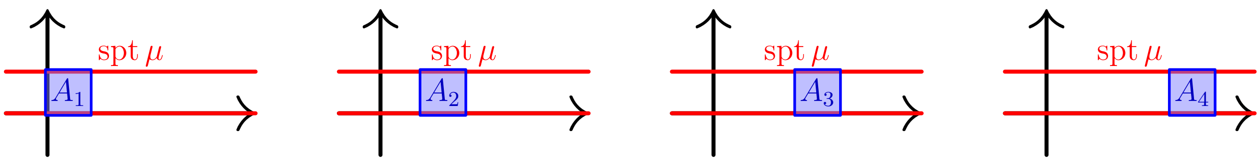

Figure 2: The sets A_k, which converge *locally* in measure on $\mathbb{R}^2$ to $\emptyset$, in case $n=2$, $B=[-1,0]$, $v=1$.

For $n=1$*, essentially the same phenomenon occurs for the measure* $\mu_- := 2\mathcal{H}^0 \llcorner \mathbb{Z}$ *(with the counting measure* $\mathcal{H}^0$*) and* $A_k := I+k$ *with any bounded interval* $I \subset \mathbb{R}$ *such that* $\overline{I}$ *contains at least two integers.*

Before proceeding to the proof of the theorem we add a brief remark on technical infinite-volume variants of the assumptions in (b) and (c). While the issue is rather marginal and could also be skipped, we find it worth recording mainly for better comparability with the later Theorem 6.1.

Remark 4.3. *In the settings (b) and (c) of Theorem 4.1 we may replace the requirements* $A_k, A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ *by* $A_k^{\mathrm{c}}, A_\infty^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$ *together with* $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ *and* $\min\{\mu_+(A_\infty^1), \mu_-(A_\infty^+)\} < \infty$*.*

Proof. From $\mathrm{P}(A_k^{\mathrm{c}}) = \mathrm{P}(A_k)$ and $\mathrm{P}(A_\infty^{\mathrm{c}}) = \mathrm{P}(A_\infty)$ we see that $\mathscr{P}_{\mu_+,\mu_-}[A_k]$ and $\mathscr{P}_{\mu_+,\mu_-}[A_\infty]$ are still well-defined. With the result for the setting (a) at hand it suffices to consider the case $\mu_-(\mathbb{R}^n) = \infty$. Then, starting from $|A_k^{\mathrm{c}}| < \infty$ and using Lemma 3.3 we infer first $\mu_-((A_k^+)^{\mathrm{c}}) \le \mu_-((A_k^{\mathrm{c}})^+) < \infty$, then $\mu_-(A_k^+) = \infty$, then $\mu_+(A_k^1) < \infty$, and finally $\mathscr{P}_{\mu_+,\mu_-}[A_k] = -\infty$ for $k \gg 1$. As in the same way we get $\mathscr{P}_{\mu_+,\mu_-}[A_\infty] = -\infty$, the semicontinuity inequality (4.3) is trivially valid with $-\infty$ on both sides. ☐

For $n \ge 2$*, in view of Theorem 2.6 we may express that either* $A_k, A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ *(as in the theorem) or* $A_k^{\mathrm{c}}, A_\infty^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$ *(as in this remark) holds by requiring the unifying condition* $|A_k \Delta A_\infty| + \mathrm{P}(A_k) + \mathrm{P}(A_\infty) < \infty$*. For* $n=1$*, the condition* $|A_k \Delta A_\infty| + \mathrm{P}(A_k) + \mathrm{P}(A_\infty) < \infty$ *includes further cases, but still semicontinuity remains valid in all of these (as it can be read off from the later proofs or the refined results in Theorems 6.1, 9.1, 9.6 and, in fact, in the one-dimensional situation can also be proved by much simpler means).*

The proof of Theorem 4.1 is approached step by step and will be finalized only at the end of this section. We start by establishing an approximation lemma, which is illustrated in Figure 3 and plays a key role.

Lemma 4.4 (good exterior approximation)**.** *For a non-negative Radon measure* μ *on* $\mathbb{R}^n$ *with* $\mu(N) = 0$ *for all* $\mathcal{H}^{n-1}$*-negligible* $N \in \mathcal{B}(\mathbb{R}^n)$*, assume that condition (3.2) holds in* $\Omega = \mathbb{R}^n$ *for some fixed choice of* $\varepsilon > 0$*,* $\delta > 0$*, and* $C \in [0,\infty)$*. Then, if a sequence* $(A_k)_{k\in\mathbb{N}}$ *in* $\mathcal{BV}(\mathbb{R}^n)$ *converges globally in measure on* $\mathbb{R}^n$ *to* $A_\infty \in \mathcal{BV}(\mathbb{R}^n)$*, there exists a Borel set* $S \in \mathcal{BV}(\mathbb{R}^n)$ *such that we have*

$$A_\infty^+ \subset \mathrm{int}(S)\,, \qquad \mu(\overline{S}) < \mu(A_\infty^+) + 3\varepsilon\,, \qquad \textit{and} \qquad \liminf_{k\to\infty} \mathrm{P}(S, A_k^+) < \varepsilon\,.$$

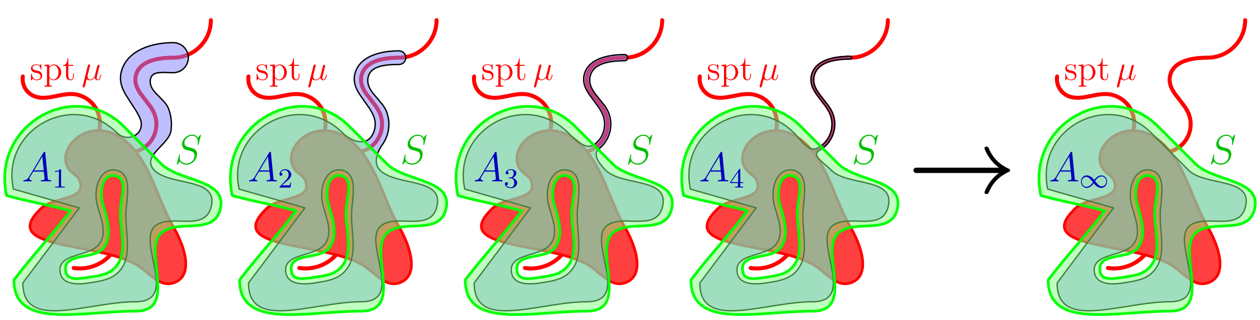

Figure 3: A set S which cuts off the tentacle of Figure 1 in the sense of Lemma 4.4 (for mildly small ε).

Proof of Lemma 4.4. We first treat the main case $n \geq 2$. Applying Lemma 2.18 to $\mathbb{1}_{A_\infty} \in \mathrm{BV}(\mathbb{R}^n)$, we find $v_\ell \in \mathrm{W}^{1,1}(\mathbb{R}^n)$ such that $1 \geq v_1 \geq v_2 \geq v_3 \geq \ldots$ holds a.e. on $\mathbb{R}^n$ and v_ℓ^* converge $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^n$ to $\mathbb{1}_{A_\infty^+}$. By assumption on μ, this convergence holds also μ-a.e. on $\mathbb{R}^n$. Next, possibly decreasing $\delta > 0$ from the statement, we can assume $C(\delta/\Gamma_n)^{\frac{n-1}{n}} \leq \varepsilon$ for the constant Γ_n of Theorem 2.6. Lemma 2.17 then gives open sets E_ℓ in $\mathbb{R}^n$ with $\mathrm{Cap}_1(E_\ell) < (\delta/\Gamma_n)^{\frac{n-1}{n}}$ (and in particular $|E_\ell| < \infty$) such that v_ℓ^* is defined and continuous on $\mathbb{R}^n \setminus E_\ell$. From Proposition 2.15 we further obtain $H_\ell \in \mathcal{BV}(\mathbb{R}^n)$ with $\mathrm{P}(H_\ell) < (\delta/\Gamma_n)^{\frac{n-1}{n}}$ such that $E_\ell \subset H_\ell^+$. By the isoperimetric estimate of Theorem 2.6 we infer $|H_\ell| \leq \Gamma_n \mathrm{P}(H_\ell)^{\frac{n}{n-1}} < \delta$, and via (3.2) we end up with $\mu(H_\ell^+) \leq C\mathrm{P}(H_\ell) + \varepsilon < C(\delta/\Gamma_n)^{\frac{n-1}{n}} + \varepsilon \leq 2\varepsilon$. For the following we can thus record

$$\mu(H_\ell^+) < 2\varepsilon \qquad \text{and} \qquad \mathrm{P}(H_\ell) < \varepsilon\,. \tag{4.4}$$

Next we observe that $\mathrm{P}(\{v_\ell > t\}) < \infty$ holds for a.e. $t \in (0,1)$ by Theorem 2.5. Furthermore, with the help of Fatou's lemma, again Theorem 2.5, $|A_k^+ \Delta A_k| = 0$, $\lim_{k\to\infty} |A_k \Delta A_\infty| = 0$, and $v_\ell \equiv 1$, $\nabla v_\ell \equiv 0$ a.e. on A_∞ we obtain

$$\int_0^1 \liminf_{k\to\infty} \mathrm{P}(\{v_\ell > t\}, A_k^+)\,\mathrm{d}t \leq \liminf_{k\to\infty} \int_0^1 \mathrm{P}(\{v_\ell > t\}, A_k^+)\,\mathrm{d}t = \liminf_{k\to\infty} \int_{A_k} |\nabla v_\ell|\,\mathrm{d}x = \int_{A_\infty} |\nabla v_\ell|\,\mathrm{d}x = 0\,.$$

As a consequence, we have $\liminf_{k\to\infty} \mathrm{P}(\{v_\ell > t\}, A_k^+) = 0$ for a.e. $t \in (0,1)$. Therefore, we can first choose $t_1 \in (0,1)$ and then $t_\ell \in [t_1, 1)$ with $\ell \geq 2$ such that we have

$$\mathrm{P}(\{v_\ell > t_\ell\}) < \infty \qquad \text{for all } \ell \in \mathbb{N}$$

and

$$\liminf_{k\to\infty} \mathrm{P}(\{v_\ell > t_\ell\}, A_k^+) = 0 \qquad \text{for all } \ell \in \mathbb{N}\,. \tag{4.5}$$

Moreover, since the measure μ has positive mass on at most countably many level sets[3] $\{v_\ell^* = t\}$ with $t \in (0,1)$, the choices can be made such that $\mu(\{v_\ell^* = t_\ell\}) = 0$ for all $\ell \in \mathbb{N}$. Now we introduce the sets[4]

$$U_\ell := \{v_\ell^* > t_\ell\} \setminus E_\ell\,.$$

We observe that U_ℓ are relatively open in $\mathbb{R}^n \setminus E_\ell$ with $\overline{U_\ell} \subset \{v_\ell^* \geq t_\ell\} \setminus E_\ell$ by the openness of E_ℓ and the continuity of v_ℓ^* outside E_ℓ. Additionally taking into account $\mu(\{v_\ell^* = t_\ell\}) = 0$ and $t_\ell \geq t_1$ we can estimate

$$\mu(\overline{U_\ell}) \leq \mu(\{v_\ell^* \geq t_\ell\}) = \mu(\{v_\ell^* > t_\ell\}) \leq \mu(\{v_\ell^* > t_1\})\,. \tag{4.6}$$

Here, in view of $v_1 \in \mathrm{L}^1(\mathbb{R}^n)$ and $\mathrm{P}(\{v_1 > t_1\}) < \infty$, we infer $\mu(\{v_1^* > t_1\} \setminus E_1) \leq \mu(\{v_1 > t_1\}^+) < \infty$ from Lemma 3.3 and then deduce also $\mu(\{v_1^* > t_1\}) \leq \mu(\{v_1^* > t_1\} \setminus E_1) + \mu(H_1^+) < \infty$. Combining this with the μ-a.e. monotone convergence $v_\ell^* \to \mathbb{1}_{A_\infty^+}$, we infer that the right-hand side $\mu(\{v_\ell^* > t_1\})$ in (4.6) converges to $\mu(\{\mathbb{1}_{A_\infty^+} > t_1\}) = \mu(A_\infty^+)$ for $\ell \to \infty$. Therefore, for a suitably large $\ell \in \mathbb{N}$, which we fix at this point for the remainder of the proof, we have

$$\mu(\overline{U_\ell}) < \mu(A_\infty^+) + \varepsilon\,. \tag{4.7}$$

Now we are ready to introduce

$$S := U_\ell \cup H_\ell^+\,,$$

and using $E_\ell \subset H_\ell^+$ we see

$$A_\infty^+ \subset \{v_\ell^* = 1\} \cup E_\ell \subset U_\ell \cup E_\ell \subset S\,.$$

Since U_ℓ is relatively open in $\mathbb{R}^n \setminus E_\ell$ and E_ℓ is open in $\mathbb{R}^n$, also $U_\ell \cup E_\ell$ is open in $\mathbb{R}^n$, and we can deduce even

$$A_\infty^+ \subset \mathrm{int}(S)\,.$$

[3]Since v_ℓ^* is defined $\mathcal{H}^{n-1}$-a.e. and then by assumption also μ-ae, the level sets $\{v_\ell^* = t\}$ are defined up to μ-negligible sets, and this will suffice for our purposes. Clearly, one may also agree on a concrete convention such as simply excluding the non-existence points of v_ℓ^* from the level sets.

[4]The sets U_ℓ are defined up to single points, since the non-existence set of v_ℓ^* is contained in E_ℓ.

Furthermore, from (4.7) and (4.4) we infer

$$\mu(\overline{S}) \le \mu(\overline{U_\ell}) + \mu(H_\ell^+) \le \mu(A_\infty^+) + 3\varepsilon\,.$$

Finally, we observe $S = \{v_\ell > t_\ell\} \cup H_\ell$ up to negligible sets with $\{v_\ell > t_\ell\}, H_\ell \in \mathcal{BV}(\mathbb{R}^n)$. Thus, by Lemma 2.10 we obtain $S \in \mathcal{BV}(\mathbb{R}^n)$ and $\mathrm{P}(S,\,\cdot\,) \le \mathrm{P}(\{v_\ell > t_\ell\},\,\cdot\,) + \mathrm{P}(H_\ell,\,\cdot\,)$. Therefore, involving also (4.5) and (4.4) we can estimate

$$\liminf_{k\to\infty} \mathrm{P}(S, A_k^+) \le \liminf_{k\to\infty} \mathrm{P}(\{v_\ell > t_\ell\}, A_k^+) + \mathrm{P}(H_\ell) < \varepsilon\,.$$

At this point, all claims on S are verified.

Finally, in the simpler case $n = 1$ the previous reasoning applies with major simplifications, which are mostly due to the full continuity of $\mathrm{W}^{1,1}(\mathbb{R})$ functions. In particular there is no need to construct E_ℓ and H_ℓ, which can be replaced with $\emptyset$, and one can directly obtain an open set $S = U_\ell = \{v_\ell^* > t_\ell\}$. □

With the lemma at hand, we now proceed to a comparably quick proof of Theorem 1.2 from the introduction, which corresponds to the setting (c) in Theorem 4.1 in the special case $\mu_+ \equiv 0$ and which is here restated as follows.

Proposition 4.5 (L^1 lower semicontinuity in case $\mu_+ \equiv 0$)**.** *Consider a non-negative Radon measure μ on $\mathbb{R}^n$ which satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1. Moreover, assume that $A_k \in \mathcal{BV}(\mathbb{R}^n)$ converge* globally *in measure on $\mathbb{R}^n$ to $A_\infty \in \mathcal{BV}(\mathbb{R}^n)$. Then we have*

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k) - \mu(A_k^+)\big] \ge \mathrm{P}(A_\infty) - \mu(A_\infty^+)\,. \tag{4.8}$$

Proof. Possibly passing to a subsequence, we can assume that $\lim_{k\to\infty}\big[\mathrm{P}(A_k){-}\mu(A_k^+)\big]$ exists. We now fix an arbitrary $\varepsilon > 0$. Drawing on Lemma 3.2 and the assumed IC, we then apply Lemma 4.4 with the given ε, the corresponding δ, and $C = 1$, and we work with the corresponding set $S \in \mathcal{BV}(\mathbb{R}^n)$. We start by splitting terms in the sense of the inequality

$$\mathrm{P}(A_k) - \mu(A_k^+) \ge \mathrm{P}(A_k, \mathrm{int}(S)) - \mu(\overline{S}) + \mathrm{P}(A_k, \mathrm{int}(S)^{\mathrm{c}}) - \mu\big(A_k^+ \setminus \overline{S}\big)\,.$$

Then we use the elementary rule $\lim_{k\to\infty}[a_k{+}b_k] \ge \liminf_{k\to\infty} a_k + \limsup_{k\to\infty} b_k$ for $a_k, b_k \in \mathbb{R}$, valid whenever the limit on the left-hand side exists. In view of the initial assumption, this leads to

$$\lim_{k\to\infty} \big[\mathrm{P}(A_k) - \mu(A_k^+)\big] \ge \liminf_{k\to\infty} \mathrm{P}(A_k, \mathrm{int}(S)) - \mu(\overline{S}) + \limsup_{k\to\infty} \big[\mathrm{P}(A_k, \mathrm{int}(S)^{\mathrm{c}}) - \mu\big(A_k^+ \setminus \overline{S}\big)\big]\,. \tag{4.9}$$

The terms on the right-hand side of (4.9) are now estimated separately. For the first term, by an application of Lemma 2.3 on the open set $\mathrm{int}(S)$ and the inclusion $A_\infty^+ \subset \mathrm{int}(S)$ from Lemma 4.4, we have

$$\liminf_{k\to\infty} \mathrm{P}(A_k, \mathrm{int}(S)) \ge \mathrm{P}(A_\infty, \mathrm{int}(S)) \ge \mathrm{P}(A_\infty, A_\infty^+) = \mathrm{P}(A_\infty)\,. \tag{4.10}$$

For the second term, the estimate

$$\mu(\overline{S}) < \mu(A_\infty^+) + 3\varepsilon\,, \tag{4.11}$$

also provided by Lemma 4.4, suffices. In order to control the last term in (4.9), we first record that in view of $A_\infty^+ \subset S$ we get $|A_k \setminus S| \le |A_k \setminus A_\infty| \le |A_k \Delta A_\infty|$ and that consequently the assumed global convergence implies $\lim_{k\to\infty} |A_k \setminus S| = 0$. This permits the crucial application of the small-volume IC with constant 1 to $A_k \setminus S$ for $k \gg 1$, which is now combined with the inclusion $A_k^+ \setminus \overline{S} \subset (A_k \setminus S)^+$, Lemma 2.9, and the inclusion $S^0 \subset \mathrm{int}(S)^{\mathrm{c}}$. All in all, for $k \gg 1$, we deduce

$$\mu\big(A_k^+ \setminus \overline{S}\big) \le \mu((A_k \setminus S)^+) \le \mathrm{P}(A_k \setminus S) + \varepsilon \le \mathrm{P}(A_k, S^0) + \mathrm{P}(S, A_k^+) + \varepsilon \le \mathrm{P}(A_k, \mathrm{int}(S)^{\mathrm{c}}) + \mathrm{P}(S, A_k^+) + \varepsilon\,.$$

Now we rearrange terms in the resulting estimate and take limits. Then, also employing the last property from Lemma 4.4, we conclude

$$\limsup_{k\to\infty} \big[\mathrm{P}(A_k, \mathrm{int}(S)^{\mathrm{c}}) - \mu\big(A_k^+ \setminus \overline{S}\big)\big] \ge -\liminf_{k\to\infty} \mathrm{P}(S, A_k^+) - \varepsilon > -2\varepsilon\,. \tag{4.12}$$

Collecting the estimates (4.9), (4.10), (4.11), and (4.12) we finally arrive at

$$\lim_{k\to\infty} \left[\mathrm{P}(A_k) - \mu(A_k^+)\right] \geq \mathrm{P}(A_\infty) - \mu(A_\infty^+) - 5\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, with this we have proven the claim (4.8). ☐

Next, essentially by passing to complements, we establish a variant of Proposition 4.5 with opposite sign convention for the measure μ. This dual statement is analogous except for the fact that in the dual case we can allow for *local* convergence of sets of potentially infinite perimeter, while in the original case we cannot generally relax the corresponding global assumptions. In terms of the general Theorem 4.1 this means that we achieve a treatment of the setting (a) with $\mu_- \equiv 0$.

Proposition 4.6 ($\mathrm{L}^1_{\mathrm{loc}}$ lower semicontinuity in case $\mu_- \equiv 0$)**.** *Consider a non-negative Radon measure μ on $\mathbb{R}^n$ which satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1. Moreover, assume that $A_k \in \mathcal{M}(\mathbb{R}^n)$ converge* locally *in measure on $\mathbb{R}^n$ to $A_\infty \in \mathcal{M}(\mathbb{R}^n)$. Then we have*

$$\liminf_{k\to\infty} \left[\mathrm{P}(A_k) + \mu(A_k^1)\right] \geq \mathrm{P}(A_\infty) + \mu(A_\infty^1)\,. \tag{4.13}$$

We remark that the deduction of Proposition 4.6 from Proposition 4.5 is quite straightforward *if* A_k are uniformly bounded and thus we can simply take complements in a fixed, suitably large ball $B \subset \mathbb{R}^n$ (for which we clearly have $B \in \mathcal{BV}(\mathbb{R}^n)$ and $\mu(B) < \infty$). However, in general we are not in this situation, and thus in the following proof we need additional cut-off arguments.

Proof of Proposition 4.6. As usual we can assume that $\lim_{k\to\infty} \left[\mathrm{P}(A_k)+\mu(A_k^1)\right]$ exists and is finite. Taking into account the sign of the μ-term we can further assume $\sup_{k\in\mathbb{N}} \mathrm{P}(A_k) < \infty$, which implies $\mathrm{P}(A_\infty) < \infty$ by Lemma 2.3. Next, by a classical version of the coarea formula (which can be seen as the case $u(x) = |x|$ in either Theorem 2.5 or Theorem 2.1), for every $R_0 \in (0,\infty)$ we have

$$\begin{aligned}\int_0^{R_0} \liminf_{k\to\infty} \mathcal{H}^{n-1}((A_k^0 \Delta A_\infty^0) \cap \partial \mathrm{B}_R)\,\mathrm{d}R &\leq \liminf_{k\to\infty} \int_0^{R_0} \mathcal{H}^{n-1}((A_k^0 \Delta A_\infty^0) \cap \partial \mathrm{B}_R)\,\mathrm{d}R \\ &= \liminf_{k\to\infty} |(A_k^0 \Delta A_\infty^0) \cap \mathrm{B}_{R_0}| = 0\,,\end{aligned}$$

and thus $\liminf_{k\to\infty} \mathcal{H}^{n-1}((A_k^0 \Delta A_\infty^0) \cap \partial \mathrm{B}_R) = 0$ holds for a.e. $R \in (0,\infty)$. In addition, the Radon measures $\gamma_k := \mathrm{P}(A_k,\,\cdot\,) + \mathrm{P}(A_\infty,\,\cdot\,) + \mu$ satisfy $\gamma_k(\partial \mathrm{B}_R) = 0$ for all but at most countably many $R \in (0,\infty)$. Altogether, this allows to choose radii $R_i \in (0,\infty)$ with $\lim_{i\to\infty} R_i = \infty$ such that, for the corresponding open balls $B_i := \mathrm{B}_{R_i}$ centered at 0, we have

$$\mathrm{P}(A_k, \partial B_i) = \mathrm{P}(A_\infty, \partial B_i) = 0 \qquad \text{for all } i,k \in \mathbb{N}\,, \tag{4.14}$$

$$\mu(\partial B_i) = 0 \qquad \text{for all } i \in \mathbb{N}\,, \tag{4.15}$$

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}((A_k^0 \Delta A_\infty^0) \cap \partial \mathrm{B}_i) = 0 \qquad \text{for all } i \in \mathbb{N}\,.$$

Here, by successively passing to subsequences of A_k and using a diagonal sequence argument, the last property can be strengthened to hold with lim in place of lim inf and then also gives

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(A_k^0 \cap \partial B_i) = \mathcal{H}^{n-1}(A_\infty^0 \cap \partial B_i) \qquad \text{for all } i \in \mathbb{N}\,. \tag{4.16}$$

Now, for arbitrary $i \in \mathbb{N}$, we consider the complements $B_i \setminus A_k$, which converge for $k \to \infty$ in measure to $B_i \setminus A_\infty$. (Observe here that indeed *local* convergence in measure of A_k implies *global* convergence in measure of the bounded sets $B_i \setminus A_k$.) Hence, by an application of Proposition 4.5, we get

$$\liminf_{k\to\infty} \left[\mathrm{P}(B_i \setminus A_k) - \mu((B_i \setminus A_k)^+)\right] \geq \mathrm{P}(B_i \setminus A_\infty) - \mu((B_i \setminus A_\infty)^+)\,. \tag{4.17}$$

We now estimate and rewrite terms in (4.17). On one hand we exploit (4.14) (which can also be written as $\mathcal{H}^{n-1}(\partial^* A_k \cap \partial B_i) = \mathcal{H}^{n-1}(\partial^* A_\infty \cap \partial B_i) = 0$) in order to apply the equality case of (2.5) in Lemma 2.9. In this way we derive

$$\mathrm{P}(B_i \setminus A_k) = \mathrm{P}(A_k, B_i^+) + \mathrm{P}(B_i, A_k^0) = \mathrm{P}(A_k, B_i) + \mathcal{H}^{n-1}(A_k^0 \cap \partial B_i)\,,$$

$$\mathrm{P}(B_i \setminus A_\infty) = \mathrm{P}(A_\infty, B_i^+) + \mathrm{P}(B_i, A_\infty^0) = \mathrm{P}(A_\infty, B_i) + \mathcal{H}^{n-1}(A_\infty^0 \cap \partial B_i)\,.$$

On the other hand, keeping (4.15) in mind, we have

$$\mu((B_i \setminus A_k)^+) = \mu(B_i \setminus A_k^1) = \mu(B_i) - \mu(A_k^1 \cap B_i)\,,$$
$$\mu((B_i \setminus A_\infty)^+) = \mu(B_i \setminus A_\infty^1) = \mu(B_i) - \mu(A_\infty^1 \cap B_i)\,.$$

We plug these findings into (4.17) and are left with

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k, B_i) + \mu(A_k^1 \cap B_i) + \mathcal{H}^{n-1}(A_k^0 \cap \partial B_i)\big] - \mu(B_i)$$
$$\geq \mathrm{P}(A_\infty, B_i) + \mu(A_\infty^1 \cap B_i) + \mathcal{H}^{n-1}(A_\infty^0 \cap \partial B_i) - \mu(B_i)\,.$$

Adding the finite number $\mu(B_i)$ and subtracting the finite number in (4.16), the inequality reduces to

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k, B_i) + \mu(A_k^1 \cap B_i)\big] \geq \mathrm{P}(A_\infty, B_i) + \mu(A_\infty^1 \cap B_i)\,.$$

At this stage, we further enlarge the terms on the left-hand side and use the initial assumption on the existence of the limit to get

$$\lim_{k\to\infty} \big[\mathrm{P}(A_k) + \mu(A_k^1)\big] \geq \mathrm{P}(A_\infty, B_i) + \mu(B_i \cap A_\infty^1)\,.$$

Finally, sending $i \to \infty$ and taking into account $\lim_{i\to\infty} R_i = \infty$, we arrive at the claim (4.13). □

By combining Propositions 4.5 and 4.6 we are able to treat the global-convergence setting (c) in Theorem 4.1 in its full generality.

Proof of Theorem 4.1 under assumptions (c). For A_k and A_∞ as in the statement, we record that both $A_k \cup A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ and $A_k \cap A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ converge globally in measure to A_∞. Then, since we assumed the small-volume IC for both μ_+ and μ_-, we can apply Proposition 4.5 to $A_k \cup A_\infty$ and Proposition 4.6 to $A_k \cap A_\infty$ to deduce

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k \cup A_\infty) - \mu_-((A_k \cup A_\infty)^+)\big] \geq \mathrm{P}(A_\infty) - \mu_-(A_\infty^+)\,,$$
$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k \cap A_\infty) + \mu_+((A_k \cap A_\infty)^1)\big] \geq \mathrm{P}(A_\infty) + \mu_+(A_\infty^1)\,.$$

We now add these two inequalities and use (2.6) in the form $\mathrm{P}(A_k \cup A_\infty) + \mathrm{P}(A_k \cap A_\infty) \leq \mathrm{P}(A_k) + \mathrm{P}(A_\infty)$ together with $(A_k \cup A_\infty)^+ = A_k^+ \cup A_\infty^+ \supset A_k^+$ and $(A_k \cap A_\infty)^1 = A_k^1 \cap A_\infty^1 \subset A_k^1$. Then we end up with

$$\mathrm{P}(A_\infty) + \liminf_{k\to\infty} \big[\mathrm{P}(A_k) + \mu_+(A_k^1) - \mu_-(A_k^+)\big] \geq 2\mathrm{P}(A_\infty) + \mu_+(A_\infty^1) - \mu_-(A_\infty^+)\,,$$

which by subtraction of $\mathrm{P}(A_\infty)$ yields the claim in (4.3). □

Before treating the remaining settings and finalizing the discussion of semicontinuity on the full space, we record the following localized semicontinuity property, which comes out from the cut-off argument in the proof of Proposition 4.6 and a „dual“ variant of this argument. This localized statement will in fact be very convenient in the sequel.

Lemma 4.7 (localized semicontinuity)**.** *Consider non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ which both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1. If $A_k \in \mathcal{M}(\mathbb{R}^n)$ converge to $A_\infty \in \mathcal{M}(\mathbb{R}^n)$ locally in measure in $\mathbb{R}^n$, then, for every $R \in (0,\infty)$, we have*

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k, \mathrm{B}_R) + \mu_+(A_k^1 \cap \mathrm{B}_R) - \mu_-(A_k^+ \cap \mathrm{B}_R)\big] \geq \mathrm{P}(A_\infty, \mathrm{B}_R) + \mu_+(A_\infty^1 \cap \mathrm{B}_R) - \mu_-(A_\infty^+ \cap \mathrm{B}_R)\,.$$

Proof. We first establish the claim simultaneously for the case $\mu_- \equiv 0$, in which we set $\mu := \mu_+$, and for the case $\mu_+ \equiv 0$, in which we set $\mu := \mu_-$. For the case $\mu_- \equiv 0$ we can follow quite closely the lines of the proof of Proposition 4.6, while for the case $\mu_+ \equiv 0$ we use an analogous but dual argument based on the convergence of $A_k \cap B_i$ to $A_\infty \cap B_i$. In the sequel we only point out the relevant modifications. First of all, we now work with a fixed $R \in (0,\infty)$ and may initially assume existence and finiteness of $\lim_{k\to\infty} \big[\mathrm{P}(A_k, \mathrm{B}_R) - \mu(A_k^+ \cap \mathrm{B}_R)\big]$ and

$\lim_{k\to\infty}\big[\mathrm{P}(A_k,\mathrm{B}_R)+\mu(A_k^1\cap\mathrm{B}_R)\big]$, respectively, which leads to $\sup_{k\in\mathbb{N}}\mathrm{P}(A_k,\mathrm{B}_R)<\infty$ and $\mathrm{P}(A_\infty,\mathrm{B}_R)<\infty$ (where we have exploited $\mu(\mathrm{B}_R)<\infty$ in case $\mu_+\equiv 0$). Then, the good radii R_i are taken in $(0,R)$ with $\lim_{i\to\infty}R_i=R$, where in case $\mu_+\equiv 0$ the coarea argument is implemented with A_k^1 and A_∞^1 instead of A_k^0 and A_∞^0 to subsequently achieve $\lim_{k\to\infty}\mathcal{H}^{n-1}(A_k^1\cap\partial B_i)=\mathcal{H}^{n-1}(A_\infty^1\cap\partial B_i)$ in place of (4.16). The remainder of the reasoning stays unchanged in case $\mu_-\equiv 0$ and in case $\mu_+\equiv 0$ is done with $A_k\cap B_i$ and $A_\infty\cap B_i$ instead of $B_i\setminus A_k$ and $B_i\setminus A_\infty$ (which slightly simplifies the handling of the μ-terms). When adapting the final step in the proof of Proposition 4.6 to the case $\mu_+\equiv 0$, we may no longer pass from $-\mu(A_k^+\cap B_i)$ to $-\mu(A_k^+\cap B_R)$ on the left-hand side by simply enlarging the term, but we can still conclude, as in view of $\mu(\mathrm{B}_R)<\infty$ we have $\lim_{i\to\infty}\mu(A_k^+\cap B_i)=\mu(A_k^+\cap B_R)$ uniformly in k.

Finally, in order to reach the general case, in which both μ_+ and μ_- do not vanish, we return to the reasoning used above to prove Theorem 4.1 in the setting (c). The adaptation of this reasoning to a ball B_R is straightforward and exploits (2.6) in the form $\mathrm{P}(A_k\cup A_\infty,\mathrm{B}_R)+\mathrm{P}(A_k\cap A_\infty,\mathrm{B}_R)\le\mathrm{P}(A_k,\mathrm{B}_R)+\mathrm{P}(A_\infty,\mathrm{B}_R)$. □

We proceed by addressing the proof of semicontinuity in the settings (a) and (b) of Theorem 4.1. We only sketch the relevant arguments, since we will later provide further details in connection with even more general cases contained in Theorem 6.1.

In fact, in order to complete the treatment of the setting (a) the observation needed is essentially the one that, for finite measures, the cases $\mu_+\equiv 0$ and $\mu_-\equiv 0$ are fully dual to each other:

Sketch of proof for Theorem 4.1 under assumptions (a). In case $\mu_-\equiv 0$ the claim is covered by Proposition 4.6. Moreover, we can move back from this case to the case $\mu_+\equiv 0$ once more by taking complements. Indeed, since we are assuming $\mu_-(\mathbb{R}^n)<\infty$, this works rather straightforwardly by exploiting $\mathrm{P}(A_k^{\mathrm{c}})=\mathrm{P}(A_k)$ and $\mu_-((A_k^{\mathrm{c}})^1)=\mu_-(\mathbb{R}^n)-\mu_-(A_k^+)$ together with the analogous formulas for A_∞^{c}. Alternatively, we can obtain the claim in the case $\mu_+\equiv 0$ by passing $R\to\infty$ in the case $\mu_+\equiv 0$ of Lemma 4.7. Finally, the general case with non-zero μ_+ and μ_- can be reached by the same reasoning used under assumptions (c). □

In connection with the setting (b) the final key observation is that the strong IC for μ_- keeps cut-off terms (almost) non-negative and prevents the failure of lower semicontinuity at ∞:

Sketch of proof for Theorem 4.1 under assumptions (b). Once more the case $\mu_-\equiv 0$ is covered by Proposition 4.6, and once we manage to additionally treat the case $\mu_+\equiv 0$, the general case follows as well. Thus, we now describe yet another cut-off argument used to deal with the case $\mu_+\equiv 0$. As usual we assume that the lim inf in (4.3) is in fact a limit. By Lemma 4.7 we have

$$\liminf_{k\to\infty}[\mathrm{P}(A_k,\mathrm{B}_R)-\mu_-(A_k^+\cap\mathrm{B}_R)]\ge\mathrm{P}(A_\infty,\mathrm{B}_R)-\mu_-(A_\infty^+\cap\mathrm{B}_R)\tag{4.18}$$

for all $R\in(0,\infty)$. For arbitrary $\varepsilon>0$, we claim that we can choose balls $B_i=\mathrm{B}_{R_i}$ with $R_i\in(R_\varepsilon,\infty)$ and $\lim_{i\to\infty}R_i=\infty$ such that $\mu_-(\partial B_i)=0$ and

$$\lim_{k\to\infty}\mathcal{H}^{n-1}(A_k^1\cap\partial B_i)=\mathcal{H}^{n-1}(A_\infty^1\cap\partial B_i)<\varepsilon\tag{4.19}$$

hold for all $i\in\mathbb{N}$ and at least for a subsequence of $(A_k)_{k\in\mathbb{N}}$, to which we pass without reflecting this in notation. Indeed, the condition $\mu_-(\partial B_i)=0$ and the convergence of the $\mathcal{H}^{n-1}$-measures in (4.19) have already been discussed (see the proofs of Proposition 4.6 and Lemma 4.7), while the ε-bound in (4.19) can be achieved by writing out $|A_\infty^1|<\infty$ via the coarea formula in a similar way. From $\mu_-(\partial B_i)=0$, the almost-strong IC with constant 1 near ∞ (applicable for $A_k\cap B_i^{\mathrm{c}}$ in view of $R_i>R_\varepsilon$), and Lemma 2.9 we get

$$\begin{aligned}\mu_-\big(A_k^+\cap B_i^{\mathrm{c}}\big)&=\mu_-((A_k\cap B_i^{\mathrm{c}})^+)\\&\le\mathrm{P}(A_k\cap B_i^{\mathrm{c}})+\varepsilon\\&\le\mathrm{P}(A_k,(B_i^{\mathrm{c}})^+)+\mathrm{P}(B_i^{\mathrm{c}},A_k^1)+\varepsilon\\&=\mathrm{P}(A_k,B_i^{\mathrm{c}})+\mathcal{H}^{n-1}(A_k^1\cap\partial B_i)+\varepsilon\,.\end{aligned}$$

Rearranging terms and bringing in (4.19) then gives control on the terms cut off in the sense of

$$\liminf_{k\to\infty}[\mathrm{P}(A_k,B_i^{\mathrm{c}})-\mu_-(A_k^+\cap B_i^{\mathrm{c}})]\ge-\varepsilon-\lim_{k\to\infty}\mathcal{H}^{n-1}(A_k^1\cap\partial B_i)>-2\varepsilon\,.\tag{4.20}$$

To conclude, we add up (4.18) (for $R = R_i$, $\mathrm{B}_R = B_i$) and (4.20), send $i \to \infty$, and finally exploit the arbitrariness of ε. Then we arrive at (4.3) in the case $\mu_+ \equiv 0$. □

5 Existence with obstacles or volume-constraints

In this section we apply the preceding semicontinuity results on full $\mathbb{R}^n$ in proving the existence of minimizers in obstacle problems or volume-constrained problems for the functional $\mathscr{P}_{\mu_+,\mu_-}$ introduced in (4.1).

In fact, for obstacle problems with a.e. obstacle constraint, the existence proof is mostly straightforward and leads to the following statement.

Theorem 5.1 (existence in obstacle problems). *For sets $I, O \in \mathcal{M}(\mathbb{R}^n)$, $n \geq 2$, consider the admissible class*

$$\mathscr{G}_{I,O} := \{E \in \mathcal{BV}(\mathbb{R}^n) \,:\, I \subset E \subset O \text{ up to negligible sets}\}\,.$$

If there exists some $A_0 \in \mathscr{G}_{I,O}$ at all and if, for non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$, which both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1, ...

(a) *either, $\mu_-(O^+) < \infty$ holds,*

(b) *or, for some $R_0 \in (0,\infty)$ and some $\gamma \in (0,1]$, the measure μ_- also satisfies the strong IC[5] in $(\mathrm{B}_{R_0})^{\mathrm{c}}$ with constant $1{-}\gamma$,*

then there exists the minimum of the obstacle problem

$$\min\{\mathscr{P}_{\mu_+,\mu_-}[E] \,:\, E \in \mathscr{G}_{I,O}\}\,, \tag{5.1}$$

with a minimum value in $(-\mu_-(O^+),\infty)$ in case (a) and in $(-(1{-}\gamma)\mathrm{P}(\mathrm{B}_{R_0}){-}\mu_-(\overline{\mathrm{B}_{R_0}}),\infty)$ in case (b).

As a basic case, which illustrates the applicability of Theorem 5.1, we consider measurable obstacles $I \Subset O \subset \mathbb{R}^n$ and $(n{-}1)$-dimensional measures $\mu_\pm = \theta_\pm \mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} {\times} \{0\})$ with $\theta_+, \theta_- \in [0,\infty)$. Then indeed, the setting (a) applies for $\mu(O^+) < \infty$ (e.g. if O is bounded) and $\theta_+ \leq 2$, $\theta_- \leq 2$, while the setting (b) covers even fully arbitrary O up to $O = \mathbb{R}^n$ in case $\theta_+ \leq 2$, $\theta_- < 2$ (but now with $\theta_- = 2$ excluded). Specifically for $n = 2$, $O = \mathbb{R}^2$, $\theta_+ = 0$, one may also identify minimizers A in the obstacle problem (5.1) in a geometrically intuitive way, illustrated in Figure 4, as a certain convex hull of I with an additional θ_--dependent constraint on the angles at the intersection of ∂A and $\operatorname{spt} \mu_- = \mathbb{R}^{n-1} {\times} \{0\}$. However, we leave a more detailed considerations on such specific geometric cases for study elsewhere.

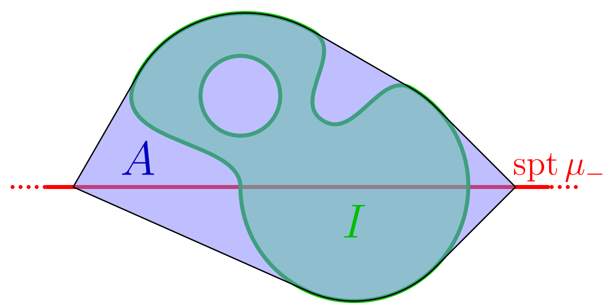

Figure 4: A minimizer A in the obstacle problem (5.1) for $n = 2$, some smooth $I \Subset \mathbb{R}^2$, $O = \mathbb{R}^2$, $\mu_+ \equiv 0$, and $\mu_- = \sqrt{2}\mathcal{H}^1 \llcorner (\mathbb{R} {\times} \{0\})$.

Here, we additionally remark that if we have $I = \emptyset$ and μ_- satisfies the strong IC even in full $\mathbb{R}^n$ with constant $1{-}\gamma$, then in view of $\mathscr{P}_{\mu_+,\mu_-}[E] \geq \gamma \mathrm{P}(E)$ for all $E \in \mathcal{BV}(\mathbb{R}^n)$ the situation of the theorem trivializes insofar that the unique minimizer up to negligible sets in (5.1) is $\emptyset$. However, our settings (a) and (b) allow for situations which do not trivialize to the same extent *even in the absence of the inner obstacle.* To demonstrate this, we consider $I := \emptyset$, an arbitrary $O \in \mathcal{M}(\mathbb{R}^n)$, any non-empty, bounded, open, convex $K \in \mathscr{G}_{I,O}$, $\mu_+ :\equiv 0$, and the finite measure $\mu_- := \theta \mathcal{H}^{n-1} \llcorner \partial K$ with $\theta \in [0,\infty)$. Then it can be checked that the obstacle problem in (5.1) has the unique minimizer $\emptyset$ in case $\theta < 1$, has both $\emptyset$ and K as minimizers in case $\theta = 1$, and has the unique minimizer K in case $\theta > 1$. Here, the measure $\mu_- = \theta \mathcal{H}^{n-1} \llcorner \partial K$ trivially satisfies the strong IC in $(\mathrm{B}_{R_0})^{\mathrm{c}}$ for R_0 large enough and by the later Theorem 8.2 satisfies the small-volume

[5]To be fully consistent with Definition 3.1, which was given on open sets, we should speak of the IC in $\big(\overline{\mathrm{B}_{R_0}}\big)^{\mathrm{c}}$ here. However, since R_0 can be increased, it does not make a difference if we work with $\overline{A} \subset \big(\overline{\mathrm{B}_{R_0}}\big)^{\mathrm{c}}$ or rather simply $A \subset (\mathrm{B}_{R_0})^{\mathrm{c}}$ instead. Thus, the slight inconsistency of writing "in $(\mathrm{B}_{R_0})^{\mathrm{c}}$" here and in the following seems justifiable.

IC in $\mathbb{R}^n$ with constant $\theta/2$, while by the later Proposition 8.1 it satisfies the strong IC in full $\mathbb{R}^n$ only with constant θ. All in all, this means that the non-trivial cases with $\theta \in [1,2]$ are indeed included in the regimes of (a) and (b) above, but would not be covered by a statement with the strong IC on full $\mathbb{R}^n$.

Proof of Theorem 5.1. We first record that $A_0 \in \mathcal{BV}(\mathbb{R}^n)$ implies $\mu_+(A_0^1) \le \mu_+(A_0^+) < \infty$ by Lemma 3.3, and thus the minimum value in (5.1) is bounded from above by $\mathrm{P}(A_0)+\mu_+(A_0^1)-\mu_-(A_0^+) < \infty$.

Now we treat the situation (a). In view of

$$\mathscr{P}_{\mu_+,\mu_-}[E] = \mathrm{P}(E)+\mu_+(E^1)-\mu_-(E^+) \ge \mathrm{P}(E) - \mu_-(O^+)$$

for all $E \in \mathscr{G}_{I,O}$, every minimizing sequence $(A_k)_{k\in\mathbb{N}}$ for $\mathscr{P}_{\mu_+,\mu_-}$ in $\mathscr{G}_{I,O}$ satisfies $\limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$. By the standard compactness and semicontinuity results from Lemmas 2.2 and 2.3, a subsequence of $(A_k)_{k\in\mathbb{N}}$ converges *locally* in measure on $\mathbb{R}^n$ to some $A_\infty \in \mathcal{M}(\mathbb{R}^n)$ with $\mathrm{P}(A_\infty) < \infty$ and $I \subset A_\infty \subset O$ up to negligible sets. Taking into account $|A_k| < \infty$, the isoperimetric estimate of Theorem 2.6 ensures $\limsup_{k\to\infty} |A_k| < \infty$, and by a basic semicontinuity property we infer $|A_\infty| < \infty$ and thus $A_\infty \in \mathscr{G}_{I,O}$. Then, Theorem 4.1(a), applied with the *finite* Radon measure $\mu_- \llcorner O^+$ instead of μ_-, ensures that the limit A_∞ is a minimizer.

Next we turn to the situation (b). Since the strong IC for μ_- in $(\mathrm{B}_{R_0})^{\mathrm{c}}$ yields

$$\begin{aligned}
\mathscr{P}_{\mu_+,\mu_-}[E] &\ge \mathrm{P}(E) - \mu_-((E\setminus \mathrm{B}_{R_0})^+) - \mu_-(\overline{\mathrm{B}_{R_0}}) \\
&\ge \mathrm{P}(E) - (1-\gamma)\mathrm{P}(E\setminus \mathrm{B}_{R_0}) - \mu_-(\overline{\mathrm{B}_{R_0}}) \\
&\ge \gamma\mathrm{P}(E) - (1-\gamma)\mathrm{P}(\mathrm{B}_{R_0}) - \mu_-(\overline{\mathrm{B}_{R_0}})
\end{aligned}$$

for all $E \in \mathscr{G}_{I,O}$, again every minimizing sequence $(A_k)_{k\in\mathbb{N}}$ for $\mathscr{P}_{\mu_+,\mu_-}$ in $\mathscr{G}_{I,O}$ satisfies $\limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$. At this stage the arguments given for the the situation (a) still yield that a subsequence of $(A_k)_{k\in\mathbb{N}}$ converges *locally* in measure on $\mathbb{R}^n$ to some $A_\infty \in \mathscr{G}_{I,O}$. Finally, by Theorem 4.1(b) we conclude that the limit A_∞ is a minimizer. □

To conclude the discussion of obstacle problems we remark that a more general point of view with thin obstacles and $\mathcal{H}^{n-1}$-a.e. obstacle constraints (compare [12, 20, 10, 6, 39], for instance) might be naturally connected to our setting, but we leave such issues for study at another point.

We now turn to volume-constrained minimization problems for $\mathscr{P}_{\mu_+,\mu_-}$, where the special case $\mu \equiv 0$ corresponds to the classical isoperimetric problem. We provide an existence statement for minimizers of $\mathscr{P}_{\mu_+,\mu_-}$ at least in case that μ_+ vanishes and μ_- is finite.

Theorem 5.2 (existence in prescribed-volume problems)**.** *Consider a non-negative Radon measure μ on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) < \infty$ and a constant $\varrho \in (0,\infty)$. If μ satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1, then there exists the minimum of the prescribed-volume problem*

$$\min\{\mathrm{P}(A)-\mu(A^+) \,:\, A \in \mathcal{BV}(\mathbb{R}^n)\,,\, |A| = \alpha_n\varrho^n\}$$

with a minimum value in $\big(-\mu(\mathbb{R}^n), n\alpha_n\varrho^{n-1}\big]$.

Here, the bounds for the minimum value leave room for improvement. For instance, estimating via the isoperimetric inequality we find that the minimum value is in fact in $\big[n\alpha_n\varrho^{n-1}-\mu(\mathbb{R}^n), n\alpha_n\varrho^{n-1}\big]$. In addition, let us point out that if μ has bounded support and ϱ is large enough such that $\operatorname{spt}\mu \subset \overline{\mathrm{B}_\varrho(x)}$ for some $x \in \mathbb{R}^n$, then $\mathrm{B}_\varrho(x)$ is a minimizer and the theorem holds trivially. In the general case, however, the result is non-trivial and the proof is somewhat involved, since (subsequences of) minimizing sequences may converge only locally, but not globally in measure, and in view of a "volume drop" at infinity the limit then violates the volume constraint and is not admissible as a minimizer. Our strategy to circumvent this phenomenon is not really new and is vaguely inspired by considerations of [22, 36], for instance. The basic idea is to suitably shift volume into a fixed ball, which in our case with $\mu_+ \equiv 0$ and $\mu_-(\mathbb{R}^n) < \infty$ can be implemented with suitable control on the values of $\mathscr{P}_{\mu_+,\mu_-}$ along the sequence. Indeed, in this way we are able to construct refined minimizing sequences with global convergence in measure and an admissible limit, which turns out to be a minimizer.

Proof. We start with the main case $n \ge 2$ and record that B_ϱ is admissible with $\mathrm{P}(\mathrm{B}_\varrho) - \mu(\mathrm{B}_\varrho^+) \le \mathrm{P}(\mathrm{B}_\varrho) = n\alpha_n\varrho^{n-1} < \infty$. Taking into account

$$\mathrm{P}(A) - \mu(A^+) \ge \mathrm{P}(A) - \mu(\mathbb{R}^n)$$

for all admissible A, it is thus clear that every minimizing sequence $(A_k)_{k\in\mathbb{N}}$ satisfies $\limsup_{k\to\infty} \mathrm{P}(A_k) < \infty$. Using compactness and semicontinuity and possibly passing to a subsequence, we get that $(A_k)_{k\in\mathbb{N}}$ converges *locally* in measure on $\mathbb{R}^n$ to some $A_\infty \in \mathcal{BV}(\mathbb{R}^n)$ with $|A_\infty| \le \alpha_n\varrho^n$.

We next choose good cut-off radii. By Fatou's lemma, the coarea formula, and the volume constraint we get

$$\int_0^\infty \liminf_{k\to\infty} \mathcal{H}^{n-1}(A_k^+ \cap \partial\mathrm{B}_R)\,\mathrm{d}R \le \lim_{k\to\infty} \int_0^\infty \mathcal{H}^{n-1}(A_k^+ \cap \partial\mathrm{B}_R)\,\mathrm{d}R = \lim_{k\to\infty} |A_k| = \alpha_n\varrho^n < \infty\,.$$

Thus, there is a sequence of radii $R_i \in (2\varrho,\infty)$ with $\lim_{i\to\infty} R_i = \infty$ and $\liminf_{k\to\infty} \mathcal{H}^{n-1}(A_k^+ \cap \partial\mathrm{B}_{R_i}) < i^{-1}$ for all $i \in \mathbb{N}$. In particular, for a suitable subsequence $(A_{k_i})_{i\in\mathbb{N}}$ of $(A_k)_{k\in\mathbb{N}}$, by the local convergence in measure and the preceding choice of radii we can achieve

$$|(A_{k_i} \Delta A_\infty) \cap \mathrm{B}_{R_i}| < i^{-1} \tag{5.2}$$

and

$$\mathcal{H}^{n-1}(A_{k_i}^+ \cap \partial\mathrm{B}_{R_i}) < i^{-1} \qquad \text{for all } i \in \mathbb{N}\,. \tag{5.3}$$

Next, since $s \mapsto |\mathrm{B}_s \setminus A_{k_i}|$ is continuous with $|\mathrm{B}_0 \setminus A_{k_i}| = 0$ (where we understand $\mathrm{B}_0 := \emptyset$ from here on) and $|\mathrm{B}_\varrho \setminus A_{k_i}| = |A_{k_i} \setminus \mathrm{B}_\varrho| \ge |A_{k_i} \setminus \mathrm{B}_{R_i}|$ (a consequence of $|A_{k_i}| = |\mathrm{B}_\varrho|$), we can also choose radii $r_i \in (0,\varrho]$ such that

$$|\mathrm{B}_{r_i} \setminus A_{k_i}| = |A_{k_i} \setminus \mathrm{B}_{R_i}| \qquad \text{for all } i \in \mathbb{N}\,,$$

and we will now attempt to produce a modified minimizing sequence without loss of volume at infinity by removing $A_{k_i} \setminus \mathrm{B}_{R_i}$ from A_{k_i} and at the same time adding $\mathrm{B}_{r_i} \setminus A_{k_i}$ for volume compensation. Indeed, this reasoning works out directly in case of

$$\mathrm{P}(A_{k_i}, \partial\mathrm{B}_{r_i}) = 0 \qquad \text{for all } i \in \mathbb{N}\,, \tag{5.4}$$

but unfortunately (5.4) cannot be ensured in general. Nonetheless, in the sequel we first complete the proof under the simplifying assumption (5.4), and we postpone the discussion how to compensate for a failure of (5.4) to the end of our reasoning. For now, we use the announced competitors

$$E_i := (A_{k_i} \cap \mathrm{B}_{R_i}) \cup \mathrm{B}_{r_i} = (A_{k_i} \cap \mathrm{B}_{R_i}) \,\dot{\cup}\, (\mathrm{B}_{r_i} \setminus A_{k_i})\,,$$

which in view of $|E_i| = |A_{k_i} \cap \mathrm{B}_{R_i}| + |\mathrm{B}_{r_i} \setminus A_{k_i}| = |A_{k_i} \cap \mathrm{B}_{R_i}| + |A_{k_i} \setminus \mathrm{B}_{R_i}| = |A_{k_i}|$ satisfy the volume constraint. In order to estimate the perimeter of E_i, we first observe

$$\mathrm{P}(E_i) \le \mathcal{H}^{n-1}((\partial\mathrm{B}_{r_i}) \setminus A_{k_i}^1) + \mathrm{P}(A_{k_i}, \mathrm{B}_{R_i} \setminus \overline{\mathrm{B}_{r_i}}) + \mathcal{H}^{n-1}(A_{k_i}^+ \cap \partial\mathrm{B}_{R_i})$$

and then continue by estimating the first term on the right-hand side. We rewrite

$$\mathcal{H}^{n-1}((\partial\mathrm{B}_{r_i}) \setminus A_{k_i}^1) = \mathrm{P}(\mathrm{B}_{r_i}) - \mathcal{H}^{n-1}(A_{k_i}^1 \cap \partial\mathrm{B}_{r_i})$$

and then on the basis of $|\mathrm{B}_{r_i}| = |\mathrm{B}_{r_i} \cap A_{k_i}| + |\mathrm{B}_{r_i} \setminus A_{k_i}| = |\mathrm{B}_{r_i} \cap A_{k_i}| + |A_{k_i} \setminus \mathrm{B}_{R_i}|$ exploit the isoperimetric inequality (2.3) to deduce

$$\mathrm{P}(\mathrm{B}_{r_i}) \le \mathrm{P}(\mathrm{B}_{r_i} \cap A_{k_i}) + \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i})\,.$$

Further we can control

$$\mathrm{P}(\mathrm{B}_{r_i} \cap A_{k_i}) \le \mathrm{P}(A_{k_i}, \mathrm{B}_{r_i}) + \mathcal{H}^{n-1}(A_{k_i}^+ \cap \partial\mathrm{B}_{r_i})\,, \qquad \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i}) \le \mathrm{P}(A_{k_i}, \mathbb{R}^n \setminus \overline{\mathrm{B}_{R_i}}) + \mathcal{H}^{n-1}(A_{k_i}^+ \cap \partial\mathrm{B}_{R_i})\,.$$

Putting together the estimates and collecting the three terms $\mathrm{P}(A_{k_i}, \mathrm{B}_{r_i})$, $\mathrm{P}(A_{k_i}, \mathrm{B}_{R_i} \setminus \overline{\mathrm{B}_{r_i}})$, $\mathrm{P}(A_{k_i}, \mathbb{R}^n \setminus \overline{\mathrm{B}_{R_i}})$ simply in $\mathrm{P}(A_{k_i})$, we arrive at

$$\mathrm{P}(E_i) \le \mathrm{P}(A_{k_i}) + \mathcal{H}^{n-1}((A_{k_i}^+ \setminus A_{k_i}^1) \cap \partial\mathrm{B}_{r_i}) + 2\mathcal{H}^{n-1}(A_{k_i}^+ \cap \partial\mathrm{B}_{R_i})\,.$$

Here, the middle term on the right-hand side can be rewritten as $\mathrm{P}(A_{k_i},\partial\mathrm{B}_{r_i})$ and vanishes under the simplifying assumption (5.4), while the last term on the right-hand side is controlled by $2i^{-1}$ through (5.3). Also bringing in that we have $\mu(E_i^+)\geq\mu(A_{k_i}^+\cap\mathrm{B}_{R_i})=\mu(A_{k_i}^+)-\mu(\mathbb{R}^n\setminus\mathrm{B}_{R_i})$, we finally arrive at

$$\mathrm{P}(E_i)-\mu(E_i^+)\leq\mathrm{P}(A_{k_i})-\mu(A_{k_i}^+)+2i^{-1}+\mu((\mathrm{B}_{R_i})^{\mathrm{c}})\,.$$

Then, crucially exploiting $\lim_{i\to\infty}R_i=\infty$ and $\mu(\mathbb{R}^n)<\infty$, we have $\lim_{i\to\infty}\mu(\mathbb{R}^n\setminus\mathrm{B}_{R_i})=0$ and can conclude that with $(A_k)_{k\in\mathbb{N}}$ also $(E_i)_{i\in\mathbb{N}}$ is a minimizing sequence in the volume-constrained problem.

Now, in view of $r_i\leq\varrho$ for all $i\in\mathbb{N}$, by passing to subsequences we can assume that $r:=\lim_{i\to\infty}r_i\in[0,\varrho]$ exists, and we finally proceed to establish that $A_\infty\cup\mathrm{B}_r$ is a minimizer in the volume-constrained problem. To this end we record that $E_i=(A_{k_i}\cap\mathrm{B}_{R_i})\cup\mathrm{B}_{r_i}$ converge *locally* in measure on $\mathbb{R}^n$ to $A_\infty\cup\mathrm{B}_r$, since in this local sense we have the convergences $A_{k_i}\to A_\infty$, $\mathrm{B}_{R_i}\to\mathbb{R}^n$, $\mathrm{B}_{r_i}\to\mathrm{B}_r$. In order to show admissibility of $A_\infty\cup\mathrm{B}_r$, for arbitrary $i\in\mathbb{N}$, we split

$$\alpha_n\varrho^n=|A_{k_i}|=|A_{k_i}\cap\mathrm{B}_{R_i}|+|A_{k_i}\setminus\mathrm{B}_{R_i}|\,,$$

and via (5.2), the choice of r_i, and the *local* convergence in measure $A_k\to A_\infty$ deduce for the right-hand volumes the convergences

$$\lim_{i\to\infty}|A_{k_i}\cap\mathrm{B}_{R_i}|=\lim_{i\to\infty}|A_\infty\cap\mathrm{B}_{R_i}|=|A_\infty|\qquad\text{and}\qquad\lim_{i\to\infty}|A_{k_i}\setminus\mathrm{B}_{R_i}|=\lim_{i\to\infty}|\mathrm{B}_{r_i}\setminus A_{k_i}|=|\mathrm{B}_r\setminus A_\infty|\,.$$

This implies that $A_\infty\cup\mathrm{B}_r$ fulfills the volume constraint $\alpha_n\varrho^n=|A_\infty|+|\mathrm{B}_r\setminus A_\infty|=|A_\infty\cup\mathrm{B}_r|$. Thus, we are in the position to finally use the semicontinuity in Theorem 4.1[6] along the minimizing sequence E_i with limit $A_\infty\cup\mathrm{B}_r$ and deduce that $A_\infty\cup\mathrm{B}_r$ is a minimizer in the volume-constrained problem.

It remains to provide an argument in case (5.4) fails. In this situation, since $\mathrm{P}(A_{k_i},\partial\mathrm{B}_q)=0$ holds for all but countably many $q\in(0,\infty)$ (and trivially for $q=0$), we can pass to ever-so-slightly-decreased good radii $q_i\in[0,r_i]$. However, in view of the volume constraint we cannot directly use $(A_{k_i}\cap\mathrm{B}_{R_i})\cup\mathrm{B}_{q_i}$ as competitors but rather need to compensate once more for the slight loss of volume. In fact, fixing arbitrary points $x_i\in(\mathrm{B}_{R_i}\setminus\overline{\mathrm{B}_{r_i}})$ with $|\mathrm{B}_\delta(x_i)\setminus A_{k_i}|>0$ for all $\delta>0$ (such points exist, since $|A_{k_i}|=\alpha_n\varrho^n\leq|\mathrm{B}_{2\varrho}\setminus\mathrm{B}_\varrho|<|\mathrm{B}_{R_i}\setminus\overline{\mathrm{B}_{r_i}}|$), for every $q_i\in[0,r_i]$, we find by continuity some $\delta_i\in[0,\infty)$ with $|\mathrm{B}_{q_i}\setminus A_{k_i}|+|\mathrm{B}_{\delta_i}(x_i)\setminus A_{k_i}|=|\mathrm{B}_{r_i}\setminus A_{k_i}|$. Moreover, if we take q_i arbitrarily close to r_i, then in view of $|\mathrm{B}_\delta(x_i)\setminus A_{k_i}|>0$ for all $\delta>0$ this results in δ_i coming arbitrarily close to 0. We can thus choose $q_i\in[0,r_i]$ with $\mathrm{P}(A_k,\partial\mathrm{B}_{q_i})=0$ close enough to r_i to ensure for a corresponding $\delta_i\in[0,\infty)$ that $\delta_i<i^{-1}$ and $\mathrm{B}_{\delta_i}(x_i)\Subset\mathrm{B}_{R_i}\setminus\overline{\mathrm{B}_{r_i}}$. Then it can be checked that

$$\widetilde{E}_i:=(A_{k_i}\cap\mathrm{B}_{R_i})\cup\mathrm{B}_{q_i}\cup\mathrm{B}_{\delta_i}(x_i)$$

satisfies the volume constraint. Moreover, we can estimate $\mathrm{P}(\widetilde{E}_i)$ essentially in the same way as $\mathrm{P}(E_i)$, just with an extra term controlled by $\mathrm{P}(\mathrm{B}_{\delta_i}(x_i))=n\alpha_n\delta_i^{n-1}<n\alpha_n i^{1-n}$. In this way we deduce

$$\mathrm{P}(\widetilde{E}_i)-\mu(\widetilde{E}_i^+)\leq\mathrm{P}(A_{k_i})-\mu(A_{k_i}^+)+2i^{-1}+n\alpha_n i^{1-n}+\mu((\mathrm{B}_{R_i})^{\mathrm{c}})\,,$$

which is still sufficient to conclude that the modified sequence $(\widetilde{E}_i)_{i\in\mathbb{N}}$ is a minimizing sequence for the volume-constrained problem. From this point onwards, taking into account $\lim_{i\to\infty}|\mathrm{B}_{\delta_i}(x_i)|=0$ the verification of the volume constraint for $A_\infty\cup\mathrm{B}_r$ with $r=\lim_{i\to\infty}r_i=\lim_{i\to\infty}q_i$ and the remainder of the reasoning work almost exactly as described before.

Finally, in the case $n=1$ a similar reasoning with major simplifications applies, where now each A_k with volume constraint $|A_k|=2\varrho$ can be represented as a union of finitely many bounded intervals and in particular satisfies $A_k^+=\overline{A_k}$ and $A_k^1=\mathrm{int}(A_k)$. Indeed, the beginning of the reasoning up to the choice of the radii R_i stays essentially unchanged with (5.3) now simplifying to $\pm R_i\notin\overline{A_{k_i}}$. However, the construction of competitors with compensated volume vastly simplifies with the need for (5.4) completely dropping out. In fact, we claim that by choice of an interval $I_i\subset\mathrm{B}_\varrho\subset\mathrm{B}_{R_i}$ (where the balls are also intervals, but for brevity we keep the B-notation) one can ensure that

$$E_i:=(A_{k_i}\cap\mathrm{B}_{R_i})\cup I_i$$

[6]More precisely, one way of reasoning at this point is to use the semicontinuity assertion from Theorem 4.1(a), which draws on the finiteness of μ and needs local convergence only. Another way is to rely only on the case covered in each of Theorem 1.2, Theorem 4.1(c), and Proposition 4.5 on the basis of the observation that the coincidence of volumes $|E_i|=\alpha_n\varrho^n=|A_\infty\cup\mathrm{B}_r|$ improves the local convergence to global convergence required in these statements.

satisfies the constraint $|E_i| = 2\varrho$ and the simple bound $\mathrm{P}(E_i) \le \mathrm{P}(A_{k_i})$. To prove this claim, first consider the case $|A_{k_i} \cap \mathrm{B}_\varrho| > 0$. Then a continuity argument gives an interval $I_i \subset \mathrm{B}_\varrho$ with $|I_i \cap A_{k_i}| > 0$ and $|I_i \setminus A_{k_i}| = |A_{k_i} \setminus \mathrm{B}_{R_i}|$, and this suffices to conclude $|E_i| = |A_{k_i}| = 2\varrho$ and $\mathrm{P}(E_i) \le \mathrm{P}(A_{k_i} \cap \mathrm{B}_{R_i}) \le \mathrm{P}(A_{k_i})$ (where the former estimate holds, since I_i intersects at least one interval of $A_{k_i} \cap \mathrm{B}_{R_i}$). In case $|A_{k_i} \cap \mathrm{B}_\varrho| = 0$ the simple choice $I_i := \mathrm{B}_{r_i}$ with $r_i := \frac{1}{2}|A_{k_i} \setminus \mathrm{B}_{R_i}| \in [0, \varrho]$ gives $|E_i| = |A_{k_i}| = 2\varrho$ and $\mathrm{P}(I_i) \le \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i})$ (as either $\mathrm{P}(I_i) = 0 = \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i})$ or $\mathrm{P}(I_i) = 2 \le \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i})$). Then in view of $\pm R_i \notin \overline{A_{k_i}}$ one still gets $\mathrm{P}(E_i) \le \mathrm{P}(A_{k_i} \cap \mathrm{B}_{R_i}) + \mathrm{P}(A_{k_i} \setminus \mathrm{B}_{R_i}) = \mathrm{P}(A_{k_i})$. With these properties of E_i and the unchanged estimate for $\mu(E_i^+)$, one directly infers that $(E_i)_{i\in\mathbb{N}}$ is a minimizing sequence in the volume-constrained problem with (after passage to a subsequence) limit $A_\infty \cup I$ for some interval $I \subset \mathrm{B}_\varrho$. As in the case $n \ge 2$ one then concludes that the convergence $E_i \to A_\infty \cup I$ looses no volume at infinity and that $A_\infty \cup I$ is a minimizer. $\square$

6 Lower semicontinuity and existence for Dirichlet problems

In this section we adapt the semicontinuity results of Section 4 to a setting with a (generalized) Dirichlet condition on the boundary of an open set $\Omega \subset \mathbb{R}^n$. To this end we prescribe the Dirichlet datum by means of a set $A_0 \in \mathcal{M}(\mathbb{R}^n)$ and consider the class

$$\begin{aligned} \mathscr{D}_{A_0}(\Omega) &:= \{E \in \mathcal{M}(\mathbb{R}^n) \,:\, \mathrm{P}(E, \overline{\Omega}) < \infty\,,\, E \setminus \Omega = A_0 \setminus \Omega\} \\ &= \{E \in \mathcal{M}(\mathbb{R}^n) \,:\, \mathrm{P}(E, \overline{\Omega}) < \infty\,,\, E \Delta A_0 \subset \Omega\}\,, \end{aligned} \tag{6.1}$$

in which sets of finite perimeter are extended from Ω to (a neighborhood of) $\overline{\Omega}$ by coincidence with the given A_0 outside Ω. In addition, we prescribe once more measures μ_+ and μ_-, which in principle live on $\overline{\Omega}$, but for which we can indeed express finiteness on all bounded sets and suitable ICs in a convenient way by considering them as a Radon measure on all of $\mathbb{R}^n$ such that $\mu_\pm \llcorner (\overline{\Omega})^{\mathrm{c}} \equiv 0$. Given the data A_0 and $\mu_\pm$ we then aim at minimizing among all $E \in \mathscr{D}_{A_0}(\Omega)$ the adaptation of the previously considered functional

$$\mathscr{P}_{\mu_+,\mu_-}[E; \overline{\Omega}] := \mathrm{P}(E, \overline{\Omega}) + \mu_+(E^1) - \mu_-(E^+)\,, \tag{6.2}$$

which is defined for $E \in \mathcal{M}(\mathbb{R}^n)$ if at least one of $\mathrm{P}(E, \overline{\Omega}) + \mu_+(E^1)$ and $\mu_-(E^+)$ is finite and specifically for $E \in \mathscr{D}_{A_0}(\Omega)$ with $\min\{\mu_+(E^1), \mu_-(E^+)\} < \infty$. Here — as customary in the $\mathcal{BV}$ setting and essentially required by the lack of weak closedness of traces — it is tolerated for $E \in \mathscr{D}_{A_0}(\Omega)$ that ∂E deviates from ∂A_0 at $\partial\Omega$, but such deviations are accounted for by taking the perimeter on $\overline{\Omega}$ and thus including $\mathrm{P}(E, \partial\Omega)$ in the functional.

With view towards non-parametric Dirichlet problems we will include — to the extent possible in a general parametric theory — unbounded domains Ω (e.g. cylinders $\Omega = D \times \mathbb{R}$ over open $D \subset \mathbb{R}^{n-1}$) and infinite measures $\mu_\pm$ (e.g. product measures $\mu_\pm = \lambda_\pm \otimes \mathcal{L}^1$ with finite Radon measures $\lambda_\pm = \lambda_\pm \llcorner \overline{D}$). Thus, the application of our results in the case $\Omega = D \times \mathbb{R}$, $\mu_\pm = \lambda_\pm \otimes \mathcal{L}^1$ is possible, but nonetheless does not directly yield a satisfactory non-parametric theory, since in this case the μ-terms in (6.2) are usually infinite on subgraphs of functions and thus do not detect the finer behavior of such non-parametric competitors. In this article, we do *not* elaborate on this technical point, but indeed we presume that it can be overcome by first looking at one-sided cases with $\Omega = D \times (z, \infty)$, $\mu_\pm = \lambda_\pm \otimes (\mathcal{L}^1 \llcorner (z, \infty))$ with $z \in \mathbb{R}$ (which are fully accessible by our means), then normalizing the μ-terms relative to a zero level or another reference configuration, and finally sending $z \to -\infty$. However, all further details of such a procedure are deferred for treatment elsewhere.

We now come back to the parametric cases under consideration here and provide our results in form of a semicontinuity theorem and an existence theorem, which both apply for the functional in (6.2) inside Dirichlet classes of type (6.1).

Theorem 6.1 (lower semicontinuity in a Dirichlet class)**.** *Consider an open set Ω in $\mathbb{R}^n$, a set $A_\infty \in \mathcal{M}(\mathbb{R}^n)$, a sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^n)$, and assume that non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ with $\mu_\pm \llcorner (\overline{\Omega})^{\mathrm{c}} \equiv 0$ both satisfies the small-volume IC in $\mathbb{R}^n$ with constant* 1*. Furthermore, assume that* one of the following sets of additional assumptions *is valid*:

(a) *The measure μ_- is* finite*, and A_k converge to A_∞* locally *in measure on $\mathbb{R}^n$ with $A_k \setminus \Omega = A_\infty \setminus \Omega$ for all $k \in \mathbb{N}$.*

(b) *The measure* μ_- *additionally satisfies the* almost-strong IC with constant 1 near ∞ *in the sense that, for every* $\varepsilon > 0$, *there exists some* $R_\varepsilon \in (0,\infty)$ *with* (4.2), *and* A_k *converge to* A_∞ locally *in measure on* $\mathbb{R}^n$ *with* $|A_k \Delta A_\infty| + \mathrm{P}(A_k, \overline{\Omega}) + \mathrm{P}(A_\infty, \overline{\Omega}) < \infty$, $A_k \setminus \Omega = A_\infty \setminus \Omega$, *and* $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ *for all* $k \in \mathbb{N}$.

(c) *The sets* A_k *converge to* A_∞ globally *in measure on* $\mathbb{R}^n$ *with* $\mathrm{P}(A_k, \overline{\Omega}) + \mathrm{P}(A_\infty, \overline{\Omega}) < \infty$, $A_k \setminus \Omega = A_\infty \setminus \Omega$, *and* $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ *for all* $k \in \mathbb{N}$.

Then we have $\min\{\mu_+(A_\infty^1), \mu_-(A_\infty^+)\} < \infty$ *and*

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k; \overline{\Omega}] \geq \mathscr{P}_{\mu_+,\mu_-}[A_\infty; \overline{\Omega}] \,. \tag{6.3}$$

Before approaching the proof of Theorem 6.1 we address some interconnected technical points.

First we remark that the hypotheses $\mathrm{P}(A_k, \overline{\Omega}) + \mathrm{P}(A_\infty, \overline{\Omega}) < \infty$ and $A_k \setminus \Omega = A_\infty \setminus \Omega$ of the situations (b) and (c) can be expressed alternatively as $A_k, A_\infty \in \mathscr{D}_{A_0}(\Omega)$ for some $A_0 \in \mathcal{M}(\mathbb{R}^n)$ or — by considering the limit A_∞ itself as the boundary datum — also as $A_k, A_\infty \in \mathscr{D}_{A_\infty}(\Omega)$. Moreover, introducing, for open $\Omega \subset \mathbb{R}^n$ and $A_0 \in \mathcal{M}(\mathbb{R}^n)$, the subclass

$$\mathscr{F}_{A_0}(\Omega) := \{E \in \mathcal{M}(\mathbb{R}^n) \,:\, |E \Delta A_0| + \mathrm{P}(E, \overline{\Omega}) < \infty \,,\, E \setminus \Omega = A_0 \setminus \Omega\}$$

of $\mathscr{D}_{A_0}(\Omega)$, we may include the additional requirement $|A_k \Delta A_\infty| < \infty$ by writing $A_k, A_\infty \in \mathscr{F}_{A_0}(\Omega)$ for some $A_0 \in \mathcal{M}(\mathbb{R}^n)$ or $A_k, A_\infty \in \mathscr{F}_{A_\infty}(\Omega)$. If there exists some $E_0 \in \mathscr{F}_{A_0}(\Omega)$ at all (e.g. if $\mathrm{P}(A_0, \overline{\Omega}) < \infty$), we can also rewrite[7]

$$\mathscr{F}_{A_0}(\Omega) = \{E \in \mathcal{M}(\mathbb{R}^n) \,:\, E \Delta E_0 \in \mathcal{BV}(\mathbb{R}^n) \,,\, E \Delta E_0 \subset \Omega\} \,.$$

Furthermore, we record the following generalization of Lemma 3.3, which is adapted for the class $\mathscr{F}_{A_0}(\Omega)$.

Lemma 6.2. *Consider an open set* $\Omega \subset \mathbb{R}^n$ *and a set* $A_0 \in \mathcal{M}(\mathbb{R}^n)$. *If a non-negative Radon measure* μ *on* $\mathbb{R}^n$ *satisfies the small-volume IC in* $\mathbb{R}^n$ *with constant* $C \in [0, \infty)$, *then* $\mu(E_0^1) < \infty$ *for* some $E_0 \in \mathscr{F}_{A_0}(\Omega)$ *implies in fact* $\mu(E^1) < \infty$ *for* all $E \in \mathscr{F}_{A_0}(\Omega)$, *and similarly* $\mu(E_0^+) < \infty$ *for* some $E_0 \in \mathscr{F}_{A_0}(\Omega)$ *implies* $\mu(E^+) < \infty$ *for* all $E \in \mathscr{F}_{A_0}(\Omega)$.

Proof. For $E, E_0 \in \mathscr{F}_{A_0}(\Omega)$, we have already recorded $E \Delta E_0 \in \mathcal{BV}(\mathbb{R}^n)$, and then by Lemma 3.3 we infer $\mu(E^1 \Delta E_0^1) \leq \mu((E \Delta E_0)^+) < \infty$ and $\mu(E^+ \Delta E_0^+) \leq \mu((E \Delta E_0)^+) < \infty$. Therefore, $\mu(E_0^1) < \infty$ implies $\mu(E^1) < \infty$, and $\mu(E_0^+) < \infty$ implies $\mu(E^+) < \infty$. □

Next some more remarks on the requirement $|A_k \Delta A_\infty| < \infty$ are in order.

Remark 6.3 (on the role of $|A_k \Delta A_\infty| < \infty$ in Theorem 6.1)**.** *While most requirements in Theorem 6.1 are natural and/or resemble features from Theorem 4.1, we find it worth pointing out that the finite-volume requirement for* $A_k \Delta A_\infty$ *of the setting* (b) *is automatically satisfied in many cases, but cannot be omitted in full generality. This is clarified by the following points, which apply for any open* $\Omega \subset \mathbb{R}^n$ *and* $A_0 \in \mathcal{M}(\mathbb{R}^n)$:

(i) *In analogy with Theorem 4.1, in the setting* (a) *the requirement* $|A_k \Delta A_\infty| < \infty$ *is simply not necessary. Moreover, in the setting* (c) *we do not require* $|A_k \Delta A_\infty| < \infty$ *explicitly, but have it implicitly (at least for* $k \gg 1$*) through the global convergence assumed there.*

(ii) *If we have* $n \geq 2$ *and* Ω *is not too close to full space in the sense of* $\mathrm{Cap}_1((\Omega^1)^{\mathrm{c}}) = \infty$ *(as it follows from* $|\Omega^{\mathrm{c}}| = \infty$*, for instance), then, for* $A, E \in \mathscr{D}_{A_0}(\Omega)$ *we always have* $|E \Delta A| < \infty$. *Thus, in this case we have* $\mathscr{F}_{A_0}(\Omega) = \mathscr{D}_{A_0}(\Omega)$ *whenever* $\mathscr{F}_{A_0}(\Omega) \neq \emptyset$, *and also in the setting* (b) *the condition* $|A_k \Delta A_\infty| < \infty$ *is automatically satisfied and need not be required explicitly.*

Proof. From $E \Delta A \subset (A \Delta A_0) \cup (E \Delta A_0) \subset \Omega$ we get $(E \Delta A)^1 \subset \Omega^1$ and $\mathrm{P}(E \Delta A) \leq \mathrm{P}(E, \overline{\Omega}) + \mathrm{P}(A, \overline{\Omega}) < \infty$. Then the isoperimetric estimate of Theorem 2.6 yields $\min\{|E \Delta A|, |(E \Delta A)^{\mathrm{c}}|\} < \infty$. In case $|(E \Delta A)^{\mathrm{c}}| < \infty$, however, observing $(\Omega^1)^{\mathrm{c}} \subset ((E \Delta A)^1)^{\mathrm{c}} = ((E \Delta A)^{\mathrm{c}})^+$ together with $(E \Delta A)^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$ we get $\mathrm{Cap}_1((\Omega^1)^{\mathrm{c}}) < \infty$ from Proposition 2.15. This leaves $|E \Delta A| < \infty$ as the sole possibility. □

[7] Indeed, the alternative characterization of $\mathscr{F}_{A_0}(\Omega)$ results from the following elementary observations (for Ω, A_0, E_0 as above). For $E \in \mathcal{M}(\mathbb{R}^n)$, we have $E \setminus \Omega = A_0 \setminus \Omega \iff E \Delta E_0 \subset \Omega$ and also $|E \Delta A_0| < \infty \iff |E \Delta E_0| < \infty$. Moreover, for $E \in \mathcal{M}(\mathbb{R}^n)$ with $E \Delta E_0 \subset \Omega$, in view of $\mathrm{P}(E \Delta E_0) = \mathrm{P}(E \Delta E_0, \overline{\Omega})$ we get $\mathrm{P}(E \Delta E_0) < \infty \iff \mathrm{P}(E, \overline{\Omega}) < \infty$.

(iii) *If we have $n \geq 2$ and Ω is close enough to full space in the sense of $\mathrm{Cap}_1((\overline{\Omega})^{\mathrm{c}}) < \infty$ (as it follows from $(\overline{\Omega})^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$, for instance), then from Proposition 2.15 we get $(\overline{\Omega})^{\mathrm{c}} \subset H^+$ for some $H \in \mathcal{BV}(\mathbb{R}^n)$, and for every $E \in \mathcal{M}(\mathbb{R}^n)$ with $\mathrm{P}(E, \overline{\Omega}) < \infty$ either E or E^{c} is in $\mathcal{BV}(\overline{\Omega})$. Specifically, for $A, E \in \mathscr{D}_{A_0}(\Omega)$, the requirement $|E \Delta A| < \infty$ then means that either $A, E \in \mathcal{BV}(\overline{\Omega})$ or $A^{\mathrm{c}}, E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$ holds, and the hypotheses of the setting (b) can be reformulated correspondingly.*

Proof that either E or E^{c} is in $\mathcal{BV}(\overline{\Omega})$. By assumption we have $\mathrm{P}(E, U) < \infty$ for an open $U \supset \overline{\Omega}$, from which we infer $\mathrm{P}(E \cup H) < \infty$, since $\mathbb{R}^n$ is covered by the open sets U and $(\overline{\Omega})^{\mathrm{c}}$ and since $E \cup H$ has finite perimeter in U and even zero perimeter in $(\overline{\Omega})^{\mathrm{c}}$. This enforces $\min\{|E \cup H|, |(E \cup H)^{\mathrm{c}}|\} < \infty$ once more by Theorem 2.6. In view of $|H| < \infty$ we deduce $\min\{|E|, |E^{\mathrm{c}}|\} < \infty$ and consequently either $E \in \mathcal{BV}(\overline{\Omega})$ or $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$. □

(iv) *In case $\mathrm{Cap}_1((\overline{\Omega})^{\mathrm{c}}) < \infty$, $\mu_-(\mathbb{R}^n) = \infty$ the explicit requirement $|A_k \Delta A_\infty| < \infty$ cannot be dropped from the setting (b), since lower semicontinuity fails with $\mathscr{P}_{\mu_+,\mu_-}[A_k; \overline{\Omega}] = -\infty$ for $k \in \mathbb{N}$, but $\mathscr{P}_{\mu_+,\mu_-}[A_\infty; \overline{\Omega}] = 0$, for instance, if we use H from point (iii) and take $A_k := (\mathrm{B}_k \cup H)^{\mathrm{c}}$ with $A_k^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$, $A_k \setminus \Omega = \emptyset$ and $A_\infty := \emptyset \in \mathcal{BV}(\mathbb{R}^n)$.*

(v) *For each open $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in view of $\Omega^1 \subset \overline{\Omega}$ at least one of the points (ii) and (iii) applies, and sometimes even both apply. For instance, the latter happens for dense open $\Omega \subset \mathbb{R}^n$ with $|\Omega^{\mathrm{c}}| = \infty$.*

Finally, we turn to the proof of the proposition.

Proof of Theorem 6.1. The subsidiary claim $\min\{\mu_+(A^1_\infty), \mu_-(A^+_\infty)\} < \infty$ is trivially satisfied in the situation (a) with finite μ_-. It is also satisfied in the situations (b) and (c), since in these we have $A_k, A_\infty \in \mathscr{F}_{A_\infty}(\Omega)$ (at least for $k \gg 1$) and since we know from Lemma 6.2 that $\mu_+(A^1_k) < \infty$ even for a single $A_k \in \mathscr{F}_{A_\infty}(\Omega)$ implies $\mu_+(A^1_\infty) < \infty$ and likewise $\mu_-(A^+_k) < \infty$ implies $\mu_-(A^+_\infty) < \infty$.

To shorten notation, in the remainder of this proof we abbreviate

$$\langle \mu_\pm \,; A \rangle := \mu_+(A^1) - \mu_-(A^+)\,,$$

and we record that, in all three situations, Lemma 4.7 yields

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k, \mathrm{B}_R) + \langle \mu_\pm \llcorner \mathrm{B}_R \,; A_k \rangle\big] \geq \mathrm{P}(A_\infty, \mathrm{B}_R) + \langle \mu_\pm \llcorner \mathrm{B}_R \,; A_\infty \rangle \qquad \text{for all } R \in (0, \infty)\,.$$

Moreover, whenever we additionally ensure $A_k, A_\infty \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ for $k \gg 1$, then in view of $A_k \setminus \Omega = A_\infty \setminus \Omega$ we may subtract $\mathrm{P}(A_k, \mathrm{B}_R \setminus \overline{\Omega}) = \mathrm{P}(A_\infty, \mathrm{B}_R \setminus \overline{\Omega}) < \infty$ on both sides to arrive at

$$\liminf_{k\to\infty} \big[\mathrm{P}(A_k, \overline{\Omega} \cap \mathrm{B}_R) + \langle \mu_\pm \llcorner \mathrm{B}_R \,; A_k \rangle\big] \geq \mathrm{P}(A_\infty, \overline{\Omega} \cap \mathrm{B}_R) + \langle \mu_\pm \llcorner \mathrm{B}_R \,; A_\infty \rangle \tag{6.4}$$

Taking these preliminary observations as a starting point, we now deal with the three situations separately, where throughout we can and do assume that $\lim_{k\to\infty} \big[\mathrm{P}(A_k, \overline{\Omega}) + \langle \mu_\pm \,; A_k \rangle\big]$ exists and is finite.

We first treat the situation (a). Since in this case μ_- is finite, we directly get $\limsup_{k\to\infty} \mathrm{P}(A_k, \overline{\Omega}) < \infty$, and then, using the lower semicontinuity of the perimeter and $A_k \setminus \Omega = A_\infty \setminus \Omega$, we infer $\mathrm{P}(A_k, U) + \mathrm{P}(A_\infty, U) < \infty$ for $k \gg 1$ on a fixed open $U \supset \overline{\Omega}$. This finding and the assumption $\mu_\pm \llcorner (\overline{\Omega})^{\mathrm{c}} = 0$ open the way to modify A_k and A_∞ away from $\overline{\Omega}$ and ensure that there is no loss of generality in assuming $A_k, A_\infty \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ for $k \gg 1$ and the validity of (6.4). Trivially estimating on the left-hand side of (6.4), we deduce, for all $R \in (0, \infty)$,

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k; \overline{\Omega}] + \mu_-((\mathrm{B}_R)^{\mathrm{c}}) \geq \mathrm{P}(A_\infty, \overline{\Omega} \cap \mathrm{B}_R) + \langle \mu_\pm \llcorner \mathrm{B}_R \,; A_\infty \rangle\,,$$

and then, sending $R \to \infty$ and crucially exploiting the finiteness of μ_-, we arrive at the claim (6.3).

Next we turn to the situation (b). From the assumptions $A_k, A_\infty \in \mathscr{D}_{A_\infty}(\Omega)$ we get $\mathrm{P}(A_k, U) + \mathrm{P}(A_\infty, U) < \infty$ for all $k \in \mathbb{N}$ on a fixed open $U \supset \overline{\Omega}$. Again this means that we may modify A_k and A_∞ away from $\overline{\Omega}$ and may assume the validity of (6.4). For arbitrary $\varepsilon > 0$, relying on cut-off arguments as in the proofs of Proposition 4.6 and Lemma 4.7 we obtain radii $R_i \in (R_\varepsilon, \infty)$ with $\lim_{i\to\infty} R_i = \infty$ and replace $(A_k)_{k\in\mathbb{N}}$ with one of its subsequences such that there hold $\mu_-(\partial \mathrm{B}_{R_i}) = 0$ and $\lim_{k\to\infty} \mathcal{H}^{n-1}((A_k \Delta A_\infty)^+ \cap \partial \mathrm{B}_{R_i}) = 0$

for all $i \in \mathbb{N}$. We exploit $\mu_-(\partial \mathrm{B}_{R_i}) = 0$ and bring in the assumptions $A_k \Delta A_\infty \subset \Omega$, $|A_k \Delta A_\infty| < \infty$ and the assumed almost-strong IC near ∞ (applicable in view of $R_i > R_\varepsilon$) in the decisive estimate

$$\begin{aligned}\mu_-((A_k^+ \Delta A_\infty^+) \setminus \mathrm{B}_{R_i}) &\le \mu_-(((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i})^+) \le \mathrm{P}((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i}) + \varepsilon = \mathrm{P}((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i}, \overline{\Omega}) + \varepsilon \\ &\le \mathrm{P}(A_k, \overline{\Omega} \setminus \mathrm{B}_{R_i}) + \mathrm{P}(A_\infty, \overline{\Omega} \setminus \mathrm{B}_{R_i}) + \mathcal{H}^{n-1}((A_k \Delta A_\infty)^+ \cap \partial \mathrm{B}_{R_i}) + \varepsilon\,. \end{aligned} \tag{6.5}$$

Taking into account $\mu_-(\mathrm{B}_{R_i}) < \infty$, the estimate (6.5) yields in particular $\mu_-(A_k^+ \Delta A_\infty^+) < \infty$ and thus leaves us with the alternative that either $\mu_-(A_k^+) = \mu_-(A_\infty^+) = \infty$ holds for all $k \in \mathbb{N}$ or $\mu_-(A_k^+)+\mu_-(A_\infty^+) < \infty$ holds for all $k \in \mathbb{N}$. In the case $\mu_-(A_k^+) = \mu_-(A_\infty^+) = \infty$, taking into account $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ and $\min\{\mu_+(A_\infty^1), \mu_-(A_\infty^+)\} < \infty$, we necessarily have $\mu_+(A_k^1)+\mu_+(A_\infty^1) < \infty$ for all $k \in \mathbb{N}$, and (6.3) is trivially satisfied with value $-\infty$ on both sides. Thus, from here on we deal with the case $\mu_-(A_k^+)+\mu_-(A_\infty^+) < \infty$ only. We rearrange the terms in (6.5), pass $k \to \infty$, and involve $\lim_{k\to\infty} \mathcal{H}^{n-1}((A_k \Delta A_\infty)^+ \cap \partial \mathrm{B}_{R_i}) = 0$ to conclude

$$\liminf_{k\to\infty} \left[\mathrm{P}(A_k, \overline{\Omega} \setminus \mathrm{B}_{R_i}) - \mu_-(A_k^+ \setminus \mathrm{B}_{R_i})\right] \ge -\mathrm{P}(A_\infty, \overline{\Omega} \setminus \mathrm{B}_{R_i}) - \mu_-(A_\infty^+ \setminus \mathrm{B}_{R_i}) - \varepsilon\,, \tag{6.6}$$

where now all the single terms are finite. Clearly, on the left-hand side we may replace $-\mu_-(A_k^+ \setminus \mathrm{B}_{R_i})$ with $\langle \mu_\pm \llcorner (\mathrm{B}_{R_i})^{\mathrm{c}}\,; A_k \rangle$, which is only larger. Adding up (6.4) (with $R = R_i$) and this slightly modified version of (6.6), we get

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\overline{\Omega}] \ge \mathrm{P}(A_\infty, \overline{\Omega} \cap \mathrm{B}_{R_i}) + \langle \mu_\pm \llcorner \mathrm{B}_{R_i}\,; A_\infty \rangle - \mathrm{P}(A_\infty, \overline{\Omega} \setminus \mathrm{B}_{R_i}) - \mu_-(A_\infty^+ \setminus \mathrm{B}_{R_i}) - \varepsilon$$

for all $i \in \mathbb{N}$. We now rewrite $\langle \mu_\pm \llcorner \mathrm{B}_{R_i}\,; A_\infty \rangle - \mu_-(A_\infty^+ \setminus \mathrm{B}_{R_i}) = \mu_+(A_\infty^1 \cap \mathrm{B}_{R_i}) - \mu_-(A_\infty^+)$ on the right-hand side, send $i \to \infty$, and exploit $\lim_{i\to\infty} R_i = \infty$. Keeping in mind that $\mathrm{P}(A_\infty, \overline{\Omega}) < \infty$ and $\mu_-(A_\infty^+) < \infty$ in the presently considered case and finally exploiting the arbitrariness of ε, we then obtain the claim (6.3) also in the situation (b).

Finally, in order to handle the situation (c) it suffices to slightly adapt the estimate (6.5) in the reasoning used for (b). Indeed, now we simply take $R_i \in (0,\infty)$ rather than $R_i \in (R_\varepsilon, \infty)$, and only eventually, given an arbitrary $\varepsilon > 0$, we exploit the global convergence $\lim_{k\to\infty} |A_k \Delta A_\infty| = 0$ assumed in (c) to find

$$\begin{aligned}\mu_-((A_k^+ \Delta A_\infty^+) \setminus \mathrm{B}_{R_i}) &\le \mu_-(((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i})^+) \le \mathrm{P}((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i}) + \varepsilon = \mathrm{P}((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i}, \overline{\Omega}) + \varepsilon \\ &\le \mathrm{P}(A_k, \overline{\Omega} \setminus \mathrm{B}_{R_i}) + \mathrm{P}(A_\infty, \overline{\Omega} \setminus \mathrm{B}_{R_i}) + \mathcal{H}^{n-1}((A_k \Delta A_\infty)^+ \cap \partial \mathrm{B}_{R_i}) + \varepsilon \end{aligned}$$

for $k \gg 1$. This is enough to establish in the limit $k \to \infty$ the estimate (6.6)[8] — now under the assumptions of (c), but still only in case $\mu_-(A_k^+)+\mu_-(A_\infty^+) < \infty$. We can thus carry out the remainder of the reasoning and establish (6.3) exactly as in the situation (b). □

Exploiting the semicontinuity result in a more or less standard way we obtain the following existence theorem for the functional in (6.2).

Theorem 6.4 (existence in Dirichlet problems)**.** *For an open set Ω in $\mathbb{R}^n$, assume that non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ with $\mu_\pm \llcorner (\overline{\Omega})^{\mathrm{c}} \equiv 0$ both satisfy the small-volume IC in $\mathbb{R}^n$ with constant* 1*. Moreover, consider $A_0 \in \mathcal{M}(\mathbb{R}^n)$ with $\mu_+(A_0^1)+\mathrm{P}(A_0, \overline{\Omega}) < \infty$, and assume that* one of the following sets of additional assumptions *is valid*:

(a) *The measure μ_- is* finite.

(b) *For some $R_0 \in (0,\infty)$ and $\gamma \in (0,1]$, the measure μ_- additionally satisfies the* strong *IC in $(\mathrm{B}_{R_0})^{\mathrm{c}}$ with constant $1{-}\gamma$.*

Then, for $n \ge 2$, there exists the minimum of the (generalized) Dirichlet problem

$$\min\{\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \,:\, E \in \mathscr{F}_{A_0}(\Omega)\}\,, \tag{6.7}$$

[8]In fact, since in the line of argument based on (c) the radii R_i do not depend on ε, one can exploit the arbitrariness of ε earlier in the argument to deduce the validity of (6.6) in fact even without the ε-term.

and moreover, in situation (a) *with* $n \ge 1$*, there also exists the minimum of the variant of the problem*

$$\min\{\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \,:\, E \in \mathscr{D}_{A_0}(\Omega)\}\,. \tag{6.8}$$

The minimum values in the situation (a) *are in* $[-\mu_-(\mathbb{R}^n),\infty)$*, and the minimum value in the situation* (b) *is in* $[-(1{-}\gamma)\mathrm{P}(A_0,\overline{\Omega}){-}(1{-}\gamma)\mathrm{P}(\mathrm{B}_{R_0}){-}\mu_-(A_0^+){-}\mu_-(\overline{\mathrm{B}_{R_0}}),\infty)$*.*

In connection with this theorem let us first set clear that the functional $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\overline{\Omega}]$ is well-defined on the admissible E. Indeed, in the situation (a) thanks to the finiteness of μ_- we evidently have $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \in (-\infty,\infty]$ for all $E \in \mathscr{D}_{A_0}(\Omega)$ and a fortiori for $E \in \mathscr{F}_{A_0}(\Omega)$. Moreover, in the situation (b) we get from the assumption $\mu_+(A_0^1){+}\mathrm{P}(A_0,\overline{\Omega}) < \infty$ and Lemma 6.2 that $\mu_+(E^1){+}\mathrm{P}(E,\overline{\Omega}) < \infty$ and consequently $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \in [-\infty,\infty)$ hold at least for all $E \in \mathscr{F}_{A_0}(\Omega)$.

We further remark that if only (b) but not (a) is satisfied (in particular $\mu_-(\mathbb{R}^n) = \infty$), we may still consider (6.8) in the form

$$\min\{\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \,:\, E \in \mathscr{D}_{A_0}(\Omega)\,,\ \mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \text{ defined}\}\,, \tag{6.9}$$

where we recall that $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}]$ is defined for $E \in \mathscr{D}_{A_0}(\Omega)$ precisely if $\min\{\mu_+(E^1),\mu_-(E^+)\} < \infty$. However, in fact this does not win much when compared to (6.7), and thus we have excluded this situation above and only comment on it briefly. Indeed, in case $n \ge 2$, $\mathrm{Cap}_1((\Omega^1)^{\mathrm{c}}) = \infty$, Remark 6.3(ii) gives $\mathscr{D}_{A_0}(\Omega) = \mathscr{F}_{A_0}(\Omega)$, and (6.9) reduces to precisely (6.7) (also keeping in mind that we have already argued for the finiteness of the μ_+-term on $\mathscr{F}_{A_0}(\Omega)$). Moreover, in case $n \ge 2$, $\mathrm{Cap}_1((\Omega^1)^{\mathrm{c}}) < \infty$ we can modify[9] A_0 inside Ω to ensure $|A_0| < \infty$ and then obtain from Remark 6.3(iii) that the sets $E \in \mathscr{D}_{A_0}(\Omega)$ split into some with $E \in \mathcal{BV}(\overline{\Omega})$ and thus $E \in \mathscr{F}_{A_0}(\Omega)$ on one hand and some with $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$ on the other hand. However, in the case considered it turns out[10] that either $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}]$ equals $-\infty$ whenever $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$ or $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}]$ is undefined whenever $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$. Thus, either (6.9) is a rather trivial extension of (6.7), or (6.9) reduces to precisely (6.7) once more.

Proof. The admissible classes in both (6.7) and (6.8) contain A_0. Thus, these classes are non-empty, and in view of $\mu_+(A_0^1){+}\mathrm{P}(A_0,\overline{\Omega}) < \infty$ the corresponding infima are in $[-\infty,\infty)$. Moreover, in view of $\mu_\pm \llcorner (\overline{\Omega})^{\mathrm{c}} \equiv 0$ the problems in (6.7) and (6.8) remain unchanged if we modify A_0 away from $\overline{\Omega}$. Hence, we can and do assume $A_0 \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, which implies that the admissible classes are contained in $\mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$.

We now focus, for a moment, on the situation (a). In view of $\mu_-(\mathbb{R}^n) < \infty$ and

$$\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \ge \mathrm{P}(E,\overline{\Omega}) - \mu_-(\mathbb{R}^n) \qquad \text{for all } E \in \mathcal{M}(\mathbb{R}^n)$$

we find that every minimizing sequence $(A_k)_{k\in\mathbb{N}}$ in either (6.7) or (6.8) satisfies $\limsup_{k\to\infty} \mathrm{P}(A_k,\overline{\Omega}) < \infty$.

Next we turn to the situation (b). We can assume $\mu_-(A_0^+) < \infty$, as otherwise A_0 with $\mathscr{P}_{\mu_+,\mu_-}[A_0;\overline{\Omega}] = -\infty$ clearly minimizes. For $E \in \mathscr{F}_{A_0}(\Omega)$, since we have $|E\Delta A_0| < \infty$ and $E\Delta A_0 \subset \Omega$, the strong IC yields

$$\begin{aligned}
\mu_-((E^+\Delta A_0^+)\setminus \overline{\mathrm{B}_{R_0}}) &\le \mu_-(((E\Delta A_0)\setminus \mathrm{B}_{R_0})^+)\\
&\le (1{-}\gamma)\mathrm{P}((E\Delta A_0)\setminus \mathrm{B}_{R_0})\\
&= (1{-}\gamma)\mathrm{P}((E\Delta A_0)\setminus \mathrm{B}_{R_0},\overline{\Omega})\\
&\le (1{-}\gamma)\mathrm{P}(E,\overline{\Omega}) + (1{-}\gamma)\mathrm{P}(A_0,\overline{\Omega}) + (1{-}\gamma)\mathrm{P}(\mathrm{B}_{R_0})\,,
\end{aligned}$$

and from this estimate we infer $\mu_-(E^+) < \infty$ and

$$\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] \ge \gamma\mathrm{P}(E,\overline{\Omega}) - (1{-}\gamma)\mathrm{P}(A_0,\overline{\Omega}) - (1{-}\gamma)\mathrm{P}(\mathrm{B}_{R_0}) - \mu_-(A_0^+) - \mu_-(\overline{\mathrm{B}_{R_0}}) \qquad \text{for all } E \in \mathscr{F}_{A_0}(\Omega)\,.$$

Thus, for every minimizing sequence $(A_k)_{k\in\mathbb{N}}$ in (6.7), we obtain once more $\limsup_{k\to\infty} \mathrm{P}(A_k,\overline{\Omega}) < \infty$.

[9] In fact, in view of $\mathrm{Cap}_1((\Omega^1)^{\mathrm{c}}) < \infty$ there exists $H \in \mathcal{BV}(\mathbb{R}^n)$ with $\Omega^{\mathrm{c}} \subset H$ up to negligible sets, and the problem under consideration stays unchanged when replacing A_0 with $A_0 \cap H$, which clearly satisfies $|A_0 \cap H| \le |H| < \infty$.

[10] The precise reasoning proceeds as follows and exploits that $H \in \mathcal{BV}(\mathbb{R}^n)$ from the previous footnote also satisfies $(\overline{\Omega})^{\mathrm{c}} \subset H^1$. In case $\mu_+(\mathbb{R}^n) < \infty = \mu_-(\mathbb{R}^n)$, from $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$ we get first $E^{\mathrm{c}} \cup H \in \mathcal{BV}(\mathbb{R}^n)$, then $\mu_-((E^+)^{\mathrm{c}}) \le \mu_-((E^{\mathrm{c}} \cup H)^+) < \infty$ via Lemma 3.3, then $\mu_-(E^+) = \infty$, and finally $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}] = -\infty$. In case $\mu_+(\mathbb{R}^n) = \infty = \mu_-(\mathbb{R}^n)$, essentially the same reasoning leads from $E^{\mathrm{c}} \in \mathcal{BV}(\overline{\Omega})$ to $\mu_-(E^+) = \mu_+(E^1) = \infty$, and thus $\mathscr{P}_{\mu_+,\mu_-}[E;\overline{\Omega}]$ is undefined.

In any of the cases considered in the statement we further proceed as follows. Fixing a minimizing sequence $(A_k)_{k\in\mathbb{N}}$, from $\limsup_{k\to\infty} \mathrm{P}(A_k,\overline{\Omega}) < \infty$ together with $A_k \setminus \Omega = A_0 \setminus \Omega$ we get $\limsup_{k\to\infty} \mathrm{P}(A_k,U) < \infty$ for some open neighborhood U of $\overline{\Omega}$ and in view of $A_0 \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ also $\limsup_{k\to\infty} \mathrm{P}(A_k,\mathrm{B}_R) < \infty$ for every $R \in (0,\infty)$. By compactness, a diagonal argument, and lower semicontinuity of the perimeter, we deduce that a subsequence of $(A_k)_{k\in\mathbb{N}}$ converges *locally* in measure on $\mathbb{R}^n$ to $A_\infty \in \mathscr{D}_{A_0}(\Omega)$ (with even $\mathrm{P}(A_\infty,U) < \infty$). In case of problem (6.7) we additionally involve the isoperimetric estimate of Theorem 2.6 to derive the subsidiary estimate $|A_k\Delta A_0| \le \Gamma_n \mathrm{P}(A_k\Delta A_0)^{\frac{n}{n-1}} \le \Gamma_n[\mathrm{P}(A_k,\overline{\Omega})+\mathrm{P}(A_0,\overline{\Omega})]^{\frac{n}{n-1}}$, which implies $|A_\infty\Delta A_0| < \infty$ also for the limit A_∞ and thus ensures the admissibility of A_∞ and $|A_k\Delta A_\infty| < \infty$ for all $k \in \mathbb{N}$. Finally, we apply Theorem 6.1(a) in situation (a) and Theorem 6.1(b) in situation (b) to conclude that the limit A_∞ is a minimizer in (6.7) and (6.8), respectively (where, as we recall, in situation (b) we consider (6.7) only). □

7 Properties and reformulations of isoperimetric conditions

In this section we take a closer look at ICs, specifically small-volume ICs, and equivalent ways to express these conditions. Most (though not really all) of the results obtained in this regard will find uses in the subsequent sections.

Remark 7.1. *Even though we will not work with the observations of this remark any further, we briefly record that the ε-δ-feature of the small-volume IC can be reformulated in the following standard way. Given a Radon measure μ on an open set $\Omega \subset \mathbb{R}^n$, the small-volume IC for μ in Ω with constant $C \in [0,\infty)$ means nothing but the existence of a modulus $\omega\colon [0,\infty] \to [0,\infty]$ with $\lim_{t\searrow 0}\omega(t) = \omega(0) = 0$ such that we have*

$$\mu(A^+) \le C\mathrm{P}(A) + \omega(|A|) \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } \overline{A} \subset \Omega\,. \tag{7.1}$$

Introducing a modified 1-capacity ${}^C\mathrm{K}_1^\omega$ by ${}^C\mathrm{K}_1^\omega(S) := \inf\{C\mathrm{P}(A)+\omega(|A|) \,:\, A \in \mathcal{M}(\mathbb{R}^n)\,,\, S \subset A^+\,,\, \overline{A} \subset \Omega\}$ (with understanding $\inf\emptyset = \infty$), one may further recast (7.1) in the (still) equivalent form

$$\mu(S) \le {}^C\mathrm{K}_1^\omega(S) \qquad \text{for all } S \in \mathcal{B}(\mathbb{R}^n)\,.$$

As shown by the next lemma, there is also some flexibility concerning the precise class of test sets for ICs.

Lemma 7.2. *Consider a Radon measure μ on an open set $\Omega \subset \mathbb{R}^n$ and $C \in [0,\infty)$. Then the following assertions (where (a) is exactly the definition of the small-volume IC in Ω with constant C) are **equivalent**:*

(a) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(A^+) \le C\mathrm{P}(A)+\varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $\overline{A} \subset \Omega$, $|A| < \delta$.*

(b) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(A^+) \le C\mathrm{P}(A)+\varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $A \Subset \Omega$, $|A| < \delta$.*

(c) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(A^+) \le C\mathrm{P}(A)+\varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $A^+ \subset \Omega$, $|A| < \delta$.*

The equivalence carries over to corresponding versions of the strong (instead of small-volume) IC.

In the sequel, from this lemma we will only need the equivalence of (a) and (b), which is trivial for bounded Ω and results from a simple cut-off argument in general. In order to prove the equivalence with (c) in the full generality stated here, we will make crucial use of the fine approximation result [44, Teorema 2] (which in turn draws on [43, 42]).

Proof of Lemma 7.2. Clearly, (c) implies (a), and (a) implies (b).

In addition, we now show that (b) implies (a). To this end, we fix $\varepsilon > 0$ and consider $A \in \mathcal{M}(\mathbb{R}^n)$ with $\overline{A} \subset \Omega$, $|A| < \delta$ for the corresponding δ. In view of $A \cap \mathrm{B}_R \Subset \Omega$, from (b) we then get

$$\mu(A^+ \cap \mathrm{B}_R) = \mu((A \cap \mathrm{B}_R)^+) \le C\mathrm{P}(A \cap \mathrm{B}_R) + \varepsilon \le C\mathrm{P}(A) + \varepsilon \qquad \text{for each } R \in (0,\infty)\,,$$

where the last estimate can be obtained from Lemmas 2.12 and 2.13, for instance. In the limit $R \to \infty$ we read off $\mu(A^+) \le C\mathrm{P}(A) + \varepsilon$.

Next we prove that (a) implies (c). For this, we fix again $\varepsilon > 0$ and consider some $A \in \mathcal{M}(\mathbb{R}^n)$ with $A^+ \subset \Omega$, $|A| < \delta$ for the corresponding δ. Clearly, we can additionally assume $\mathrm{P}(A) < \infty$. From the interior approximation result [44, Teorema 2] we then obtain a sequence of sets $A_k \in \mathcal{M}(\mathbb{R}^n)$ such that

$$A_k \subset A_{k+1} \subset A\,, \qquad \overline{A_k} = A_k^+\,, \qquad \mathrm{P}(A_k) \le \mathrm{P}(A) \qquad \text{for all } k \in \mathbb{N}$$

(where the crucial condition $\overline{A_k} = A_k^+$ is stated in [44, Teorema 2] in the equivalent form $A_k^0 \cap \partial A_k = \emptyset$) and

$$\lim_{k\to\infty} \mathrm{P}(A \setminus A_k) = 0\,.$$

In view of $\overline{A_k} = A_k^+ \subset A^+ \subset \Omega$, from (a) and the preceding properties of A_k we conclude

$$\mu(A_k^+) \le C\mathrm{P}(A_k) + \varepsilon \le C\mathrm{P}(A) + \varepsilon \qquad \text{for each } k \in \mathbb{N}\,. \tag{7.2}$$

Evidently the above conditions imply $\bigcup_{k=1}^\infty A_k^+ \subset A^+$, and we now show that, decisively, they also ensure

$$\mu\bigg(A^+ \setminus \bigcup_{k=1}^\infty A_k^+\bigg) = 0\,. \tag{7.3}$$

Indeed, observing $A^+ \setminus \bigcup_{k=1}^\infty A_k^+ \subset A^+ \setminus A_\ell^+ \subset (A \setminus A_\ell)^+$ for each $\ell \in \mathbb{N}$, from Proposition 2.15 we first infer $\mathrm{Cap}_1\big(A^+ \setminus \bigcup_{k=1}^\infty A_k^+\big) \le \lim_{\ell\to\infty} \mathrm{P}(A \setminus A_\ell) = 0$, then by Proposition 2.16 we deduce $\mathcal{H}^{n-1}\big(A^+ \setminus \bigcup_{k=1}^\infty A_k^+\big) = 0$, and finally via Lemma 3.2 we arrive at (7.3). With (7.3) at hand we can then go to the limit $k \to \infty$ in (7.2) to establish $\mu(A^+) \le C\mathrm{P}(A) + \varepsilon$ in the generality of (c).

For the strong conditions instead of the small-volume ones, the reasoning works in the same way. ☐

In the specific cases that the measure μ is finite or supported at positive distance from $\partial\Omega$, we have further characterizations of the small-volume IC for μ in Ω. Indeed, we can allow test sets A reaching up to $\partial\Omega$, can pass to the *relative* perimeter $\mathrm{P}(A,\Omega)$, or can even state the condition in a fully localized way. This is detailed in the next statement, where for notational convenience[11] we work with a Radon measure μ defined on full $\mathbb{R}^n$.

Lemma 7.3. *Consider an open set $\Omega \subset \mathbb{R}^n$, a Radon measure μ on $\mathbb{R}^n$, and $C \in [0,\infty)$. If either μ is finite with $\mu \llcorner \Omega^{\mathrm{c}} \equiv 0$ or μ satisfies $\mathrm{dist}(\mathrm{spt}\,\mu, \Omega^{\mathrm{c}}) > 0$, then the following assertions are* ***equivalent***:

(a) *The measure μ satisfies the small-volume IC in Ω with constant C.*

(b) *For every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\mu(A^+) \le C\mathrm{P}(A) + \varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A \setminus \Omega| = 0$, $|A| < \delta$.*

(c) *For every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $\mu(A^+) \le C\mathrm{P}(A,\Omega) + \varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A| < \delta$.*

In the case of finite μ with $\mu \llcorner \Omega^{\mathrm{c}} \equiv 0$ one more equivalent assertion is:

(d) *For every $x \in \Omega$, there exists $r_x > 0$ with $\mathrm{B}_{r_x}(x) \subset \Omega$ such that μ restricted to $\mathrm{B}_{r_x}(x)$ satisfies the small-volume IC in $\mathrm{B}_{r_x}(x)$ with constant C.*

Here the implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) $\Longrightarrow$ (d) are simple generalities, while the reverse implications are non-trivial and draw crucially on the assumption that μ is finite or satisfies $\mathrm{dist}(\mathrm{spt}\,\mu, \Omega^{\mathrm{c}}) > 0$. Indeed, setting $h_k := \sum_{i=1}^k \frac{1}{i} \in \mathbb{R}$, we record that (b) $\Longrightarrow$ (c) fails for the *infinite* Radon measure $\mu = 2C\sum_{k=1}^\infty \delta_{h_{3k}}$ on $\mathbb{R}$ with $C > 0$ and $\Omega = \bigcup_{k=1}^\infty (h_{3k-1}, h_{3k+1})$, while (a) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (a) fail for the same measure together with $\Omega = \bigcup_{k=1}^\infty (h_{3k-2}, h_{3k+1})$ and $\Omega = \mathbb{R}$, respectively.

In addition, also the ε-δ-nature of the small-volume IC is crucial for Lemma 7.3 insofar that the simple implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) $\Longrightarrow$ (d) carry over by analogy to a strong-IC case with ε and δ removed, while the reverse implications do not have analogs there. Indeed, the strong-IC analog of (b) $\Longrightarrow$ (c) fails for the *finite* Radon measure $\mu = 2C(\delta_{-2}+\delta_2)$ on $\mathbb{R}$ together with $\Omega = (-3,-1) \,\dot\cup\, (1,3)$, while the analoga of (a) $\Longrightarrow$ (b) and (d) $\Longrightarrow$ (a) fail for the same measure together with $\Omega = (-3,3) \setminus \{0\}$ and $\Omega = (-3,3)$, respectively.

Furthermore, all counterexamples mentioned here can be easily adapted to work in $\mathbb{R}^n$ instead of $\mathbb{R}$.

[11]Indeed, if one considers a Radon measure μ on Ω and assumes in analogy to Lemma 7.3 either finiteness of μ or $\mathrm{dist}(\mathrm{spt}\,\mu, \Omega^{\mathrm{c}}) > 0$, the extension of μ from Ω to $\mathbb{R}^n$ by zero is still a Radon measure. This goes without saying for finite μ, but is true also when requiring $\mathrm{dist}(\mathrm{spt}\,\mu, \Omega^{\mathrm{c}}) > 0$, since this condition improves local finiteness on Ω to finiteness on all *bounded* subsets of Ω and thus ensures local finiteness of the extension.

Proof of Lemma 7.3. As already observed, the implications (c)$\Longrightarrow$(b)$\Longrightarrow$(a)$\Longrightarrow$(d) are straightforward.

Next we prove that (a) implies (c). We record that $d\colon \mathbb{R}^n \to (0,\infty)$, given by $d(x) := \operatorname{dist}(x,\Omega^c)$, is Lipschitz with constant 1 and then by Rademacher's theorem satisfies $|\nabla d| \le 1$ a.e. on Ω. Moreover, since Ω is open, we have $\Omega = \bigcup_{t>0}\{d>t\}$. Now we consider an arbitrary $\varepsilon > 0$. Then, in case of finite μ with $\mu \llcorner \Omega^c \equiv 0$ we can fix a corresponding $t_0 > 0$ such that $\mu(\{d<t_0\}) < \frac{\varepsilon}{3}$ holds, while in case $\operatorname{dist}(\operatorname{spt}\mu,\Omega^c) > 0$ we are even in position to ensure $\mu(\{d<t_0\}) = 0$. In addition, we fix $\delta > 0$ such that the standard form of the small-volume IC in Ω from (a) applies with this δ and $\frac{\varepsilon}{3}$ in place of ε, and we consider $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A| < \min\{\delta, \frac{t_0\varepsilon}{3C}\}$. Via the coarea formula of Theorem 2.1 we get

$$\int_0^{t_0} \mathcal{H}^{n-1}(A^+ \cap \{d=t\})\,\mathrm{d}t = \int_{A^+\cap\{d<t_0\}} |\nabla d|\,\mathrm{d}x \le |A^+| < \frac{t_0\varepsilon}{3C}$$

and can thus choose $t \in (0,t_0)$ with

$$\mathcal{H}^{n-1}(A^+ \cap \{d=t\}) < \frac{\varepsilon}{3C} \tag{7.4}$$

(where for $C=0$ an arbitrary $t\in(0,t_0)$ suffices). We now cut off portions of A close to $\partial\Omega$ by introducing $E := A \cap \{d>t\}$, for which clearly $\overline{E} \subset \{d \ge t\} \subset \Omega$ and $|E| \le |A| < \delta$ hold. Estimating via the choice of t_0, the small-volume IC from (a) (with $\frac{\varepsilon}{3}$ in place of ε), Lemma 2.9, and (7.4), we then arrive at

$$\begin{aligned}\mu(A^+) &\le \mu(A^+\cap\{d>t\}) + \mu(\{d<t_0\}) \le \mu(E^+) + \frac{\varepsilon}{3} \le C\mathrm{P}(E) + \frac{2\varepsilon}{3}\\ &\le C\mathrm{P}(A,\Omega) + C\mathcal{H}^{n-1}(A^+\cap\{d=t\}) + \frac{2\varepsilon}{3} \le C\mathrm{P}(A,\Omega) + \varepsilon\,.\end{aligned}$$

Thus, we obtain $\mu(A^+\cap\Omega) \le C\mathrm{P}(A,\Omega) + \varepsilon$ in the setting of (c).

Finally, in case of finite μ with $\mu\llcorner\Omega^c \equiv 0$ we show that (d) implies (c). To this end we fix once more some $\varepsilon > 0$. We then apply Vitali's covering theorem (see [29, Theorem 2.8], for instance) to the family of all balls $\mathrm{B}_r(x)$ with $x \in \Omega$ and $r \le r_x$ and exploit $\mu(\Omega) < \infty$ to obtain finite number $k \in \mathbb{N}$ of disjoint balls $\mathrm{B}_{\varrho_i}(x_i)$ with $x_i \in \Omega$ and $\varrho_i \le r_{x_i}$ for $i \in \{1,2,\ldots,k\}$ such that it holds

$$\mu\bigg(\Omega \setminus \bigcup_{i=1}^k \mathrm{B}_{\varrho_i}(x_i)\bigg) \le \frac{\varepsilon}{2}\,.$$

Now the assumption (d) guarantees the validity of (a) on each of the balls $\mathrm{B}_{\varrho_i}(x_i) \subset \mathrm{B}_{r_{x_i}}(x_i)$ with $i \in \{1,2,\ldots,k\}$ in place of Ω. Since we have already shown that (a) implies (c), we also have (c) on each of these balls. Since the number of balls is finite, this in turn yields a common $\delta > 0$ such that we have

$$\mu(A^+ \cap \mathrm{B}_{\varrho_i}(x_i)) \le C\mathrm{P}(A,\mathrm{B}_{\varrho_i}(x_i)) + \frac{\varepsilon}{2k}$$

for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A| < \delta$ and all $i \in \{1,2,\ldots,k\}$. In conclusion, for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A| < \delta$, we achieve

$$\mu(A^+) \le \sum_{i=1}^k \mu(A^+\cap\mathrm{B}_{\varrho_i}(x_i)) + \mu\bigg(\Omega\setminus\bigcup_{i=1}^k \mathrm{B}_{\varrho_i}(x_i)\bigg) \le \sum_{i=1}^k \Big[C\mathrm{P}(A,\mathrm{B}_{\varrho_i}(x_i)) + \frac{\varepsilon}{2k}\Big] + \frac{\varepsilon}{2} \le C\mathrm{P}(A,\Omega) + \varepsilon\,,$$

where the disjointness of $\mathrm{B}_{\varrho_i}(x_i)$ is used in the last step. In this way we arrive at (c). □

As a rather unexpected consequence of Lemma 7.3, we next derive that the small-volume IC with a fixed constant actually carries over to the sum of two (or finitely many) mutually singular measures with still the same constant. Clearly, for the strong IC, one cannot draw an analogous conclusion in comparable generality.

Proposition 7.4 (small-volume IC for a sum of singular measures)**.** *Consider non-negative Radon measures μ_1, μ_2 on $\mathbb{R}^n$ which are singular to each other in the sense that there exists a decomposition $\mathbb{R}^n = S_1 \mathbin{\dot\cup} S_2$ into $S_1, S_2 \in \mathcal{B}(\mathbb{R}^n)$ with $\mu_1(S_1^c) = \mu_2(S_2^c) = 0$. Further suppose that either μ_1 is finite or $\operatorname{dist}(\operatorname{spt}\mu_1,\operatorname{spt}\mu_2) > 0$ holds. Then, if μ_1 and μ_2 both satisfy the small-volume IC on $\mathbb{R}^n$ with constant $C \in [0,\infty)$, also $\mu_1+\mu_2$ satisfies the small-volume IC on $\mathbb{R}^n$ with the same constant C.*

From the example in the later Remark 8.3(ii) it becomes clear that the extra assumptions in the proposition (either one measure is finite or supports at positive distance) cannot be dropped.

Proof. We start with the case that μ_1 is finite. Given an arbitrary $\varepsilon > 0$, the finiteness of μ_1 together with $\mu_1(S_1^c) = \mu_2(S_2^c) = 0$ yields the existence of a compact set $K_1 \subset S_1$ and a closed set $C_2 \subset S_2$ such that $\mu_1(K_1^c)+\mu_2(C_2^c) \le \varepsilon$. In view of $\operatorname{dist}(K_1, C_2) > 0$ we can choose *disjoint* open sets $O_1 \supset K_1$ and $O_2 \supset C_2$ and can also ensure $\operatorname{dist}(C_2, O_2^c) > 0$. Since the closedness of C_2 yields $\operatorname{spt}(\mu_2 \llcorner C_2) \subset C_2$, we can then apply (a)$\Longrightarrow$(c) from Lemma 7.3 on one hand for the finite measure $\mu_1 \llcorner K_1$, on the other hand for the possibly infinite measure $\mu_2 \llcorner C_2$ with $\operatorname{dist}(\operatorname{spt}(\mu_2 \llcorner C_2), O_2^c) > 0$ to obtain some $\delta > 0$ such that we have $\mu_1(A^+ \cap K_1) \le C\mathrm{P}(A, O_1)+\varepsilon$ and $\mu_2(A^+ \cap C_2) \le C\mathrm{P}(A, O_2)+\varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $|A| < \delta$. Consequently, for such sets we also get

$$(\mu_1+\mu_2)(A^+) \le \mu_1(A^+ \cap K_1) + \mu_2(A^+ \cap C_2) + \varepsilon \le C\mathrm{P}(A, O_1) + C\mathrm{P}(A, O_2) + 3\varepsilon \le C\mathrm{P}(A) + 3\varepsilon\,,$$

which yields the claim.

The case of $\operatorname{dist}(\operatorname{spt}\mu_1, \operatorname{spt}\mu_2) > 0$ is a bit simpler, since we can directly choose disjoint open sets $O_1 \supset \operatorname{spt}\mu_1$ and $O_2 \supset \operatorname{spt}\mu_2$ with $\operatorname{dist}(\operatorname{spt}\mu_1, O_1^c) > 0$ and $\operatorname{dist}(\operatorname{spt}\mu_2, O_2^c) > 0$. Then, we can apply (a)$\Longrightarrow$(c) from Lemma 7.3 to both $\mu_1 = \mu_1 \llcorner O_1$ and $\mu_2 = \mu_2 \llcorner O_2$ and conclude the reasoning as before. ☐

In the sequel we record that ICs can be expressed not only with test sets, but also with test functions and partially in a distributional way. This is detailed in the following (almost) twin theorems, where the one for the strong-IC case is a minor variant of known results from [30, Theorem 4.7], [47, Theorem 5.12.4], [14, Section 2], [34, Theorem 3.3, Theorem 3.5], [35, Theorem 4.4], while the adaptation to the small-volume case does not seem to have direct predecessors in the literature. As a side benefit it turns out in this context that the measure-theoretic closure A^+ can be replaced with the measure-theoretic interior A^1 in the formulation of both types of ICs.

Theorem 7.5 (characterizations of the strong IC)**.** *For a Radon measure μ on an open set $\Omega \subset \mathbb{R}^n$ and a constant $C \in [0,\infty)$, the following assertions are **equivalent** with each other:*

(a) *The strong IC holds for μ in Ω with constant C.*

(b) *We have $\mu(A^1) \le C\mathrm{P}(A)$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $\overline{A} \subset \Omega$ and $|A| < \infty$.*

(c) *We have $\int_\Omega \eta \,\mathrm{d}\mu \le C \int_\Omega |\nabla\eta| \,\mathrm{d}x$ for all non-negative functions $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$.*

(d) *We have $\mu(N) = 0$ for all $\mathcal{H}^{n-1}$-negligible $N \in \mathcal{B}(\Omega)$ and $\int_\Omega |v^*| \,\mathrm{d}\mu \le C \int_\Omega |\nabla v| \,\mathrm{d}x$ for all $v \in \mathrm{W}^{1,1}_0(\Omega)$.*

(e) *We have $\mu = \operatorname{div}\sigma$ in the sense of distributions on Ω for some vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le C$.*

Theorem 7.6 (characterizations of the small-volume IC)**.** *For a Radon measure on an open set $\Omega \subset \mathbb{R}^n$ and a constant $C \in [0,\infty)$, the following assertions are **equivalent** with each other:*

(a) *The small-volume IC holds for μ in Ω with constant C.*

(b) *For every $\varepsilon > 0$, there exists some $\delta > 0$ such that we have $\mu(A^1) \le C\mathrm{P}(A) + \varepsilon$ for all $A \in \mathcal{M}(\mathbb{R}^n)$ with $\overline{A} \subset \Omega$ and $|A| < \delta$.*

(c) *There exists a modulus $\omega\colon [0,\infty) \to [0,\infty]$ with $\lim_{t\searrow 0}\omega(t) = \omega(0) = 0$ such that we have $\int_\Omega \eta \,\mathrm{d}\mu \le C \int_\Omega |\nabla\eta| \,\mathrm{d}x + \omega\big(|\operatorname{spt}\eta|\big)$ for all $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $0 \le \eta \le 1$ on Ω.*

(d) *We have $\mu(N) = 0$ for all $\mathcal{H}^{n-1}$-negligible $N \in \mathcal{B}(\Omega)$, and, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that we have $\int_\Omega |v^*| \,\mathrm{d}\mu \le C \int_\Omega |\nabla v| \,\mathrm{d}x + \varepsilon \sup_\Omega |v|$ for all $v \in \mathrm{W}^{1,1}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $|\{v \ne 0\}| < \delta$.*

*In addition, the **subsequent property at least implies** each of the preceding ones:*

(e) *We have $\mu = H\mathcal{L}^n + \operatorname{div}\sigma$ in the sense of distributions on Ω for some vector field $\sigma \in \mathrm{L}^\infty(\Omega, \mathbb{R}^n)$ with $\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le C$ and some function $H \in \mathrm{L}^1(\Omega)$.*

Here, the extra terms distinguishing Theorem 7.6 from Theorem 7.5 have been incorporated in slightly different forms, but indeed the formulations are to some extent interchangable. However, a subtlety related to Lemma 2.22 is that in condition (d) it seems decisive to require smallness for $|\{v \neq 0\}|$ (or alternatively for any L^p norm of v), but *not* in fact for $|\operatorname{spt} v|$.

In the sequel we first detail the proof of Theorem 7.6 and then comment on the necessary adaptations needed to cover the case of Theorem 7.5 as well.

Proof of Theorem 7.6. Since we have $A^1 \subset A^+$ by definition, it is clear that (a) implies (b).

We start by proving that (b) implies (c). We denote by $\delta_i > 0$ the value of δ which corresponds to $\varepsilon = \frac{1}{i}$ in (b), we assume $\delta_{i+1} < \delta_i$ for $i \in \mathbb{N}$, and we choose the modulus $\omega := \sum_{i=1}^{\infty} \frac{1}{i} \mathbb{1}_{[\delta_{i+1},\delta_i)} + \infty \mathbb{1}_{[\delta_1,\infty)}$. We now consider $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $0 \le \eta \le 1$ on Ω. If η vanishes identically or we have $|\operatorname{spt}\eta| \ge \delta_1$, the claim is trivially valid. Otherwise we henceforth fix $i \in \mathbb{N}$ with $|\operatorname{spt}\eta| \in [\delta_{i+1},\delta_i)$ and thus $\omega(|\operatorname{spt}\eta|) = \frac{1}{i}$. We observe that $\{\eta > t\}$ is open and thus $\{\eta > t\} \subset \{\eta > t\}^1$ holds for all $t \in \mathbb{R}$. Then, via a layer-cake type rewriting, the estimate from (b) for $\{\eta > t\} \Subset \Omega$ with $|\{\eta > t\}| < \delta_i$, and the coarea formula of Theorem 2.5 we get

$$\int_\Omega \eta \,\mathrm{d}\mu = \int_0^1 \mu(\{\eta > t\}) \,\mathrm{d}t \le \int_0^1 \mu(\{\eta > t\}^1) \,\mathrm{d}t \le \int_0^1 \Big[\mathrm{P}(\{\eta > t\}) + \tfrac{1}{i}\Big] \,\mathrm{d}t = \int_\Omega |\nabla \eta| \,\mathrm{d}x + \omega(|\operatorname{spt}\eta|)\,.$$

This gives the property (c).

Next we verify that (c) implies (d). In order to show $\mu(N) = 0$ for an $\mathcal{H}^{n-1}$-negligible $N \in \mathcal{B}(\Omega)$, we slightly adapt the proof of Lemma 3.2. Indeed, we can assume $N \Subset \Omega$. Given $\varepsilon > 0$, Lemma 2.7 yields an open A with $N \subset A \Subset \Omega$, $|A| < \varepsilon$, $\mathrm{P}(A) < \varepsilon$, and by mollifying the $\mathbb{1}_A$ we obtain $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $\mathbb{1}_N \le \eta \le 1$ on Ω, $|\operatorname{spt}\eta| < \varepsilon$, and $\int_\Omega |\nabla\eta| < \varepsilon$. Exploiting the estimate from (c) for this η, we find $\mu(N) < C\varepsilon + \sup_{[0,\varepsilon)} \omega$. As $\varepsilon > 0$ is arbitrary, we end up with $\mu(N) = 0$. We now derive the main inequality in (d). Given $\varepsilon > 0$ we fix $\delta > 0$ such that $\sup_{[0,\delta)} \omega \le \varepsilon$. We consider $v \in \mathrm{W}^{1,1}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $|\{v \neq 0\}| < \delta$ and may additionally assume $\sup_\Omega |v| = 1$. We record $|v| \in \mathrm{W}^{1,1}_0(\Omega) \cap \mathrm{L}^\infty(\Omega)$ with $|\nabla|v|| = |\nabla v|$ a.e. and choose $\eta_k \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $0 \le \eta_k \le 1$ on Ω such that η_k converge to $|v|$ in $\mathrm{W}^{1,1}(\Omega)$. Involving $|\{|v| > 0\}| = |\{v \neq 0\}| < \delta$ and drawing on Lemma 2.22 we can modify the sequence $(\eta_k)_{k\in\mathbb{N}}$ such that we additionally have $|\operatorname{spt}\eta_k| < \delta$ for all $k \in \mathbb{N}$. Moreover, we infer from Lemma 2.19 that η_k converge to $|v|^* = |v^*|$ also $\mathcal{H}^{n-1}$-a.e. on Ω, and by the preceding this convergence holds μ-a.e. on Ω as well. Hence, via Fatou's lemma and the estimate in (c) we find

$$\int_\Omega |v^*| \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_\Omega \eta_k \,\mathrm{d}\mu \le \liminf_{k\to\infty} \Big[C \int_\Omega |\nabla\eta_k| \,\mathrm{d}x + \omega(|\operatorname{spt}\eta_k|)\Big] \le C \int_\Omega |\nabla v| \,\mathrm{d}x + \varepsilon\,.$$

This completes the derivation of (d).

We turn to the implication from (d) back to (a). We consider $\varepsilon > 0$, the corresponding δ from (d), and a set $A \in \mathcal{BV}(\mathbb{R}^n)$ with $A \Subset \Omega$ and $|A| < \delta$. Then, by Lemma 2.21 applied with $u = \mathbb{1}_A$, we can find $v_k \in \mathrm{W}^{1,1}_0(\Omega)$ with $\mathbb{1}_A \le v_k \le 1$ a.e. on Ω for all $k \in \mathbb{N}$ such that v_k converge strictly in $\mathrm{BV}(\Omega)$ to $\mathbb{1}_A$. Observing $|\{\mathbb{1}_A > 0\}| = |A| < \delta$, we next apply Lemma 2.22 with $u = \mathbb{1}_A$ to modify the sequence and achieve additionally $|\{v_k > 0\}| < \delta$ for all $k \in \mathbb{N}$. Taking into account that $\eta_k^* \ge (\mathbb{1}_A)^+ = \mathbb{1}_{A^+}$ holds $\mathcal{H}^{n-1}$-a.e., we deduce

$$\mu(A^+) \le \liminf_{k\to\infty} \int_\Omega \eta_k^* \,\mathrm{d}\mu \le \lim_{k\to\infty} \Big[C \int_\Omega |\nabla\eta_k| \,\mathrm{d}x + \varepsilon \sup_\Omega |\eta_k|\Big] \le C\mathrm{P}(A) + \varepsilon\,.$$

By Lemma 7.2 this suffices to ensure the small-volume IC in Ω with constant C

Finally, we prove that (e) implies (c). Given σ and H as in (e), by absolute continuity of the integral, there exists $\omega \colon [0,\infty] \to [0,\infty]$ with $\lim_{t\searrow 0} \omega(t) = \omega(0) = 0$ such that $\int_A |H| \,\mathrm{d}x \le \omega(|A|)$ holds for all $A \in \mathcal{B}(\Omega)$. Using this together with the definition of the distributional divergence, we estimate

$$\int_\Omega \eta \,\mathrm{d}\mu = -\int_\Omega \sigma \cdot \nabla\eta \,\mathrm{d}x + \int_\Omega \eta H \,\mathrm{d}x \le \int_\Omega |\sigma|\,|\nabla\eta| \,\mathrm{d}x + \int_{\operatorname{spt}\eta} |H| \,\mathrm{d}x \le C \int_\Omega |\nabla\eta| \,\mathrm{d}x + \omega(|\operatorname{spt}\eta|)$$

for every $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$ with $0 \le \eta \le 1$ on Ω. □

Theorem 7.5 is in most regards a special case of Theorem 7.6, the only true addition being the fact that we can also get back from (a), (b), (c), (d) to (e). Consequently, we can keep the proof comparably brief:

Proof of Theorem 7.5. The implications (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d), (d) $\Longrightarrow$ (a), (e) $\Longrightarrow$ (c) in Theorem 7.5 can be proved along the lines of the corresponding implications in Theorem 7.6. In fact, one can drop from the reasoning all arguments and terms with ε, ω, H as well as the requirements $\eta \le 1$, $v \in \mathrm{L}^\infty(\Omega)$, while at the same time weakening all δ-smallness conditions to merely finiteness conditions. This leads to some simplifications, for instance, Lemma 2.22 is no longer needed. However, we refrain from discussing any further details in this regard.

Rather to conclude the proof we address the implication (d) $\Longrightarrow$ (e), which follows from (a homogeneous version of) the duality $(\mathrm{W}_0^{1,1})^* = \mathrm{W}^{-1,\infty}$ and, in more concrete terms, from the following reasoning. Consider the closed subspace $X := \{\nabla\eta \,:\, \eta \in \mathrm{W}_0^{1,1}(\Omega)\}$ of $\mathrm{L}^1(\Omega,\mathbb{R}^n)$ with the L^1-norm. Then the assumption (d) gives that the linear functional $\nabla\eta \mapsto \int_\Omega \eta^*\,\mathrm{d}\mu$ is an element of norm $\le C$ in the dual X^*. By the Hahn-Banach theorem, this functional extends to an element of norm $\le C$ in $\mathrm{L}^1(\Omega,\mathbb{R}^n)^*$, and by the Riesz duality $(\mathrm{L}^1)^* = \mathrm{L}^\infty$ there exists some $\sigma \in \mathrm{L}^\infty(\Omega,\mathbb{R}^n)$ with $\|\sigma\|_{\mathrm{L}^\infty(\Omega,\mathbb{R}^n)} \le C$ such that

$$\int_\Omega \eta^*\,\mathrm{d}\mu = -\int_\Omega \sigma\cdot\nabla\eta\,\mathrm{d}x \qquad \text{holds for all } \eta \in \mathrm{W}_0^{1,1}(\Omega)\,.$$

Specifying this conclusion to $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\Omega)$, we obtain $\mu = \operatorname{div}\sigma$ in the sense of distributions on Ω. □

8 Isoperimetric conditions for perimeter measures and rectifiable measures

We begin this section by checking the validity of the strong IC in an already-mentioned basic case, namely for the perimeter measure of a pseudoconvex set. In view of the preceding results this can be implemented conveniently by checking the variant of the IC with the representative A^1 instead of A^+.

Proposition 8.1 (strong IC for perimeters measures of pseudoconvex sets)**.** *For every pseudoconvex set $K \in \mathcal{BV}(\mathbb{R}^n)$, the perimeter measure $\mathcal{H}^{n-1}\llcorner\partial^*K$ satisfies the strong IC in $\mathbb{R}^n$ with constant 1 and in case $|K| > 0$ does not satisfy the strong IC in $\mathbb{R}^n$ with any smaller constant.*

Proof. By Theorems 2.4 and 2.8 together with Lemma 2.12, we infer

$$(\mathcal{H}^{n-1}\llcorner\partial^*K)(A^1) = \mathcal{H}^{n-1}(A^1\cap K^{\frac12}) \le \mathcal{H}^{n-1}((A\cap K)^{\frac12}) = \mathrm{P}(A\cap K) \le \mathrm{P}(A)$$

By Theorem 7.5 this means that $\mathcal{H}^{n-1}\llcorner\partial K$ satisfies the strong IC in $\mathbb{R}^n$ with constant 1. As moreover the equality $(\mathcal{H}^{n-1}\llcorner\partial K)(K^+) = \mathrm{P}(K)$ occurs for the test set K itself, the constant 1 is optimal in case $|K| > 0$ (in which we have $\mathrm{P}(K) > 0$ as well). □

We stress that the pseudoconvexity assumption in Proposition 8.1 cannot be dropped, as already for $n = 2$ and a bounded, smooth, open, but non-convex $K \subset \mathbb{R}^2$ one finds with $(\mathcal{H}^1\llcorner\partial K)(\mathrm{C}(K)) = \mathrm{P}(K) > \mathrm{P}(\mathrm{C}(K))$ for the closed convex hull $\mathrm{C}(K)$ of K that the strong IC fails for $\mathcal{H}^1\llcorner\partial K$. In contrast to this, however, we show with the next (and much more interesting) results that the small-volume IC is independent of geometric properties such as convexity of an underlying set and indeed admits a much wider class of admissible measures.

Theorem 8.2 (small-volume IC for general perimeter measures)**.** *For every $E \in \mathcal{M}(\mathbb{R}^n)$ with $\mathrm{P}(E) < \infty$, the double perimeter measure*

$$\mu := 2\mathrm{P}(E,\,\cdot\,) = 2|\mathrm{D}\mathbb{1}_E| = 2\mathcal{H}^{n-1}\llcorner\partial^*E$$

can be expressed in the form $\mu = H\mathcal{L}^n + \operatorname{div}\sigma$ in $\mathscr{D}'(\mathbb{R}^n)$ with a sub-unit L^∞ vector field σ on $\mathbb{R}^n$ and a function $H \in \mathrm{L}^1(\mathbb{R}^n)$. Consequently, μ satisfies all properties in Theorem 7.6 on $\Omega = \mathbb{R}^n$ and in particular satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1, that is, for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$2\mathcal{H}^{n-1}(A^+\cap\partial^*E) \le \mathrm{P}(A) + \varepsilon \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } |A| < \delta\,. \tag{8.1}$$

We would like to highlight that the small-volume IC reached in the theorem trivially carries over to $\mu = 2\mathcal{H}^{n-1}\llcorner S$ with any subset $S \in \mathcal{B}(\partial^*E)$ and even more generally to $\mu = \alpha\mathcal{H}^{n-1}\llcorner\partial^*E$ with any $[0,2]$-valued Borel density $\alpha\colon \partial^*E \to [0,2]$ on ∂^*E. Thus, we have identified a reasonably broad class of $(n{-}1)$-dimensional measures for which the central assumption of our semicontinuity and existence results holds. Beyond that a further broadening of the class will be achieved in Corollary 8.4, and the optimality of the upper bound 2 for the density α will be established in Proposition 8.5.

Proof. In the case $n = 1$, the boundary $\partial^* E$ consists of finitely many points. Then, for $\mu = 2\mathcal{H}^0 \llcorner \partial^* E$, the claim $\mu = H\mathcal{L}^1 + \sigma'$ follows trivially by taking any sub-unit $\sigma \in \mathrm{BV}(\mathbb{R})$ which is smooth on $(\partial^* E)^{\mathrm{c}}$ and jumps from -1 to 1 at each point of $\partial^* E$ so that $\sigma' = -H\mathcal{L}^1 + 2\mathcal{H}^0 \llcorner \partial^* E$ with $H \in \mathrm{L}^1(\mathbb{R})$. (In fact, if $\partial^* E \subset (a, b)$ for a bounded interval (a, b), one may take σ linear on each component of $(a, b) \setminus \partial^* E$ and $\sigma \equiv 0$ on $(a, b)^{\mathrm{c}}$.)

In the case $n \geq 2$, from Theorem 2.6 we get $E \in \mathcal{BV}(\mathbb{R}^n)$ or $E^{\mathrm{c}} \in \mathcal{BV}(\mathbb{R}^n)$, where in view of $\mathrm{P}(E^c, \cdot\,) = \mathrm{P}(E, \cdot\,)$ and $\partial^* E^{\mathrm{c}} = \partial^* E$ it suffices to treat the case $E \in \mathcal{BV}(\mathbb{R}^n)$. By results of Barozzi & Gonzalez & Tamanini [3] and Barozzi [2] (see specifically [2, Remark 2.1, Theorem 2.1] or alternatively [23, Section 2]), there exists an optimal L^1 variational mean curvature H_E of E, that is, a function $H_E \in \mathrm{L}^1(\mathbb{R}^n)$ with $\int_E H_E \,\mathrm{d}x = \mathrm{P}(E) = -\int_{E^{\mathrm{c}}} H_E \,\mathrm{d}x$ and thus $\int_{\mathbb{R}^n} H_E \,\mathrm{d}x = 0$ such that

$$\mathrm{P}(E) - \int_E H_E \,\mathrm{d}x \leq \mathrm{P}(F) - \int_F H_E \,\mathrm{d}x \qquad \text{for all } F \in \mathcal{M}(\mathbb{R}^n) \text{ with } \mathrm{P}(F) < \infty\,.$$

We apply this to F and F^{c} and exploit $\mathrm{P}(F^{\mathrm{c}}) = \mathrm{P}(F)$ and $\int_{F^{\mathrm{c}}} H_E \,\mathrm{d}x = -\int_F H_E \,\mathrm{d}x$ to deduce

$$\left| \int_F H_E \,\mathrm{d}x \right| \leq \mathrm{P}(F) \qquad \text{for all } F \in \mathcal{M}(\mathbb{R}^n) \text{ with } \mathrm{P}(F) < \infty\,.$$

This estimate can be read as a strong IC for $H_E\mathcal{L}^n$, but at this point is not perfectly in line with the previous considerations in this paper, which would rather require separate conditions on $(H_E)_+\mathcal{L}^n$ and $(H_E)_-\mathcal{L}^n$. Nonetheless, most of the arguments used for Theorems 7.5 and 7.6 still apply, and we now give a brief rereading in the present situation in order to eventually reach a divergence structure $H_E = \operatorname{div} \sigma_E$. Indeed, for $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$, with the help of a layer-cake formula and the coarea formula of Theorem 2.5 we find $\mathrm{P}(\{\eta > t\}) < \infty$ for a.e. $t \in \mathbb{R}$ and

$$\left| \int_{\mathbb{R}^n} \eta H_E \,\mathrm{d}x \right| = \left| \int_{\mathbb{R}} \int_{\{\eta > t\}} H_E \,\mathrm{d}x \,\mathrm{d}t \right| \leq \int_{\mathbb{R}} \left| \int_{\{\eta > t\}} H_E \,\mathrm{d}x \right| \mathrm{d}t \leq \int_{\mathbb{R}} \mathrm{P}(\{\eta > t\}) \,\mathrm{d}t = \int_{\mathbb{R}^n} |\nabla \eta| \,\mathrm{d}x\,.$$

Consequently, if we consider the subspace $X := \{\nabla \eta : \eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\}$ of $\mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^n)$ with the L^1-norm, the functional $\nabla \eta \mapsto \int_{\mathbb{R}^n} \eta H_E \,\mathrm{d}x$ is a sub-unit element in X^* and extends to a sub-unit element in $\mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^n)^*$ by virtue of the Hahn-Banach theorem. The duality $(\mathrm{L}^1)^* = \mathrm{L}^\infty$ then yields some $\sigma_E \in \mathrm{L}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\sigma_E\|_{\mathrm{L}^\infty(\mathbb{R}^n, \mathbb{R}^n)} \leq 1$ such that $\int_{\mathbb{R}^n} \eta H_E \,\mathrm{d}x = -\int_{\mathbb{R}^n} \sigma_E \cdot \nabla \eta \,\mathrm{d}x$ holds for all $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$, in other words, it gives a sub-unit L^∞ vector field σ_E on $\mathbb{R}^n$ with

$$\operatorname{div} \sigma_E = H_E \qquad \text{in the sense of distributions on } \mathbb{R}^n\,.$$

Exploiting $E \in \mathcal{BV}(\mathbb{R}^n)$ and the Gauss-Green formula (2.13) we then infer

$$\mathcal{H}^{n-1}(\partial^* E) = \mathrm{P}(E) = \int_E H_E \,\mathrm{d}x = \int_E \operatorname{div} \sigma_E \,\mathrm{d}x = \int_{\partial^* E} \sigma_E \cdot \nu_E \,\mathrm{d}\mathcal{H}^{n-1}$$

for the generalized normal trace $\sigma_E \cdot \nu_E$ introduced in Definition 2.25. This improves the $\mathcal{H}^{n-1}$-a.e. *in*equality $|\sigma_E \cdot \nu_E| \leq 1$ on $\partial^* E$ to the $\mathcal{H}^{n-1}$-a.e. equality

$$\sigma_E \cdot \nu_E = 1 \qquad \text{on } \partial^* E\,.$$

We next introduce the modifications

$$\sigma := \begin{cases} -\sigma_E & \text{on } E \\ \sigma_E & \text{on } E^{\mathrm{c}} \end{cases}$$

and

$$H := \begin{cases} H_E & \text{on } E \\ -H_E & \text{on } E^{\mathrm{c}} \end{cases}$$

of σ_E and H_E and record that σ and H are still a sub-unit L^∞ vector field and an L^1 function on $\mathbb{R}^n$. Then, for arbitrary $\varphi \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$, the Gauss-Green formulas

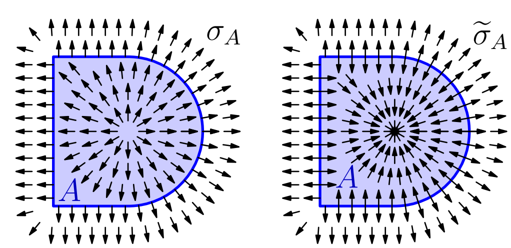

Figure 5: An illustration of σ_A and $\widetilde{\sigma}_A$, which differ by reversing the arrows inside A.

(2.11), (2.12) (here used for σ_E with $\operatorname{div}\sigma_E = H_E \in \mathrm{L}^1(\mathbb{R}^n)$ on $\Omega = \mathbb{R}^n$) yield

$$\begin{aligned}\int_{\mathbb{R}^n} \sigma\cdot\nabla\varphi\,\mathrm{d}x &= -\int_E \sigma_E\cdot\nabla\varphi\,\mathrm{d}x + \int_{E^c}\sigma_E\cdot\nabla\varphi\,\mathrm{d}x\\ &= \int_E \varphi(\operatorname{div}\sigma_E)\,\mathrm{d}x - \int_{E^c}\varphi(\operatorname{div}\sigma_E)\,\mathrm{d}x - 2\int_{\partial^*E}\varphi\,\sigma\cdot\nu_E\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_E \varphi H_E\,\mathrm{d}x - \int_{E^c}\varphi H_E\,\mathrm{d}x - 2\int_{\partial^*E}\varphi\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_{\mathbb{R}^n}\varphi\,\mathrm{d}(H\mathcal{L}^n - 2\mathcal{H}^{n-1}\llcorner\partial^*E)\,.\end{aligned}$$

In conclusion we have

$$-\operatorname{div}\sigma = H\mathcal{L}^n - 2\mathcal{H}^{n-1}\llcorner\partial^*E \qquad\qquad \text{in the sense of distributions on } \mathbb{R}^n$$

or in other words $\mu = H\mathcal{L}^n + \operatorname{div}\sigma$ in the sense of distributions on $\mathbb{R}^n$. Thus, all the claims follow directly from Theorem 7.6. □

Remark 8.3 (on infinite perimeter measures)**.** *If $E \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)\setminus\mathcal{BV}(\mathbb{R}^n)$ has only locally finite, but not finite perimeter, the following examples show that* $2\mathrm{P}(E,\,\cdot\,)$ may or may not *satisfy the small-volume IC with constant* 1*.*

(i) *On one hand, if E is a half-space or the infinite strip between two parallel hyperplanes, for instance, then $2\mathrm{P}(E,\,\cdot\,)$ satisfies the small-volume IC with constant* 1*; see Proposition* A.3*.*

(ii) *On the other hand, if we consider $n=1$ and the union of intervals $E_\ell := \bigcup_{k=2\ell}^\infty\bigcup_{i=1}^\ell\,(k+\frac{2i-1}{k},k+\frac{2i}{k})$, with arbitrary fixed $\ell\in\mathbb{N}$, then $\mathrm{P}(E_\ell,\,\cdot\,)$ consists of groups of 2ℓ Dirac measures concentrated on shorter and shorter intervals, and thus $2\mathrm{P}(E_\ell,\,\cdot\,)$ satisfies the small-volume IC with constant $\frac{2}{\ell}$ at most* (*but no larger constant*)*. This example can be adapted to higher dimensions either simply by taking $E_\ell{\times}(0,1)^{n-1}\subset\mathbb{R}^n$ or by considering $\bigcup_{i=1}^\ell\{(x',x_n)\in\mathbb{R}^{n-1}{\times}\mathbb{R}\,:\,f_{2i-1}(x')<x_n<f_{2i}(x')\}$, where $f_1<f_2<\ldots<f_{2\ell}$ are smooth functions $\mathbb{R}^{n-1}\to\mathbb{R}$ with $\lim_{|x'|\to\infty}f_j(x')=0$.*

Next, as announced, we address a further extension of Theorem 8.2:

Corollary 8.4 (small-volume IC for rectifiable $\mathcal{H}^{n-1}$-measures)**.** *If $S\in\mathcal{B}(\mathbb{R}^n)$ is $\mathcal{H}^{n-1}$-finite and countably $\mathcal{H}^{n-1}$-rectifiable* (*in the sense that $\mathcal{H}^{n-1}(S)<\infty$ and $\mathcal{H}^{n-1}(S\setminus\bigcup_{j=1}^\infty f_j(\mathbb{R}^{n-1}))=0$ for Lipschitz mappings $f_j\colon\mathbb{R}^{n-1}\to\mathbb{R}^n$*)*, then the measure $2\mathcal{H}^{n-1}\llcorner S$ satisfies the small-volume IC in $\mathbb{R}^n$ with constant* 1*.*

Proof. It follows from [1, Proposition 2.76] that we have $\mathcal{H}^{n-1}(S\setminus\bigcup_{j=1}^\infty K_j)=0$ for countably many compact subsets $K_j\subset\Gamma_j$ of Lipschitz-$(n{-}1)$-graphs Γ_j in the sense of [1, Example 2.58]. Clearly, we have $K_j\subset\partial^*E_j$ for some $E_j\in\mathcal{BV}(\mathbb{R}^n)$ (which can be obtained by suitably cutting off the subgraphs of the Lipschitz functions, for instance). From Theorem 8.2 we have that $2\mathcal{H}^{n-1}\llcorner K_j'$ with $K_j' := K_j\setminus\bigcup_{i=1}^{j-1}K_i$ for $j\in\mathbb{N}$ satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1. In a next step we use Proposition 7.4 and the finiteness of these measures to conclude that $2\mathcal{H}^{n-1}\llcorner\bigcup_{j=1}^k K_j = \sum_{j=1}^k 2\mathcal{H}^{n-1}\llcorner K_j'$ with $k\in\mathbb{N}$ satisfies this condition as well. Given an arbitrary $\varepsilon>0$, in view of $\mathcal{H}^{n-1}(S)<\infty$ we can fix first $k\in\mathbb{N}$ with $\mathcal{H}^{n-1}(S\setminus\bigcup_{j=1}^k K_j)\le\frac{\varepsilon}{2}$ and then $\delta>0$ such that $2\mathcal{H}^{n-1}(A^+\cap\bigcup_{j=1}^k K_j)\le\mathrm{P}(A){+}\frac{\varepsilon}{2}$ holds for all $A\in\mathcal{M}(\mathbb{R}^n)$ with $|A|<\delta$. By combination of these properties we obtain in fact $2\mathcal{H}^{n-1}(A^+\cap S)\le\mathrm{P}(A){+}\varepsilon$, that is, the small-volume IC holds for $2\mathcal{H}^{n-1}\llcorner S$ in $\mathbb{R}^n$ with constant 1. □

Finally, we establish a converse to Theorem 8.2 and Corollary 8.4.

Proposition 8.5 (necessity of the upper density bound 2 for the small-volume IC)**.** *If $S\in\mathcal{B}(\mathbb{R}^n)$ is countably $\mathcal{H}^{n-1}$-rectifiable and $\alpha\mathcal{H}^{n-1}\llcorner S$ with $\alpha\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n;\mathcal{H}^{n-1}\llcorner S)$ satisfies the small-volume IC with constant* 1*, then necessarily $\alpha\le 2$ holds $\mathcal{H}^{n-1}$-a.e. on S.*

Proof. We assume, for a proof by contradiction, that $\alpha > 2$ holds on a non-$\mathcal{H}^{n-1}$-negligible subset of S, and similar to the preceding proof we infer from [1, Proposition 2.76] that $\mathcal{H}^{n-1}(S \setminus \bigcup_{j=1}^\infty \Gamma_j) = 0$ holds for countably many Lipschitz-$(n-1)$-graphs Γ_j over hyperplanes π_j in $\mathbb{R}^n$. Then, we can also find a compact subset G of $S \cap \Gamma_{j_0}$, for some fixed $j_0 \in \mathbb{N}$, with $\mathcal{H}^{n-1}(G) > 0$ such that $\alpha \geq 2+4\varepsilon/\mathcal{H}^{n-1}(G)$ holds $\mathcal{H}^{n-1}$-a.e. on G for some $\varepsilon > 0$. Since G is compact, there exists an open neighborhood U of G in Γ_{j_0} such that U is a Lipschitz-$(n-1)$-graph over an open $\mathcal{BV}$ set in the hyperplane π_{j_0} with $\mathcal{H}^{n-1}(U) < \mathcal{H}^{n-1}(G)+\varepsilon$. Next, for the $\varepsilon > 0$ already fixed, we consider the corresponding $\delta > 0$ from the IC, and we choose $\ell > 0$ small enough that the "width-2ℓ thickening" $A := \bigcup_{t\in(-\ell,\ell)}(U+t\nu_{j_0}) \in \mathcal{BV}(\mathbb{R}^n)$ of U in the normal direction ν_{j_0} of π_{j_0} satisfies $|A| < \delta$ and $\mathrm{P}(A) < 2\mathcal{H}^{n-1}(U)+\varepsilon$. Then the previous estimates combine to $\mathrm{P}(A) < 2\mathcal{H}^{n-1}(G)+3\varepsilon$, and in view of $G \subset S$ and $G \subset U \subset A^+$ we arrive at

$$(\alpha\mathcal{H}^{n-1} \llcorner S)(A^+) \geq (\alpha\mathcal{H}^{n-1})(G) \geq 2\mathcal{H}^{n-1}(G)+4\varepsilon > \mathrm{P}(A)+\varepsilon\,.$$

This, however, contradicts the assumed small-volume IC for $\alpha\mathcal{H}^{n-1} \llcorner S$. ☐

9 Lower semicontinuity on general domains

Once more we consider non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ and define a functional of the previously considered type over arbitrary $D \in \mathcal{B}(\mathbb{R}^n)$ by setting

$$\mathscr{P}_{\mu_+,\mu_-}[A;D] := \mathrm{P}(A,D) + \mu_+(A^1) - \mu_-(A^+) \tag{9.1}$$

whenever for $A \in \mathcal{M}(\mathbb{R}^n)$ at least one of $\mathrm{P}(A,D)+\mu_+(A^1)$ and $\mu_-(A^+)$ is finite. Our aim in this section is to complement the semicontinuity results of Section 4 for the full-space functional $\mathscr{P}_{\mu_+,\mu_-} = \mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\mathbb{R}^n]$ and the ones of Section 6 for (generalized) Dirichlet classes with local semicontinuity results, which do not involve boundary conditions and apply for $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;D]$ with $\mu_\pm \llcorner D^{\mathrm{c}} \equiv 0$ over arbitrary (measure-theoretically) open sets D.

In order to single out basic lines of our approach we point out directly that in spite of requiring $\mu_\pm \llcorner D^{\mathrm{c}} \equiv 0$ we keep working with Radon measures $\mu_\pm$ on all of $\mathbb{R}^n$ and impose ICs on these measures in all of $\mathbb{R}^n$ rather than using ICs in the sense of Definition 3.1 on open domains $D = \Omega$. In particular, our measures $\mu_\pm$ are necessarily finite in cases with bounded D (by definition of a Radon measure on $\mathbb{R}^n$) and more generally whenever $\mathrm{Cap}_1(D) < \infty$ (by Proposition 2.15 and Lemma 3.3). One reason for proceeding in this way is that the full-space viewpoint is convenient in order to apply the previously achieved results and at least in case of *finite* measures $\mu_\pm$ on open $\Omega = D$ is not truly restrictive, as in fact the small-volume ICs in Ω and in $\mathbb{R}^n$ are even equivalent by Lemma 7.3. Moreover, for cases with *infinite* measures μ_- concentrated on domains D with $\mathrm{Cap}_1(D) = \infty$ the following example suggests that working with ICs in all of $\mathbb{R}^n$ is even more appropriate for semicontinuity. Indeed, we consider for $n = 1$ an open domain $\Omega = \bigcup_{m=1}^\infty \Omega_m$, where Ω_m are disjoint and each Ω_m is itself a disjoint union of a group of m open intervals all placed inside an interval of length 2^{-m}. Correspondingly we consider a countable set $S \subset \mathbb{R}$ which contains precisely one point in each interval of each group Ω_m and the infinite Borel measure $\mu_- = 2\sum_{x\in S}\delta_x = 2\mathcal{H}^0 \llcorner S$ on Ω. Then μ_- satisfies even the strong IC[12] with constant 1 in Ω, but does not satisfy the strong or even small-volume IC with any constant in all of $\mathbb{R}$. Moreover, $\mathscr{P}_{0,\mu_-}[\,\cdot\,;\Omega]$ is *not* lower semicontinuous, since $\bigcup_{m=k}^\infty \Omega_m$ converge globally in measure to $\emptyset$ with $\mathscr{P}_{0,\mu_-}\big[\bigcup_{m=k}^\infty \Omega_m;\Omega\big] = -\infty$ for all $k \in \mathbb{N}$, but clearly $\mathscr{P}_{0,\mu_-}[\emptyset;\Omega] = 0$. We remark that by suitably placing the groups Ω_m and possibly adding to Ω an additional unbounded interval with zero μ_--measure, this examples covers bounded or unbounded Ω and finite or infinite volumes $|\Omega|$. Moreover, analogous configurations can also be arranged with absolutely continuous measures (by "spreading out" the Dirac measures a bit) and in arbitrary dimension $n \in \mathbb{N}$ (e.g. by placing measures in thin annuli instead of short intervals). Thus, as foreshadowed above, an IC in open $D = \Omega$ in the sense of Definition 3.1 does not necessarily yield semicontinuity, while ICs in full $\mathbb{R}^n$ will lead in the sequel to general semicontinuity results. Nonetheless, we also point out that our full-space viewpoint, for infinite measures $\mu_\pm$, does more or less automatically lead to considering, if not ICs in D, then still ICs *relative to* D with the relative perimeter occurring in essentially the same way as in the condition of Lemma 7.3(c).

[12] In some cases (actually whenever Ω is bounded and, more generally, whenever S has an accumulation point in $\mathbb{R}$), the given measure μ_- is infinite already on some bounded sets and hence is only a Borel measure, but not a Radon measure on all of $\mathbb{R}$. Regardless of that we here understand that ICs for this measure are defined just as usual via (3.1) and (3.2) from Definition 3.1.

Before reaching semicontinuity on arbitrary open sets $D = \Omega$ in the later Theorem 9.6, we provide a first semicontinuity statement, which applies on the measure-theoretic interior $D = \Omega^1$ of a set Ω of locally finite perimeter and in fact seems illustrative and interesting in its own right. We remark that at this point we apply the notions of local and global convergence in measure from (2.1) and (2.2) on the possibly non-open set Ω^1.

Theorem 9.1 (lower semicontinuity on a domain of locally finite perimeter). *Consider a set $\Omega \in \mathcal{M}(\mathbb{R}^n)$, a set $A_\infty \in \mathcal{M}(\mathbb{R}^n)$, a sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^n)$, and non-negative Radon measures μ_+ and μ_- on $\mathbb{R}^n$ with $\mu_\pm \llcorner (\Omega^1)^c \equiv 0$ such that one of the following sets of assumptions is valid:*

(a) *We have $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, the measure μ_- is* finite*, the measures μ_+ and μ_- both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1, and A_k converge to A_∞* locally *in measure on Ω.*

(b) *We have $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, the measures μ_+ and $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1, the measure $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ additionally satisfies the* almost-strong *IC from (4.2) with constant 1 near ∞, and A_k converge to A_∞* locally *in measure on Ω with $|(A_k \Delta A_\infty) \cap \Omega| + \mathrm{P}(A_k \cap \Omega) + \mathrm{P}(A_\infty \cap \Omega) < \infty$ for all $k \in \mathbb{N}$.*

(c) *We have $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$, the measures μ_+ and $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ both satisfy the small-volume IC in $\mathbb{R}^n$ with constant 1, and A_k converge to A_∞* globally *in measure on Ω with $\mathrm{P}(A_k \cap \Omega) + \mathrm{P}(A_\infty \cap \Omega) < \infty$ for all $k \in \mathbb{N}$.*

If furthermore $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ holds for all $k \in \mathbb{N}$, then we have $\min\{\mu_+(A_\infty^1), \mu_-(A_\infty^+)\} < \infty$ and

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega^1] \geq \mathscr{P}_{\mu_+,\mu_-}[A_\infty;\Omega^1]\,. \tag{9.2}$$

Since (all representatives of) a set $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ with $|\Omega| > 0$ may have empty interior, the previous statement differs from the more usual semicontinuity on open sets, and indeed semicontinuity on $D = \Omega^1$ does not to seem to be well known even in case $\mu_\pm \equiv 0$, that is, for the perimeter itself. Therefore, we explicitly record as a subcase of Theorem 9.1:

Corollary 9.2 (lower semicontinuity of the perimeter on a measure-theoretic interior). *Consider a set $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$. If a sequence $(A_k)_{k\in\mathbb{N}}$ in $\mathcal{M}(\mathbb{R}^n)$ converges to $A_\infty \in \mathcal{M}(\mathbb{R}^n)$ locally in measure on Ω, then we have*

$$\liminf_{k\to\infty} \mathrm{P}(A_k, \Omega^1) \geq \mathrm{P}(A_\infty, \Omega^1)\,.$$

Interestingly, when specializing the subsequent proof of Theorem 9.1(a) to the case $\mu_\pm \equiv 0$ of the corollary, it turns out that even in this case the approach does rely on the theory of the previous sections with $\mu_\pm \not\equiv 0$ and indeed plugs in the perimeter measure $\mathrm{P}(\Omega,\,\cdot\,)$ in place of either μ_+ or μ_-. Alternatively, however, Corollary 9.2 can be derived as a special case of a recent result of Lahti [24]. Indeed, [24, Theorem 4.5] guarantees lower semicontinuity of the perimeter even on every Cap_1-quasi-open set in a general metric-space setting, while it follows from [6, Theorem 2.5] that Ω^1 is Cap_1-quasi-open for every $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$.

Next, we provide a refined discussion of the different settings in Theorem 9.1, where once more the differences concern the handling of the μ_--term only.

First of all we emphasize that the statement under assumptions (a) with *finite* μ_- should be considered as the most basic, but also central point of the theorem and will be sufficient in order to eventually move on to semicontinuity on arbitrary open sets. Exemplary cases covered by (a) are finite perimeter measures $\mu_- = 2\mathcal{H}^{n-1} \llcorner \partial^* E$ of $E \in \mathcal{BV}(\mathbb{R}^n)$ considered on any open $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ with $\partial^* E \subset \Omega$, since for these Theorem 8.2 gives the small-volume IC with constant 1.

The settings (b) and (c) of Theorem 9.1 improve on (a) in case of infinite measures μ_-, as seen similarly in Theorems 4.1 and 6.1. An exemplary case covered by (b), but not by (a) is $\mu_- = 2\mathcal{H}^{n-1} \llcorner ((0,\infty) \times \mathbb{R}^{n-2} \times \{0\})$ on $\Omega = (0,\infty) \times \mathbb{R}^{n-1}$ with $n \geq 2$, for which $\mathrm{P}(\Omega) = \infty$ holds, but still $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ satisfies even the strong IC on full $\mathbb{R}^n$ with constant 1. While the exemplary cases mentioned so far are covered also by the setting (c), from (c) we get the semicontinuity conclusion only along sequences with *global* convergence. Additional exemplary cases which are covered by (c) only and come merely with global-convergence semicontinuity are given by the infinite measures $\mu_- = 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{0,1\})$ on $\Omega = \mathbb{R}^n$ and $\mu_- = 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{1\})$ on

$\Omega = \mathbb{R}^{n-1}\times(0,\infty)$. In both these cases, Proposition A.3 implies the small-volume IC with constant 1 for $\mu_-+\mathrm{P}(\Omega,\,\cdot\,)$, but this measure does not satisfy the almost-strong IC required in (b).

We add one specific remark on the assumptions of the theorem:

Remark 9.3 (on the finite-perimeter assumptions in Theorem 9.1)**.** *The assumption* $\mathrm{P}(A_k\cap\Omega)<\infty$*, which occurs in parts* (b) *and* (c) *of Theorem* 9.1*, follows from the more local and thus slightly more natural assumption* $\mathrm{P}(A_k,\Omega^1)<\infty$ *together with* $\mathrm{P}(\Omega)<\infty$*. Clearly,* $\mathrm{P}(A_\infty\cap\Omega)<\infty$ *follows from* $\mathrm{P}(A_\infty,\Omega^1)<\infty$ *together with* $\mathrm{P}(\Omega)<\infty$ *in the same way.*

Proof. By distinguishing between points inside Ω^1 and outside Ω^1 it is not difficult to verify the inclusion $\partial^{\mathrm{e}}(A_k\cap\Omega)\subset(\partial^{\mathrm{e}}A_k\cap\Omega^1)\cup\partial^{\mathrm{e}}\Omega$. By Theorems 2.4 and 2.8 we infer $\mathcal{H}^{n-1}(\partial^{\mathrm{e}}(A_k\cap\Omega))\le\mathrm{P}(A_k,\Omega^1)+\mathrm{P}(\Omega)<\infty$, and then Federer's criterion (see [15, Theorem 5.23], for instance) yields $\mathrm{P}(A_k\cap\Omega)<\infty$. ☐

Now we turn to the proof of the theorem, where the essential strategy is to apply the full-space or Dirichlet results and to include in μ_- a boundary term $\mathrm{P}(\Omega,\,\cdot\,)$, which eventually cancels out with the boundary contribution $\mathrm{P}(\,\cdot\,,\partial^*\Omega)$ of the perimeter.

Proof of Theorem 9.1*.* In a first step we establish the result for the setting (a) with additional requirement $\mathrm{P}(\Omega)<\infty$ and for the settings (b), (c). We introduce

$$S_k := A_k\cap\Omega\,,\qquad S_\infty := A_\infty\cap\Omega\,,\qquad \mu_-^\Omega := \mu_- + \mathrm{P}(\Omega,\,\cdot\,)\,,$$

and observe that the present assumptions imply the ones of the corresponding setting in Theorem 4.1 or its extension due to Remark 4.3 with S_k, S_∞, μ_+, μ_-^Ω in place of A_k, A_∞, μ_+, μ_-. (As an alternative, we could also take into account $S_k\setminus\Omega=\emptyset=S_\infty\setminus\Omega$ and use Theorem 6.1 as our reference here.) However, while in assumptions (b) and (c) the relevant IC on μ_-^Ω is explicitly included, under (a) with additionally $\mathrm{P}(\Omega)<\infty$ it remains to justify that μ_-^Ω satisfies the small-volume IC on $\mathbb{R}^n$ with constant 1. To this end we first argue that in view of the requirement $\mathrm{P}(\Omega)<\infty$ in (a) the small-volume IC with constant 1 holds for $\mathrm{P}(\Omega,\,\cdot\,)$ by Theorem 8.2 (where we have even discarded a factor 2). Moreover, in view of $\mu_-\llcorner(\Omega^1)^{\mathrm{c}}\equiv 0$ and specifically $\mu_-\llcorner\partial^*\Omega\equiv 0$ the measures μ_- and $\mathrm{P}(\Omega,\,\cdot\,)=\mathcal{H}^{n-1}\llcorner\partial^*\Omega$ are singular to each other and under the present assumptions are both finite. Thus, by Proposition 7.4 the small-volume IC with constant 1 carries over from these two measures to their sum μ_-^Ω. After this justification we are in position to apply Theorem 4.1, which yields

$$\liminf_{k\to\infty}\mathscr{P}_{\mu_+,\mu_-^\Omega}[S_k]\ge\mathscr{P}_{\mu_+,\mu_-^\Omega}[S_\infty] \tag{9.3}$$

for the full-space functional defined in (4.1), but now with μ_-^Ω in place of μ_-. In order to rewrite the perimeter term in this functional we next deduce from the equality case of (2.4) in Lemma 2.9 that we have

$$\mathrm{P}(S_k)=\mathrm{P}(S_k,\Omega^1)+\mathrm{P}(\Omega,S_k^+)\,.$$

We use this equality in conjunction with the definition of μ_-^Ω and the observations $\mathrm{P}(A\cap\Omega,\Omega^1)=\mathrm{P}(A,\Omega^1)$ and $\mu_\pm\llcorner(\Omega^1)^{\mathrm{c}}\equiv 0$. Arguing in this way we end up with

$$\mathscr{P}_{\mu_+,\mu_-^\Omega}[S_k]=\mathrm{P}(S_k)+\mu_+(S_k^1)-\mu_-^\Omega(S_k^+)=\mathrm{P}(S_k,\Omega^1)+\mu_+(S_k^1)-\mu_-(S_k^+)=\mathscr{P}_{\mu_+,\mu_-}[S_k;\Omega^1]=\mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega^1]\,.$$

Since we can analogously rewrite $\mathscr{P}_{\mu_+,\mu_-^\Omega}[S_\infty]=\mathscr{P}_{\mu_+,\mu_-}[A_\infty;\Omega^1]$, the semicontinuity property obtained in (9.3) directly transforms into the one claimed in (9.2).

In a second step, it remains to remove in case of the setting (a) the additional assumption $\mathrm{P}(\Omega)<\infty$ which we have imposed so far. To this end we consider the general case of (a) with merely $\Omega\in\mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ and apply the result achieved on the cut-offs $\Omega_R:=\Omega\cap\mathrm{B}_R\in\mathcal{BV}(\mathbb{R}^n)$ with $\mu_\pm\llcorner\Omega_R^1$ in place of $\mu_\pm$ to establish

$$\liminf_{k\to\infty}\big[\mathrm{P}(A_k,\Omega_R^1)+\mu_+(A_k^1\cap\Omega_R^1)-\mu_-(A_k^+\cap\Omega_R^1)\big]\ge\mathrm{P}(A_\infty,\Omega_R^1)+\mu_+(A_\infty^1\cap\Omega_R^1)-\mu_-(A_\infty^+\cap\Omega_R^1)$$

for every $R\in(0,\infty)$. Using $\Omega_R^1\subset\Omega^1$ and elementary estimations we deduce

$$\liminf_{k\to\infty}\mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega^1]+\mu_-((\Omega_R^1)^{\mathrm{c}})\ge\mathrm{P}(A_\infty,\Omega_R^1)+\mu_+(A_\infty^1\cap\Omega_R^1)-\mu_-(A_\infty^+)\,,$$

from which we obtain the claim (9.2) also in the general case of (a) by sending $R\to\infty$, by taking into account pointwise monotone convergence of Ω_R^1 to Ω^1 and the assumption $\mu_\pm\llcorner(\Omega^1)^c\equiv 0$, and finally by crucially exploiting the finiteness of μ_-. ☐

Next, even though these are side issues, we add remarks on a modified strategy for proving Theorem 9.1 and on a refined version of the theorem, which gives the semicontinuity conclusion (9.2) for $\mathscr{P}_{\mu_+,\mu_-}[\,\cdot\,;\Omega^1]$ even for some measures $\mu_\pm$ which merely satisfy $\mu \llcorner (\Omega^+)^{\rm c} \equiv 0$ and thus include boundary terms on $\partial^*\Omega$.

Remark 9.4 (on a modified proof of Theorem 9.1 and a variant with boundary measures)**.**

(i) *Imposing* $\mathrm{P}(\Omega) < \infty$ *as a decisive additional assumption, the conclusion of Theorem* 9.1 *can also be established by modified strategy. In case of the setting* (a) *this strategy bypasses Proposition* 7.4*, and in case of the settings* (b) *and* (c) *it requires the ICs imposed on* $\mu_-{+}\mathrm{P}(\Omega,\,\cdot\,)$ *now merely for* μ_- *itself. One may wonder whether the latter point partially improves on the statement of the theorem, but actually it does not, since in case* $\mathrm{P}(\Omega) < \infty$ *the relevant ICs for* μ_- *imply the ones for* $\mu_-{+}\mathrm{P}(\Omega,\,\cdot\,)$ *(possibly with increased* R_ε *and decreased* δ*); compare with points* (i) *and* (ii) *of Remark* 9.5 *below. Nonetheless, we believe that the modified strategy is of some intrinsic interest, and thus we explicate it here.*

Modified proof of Theorem 9.1 *in case* $\mathrm{P}(\Omega) < \infty$. We first record that $\mathrm{P}(\Omega) < \infty$ implies, by Theorem 8.2, the small-volume IC with constant 1 for the finite measure $\pi^\Omega := \mathrm{P}(\Omega,\,\cdot\,)$. Arguing as in the preceding proof, but with application of Theorem 4.1 to $\mathscr{P}_{\mu_+,\pi^\Omega}$ (and thus no need for having or checking ICs for $\mu_-{+}\pi^\Omega$), we end up with

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,0}[A_k;\Omega^1] \ge \mathscr{P}_{\mu_+,0}[A_\infty;\Omega^1]\,. \tag{9.4}$$

We can now complement this with a similar, but „dual“ reasoning. To this end we work with

$$U_k := A_k \cup \Omega^{\rm c} \qquad \text{and} \qquad U_\infty := A_\infty \cup \Omega^{\rm c}$$

(which under (b) or (c) with $\mathrm{P}(\Omega) < \infty$ are finite-perimeter sets) and deduce by an application of Theorem 4.1 to $\mathscr{P}_{\pi^\Omega,\mu_-}$ (still with $\pi^\Omega = \mathrm{P}(\Omega,\,\cdot\,)$) the semicontinuity property

$$\liminf_{k\to\infty} \mathscr{P}_{\pi^\Omega,\mu_-}[U_k] \ge \mathscr{P}_{\pi^\Omega,\mu_-}[U_\infty]\,. \tag{9.5}$$

Crucially exploiting $\mathrm{P}(\Omega) < \infty$ once more, we can rewrite $\mathrm{P}(U_k) = \mathrm{P}(U_k,\Omega^1) + \mathrm{P}(\Omega,(U_k^1)^{\rm c}) = \mathrm{P}(A_k,\Omega^1) + \mathrm{P}(\Omega) - \mathrm{P}(\Omega,U_k^1)$ and consequently $\mathscr{P}_{\pi^\Omega,\mu_-}[U_k] = \mathscr{P}_{0,\mu_-}[A_k;\Omega^1] + \mathrm{P}(\Omega)$. With this and the analogous formula for U_∞ and A_∞ we go into (9.5) and, after canceling the $\mathrm{P}(\Omega)$-terms, then find

$$\liminf_{k\to\infty} \mathscr{P}_{0,\mu_-}[A_k;\Omega^1] \ge \mathscr{P}_{0,\mu_-}[A_\infty;\Omega^1]\,. \tag{9.6}$$

Since (9.4) and (9.6) apply also with $A_k \cap A_\infty$ and $A_k \cup A_\infty$, respectively, in place of A_k, we can combine these two semicontinuity properties by the strategy from the proof of Theorem 4.1(c). Thus, we indeed arrive at the full claim (9.2) which includes both the μ_+- and μ_--terms. ☐

(ii) *If we add again* $\mathrm{P}(\Omega) < \infty$ *to the assumptions of Theorem* 9.1 *and require also those ICs imposed in the original statement on* $\mu_\pm$ *now even for* $\mu_\pm{+}\mathrm{P}(\Omega,\,\cdot\,)$*, then we can weaken the requirement* $\mu_\pm \llcorner (\Omega^1)^{\rm c} \equiv 0$ *from the original statement to merely* $\mu_\pm \llcorner (\Omega^+)^{\rm c} \equiv 0$ *and still obtain the semicontinuity conclusion for the up-to-the-boundary functional*

$$A \mapsto \mathrm{P}(A,\Omega^1) + \mu_+((A\cup\Omega^{\rm c})^1) - \mu_-((A\cap\Omega)^+)\,.$$

Here, in order to better classify the terms we record that

$$\mu_+((A\cup\Omega^{\rm c})^1) = \mu_+(A^1\cap\Omega^1) + \mu_+((A\cup\Omega^{\rm c})^1\cap\partial^*\Omega)\,,$$
$$\mu_-((A\cap\Omega)^+) = \mu_-(A^+\cap\Omega^1) + \mu_-((A\cap\Omega)^+\cap\partial^*\Omega)$$

split into an interior portion on Ω^1 *and a boundary portion on* $\partial^*\Omega$*, where the latter is evaluated via the interior traces* $(A\cap\Omega)^+\cap\partial^*\Omega$ *and* $(A\cup\Omega^{\rm c})^1\cap\partial^*\Omega$ *of* A *on* $\partial^*\Omega$ *and where these two traces actually coincide up to* $\mathcal{H}^{n-1}$*-negligible sets at least in case* $\mathrm{P}(A,\partial^*\Omega) < \infty$ *of finite perimeter up to* $\partial^*\Omega$*.*

The proof of the semicontinuity just claimed is still a variant of the preceding ones. Indeed, setting again $\pi^\Omega := \mathrm{P}(\Omega,\,\cdot\,)$*, we recall that in the original proof we applied Theorem 4.1 directly for* $\mathscr{P}_{\mu_+,\mu_-+\pi^\Omega}[S_k]$*, while in the variant of the preceding point (i) we applied it for* $\mathscr{P}_{\mu_+,\pi^\Omega}[S_k]$ *and* $\mathscr{P}_{\pi^\Omega,\mu_-}[U_k]$*. We now follow closely the latter strategy, where the only essential modification is that in order to come out with non-trivial boundary terms we cannot anymore „decouple“* $\mu_\pm$ *and* $\pi^\Omega = \mathrm{P}(\Omega,\,\cdot\,)$*, but rather now apply Theorem 4.1 for* $\mathscr{P}_{0,\mu_-+\pi^\Omega}[S_k]$ *and* $\mathscr{P}_{\mu_++\pi^\Omega,0}[U_k]$*.*

Among the assumptions mentioned above, we single out and discuss the case of the basic setting (a) with $\mu_\pm \llcorner (\Omega^+)^{\mathrm{c}} \equiv 0$ *and the small-volume IC with constant 1 for the finite measures* $\mu_\pm + \mathrm{P}(\Omega,\,\cdot\,) = \mu_\pm + \mathcal{H}^{n-1} \llcorner \partial^*\Omega$*. In this case, once more by Proposition 7.4, the IC splits into separate ICs for* $\mu_\pm \llcorner \Omega^1$ *and* $(\mu_\pm + \mathcal{H}^{n-1}) \llcorner \partial^*\Omega$*, and then Theorem 8.2 identifies a wide class of admissible measures. Indeed,* $\mu_\pm$ *will be admissible if the interior portion* $\mu_\pm \llcorner \Omega^1$ *has the form* $\alpha\mathcal{H}^{n-1} \llcorner (\Omega^1 \cap \partial^*E)$ *with* $E \in \mathcal{M}(\mathbb{R}^n)$*,* $\mathrm{P}(E) < \infty$ *and weight function* α *bounded by 2 and if the boundary portion* $\mu_\pm \llcorner \partial^*\Omega$ *has the form* $\beta\mathcal{H}^{n-1} \llcorner \partial^*\Omega$ *with boundary weight function* β *bounded by 1 (so that the resulting weight for* $(\mu_\pm + \mathcal{H}^{n-1}) \llcorner \partial^*\Omega$ *is again bounded by 2). We actually consider this part of the outcome with the bound 2 on* Ω^1 *and the bound 1 on* $\partial^*\Omega$ *as a natural and very plausible manifestation of the „one-sided accessibility“ of* $\partial^*\Omega$ *only from inside* Ω*.*

The next remark uncovers that the ICs for $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ in Theorem 9.1 may in fact be understood as a kind of domain-adapted ICs. This also motivates the usage of very similar ICs in the subsequent semicontinuity statement of Theorem 9.1 on general open sets.

Remark 9.5 (on the interpretation of the ICs for $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$ in Theorem 9.1)**.** *Consider* $\Omega \in \mathcal{B}(\mathbb{R}^n)$ *and a Radon measure* μ_- *on* $\mathbb{R}^n$*.*

(i) *If we assume* $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $\mu_- \llcorner (\Omega^1)^{\mathrm{c}} \equiv 0$*, then the almost-strong IC near* ∞ *with constant 1 for* $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$*, as it occurs in (b), implies that, for every* $\varepsilon > 0$ *with its corresponding* R_ε*, we have*

$$\mu_-(E^+) \le \mathrm{P}(E,\Omega^1) + \varepsilon \qquad \text{for all } E \in \mathcal{M}(\mathbb{R}^n) \text{ with } |E \cap \mathrm{B}_{R_\varepsilon}| = 0 \text{ and } |E| < \infty\,. \tag{9.7}$$

This can be understood as version of the same type of IC only for μ_- *but* relative to the domain Ω^1.

Proof. It suffices to verify (9.7) for $E \in \mathcal{M}(\mathbb{R}^n)$ with $|E \cap \mathrm{B}_{R_\varepsilon}| = 0$ and $|E| + \mathrm{P}(E,\Omega^1) < \infty$. To this end, we consider $R \in (R_\varepsilon,\infty)$, abbreviate $\Omega_R := \Omega \cap \mathrm{B}_R$, use $\mu_- \llcorner (\Omega^1)^{\mathrm{c}} \equiv 0$, and test the IC with $E \cap \Omega_R$. In this way we find $\mu_-(E^+ \cap \mathrm{B}_R) + \mathrm{P}(\Omega,(E \cap \Omega_R)^+) \le \mu_-((E \cap \Omega_R)^+) + \mathrm{P}(\Omega,(E \cap \Omega_R)^+) \le \mathrm{P}(E \cap \Omega_R) + \varepsilon$. Next we derive a slightly sharpened variant of the quality case in (2.4). By distinguishing between points in Ω^1_R and $\partial^{\mathrm{e}}\Omega_R$ we verify $\partial^{\mathrm{e}}(E \cap \Omega_R) = (\Omega^1_R \cap \partial^{\mathrm{e}}E) \,\dot{\cup}\, ((E \cap \Omega_R)^+ \cap \partial^{\mathrm{e}}\Omega_R)$, and then via Theorems 2.4, 2.8, and (2.4) we arrive at $\mathrm{P}(E \cap \Omega_R) = \mathrm{P}(E,\Omega^1_R) + \mathrm{P}(\Omega_R,(E \cap \Omega_R)^+) \le \mathrm{P}(E,\Omega^1) + \mathrm{P}(\Omega,(E \cap \Omega_R)^+) + \mathcal{H}^{n-1}(E^+ \cap \partial\mathrm{B}_R)$ for $R \in (0,\infty)$. When combining this with the previous estimate, the terms $\mathrm{P}(\Omega,(E \cap \Omega_R)^+)$ cancel out, and we obtain $\mu_-(E^+ \cap \mathrm{B}_R) \le \mathrm{P}(E,\Omega^1) + \mathcal{H}^{n-1}(E^+ \cap \partial\mathrm{B}_R) + \varepsilon$. Exploiting once more $|E^+| = |E| < \infty$ in a coarea argument, we have $\liminf_{R\to\infty} \mathcal{H}^{n-1}(E^+ \cap \partial\mathrm{B}_R) = 0$, and in the limit $R \to \infty$ we arrive at (9.7). (In case of $\mathrm{P}(\Omega,(E \cap \Omega)^+) < \infty$ this argument also works more directly with $E \cap \Omega$ in place of $E \cap \Omega_R$. However, we cannot exclude $\mathrm{P}(\Omega,(E \cap \Omega)^+) = \infty$ in general.) ☐

Moreover, in case of $\mathrm{P}(\Omega) < \infty$ *and with a possible increase of the radii* R_ε*, we can also get back from (9.7) to the original almost-strong IC near* ∞ *for* $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$*. This simply works by trivially enlarging the right-hand side in (9.7) to* $\mathrm{P}(E) + \varepsilon$ *and using* $\lim_{R\to\infty} \mathrm{P}(\Omega,(\mathrm{B}_R)^{\mathrm{c}}) = 0$ *to estimate* $\mathrm{P}(\Omega,\,\cdot\,)$ *outside large balls by* ε*. In case* $\mathrm{P}(\Omega) = \infty$*, however, this backwards implication is false even if, in addition to (9.7) for* μ_-*, both* μ_- *and* $\mathrm{P}(\Omega,\,\cdot\,)$ *separately satisfy the strong IC with constant 1. This is demonstrated, for* $n \ge 2$*, by* $\mu_- = \mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{-2,2\})$ *on* $\Omega = \mathbb{R}^{n-1} \times [-1,1]^{\mathrm{c}}$*, which has the announced properties.*

(ii) *If we assume once more* $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $\mu_- \llcorner (\Omega^1)^{\mathrm{c}} \equiv 0$*, then the small-volume IC with constant 1 for* $\mu_- + \mathrm{P}(\Omega,\,\cdot\,)$*, as it occurs in (c), implies, for every* $\varepsilon > 0$*, the existence of* $\delta > 0$ *such that*

$$\mu_-(E^+) \le \mathrm{P}(E,\Omega^1) + \varepsilon \qquad \text{for all } E \in \mathcal{M}(\mathbb{R}^n) \text{ with } |E| < \delta\,. \tag{9.8}$$

This can be seen as a small-volume IC for μ_- relative to the domain Ω^1*, and the implication can be proved by straightforward adaptation of the reasoning in the preceding point (i). Moreover, if we* *assume* $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$*,* $\mu_- \llcorner (\partial^*\Omega)^{\mathrm{c}} \equiv 0$*, the small-volume IC with constant* 1 *on* $\mathbb{R}^n$ *for* $\mathrm{P}(\Omega,\,\cdot\,)$ *(as it is generally satisfied in case* $\mathrm{P}(\Omega) < \infty$ *by Theorem 8.2), and that either* μ_- *is finite or the supports of* μ_- *and* $\mathrm{P}(\Omega,\,\cdot\,)$ *have positive distance, then we can also get back from (9.8) to the small-volume IC for* $\mu_+\mathrm{P}(\Omega,\,\cdot\,)$ *with constant* 1 *by using Proposition 7.4. In connection with this last claim, it is easy to see that the assumptions* $\Omega \in \mathcal{BV}_{\mathrm{loc}}(\mathbb{R}^n)$*,* $\mu_- \llcorner (\partial^*\Omega)^{\mathrm{c}} \equiv 0$*, and the small-volume IC for* $\mathrm{P}(\Omega,\,\cdot\,)$ *cannot be dropped. Moreover, the example given, for* $n=2$*, by* $\mu_- = \mathcal{H}^1 \llcorner (\mathbb{R}{\times}\{0\})$ *on* $\Omega = \mathbb{R}^2 \setminus \bigcup_{i=1}^\infty \big([2i{-}1,2i]{\times}\big[\frac{1}{2i},\frac{1}{i}\big]\big)$ *demonstrates that also requirement of finiteness of* μ_- *or supports at positive distance is indeed necessary for the backwards implication (even if, as it happens here, both* μ_- *and* $\mathrm{P}(\Omega,\,\cdot\,)$ *separately satisfy the strong IC with constant* 1*).*

At this point we finally turn to the second main statement of this section, which complements the previous result on the measure-theoretic interior of $\mathcal{BV}_{(\mathrm{loc})}$ sets with a version on arbitrary open sets $D = \Omega$ in $\mathbb{R}^n$. So, in comparison with Theorem 9.1 we drop any regularity of the domain, but require openness in the stronger topological sense.

Theorem 9.6 (lower semicontinuity on a general open set)**.** *Consider an open set* Ω *in* $\mathbb{R}^n$*, a set* $A_\infty \in \mathcal{M}(\mathbb{R}^n)$*, a sequence* $(A_k)_{k\in\mathbb{N}}$ *in* $\mathcal{M}(\mathbb{R}^n)$*. For non-negative Radon measures* μ_+ *and* μ_- *on* $\mathbb{R}^n$ *with* $\mu_\pm \llcorner \Omega^{\mathrm{c}} \equiv 0$*, assume that both* μ_+ *and* μ_- *satisfy the small-volume IC in* $\mathbb{R}^n$ *with constant* 1 *and that* one of the following sets of additional assumptions *is valid*:

(a) *The measure* μ_- *is* finite*, and* A_k *converge to* A_∞ locally *in measure on* Ω*.*

(b) *The measure* μ_- *satisfies an* almost-strong *IC near* ∞ relative *to* Ω *with constant* 1 *in the sense that, for every* $\varepsilon > 0$*, there exists some* $R_\varepsilon \in (0,\infty)$ *such that*

$$\mu_-(A^+) \le \mathrm{P}(A,\Omega) + \varepsilon \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } |A \cap \mathrm{B}_{R_\varepsilon}| = 0 \text{ and } |A| < \infty\,, \tag{9.9}$$

and A_k *converge to* A_∞ locally *in measure on* Ω *with* $|(A_k \Delta A_\infty)\cap\Omega| + \mathrm{P}(A_k,\Omega) + \mathrm{P}(A_\infty,\Omega) < \infty$ *for all* $k \in \mathbb{N}$*.*

(c) *The measure* μ_- *satisfies the small-volume IC* relative *to* Ω *with constant* 1 *in the sense that, for every* $\varepsilon > 0$*, there is some* $\delta > 0$ *such that*

$$\mu_-(A^+) \le \mathrm{P}(A,\Omega) + \varepsilon \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n) \text{ with } |A| < \delta\,, \tag{9.10}$$

and A_k *converge to* A_∞ globally *in measure on* Ω *with* $\mathrm{P}(A_k,\Omega) + \mathrm{P}(A_\infty,\Omega) < \infty$ *for all* $k \in \mathbb{N}$*.*

If furthermore $\min\{\mu_+(A_k^1), \mu_-(A_k^+)\} < \infty$ *holds for all* $k \in \mathbb{N}$*, then we have* $\min\{\mu_+(A_\infty^1), \mu_-(A_\infty^+)\} < \infty$ *and*

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega] \ge \mathscr{P}_{\mu_+,\mu_-}[A_\infty;\Omega]\,. \tag{9.11}$$

Since the different cases in Theorem 9.6 are still illustrated well by the examples given in connection with Theorem 9.1, we now keep the discussion brief. Once more, the setting (a) concerns finite measures μ_-, and this part of Theorem 9.6 will be deduced from the corresponding assertion for finite-perimeter domains by a simple exhaustion argument, which closely resembles the last step in the proof of Theorem 9.1 and crucially draws on the finiteness of μ_-. The improvements for infinite measures provided by (b) and (c) involve essentially the same relative ICs found in (9.7) and (9.8). Despite this similarity, under (b) or (c) with possibly infinite μ_- we cannot derive the result directly from the counterparts in Theorem 9.1 by exhaustion, but rather will implement a deduction from the result in the setting (a) by cut-off arguments widely analogous to the proof of Theorem 6.1.

The difference between (b) and (c) can again be underpinned with concrete examples: On one hand, the cases $n \ge 2$, $\mu = 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1}{\times}\{0\})$, $\Omega = \mathbb{R}^{n-1}{\times}(-1,1)$ and $n=1$, $\mu = 2\mathcal{H}^0 \llcorner \mathbb{Z}$, $\Omega = \mathbb{R}$ are included in (c), but not in (b). On the other hand, both (b) and (c) apply in the cases $n=2$, $\mu = 2\mathcal{H}^1 \llcorner (\mathbb{R}{\times}\{0\})$, $\Omega = \{(x,y) \in \mathbb{R}^2 : |y| < |x|\}$ and $n=1$, $\mu = 2\mathcal{H}^0 \llcorner (2\mathbb{Z}{-}1)$, $\Omega = \mathbb{R}\backslash 2\mathbb{Z}$, where, however, only (b) gives semicontinuity even with respect to *local* convergence in measure.

We also record in connection with both Theorem 9.1 and Theorem 9.6 and the corresponding examples:

Remark 9.7 (on the settings of Theorems 9.1 and 9.6)**.** *In Theorem 9.6, the settings* (b) *and* (c) *improve on* (a) *only in the infinite-volume case $|\Omega| = \infty$, since indeed the IC from* (9.9) *or* (9.10) *for a Radon measure μ_- on $\mathbb{R}^n$ and open $\Omega \subset \mathbb{R}^n$ together with $|\Omega| < \infty$ and $\mu_- \llcorner (\Omega^+)^c \equiv 0$ already enforces finiteness of μ_-.*

Proof. In case $|\Omega| < \infty$ we may test (9.9) with $\Omega \setminus \mathrm{B}_{R_1}$ to infer $\mu_-(\Omega^+ \setminus \overline{\mathrm{B}_{R_1}}) \le \mu_-((\Omega \setminus \mathrm{B}_{R_1})^+) \le \mathrm{P}(\Omega \setminus \mathrm{B}_{R_1}, \Omega) + 1 \le \mathrm{P}(\mathrm{B}_{R_1}) + 1 < \infty$. Similarly, if we fix $\delta > 0$ such that (9.10) applies with $\varepsilon = 1$, then in view of $|\Omega| < \infty$ we have $|\Omega \setminus \mathrm{B}_{R_1}| < \delta$ for some suitably large $R_1 \in (0, \infty)$, and by testing (9.10) with $\Omega \setminus \mathrm{B}_{R_1}$ we deduce exactly the same estimate. Clearly, taking into account local finiteness of μ_- and $\mu_- \llcorner (\Omega^+)^c \equiv 0$, the estimate yields finiteness of μ_- in both cases. □

Also in the earlier Theorem 9.1, the settings (b) *and* (c) *improve on* (a) *only in case $|\Omega| = \infty$. This follows by the same reasoning, which also works with* (9.7) *and* (9.8) *in place of* (9.9) *and* (9.10)*.*

Finally, let us point out that the additional *relative* IC (9.10) of Theorem 9.6(c) could in fact be required only near ∞ by adding a condition $|A \cap \mathrm{B}_{R_\varepsilon}| = 0$, as it was included in all our settings of type (b). However, while for the strong-type setting (b) the near-∞ feature does win some generality, in the small-volume setting (c) an adaptation of Proposition 7.4 shows that it does not, and therefore we have in fact decided to stick to the formulation of Theorem 9.6(c) given above.

Now we proceed to the final semicontinuity proof of this paper.

Proof of Theorem 9.6. Throughout the proof we assume that $\lim_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega]$ exists and is finite. In addition, in view of $\mu_\pm \llcorner \Omega^c \equiv 0$ the values $\mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega]$, $\mathscr{P}_{\mu_+,\mu_-}[A_\infty;\Omega]$ and all assumptions depend only on the portions $A_k \cap \Omega$ and $A_\infty \cap \Omega$ of A_k and A_∞. Hence we may and do assume

$$A_k \subset \Omega \qquad \text{and} \qquad A_\infty \subset \Omega\,,$$

which allows to rewrite the assumption $|(A_k \Delta A_\infty) \cap \Omega| < \infty$ of (b) as $|A_k \Delta A_\infty| < \infty$ and to consider the global convergence on Ω in (c) as global convergence on $\mathbb{R}^n$.

In order to treat the situation (a) we observe that the open set Ω can be exhausted by smooth open sets $\Omega_\ell \Subset \Omega$ with $\ell \in \mathbb{N}$ in the sense that $\Omega_\ell \subset \Omega_{\ell+1}$ for all $\ell \in \mathbb{N}$ and $\bigcup_{\ell=1}^\infty \Omega_\ell = \Omega$. Applying Theorem 9.1(a) on Ω_ℓ (which in particular satisfies $\Omega_\ell \in \mathcal{BV}(\mathbb{R}^n)$ and $\Omega_\ell^1 = \Omega_\ell$) with the measures $\mu_\pm \llcorner \Omega_\ell$ we find

$$\liminf_{k\to\infty} \left[\mathrm{P}(A_k, \Omega_\ell) + \mu_+(A_k^1 \cap \Omega_\ell) - \mu_-(A_k^+ \cap \Omega_\ell)\right] \ge \mathrm{P}(A_\infty, \Omega_\ell) + \mu_+(A_\infty^1 \cap \Omega_\ell) - \mu_-(A_\infty^+ \cap \Omega_\ell)\,.$$

Using $\Omega_\ell \subset \Omega$ and elementary estimations we deduce

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega] + \mu_-(\Omega_\ell^c) \ge \mathrm{P}(A_\infty, \Omega_\ell) + \mu_+(A_\infty^1 \cap \Omega_\ell) - \mu_-(A_\infty^+)\,,$$

from which we obtain the claim (9.11) in the generality of the situation (a) by sending $\ell \to \infty$, by taking into account the pointwise monotone convergence of Ω_ℓ to Ω and the assumption $\mu_\pm \llcorner \Omega^c \equiv 0$, and finally by crucially exploiting the finiteness of μ_-.

In view of the analogy to the proof of Theorem 6.1(b) we only sketch the arguments relevant for the present setting (b). As in the earlier proof, given an arbitrary $\varepsilon > 0$, we first choose a sequence of radii $R_i \in (R_\varepsilon, \infty)$ with $\lim_{i\to\infty} R_i = \infty$ and pass to a subsequence of $(A_k)_{k\in\mathbb{N}}$ in order to ensure $\mu_-(\partial \mathrm{B}_{R_i}) = 0$ and $\lim_{k\to\infty} \mathcal{H}^{n-1}((A_k \Delta A_\infty)^+ \cap \partial \mathrm{B}_{R_i}) = 0$. We then apply the already established part (a) of the present theorem on $\Omega \cap \mathrm{B}_{R_i}$ with the *finite* measures $\mu_\pm \llcorner (\Omega \cap \mathrm{B}_{R_i})$, which inherit the small-volume IC from $\mu_\pm$, to infer

$$\liminf_{k\to\infty} \left[\mathrm{P}(A_k, \Omega \cap \mathrm{B}_{R_i}) + \mu_+(A_k^1 \cap \mathrm{B}_{R_i}) - \mu_-(A_k^+ \cap \mathrm{B}_{R_i})\right] \ge \mathrm{P}(A_\infty, \Omega \cap \mathrm{B}_{R_i}) + \mu_+(A_\infty^1 \cap \mathrm{B}_{R_i}) - \mu_-(A_\infty^+ \cap \mathrm{B}_{R_i})\,.$$

In order to estimate the terms cut off we follow closely the derivation around (6.5) and (6.6), where now we take perimeters in the open domain Ω and rely on the relative version (9.9) of the almost-strong IC in the form $\mu_-(((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i})^+) \le \mathrm{P}((A_k \Delta A_\infty) \setminus \mathrm{B}_{R_i}, \Omega) + \varepsilon$ (which does apply, since $R_i \ge R_\varepsilon$). Arguing as described we find that either the claim (9.11) holds trivially or we have $\mu_-(A_k^+) + \mu_-(A_\infty^+) < \infty$ for all $k \in \mathbb{N}$ together with

$$\liminf_{k\to\infty} \left[\mathrm{P}(A_k, \Omega \setminus \mathrm{B}_{R_i}) - \mu_-(A_k^+ \setminus \mathrm{B}_{R_i})\right] \ge -\mathrm{P}(A_\infty, \Omega \setminus \mathrm{B}_{R_i}) - \mu_-(A_\infty^+ \setminus \mathrm{B}_{R_i}) - \varepsilon\,. \tag{9.12}$$

By addition of the last two displayed equations and elementary estimation we arrive at

$$\liminf_{k\to\infty} \mathscr{P}_{\mu_+,\mu_-}[A_k;\Omega] \geq \mathrm{P}(A_\infty,\Omega\cap \mathrm{B}_{R_i}) - \mathrm{P}(A_\infty,\Omega\setminus \mathrm{B}_{R_i}) + \mu_+(A_\infty^1\cap \mathrm{B}_{R_i}) - \mu_-(A_\infty^+) - \varepsilon\,.$$

Going to the limit $i\to\infty$ ans using the arbitrariness of ε, we obtain the claim (9.11) in the generality of the situation (b).

The proof in the setting (c) is an adaptation of the one in the setting (b), precisely as Theorem 6.1(c) was obtained by adapting the argument given for Theorem 6.1(b). Indeed, for an arbitrary $\varepsilon>0$, we can exploit $\lim_{k\to\infty}|A_k\Delta A_\infty|=0$ in order to apply the relative version (9.10) of the small-volume IC in the form $\mu_-(((A_k\Delta A_\infty)\setminus \mathrm{B}_{R_i})^+)\leq \mathrm{P}((A_k\Delta A_\infty)\setminus \mathrm{B}_{R_i},\Omega)+\varepsilon$ at least for $k\gg 1$. In the limit $k\to\infty$ we still arrive at the estimate (9.12) and in conclusion can deduce the claim (9.11) also in the generality of the situation (c). □

We conclude this section by pointing out that, as it was on $\Omega=\mathbb{R}^n$, also on arbitrary Ω the relative small-volume IC (9.10) on μ_- is in fact optimal. This will go hand in hand with recording further connections between the standard small-volume IC, its variant in (9.10), and semicontinuity properties of the functional, and will now be explicated for the case $\mu_+\equiv 0$, $\mu_-=\mu$:

Remark 9.8 (on optimality of the relative IC (9.10) and more connections between ICs and semicontinuity)**.** *We here consider an open set $\Omega\subset\mathbb{R}^n$ and a non-negative Radon measure μ on $\mathbb{R}^n$ with $\mu\llcorner\Omega^{\mathrm{c}}\equiv 0$.*

(i) *If $\mathscr{P}_{0,\mu}[\,\cdot\,;\Omega]$ is lower semicontinuous on $\mathcal{BV}(\Omega)$ with respect to global convergence in measure on Ω, then for every $\varepsilon>0$, there is some $\delta>0$ such that (9.10) holds for μ, that is, $\mu(A^+)\leq \mathrm{P}(A,\Omega)+\varepsilon$ for all $A\in\mathcal{M}(\mathbb{R}^n)$ with $|A|<\delta$.*

Proof. If (9.10) fails for some $\varepsilon>0$ and all $\delta>0$, in particular, for each $k\in\mathbb{N}$, there exists $A_k\in\mathcal{M}(\mathbb{R}^n)$ with $|A_k|<\frac{1}{k}$ and $\mu(A_k^+)>\mathrm{P}(A_k,\Omega)+\varepsilon$. However, then $A_k\in\mathcal{BV}(\Omega)$ converge to $\emptyset$ in measure on Ω with $\mathscr{P}_{0,\mu}[A_k;\Omega]<-\varepsilon$, and $\mathscr{P}_{0,\mu}[\,\cdot\,;\Omega]$ is not lower semicontinuous. □

Thus, at least in case $\mu_+\equiv 0$ the assumption (9.10) on μ_- in Theorem 9.6(c) is also necessary for lower semicontinuity of $\mathscr{P}_{0,\mu_-}[\,\cdot\,;\Omega]$ and thus optimal.

(ii) *Consider the following assertions*[13]:

(1) *The measure μ is finite and satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1.*

(2) *For every $A_0\in\mathcal{M}(\mathbb{R}^n)$ with $\mathrm{P}(A_0,\Omega)<\infty$, the functional $\mathscr{P}_{0,\mu}[\,\cdot\,;\Omega]$ is lower semicontinuous on $\{A\in\mathcal{M}(\mathbb{R}^n) : A\Delta A_0\in\mathcal{BV}(\Omega)\}$ with respect to* local *convergence in measure on Ω.*

(3) *For every $A_0\in\mathcal{M}(\mathbb{R}^n)$ with $\mathrm{P}(A_0,\Omega)<\infty$, the functional $\mathscr{P}_{0,\mu}[\,\cdot\,;\Omega]$ is lower semicontinuous on $\{A\in\mathcal{M}(\mathbb{R}^n) : A\Delta A_0\in\mathcal{BV}(\Omega)\}$ with respect to* global *convergence in measure on Ω.*

(4) *The functional $\mathscr{P}_{0,\mu}[\,\cdot\,;\Omega]$ is lower semicontinuous on $\mathcal{BV}(\Omega)$ with respect to* global *convergence in measure on Ω.*

(5) *For every $\varepsilon>0$, there is some $\delta>0$ such that μ satisfies small-volume IC (9.10)* relative *to Ω.*

Then, we claim that the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) are generally valid. Indeed, (1) $\Longrightarrow$ (2) holds by Theorem 9.6(a), the implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are trivial, (4) $\Longrightarrow$ (5) has been established in the preceding point (i), and (5) $\Longrightarrow$ (3) holds by Theorem 9.6(c).

We could in fact formulate even more equivalent statements, for instance, one such statement is given by the localized IC variant of Lemma 7.3(d) together with finiteness of μ.

[13] Here, for the local-convergence semicontinuity (2), we need to restrict to subclasses of $\mathcal{BV}(\Omega)$ which exclude convergence of A_k to A with $|(A_k\Delta A)\cap\Omega|=\infty$ for all $k\in\mathbb{N}$. In contrast, the global-convergence statement (3) could equivalently be stated on all of $\{A\in\mathcal{M}(\mathbb{R}^n):\mathrm{P}(A,\Omega)<\infty\}$, since global convergence of A_k to A anyway yields $|(A_k\Delta A)\cap\Omega|<\infty$ for $k\gg 1$.

(iii) *In general, the implication* (1) $\Longrightarrow$ (2) *from point* (ii) *cannot be reversed. To see this, for* $n \geq 2$, *we consider* $\mu = 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{0\})$ *on* $\Omega = \mathbb{R}^n$ *or alternatively* $\mu = \mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{0\})$ *on any open* $\Omega \subset \mathbb{R}^n$ *with* $\mathbb{R}^{n-1} \times [0, \infty) \subset \Omega$. *Then, it can be checked that* μ *satisfies* (9.9). *Thus, Theorem* 9.6(b) *gives the validity of* (2), *while* (1) *fails in view of the infiniteness of* μ. (*The specific case* $n = 1$ *is different, and for this case one can in fact show that the validity of* (2) *requires finiteness of* μ *and that* (1) $\Longleftrightarrow$ (2) *holds.*)

Also the implication (2) $\Longrightarrow$ (3) *cannot be reversed in general. Here, for* $n \geq 2$ *we consider the infinite measure* $\mu = 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1} \times \{0, 1\})$ *on any open* $\Omega \subset \mathbb{R}^n$ *with* $\operatorname{dist}(\mathbb{R}^{n-1} \times \{0, 1\}, \Omega^c) > 0$. *Then, by adapting the proof of Proposition* A.3 *one checks that* μ *satisfies* (9.10) *for all these* Ω. *Hence, Theorem* 9.6(c) *gives the validity of* (3), *while* $A_k := [k, k+n]^{n-1} \times [0, 1] \in \mathcal{BV}(\mathbb{R}^n)$ *converge locally in measure on* Ω *to* $\emptyset$ *with* $\mathscr{P}_{0,\mu}[A_k; \Omega] \leq \mathscr{P}_{0,\mu}[A_k; \mathbb{R}^n] = -2n^{n-2} < 0$ *and thus demonstrate that* (2) *fails in this case. For* $n = 1$, *the same phenomenon occurs for* $\mu = 2\mathcal{H}^0 \llcorner \mathbb{Z}$ *on any open* $\Omega \subset \mathbb{R}$ *with* $\operatorname{dist}(\mathbb{Z}, \Omega^c) > 0$.

(iv) *However, if we impose as an additional assumption*

$$\text{either } |\Omega| < \infty \text{ or } \mu(\Omega) < \infty\,,$$

it turns out that the ***five assertions*** *of point* (ii) *are in fact* ***all equivalent***. *In order to justify this claim we recall from Remark* 9.7 *that* (9.10) *and* $|\Omega| < \infty$ *together enforce finiteness of* μ.. *Since moreover* (9.10) *is stronger than the usual small-volume IC, this means that under the additional assumption we also have the backwards implication* (5) $\Longrightarrow$ (1).

In particular, we record that for the (*counter*)*examples of point* (iii) *it was inevitable to have both* $|\Omega| = \infty$ *and* $\mu(\Omega) = \infty$.

A Isoperimetric conditions for infinite model measures

In this appendix we justify the validity of ICs for basic infinite model measures concentrated on hyperplanes by suitable capacity computations. We start with an auxiliary lemma, which determines the 1-capacity of sets in a hyperplane and is not at all surprising. Still, since we are not aware of a custom-fit reference for this statement, we also include a proof.

Lemma A.1 (1-capacity on hyperplanes)**.** *For* $n \geq 2$, *every* $S \in \mathcal{B}(\mathbb{R}^{n-1})$, *and* $t \in \mathbb{R}$, *we have*

$$\operatorname{Cap}_1(S \times \{t\}) = 2|S|\,.$$

In different words, this means $\operatorname{Cap}_1(A) = 2\mathcal{H}^{n-1}(A)$ *for every* $A \in \mathcal{B}(\mathbb{R}^{n-1} \times \{t\})$ *with* $t \in \mathbb{R}$.

Proof. We prove the inequalities „$\leq$“ and „$\geq$“ separately.

We consider an open $U \in \mathcal{BV}(\mathbb{R}^{n-1})$ and the open cylinder $U^\delta := U \times (t-\delta, t+\delta)$ with $\delta > 0$. One verifies $|U^\delta| = 2\delta|U| < \infty$, $U \times \{t\} \subset U^\delta \subset (U^\delta)^+$, and $\mathrm{P}(U^\delta) = 2|U| + 2\delta \mathrm{P}(U)$. Therefore, Proposition 2.15 gives $\operatorname{Cap}_1(U \times \{t\}) \leq \operatorname{Cap}_1(U^\delta) \leq 2|U| + 2\delta \mathrm{P}(U)$ for arbitrary $\delta > 0$, and we get $\operatorname{Cap}_1(U \times \{t\}) \leq 2|U|$. Now, an arbitrary open set in $\mathbb{R}^{n-1}$ is the union of an increasing sequence of bounded open sets with smooth boundaries, thus in particular of open sets from $\mathcal{BV}(\mathbb{R}^{n-1})$. (This claim can be proved essentially by mollifying $\mathbb{1}_K$ with compact $K \subset U$ and then choosing good superlevel sets of the mollifications via Sard's theorem.) By [15, Theorem 4.15(viii)] one can pass to the limit along such a sequence to deduce that $\operatorname{Cap}_1(U \times \{t\}) \leq 2|U|$ stays valid for arbitrary open $U \subset \mathbb{R}^{n-1}$. For arbitrary $S \in \mathcal{B}(\mathbb{R}^{n-1})$, one then concludes

$$\operatorname{Cap}_1(S \times \{t\}) \leq \inf\{\operatorname{Cap}_1(U \times \{t\}) : U \text{ open in } \mathbb{R}^{n-1}\,,\, S \subset U\} \leq \inf\{2|U| : U \text{open in } \mathbb{R}^{n-1}\,,\, S \subset U\} = 2|S|\,.$$

From Definition 2.14 one obtains in a standard way (essentially by mollification and multiplication with cut-off functions) the equality

$$\operatorname{Cap}_1(K) = \inf\left\{\int_{\mathbb{R}^n} |\nabla\eta|\,\mathrm{d}x \,:\, \eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)\,,\, \eta \geq 1 \text{ on } K\right\} \qquad \text{for compact } K \subset \mathbb{R}^n\,. \tag{A.1}$$

Now, if H is compact in $\mathbb{R}^{n-1}$, for every $\eta \in \mathrm{C}^\infty_{\mathrm{cpt}}(\mathbb{R}^n)$ with $\eta \geq 1$ on $H{\times}\{t\}$, one has

$$\int_{\mathbb{R}^n} |\nabla\eta|\,\mathrm{d}x \geq \int_H \left[\left|\int_{-\infty}^t \partial_n\eta(x',x_n)\,\mathrm{d}x_n\right| + \left|\int_t^\infty \partial_n\eta(x',x_n)\,\mathrm{d}x_n\right|\right]\mathrm{d}x' = \int_H 2|\eta(x',t)|\,\mathrm{d}x' \geq 2|H|\,,$$

and by (A.1) this implies $\mathrm{Cap}_1(H{\times}\{t\}) \geq 2|H|$. For arbitrary $S \in \mathcal{B}(\mathbb{R}^{n-1})$, one then concludes

$$\mathrm{Cap}_1(S{\times}\{t\}) \geq \sup\{\mathrm{Cap}_1(H{\times}\{t\}) : H \text{ compact}\,,\ H \subset S\} \geq \sup\{2|H| : H \text{ compact}\,,\ H \subset S\} = 2|S|\,,$$

which completes the proof. □

The following results now identify two infinite measures, which satisfy the strong IC with constant 1 and the small-volume IC with constant 1, respectively.

Proposition A.2 (strong IC for $\mathcal{H}^{n-1}$ on a single hyperplane)**.** *For $n \geq 2$, the non-negative Radon measure*

$$\mu := 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1}{\times}\{0\}) \qquad \text{on } \mathbb{R}^n$$

satisfies the strong IC in $\mathbb{R}^n$ with constant 1.

Proof. For $A \in \mathcal{BV}(\mathbb{R}^n)$, from Lemma A.1 and Proposition 2.15 we obtain

$$\mu(A^+) = 2\mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{0\})) = \mathrm{Cap}_1(A^+ \cap (\mathbb{R}^{n-1}{\times}\{0\})) \leq \mathrm{P}(A)\,.$$

Since the resulting estimate trivially holds in case $\mathrm{P}(A) = \infty$ as well, we have verified the claimed IC. □

Proposition A.3 (small-volume IC for $\mathcal{H}^{n-1}$ on two parallel hyperplanes)**.** *For $n \geq 2$, the non-negative Radon measure*

$$\mu := 2\mathcal{H}^{n-1} \llcorner (\mathbb{R}^{n-1}{\times}\{0,1\}) \qquad \text{on } \mathbb{R}^n$$

satisfies the small-volume IC in $\mathbb{R}^n$ with constant 1, and more precisely we have in fact

$$\mu(A^+) \leq \mathrm{P}(A) + 2|A| \qquad \text{for all } A \in \mathcal{M}(\mathbb{R}^n)\,.$$

Proof. The validity of the IC follows by combining Proposition A.2 and Proposition 7.4. However, we now carry out an alternative and self-contained proof, which also yields the explicit estimate claimed. Clearly we can assume $A \in \mathcal{BV}(\mathbb{R}^n)$. In view of $\int_0^1 \mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{t\}))\,\mathrm{d}t \leq |A^+| = |A|$ we can find and fix some $t \in (0,1)$ with

$$\mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{t\})) \leq |A|\,.$$

Introducing $A_0 := A \cap (\mathbb{R}^{n-1}{\times}(-\infty,t))$ with $|A_0| \leq |A| < \infty$, by an application[14] of (2.4) we get

$$\mathrm{P}(A_0) \leq \mathrm{P}(A, \mathbb{R}^{n-1}{\times}(-\infty,t)) + \mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{t\})) \leq \mathrm{P}(A, \mathbb{R}^{n-1}{\times}(-\infty,t)) + |A|\,.$$

Via Lemma A.1 and Proposition 2.15 (the latter applied in view of $A^+ \cap (\mathbb{R}^{n-1}{\times}\{0\}) \subset A_0^+$) we infer

$$2\mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{0\})) = \mathrm{Cap}_1(A^+ \cap (\mathbb{R}^{n-1}{\times}\{0\})) \leq \mathrm{P}(A_0) \leq \mathrm{P}(A, \mathbb{R}^{n-1}{\times}(-\infty,t)) + |A|\,.$$

With the help of $A_1 := A \cap (\mathbb{R}^{n-1}{\times}(t,\infty))$, we analogously obtain the estimate

$$2\mathcal{H}^{n-1}(A^+ \cap (\mathbb{R}^{n-1}{\times}\{1\})) = \mathrm{Cap}_1(A^+ \cap (\mathbb{R}^{n-1}{\times}\{1\})) \leq \mathrm{P}(A_1) \leq \mathrm{P}(A, \mathbb{R}^{n-1}{\times}(t,\infty)) + |A|\,.$$

Adding up the two estimates gives the claim $\mu(A^+) \leq \mathrm{P}(A) + 2|A|$, from which the IC is immediate. □

We remark that the preceding propositions formally extend to the case $n = 1$, where they correspond to the much simpler estimates $2\delta_0(A^+) \leq \mathrm{P}(A)$ for $A \in \mathcal{B}(\mathbb{R})$ with $|A| < \infty$ and $2(\delta_0{+}\delta_1)(A^+) \leq \mathrm{P}(A){+}2|A|$ for arbitrary $A \in \mathcal{B}(\mathbb{R})$, with the Dirac measures δ_0 and δ_1 at 0 and 1. However, the measures δ_0 and $\delta_0{+}\delta_1$ are clearly finite, and indeed, for $n = 1$, measures with strong IC are necessarily finite, while the small-volume IC with constant 1 still admits infinite examples such as the measure $2\mathcal{H}^0 \llcorner \mathbb{Z} = 2\sum_{z\in\mathbb{Z}} \delta_z$, for instance.

[14] If we stick to the precise statement of Lemma 2.9, then in view of $\mathrm{P}(\mathbb{R}^{n-1}{\times}(-\infty,t)) = \infty$ we cannot use (2.4) directly for $\mathbb{R}^{n-1}{\times}(-\infty,t)$ and $G = \mathbb{R}^n$, but clearly we can circumvent this by applying (2.4) with $G = \mathrm{B}_R$ first and then passing $R \to \infty$.

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