$\mathcal{A}$-quasiconvexity: weak-star convergence and the gap

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Abstract

Lower semicontinuity results with respect to weak-$*$ convergence in the sense of measures and with respect to weak convergence in L^p are obtained for functionals

$$v \in L^1(\Omega; \mathbb{R}^m) \mapsto \int_\Omega f(x, v(x))\, dx,$$

where admissible sequences $\{v_n\}$ satisfy a first order system of PDEs $\mathcal{A}v_n = 0$. We suppose that $\mathcal{A}$ has constant rank, f is $\mathcal{A}$-quasiconvex and satisfies the non standard growth conditions

$$\frac{1}{C}(|v|^p - 1) \leq f(v) \leq C(1 + |v|^q)$$

with $q \in [p, pN/(N-1))$ for $p \leq N-1$, $q \in [p, p+1)$ for $p > N-1$. In particular, our results generalize earlier work where $\mathcal{A}v = 0$ reduced to $v = \nabla^s u$ for some $s \in \mathbb{N}$.

Keywords. $\mathcal{A}$-quasiconvexity, gap, non standard growth conditions, lower semicontinuity, Sobolev Embedding Theorem, Radon-Nikodym Decomposition Theorem

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1 Introduction

It is well known that quasiconvexity is a necessary and sufficient condition for lower semicontinuity with respect to strong convergence in L^1 of functionals of the form

$$u \in W^{1,1}(\Omega; \mathbb{R}^m) \mapsto \int_\Omega f(\nabla u(x))\, dx, \tag{1.1}$$

where the integrand $f = f(\nabla u)$ is nonnegative and has linear growth. More precisely, the following result holds:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $f : \mathbb{R}^{m\times N} \to [0, \infty)$ be a quasiconvex function such that*

$$0 \leq f(\xi) \leq C(1 + |\xi|) \tag{1.2}$$

for all $\xi \in \mathbb{R}^{m\times N}$ and for some constant $C > 0$. Then

$$\int_\Omega f(\nabla u(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega f(\nabla u_n(x))\, dx$$

for any sequence $\{u_n\} \subset W^{1,1}(\Omega; \mathbb{R}^m)$ strongly convergent in $L^1(\Omega; \mathbb{R}^m)$ to some $u \in BV(\Omega; \mathbb{R}^m)$ if and only if f is quasiconvex.

The proof of the necessity is due to Morrey [39], while the sufficiency relies on De Giorgi's Slicing Lemma (see e.g. [6]; see also [25], [26], [35]). In the Appendix we present a another argument based on Gagliardo's Trace Theorem for $W^{1,1}(\Omega; \mathbb{R}^m)$ (see [29]). It is interesting to observe that the idea behind the proofs using either De Giorgi's Slicing Lemma or Gagliardo's Trace Theorem is actually the same.

In the scalar case, that is when $m = 1$, it has been proved by Serrin [44] that Theorem 1.1 continues to hold without assuming the upper bound in (1.2). This is due to the fact that when $m = 1$ quasiconvexity reduces to convexity. Since any nonnegative convex function is the supremum of a sequence of linear functions, trivially satisfying (1.2), lower semicontinuity results for this type of integrands do not require apriori growth conditions. The situation is completely different in the vectorial case $m > 1$, where Theorem 1.1 fails in general if f has superlinear growth. Indeed, Acerbi, Buttazzo and Fusco [2] proved that when $N = m = 2$ the functional

$$u \in W^{1,2}\left(\Omega; \mathbb{R}^2\right) \mapsto \int_\Omega |\det \nabla u|\, dx$$

is not lower semicontinuous with respect to strong convergence in $L^p\left(\Omega;\mathbb{R}^2\right)$ for any $1 \leq p < \infty$.

This striking difference in lower semicontinuity properties between functionals with integrands with linear growth of the type (1.2) and integrands with superlinear growth such as

$$0 \leq f(\xi) \leq C\left(1+|\xi|^q\right), \quad q>1,$$

maybe explained in part by the profound disparity in the characterization and properties of the trace space of $W^{1,q}(\Omega;\mathbb{R}^m)$ when $q=1$ and $q>1$. If Ω is a Lipschitz domain then the trace space of $W^{1,1}(\Omega;\mathbb{R}^m)$ is $L^1(\partial\Omega;\mathbb{R}^m)$, and thus strong convergence in $L^1(\Omega;\mathbb{R}^m)$ implies (up to a subsequence) strong convergence of the traces in $L^1(\partial\Omega_t;\mathbb{R}^m)$ where Ω_t is a smooth domain arbitrarily "close" do Ω and hence there exists an extension which converges in $W^{1,1}(\Omega;\mathbb{R}^m)$. On the other hand, when $q>1$ the trace space of $W^{1,q}(\Omega;\mathbb{R}^m)$ is the fractional Sobolev space $W^{1-\frac{1}{q},q}(\partial\Omega_t;\mathbb{R}^m)$, therefore strong convergence alone in $L^p(\Omega;\mathbb{R}^m)$ for *any* $1 \leq p < \infty$ does not necessarily entail strong convergence of the traces in $W^{1-\frac{1}{q},q}(\partial\Omega;\mathbb{R}^m)$. By virtue of Sobolev's Imbedding Theorem this is guaranteed, however, if the integrand f satisfies a coercivity condition of the form

$$f(\xi) \geq \frac{1}{C}\left(|\xi|^p-1\right),$$

with

$$1 \leq p \leq q < \frac{N}{N-1}p. \tag{1.3}$$

As a consequence, the following result holds:

Theorem 1.2 *Assume that p,q satisfy* (1.3)*. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $f:\mathbb{R}^{m\times N} \to [0,\infty)$ be a quasiconvex function such that*

$$\frac{1}{C}\left(|\xi|^p-1\right) \leq f(\xi) \leq C\left(1+|\xi|^q\right) \tag{1.4}$$

for all $\xi \in \mathbb{R}^{m\times N}$. Then

$$\int_\Omega f(\nabla u)\,dx \leq \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\,dx$$

for any sequence $\{u_n\} \subset W^{1,q}(\Omega;\mathbb{R}^m)$ which converges to $u \in BV(\Omega;\mathbb{R}^m)$ strongly in $L^1(\Omega;\mathbb{R}^m)$.

In this generality Theorem 1.2 was proved by Fonseca e Malý [23] for $p>1$, and by Kristensen [32] when $p=1$ (see the bibliography therein for previous partial results). For the convenience of the reader we present a proof of Theorem 1.2 in the Appendix.

Observe that we take admissible converging sequences $\{u_n\}$ in the space $W^{1,q}(\Omega;\mathbb{R}^d)$, otherwise not only we would be unable to guarantee finiteness of the energy, but also, since f is quasiconvex and $f(\xi) \leq C\left(1+|\xi|^q\right)$, f is

$W^{1,q}$-quasiconvex but it may fail to be $W^{1,p}$-quasiconvex (see [8]). In addition, note that by (1.4) if $p > 1$ and if $\liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\, dx < \infty$ then, necessarily, $u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

The proof of Theorem 1.2 strongly hinges on the properties of a linear, compact, lifting operator

$$\begin{aligned} E : W^{1,p}(\partial\Omega; \mathbb{R}^m) &\to W^{1,q}(\Omega; \mathbb{R}^m) \\ v &\mapsto E(v) \end{aligned}$$

such that v is the trace of $E(v)$. The existence of such an operator follows from standard Sobolev trace and compact embedding theorems when $q < \frac{N}{N-1}p$. The exponent

$$q_c = \frac{N}{N-1}p$$

is critical for the existence of the operator E, and, not surprisingly, also for lower semicontinuity of functionals of the type (1.1). Indeed, Malý [33] proved that the functional

$$u \in W^{1,N}\left(\Omega; \mathbb{R}^N\right) \mapsto \int_\Omega |\det \nabla u| \, dx$$

is not lower semicontinuous with respect to weak convergence in $W^{1,p}\left(\Omega; \mathbb{R}^N\right)$ for any $p < N - 1$.

Lower semicontinuity of (1.1) in the borderline case where (1.4) holds for $q = \frac{N}{N-1}p$ is still unknown (see [24], [32], [34] for some partial results), except for the special case where $m = N$ and

$$f(\xi) := |\xi|^{N-1} + g(\det \xi). \tag{1.5}$$

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $g : \mathbb{R} \to [0, \infty]$ be a lower semicontinuos convex function such that $g(0) < \infty$. Let $\{u_n\}$ be a sequence of functions in $W^{1,N}(\Omega; \mathbb{R}^N)$ which converges to $u \in BV(\Omega; \mathbb{R}^N)$ in $L^1(\Omega; \mathbb{R}^N)$, and such that*

$$\sup_n \int_\Omega |\nabla u_n|^{N-1} \, dx < \infty.$$

Then

$$\int_\Omega g(\det \nabla u)\, dx \le \liminf_{n\to\infty} \int_\Omega g(\det \nabla u_n)\, dx.$$

Theorem 1.3 was proved by Celada and Dal Maso [13] using cartesian current (see also [22] for a new proof).

Functions of the form (1.5) may be viewed as prototypes of integrands $f = f(x, u, \nabla u)$ satisfying a "limiting" non standard growth condition (1.4) and whose importance stems from the study of cavitation and related issues in nonlinear elasticity and continuum mechanics. For further results in related subjects we refer the reader to [1], [3], [7], [9], [13], [17], [20], [23], [24], [32], [33], [34], [35], [36], [53], [54].

The purpose of this paper is to extend Theorems 1.1 and 1.2 to the general setting of $\mathcal{A}$-quasiconvexity, which has been introduced by Dacorogna [14] and further developed by Fonseca and Müller in [27] (see also [10]). Here, and following [40],

$$\mathcal{A} : L^q(\Omega; \mathbb{R}^d) \to W^{-1,q}(\Omega; \mathbb{R}^l), \quad \mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i},$$

is a constant–rank (see (2.1)), first order linear partial differential operator, with $A^{(i)} : \mathbb{R}^d \to \mathbb{R}^l$ linear transformations, $i = 1, \ldots, N$. We recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(\xi) \leq \int_Q f(\xi + w(y))\, dy \tag{1.6}$$

for all $\xi \in \mathbb{R}^d$ and all $w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$, where Q denotes the unit cube in $\mathbb{R}^N$, and the space $C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d)$ is introduced in Section 2.

The relevance of this general framework, as emphasized by Tartar (see [47, 48, 49, 50, 51, 52]), lies on the fact that in continuum mechanics and in electromagnetism PDEs other than $\operatorname{curl} v = 0$ arise naturally and are physically relevant, and this calls for a relaxation theory which encompasses PDE constraints of the type $\mathcal{A}v = 0$. Some important examples included in this general setting are given by:

(a) [Unconstrained Fields]

$$\mathcal{A}v \equiv 0.$$

Here, due to Jensen's inequality, A-quasiconvexity reduces to convexity.

(b) [Divergence Free Fields]

$$\mathcal{A}v = \operatorname{div} v = 0,$$

where $v : \Omega \subset \mathbb{R}^N \to \mathbb{R}^N$ (see [41]).

(c) [Maxwell's Equations]

$$\mathcal{A}\begin{pmatrix} m \\ h \end{pmatrix} := \begin{pmatrix} \operatorname{div}(m+h) \\ \operatorname{curl} h \end{pmatrix} = 0,$$

where $m : \mathbb{R}^3 \to \mathbb{R}^3$ is the *magnetization* and $h : \mathbb{R}^3 \to \mathbb{R}^3$ is the *induced magnetic field* (see [18, 51]).

(d) [Gradients]

$$\mathcal{A}v = \operatorname{curl} v = 0.$$

Note that $w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^d)$ is such that curl $w = 0$ and $\int_Q w(y)\, dy = 0$ if and only if there exists $\varphi \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^m)$ such that $\nabla\varphi = v$, where $d = m \times N$. In

this case, (1.6) reduces to the well-known notion of *quasiconvexity* introduced by Morrey [39].
(e) [Higher Order Gradients]
Replacing the target space $\mathbb{R}^d$ by an appropriate finite dimensional vector space E_s^m of m-tuples of symmetric s-linear maps on $\mathbb{R}^N$, it is possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in L^p(\Omega; E_s^m)$ and $\mathcal{A}v = 0$ if and only if there exists $\varphi \in W^{s,q}(\Omega; \mathbb{R}^m)$ such that $v = \nabla^s \varphi$ (see Theorem 1.8). In this case, (1.6) reduces to the notion of $s-$*quasiconvexity* introduced by Meyers [38].

The first main result of the paper is given by the following theorem:

Theorem 1.4 *Let*

$$1 \leq q < \frac{N}{N-1}. \tag{1.7}$$

Let $\Omega \subset \mathbb{R}^N$ *be an open bounded set, and let* $f : \Omega \times \mathbb{R}^d \to [0, \infty)$ *be a Borel measurable,* $\mathcal{A}$*-quasiconvex function such that*

$$|f(x,\xi) - f(x,\xi_1)| \leq C\left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right)|\xi - \xi_1| \tag{1.8}$$

for all $x \in \Omega$ *and all* $\xi, \xi_1 \in \mathbb{R}^d$*, and for some* $C > 0$*. Assume further that for all* $x_0 \in \Omega$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$f(x_0,\xi) - f(x,\xi) \leq \varepsilon(1 + f(x,\xi)) \tag{1.9}$$

for all $x \in \Omega$ *with* $|x - x_0| \leq \delta$ *and for all* $\xi \in \mathbb{R}^d$*. Then*

$$\int_\Omega f\left(x, \frac{d\lambda}{d\mathcal{L}^N}(x)\right) dx \leq \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))\, dx$$

for any sequence $\{v_n\} \subset L^q\left(\Omega; \mathbb{R}^d\right) \cap \ker \mathcal{A}$ *weakly-$*$ converging in the sense of measures to some* $\mathbb{R}^d-$*valued Radon measure* $\lambda \in \mathcal{M}\left(\Omega; \mathbb{R}^d\right)$.

Lower semicontinuity properties of the constrained functional

$$\int_\Omega f(x, v(x))\, dx \quad \text{with} \quad \mathcal{A}v = 0,$$

have been proved by Fonseca and Müller in [27] with respect to weak convergence in $L^1\left(\Omega; \mathbb{R}^d\right)$. Note, however, that for integrands with linear growth weak-$*$ convergence in the sense of measures is more natural in view of the lack of reflexivity of the space $L^1(\Omega; \mathbb{R}^d)$.

Also, in the case (d) of gradients, that is, when

$$\mathcal{A}v = \operatorname{curl} v = 0,$$

Theorem 1.4 includes Theorem 1.1 for integrands which satisfy the additional coercivity assumption

$$f(\xi) \geq \frac{1}{C}|\xi| - C. \tag{1.10}$$

Indeed, condition (1.10) implies that the sequence $\{\nabla u_n\}$ is uniformly bounded in $L^1\left(\Omega;\mathbb{R}^{m\times N}\right)$, and thus a subsequence weakly-$*$ converges in the sense of measures.

We do not know if Theorem 1.4 continues to hold under a convergence weaker than weak-$*$ convergence in the sense of measures. On one hand, Theorem 1.1 certainly seems to point in that direction, but on the other hand, even for higher order gradients (also contemplated within the $\mathcal{A}$-quasiconvexity framework; see example (e) above) the situation is far from clear. Indeed, it is still an open problem to determine whether the functional

$$u \in W^{2,1}(\Omega;\mathbb{R}^m) \mapsto \int_\Omega f(\nabla^2 u)\,dx,$$

where $f : E_2^d \to [0,\infty)$ is a 2-quasiconvex function satisfying

$$0 \leq f(\xi) \leq C\,(1+|\xi|)$$

for all $\xi \in E_2^d$, is lower semicontinuous with respect to strong convergence in $W^{1,1}(\Omega;\mathbb{R}^m)$. Note that if $u \in W^{2,1}(\Omega;\mathbb{R}^m)$, then (see [11], [19], [37])

$$\left.\left(u, \frac{\partial u}{\partial \nu}\right)\right|_{\partial\Omega} \in B^{1,1}\left(\partial\Omega;\mathbb{R}^m\right) \times L^1\left(\partial\Omega;\mathbb{R}^m\right),$$

and strong convergence in $W^{1,1}(\Omega;\mathbb{R}^m)$ implies strong convergence of the normal derivatives $\left\{\dfrac{\partial u_n}{\partial \nu}\right\}$ in $L^1\left(\partial\Omega_t;\mathbb{R}^m\right)$ where Ω_t is a smooth domain arbitrarily "close" do Ω. However, this does not necessarily guarantee strong convergence of the traces in the Besov space $B^{1,1}\left(\partial\Omega_t;\mathbb{R}^m\right)$. This suggests that lower semicontinuity might not hold under strong convergence in $W^{1,1}(\Omega;\mathbb{R}^m)$ and that a stronger notion of convergence is needed. We do not know how to prove or disprove this.

Condition (1.9) is satisfied in the important special case where the integrand $f(x,\xi)$ is a decoupled product. Indeed we have the following

Corollary 1.5 *Let* $1 \leq q < \infty$ *satisfy* (1.7), *let* $g : \mathbb{R}^N \to [0,\infty)$ *be an* $\mathcal{A}$-*quasiconvex function such that*

$$|g(\xi) - g(\xi_1)| \leq C\left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right)|\xi - \xi_1|$$

for all $\xi, \xi_1 \in \mathbb{R}^d$, *and for some* $C > 0$, *and let* $h : \Omega \times \mathbb{R} \to [0,\infty]$ *be a lower semicontinuous function. Then*

$$\int_\Omega h(x) g\left(\frac{d\lambda}{d\mathcal{L}^N}(x)\right)dx \leq \liminf_{n\to\infty} \int_\Omega h(x) g\left(v_n(x)\right)dx$$

for any sequence $\{v_n\} \subset L^1\left(\Omega;\mathbb{R}^d\right) \cap \ker\mathcal{A}$ *weakly-$*$ converging in the sense of measures to some* $\mathbb{R}^d-$*valued Radon measure* $\lambda \in \mathcal{M}\left(\Omega;\mathbb{R}^d\right)$.

The second main result of the paper partially extends Theorem 1.2 to the realm of $\mathcal{A}$-quasiconvexity:

Theorem 1.6 *Let $1 < p \leq q < \infty$, and assume that*

$$q < \begin{cases} \dfrac{N}{N-1} p & \text{if } p \leq N-1, \\ p+1 & \text{if } p > N-1. \end{cases} \tag{1.11}$$

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $f : \Omega \times \mathbb{R}^d \to [0, \infty)$ be an $\mathcal{A}$-quasiconvex function such that

$$|f(x,\xi) - f(x,\xi_1)| \leq C \left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right) |\xi - \xi_1| \tag{1.12}$$

for all $x \in \Omega$ and all $\xi, \xi_1 \in \mathbb{R}^d$, and for some $C > 0$. Assume that f satisfies condition (1.9)*. Then*

$$\int_\Omega f(x, v(x))\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, v_n(x))\, dx$$

for any sequence $\{v_n\} \subset L^q(\Omega; \mathbb{R}^d) \cap \ker \mathcal{A}$ weakly converging in $L^p(Q; \mathbb{R}^d)$ to some $v \in L^p(\Omega; \mathbb{R}^d)$.

Note that, unlike the case where $p = q$ (see [4], [15]), in general one may not take f to be a Carathéodory function, and some kind of regularity is needed in the x variable. Indeed, Gangbo [30] has proved that the functional

$$u \in W^{1,q}(\Omega; \mathbb{R}^N) \mapsto \int_\Omega \chi_K(x) |\det \nabla u(x)|\, dx,$$

where $K \subset \mathbb{R}^N$ is a compact set, is lower semicontinuous with respect to weak convergence in $W^{1,p}(\Omega; \mathbb{R}^N)$ for some $N-1 < p < N$ if and only if

$$\mathcal{L}^N(\partial K) = 0.$$

Here, again, one witnesss the intrinsic differences between the convex and the quasiconvex frameworks, as it has been shown by Acerbi, Bouchitté and Fonseca [1] that Theorem 1.6 still holds for Carathéodory functions f and with $\mathcal{A}v = 0$ if and only if $\operatorname{curl} v = 0$, provided $f(x, \cdot)$ is convex, and without requiring condition (1.12).

The analog of Corollary 1.5 is now:

Corollary 1.7 *Let $1 < p \leq q < \infty$ satisfy* (1.11)*, let $g : \mathbb{R}^N \to [0, \infty)$ be an $\mathcal{A}$-quasiconvex function such that*

$$|g(\xi) - g(\xi_1)| \leq C \left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right) |\xi - \xi_1|$$

for all $\xi, \xi_1 \in \mathbb{R}^d$, and for some $C > 0$, and let $h : \Omega \times \mathbb{R} \to [0, \infty]$ be a lower semicontinuous function. Then

$$\int_\Omega h(x) g\left(v\left(x\right)\right)\, dx \leq \liminf_{n\to\infty} \int_\Omega h(x) g\left(v_n\left(x\right)\right)\, dx$$

for any sequence $\{v_n\} \subset L^q\left(\Omega; \mathbb{R}^d\right) \cap \ker \mathcal{A}$ weakly converging in $L^p\left(Q; \mathbb{R}^d\right)$ to some $v \in L^p\left(\Omega; \mathbb{R}^d\right)$.

In the case of first or higher order gradients (d) and (e), the Lipschitz condition (1.12) follows from the s–quasiconvexity of the integrand $f(x, \xi)$ together with the growth condition (1.13) below. For first order gradients, this was shown by Marcellini [35]. The case $s = 2$ was treated by Guidorzi and Poggiolini [31], while the general case was studied by Santos and Zappale [43]. More generally, it can be shown that if the span of the characteristic cone

$$\Lambda := \bigcup_{w\in S^{N-1}} \mathbb{A}(w),$$

where $\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)}$, has dimension d then $\mathcal{A}$-quasiconvexity, together with (1.13) below, implies (1.12).

As a corollary of Theorem 1.6 we obtain the following result:

Theorem 1.8 *Let $1 < p, q < \infty$ satisfy (1.11), let $s \in \mathbb{N}$, and suppose that $f : \Omega \times E_s^m \to [0, \infty)$ is a Borel integrand satisfying (1.9), and*

$$0 \leq f(x, \xi) \leq C\left(1 + |\xi|^q\right) \tag{1.13}$$

for a.e. $x \in \Omega$ and all $\xi \in E_s^m$, where $C > 0$. Assume that for a.e. $x \in \Omega$ the function $f(x, \cdot)$ is $s-$quasiconvex, that is for all $\xi \in E_s^m$

$$f(x, \xi) = \inf\left\{\int_Q f(x, \xi + \nabla^s w(y))\, dy : w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^m)\right\}.$$

If $\{u_n\} \subset W^{s,q}(\Omega; \mathbb{R}^m)$ converges weakly to u in $W^{s,p}(\Omega; \mathbb{R}^m)$ then

$$\int_\Omega f(x, \nabla^s u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, \nabla^s u_n)\, dx.$$

Here E_s^m stands for the space of m-tuples of symmetric s-linear maps on $\mathbb{R}^N$. Theorem 1.8 was proved by Esposito and Mingione (see Theorem 4.1 in [21]) under the assumptions

$$q < \frac{N\left(s-1\right)}{N\left(s-1\right)-1}\, p$$

when $s \geq 2$.

2 Preliminaries

We start with some notation. Here Ω is an open, bounded subset of $\mathbb{R}^N$, $\mathcal{L}^N$ is the N dimensional Lebesgue measure, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere, and $Q := (-1/2, 1/2)^N$ the unit cube centered at the origin. For $r > 0$ and $x_0 \in \mathbb{R}^N$ we set $Q_r := rQ$ and $Q(x_0, r) := x_0 + rQ$. A function $w \in L^q_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^d)$ is said to be *Q–periodic* if $w(x + e_i) = w(x)$ for a.e. $x \in \mathbb{R}^N$ and every $i = 1, \ldots, N$, where $\{e_1, \ldots, e_N\}$ is the canonical basis of $\mathbb{R}^N$, and we write $w \in L^q_{\text{per}}(Q; \mathbb{R}^d)$. Also $C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ will stands for the space of Q–periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$. The *Fourier coefficients* of a function $w \in L^q_{\text{per}}(Q; \mathbb{R}^d)$ are defined by

$$\hat{w}(\lambda) := \int_Q w(x)\, e^{-2\pi i x\cdot\lambda} dx, \quad \lambda \in \mathbb{Z}^N.$$

If $1 < q \leq \infty$ then $W^{-1,q}(\Omega; \mathbb{R}^l)$ is the dual of $W^{1,q'}_0(\Omega; \mathbb{R}^l)$, where q' is the Hölder conjugate exponent of q, that is $1/q + 1/q' = 1$. It is well known that $F \in W^{-1,q}(\Omega; \mathbb{R}^l)$ if and only if there exist $g_1, \ldots, g_N \in L^q(\Omega; \mathbb{R}^l)$ such that

$$\langle F, w\rangle = \sum_{i=1}^N \int_\Omega g_i \cdot \frac{\partial w}{\partial x_i}\, dx \qquad \text{for all } w \in W^{1,q'}_0(\Omega; \mathbb{R}^l).$$

Consider a collection of linear operators $A^{(i)} : \mathbb{R}^d \to \mathbb{R}^l$, $i = 1, \ldots, N$, and define the differential operator

$$\begin{aligned} \mathcal{A}_\Omega : L^q(\Omega; \mathbb{R}^d) &\longrightarrow W^{-1,q}(\Omega; \mathbb{R}^l) \\ v &\longmapsto \mathcal{A}v \end{aligned}$$

as follows:

$$\begin{aligned} \langle \mathcal{A}_\Omega v, w\rangle &:= \left\langle \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, w \right\rangle \\ &= -\sum_{i=1}^N \int_\Omega A^{(i)} v \,\frac{\partial w}{\partial x_i}\, dx \quad \text{for all } w \in W^{1,q'}_0(\Omega; \mathbb{R}^l). \end{aligned}$$

Even though the operator $\mathcal{A}_\Omega$ so defined depends on Ω, we will omit reference to the underlying domain whenever it is clear from the context, and we will write simply $\mathcal{A}$ in place of $\mathcal{A}_\Omega$. In particular, if $v \in L^q_{\text{per}}(Q; \mathbb{R}^d)$ then we will say that $v \in \ker \mathcal{A}$ if $\mathcal{A}v = 0$ in $W^{-1,q}_{\text{per}}(Q; \mathbb{R}^l)$, i.e. we consider test functions $w \in W^{1,q'}_{\text{per}}(Q; \mathbb{R}^l)$.

In the sequel we will assume that $\mathcal{A}$ satisfies the *constant-rank* property (see [40]), precisely there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}(w) = r \quad \text{for all } w \in S^{N-1} \tag{2.1}$$

where

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)}, \qquad w \in \mathbb{R}^N.$$

For each $w \in \mathbb{R}^N$ the operator $\mathbb{P}(w) : \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection of $\mathbb{R}^d$ onto $\ker \mathbb{A}(w)$, and $\mathbb{S}(w) : \mathbb{R}^l \to \mathbb{R}^d$ is defined by $\mathbb{S}(w)\mathbb{A}(w)z := z - \mathbb{P}(w)z$ for $z \in \mathbb{R}^d$ and $\mathbb{S} \equiv 0$ on $(\text{range}(\mathbb{A}(\text{w}))^{\perp}$. It may be shown that $\mathbb{P} : \mathbb{R}^N \setminus \{0\} \to \text{Lin}(\mathbb{R}^d;\mathbb{R}^d)$ is smooth and homogeneous of degree zero and $\mathbb{S} : \mathbb{R}^N \setminus \{0\} \to \text{Lin}(\mathbb{R}^l;\mathbb{R}^d)$ is smooth and homogeneous of degree -1 (see [27]).

For $q > 1$ we define the operator

$$\mathcal{S}_q : L^q_{\text{per}}(Q;\mathbb{R}^d) \to W^{1,q}_{\text{per}}(Q;\mathbb{R}^l),$$

by

$$\mathcal{S}_q v(x) := \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{S}(\lambda)\, \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda} \tag{2.2}$$

whenever $v \in L^q_{\text{per}}(Q;\mathbb{R}^d)$ can be written as

$$v(x) := \sum_{\lambda \in \mathbb{Z}^N} \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda}. \tag{2.3}$$

Using (2.2) and (2.3) we may write

$$\mathcal{S}_q v(x) := \int_Q K(x-y)\, v(y)\, dy,$$

where the periodic kernel K is given by the Fourier series

$$K(x) := \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{S}(\lambda)\, e^{2\pi i x \cdot \lambda},$$

which converges in the sense of distributions.

For any function w defined on $\mathbb{R}^N$ and for every $k = 1, \ldots, N$ and any positive integer $s \in \mathbb{N}$ we define

$$\frac{(\partial^{\pm})^s w}{\partial x_k^s}(x) = \underbrace{\frac{\partial^{\pm}}{\partial x_k}\left(\frac{\partial^{\pm}}{\partial x_k}\left(\cdots\left(\frac{\partial^{\pm} w}{\partial x_k}\right)\right)\right)}_{s \text{ times}},$$

where the difference quotient $\dfrac{\partial^{\pm} w}{\partial x_k}$ is given by

$$\frac{\partial^{\pm} w}{\partial x_k}(x) := w(x \pm e_k) - w(x).$$

Moreover for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, we use the notation

$$(\partial^{\pm})^{\alpha} w(x) := \frac{(\partial^{\pm})^{\alpha_1}}{\partial x_1^{\alpha_1}}\left(\frac{(\partial^{\pm})^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\cdots\left(\frac{(\partial^{\pm})^{\alpha_N} w(x)}{\partial x_N^{\alpha_N}}\right)\right)\right).$$

Proposition 2.1 *There exists $C>0$ such that*

$$|K(x)| \leq C|x|^{1-N} \tag{2.4}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$.

Proof. Although the result is well-known to experts, we include a proof for the convenience of the reader. It suffices to prove that

$$|\nabla K(x)| \leq C|x|^{-N} \tag{2.5}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$. Let $l \in \{1, \cdots, N\}$ and let

$$\begin{aligned}\tilde{K}(x) = \frac{\partial K}{\partial x_l}(x) &:= \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} 2\pi i \lambda_l \mathbb{S}(\lambda)\, e^{2\pi i x \cdot \lambda} \\ &= \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} m(\lambda)\, e^{2\pi i x \cdot \lambda} = \sum_{\lambda \in \mathbb{Z}^N} m(\lambda)\, e^{2\pi i x \cdot \lambda}\end{aligned}$$

with

$$m(\lambda) := 2\pi i \lambda_l \mathbb{S}(\lambda)$$

and where we have used the fact $m(0) = 0$.

We consider the following dyadic decomposition (cf. [45], page 241). Let $\varphi \in C_c^\infty(B(0,2);[0,1])$, $\varphi = 1$ in $B(0,1)$, and define $\delta(x) := \varphi(x) - \varphi(2x)$. Observe that $\delta = 0$ if $|x| \leq \frac{1}{2}$ and $|x| \geq 2$. It turns out that

$$\sum_{j=-\infty}^{\infty} \delta\left(\frac{x}{2^j}\right) = 1$$

for all $x \in \mathbb{R}^N \setminus \{0\}$. Hence

$$\sum_{j=-n}^{n} K_j \to \tilde{K} \tag{2.6}$$

in the sense of distributions, where

$$K_j(x) := \sum_{\lambda \in \mathbb{Z}^N} m_j(\lambda)\, e^{2\pi i x \cdot \lambda}$$

and $m_j(\lambda) := m(\lambda)\, \delta\left(\frac{\lambda}{2^j}\right)$. Since

$$m_j(\lambda) \neq 0 \text{ only if } 2^{j-1} \leq |\lambda| \leq 2^{j+1}, \tag{2.7}$$

it is clear that K_j reduces to a finite sum. Note that if $j \leq -2$ clearly no integer satisfies $2^{j-1} \leq |\lambda| \leq 2^{j+1}$ and so $m_j \equiv 0$ for all $j \leq -2$.

We claim that for every $M \in \mathbb{N}$

$$|K_j(x)| \leq C_M \frac{1}{|x|^M} 2^{j(N-M)} \tag{2.8}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$. This, together with (2.6), yields the result. Indeed, fix $x \in \mathbb{R}^N \setminus \{0\}$, and note first that (2.8) with $M = 0$ reduces to

$$|K_j(x)| \le C_0 2^{jN}. \tag{2.9}$$

Choose $M > N$. We have

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |K_j(x)| &\le \sum_{2^j \le |x|^{-1}} |K_j(x)| + \sum_{2^j > |x|^{-1}} |K_j(x)| \\ &\le C_0 \sum_{2^j \le |x|^{-1}} 2^{jN} + C_M \frac{1}{|x|^M} \sum_{2^j > |x|^{-1}} 2^{j(N-M)}. \end{aligned} \tag{2.10}$$

In the latter expression the first sum can be bounded above by

$$\sum_{2^j \le |x|^{-1}} 2^{jN} \le \frac{\left(2^N\right)^{1+\log_2 |x|^{-1}} - 1}{2^N - 1} \le \frac{1}{|x|^N} \frac{2^N}{2^N - 1},$$

while the second term in (2.10) may be estimated by

$$C_M \frac{1}{|x|^M} \sum_{2^j > |x|^{-1}} 2^{j(N-M)} \le C_M \frac{1}{|x|^M} \frac{1}{|x|^{N-M}} \frac{1}{1-2^{N-M}} \frac{1}{2^{N-M}} \le C_M \frac{1}{|x|^N}.$$

To conclude the proof, it remains to establish (2.8). By means of a summation by parts and by the Mean Value Theorem, for any $k = 1, \cdots, N$, we have

$$\begin{aligned}\left(e^{2\pi i x_k} - 1\right) K_j(x) &= \sum_{\lambda \in \mathbb{Z}^N} m_j(\lambda) \left(e^{2\pi i x\cdot(\lambda + e_k)} - e^{2\pi i x \cdot \lambda}\right) \\ &= \sum_{\lambda \in \mathbb{Z}^N} \left(m_j(\lambda - e_k) - m_j(\lambda)\right) e^{2\pi i x\cdot\lambda} \\ &= \sum_{\lambda \in \mathbb{Z}^N} \frac{\partial^- m_j}{\partial \lambda_k}(\lambda)\, e^{2\pi i x\cdot\lambda} \\ &= \sum_{\lambda \in \mathbb{Z}^N} \frac{\partial m_j}{\partial \lambda_k}\left(\lambda + \theta_k^{(1)} e_k\right) e^{2\pi i x\cdot\lambda},\end{aligned}$$

for some $\theta_k^{(1)} \in (0,1)$. By replacing m_j with $\dfrac{\partial m_j}{\partial \lambda_l}$ in the previous identity, we obtain respectively

$$\left(e^{2\pi i x_l} - 1\right)\left(e^{2\pi i x_k} - 1\right) K_j(x) = \sum_{\lambda \in \mathbb{Z}^N} \frac{\partial^2 m_j}{\partial \lambda_l \partial \lambda_k}\left(\lambda + \theta_k^{(1)} e_k + \theta_l^{(1)} e_l\right) e^{2\pi i x\cdot\lambda}$$

if $l \ne k$,

$$\left(e^{2\pi i x_k} - 1\right)^2 K_j(x) = \sum_{\lambda \in \mathbb{Z}^N} \frac{\partial^2 m_j}{\partial \lambda_k^2}\left(\lambda + \left(\theta_k^{(1)} + \theta_k^{(2)}\right) e_k\right) e^{2\pi i x\cdot\lambda}$$

if $l=k$, where we have used the fact that partial derivatives and difference quotients commute, i.e.

$$\frac{\partial^-}{\partial\lambda_l}\left(\frac{\partial m_j}{\partial\lambda_k}\right)=\frac{\partial}{\partial\lambda_k}\left(\frac{\partial^- m_j}{\partial\lambda_l}\right),$$

and, once again, we have invoked the Mean Value Theorem.

In turn, if α is a multi-index with $|\alpha|=M$, we have

$$\prod_{k=1}^{N}\left(e^{2\pi i x_k}-1\right)^{\alpha_k}K_j(x)=\sum_{\lambda\in\mathbb{Z}^N}\frac{\partial^{|\alpha|}m_j}{\partial\lambda^\alpha}\left(\lambda+\sum_{k=1}^{N}\left(\theta_k^{(1)}+\cdots+\theta_k^{(\alpha_k)}\right)e_k\right)e^{2\pi i x\cdot\lambda},$$

where $\theta_k^{(1)},\cdots,\theta_k^{(\alpha_k)}\in(0,1)$. By the Mean Value Theorem we derive

$$(2\pi)^{|\alpha|}\left|x^\alpha\right|\left|K_j(x)\right|\leq\sum_{\lambda\in\mathbb{Z}^N}\left|\frac{\partial^{|\alpha|}m_j}{\partial\lambda^\alpha}\left(\lambda+\sum_{k=1}^{N}\left(\theta_k^{(1)}+\cdots+\theta_k^{(\alpha_k)}\right)e_k\right)\right|,$$

which, togethere with (2.7), yields

$$\begin{aligned}\left|K_j(x)\right|&\leq\frac{C}{|x|^M}\sum_{2^{j-1}-|\alpha|\leq|\lambda|\leq2^{j+1}+|\alpha|}\left|\frac{\partial^{|\alpha|}m_j}{\partial\lambda^\alpha}\left(\lambda+\sum_{k=1}^{N}\left(\theta_k^{(1)}+\cdots+\theta_k^{(\alpha_k)}\right)e_k\right)\right|\\&\leq\frac{C2^{-jM}}{|x|^M}\sum_{2^{j-1}-|\alpha|\leq|\lambda|\leq2^{j+1}+|\alpha|}1\\&\leq C\frac{2^{-jM+jN}}{|x|^M}.\end{aligned}$$

Note that here we have used the fact that

$$\left|\frac{\partial^{|\alpha|}m_j(\lambda)}{\partial\lambda^\alpha}\right|\leq C_\alpha 2^{-jM},$$

which results directly from the homogeneity of degree zero of the function m, yielding

$$\left|\frac{\partial^{|\alpha|}m(\lambda)}{\partial\lambda^\alpha}\right|\leq C_\alpha|\lambda|^{-M},$$

and from

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^\alpha}\left(\delta\left(\frac{\lambda}{2^j}\right)\right)\right|=\left(\frac{1}{2^j}\right)^M\left|\frac{\partial^{|\alpha|}\delta}{\partial\lambda^\alpha}\left(\frac{\lambda}{2^j}\right)\right|\leq C_\alpha|\lambda|^{-M},$$

where we took into account the fact $\operatorname{supp}\delta\left(\frac{\cdot}{2^j}\right)\subset\left[2^{j-1},2^{j+1}\right]$. ■

It is clear that $\mathcal{S}_q$ may be extended as an homogeneous operator of degree -1 from $W^{-1,q}_{\mathrm{per}}(Q;\mathbb{R}^d)$ into $L^q_{\mathrm{per}}(Q;\mathbb{R}^d)$. Indeed, as it is usual, using duality principles, if $L\in W^{-1,q}_{\mathrm{per}}(Q;\mathbb{R}^d)$ and if $\varphi\in L^{q'}_{\mathrm{per}}(Q;\mathbb{R}^d)$ then

$$\left\langle\mathcal{S}_qL,\varphi\right\rangle:=\left\langle L,\mathcal{S}^*_{q'}\varphi\right\rangle,\tag{2.11}$$

where for $f, g \in C^{\infty}_{\mathrm{per}}(Q;\mathbb{R}^d)$ the duality pair is defined by

$$\langle f, g\rangle := \sum_{\lambda\in\mathbb{Z}^N} \overline{\hat{f}\,(\lambda)}\,\hat{g}\,(\lambda) = \int_Q \overline{f\,(x)}\,g\,(x)\;dx,$$

and where the operator

$$\mathcal{S}^*_{q'} : L^{q'}_{\mathrm{per}}\left(Q;\mathbb{R}^d\right) \to W^{1,q'}_{\mathrm{per}}\left(Q;\mathbb{R}^l\right)$$

is defined by

$$\mathcal{S}^*_{q'} v\,(x) := \sum_{\lambda\in\mathbb{Z}^N\setminus\{0\}} \overline{\mathbb{S}\,(\lambda)}\hat{v}\,(\lambda)\,e^{2\pi i x\cdot\lambda}$$

whenever $v \in L^{q'}_{\mathrm{per}}(Q;\mathbb{R}^d)$.

In particular, consider

$$1 < q < \frac{N}{N-1}.$$

Since the space of all Q-periodic $\mathbb{R}^l$-valued Radon measures $\mathcal{M}_{\mathrm{per}}\left(Q;\mathbb{R}^l\right)$ is contained in $W^{-1,q}_{\mathrm{per}}(Q;\mathbb{R}^d)$, if $\mu \in \mathcal{M}_{\mathrm{per}}\left(Q;\mathbb{R}^l\right)$ then in view of (2.11), $\mathcal{S}_q\mu$ is well defined, and using Fubini's Theorem we may find the representation

$$\mathcal{S}_q\mu\,(x) = \int_Q K\,(x-y)\;d\mu\,(y)\,. \tag{2.12}$$

Indeed, if $\varphi \in C^{\infty}_{\mathrm{per}}(Q;\mathbb{R}^d)$ we have

$$\begin{aligned}\int_Q \overline{(\mathcal{S}_q\mu)\,(x)}\,\varphi\,(x)\;dx = \langle \mathcal{S}_q\mu, \varphi\rangle = \left\langle \mu, \mathcal{S}^*_{q'}\varphi\right\rangle &= \int_Q \mathcal{S}^*_{q'}\varphi\,(y)\;d\overline{\mu}\,(y)\\ &= \int_Q \sum_{\lambda\in\mathbb{Z}^N\setminus\{0\}} \overline{\mathbb{S}\,(\lambda)}\,\hat{\varphi}\,(\lambda)\,e^{2\pi i y\cdot\lambda}\,d\overline{\mu}\,(y)\\ &= \int_Q \left(\int_Q \sum_{\lambda\in\mathbb{Z}^N\setminus\{0\}} \overline{\mathbb{S}\,(\lambda)}\,e^{2\pi i (y-x)\cdot\lambda}\,d\overline{\mu}\,(y)\right)\varphi\,(x)\;dx\\ &= \int_Q \left(\overline{\int_Q K\,(x-y)\,d\mu\,(y)}\right)\varphi\,(x)\;dx,\end{aligned}$$

thus asserting (2.12).

We can now define the operator

$$\mathcal{T}_q : L^q_{\mathrm{per}}(Q;\mathbb{R}^d) \to L^q_{\mathrm{per}}(Q;\mathbb{R}^d)$$

as follows

$$\mathcal{T}_q v\,(x) := v - \mathcal{S}_q\mathcal{A}v.$$

When there is no possibility of confusion we write simply $\mathcal{S}$ and $\mathcal{T}$ in place of $\mathcal{S}_q$ and $\mathcal{T}_q$, respectively.

The following proposition may be found in [27].

Proposition 2.2 *$\mathcal{T}: L^q_{\mathrm{per}}(Q;\mathbb{R}^d) \to L^q_{\mathrm{per}}(Q;\mathbb{R}^d)$ is a bounded linear operator and $\mathcal{S}: W^{-1,q}_{\mathrm{per}}\left(Q;\mathbb{R}^l\right) \to L^q_{\mathrm{per}}\left(Q;\mathbb{R}^d\right)$ is a pseudo differential bounded operator of order -1 such that*

(i) if $v \in L^q_{\mathrm{per}}(Q;\mathbb{R}^d)$ then $\mathcal{T}\circ\mathcal{T}v = \mathcal{T}\,v$ and $\mathcal{A}\left(\mathcal{T}\,v\right) = 0$;

(ii) $||v-\mathcal{T}v||_{L^q} \leq C_q||\mathcal{A}\left(v\right)||_{W^{-1,q}}$ for all $v \in L^q_{\mathrm{per}}(Q;\mathbb{R}^d)$ such that $\int_Q v\,dx = 0$, for some $C_q > 0$;

(iii) $v - \mathcal{T}\,v = \mathcal{S}\mathcal{A}v$.

The next result is well-known to experts. We include a proof for the convenience of the reader.

Proposition 2.3 *Let $1 \leq p < \infty$, let $h \in L^p\left(\partial Q_r;\mathbb{R}^d\right)$, where $r \in \left(\frac{3}{4},1\right)$, and consider the measure*

$$\mu = h\ \mathcal{H}^{N-1}\lfloor_{\partial Q_r}.$$

Then for $s \in (0,1)$, $0 < \alpha \leq 1$, $\alpha \neq \frac{N-1}{p}$, we have

$$||\mathcal{S}\,\mu||_{L^t(\partial Q_s)} \leq C\left|s-r\right|^{-\alpha}||h||_{L^p(\partial Q_r)},$$

where

$$t := \begin{cases} \frac{p(N-1)}{N-1-\alpha p} & \text{if } \frac{1}{p}-\frac{\alpha}{N-1} > 0, \\ \infty & \text{if } \frac{1}{p}-\frac{\alpha}{N-1} < 0. \end{cases}$$

Proof. Consider now

$$\mu = h\ \mathcal{H}^{N-1}\lfloor_{\partial Q_r}.$$

We have

$$\mathcal{S}\mu\left(x\right) = \int_Q K\left(x-y\right)\,d\mu\left(y\right) = \int_{\partial Q_r} K\left(x-y\right)h\left(y\right)\,d\mathcal{H}^{N-1}\left(y\right).$$

For any $\alpha \in (0,1]$ there exists a constant $C > 0$ such that

$$\left|x-y\right|^{N-1} \geq C\left|r-s\right|^{\alpha}\left|\xi-\xi'\right|^{N-1-\alpha}$$

for all $x = s\xi \in \partial Q_s$ and $y = r\xi' \in \partial Q_r$, where $\xi, \xi' \in \partial Q$ (recall that $r \in \left(\frac{3}{4},1\right)$). Thus for $x = s\xi \in \partial Q_s$ we have

$$\left|\mathcal{S}\mu\left(x\right)\right| \leq C\left|r-s\right|^{-\alpha}\int_{\partial Q}\frac{\left|h_r\left(\xi'\right)\right|}{\left|\xi-\xi'\right|^{N-1-\alpha}}\,d\mathcal{H}^{N-1}\left(\xi'\right),$$

where

$$h_r\left(\xi'\right) = r^{N-1}h\left(r\xi'\right),$$

and we used (2.4). The conclusion follows from the standard convolution inequality for fractional integrals applied to the $(N-1)$-dimensional Lipschitz manifold ∂Q equipped with the distance induced by $\mathbb{R}^N$; see [45], I§8.21, for a very general version of fractional integration. For the case at hand one can of course use the classical argument on local charts (see also Hardy-Littlewood-Sobolev inequality in $\mathbb{R}^{N-1}$ [45], page 354). ■

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *$\mathcal{A}$-quasiconvex* if

$$f(\xi) \leq \int_Q f(\xi + w(y))\, dy$$

for all $\xi \in \mathbb{R}^d$ and all $w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$.

As it is usual, the regularity of the test function w maybe relaxed if f satisfies appropriate growth conditions.

Proposition 2.4 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an upper semicontinuous, $\mathcal{A}$-quasiconvex function, such that*

$$f(\xi) \leq C(1 + |\xi|^q) \tag{2.13}$$

for all $\xi \in R^d$, and for some $1 < q < \infty$ and $C > 0$. Then

$$f(\xi) \leq \int_Q f(\xi + w(y))\, dy$$

for all $\xi \in \mathbb{R}^d$ and all $w \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$.

Proof. Fix $\xi \in \mathbb{R}^d$ and let $w \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ be such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$. Then the functions

$$w_\varepsilon := \rho_\varepsilon * w - \int_Q \rho_\varepsilon * w\, dy$$

are in $C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$, $\mathcal{A}w_\varepsilon = 0$ and $\int_Q w_\varepsilon(y)\, dy = 0$. In view of (2.13), Fatou's Lemma and the upper semicontinuity of f imply

$$\liminf_{\varepsilon \to 0^+} \int_Q [C(1 + |w_\varepsilon|^q) - f(\xi + w_\varepsilon)]\, dy \geq \int_Q [C(1 + |w|^q) - f(\xi + w)]\, dy.$$

Since f is $\mathcal{A}$-quasiconvex it follows that

$$\int_Q f(\xi + w(y))\, dy \geq \limsup_{\varepsilon \to 0^+} \int_Q f(\xi + w_\varepsilon)\, dy \geq f(\xi),$$

and the proof is complete. ■

3 Proof of Theorem 1.4

In this section we prove Theorem 1.4 using the blow-up method. As it is usual, the main effort will target the case where the limit function v reduces to a constant.

Proposition 3.1 *Let $g : \mathbb{R}^N \to [0, \infty)$ be an $\mathcal{A}$-quasiconvex function such that*

$$|g(\xi) - g(\xi_1)| \leq C\left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right)|\xi - \xi_1|, \tag{3.1}$$

for all $\xi, \xi_1 \in \mathbb{R}^d$ and for some $C > 0$, where $1 \leq q < \infty$ satisfies (1.7). Then

$$g(0) \leq \liminf_{n\to\infty} \int_Q g(v_n(x))\, dx$$

for any sequence $\{v_n\} \subset L^1(Q; \mathbb{R}^d) \cap \ker \mathcal{A}$ converging weakly-$*$ to zero in the sense of measures.

Proof. By a simple mollification argument and by passing to a subsequence if necessary, without loss of generality we may assume that $\{v_n\} \subset C^\infty\left(\overline{Q}; \mathbb{R}^d\right) \cap \ker \mathcal{A}$,

$$C_0 := \sup_n \int_Q |v_n(x)|\, dx < \infty, \tag{3.2}$$

$$\liminf_{n\to\infty} \int_Q g(v_n(x))\, dx = \lim_{n\to\infty} \int_Q g(v_n(x))\, dx, \tag{3.3}$$

and there exists a nonnegative Radon measure μ such that

$$|v_n(x)|\, \mathcal{L}^N \lfloor Q \overset{*}{\rightharpoonup} \mu \tag{3.4}$$

as $n \to \infty$, weakly $*$ in the sense of measures. Fix $\delta > 0$. By $(3.2)_1$ there exist

$$E_n \subset (1 - 2\delta, 1 - \delta), \quad \mathcal{L}^1(E_n) = \frac{\delta}{2}$$

such that

$$\mu(\partial Q_r) = 0 \tag{3.5}$$

and

$$\int_{\partial Q_r} |v_n|\, d\mathcal{H}^{N-1} \leq C_1(\delta) := \frac{4C_0}{\delta} \tag{3.6}$$

for all $r \in E_n$. Fix $r \in E_n$ and let

$$w_{n,r} := \chi_r v_n - \int_{Q_r} v_n\, dy$$

By the $\mathcal{A}$-quasiconvexity of g and as $g \geq 0$, we have

$$\begin{aligned}
\int_Q g(v_n)\, dx &\geq \int_Q g(\mathcal{T}(w_{n,r}))\, dx + \int_{Q_r} g(v_n)\, dx - \int_Q g(\mathcal{T}(w_{n,r}))\, dx \\
&\geq g(0)\,|Q| + \int_{Q_r} g(v_n)\, dx - \int_Q g(\mathcal{T}(w_{n,r}))\, dx \\
&= g(0)\,|Q_r| + \int_Q [g(\chi_r v_n) - g(\mathcal{T}(w_{n,r}))]\, dx,
\end{aligned} \tag{3.7}$$

where χ_r is the characteristic function of the set Q_r and where we have used Proposition 2.4. By (3.1) we have

$$\begin{aligned}&\int_Q [g(\chi_r v_n) - g(\mathcal{T}(w_{n,r}))]\,dx\\&\quad\leq C\int_Q \left(1+|\chi_r v_n|^{q-1}+|\mathcal{T}(w_{n,r})|^{q-1}\right)|\chi_r v_n - \mathcal{T}(w_{n,r})|\,dx\\&\quad\leq C\int_Q |\chi_r v_n - \mathcal{T}(w_{n,r})|^q\,dx + C\int_Q\left(1+|\chi_r v_n|^{q-1}\right)|\chi_r v_n - \mathcal{T}(w_{n,r})|\,dx.\end{aligned}$$

Hence from (3.7) we have

$$\begin{aligned}\int_Q g(v_n)\,dx \geq g(0)\,|Q_{1-2\delta}| &- C\int_Q |\chi_r v_n - \mathcal{T}(w_{n,r})|^q\,dx\\&- C\int_Q\left(1+|\chi_r v_n|^{q-1}\right)|\chi_r v_n - \mathcal{T}(w_{n,r})|\,dx.\end{aligned}$$

Multiply the previous inequality by χ_{E_n} and integrate in r to obtain

$$\begin{aligned}\frac{\delta}{2}\int_Q g(v_n)\,dx \geq & g(0)\,|Q_{1-2\delta}|\,\frac{\delta}{2}\\&- C\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_Q |\chi_r v_n - \mathcal{T}(w_{n,r})|^q\,dx\,dr\\&- C\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_Q\left(1+|\chi_r v_n|^{q-1}\right)|\chi_r v_n - \mathcal{T}(w_{n,r})|\,dx\,dr,\end{aligned}\tag{3.8}$$

where we have used the fact that $\mathcal{L}^1(E_n) = \frac{\delta}{2}$. By (1.7) we may choose q_1 such that

$$q_1 := \begin{cases} q & \text{if } q > 1,\\ \in \left(1, \frac{N}{N-1}\right) & \text{if } q = 1.\end{cases}\tag{3.9}$$

We claim that

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\left|\int_{Q_r} v_n\,dy\right|^{q_1}\,dr = 0.\tag{3.10}$$

Indeed, fix $r \in (1-2\delta, 1-\delta)$. If $\mu(\partial Q_r) > 0$ then $\chi_{E_n}(r) = 0$ for all n by (3.5), while if $\mu(\partial Q_r) = 0$ then $\int_Q \chi_r v_n dy \to 0$ by Theorem 1.62 in [5], and because $v_n \overset{*}{\rightharpoonup} 0$ in the sense of measures. The claim now follows from (3.2) by Lebesgue Dominated Convergence Theorem.

Next we show that

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_Q |\chi_r v_n - \mathcal{T}(w_{n,r})|^{q_1}\,dx\,dr = 0,\tag{3.11}$$

or, equivalently by (3.10),

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_Q |w_{n,r} - \mathcal{T}(w_{n,r})|^{q_1}\,dx\,dr = 0.\tag{3.12}$$

By Proposition 2.2(ii) we have

$$\left\| w_{n,r} - \mathcal{T}\left(w_{n,r}\right)\right\|_{L^{q_1}(Q)} \leq C \left\| \mathcal{A}\left(v_n \chi_r\right)\right\|_{W^{-1,q_1}(Q)},$$

and thus to prove (3.12) it suffices to show that

$$\int_{1-2\delta}^{1-\delta} \chi_{E_n} \left\| \mathcal{A}\left(v_n \chi_r\right)\right\|_{W^{-1,q_1}(Q)}^{q_1} dr \to 0. \tag{3.13}$$

Fix $\psi \in C_c^\infty(Q;\mathbb{R}^l)$. Using (3.6), if $r \in E_n$ then we deduce that

$$\begin{aligned} \left|\left\langle \mathcal{A}\left(\chi_r v_n\right), \psi\right\rangle\right| &= \left| \sum_{i=1}^N \int_Q A^{(i)} \chi_r v_n \frac{\partial \psi}{\partial x_i}\, dx \right| \\ &= \left| - \int_{\partial Q_r} \mathbb{A}\left(\nu_r\right) v_n \psi \, d\mathcal{H}^{N-1} \right| \\ &\leq C \int_{\partial Q_r} \left|v_n\right| d\mathcal{H}^{N-1} \left\|\psi\right\|_{L^\infty(Q;\mathbb{R}^l)} \leq C \left\|\psi\right\|_{L^\infty(Q;\mathbb{R}^l)}. \end{aligned} \tag{3.14}$$

Hence

$$\chi_{E_n} \left\| \mathcal{A}\left(\chi_r v_n\right)\right\|_{\mathcal{M}(Q;\mathbb{R}^l)} \leq C \tag{3.15}$$

for all n and $r \in (1-2\delta, 1-\delta)$. We now show that

$$\chi_{E_n} \mathcal{A}\left(\chi_r v_n\right) \overset{*}{\rightharpoonup} 0$$

in the sense of measures. Fix $\psi \in C_c^\infty(Q;\mathbb{R}^l)$ and $r \in (1-2\delta, 1-\delta)$. If $\mu(\partial Q_r) > 0$ then $\chi_{E_n}(r) = 0$ for all n by (3.5). Thus assume $\mu(\partial Q_r) = 0$. Since $v_n \in \ker \mathcal{A}$, we have, by Theorem 1.62 in [5] and the fact that $v_n \overset{*}{\rightharpoonup} 0$ in the sense of measures,

$$\left\langle \mathcal{A}\left(\chi_r v_n\right), \psi\right\rangle = -\sum_{i=1}^N \int_Q A^{(i)} \chi_r v_n \frac{\partial \psi}{\partial x_i}\, dx \to 0. \tag{3.16}$$

Therefore, $\{\chi_{E_n} \mathcal{A}\left(\chi_r v_n\right)\}$ is a bounded sequence of $\mathbb{R}^l$-valued Radon measures converging weak-$*$ to zero. Since $\mathcal{M}\left(Q;\mathbb{R}^l\right)$, the space of all $\mathbb{R}^l$-valued Radon measures, is compactly embedded in $W^{-1,q_1}(Q;\mathbb{R}^l)$, we deduce that

$$\chi_{E_n} \mathcal{A}\left(\chi_r v_n\right) \to 0 \text{ in } W^{-1,q_1}(Q;\mathbb{R}^l) \text{ as } n \to \infty$$

for all $r \in (1-2\delta, 1-\delta)$, with

$$\chi_{E_n} \left\| \mathcal{A}\left(v_n \chi_r\right)\right\|_{W^{-1,q_1}(Q)} \leq C$$

for all n and $r \in (1-2\delta, 1-\delta)$. By Lebesgue Dominated Convergence Theorem, we obtain (3.13), and, in turn, (3.11).

Finally, we prove that

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_{Q_r}|v_n|^{q-1}\,|v_n-\mathcal{T}\,(w_{n,r})|\;dx\,dr=0. \tag{3.17}$$

If $q=1$ then this is a consequence of (3.11). Thus, without loss of generality, we may assume that $q>1$. We begin by showing that

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\left(\int_{Q_r}|v_n|^{q-1}\,dx\right)\left|\int_{Q_r}v_n\,dy\right|\;dx\,dr=0. \tag{3.18}$$

Indeed, since $q\leq 2$, by $(3.2)_1$ we have

$$\int_{1-2\delta}^{1-\delta}\chi_{E_n}\left(\int_{Q_r}|v_n|^{q-1}\,dx\right)\left|\int_{Q_r}v_n\,dy\right|\;dx\,dr\leq C\int_{1-2\delta}^{1-\delta}\chi_{E_n}\left|\int_{Q_r}v_n\,dy\right|\;dx\,dr,$$

and thus (3.18) follows from (3.10). In view of (3.18), proving (3.17) is equivalent to showing that

$$\lim_{n\to\infty}\int_{1-2\delta}^{1-\delta}\chi_{E_n}\int_{Q_r}|v_n|^{q-1}\,|w_{n,r}-\mathcal{T}\,(w_{n,r})|\;dx\,dr=0. \tag{3.19}$$

Now, if $\varepsilon\in(0,1)$ then we have

$$\begin{aligned}
&\int_{E_n}\int_{Q_r}|v_n|^{q-1}\,|w_{n,r}-\mathcal{T}\,(w_{n,r})|\;dx\,dr\\
&\quad=\int_{E_n}\int_{Q_r}|v_n|^{q-1}|w_{n,r}-\mathcal{T}\,(w_{n,r})\,|^{1-\varepsilon}|w_{n,r}-\mathcal{T}\,(w_{n,r})\,|^{\varepsilon}\,dx\,dr \qquad (3.20)\\
&\quad\leq\left(\int_{E_n}\int_{Q_r}|v_n|^{\frac{q-1}{1-\varepsilon}}|w_{n,r}-\mathcal{T}\,(w_{n,r})\,|\,dx\,dr\right)^{1-\varepsilon}\\
&\quad\quad\times\left(\int_{E_n}\int_{Q_r}|w_{n,r}-\mathcal{T}\,(w_{n,r})\,|\,dx\,dr\right)^{\varepsilon},
\end{aligned}$$

where we used Hölder's inequality with exponents $1/(1-\varepsilon)$ and $1/\varepsilon$. By (3.12) the second factor on the right hand side of the previous inequality converges to zero as $n\to\infty$, hence to prove (3.19), and thus (3.17), it remains to show that

$$\sup_n\int_{E_n}\int_{Q_r}|v_n|^{\frac{q-1}{1-\varepsilon}}|w_{n,r}-\mathcal{T}\,(w_{n,r})\,|\,dx\,dr<\infty.$$

In light of (3.14), and since $\mathcal{A}\,(v_n)=0$, we may identify $\mathcal{A}\,(w_{n,r})=\mathcal{A}\,(\chi_r v_n)$ with the measure

$$\mu_{r,n}:=-\mathbb{A}\,(\nu_r)\,v_n\;\;\mathcal{H}^{N-1}\lfloor_{\partial Q_r}.$$

Hence by Proposition 2.2(iii)

$$w_{n,r}-\mathcal{T}\;(w_{n,r})=\mathcal{S}\mathcal{A}\,(\chi_r v_n)=\mathcal{S}\mu_{r,n}. \tag{3.21}$$

Note that $0 < (q-1)(N-1) < 1$ and let

$$\alpha \in ((q-1)(N-1), 1), \quad t := \frac{N-1}{N-1-\alpha}. \tag{3.22}$$

Then

$$t > \frac{N-1}{N-1-(q-1)(N-1)} = \frac{1}{2-q}, \quad t' < \frac{1}{q-1}.$$

Using Hölder's inequality, Proposition 2.3 with $p=1$, (3.6), and (3.21), we have

$$\begin{aligned}
&\int_{E_n} \int_{Q_r} |v_n|^{\frac{q-1}{1-\varepsilon}} |w_{n,r} - \mathcal{T}(w_{n,r})| \, dx \, dr \\
&\quad = \int_{E_n} \int_0^r \int_{\partial Q_s} |v_n|^{\frac{q-1}{1-\varepsilon}} |w_{n,r} - \mathcal{T}(w_{n,r})| \, d\mathcal{H}^{n-1} \, ds \, dr \\
&\quad \le \int_{E_n} \int_0^r \left\| |v_n|^{\frac{q-1}{1-\varepsilon}} \right\|_{L^{t'}(\partial Q_s)} \|w_{n,r} - \mathcal{T}(w_{n,r})\|_{L^t(\partial Q_s)} \, ds \, dr \\
&\quad \le C \int_{E_n} \int_0^r \left\| |v_n|^{\frac{q-1}{1-\varepsilon}} \right\|_{L^{t'}(\partial Q_s)} (r-s)^{-\alpha} ||v_n||_{L^1(\partial Q_r)} \, ds dr \\
&\quad \le C \int_0^1 \left\| |v_n|^{\frac{q-1}{1-\varepsilon}} \right\|_{L^{t'}(\partial Q_s)} \int_{E_n \cap [s,1]} (r-s)^{-\alpha} \, dr \, ds \\
&\quad \le C \int_0^1 \left\| |v_n|^{\frac{q-1}{1-\varepsilon}} \right\|_{L^{t'}(\partial Q_s)} \, ds \\
&\quad \le C \left(\int_0^1 \left\| |v_n|^{\frac{q-1}{1-\varepsilon}} \right\|_{L^{t'}(\partial Q_s)}^{t'} \, ds \right)^{1/t'} = C \left(\int_Q |v_n|^{\frac{q-1}{1-\varepsilon} t'} \, dx \right)^{1/t'},
\end{aligned}$$

which remains bounded as $n \to \infty$, since $(q-1)\,t' < 1$ we may choose $\varepsilon := 1 - (q-1)\,t'$. Hence (3.17) holds.

By (3.11) and (3.17), letting $n \to \infty$ in (3.8) yields

$$\frac{\delta}{2} \liminf_{n \to \infty} \int_Q g(v_n) \, dx \ge g(0) \, |Q_{1-2\delta}| \, \frac{\delta}{2},$$

and to conclude the proof it suffices to divide the previous inequality by $\frac{\delta}{2}$ and then let $\delta \to 0^+$. ■

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Without loss of generality we may assume that

$$\liminf_{n \to \infty} \int_\Omega f(x, v_n(x)) \, dx = \lim_{n \to \infty} \int_\Omega f(x, v_n(x)) \, dx < \infty.$$

Passing to a subsequence, if necessary, we find a nonnegative Radon measure μ such that

$$f(x, v_n(x)) \, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly $*$ in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{r \to 0^+} \frac{\mu(Q(x_0, r))}{r^N} \geq f(x_0, v(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega. \tag{3.23}$$

If (3.23) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega; \mathbb{R})$, $0 \leq \varphi \leq 1$. Since

$$\mu = \frac{d\mu}{d\mathcal{L}^N} \mathcal{L}^N + \mu_s$$

where $\mu_s \geq 0$, we have

$$\begin{aligned} \lim_{n \to \infty} \int_\Omega f(x, v_n)\, dx &\geq \liminf_{n \to \infty} \int_\Omega \varphi(x)\; f(x, v_n)\, dx = \int_\Omega \varphi\, d\mu \\ &\geq \int_\Omega \varphi \frac{d\mu}{d\mathcal{L}^N}\, dx \geq \int_\Omega \varphi\, f(x, v)\, dx. \end{aligned}$$

By letting $\varphi \to 1^-$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem, it suffices to show (3.23).

Let

$$v := \frac{d\lambda}{d\mathcal{L}^N} \in L^1\left(\Omega; \mathbb{R}^d\right),$$

and fix $x_0 \in \Omega$ such that

$$\begin{gathered} \frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{r \to 0^+} \frac{\mu(Q(x_0, r))}{r^N} < \infty, \\ \lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0, r)} |v(x) - v(x_0)|\, dx = 0, \qquad \lim_{r \to 0^+} \frac{|\lambda_s|(Q(x_0, r))}{r^N} = 0. \end{gathered} \tag{3.24}$$

Choosing $r_k \searrow 0$ such that $\mu(\partial Q(x_0, r_k)) = 0$, we have

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k \to \infty} \frac{\mu(Q(x_0, r_k))}{r_k^N} = \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{r_k^N} \int_{Q(x_0, r_k)} f(x, v_n)\, dx \\ &= \lim_{k \to \infty^+} \lim_{n \to \infty} \int_Q f(x_0 + r_k y, v(x_0) + w_{n,k}(y))\, dy, \end{aligned}$$

where $w_{n,k}(y) := v_n(x_0 + r_k y) - v(x_0)$. Clearly $w_{n,k} \in \ker \mathcal{A}$, and we claim that $w_{n,k} \overset{*}{\rightharpoonup} 0$ weakly-$*$ in the sense of measures if we first let $n \to \infty$ and then $k \to \infty$. Indeed, fix $\varphi \in C_c(Q; \mathbb{R}^d)$. After a change of variables, we get

$$\begin{aligned} \int_Q \varphi(y)\, w_{n,k}(y)\, dy &= \int_Q \varphi(y)(v_n(x_0 + r_k y) - v(x_0))\, dy \\ &= \frac{1}{r_k^N} \int_{Q(x_0, r_k)} \varphi\left(\frac{x - x_0}{r_k}\right)(v_n(x) - v(x_0))\, dx. \end{aligned}$$

If we now let $n \to \infty$, and use the facts that $v_n \overset{*}{\rightharpoonup} \lambda$ weakly-$*$ in the sense of measures and that

$$\lambda = v\,\mathcal{L}^N \lfloor_{\Omega} + \lambda_s,$$

we obtain that

$$\begin{aligned}\lim_{n\to\infty} \int_Q \varphi(y)\, w_{n,k}(y)\, dy =& \frac{1}{r_k^N} \int_{Q(x_0,r_k)} \varphi\left(\frac{x-x_0}{r_k}\right)(v(x)-v(x_0))\, dx \\ &+ \frac{1}{r_k^N} \int_{Q(x_0,r_k)} \varphi\left(\frac{x-x_0}{r_k}\right) d\lambda_s.\end{aligned}$$

Hence

$$\begin{aligned}\left|\lim_{n\to\infty} \int_Q \varphi(y)\, w_{n,k}(y)\, dy\right| \leq& ||\varphi||_{L^\infty(Q)} \frac{1}{r_k^N} \int_{Q(x_0,r_k)} |v(x)-v(x_0)|\, dx \\ &+ ||\varphi||_{L^\infty(Q)} \frac{|\lambda_s|\,(Q(x_0,r_k))}{r_k^N}.\end{aligned}$$

The claim then follows by letting $k \to \infty$ and by using (3.24). Diagonalize to get $w_k \in L^1(Q;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $w_k \overset{*}{\rightharpoonup} 0$ weakly-$*$ in the sense of measures, and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty} \int_Q f(x_0 + r_k y, v(x_0) + w_k(y))\, dy$$

where $r_k \to 0$. Fix $\varepsilon > 0$. By (1.9) and Proposition 3.1 we have

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \frac{1}{1+\varepsilon} \lim_{k\to\infty} \int_Q f(x_0, v(x_0) + w_k(y))\, dy - \frac{\varepsilon}{1+\varepsilon} \\ &\geq \frac{1}{1+\varepsilon} f(x_0, v(x_0)) - \frac{\varepsilon}{1+\varepsilon}.\end{aligned}$$

It now suffices to let $\varepsilon \to 0^+$. ■

4 Proof of Theorem 1.6

In this section we prove Theorem 1.6. We begin with the following

Proposition 4.1 *Let $g : \mathbb{R}^N \to [0,\infty)$ be an $\mathcal{A}$-quasiconvex function such that*

$$|g(\xi) - g(\xi_1)| \leq C\left(1 + |\xi|^{q-1} + |\xi_1|^{q-1}\right)|\xi - \xi_1|, \tag{4.1}$$

for all $\xi, \xi_1 \in \mathbb{R}^d$, and for some $1 < q < \infty$ and $C > 0$. If $\{v_n\} \subset L^q\left(Q;\mathbb{R}^d\right) \cap \ker \mathcal{A}$ converges weakly to zero in $L^p\left(Q;\mathbb{R}^d\right)$, where $1 < p < \infty$ satisfies (1.11), then

$$g(0) \leq \liminf_{n\to\infty} \int_Q g(v_n(x))\, dx.$$

Proof. The proof of this proposition follows closely that of Proposition 3.1, therefore we indicate only the main modifications. Condition $(3.2)_1$ should be replaced by

$$C_0 := \sup_n \int_Q |v_n(x)|^p \, dx < \infty,$$

and, correspondingly, (3.6) by

$$\int_{\partial Q_r} |v_n|^p \, d\mathcal{H}^{N-1} \le C_1(\delta) \tag{4.2}$$

for all $r \in E_n$. Conditions (3.4) and (3.5) are no longer needed, while the exponent q_1 in (3.9) is set to be equal to q. Equality (3.10) now follows immediately since $\int_Q \chi_r v_n dy \to 0$ as $v_n \rightharpoonup 0$ in $L^p\left(Q;\mathbb{R}^d\right)$ for any $r \in (1-2\delta, 1-\delta)$. To prove (3.13), fix $\psi \in C_c^\infty(Q;\mathbb{R}^l)$. Since $v_n \in \ker\mathcal{A}$ and $v_n \rightharpoonup 0$ in $L^p\left(Q;\mathbb{R}^d\right)$, we have

$$\langle \mathcal{A}(\chi_r v_n), \psi\rangle = \sum_{i=1}^N \int_Q A^{(i)} \chi_r v_n \frac{\partial \psi}{\partial x_i}\, dx \to 0, \tag{4.3}$$

and if $r \in E_n$, and by (4.2),

$$\begin{aligned} |\langle \mathcal{A}(\chi_r v_n), \psi\rangle| &= \left| \sum_{i=1}^N \int_Q A^{(i)} \chi_r v_n \frac{\partial \psi}{\partial x_i}\, dx \right| = \left| -\int_{\partial Q_r} \mathbb{A}(\nu_r)\, v_n \psi \, d\mathcal{H}^{N-1} \right| \\ &\le C \left(\int_{\partial Q_r} |v_n|^p \, d\mathcal{H}^{N-1} \right)^{1/p} \|\psi\|_{L^{p'}(\partial Q_r;\mathbb{R}^l)} \le C \|\psi\|_{L^{p'}(\partial Q_r;\mathbb{R}^l)}. \end{aligned}$$

Hence,

$$\chi_{E_n} \|\mathcal{A}(\chi_r v_n)\|_{L^p(\partial Q_r;\mathbb{R}^l)} \le C \tag{4.4}$$

for all n and $r \in (1-2\delta, 1-\delta)$.

We recall that Sobolev Compact Embedding Theorem we have

$$W_0^{1,q'}\left(Q;\mathbb{R}^l\right) \hookrightarrow L^s\left(\partial Q_r;\mathbb{R}^l\right),$$

where

$$s < \begin{cases} \dfrac{(N-1)\,q'}{N-q'} & \text{if } q' < N, \\ \infty & \text{if } q' \ge N. \end{cases}$$

Thus, (1.11) yields that the Sobolev space $W_0^{1,q'}\left(Q;\mathbb{R}^l\right)$ is compactly embedded in $L^{p'}\left(\partial Q_r;\mathbb{R}^l\right)$, and by duality we have $L^p\left(\partial Q_r;\mathbb{R}^l\right)$ compactly embedded in $W^{-1,q}\left(Q;\mathbb{R}^l\right)$, which, together with (4.3) and (4.4) implies that

$$\chi_{E_n} \mathcal{A}(\chi_r v_n) \to 0 \text{ in } W^{-1,q}(Q;\mathbb{R}^l) \text{ as } n \to \infty$$

for all $r \in (1-2\delta, 1-\delta)$, with

$$\chi_{E_n} \|\mathcal{A}(v_n \chi_r)\|_{W^{-1,q}(Q)} \le C$$

for all n and $r \in (1-2\delta, 1-\delta)$. By Lebesgue Dominated Convergence Theorem we obtain (3.13), and, in turn, (3.11).

To prove (3.17), in place of (3.22) we take

$$\alpha \in \left(\frac{(q-p)\,(N-1)}{p}, 1 \right), \quad t := \frac{p\,(N-1)}{N-1-\alpha p},$$

where, without loss of generality, we are assuming $q > p$ (see [27] for the case $p = q$). Then

$$t > \frac{p}{p-q+1}, \quad t' < \frac{p}{q-1},$$

where we have used the fact that $p - q + 1 > 0$ by (1.11).

We may now proceed exactly as before, with the only exception that now we have $(q-1)\,t' < p$. Hence taking

$$1 - \varepsilon = \frac{q-1}{p} t'$$

we conclude. ■

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. We proceed as in the proof of Theorem 1.4, until (3.24) which should be replaced by

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{r\to 0^+} \frac{\mu(Q(x_0,r))}{r^N} < \infty, \quad \lim_{r\to 0^+} \frac{1}{r^N} \int_{Q(x_0,r)} |v(x) - v(x_0)|^p dx = 0. \tag{4.5}$$

As in Theorem 1.4 we let $w_{n,k}(y) := v_n(x_0 + r_k y) - v(x_0)$. We claim that $w_{n,k} \rightharpoonup 0$ in $L^p(Q;\mathbb{R}^d)$ if we first let $n \to \infty$ and then $k \to \infty$. Indeed, fix $\varphi \in L^{p'}(Q;\mathbb{R}^d)$, where p' is the Hölder conjugate exponent of q. Using Hölder's inequality and then making a change of variables, we get

$$\begin{aligned}
\left| \int_Q \varphi(y)\, w_{n,k}(y)\, dy \right| &\le \left| \int_Q \varphi(y)(v_n(x_0 + r_k y) - v(x_0 + r_k y))\, dy \right| \\
&\quad + \left| \int_Q \varphi(y)(v(x_0 + r_k y) - v(x_0))\, dy \right| \\
&\le \left| \frac{1}{r_k^N} \int_{Q(x_0,r_k)} \varphi\left(\frac{x - x_0}{r_k} \right) (v_n(x) - v(x))\, dx \right| \\
&\quad + ||\varphi||_{L^{p'}(Q)} \left(\frac{1}{r_k^N} \int_{Q(x_0,r_k)} |v(x) - v(x_0)|^p\, dx \right)^{1/p}.
\end{aligned}$$

If we now let $n \to \infty$ the first integral tends to zero due to the fact that $v_n \rightharpoonup v$ in $L^p(Q(x_0, r_k);\mathbb{R}^d)$. The claim then follows by letting $k \to \infty$ and by using (4.5). Diagonalize to get $w_k \in L^q(Q;\mathbb{R}^d) \cap \ker \mathcal{A}$ such that $w_k \rightharpoonup 0$ in $L^q(Q;\mathbb{R}^d)$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty} \int_Q f(x_0 + r_k y, v(x_0) + w_k(y))\, dy$$

where $r_k \to 0$. We may now continue as in the proof of Theorem 1.4 using Proposition 4.1 in place of Proposition 3.1. ■

5 Proof of Theorem 1.8

Finally, we prove Theorems 1.8.

Proof. For any function $v \in L^p(\Omega; E_s^m)$ consider the differential operator $\mathcal{A}$

$$\mathcal{A}v := \left(\frac{\partial}{\partial x_i} v_{i_1 \ldots i_h j i_{h+2} \ldots i_s} - \frac{\partial}{\partial x_j} v_{i_1 \ldots i_h i i_{h+2} \ldots i_s} \right)_{0 \le h \le s-1,\, 1 \le i, j, i_1 \ldots i_s \le N}.$$

Here $h = 0$ and $h = s-1$ correspond to the multi-indexes $j i_2 \ldots i_s$ and $i_1 \ldots i_{s-1} j$. As shown in [27],

$$\left\{ w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; E_s^m) : \mathcal{A}w = 0, \quad \int_Q w \, dx = 0 \right\} = \left\{ \nabla^s \varphi : \varphi \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^m) \right\}. \tag{5.1}$$

Since for a.e. $x \in \Omega$ and for all $v \in E_s^m$,

$$f(x, v) = \inf \left\{ \int_Q f(x, v + \nabla^s \varphi(y)) \, dy : \varphi \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^N) \right\},$$

it follows from (5.1) that

$$f(x, v) = \inf \left\{ \int_Q f(x, v + w(y)) \, dy : w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N; E_s^m) \cap \ker \mathcal{A}, \int_Q w(y) \, dy = 0 \right\}$$

and thus f is $\mathcal{A}$-quasiconvex. Let $\{u_k\} \subset W^{s,q}(\Omega; \mathbb{R}^m)$ be any sequence such that $u_k \rightharpoonup u$ in $W^{s,p}(\Omega; \mathbb{R}^m)$. Again by (5.1) $\mathcal{A}\nabla^s u_k = 0$, and so we may apply Theorem 1.6, where the target space $\mathbb{R}^d$ is replaced by the finite dimensional Euclidean vector space E_s^m, to obtain

$$\int_\Omega f(x, \nabla^s u) \, dx \le \liminf_{k \to \infty} \int_\Omega f(x, \nabla^s u_k) \, dx.$$

■

6 Appendix

Proof of Theorem 1.1. Using the blow-up method as in Theorem 1.4, we may assume, without loss of generality, that

$$\Omega = Q := \left(-\frac{1}{2}, \frac{1}{2} \right)^N \quad \text{and} \quad u(x) \equiv 0.$$

As $u_n \to 0$ in $L^1\left(Q;\mathbb{R}^d\right)$, by Egoroff's and Fubini's Theorems for any $\delta \in (0,1)$ we may find a subsequence of $\{u_n\}$ (not relabelled) such that for a.e. $r \in (\delta, 1)$

$$\lim_{n\to\infty} \int_{\partial Q_r} |u_n| \, d\mathcal{H}^{N-1} = 0.$$

Since $L^1\left(\partial Q_r \cup \partial Q;\mathbb{R}^d\right)$ is the trace space of $W^{1,1}\left(Q\backslash Q_r;\mathbb{R}^d\right)$, we may find $\{v_n\} \subset W^{1,1}\left(Q\backslash Q_r;\mathbb{R}^d\right)$ such that $v_n = u_n$ on ∂Q_r and $v_n = 0$ on ∂Q (in the sense of traces) and

$$\|v_n\|_{W^{1,1}(Q\backslash Q_r;\mathbb{R}^d)} \leq K_r \|u_n\|_{L^1(\partial Q_r;\mathbb{R}^d)}$$

for some constant $K_r > 0$. We have

$$\begin{aligned}\int_{Q\backslash Q_r} f(\nabla v_n)\, dx &\leq \int_{Q\backslash Q_r} C\left(1 + |\nabla v_n|\right) dx \\ &\leq C\mathcal{L}^N\left(Q\backslash Q_r\right) + CK_r \|u_n\|_{L^1(\partial Q_r;\mathbb{R}^d)}.\end{aligned}$$

If we define v_n to be u_n in Q_r then $\{v_n\} \subset W_0^{1,1}\left(Q;\mathbb{R}^d\right)$ and thus by the quasiconvexity of f we have

$$\begin{aligned}f(0) &\leq \int_Q f(\nabla v_n)\, dx = \int_{Q\backslash Q_r} f(\nabla v_n)\, dx + \int_{Q_r} f(\nabla u_n)\, dx \\ &\leq C\mathcal{L}^N\left(Q\backslash Q_r\right) + CK_r \|u_n\|_{L^1(\partial Q_r;\mathbb{R}^d)} + \int_Q f(\nabla u_n)\, dx\end{aligned}$$

and letting $n \to \infty$ we conclude that

$$f(0) \leq C\mathcal{L}^N\left(Q\backslash Q_r\right) + \liminf_{n\to\infty} \int_Q f(\nabla u_n)\, dx.$$

It now suffices to let $\delta \to 1^-$ (and hence r). ∎

Proof of Theorem 1.2. We consider only the case $1 < p < \infty$. As in the previous proof, we may assume, without loss of generality, that

$$\Omega = Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^N \quad \text{and} \quad u(x) \equiv 0,$$

and

$$\liminf_{n\to\infty} \int_Q f(\nabla u_n)\, dx = \lim_{n\to\infty} \int_Q f(\nabla u_n)\, dx < \infty,$$

so that by condition (1.4)

$$K := \sup_n \int_Q |\nabla u_n|^p \, dx < \infty.$$

Fix $\delta \in (0,1)$. By Egoroff's and Fubini's Theorems, we may find a subsequence (not relabeled) such that for a.e. $r \in (0,1)$

$$\lim_{n\to\infty} \int_{\partial Q_r} |u_n|^p \, d\mathcal{H}^{N-1} = 0.$$

Define

$$\mathcal{R} := \left\{ r \in (\delta,1) : \lim_{n\to\infty} \int_{\partial Q_r} |u_n|^p \, d\mathcal{H}^{N-1} = 0, \right.$$
$$\left. \liminf_{n\to\infty} \int_{\partial Q_r} |\nabla u_n|^p \, d\mathcal{H}^{N-1} \leq \frac{2K}{1-\delta} \right\}.$$

Note that by Fatou's Lemma

$$\begin{aligned} K &\geq \liminf_{n\to\infty} \int_{(\delta,1)\setminus\mathcal{R}} \int_{\partial Q_r} |\nabla u_n|^p \, d\mathcal{H}^{N-1} dr \\ &\geq \int_{(\delta,1)\setminus\mathcal{R}} \liminf_{n\to\infty} \int_{\partial Q_r} |\nabla u_n|^p \, d\mathcal{H}^{N-1} dr \\ &\geq \mathcal{L}^1((\delta,1)\setminus\mathcal{R}) \frac{2K}{1-\delta}, \end{aligned}$$

and so $\mathcal{L}^1(\mathcal{R}) \geq (1-\delta)/2$.

Fix $r \in \mathcal{R}$. Since $p > \frac{N-1}{N} q$, standard Sobolev trace and compact embedding theorems guarantee the existence of a lifting linear and compact operator

$$\begin{aligned} E :& W^{1,p}(\partial Q_r;\mathbb{R}^d) \to W^{1,q}(Q;\mathbb{R}^d) \\ & v \longmapsto E(v) \end{aligned}$$

such that v is the trace of $E(v)$. Define $\{v_n\} \subset W_0^{1,p}(Q;\mathbb{R}^d)$ by

$$v_n(x) := \begin{cases} u_n(x) & \text{if } x \in Q_r \\ \varphi(x) E(u_n)(x) & \text{if } x \in Q\setminus Q_r, \end{cases}$$

where $\varphi \in C_c^1(Q;[0,1])$ is such that $\varphi(x) \equiv 1$ in Q_r and $|\nabla\varphi| \leq \frac{C}{1-r}$. As $E(u_n) \to 0$ in $W^{1,q}(Q\setminus Q_r;\mathbb{R}^d)$, by condition (1.4) we have

$$\lim_{n\to\infty} \int_{Q\setminus Q_r} f(\nabla v_n)\, dx = \int_{Q\setminus Q_r} f(0)\, dx.$$

Hence, using the quasiconvexity of f at 0 we obtain

$$\begin{aligned} \liminf_{n\to\infty} \int_Q f(\nabla u_n)\, dx &\geq \liminf_{n\to\infty} \int_{Q_r} f(\nabla v_n)\, dx \\ &= \liminf_{n\to\infty} \int_Q f(\nabla v_n)\, dx - \lim_{n\to\infty} \int_{Q\setminus Q_r} f(\nabla v_n)\, dx \\ &\geq f(0) - \mathcal{L}^N(Q\setminus Q_r) f(0) \\ &= \mathcal{L}^N(Q_r) f(0), \end{aligned}$$

and the proof is complete if we let $\delta \to 1^-$ (and hence r). ■

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References

[1] E. Acerbi, G. Bouchitté and I. Fonseca, Relaxation of convex functionals and the Lavrentiev phenomenon, to appear.

[2] E. Acerbi, G. Buttazzo and N. Fusco, Semicontinuity and relaxation for integrals depending on vector valued functions, *J. Math. Pures et Appl.* **62** (1983), 371–387.

[3] E. Acerbi and G. Dal Maso, New lower semicontinuity results for polyconvex integrals case, *Calc. Var.* **2** (1994), 329–372.

[4] E. Acerbi and N. Fusco, Semicontinuity problems in the calculus of variations, *Arch. Rat. Mech. Anal.* **86** (1984), 125–145.

[5] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Mathematical Monographs, Oxford University Press, 2000.

[6] L. Ambrosio and G. Dal Maso, On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasi–convex integrals, *J. Funct. Anal.* **109** (1992), 76–97.

[7] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rat. Mech. Anal.* **63** (1977), 337–403.

[8] J. M. Ball and F. Murat, $W^{1,p}$ quasiconvexity and variational problems for multiple integrals, *J. Funct. Anal.* **58** (1984), 225–253.

[9] G. Bouchitté, I. Fonseca and J. Malý, Relaxation of multiple integrals below the growth exponent, *Proc. Royal Soc. Edin.* **128A** (1998), 463–479.

[10] A. Braides, I. Fonseca and G. Leoni, A-quasiconvexity: relaxation and homogenization, *ESAIM:COCV* **5** (2000), 539–577.

[11] V. I. Burenkov, *Sobolev Spaces on Domains*, Teuber, Stuttgart, 1998.

[12] M. Carbone L. and R. De Arcangelis, Further results on Γ-convergence and lower semicontinuity of integral functionals depending on vector-valued functions, *Richerche Mat.* **39** (1990), 99–129.

[13] P. Celada and G. Dal Maso, Further remarks on the lower semicontinuity of polyconvex integrals, *Ann. Inst. H. Poincaré, Analyse non Linéaire* **11** (1994), 661–691.

[14] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity for Nonlinear Functionals*, Springer Lecture Notes in Mathematics **922**, 1982.

[15] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, New York, 1989.

[16] B.Dacorogna and P. Marcellini, Semicontinuité pour des intégrandes polyconvexes sans continuité des determinants, *C. R. Acad. Sci. Paris Sér. I Math.* **311, 6** (1990), 393–396.

[17] G. Dal Maso and C. Sbordone, Weak lower semicontinuity of polyconvex integrals: a borderline case, *Math. Z.* **218** (1995), 603–609.

[18] A. De Simone, Energy minimizers for large ferromagnetic bodies, *Arch. Rat. Mech. Anal.* **125** (1993), 99–143.

[19] F. Demengel, Fonctions à hessien borné, *Ann. Inst. Fourier* **34** (1984), 155–190.

[20] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with (p,q) growth, preprint.

[21] L. Esposito and G. Mingione, Relaxation results for higher order integrals below the natural growth exponent, to appear in *Differential Integral Equations*.

[22] I. Fonseca, G. Leoni and J. Malý, Weak continuity and lower semicontinuity results for determinants, in preparation.

[23] I. Fonseca and J. Malý, Relaxation of multiple integrals below the growth exponent, *Ann. Inst. H. Poincaré, Analyse non Linéaire* **14** (1997), 308–338.

[24] I. Fonseca and P. Marcellini, Relaxation of multiple integrals in subcritical Sobolev spaces, *J. Geom. Anal.* **7** (1997), 57–81.

[25] I. Fonseca and S. Müller, Quasiconvex integrands and lower semicontinuity in L^1, *SIAM J. Math. Anal.* **23** (1992), 1081–1098.

[26] I. Fonseca and S. Müller, Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$, *Arch. Rat. Mech. Anal.* **123** (1993), 1–49.

[27] I. Fonseca and S. Müller, A-quasiconvexity, lower semicontinuity and Young measures, *SIAM J. Math. Anal.* **30** (1999), 1355–1390.

[28] N. Fusco and J. E. Hutchinson, A direct proof for lower semicontinuity of polyconvex functionals, *Manuscripta Math.* **85** (1995), 35–50.

[29] E. Gagliardo, Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili, *Rend. Sem. Mat. Padova* **27** (1957), 283–305.

[30] W. Gangbo, On the weak lower semicontinuity of energies with polyconvex integrands, *J. Math. Pures et Appl.* **73** (1994), 455–469.

[31] M. Guidorzi and L. Poggiolini, Lower semicontinuity for quasiconvex integrals of higher order, *NoDEA* **6** (1999), 227–246.

[32] J. Kristensen, Lower semicontinuity of quasi-convex integrands in BV , *Calc. Var.* **7** (1998), 249–261.

[33] J. Malý, Weak lower semicontinuity of polyconvex integrals, *Proc. Royal Soc. Edin.* **123A** (1993), 681–691.

[34] J. Malý, Lower semicontinuity of quasiconvex integrals, *Manuscripta Math.* **85** (1994), 419–428.

[35] P. Marcellini, Approximation of quasiconvex functions and lower semicontinuity of multiple integrals quasiconvex integrals, *Manuscripta Math.* **51** (1985), 1–28.

[36] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, *Ann. Inst. H. Poincaré, Analyse non Linéaire* **3** (1986), 391–409.

[37] V.G. Maz'ja, Sobolev spaces, Springer-Verlag, Berlin, 1985.

[38] N. G. Meyers, Quasi-convexity and lower semi-continuity of multiple variational integrals of any order, *Trans. Amer. Math. Soc.* **119** (1965), 125-149.

[39] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.

[40] F. Murat, Compacité par compensation : condition necessaire et suffisante de continuité faible sous une hypothése de rang constant, *Ann. Sc. Norm. Sup. Pisa* (4)8 (1981), 68–102.

[41] P. Pedregal, *Parametrized Measures and Variational Principles*, Birkhäuser, Boston, 1997.

[42] W. Rudin, *Functional Analysis*, McGraw Hill, New York, 1991.

[43] P. Santos and E. Zappale, in preparation.

[44] J. Serrin, On the definition and properties of certain variational integrals, *Trans. Amer. Math. Soc.* **161** (1961), 139–167.

[45] E. M. Stein, *Harmonic Analysis,* Princeton University Press, 1993.

[46] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton University Press, 1971.

[47] L. Tartar, Compensated compactness and applications to partial differential equations, in *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, R. Knops, ed., vol. IV, Pitman Res. Notes Math. Vol 39, 1979, 136–212.

[48] L. Tartar, The compensated compactness method applied to systems of conservation laws, in *Systems of Nonlinear Partial Differential Eq.*, J. M. Ball, ed., Riedel, 1983.

[49] L. Tartar, Étude des oscillations dans les équations aux dérivées partielles nonlinéaires, Springer Lectures Notes in Physics, Springer-Verlag, Berlin, 195 (1984), 384–412.

[50] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edin.* **115A** (1990), 193–230.

[51] L. Tartar, On mathematical tools for studying partial differential equations of continuum physics: H-measures and Young measures, in *Developments in Partial Differential Equations and Applications to Mathematical Physics*, Buttazzo, Galdi, Zanghirati, eds., Plenum, New York, 1991.

[52] L. Tartar, Some remarks on separately convex functions, in Microstructure and Phase Transitions, D. Kinderlehrer, R. D. James, M. Luskin and J. L. Ericksen, eds., Vol. 54, IMA Vol. Math. Appl., Springer-Verlag, 1993, 191–204.

[53] V. V. Zhikov, On Lavrentiev's phenomenon, *Russian J. Math. Phys.* **3** (1995), 249–269.

[54] V. V. Zhikov, On some variational problems, *Russian J. Math. Phys.* **5** (1997), 105–116.