COMPACTNESS OF SINGULAR SOLUTIONS TO THE SIXTH ORDER GJMS EQUATION

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Abstract. We study compactness properties of the set of conformally flat singular metrics with constant, positive sixth order Q-curvature on a finitely punctured sphere. Based on a recent classification of the local asymptotic behavior near isolated singularities, we introduce a notion of necksize for these metrics in our moduli space, which we use to characterize compactness. More precisely, we prove that if the punctures remain separated and the necksize at each puncture is bounded away from zero along a sequence of metrics, then a subsequence converges with respect to the Gromov–Hausdorff metric. Our proof relies on an upper bound estimate which is proved using moving planes and a blow-up argument. This is combined with a lower bound estimate which is a consequence of a removable singularity theorem. We also introduce a homological invariant which may be of independent interest for upcoming research.

1. Introduction

In recent years, there has been active research into analogs of the Yamabe problem and its singular counterpart. In each of these problems, one seeks a representative of a conformal class with constant curvature of some type, scalar curvature in the classical case, and some σ_k-curvature or one of Branson's Q^{2m}-curvatures in more modern examples. Conformal invariance (or, more generally, covariance) often complicates these problems, leading to singular solutions and the lack of compactness in the space of solutions. For this reason, it is always appealing to characterize which geometric properties in the solution space imply compactness.

In the present paper, we study the moduli space of complete, conformally flat metrics with constant sixth order Q^6-curvature on a finitely punctured sphere. Our main result generalizes a theorem of Pollack [25] in the scalar curvature setting, stating that so long as the punctures remain separated and certain geometric necksizes bounded away from zero, the the corresponding subset of moduli space is compact in the Gromov-Hausdorff topology.

Let $n \geqslant 7$ and denote the n-dimensional sphere by $\mathbb{S}^n$. For $N \in \mathbb{N}$ we let $\Lambda = \{p_1, \ldots, p_N\} \subset \mathbb{S}^n$ be a finite subset and seek complete metrics on $\Omega := \mathbb{S}^n \backslash \Lambda$ of the form $g = U^{4/n-6} g_0$, where g_0 is the standard round metric. The fact that g is complete on Ω forces $\liminf_{p \to p_i} U(p) = \infty$ for each $i = 1, \ldots, N$. Furthermore, we prescribe the resulting metric to have constant Q^6-curvature, which we normalize to be

$$Q_n = Q^6(g_0) = \frac{n(n^4 - 20n^2 + 64)}{2^5}. \tag{1.1}$$

We define $Q^6(g)$ the quantity in Definition A.3 for any smooth metric.

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The Q-curvature Q^6 behaves well under a conformal change of metric. More precisely, the condition that $g = U^{4/(n-6)}g_0$ satisfies $Q^6(g) = Q_n$ on $\Omega = \mathbb{S}^n \backslash \Lambda$ is equivalent to the PDE

$$P^6_{g_0}U = c_n U^{\frac{n+6}{n-6}} \quad \text{on} \quad \Omega, \tag{$\mathcal{Q}_{6,g_0,N}$}$$

where $c_n = \frac{n-6}{2}Q_n$ is a normalizing constant. The operator on the left-hand side is the sixth order GJMS operator on the sphere defined by

$$P^6_{g_0} = \left(-\Delta_{g_0} + \frac{(n-6)(n+4)}{4}\right)\left(-\Delta_{g_0} + \frac{(n-4)(n+2)}{4}\right)\left(-\Delta_{g_0} + \frac{n(n-2)}{4}\right), \tag{1.2}$$

and after a conformal change of metric $g = U^{4/n-6}g_0$, it transforms as

$$P^6_g\phi = U^{-\frac{n+6}{n-6}}P^6_{g_0}(U\phi) \quad \text{for all} \quad \phi \in \mathcal{C}^\infty(\Omega). \tag{1.3}$$

For more details on this subject, we refer the interested reader to [7, 10, 12, 19].

In [14] Graham, Jenne, Mason and Sparling constructed conformally covariant differential operators P^{2m}_g on a compact n-dimensional Riemannian manifold (M^n, g) for any $m \in \mathbb{N}$ such the leading order term of P^{2m}_g is $(-\Delta_g)^m$. One can then construct the associated Q-curvature of order $2m$ by $Q^{2m}_g = P^{2m}_{g_0}(1)$. In the special case $m = 1$, one recovers the conformal Laplacian

$$P^2_g = -\Delta_g + \frac{n-2}{4(n-1)}R_g \quad \text{with} \quad Q^2_g = \frac{n-2}{4(n-1)}R_g,$$

where Δ_g is the Laplace-Beltrami operator of g and R_g is its scalar curvature. Subsequently, Grahan and Zworski [15] and Chang and González [8] extended these definitions in the case the background metric is the round metric on the sphere to obtain (nonlocal) operators $P^\sigma_{g_0}$ of any order $\sigma \in (0, n/2)$ as well as the corresponding Q-curvatures of order σ. Once again, the leading order part of $P^\sigma_{g_0}$ is $(-\Delta_{g_0})^\sigma$, understood as the principal value of a singular integral operator. We write the formulae for P^2_g, P^4_g and P^6_g explicitly in Definitions A.2 and A.4. Nevertheless, the expressions for P^σ_g and Q^σ_g for a general $\sigma \in \mathbb{R}_+$ are more complicated (see for instance [11]).

We remark that the nonlinearity on the right-hand side of ($\mathcal{Q}_{6,g_0,N}$) has critical growth with respect to the Sobolev embedding $W^{3,2}(\mathbb{R}^n) \hookrightarrow L^{2^\#}(\mathbb{R}^n)$, where $2^\# = \frac{2n}{n-6}$. It is well known that this embedding is not compact, reflecting the conformal invariance of the PDE ($\mathcal{Q}_{6,g_0,N}$).

It will be convenient to transfer the PDE ($\mathcal{Q}_{6,g_0,N}$) to Euclidean space, which we can do using the standard stereographic projection (with the north pole in Ω, and thus a regular point of any of the metrics we consider). After stereographic projection, we can write

$$g_0 = u_{\rm sph}^{\frac{4}{n-6}}\delta, \qquad u_{\rm sph}(x) = \left(\frac{1+|x|^2}{2}\right)^{\frac{6-n}{2}},$$

where δ is the Euclidean metric. In these coordinates our conformal metric takes the form $g = U^{4/(n-6)}g_0 = (U \cdot u_{\rm sph})^{4/(n-6)}\delta$. Thus, $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma)$ given by $u = U \cdot u_{\rm sph}$ is a positive singular solution to the transformed equation

$$(-\Delta)^3 u = c_n u^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{R}^n \setminus \Gamma, \tag{$\mathcal{Q}_{6,\delta,N}$}$$

where Δ is the usual flat Laplacian and Γ is the image of the singular set Λ under the stereographic projection. As a notational shorthand, we adopt the convention that U refers to a conformal factor relating the metric g to the round metric, *i.e.* $g = U^{4/(n-6)}g_0$, while u refers to a conformal factor relating the metric g to the Euclidean metric, *i.e.* $g = u^{4/(n-6)}\delta$, with the two related by $u = U u_{\rm sph}$.

Remark 1.1. *In this Euclidean setting, the transformation law* (1.3) *in particular implies the scaling law for* ($\mathcal{Q}_{6,\delta,N}$)*, namely if u solves* ($\mathcal{Q}_{6,\delta,N}$) *then so does* $u_\lambda(x) := \lambda^{\frac{n-6}{2}}u(\lambda x)$ *for any* $\lambda > 0$.

We study the compactness properties of both the unmarked and the marked moduli spaces of admissible constant sixth Q-curvature metrics. We define the unmarked moduli space as

$$\mathcal{M}_N^6 = \left\{ g \in [g_0] : g \text{ is complete on } \mathbb{S}^n \setminus \Lambda \text{ with } \#\Lambda = N,\ Q_g^6 \equiv Q_n \right\}, \tag{1.4}$$

and the marked moduli space as

$$\mathcal{M}_\Lambda^6 = \left\{ g \in [g_0] : g \text{ is complete on } \mathbb{S}^n \setminus \Lambda,\ Q_g^6 \equiv Q_n \right\}.$$

Intuitively, in the unmarked moduli space we fix only the number of punctures, whereas in the marked moduli space, we fix the punctures themselves. We place the Gromov-Hausdorff topology on both the marked and unmarked moduli spaces.

The first step to understanding the properties of the marked moduli space $\mathcal{M}_N^6$ is to study the conformally flat equation

$$(-\Delta)^3 u = c_n u^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{B}_R^*, \tag{$\mathcal{P}_{6,R}$}$$

where $\mathbb{B}_R^* := \{x \in \mathbb{R}^n : 0 < |x| < R\}$ is the punctured ball for $R < +\infty$. Allowing $R \to +\infty$ turns ($\mathcal{P}_{6,R}$) into the following PDE on the punctured space

$$(-\Delta)^3 u = c_n u^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{$\mathcal{P}_{6,\infty}$}$$

On this subject, the classification of non-singular solutions to ($\mathcal{P}_{6,\infty}$) is provided in [29]. Later on, in [18] it is proved that blow-up limit solutions do exist. Recently, based on a topological shooting method, the first and last authors classified all possible solutions to this limit equation [3].

One can merge these classification results into the statement below

Theorem A. *Let u be a positive solution to ($\mathcal{P}_{6,\infty}$). Assume that*

(a) *the origin is a removable singularity, then there exists $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that u is radially symmetric about x_0 and, up to a constant, is given by*

$$u_{x_0,\varepsilon}(x) = \left(\frac{2\varepsilon}{1 + \varepsilon^2 |x - x_0|^2} \right)^{\frac{n-6}{2}}. \tag{1.5}$$

These are called the (sixth order) spherical solutions (or bubbles).

(b) *the origin is a non-removable singularity, then u is radially symmetric about the origin. Moreover, there exist $\varepsilon_0 \in (0, \varepsilon_n^*]$ and $T \in (0, T_{\varepsilon_0}]$ such that*

$$u_{\varepsilon,T}(x) = |x|^{\frac{6-n}{2}} v_\varepsilon(\ln |x| + T). \tag{1.6}$$

Here $\varepsilon_n^* = K_0^{(n-6)/6}$, $T_\varepsilon \in \mathbb{R}$ is the fundamental period of the unique T-periodic bounded solution v_T to the following sixth order IVP,

$$\begin{cases} v^{(6)} - K_4 v^{(4)} + K_2 v^{(2)} - K_0 v = c_n v^{\frac{n+6}{n-6}} \\ v(0) = \varepsilon_0,\ v^{(2)}(0) = \varepsilon_2,\ v^{(4)}(0) = \varepsilon_4,\ v^{(1)}(0) = v^{(3)}(0) = v^{(5)}(0) = 0, \end{cases}$$

where $K_4, K_2, K_0, \varepsilon_n^*$ are dimensional constants $\varepsilon_0 \in (0, \varepsilon_n^*]$ (See (2.2)). These are called (sixth order) Emden–Fowler solutions.

In [18], it is shown that solutions to ($\mathcal{P}_{6,R}$) with $R < +\infty$ satisfy a priori bound near the isolated singularity, which implies that they behave like the solutions to the limit equation near the isolated singularity

Theorem B. *Let u be a positive singular solution to ($\mathcal{P}_{6,R}$). Suppose that $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$. Then*

$$u(x) = (1 + \mathrm{o}(1)) u_{\varepsilon,T}(|x|) \quad \text{as} \quad x \to 0, \tag{1.7}$$

where $u_{\varepsilon,T}$ belongs to the family (1.6).

These two results combined motivate the following definition

Definition 1.2. *Let $g \in \mathcal{M}_N$ with a singular set $\Lambda \subset \mathbb{S}^n$, $\#\Lambda = N$, and let $p_j \in \Lambda$. Let $g = U^{4/(n-6)}g_0 = u^{4/(n-6)}\delta$ where we choose stereographic coordinates centered at p_j. By (1.7) we know $u(x) = u_{\varepsilon_j,T_j}(|x|)(1+\mathrm{o}(|x|))$ for some $\varepsilon_j \in (0,\varepsilon_n^*]$. This ε_j is the asymptotic necksize of the metric g at the puncture p_j.*

Now we have conditions to state our main compactness theorem for the unmarked moduli space

Theorem 1.3. *Let $N \geqslant 3$ and let $0 < \delta_1, \delta_2 < 1$ be positive real numbers. Then the set*

$$\mathcal{Q}^6_{\delta_1,\delta_2} = \left\{ g \in \mathcal{M}^6_N : \mathrm{d}_{g_0}(p_j,p_\ell) \geqslant \delta_1 \text{ for each } j \neq \ell \text{ and } \varepsilon_j(g) \geqslant \delta_2 \right\}.$$

is sequentially compact with respect to the Gromov–Hausdorff topology.

Remark 1.4. *Notice that as a consequence of Theorem A (a), it follows that $\mathcal{M}_1 = \varnothing$. Also, from Theorem A (b), we have that $\mathcal{M}_{p_1,p_2} = (0,\varepsilon_n^*]$ for any $p_1 \neq p_2$, where $\varepsilon_n^* \in (0,1)$. Moreover, it follows that $\mathcal{M}_2 = (0,\varepsilon_n^*] \times ((\mathbb{S}^n \times \mathbb{S}^n \setminus \mathrm{diag})/SO(n+1,1))$, where the group $SO(n+1,1)$ of conformal transformations acts on each $\mathbb{S}^n$ factor simultaneously. These metrics are called the Delaunay metrics. Furthermore, they all correspond to a doubly punctured sphere and are rotationally invariant.*

Remark 1.5. *It is worthwhile to now describe the possible degenerations of a sequence of metrics in $\mathcal{M}^6_N$. Let $\{g_k = (U_k)^{4/n-6}g_0\} \in \mathcal{M}^6_N$ be a sequence that leaves every compact subset. We denote the singular set of g_k by $\Lambda_k = \{p_{1,k},\ldots,p_{N,k}\}$ and the asymptotic necksize of g_k at the puncture $p_{j,k}$ as $\varepsilon_{j,k}$. Then either $\lim_{k\to\infty}\varepsilon_{j,k} = 0$ for some j or $\lim_{k\to\infty} p_{j,k} = \lim_{k\to\infty} p_{j',k}$ for some $j \neq j'$. We sketch these two degenerations in Figure 1. (It is possible that both degenerations happen simultaneously.) In either case, in the limit one obtains a metric $g_\infty \in \mathcal{M}^6_{N'}$ for some $N' < N$. In this way, one can compactify the moduli space $\mathcal{M}^6_N$ by gluing copies of $\mathcal{M}^6_{N'}$ for $N' < N$ to $\partial\mathcal{M}^6_N$. We speculate that this compactification would not give a smooth manifold with boundary, but rather that $\partial\mathcal{M}^6_N$ is in general a stratified space.*

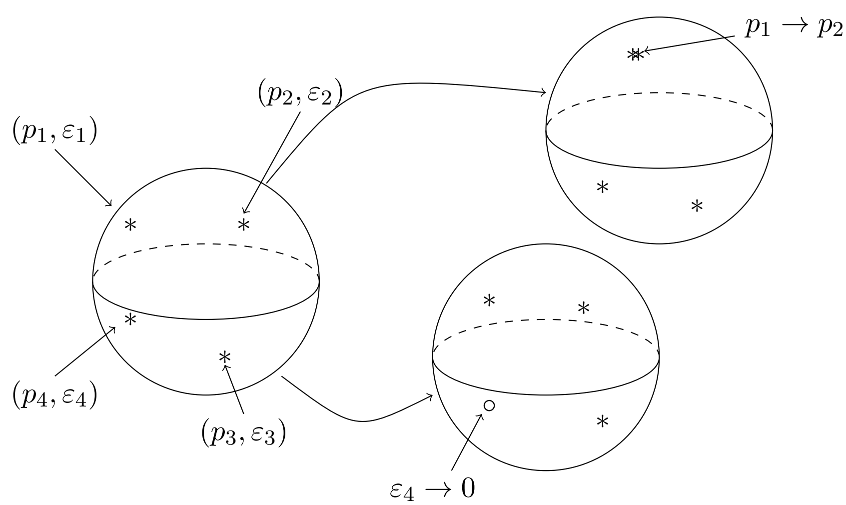

FIGURE 1. The two possible degenerations in the moduli space $\mathcal{M}^6_4$.

Let us compare our main results with the second and fourth order analogs. In the same spirit as our main result, it was proved in [25] and [2] that the moduli sets below

$$\mathcal{Q}^2_{\delta_1,\delta_2} \subset \mathcal{M}^2_N = \left\{ g \in [g_0] : g \text{ is complete and } R_g \equiv 2^{-1}(n-4) \right\}. \tag{1.8}$$

and

$$\mathcal{Q}^4_{\delta_1,\delta_2} \subset \mathcal{M}^4_N = \left\{ g \in [g_0] : \text{g is complete, } R_g \geqslant 0, \text{ and } Q^4_g \equiv 2^{-3}n(n^2-4) \right\}. \tag{1.9}$$

are also sequentially compact.

Based on classifications results like Theorem A and Theorem B, much more is known about the moduli spaces in (1.8) and (1.9). In fact, in some classical works of Mazzeo and Pacard [21] used gluing techniques to prove that there exists a family of solutions in (1.8). Furthermore, Mazzeo, Pollack, and Uhlenbeck [22], this space turns out to be a finite-dimensional analytic submanifold furnished with a natural Lagrangian structure. On the moduli space (1.9), less is known; it is not proved yet whether this is non-empty. Some of the authors in [1] proved that this property holds for non-degenerate manifolds with a suitable hypothesis on the vanishing of the Weyl tensor, However, the standard round sphere is not included in this class.

Inspired by the arguments in [25], the proof of Theorem 1.3 is divided into three parts that we describe as follows. First, we need to introduce the so-called sixth order geometric Pohozaev invariant, which is related to the Hamiltonian energy of the limiting ODE [24, 27]. Second, we obtain an *a priori* upper and for positive singular solutions to ($\mathcal{Q}_{6,g_0,N}$), estimates which are accomplished by combining a sliding method, a blow-up argument, and a Harnack inequality. From this, we obtain uniform bound on certain Hölder norms, which by compactness, allows us to extract a limit, up to subsequence. Third, we use the first order asymptotic expansion for the Green function of the sixth order GJMS operator near the pole and the fact the necksizes are away from zero shows that this limit is non-trivial. Finally, one can apply a removable singularity theorem to conclude the proof.

The rest of the paper is divided as follows. In Section 2, we define the logarithmic cylindrical change of variables and we use the conformal invariance between the punctured space and the cylinder to transform ($\mathcal{Q}_{6,g_0,N}$) into a PDE on the cylinder. In Section 3, we describe all singular solutions on a doubly punctured sphere. These Delaunay metrics are especially important because they provide asymptotic models for the metrics in $\mathcal{M}^6_N$ near a given puncture point. In Section 4, we define the sixth order Pohozaev invariants associated with ($\mathcal{Q}_{6,g_0,N}$). In Section 5, we prove *a priori* upper and lower bound estimates for positive singular solutions to ($\mathcal{Q}_{6,g_0,N}$). In Section 6, we prove the compactness statement in Theorem 1.3.

Remark 1.6. *Several of our supporting results below generalize to the Paneitz operators and Q-curvatures of any order $\sigma \in (0, n/2)$, at least in the conformally flat setting. In particular, the convexity result of Lemma 5.1 and the upper bound of Proposition 5.2 both generalize, and may be of independent interest. On the other hand, some parts of the proof of Theorem 1.3 do not carry over. In particular, at this time we cannot classify all two-ended constant Q^σ-curvature metrics on the sphere, which is very important for our proof.*

2. Cylindrical coordinates

This section is devoted to constructing a change of variables that transforms the local singular PDE ($\mathcal{P}_{6,R}$) problem into a nice ODE problem with constant coefficients. This is the conformally flat problem associated with ($\mathcal{Q}_{6,g_0,N}$).

Definition 2.1. *We define the sixth order autonomous Emden-Fowler change of variables as follows. Let $R > 0$ and $T = -\ln R$ and $\mathcal{C}_T = (T, \infty) \times \mathbb{S}^{n-1}$. We then define*

$$\mathfrak{F} : \mathcal{C}^\infty(B^*_R) \to \mathcal{C}^\infty(\mathcal{C}_T), \qquad \mathfrak{F}(u)(t,\theta) = e^{-\gamma_n t} u(e^{-t}\theta) = v(t,\theta), \tag{2.1}$$

where $\gamma_n = \frac{n-6}{2}$.

It is easy to show the inverse transform is given by

$$\mathfrak{F}^{-1}: \mathcal{C}^\infty(\mathcal{C}_T) \to \mathcal{C}^\infty(B_R^*), \qquad \mathfrak{F}^{-1}(v)(x) = |x|^{-\gamma_n} v(-\ln|x|, x/|x|) = u(x).$$

Using $\mathfrak{F}$ and performing a lengthy computation we arrive at the following sixth order nonlinear PDE on $\mathcal{C}_T$:

$$-P^6_{\rm cyl} v = c_n v^{\frac{n+6}{n-6}} \quad \text{on} \quad \mathcal{C}_T. \tag{$\mathcal{C}_T$}$$

Here $P^6_{\rm cyl}$ is the sixth order GJMS operator associated to the cylindrical metric $g_{\rm cyl} = dt^2 + d\theta^2$ on $\mathbb{R} \times \mathbb{S}^{n-1}$, and it is given by

$$P^6_{\rm cyl} := P^6_{\rm rad} + P^6_{\rm ang},$$

where

$$P^6_{\rm rad} := \partial_t^{(6)} - K_4 \partial_t^{(4)} + K_2 \partial_t^{(2)} - K_0$$

and

$$P^6_{\rm ang} := 2\partial_t^{(4)}\Delta_\theta - J_3\partial_t^{(3)}\Delta_\theta + J_2\partial_t^{(2)}\Delta_\theta - J_1\partial_t\Delta_\theta + J_0\Delta_\theta + 3\partial_t^{(2)}\Delta_\theta^2 - L_0\Delta_\theta^2 + \Delta_\theta^3$$

with

$$\begin{aligned}
K_0 &= 2^{-8}(n-6)^2(n-2)^2(n+2)^2\\
K_2 &= 2^{-4}(3n^4 - 24n^3 + 72n^2 - 96n + 304)\\
K_4 &= 2^{-2}(3n^2 - 12n + 44)\\
J_0 &= 2^{-3}(3n^4 - 18n^3 - 192n^2 + 1864n - 3952)\\
J_1 &= 2^{-1}(3n^3 + 3n^2 - 244n + 620)\\
J_2 &= 2n^2 + 13n - 68\\
J_3 &= 2(n+1)\\
L_0 &= 2^{-2}(3n^2 - 12n - 20)
\end{aligned} \tag{2.2}$$

dimensional constants.

Remark 2.2. *The following decomposition holds*

$$P^6_{\rm rad} = L_{\lambda_1} \circ L_{\lambda_2} \circ L_{\lambda_3},$$

where $L_{\lambda_j} := -\partial_t^2 + \lambda_j$ *for* $j = 1, 2, 3$ *with*

$$\lambda_1 = \frac{n-6}{2}, \quad \lambda_2 = \frac{n-2}{2}, \quad \text{and} \quad \lambda_3 = \frac{n+2}{2}.$$

We refer the reader to [3, Proposition 2.7] for the proof.

3. Spherical and Delaunay metrics

In this section, we present some particular model metrics on the moduli space. Let $p_1, p_2 \in \mathbb{S}^n$, which without loss of generality can be chosen such that $p_1 = \mathbf{e}_n$ is the north pole and $p_2 = -p_1$ is the south pole. The conformal factor $U : \mathbb{S}^n \backslash \{p_1, p_2\} \to (0, \infty)$ determines a metric $g \in \mathcal{M}_{p_1,p_2}$ and after composing with a stereographic projection it corresponds to a singular solution to ($\mathcal{P}_{6,\infty}$)

Applying the cylindrical transform (2.1) to this PDE in turn yields

$$-P^6_{\rm cyl} v = c_n v^{\frac{n+6}{n-6}} \quad \text{on} \quad \mathcal{C}_\infty := \mathbb{R} \times \mathbb{S}^n.$$

Next, using those solutions to ($\mathcal{Q}_{6,g_0,N}$) are radially symmetric with respect to the origin, ($\mathcal{C}_T$) reduces to a sixth order ODE problem

$$-v^{(6)} + K_4 v^{(4)} - K_2 v^{(2)} + K_0 v = c_n v^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{R}. \tag{$\mathcal{O}_{6,\infty}$}$$

From this last formulation, we quickly compute the cylindrical solution

$$v_{\mathrm{cyl}}(t) = \left(\frac{K_0}{c_n}\right)^{\frac{12}{n-6}} = \left(\frac{K_0}{c_n}\right)^{\frac{6}{\gamma_n}} = \varepsilon_n^* > 0,$$

which is the only constant solution. Transforming back from the cylinder to $\mathbb{R}^n \backslash \{0\}$ we see

$$u_{\mathrm{cyl}}(x) = \mathfrak{F}^{-1}(v_{\mathrm{cyl}}) = \left(\frac{K_0}{c_n}\right)^{\frac{12}{n-6}} |x|^{-\gamma_n}, \quad g_{\mathrm{cyl}} = u_{\mathrm{cyl}}^{\frac{4}{n-6}} \delta.$$

We have already encountered the spherical solution, given by

$$u_{\mathrm{sph}}(x) = \left(\frac{1+|x|^2}{2}\right)^{-\gamma_n} \quad \text{and} \quad g_{\mathrm{sph}} = u_{\mathrm{sph}}^{4/(n-6)} \delta, \tag{3.1}$$

which is the particular case of (1.5) with $\epsilon = 1$ and $x_0 = 0$. Applying the Emden-Fowler change of variables to u_{sph} we obtain

$$v_{\mathrm{sph}}(t,\theta) = \mathfrak{F}(u_{\mathrm{sph}})(t,\theta) = (\cosh t)^{-\gamma_n}.$$

In this setting, Theorem A classifies all positive solutions $v_{\varepsilon_0} \in \mathcal{C}^6(\mathbb{R})$ to ($\mathcal{O}_{6,\infty}$) in terms of the necksize $\varepsilon_0 \in (0, \varepsilon_n^*]$, where $\varepsilon_0 = \min_{\mathbb{R}} v \in (0, \varepsilon_n^*]$. Varying the parameter ε from its maximal value of ε_n^* to 0, we see that the Delaunay solutions in Theorem A (b) interpolate between the cylindrical solution v_{cyl} and the spherical solution v_{sph}. We denote the minimal period of v_ε by T_ε.

Definition 3.1. *For each $\varepsilon \in (0, \varepsilon_n^*]$ the Delaunay metric of necksize ε is*

$$g_\varepsilon = v_\varepsilon^{\frac{4}{n-6}}(\mathrm{d}t^2 + \mathrm{d}\theta^2) = u_\varepsilon^{\frac{4}{n-6}} \delta,$$

where $u_\varepsilon = \mathfrak{F}^{-1}(v_\varepsilon)$. Observe that we have equivalently defined g_ε as a metric on $\mathcal{C}_{-\infty}$, using v_ε as the conformal factor, and on $\mathbb{R}^n \backslash \{0\}$, using $u_\varepsilon = \mathfrak{F}^{-1}(v_\varepsilon)$ as the conformal factor.

We can reformulate the expansion (1.7) to read

Proposition 3.2. *Let $g \in \mathcal{M}_N^6$ with the singular set Λ and let $p \in \Lambda$. Then there exists a Delaunay solution u_ε such that in stereographic coordinates centered at p the asymptotic expansion*

$$g = ((1 + \mathrm{o}(|x|)) u_{\varepsilon,R}(x))^{\frac{4}{n-6}} \delta, \qquad u_{\varepsilon,R}(x) = u_\varepsilon(Rx).$$

We can restate this asymptotic expansion as

$$g \;=\; ((1+\mathrm{o}(|x|))\mathfrak{F}^{-1}(v_\varepsilon(\cdot + T))(x))^{\frac{4}{n-6}} \delta = ((1+\mathrm{o}(e^{-t})) v_\varepsilon(t+T))^{\frac{4}{n-6}} (\mathrm{d}t^2 + \mathrm{d}\theta^2).$$

In other words, any admissible metric is asymptotic to a translated Delaunay metric near a puncture. In the formulae above R and T are related by $R = -\ln T$.

4. Pohozaev invariants

We now turn to a discussion of the existence and specific form of a family of homological integral invariants of solutions of equation ($\mathcal{Q}_{6,g_0,N}$). These homological invariants were discovered in their simplest form by S. Pohozaev [24], and generalized by R. Schoen [27] for the Riemannian setting.

As a starting point, we define the energy $\mathcal{H}_{\rm cyl}$ by

$$\mathcal{H}_{\rm cyl}(v) := \mathcal{H}_{\rm rad}(v) + \mathcal{H}_{\rm ang}(v) + F(v), \tag{4.1}$$

where

$$\mathcal{H}_{\rm rad}(v) := \frac{1}{2}v^{(3)2} + \frac{K_4}{2}v^{(2)2} + \frac{K_2}{2}v^{(1)2} - \frac{K_0}{2}v^2 + v^{(5)}v^{(1)} - v^{(4)}v^{(2)} - K_4 v^{(3)}v^{(1)},$$

is the radial part,

$$\begin{aligned}\mathcal{H}_{\rm ang}(v) := &-J_4\left(\partial_t^{(3)}\nabla_\theta v \partial_t \nabla_\theta v - |\partial_t^{(2)}\nabla_\theta v|^2\right) - \frac{J_2}{2}|\partial_t^{(2)}\nabla_\theta v|^2 - \frac{J_1}{2}|\partial_t^{(2)}\nabla_\theta v|^2 \\ &- \frac{J_0}{2}|\nabla_\theta v|^2 + \frac{L_2}{2}|\partial_t^{(2)}\Delta_\theta v|^2 + \frac{L_0}{2}|\partial_t^{(2)}\Delta_\theta v|^2 + \frac{1}{2}|\Delta_\theta v|^2.\end{aligned}$$

is the angular part, and

$$F(v) := \frac{c_n(n-6)}{2n}|v|^{\frac{2n}{n-6}}$$

is the nonlinear term.

Evaluating a derivative, one can easily verify $\mathcal{H}_{\rm cyl}(v)$ is constant for any solution v of the PDE ($\mathcal{C}_T$). We further observe that the last term F in (4.1) is homogeneous of degree $\frac{2n}{n-6}$ while the remaining terms are all homogeneous of degree 2.

Definition 4.1. *Let $v \in \mathcal{C}^6(\mathcal{C}_T)$ be a positive solution to ($\mathcal{C}_T$). We define its cylindrical Pohozaev invariant as*

$$\mathcal{P}_{\rm cyl}(v) := \int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}_{\rm cyl}(v){\rm d}\theta$$

for any $t > T$. Observe that this integral does not, in fact, depend on t.

In light of the cylindrical transformation from Definition 2.1, we can define this invariant in spherical coordinates

Definition 4.2. *Let $u \in \mathcal{C}^6(\mathbb{B}_R^*)$ be a positive solution to ($\mathcal{P}_{6,R}$). We define its spherical Pohozaev invariant as*

$$\mathcal{P}_{\rm sph}(u) := (\mathcal{P}_{\rm cyl}\circ\mathfrak{F}^{-1})(u) = \int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}_{\rm cyl}(\mathfrak{F}^{-1}(u)){\rm d}\theta.$$

Finally, in terms of conformal metrics, we have the following definition of an invariant associated with metrics in the moduli space.

Definition 4.3. *Let $g \in \mathcal{M}_N^6$ and $p_j \in \Lambda$. We define its radial (or dilational) Pohozaev invariant at the puncture p_j as follows. Choose stereographic coordinates sending p_j to the origin and write $g = u^{\frac{4}{n-6}}\delta$ in these coordinates. Then define*

$$\mathcal{P}_{\rm rad}(g,p_j) := \mathcal{P}_{\rm sph}(u) = \int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}_{\rm cyl}(\mathfrak{F}^{-1}(u)){\rm d}\theta.$$

The most important result of this section states that bounding the radial Pohozaev invariants away from zero is equivalent to bounding the necksizes of the Delaunay asymptotes away from zero.

Proposition 4.4. *Let $g \in \mathcal{M}_N^6$ and $p_j \in \Lambda$. Then $\mathcal{P}_{rad}(g, p_j)$ is well-defined, negative and depends only on the necksize ε_j of the Delaunay asymptote at $p_j \in \Lambda$. Moreover, decreasing ε_j will increase $\mathcal{P}_{rad}(g, p_j)$ and if $\varepsilon_j \searrow 0$ then $\mathcal{P}_{rad}(g, p_j) \nearrow 0$.*

Proof. By construction, the integral defining $\mathcal{P}_{\mathrm{rad}}(g, p_j)$ does not depend on which sphere $\{t\}\times\mathbb{S}^{n-1}$ we choose, so long as t is sufficiently large, and therefore $\mathcal{P}_{\mathrm{rad}}$ is well-defined. By the asymptotics in Theorem B we know that the conformal factor is asymptotic to a Delaunay solution u_ε, and so letting $t \to \infty$ we see

$$\mathcal{P}_{\mathrm{rad}}(g, p_j) = \lim_{t\to\infty} \int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}_{\mathrm{cyl}}(\mathfrak{F}^{-1}(u))\mathrm{d}\theta = \lim_{t\to\infty} \int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}_{\mathrm{cyl}}(v_\varepsilon)\mathrm{d}\theta < 0.$$

The remaining properties follow directly from energy ordering of the Delaunay solutions as described in [3, Lemma 4.14]. □

Remark 4.5. *One often finds integral invariants in geometric variational problems. For more details on a class of general higher order conformally invariant locally conserved tensors, we cite [13]. These invariants arise from the conformal invariance of ($\mathcal{Q}_{6,g_0,N}$), by Noether's famous conservation theorem.*

For our later applications we will need a slight refinement of Proposition 4.4.

Proposition 4.6. *Let $v \in \mathcal{C}^6(\mathcal{C}_T)$ be a positive solution to the following rescaled equation*

$$-P^6_{\mathrm{cyl}} v = A v^{\frac{n+6}{n-6}}$$

for some constant A and let

$$\mathcal{H}^A_{\mathrm{cyl}}(v) = \mathcal{H}_{\mathrm{rad}}(v) + \mathcal{H}_{\mathrm{ang}}(v) + \frac{(n-6)A}{2n}|v|^{\frac{2n}{n-6}}.$$

Then

$$\int_{\{t\}\times\mathbb{S}^{n-1}} \mathcal{H}^A_{\mathrm{cyl}}(v)\mathrm{d}\theta$$

is independent of t.

Proof. The proposition follows from taking the derivative with respect to t and integrating by parts. □

5. Uniform estimates

This section is devoted to proving uniform upper and lower estimates near the singular set for positive singular solutions to ($\mathcal{Q}_{6,g_0,N}$).

We begin by quoting a superharmonicity result of Ngô and Ye [23]. We also remark a similar superharmonicity result for a related integral equation Ao et al. [4].

Proposition A. *Let $u \in \mathcal{C}^\infty(\mathbb{R}^n\setminus\Gamma)$ be a positive solution to ($\mathcal{Q}_{6,\delta,N}$). Then additionally $-\Delta u \geqslant 0$ and $\Delta^2 u \geqslant 0$ in $\mathbb{R}^n \setminus \Gamma$.*

Proof. Following [23, Proposition 1.5] we see that u is both weakly superharmonic and weakly superbiharmonic in $\mathbb{R}^n$. In other words, for a smooth test function ϕ compactly supported in $\mathbb{R}^n \setminus \Gamma$, we have

$$\int_{\mathbb{R}^n} u(-\Delta)\phi \mathrm{d}x \geqslant 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u(-\Delta)^2\phi \mathrm{d}x \geqslant 0.$$

Standard elliptic regularity then implies u is superharmonic and superbiharmonic where it is smooth, namely in $\mathbb{R}^n \setminus \Gamma$. □

The first step is a sixth order version of the convexity result [26, Proposition 1], which is proved using the Alexandrov's moving planes (see also [9, Theorem 4.1] for a fourth order version).

Lemma 5.1. *Let $g = U^{4/(n-6)} g_0$ be a complete metric on $\Omega = \mathbb{S}^n \backslash \Lambda$ which is conformal to the round metric, such that Q^6_g is a positive constant. Then the boundary of any (spherically) round ball in Ω has a non-negative definite second fundamental form with respect to g.*

Proof. We let $\mathcal{B}$ be a geodesic ball with respect to the round metric such that $\overline{\mathcal{B}} \subset \Omega$ and choose a stereographic projection that sends $\mathcal{B}$ to the half-space $\{x \in \mathbb{R}^n : x_1 < 0\}$. As before, we denote the image of the singular set Λ under this stereographic projection by Γ. With respect to these stereographic coordinates the metric takes the form $g = u^{4/(n-6)}\delta$ where $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma)$ satisfies ($\mathcal{Q}_{6,\delta,N}$), namely $u : \mathbb{R}^n \setminus \Gamma \to (0, \infty)$ satisfy

$$(-\Delta)^3 u = c_n u^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{R}^n \setminus \Gamma.$$

Furthermore, the boundary of our round ball is $\partial\mathcal{B} = \{x \in \mathbb{R}^n : x_1 = 0\}$ and is oriented by the inward unit normal $\eta_g = u^{-2/(n-6)}\partial_{x_1}$. It follows that the second fundamental form II and mean curvature H of $\partial\mathcal{B}$ are given by

$$II_{ij} = -\langle \nabla_{\partial x_j}\eta, \partial_{x_i}\rangle = \frac{2}{n-6}\delta_{ij} u^{\frac{8-n}{n-6}}\partial_{x_1}u, \quad H = \frac{2n}{n-6}u^{\frac{8-n}{n-6}}\partial_{x_1}u.$$

Therefore, (weak) convexity of $\partial\mathcal{B}$ follows once we show $\partial_{x_1}u \geqslant 0$ along the hyperplane $\{x_1 = 0\}$.

By Proposition A, we have

$$-\Delta u \geqslant 0 \quad \text{and} \quad (-\Delta)^2 u \geqslant 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Gamma.$$

We now rewrite ($\mathcal{Q}_{6,\delta,N}$) as a second order system, letting

$$\mathrm{u}_0 = u, \quad \mathrm{u}_1 = -\Delta u \quad \text{and} \quad \mathrm{u}_2 = (-\Delta)^2 u.$$

so that we obtain $\mathrm{u}_i : \mathbb{R}^n \setminus \Gamma \to (0,\infty)$ for $i = 0, 1, 2$ satisfy

$$\begin{cases} -\Delta \mathrm{u}_0 &= \mathrm{u}_1 \geqslant 0 \\ -\Delta \mathrm{u}_1 &= \mathrm{u}_2 \geqslant 0 \\ -\Delta \mathrm{u}_2 &= c_n u_0^{\frac{n+6}{n-6}} \geqslant 0. \end{cases} \tag{5.1}$$

It follows from [28, Theorem 2.7] that the Newtonian capacity of the singular set vanishes, *i.e.* $\mathrm{cap}(\Gamma) = 0$. As a consequence, one can find $a_0 > 0$ and $a_j \in \mathbb{R}$ for $j = 1, \ldots, n$ such that

$$\begin{cases} \mathrm{u}_0(x) &= a_0|x|^{6-n} + \sum_{j=1}^n a_j x_j |x|^{4-n} + \mathcal{O}\left(|x|^{4-n}\right) \\ \partial_{x_i}\mathrm{u}_0(x) &= -(n-6)a_0 x_i |x|^{4-n} + \mathcal{O}\left(|x|^{4-n}\right) \\ \partial^2_{x_i x_j}\mathrm{u}_0(x) &= \mathcal{O}\left(|x|^{4-n}\right), \end{cases} \tag{5.2}$$

which, by differentiating further, yields

$$\begin{cases} \mathrm{u}_1(x) &= b_0|x|^{4-n} + \sum_{j=1}^n b_j x_j |x|^{2-n} + \mathcal{O}\left(|x|^{2-n}\right) \\ \partial_{x_i}\mathrm{u}_1(x) &= -(n-4)b_0 x_i |x|^{-n} + \mathcal{O}\left(|x|^{2-n}\right) \\ \partial^2_{x_i x_j}\mathrm{u}_1(x) &= \mathcal{O}\left(|x|^{2-n}\right) \end{cases} \tag{5.3}$$

and

$$\begin{cases} \mathrm{u}_2(x) &= c_0|x|^{2-n} + \sum_{j=1}^n c_j x_j |x|^{-n} + \mathcal{O}\left(|x|^{-n}\right) \\ \partial_{x_i}\mathrm{u}_2(x) &= -(n-2)c_0 x_i |x|^{-n} + \mathcal{O}\left(|x|^{2-n}\right) \\ \partial^2_{x_i x_j}\mathrm{u}_2(x) &= \mathcal{O}\left(|x|^{-n}\right) \end{cases} \tag{5.4}$$

as $|x| \to 0$, where $b_0, c_0 > 0$ and $b_j, c_j \in \mathbb{R}$ for $j = 1, \ldots, n$.

We are now ready to set up the method of moving planes applied to the triple of functions $(\mathrm{u}_0, \mathrm{u}_1, \mathrm{u}_2)$. For any $\lambda \in \mathbb{R}$, we let $\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 > \lambda\}$ and $T_\lambda = \partial\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 = \lambda\}$. We also set $\Sigma'_\lambda = \Sigma_\lambda \setminus \Gamma$. For any $x \in \Sigma'_\lambda$, we let

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$$

be the reflection of x across the hyperplane $T_\lambda = \{x_1 = \lambda\}$. Finally, our goal in moving planes is to show that for any $\lambda \leqslant 0$ and $i = 0, 1, 2$, we have

$$\mathrm{w}_i^\lambda(x) > 0 \quad \text{for} \quad i = 0, 1, 2, \tag{5.5}$$

where $\mathrm{w}_i^\lambda : \Sigma'_\lambda \to \mathbb{R}$ is given by

$$\mathrm{w}_i^\lambda(x) = \mathrm{u}_i(x) - \mathrm{u}_i(x^\lambda).$$

Once we establish (5.5), letting $\lambda \nearrow 0$ the first inequality implies $\partial_{x_1}\mathrm{u} \geqslant 0$ on $T_0 = \partial\mathcal{B}$, completing our proof.

Observe that the expansion (5.4) implies u_2 is not identically zero. Thus, using the strong maximum principle and the last equation in (5.1), we see that $\mathrm{u}_2 > 0$ on $\mathbb{R}^n \backslash \Gamma$. Working backwards, the inequality $\mathrm{u}_2 > 0$ and the same reasoning implies $\mathrm{u}_1 > 0$ on $\mathbb{R}^n \backslash \Gamma$, which then in turn gives us $\mathrm{u}_0 > 0$ on $\mathbb{R}^n \backslash \Gamma$.

The singular set Γ is compact, so there exists $R_0 > 0$ such that $\Gamma \subset \mathbb{B}_{R_0}(0)$. We use the extended maximum principle [20, Theorem 3.4] to conclude there exists $\delta > 0$, depending on $R > R_0$, such that

$$\mathrm{u}_0|_{\mathbb{B}_R(0)\backslash\Gamma} \geqslant \delta, \quad \mathrm{u}_1|_{\mathbb{B}_R(0)\backslash\Gamma} \geqslant \delta, \quad \text{and} \quad \mathrm{u}_2|_{\mathbb{B}_R(0)\backslash\Gamma} \geqslant \delta. \tag{5.6}$$

Combining our expansion (5.4) with [5, Lemma 2.3] there exists $R_1 > 0$ and $\lambda_1 \leqslant \lambda_0$ such that for each $\lambda < \lambda_1$ we have

$$\mathrm{w}_0^\lambda(x) > 0, \quad \mathrm{w}_1^\lambda(x) > 0, \quad \text{and} \quad \mathrm{w}_2^\lambda(x) > 0 \quad \text{for} \quad x \in \Sigma_\lambda \quad \text{and} \quad |x| > R.$$

Using this inequality together with (5.6) then implies that there exists $\lambda_2 \leqslant \lambda_1$ such that

$$\mathrm{w}_0^\lambda(x) > 0, \quad \mathrm{w}_1^\lambda(x) > 0, \quad \text{and} \quad \mathrm{w}_2^\lambda(x) > 0 \quad \text{on} \quad \Sigma'_\lambda \quad \text{for each} \quad \lambda \leqslant \lambda_2.$$

By construction

$$\Delta\mathrm{w}_2^\lambda(x) = c_n \left(\mathrm{u}_0(x^\lambda)^{\frac{n+6}{n-6}} - \mathrm{u}_0(x)^{\frac{n+6}{n-6}}\right) < 0 \quad \text{on} \quad \Sigma'_\lambda \quad \text{for each} \quad \lambda \leqslant \lambda_2. \tag{5.7}$$

On the other hand, the asymptotic expansion (5.4) implies

$$\mathrm{w}_2^\lambda(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{5.8}$$

Putting together (5.7), (5.8) and $\mathrm{w}_2^\lambda\big|_{T_\lambda} = 0$, we see by the maximum principle that $\mathrm{w}_2^\lambda(x) \geqslant 0$ for each $x \in \Sigma'_\lambda$ and $\lambda \leqslant \lambda_2$. However, by the completeness of the metric g on Ω we know that w_2^λ is not identically zero on Σ'_λ, so again the maximum principle actually implies $\mathrm{w}_2^\lambda(x) > 0$ for each $x \in \Sigma'_\lambda$ and $\lambda \leqslant \lambda_2$. Once again, analogous arguments imply $\mathrm{w}_1^\lambda > 0$ and $\mathrm{w}_0^\lambda > 0$ on Σ'_λ for each $\lambda \leqslant \lambda_2$.

At this point, we define

$$\lambda^* = \sup\{\lambda \leqslant 0 : \mathrm{w}_i^\mu(x) > 0 \text{ for each } \mu \leqslant \lambda \text{ and } i = 0, 1, 2\}$$

and prove that $\lambda^* = 0$. Following our definitions, we have

$$\Delta\mathrm{w}_0^\lambda(x) = -\Delta\mathrm{u}_0(x) + \Delta\mathrm{u}_0(x^\lambda) < 0$$

for each $x \in \Sigma'_\lambda$ and $\lambda < \lambda^*$, and so $\Delta\mathrm{w}_0^{\lambda^*} \leqslant 0$ on Σ'_{λ^*}. By similar arguments, we also have

$$\Delta\mathrm{w}_1^{\lambda^*} \leqslant 0 \quad \text{and} \quad \Delta\mathrm{w}_2^{\lambda^*} \leqslant 0 \quad \text{on} \quad \Sigma'_{\lambda^*}.$$

Now suppose $\lambda^* < 0$ and let $x^* \in \overline{\Sigma}'_{\lambda^*}$ such that $\mathrm{w}_i^{\lambda^*}(x^*) = 0$ for some $i = 0, 1, 2$. If $x^{\lambda^*} \in \Sigma'_{\lambda^*}$ is an interior point then the maximum principle implies $\mathrm{w}_i^{\lambda^*} \equiv 0$, which in turn means u_i is symmetric about the hyperplane T_{λ^*}. This is impossible because the singular set Γ lies to one side of T_{λ^*}. On the other hand, if $x^* \in T_{\lambda^*}$ then by the Hopf boundary lemma (together with the fact that $\mathrm{w}_i^{\lambda^*}$ may not be constant in Σ'_{λ^*}) we have

$$0 < \partial_{x_1} \mathrm{w}_i^{\lambda^*}(x^*) = 2\partial_{x_1} \mathrm{u}_i(x^*). \tag{5.9}$$

However, the asymptotic expansions (5.2), (5.3) and (5.4) combined with $\lambda^* < 0$ tells us

$$\mathrm{u}_i(x) - \mathrm{u}_i(x^{\lambda^*}) \geqslant \delta_3 \quad \text{for} \quad |x| > R_2 \quad \text{and} \quad x_1 = \lambda^* \tag{5.10}$$

for some positive numbers δ_3 and R_2. Combining (5.9) and (5.10) implies the inequality w_i^{λ} continues to hold for some small value $\lambda < \lambda^*$, contradicting the definition of λ^*. □

First, we prove the upper bound estimate. Our proof borrows from Pollack's proof of the corresponding upper bound in the scalar curvature case.

Proposition 5.2. *Let $u \in \mathcal{C}^\infty(\Omega)$ be a positive singular solution to ($\mathcal{Q}_{6,g_0,N}$). There exists $C_1 > 0$ depending only on n and d satisfying*

$$u(x) \leqslant C_1 \mathrm{d}_{g_0}(x, \Lambda)^{-\gamma_n}.$$

Proof. Let $p_0 \notin \Lambda$, and $\rho > 0$ such that $\mathcal{B}_\rho(p_0) \subset \Omega$, where $\mathcal{B}_\rho(p_0)$ is a geodesic ball with respect to the round metric. We define the auxiliary function $\psi_\rho : \mathcal{B}_\rho(p_0) \to \mathbb{R}$ given by

$$\psi_\rho(x) = (\rho - \mathrm{d}_{g_0}(x, x_0))^{\gamma_n} u(x).$$

Notice that choosing $\rho = \frac{1}{2}\mathrm{d}_{g_0}(x_0, \Lambda)$, it follows

$$\psi_\rho(x_0) = \rho^{\gamma_n} u(x_0) = 2^{-\gamma_n} \mathrm{d}_{g_0}(x_0, \Lambda)^{\gamma_n} u(x_0).$$

We claim that there exists $C > 0$ depending only on n such that $\psi_\rho(x) \leqslant C$ for all admissible choices of λ, u, x_0, and ρ. We suppose by contradiction that one can find sequences $\{\Lambda_k\}_{k\in\mathbb{N}}$, $\{u_k\}_{k\in\mathbb{N}}$, $\{p_{0,k}\}_{k\in\mathbb{N}}$, and $\{\rho_k\}_{k\in\mathbb{N}}$ of admissible parameters satisfying

$$M_k = \psi_\rho(p_{1,k}) = \sup_{x \in \mathcal{B}_{\rho_k}(p_{0,k})} \psi_\rho(x) \to +\infty.$$

Also, we observe $\psi_\rho|_{\partial\mathcal{B}_{\rho_k}(p_{0,k})} = 0$, so $p_{1,k} \in \operatorname{int}(\mathcal{B}_{\rho_k}(p_{0,k}))$. Next, by taking $r_k = \rho_k - \mathrm{d}_{g_0}(p_{1,k}, p_{0,k})$, and defining be geodesic normal coordinates centered at $p_{1,k}$, denoted by y, we set

$$\lambda_k = 2u_k(p_{1,k})^{-\gamma_n} \quad \text{and} \quad R_k = r_k\lambda_k^{-1} = 2^{-1} r_k (u_k(p_{1,k}))^{-\gamma_n} = 2^{-1} M_k^{1/\gamma_n}.$$

We now construct a blow-up sequence $\{w_k\}_{k\in\mathbb{N}} \subset \mathcal{C}^{6,\alpha}(\mathbb{B}_{R_k})$ for some $\alpha \in (0,1)$ by $w_k : \mathbb{B}_{R_k}(0) \to \mathbb{R}$ is such that

$$w_k(y) = \lambda_k^{\gamma_n} u_k(\lambda y) \quad \text{for all} \quad k \in \mathbb{N}.$$

Whence, using the conformal invariance in Remark 1.1, one can verify that the function $w_k \in \mathcal{C}^{6,\alpha}(\mathbb{B}_{R_k})$ satisfies

$$P^6_{\lambda g_k} w_k = c_n w_k^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{B}_{R_k}.$$

Moreover, by construction, one has

$$2^{\gamma_n} = w_k(0) = \sup_{\mathbb{B}_{R_k}(0)} w_k(x) \quad \text{for all} \quad k \in \mathbb{N},$$

which, by Arzela–Ascoli theorem, means there exists subsequence that converges uniformly on compacts.

In addition, it is not hard to check that the rescaled metrics λg_0 converge to the classical Euclidean metric δ as $k \to \infty$. Therefore, by taking the limit of the blow-up sequence, we obtain a positive function $w_\infty \in \mathcal{C}^{6,\alpha}(\mathbb{R}^n)$ satisfying $w_\infty(0) = \sup w_\infty = 2^{\gamma_n}$ and

$$(-\Delta)^3 w_\infty = c_n w_\infty^{\frac{n+6}{n-6}} \quad \text{in} \quad \mathbb{R}^n.$$

By the classification theorem in Theorem A (a), we must have

$$w_\infty(x) := 2^{-\gamma_n}\left(1+|x|^2\right)^{-\gamma_n} = 2^{-\gamma_n} u_{\rm sph}(x).$$

Thus each solution u_k has a bubble for $k \gg 1$ sufficiently large. In other terms, a small neighborhood of $p_{1,k}$ is close (in $\mathcal{C}^{6,\alpha}$-norm) to the round metric, and hence has a concave boundary, for $k \gg 1$ sufficiently large.

We verify this by computing the mean curvature of a geodesic sphere explicitly. Using $g_0 = 4(1+|x|^2)^{-2}\delta$, a direct computation shows the mean curvature of a hypersurface is given by $H_\Sigma = -\operatorname{tr}_g \langle \nabla_{\partial \ell} \nu_\Sigma, \partial_m \rangle$, where ν_Σ is the unit inward normal vector of Σ.

A geodesic sphere centered at $p = 0$ coincides with a Euclidean round sphere centered at the origin (with a different radius), and so

$$\nu_\Sigma = -\left(\frac{1+|x|^2}{2|x|}\right) x^\ell \partial_{x_\ell}.$$

A straightforward computation yields

$$H_\Sigma = -2n|x|(1+|x|^2) + \frac{n-1+n|x|^2}{|x|},$$

which is negative when $|x| > 3$. Additionally, since

$$\lim_{k\to\infty} \|w_k - w_\infty\|_{\mathcal{C}^{6,\alpha}(B_{3R_k/4}(0))} = 0,$$

it holds that $\partial B_{3R_k/4}(0)$ is also mean concave with respect to the metric $\hat{g}_k \in \mathrm{Met}^\infty(B_{3R_k/4}(0))$ defined as $\hat{g}_k = w_k^{4/(n-6)}\delta_{\ell m}$, which in turn implies $\partial \mathbb{B}_{3|p_{1,k}|/8}(p_{1,k})$ is mean concave with respect to the metric $g_k \in \mathrm{Met}^\infty(\Omega)$ given by $\hat{g}_k = u_k^{4/(n-6)}\delta_{\ell m}$. This is contradiction with Lemma 5.1, which proves the claim. $\square$

Second, we obtain a lower bound estimate.

Proposition 5.3. *Let $u \in \mathcal{C}^\infty(\Omega)$ be a positive singular solution to ($\mathcal{Q}_{6,g_0,N}$). There exists $C_2 > 0$ depending only on u satisfying*

$$C_2 \min_{j\in I_N} \mathrm{d}_{g_0}(x,p_j)^{-\gamma_n} \leqslant u(x).$$

Proof. Indeed, notice that by applying [17, Theorem 1.3] in cylindrical coordinates $v = \mathfrak{F}(u)$, we obtain that $\mathcal{P}_{\rm cyl}(v) \leqslant 0$ with equality if and only if

$$\liminf_{t\to\infty} v(t,\theta) = \limsup_{t\to\infty} v(t,\theta) = \lim_{t\to\infty} v(t,\theta) = 0.$$

Otherwise, if $\mathcal{P}_{\rm cyl}(v) < 0$, there exists $C_2 > 0$, which depends on the solution v, such that $v(t,\theta) \geqslant C_2$. This proves the proposition. $\square$

Third, we have a version of Harnack inequality for our setting, which will be important in the proof of our main result.

Proposition 5.4. *Let $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{C}^\infty(\Omega)$. Assume that $-\Delta u \geqslant 0$, $\Delta^2 u \geqslant 0$, and*

$$(-\Delta)^3 u = f(u),$$

where f is either linear or superlinear and $f(0) = 0$. Then, there exists $\rho_0 > 0$ such that for $\rho \in (0, \rho_0]$ and $C_3 > 0$ depending only on Ω, f, and ρ, it holds

$$\sup_{\mathcal{B}_\rho(0)} u \leqslant C_3 \inf_{\mathcal{B}_\rho(0)} u.$$

Proof. The proof is a straightforward adaptation of [6, Theorem 3.6]. □

6. Compactness result

In this section, we prove the main result of the manuscript.

Before proceeding to the proof, we need to obtain the existence of a positive Green function for the sixth order GJMS of the round sphere with a prescribed asymptotic rate near a pole given by the fundamental solution to the flat tri-Laplacian.

Proposition 6.1. *Let $p \in \Lambda \subset (\mathbb{S}^n, g_0)$ be a point on the standard round sphere. There exists a Green function with pole at p, denoted by $G_p : \mathbb{S}^n \setminus \{p\} \to (0, \infty)$, that satisfies*

$$P_{g_0} G_p = \delta_p,$$

where P_{g_0} is the sixth order GJMS operator of the round metric given by (1.2) and δ_p is the Dirac function concentrated at p. Furthermore, there exists $C_n > 0$ depending only on n such that

$$G_p(x) = C_n \mathrm{d}_{g_0}(x,p)^{6-n} + \mathcal{O}(1) \tag{6.1}$$

in conformal normal coordinates.

Proof. This is a direct application of [10, Proposition 2.1] for the standard round sphere $(\mathbb{S}^n, g_0)$. □

Proof of Theorem 1.3. Let $\{g_k\}_{k\in\mathbb{N}} = \{(U_k)^{4/n-6} g_0\} \subset \mathcal{M}^6_N$ be a sequence of admissible metrics, each of which is a complete, conformally flat metric on $\Omega_k = \mathbb{S}^n \backslash \Lambda_k$ with $Q^6_{g_k} \equiv Q_n = \frac{n(n^4-20n^2+64)}{32}$. We denote the punctures of g_k by

$$\Lambda_k := \mathrm{sing}(U_k) = \{p_{1,k}, \dots, p_{N,k}\} \subset \mathbb{S}^n.$$

The proof will be divided into a sequence of steps.

The first step will simplify our later analysis since it allows us to assume the singular points are fixed.

Step 1. After passing to a subsequence, we may assume that for $k \gg 1$ sufficiently large each U_k is non-singular on the set $K_1 := \mathbb{S}^n \setminus (\cup_{i=1}^N \mathcal{B}_{\delta_1/2}(p_{j,i}))$.

Indeed, for $0 < \delta_1$ small enough, the set

$$(\mathbb{S}^n)^N \setminus \left\{(q_1, \dots, q_k) \in (\mathbb{S}^n)^N : \mathrm{d}_{g_0}(q_j, q_\ell) \geqslant \delta_1 \text{ for each } j \neq \ell\right\}$$

is compact and contains each singular set Λ_k for all $k \in \mathbb{N}$. Thus, there exits $\{p_{1,\infty}, \dots, p_{N,\infty}\} \subset \mathbb{S}^n$, and a convergent subsequence such that $p_{j,k} \to p_{j,\infty}$ as $k \to +\infty$, proving Step 1.

To set notation, we define the compact sets

$$K_\ell := \mathbb{S}^n \setminus \left(\cup_{j=1}^N \mathcal{B}_{2^{-\ell}\delta_1}(p_{j,\infty})\right) \quad \text{for each} \quad \ell \in \mathbb{N}$$

Notice that by construction the family $\{K_\ell\}_{\ell\in\mathbb{N}}$ is a compact exhaustion of the limit singular set

$$\Omega_\infty := \mathbb{S}^n \setminus \Lambda_\infty, \quad \text{where} \quad \Lambda_\infty := \{p_{1,\infty} \dots, p_{k,\infty}\}.$$

Furthermore, by the convergence $p_{j,k} \to p_{j,\infty}$ as $k \to +\infty$, for each fixed $\ell \in \mathbb{N}$ there exists $k_0 \gg 1$ such that $k \geqslant k_0$ implies U_k is smooth in K_ℓ.

The second step is based on the uniform upper bound and states that we can extract a limit.
Step 2. The exists $U_\infty \in \mathcal{C}^\infty(\Omega_\infty)$ solving ($\mathcal{Q}_{6,g_0,N}$) such that

$$\lim_{k\to+\infty} \|U_\infty - U_k\|_{\mathcal{C}^\infty_{\mathrm{loc}}(\Omega_\infty)} = 0. \tag{6.2}$$

In fact, using the upper bound in Proposition 5.2, one has that for each compact subset $K \subset \Omega_\infty$, there exists $\alpha \in (0,1)$ and $C_1 > 0$ depending only on n, Ω, and α such that

$$\|U_k\|_{\mathcal{C}6,\alpha(K)} \leqslant C_1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Therefore, as a consequence of the Arzela–Ascoli theorem, one can find a limit $U_\infty \in \mathcal{C}^{6,\alpha}(K)$ a convergent subsequence, which we again denote the same, such that

$$\lim_{k\to+\infty} \|U_\infty - U_k\|_{\mathcal{C}^{6,\alpha}_{\mathrm{loc}}(\Omega_\infty)} = 0.$$

Furthermore, by applying standard elliptic regularity, we directly obtain that (6.2) holds, and so Step 2 is proved.

The next step is to show that this limit is non-trivial.
Step 3. $U_\infty > 0$ on Ω_∞.
If this step were false, there would exist $p_* \in \Omega_\infty$ such that

$$0 = U_\infty(p_*) = \lim_{k\to+\infty} U_k(p_*).$$

For each $k \in \mathbb{N}$, we define $\varepsilon_k = U_k(p_*)$ and the rescaled function $\widehat{U}_k \in \mathcal{C}^\infty(\Omega_k)$ given by

$$\widehat{U}_k(x) = {\varepsilon_k}^{-1} U_k(x) \quad \text{for all} \quad k \in \mathbb{N}.$$

As a consequence of Remark 1.1, it follows

$$P_{g_0}\widehat{U}_k = \varepsilon_k^{\frac{12}{n-6}} c_n \widehat{U}_k^{\frac{n+6}{n-6}} \quad \text{in} \quad \Omega_k \quad \text{for all} \quad k \in \mathbb{N}.$$

In addition, by construction, the sequence $\{\widehat{U}_k\}_{k\in\mathbb{N}}$ satisfy the normalization

$$\widehat{U}_k(p_*) = 1 \quad \text{for all} \quad k \in \mathbb{N}. \tag{6.3}$$

By the Harnack inequality of Lemma 5.4 there exists a positive constant C_1 depending only on n and ℓ such that

$$\sup_{K_\ell} |u_{\mathrm{sph}} \widehat{U}_k| \leqslant C_1. \tag{6.4}$$

However, there is another positive constant C_2, again depending only on n and ℓ, such that

$$C_2 \leqslant u_{\mathrm{sph}} \leqslant 2^{\gamma_n}. \tag{6.5}$$

Combining (6.4) and (6.5) there exists a uniform constant C_3 such that

$$\sup_{K_\ell} \widehat{U}_k \leq C_3,$$

and so by the Arzela-Ascoli theorem we may pass to a subsequence $\widehat{U}_k$ that converges uniformly on compact subsets of Ω_∞ to a smooth function $\widehat{U}_\infty$.

This limit function $\widehat{U}_\infty : \Omega_\infty \to \mathbb{R}$ satisfies

$$P_{g_0}\widehat{U}_\infty = 0 \quad \text{in} \quad \Omega_\infty$$

and so it has the form

$$\widehat{U}_\infty = \sum_{j=1}^{N} \beta_j G_{p_{j,\infty}}$$

for some collection of real numbers $\beta_1,\dots,\beta_N$. The normalization (6.3) implies one of the coefficients β_{j_0} is positive, so after possibly relabeling the punctures we may assume $\beta_1>0$.

We now choose a stereographic projection sending $p_{1,\infty}$ to the origin and perform the Emden-Fowler change of coordinates in Definition 2.1, which yield the functions

$$v_k:=\mathfrak{F}(u_{\rm sph}U_k)\quad\text{and}\quad \widehat{v}_k:=\mathfrak{F}(u_{\rm sph}\widehat{U}_k)$$

and their respective limits

$$v_\infty:=\mathfrak{F}(u_{\rm sph}U_\infty)\quad\text{and}\quad \widehat{v}_\infty:=\mathfrak{F}(u_{\rm sph}\widehat{U}_\infty).$$

The expansion (6.1) implies

$$\widehat{v}_\infty(t,\theta)=e^{-\gamma_n t}(\cosh t)^{-\gamma_n}(C_ne^{-\gamma_n t}+\mathcal{O}(1))=C_n+\mathcal{O}(e^{(6-n)t})\quad\text{as}\quad t\to+\infty. \tag{6.6}$$

Also, observe that $\widehat{v}_k\in\mathcal{C}^6(\mathcal{C}_T)$ satisfies the PDE

$$P^6_{\rm cyl}\widehat{v}_k=\varepsilon_k^{\frac{12}{n-6}}c_n\widehat{v}_k^{\frac{n+6}{n-6}}\quad\text{in}\quad \mathcal{C}_{T_k}\quad\text{for all}\quad k\in\mathbb{N},$$

which we combine with (6.6) and Proposition 4.6 and the convergence $\widehat{v}_k\to\widehat{v}_\infty$ to see that for t sufficiently large

$$\begin{aligned}\int_{\{t\}\times\mathbb{S}^{n-1}}\mathcal{H}_{\rm cyl}^{\varepsilon_k^{\frac{12}{n-6}}c_n}{\rm d}\theta &= \int_{\{t\}\times\mathbb{S}^{n-1}}\mathcal{H}_{\rm rad}(\widehat{v}_k)+\mathcal{H}_{\rm ang}(\widehat{v}_k)+\frac{n-6}{2n}\varepsilon^{\frac{12}{n-6}}c_n|\widehat{v}_k|^{\frac{2n}{n-6}}{\rm d}\theta\\ &= -\widetilde{C}_n\beta_1^2+\mathcal{O}(e^{(6-n)t}).\end{aligned}\tag{6.7}$$

for some $\tilde{C}_n>0$. On the other hand, by our construction we have

$$\begin{aligned}\mathcal{P}_{\rm cyl}(v_k) &= \int_{\{t\}\times\mathbb{S}^{n-1}}\mathcal{H}_{\rm rad}(v_k)+\mathcal{H}_{\rm ang}(v_k)+F(v_k){\rm d}\theta\\ &= \int_{\{t\}\times\mathbb{S}^{n-1}}\varepsilon_k^2\left(\mathcal{H}_{\rm rad}(\widehat{v}_k)+\mathcal{H}_{\rm ang}(\widehat{v}_k)\right)+\varepsilon_k^{\frac{2n}{n-6}}F(\widehat{v}_k){\rm d}\theta\to 0\end{aligned}\tag{6.8}$$

From (6.7) and (6.8), we find

$$\lim_{k\to+\infty}\mathcal{P}_{\rm rad}(g_k,p_{1,k})=0,$$

which, together with Proposition 4.4, implies $\lim_{k\to+\infty}\varepsilon_1(g_k)=0$. This contradicts the hypothesis that the necksizes are bounded away from zero, that is, $\varepsilon_j(g_k)>\delta_1$ for some $0<\delta_1\ll 1$.

At last, we can complete our argument

Step 4. The metric $g_\infty=U_\infty^{\frac{4}{n-6}}g_0$ is a complete metric on Ω_∞.

Indeed, suppose by contradiction that is g_∞ is incomplete. Then there exists an index $j\in\{1,\dots,N\}$ such that $\liminf_{x\to p_{j,\infty}}U_\infty(x)<\infty$. In this case, the removable singularity result in Proposition 5.3 implies

$$\mathcal{P}_{\rm rad}(g_\infty,p_{j,\infty})=0.$$

However, by construction

$$0=\mathcal{P}_{\rm rad}(g_\infty,p_{j,\infty})=\lim_{k\to+\infty}\mathcal{P}_{\rm rad}(g_k,p_{j,k})\geqslant\delta_2,$$

which, by Proposition 4.4 implies $\varepsilon_j(g_k)\geqslant\delta_2$, which is contradiction with the fact $g_k\in\mathcal{Q}^6_{\delta_1,\delta_2}$.

By putting all these steps together, our main theorem is proved. □

Appendix A. Higher order curvature tensors

Let (M^n, g) is a Riemannian manifold with $n \geqslant 2$. In what follows, we will always be using Einstein's summation convection. In a local coordinate frame, denoted by $\{\partial_i\}_{i=1}^n$, we let $\mathrm{Rm}_g \in \mathfrak{T}_1^3(M)$ be the Riemannian curvature tensor, $\mathring{\mathrm{Rm}}_g \in \mathfrak{T}_0^4(M)$ be covariant Riemann curvature tensor, and the Ricci curvature tensor $\mathrm{Ric}_g = \mathrm{tr}_g \mathring{\mathrm{Rm}}_g \in \mathfrak{T}_0^2(M)$, which can be expressed as $\mathrm{Ric}_{jk} = \mathring{\mathrm{Rm}}^i_{ijk} = g^{i\ell}\mathring{\mathrm{Rm}}_{ijk\ell}$. We also consider the scalar curvature $R_g = \mathrm{tr}_g \mathrm{Ric}_g \in \mathfrak{T}_0^0(M)$, defined by $R = g^{ij}\mathrm{Ric}_{ij}$, where $\mathfrak{T}_s^r(M)$ stands for the set of (r,s)-type tensor over M with $\mathfrak{T}_0^0(M) = \mathcal{C}^\infty(M)$ and $\mathrm{tr}_g : \mathfrak{T}_s^r(M) \to \mathfrak{T}_s^{r-2}(M)$. Also for the Laplace–Beltrami operator, we simply denote $\Delta_g := g^{ij}\nabla_i\nabla_j$, where ∇_g the Levi–Civita connection associated to g.

It is also convenient to define some operations involving two tensors.

Definition A.1. *First, let us introduce the cross product* $\times : \mathrm{Sym}_2(M) \times \mathrm{Sym}_2(M) \to \mathrm{Sym}_2(M)$ *is given by*

$$(h_1 \times h_2)_{ij} := g^{k\ell} h_{1,ik} h_{2,j\ell} = h^\ell_{1,i} h_{2,\ell j}.$$

Second, we define a dot product $\times : \mathrm{Sym}_2(M) \times \mathrm{Sym}_2(M) \to \mathbb{R}$, *given by*

$$h_1 \cdot h_2 := \mathrm{tr}_g(h_1 \times h_2) = g^{ij} g^{k\ell} h_1^{ik} h_{2,j\ell} = h_1^{jk} h_{2,jk}.$$

Third, we also recall the Kulkarni–Nomizu product $\owedge : \mathrm{Sym}_2(M) \times \mathrm{Sym}_2(M) \to \mathfrak{T}_0^4(M)$

$$(h_1 \owedge h_2)_{ijk\ell} := h_{1,i\ell} h_{2,jk} + h_{1,jk} h_{2,i\ell} - h_{1,ik} h_{2,j\ell} - h_{1,j\ell} h_{2,ik}.$$

At last, we consider $\cdot : \mathrm{Sym}_2(M) \to \mathrm{Sym}_2(M)$ *and* $\delta_g : \mathrm{Sym}_2(M) \to \mathbb{R}$,

$$(\mathring{\mathrm{Rm}} \cdot h)_{jk} := R_{ijk\ell} h^{i\ell} \quad \textit{and} \quad (\delta_g h)_i := -(\mathrm{div}_g\, h)_i = -\nabla^j h_{ij},$$

where the latter one is the L^2-formal adjoint of Lie derivative (up to scalar multiple).

Definition A.2. *Let us define the Schouten tensor, Weyl tensor, Bach tensor, and nameless tensor, respectively, by*

$$\begin{aligned}
A_g &:= \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{1}{2(n-1)} R_g g\right)\\
W_g &:= \mathring{\mathrm{Rm}}_g - A_g \owedge g\\
B_g &:= \Delta_g A_g - \nabla_g^2 \,\mathrm{tr}_g\, A_g + 2\mathring{\mathrm{Rm}}_g \cdot A_g - (n-4) A_g \times A_g - |A_g|^2 g - 2(\mathrm{tr}_g\, A_g) A_g, A
\end{aligned}$$

where these expressions are written in an abstract index-free manner.

From this, we introduce the following tensors

$$\begin{aligned}
T_g^2 &:= (n-2)\sigma_1(A_g) g - 8A_g,\\
T_g^4 &:= -\frac{3n^2 - 12n - 4}{4}\sigma_1(A_g)^2 g + 4(n-4)|A|_g^2 g + 8(n-2)\sigma_1(A_g) A_g\\
&\quad + (n-6)\Delta_g \sigma_1(A_g) g + 48 A_g^2 - \frac{16}{n-4} B_g,\\
T_g^6 &:= -\frac{1}{8}\sigma_3(A_g) - \frac{1}{24(n-4)}\langle B_g, A_g\rangle_g,
\end{aligned}$$

where σ_k is the k-th elementary symmetric function for each $k \in \mathbb{N}$.

Based on this notation, we introduce the concept of higher order curvatures as follows

Definition A.3. *For any $g \in \mathrm{Met}^\infty(\Omega)$, let us define the $2m$th order Q-curvature Q_g^{2m} for $m=1,2,3$, respectively, by*

$$\begin{aligned}
Q_g^2 &:= R_g \\
Q_g^4 &:= -\frac{1}{2(n-1)}\Delta R_g - \frac{2}{(n-2)^2}|\operatorname{Ric}_g|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_g^2, \\
Q_g^6 &:= -3!2^6 T_g^6 - \frac{n+2}{2}\Delta_g(\sigma_1(A_g)^2) + 4\Delta_g|A|_g^2 - 8\delta\left(A_g \mathrm{d}\sigma_1(A_g)\right) + \Delta_g^2\sigma_1(A_g) \\
&\quad - \frac{n-6}{2}\sigma_1(A_g)\Delta_g\sigma_1(A_g) - 4(n-6)\sigma_1(A_g)|A|_g^2 + \frac{(n-6)(n+6)}{4}\sigma_1(A_g)^3.
\end{aligned}$$

Associated with these curvatures, we have the following conformally invariant operators

Definition A.4. *For any $g \in \mathrm{Met}^\infty(\Omega)$, let us define the Nth order GJMS operator P_g^{2m} for $m=1,2,3$, respectively, by*

$$\begin{aligned}
P_g^2 &:= -\Delta_g + \frac{n-2}{2}R_g \\
P_g^4 &:= \Delta_g^2 - \operatorname{div}\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}R_g g - \frac{4}{n-2}\operatorname{Ric}_g\right)\mathrm{d} + \frac{n-4}{2}Q_g^4 \\
P_g^6 &:= -\Delta_g^3 - \Delta_g\delta T_2\mathrm{d} - \delta T_2\mathrm{d}\Delta_g - \frac{n-2}{2}\Delta_g\left(\sigma_1(A_g)\Delta_g\right) - \delta T_4\mathrm{d} + \frac{n-6}{2}Q_g^6.
\end{aligned}$$

When $m=1$, the operator $P_g^2 = L_g$ is the so-called conformal Laplacian.

Appendix B. Modica estimates

In this appendix, we discuss possible pointwise estimates for positive smooth solutions to ($\mathcal{P}_{6,\infty}$). These estimates have strong geometric implications in terms of the associated conformally flat metric.

In [16, Theorem 1.4], it is proved that positive smooth solutions to

$$\Delta^2 u = \frac{n(n-4)(n^2-4)}{16}u^{\frac{n+4}{n-4}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

satisfies the following pointwise inequality

$$-\Delta u - \frac{4}{n-2}\frac{|\nabla u|^2}{u} \geqslant \sqrt{\frac{n-4}{n}}u^{\frac{n}{n-4}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

This implies in particular that the scalar curvature Q_g^2 of the conformally flat metric $g = u^{4/(n-4)}\delta$ is positive. This type of result is known in the literature as Modica-type estimates.

In our situation, we start by writing the metric $g \in [g_0]$ as $g = (u^{\frac{n-2}{n-6}})^{\frac{4}{n-2}}\delta$, we see

$$Q_g^2 = -\frac{4(n-1)}{n-2}u^{\frac{-(n+2)}{n-6}}\Delta\left(u^{\frac{n-2}{n-6}}\right) = -\frac{4(n-1)}{n-6}u^{-\frac{n-2}{n-6}}\left(\Delta u + \frac{4}{n-6}\frac{|\nabla u|^2}{u}\right). \tag{B.1}$$

and

$$-\Delta\left(u^{\frac{n-2}{n-6}}\right) = -\Delta u - \frac{4}{n-6}\frac{|\nabla u|^2}{u}.$$

From this, we conclude that $Q_g^2 \geqslant 0$ implies $-\Delta u \geqslant 0$, and in fact is a stronger condition. Similarly, writing $g = (u^{\frac{n-4}{n-6}})^{\frac{4}{n-4}}\delta$, it follows

$$Q_g^4 = \frac{2}{n-4}u^{-\frac{n+4}{n-6}}\Delta^2\left(u^{\frac{n-4}{n-6}}\right). \tag{B.2}$$

Furthermore, a long computation shows

$$\begin{aligned}(-\Delta)^2\left(u^{\frac{n-4}{n-6}}\right) &= \frac{n-4}{n-6}u^{\frac{2}{n-6}}\Delta^2 u+\frac{8(n-4)}{(n-6)^2}u^{\frac{8-n}{n-6}}\langle\nabla u,\nabla\Delta u\rangle\\ &\quad+\frac{4(n-4)}{(n-6)^2}u^{\frac{8-n}{n-6}}|D^2u|^2+\frac{8(n-4)(8-n)}{(n-6)^3}u^{\frac{-2(n-7)}{n-6}}D^2u(\nabla u,\nabla u)\\ &\quad+\frac{4(n-4)(8-n)}{(n-6)^3}u^{\frac{-2(n-7)}{n-6}}|\nabla u|^2\Delta u+\frac{2(n-7)(n-8)}{(n-6)^4}u^{\frac{20-3n}{n-6}}|\nabla u|^4,\end{aligned}$$

where

$$|D^2u|^2=\sum_{i,j=1}^n u_{x_ix_j}^2 \quad\text{and}\quad D^2u(\nabla u,\nabla u)=\sum_{i,j=1}^n u_{x_ix_j}u_{x_i}u_{x_j}.$$

Hence, the conditions $Q_g^2\geqslant 0$ and $Q_g^4\geqslant 0$ are not enough to guarantee that $\Delta^2 u\geqslant 0$ directly.

Based on this, it is natural to ask whether the following result holds.

Conjecture B.1. *Let $u\in\mathcal{C}^\infty(\mathbb{R}^n\setminus\{0\})$ be a positive solutions to ($\mathcal{P}_{6,\infty}$). Then, the conformally flat metric given by $g=u^{4/(n-6)}\delta$ satisfies the following pointwise estimate*

$$Q_2(u)\geqslant\sqrt{\frac{n-6}{n}}u^{\frac{n}{n-6}} \quad\text{and}\quad Q_4(u)\geqslant\sqrt{\frac{n-6}{n}}u^{\frac{n}{n-6}} \quad\text{in}\quad \mathbb{R}^n\setminus\{0\}$$

where

$$Q_2(u):=-\Delta u-\frac{4}{n-6}\frac{|\nabla u|^2}{u}.$$

and

$$\begin{aligned}Q_4(u):=\Delta^2u-\frac{8}{(n-6)}u^{\frac{8-n}{2}}\langle\nabla u,\nabla\Delta u\rangle-\frac{4}{(n-6)}u^{\frac{8-n}{2}}|D^2u|^2-\frac{8(8-n)}{(n-6)^2}u^{7-n}D^2u(\nabla u,\nabla u)\\ -\frac{4(8-n)}{(n-6)^2}u^{7-n}|\nabla u|^2\Delta u-\frac{2(n-7)(n-8)}{(n-6)^3(n-4)}u^{\frac{20-3n}{2}}|\nabla u|^4.\end{aligned}$$

In particular, it follows that the curvatures Q_g^2 and Q_g^4 associated with the conformally flat metric $g=u^{4/(n-6)}\delta$ are both positive.

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