GENERIC UNIQUENESS FOR THE PLATEAU PROBLEM

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ABSTRACT. Given a complete Riemannian manifold $\mathcal{M} \subset \mathbb{R}^d$ which is a Lipschitz neighbourhood retract of dimension $m+n$, of class $C^{h,\beta}$ and an oriented, closed submanifold $\Gamma \subset \mathcal{M}$ of dimension $m-1$, which is a boundary in integral homology, we construct a complete metric space $\mathcal{B}$ of $C^{h,\alpha}$-perturbations of Γ inside $\mathcal{M}$, with $\alpha < \beta$, enjoying the following property. For the typical element $b \in \mathcal{B}$, in the sense of Baire categories, there exists a unique m-dimensional integral current in $\mathcal{M}$ which solves the corresponding Plateau problem and it has multiplicity one.

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CONTENTS

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1 INTRODUCTION

In the following let $n, m \geq 1$, $\beta \in [0,1]$ and let $\mathcal{M} \subset \mathbb{R}^d$ be a complete Riemannian manifold (without boundary), which is a Lipschitz neighbourhood retract[1] of dimension $m+n$, of class $C^{h,\beta}$, with $h+\beta > 3$. For every $k = 0, \dots, m+n$, we denote by $\mathscr{D}_k(\mathcal{M})$ the set of k-dimensional currents with support in $\mathcal{M}$ and by $\mathscr{I}_k(\mathcal{M})$ the subgroup of k-dimensional integral currents. We refer to Section 2 for the relevant definitions. We denote by $\mathbf{AMC}(b)$ the set of area-minimizing integral currents in $\mathcal{M}$ with boundary b, namely

$$\mathbf{AMC}(b) := \{T \in \mathscr{I}_m(\mathcal{M}) : \partial T = b, \mathbb{M}(T) \leq \mathbb{M}(S) \text{ for every } S \in \mathscr{I}_m(\mathcal{M}) \text{ with } \partial S = b\}.$$

We denote the set of $(m-1)$-dimensional boundaries in $\mathcal{M}$ by

$$\mathscr{B}_{m-1}(\mathcal{M}) := \{b \in \mathscr{D}_{m-1}(\mathcal{M}) : b = \partial T \text{ for some } T \in \mathscr{D}_m(\mathcal{M})\}.$$

Let $\Gamma \subset \mathcal{M}$ be an oriented, closed (*i.e.* compact and without boundary) submanifold of dimension $m-1$ and of class $C^{\ell,\alpha}$, with $\ell + \alpha < h + \beta$. Let $b_0 := [\![\Gamma]\!]$ be the associated current and assume that $b_0 \in \mathscr{B}_{m-1}(\mathcal{M})$. For every $P \in \Gamma$ there exists a connected, open set $U \subset \mathbb{R}^{m+n}$, a diffeomorphism $\mathbf{\Phi} : U \to \mathbf{\Phi}(U) \subseteq \mathcal{M}$ of class $C^{h,\beta}$ such that $P \in \mathbf{\Phi}(U)$, a relatively open, connected, bounded set $\Omega \subset \mathbb{R}^{m-1} = \langle e_1, \dots, e_{m-1}\rangle$, and a function $f : \Omega \to \mathbb{R}^{n+1}$ of class $C^{\ell,\alpha}$ such that

$$gr(f) := \{(x,y) \in \Omega \times \langle e_m, \dots, e_{m+n}\rangle : y = f(x)\},$$

satisfies $gr(f) \subset U$ and such that

$$\Gamma \cap \mathbf{\Phi}(U) = \mathbf{\Phi}(gr(f)). \tag{1.1}$$

Observe that since Ω is connected, then (1.1) implies that $\Gamma \cap \mathbf{\Phi}(U)$ is also connected.

Given a connected open set Ω' compactly contained in Ω and $\varepsilon > 0$, we let

$$X_\varepsilon(P) := \{u \in C^{\ell,\alpha}(\Omega, \mathbb{R}^{n+1}) : f - u \equiv 0 \text{ on } \Omega \setminus \Omega', \|f-u\|_{C^{\ell,\alpha}} \leq \varepsilon\}. \tag{1.2}$$

By (1.1) there exists $\varepsilon > 0$ such that

$$gr(u) \subseteq U \text{ for every } u \in X_\varepsilon(P). \tag{1.3}$$

We endow $X_\varepsilon(P)$ with the norm $\|\cdot\|_{C^{\ell,\alpha}}$, which makes it a complete metric space, see Lemma 3.1.

For $i = 1, \dots, N$, we select one point p_i on each connected component of Γ and we assume that the definition of $U_i, \mathbf{\Phi}_i, \Omega_i$, f_i and ε_i as in (1.3) is understood. We assume that $\mathbf{\Phi}_i(U_i)$ are disjoint and we denote

$$\eta := \min\{1; \min_{i=1,\dots,N} \varepsilon_i\}. \tag{1.4}$$

1 This assumption is satisfied for instance if $\mathcal{M}$ is a closed Riemannian manifold or if $\mathcal{M} = \mathbb{R}^{m+n}$.

Further restrictions on η will be specified in Lemma 4.1. We denote by $\mathbf{X}_\eta$ the product space

$$\mathbf{X}_\eta := \prod_{i=1}^{N} X_\eta(p_i), \tag{1.5}$$

endowed with the 1-product distance, namely the distance induced by the norm

$$\|(u_1, \dots, u_N)\| := \sum_{i=1}^{N} \|u_i\|_{C^{\ell,\alpha}(\Omega_i)}. \tag{1.6}$$

We define a map $\Psi : \mathbf{X}_\eta \to \mathscr{B}_{m-1}(\mathcal{M})$ as follows

$$\Psi(u_1, \dots, u_N) := \sum_{i=1}^{N} [\![\mathbf{\Phi}_i(gr(u_i))]\!] + b_0 \llcorner (\mathcal{M} \setminus \bigcup_{i=1}^{N} \mathbf{\Phi}_i(U_i)). \tag{1.7}$$

We observe that Ψ is injective and $\Psi(u_1, \dots, u_N)$ and b_0 are in the same homology class for every $(u_1, \dots, u_N) \in \mathbf{X}_\eta$, see Lemma 2.2. We define the space of boundaries associated to $\mathbf{X}_\eta$ as

$$\mathcal{B}_\eta := \Psi(\mathbf{X}_\eta). \tag{1.8}$$

We naturally endow $\mathcal{B}_\eta$ with the distance d induced by the map Ψ. More precisely, for every $b \in \mathbf{X}_\eta$ we denote

$$(u_1(b), \dots, u_N(b)) := \Psi^{-1}(b) \tag{1.9}$$

and we define

$$d(b, \bar{b}) := \sum_{i=1}^{N} \|u_i(b) - u_i(\bar{b})\|_{C^{\ell,\alpha}(\Omega_i)}. \tag{1.10}$$

Observe that $(\mathcal{B}_\eta, d)$ is also a complete metric space, because Ψ is by definition an isometry. Roughly speaking, the space $\mathcal{B}_\eta$ consists of $C^{\ell,\alpha}$-perturbations of the boundary Γ that allow us to deform each connected component of Γ, locally around a point. We are ready to state the main results of this paper which we prove in Section 4.

Theorem 1.1. *For the typical boundary $b \in \mathcal{B}_\eta$, any area-minimizing integral current T with $\partial T = b$ has multiplicity one $\|T\|$-a.e.*

In codimension $n = 1$ the previous theorem has the following interesting consequence.

Corollary 1.2. *If $n = 1$, then for the typical boundary $b \in \mathcal{B}_\eta$, any area-minimizing integral current T with $\partial T = b$ has density $1/2$ at every point of the support of b.*

We also deduce the following general result.

Theorem 1.3. *For the typical boundary $b \in \mathcal{B}_\eta$, there is a unique area-minimizing integral current T with $\partial T = b$.*

Since the intersection of residual sets is a residual set, then for the typical boundary $b \in \mathcal{B}_\eta$ both the conclusion of Theorem 1.1 and the conclusion of Theorem 1.3 are satisfied.

1.1 *Content of the paper*

In Section 2, we introduce the notation for currents and prove preliminary properties of our space of boundaries. In Section 3 we play a Banach-Mazur game in the following context. The main idea behind Theorem 1.1 is that for an area-minimzing integral current, regular two-sided boundary points are contained in the support of the current which locally is smooth. However, the typical boundary is not contained in any smooth submanifold of higher regularity (proof of this can be found in Section 3) and thus, the typical boundary does not allow for area-minimizing currents with regular two-sided boundary points, which we prove in Section 4. We deduce Theorem 1.3 exploiting a technique introduced in [23]. In Section 5 we prove a similar result, which is more general in terms of the class of boundaries that we consider; on the other hand the term *typical* is referred to a weaker notion of distance between boundaries. More precisely, we prove that in a natural complete metric space (of boundaries) metrized by the flat norm, the typical integral current admits a unique solution to the Plateau's problem, see Theorem 5.2. Even if the result is arguably weaker than Theorem 1.3, the strategy is quite flexible and can be adapted to variational problems in which the singular set of minimizers is so large that it can disconnect the regular part, see [5].

1.2 *Previous results on generic properties of area-minimizing currents*

Generic properties, in the sense of Baire categories, are of fundamental importance in the study of the well-posedness of solutions to geometric variational problems. Fine results have been derived when the ambient manifold is endowed with a C^∞-generic metric, such as density, equidistribution, multiplicity one and Morse index estimates of min-max minimal hypersurfaces, see [18, 21, 35, 20]. Recent generic regularity results have been obtained for locally stable minimal hypersurfaces in 8-dimensional closed Riemannian manifolds and for minimizing hypersurfaces in ambient manifolds of dimension 9 and 10, see [19, 6] and [7] respectively. In other words, it has been shown that singularities can be "perturbed away" for generic ambient metrics or for slight perturbations of the boundary, leading to generic smoothness of solutions. Generic regularity for higher dimensional hypersurfaces is still an open problem and not much is known about generic regularity of minimal submanifolds with codimension higher than one, see [34, 33].

Another question occurring naturally in connection with the Plateau problem is that of uniqueness of solutions: it goes back at least to the first decades of the twentieth century, to works by many authors, see [8, 13, 14, 27, 30]. There are many examples of curves admitting several different minimizers, see [22, 28]. However, the presence of many symmetries motivated the question whether uniqueness is a generic property itself, see [3, Section I.11, (3)].

Morgan proved in [23] that almost every curve in $\mathbb{R}^3$ (with respect to a suitable measure) bounds a unique area-minimizing surface. The result has been later generalized by the same author to elliptic integrands and to any dimension and codimension, see [24, 25]. Morgan's works deeply rely on Allard's boundary regularity theorem, see [1, 2], proving that if a boundary Γ is contained in the boundary of a uniformly convex set, then every boundary point $p \in \Gamma$ has density $1/2$ and is regular, see [23, Proposition 6.1] and [2, §4]. This assumption allows the author to rule out the existence of *two-sided* regular boundary points, namely regular boundary points at which the current "crosses" the boundary, see [9, Example 1.3].

Hardt and Simon proved in [16] that, for codimension one area-minimizing currents in the Euclidean space, every boundary point is regular (possibly two-sided) without assuming the convexity condition. More recently, the fourth-named author extended this result to codimension one Riemannian ambient manifolds, see [32]. Moreover, a recent result by De Lellis, De Philippis, Hirsch and Massaccesi, see [9], proves the first general boundary regularity theorem with no restrictions on the codimension, showing that the set of regular boundary points (possibly two-sided) is dense, see also [26] for a 2-dimensional analogue allowing for arbitrary boundary multiplicity.

In this article we prove generic uniqueness and the multiplicity-one property of area-minimizing integral currents in full generality, *i.e.* for general ambient manifolds $\mathcal{M}$ of any dimension, for any codimension and with no convexity assumption on the geometry of the boundary Γ. Our result relies on the aforementioned boundary regularity theorem in [9], as our main goal is to prove the generic absence of two-sided boundary points.

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2 NOTATION AND PRELIMINARIES

We briefly recall the relevant definitions of the theory of currents and we refer the reader to [15, 31] for a complete treatment of the subject. A k-dimensional *current* on $\mathbb{R}^d$ ($k \le d$) is a continuous linear functional on the space $\mathscr{D}^k(\mathbb{R}^d)$ of smooth and compactly supported differential k-forms in $\mathbb{R}^d$. The space of k-dimensional currents in $\mathbb{R}^d$ is denoted by $\mathscr{D}_k(\mathbb{R}^d)$. The *boundary* of a current $T \in \mathscr{D}_k(\mathbb{R}^d)$ is the current $\partial T \in \mathscr{D}_{k-1}(\mathbb{R}^d)$ such that

$$\partial T(\varphi) = T(d\varphi), \quad \text{for every } \varphi \in \mathscr{D}^{k-1}(\mathbb{R}^d),$$

where as usual d denotes the exterior differential. Given $T \in \mathscr{D}_k(\mathbb{R}^d)$, the *mass* of T is denoted by $\mathbb{M}(T)$ and is defined as the supremum of $T(\omega)$ over all forms ω with $|\omega(x)| \le 1$ for all $x \in \mathbb{R}^d$. The *support* of a current T, denoted $\mathrm{supp}(T)$, is the intersection of all closed sets C in $\mathbb{R}^d$ such that $T(\omega) = 0$ whenever $\omega \equiv 0$ on C. For every closed subset K of $\mathbb{R}^d$, we will denote by $\mathscr{D}_k(K)$ the set

$$\mathscr{D}_k(K) := \{T \in \mathscr{D}_k(\mathbb{R}^d) \mid \mathrm{supp}(T) \subset K\}.$$

Given a smooth, proper map $f : \mathbb{R}^d \to \mathbb{R}^{d'}$ and a k-current T in $\mathbb{R}^d$, the *push-forward* of T according to the map f is the k-current $f_\sharp T$ in $\mathbb{R}^{d'}$ defined by

$$f_\sharp T(\omega) := T(f^\sharp \omega), \quad \text{for every } \omega \in \mathscr{D}^k(\mathbb{R}^{d'}), \tag{2.1}$$

where $f^\sharp \omega$ denotes the pullback of ω through f. If T has finite mass and compact support, then the previous definition can be extended to any f of class C^1.

We say that a current $T \in \mathscr{D}_k(\mathbb{R}^d)$ is *integer rectifiable* and we write $T \in \mathscr{R}_k(\mathbb{R}^d)$ if we can identify T with a triple (E, τ, θ), where $E \subset K$ is a k-rectifiable set, $\tau(x)$ is a unit k-vector spanning the tangent space $T_x E$ at $\mathscr{H}^k$-a.e. x and $\theta \in L^1(\mathscr{H}^k \llcorner E, \mathbb{Z})$ is an integer-valued multiplicity, where the identification means that the action of T can be expressed by

$$T(\omega) = \int_E \langle \omega(x), \tau(x) \rangle \theta(x) d\mathscr{H}^k(x), \quad \text{for every } \omega \in \mathscr{D}^k(\mathbb{R}^d). \tag{2.2}$$

If T is as in (2.2), we denote it by $T = [\![E, \tau, \theta]\!]$. We denote by $\mathscr{I}_k(\mathbb{R}^d)$ the subgroup of k-dimensional *integral currents*, that is the set of currents $T \in \mathscr{R}_k(\mathbb{R}^d)$ with $\partial T \in \mathscr{R}_{k-1}(\mathbb{R}^d)$. If $T = [\![E, \tau, \theta]\!] \in \mathscr{R}_k(\mathbb{R}^d)$ and $B \subset \mathbb{R}^d$ is a Borel set, we denote the *restriction* of T to B by setting $T \llcorner B := [\![E \cap B, \tau, \theta]\!]$. The set of integer rectifiable (respectively integral) k-currents with support in a closed set K is denoted by $\mathscr{R}_k(K)$ (respectively $\mathscr{I}_k(K)$).

We recall that the *(integral) flat norm* $\mathbb{F}(T)$ of an integral current $T \in \mathscr{I}_k(K)$, with K compact, is defined by:

$$\mathbb{F}(T) := \min\{\mathbb{M}(R) + \mathbb{M}(S) \mid T = R + \partial S,\, R \in \mathscr{I}_k(K),\, S \in \mathscr{I}_{k+1}(K)\}. \tag{2.3}$$

A k-dimensional *polyhedral* current is a current P of the form

$$P := \sum_{i=1}^N \theta_i [\![\sigma_i]\!], \tag{2.4}$$

where $\theta_i \in \mathbb{R}$, σ_i are non-overlapping k-dimensional simplexes in $\mathbb{R}^d$, oriented by (constant) k-vectors τ_i and $[\![\sigma_i]\!] = [\![\sigma_i, \tau_i, 1]\!]$ is the multiplicity-one current naturally associated to σ_i. A polyhedral current with integer coefficients θ_i is called *integer polyhedral* and we denote the subgroup of integer polyhedral currents with support in K by $\mathscr{P}_k(K)$.

Lemma 2.1. *There exists a constant $C > 0$ such that $\mathbb{F}(b - \bar{b}) \le C\, d(b, \bar{b})$, for every $b, \bar{b} \in \mathcal{B}_\eta$.*

Proof. It is sufficient to prove the lemma for $N = 1$. Indeed, denoting for every $b \in \mathcal{B}_\eta$ and for $i = 1, \dots, N$ the boundary $b^i \in \mathcal{B}_\eta$ defined by

$$b^i := b \llcorner (\mathbf{\Phi}_i(U_i)) + [\![\Gamma]\!] \llcorner (\mathcal{M} \setminus \mathbf{\Phi}_i(U_i)),$$

we have

$$\bar{b} - b = \sum_{i=1}^N \bar{b}^i - b^i,$$

so that

$$\mathbb{F}(\bar{b}-b) \leq \sum_{i=1}^{N} \mathbb{F}(\bar{b}^i - b^i) \leq N \max_{i=1,\dots,N} \mathbb{F}(\bar{b}^i - b^i).$$

Hence we can assume that $N = 1$ and for $w \in \mathbf{X}_\eta$ we define $\mathbf{w} : \Omega \to \mathbb{R}^{m+n}$ by

$$\mathbf{w}(x) := (x, w(x)). \tag{2.5}$$

Let $u := \Psi^{-1}(b)$ and $\bar{u} := \Psi^{-1}(\bar{b})$ and we denote $I := [\![[0,1]]\!] \in \mathscr{I}_1(\mathbb{R})$ and we let $F : [0,1] \times \Omega \to \mathbb{R}^{m+n}$ be the linear homotopy

$$F(t,x) = (1-t)\mathbf{u}(x) + t\bar{\mathbf{u}}(x).$$

Denote $S := F_\sharp(I \times [\![\Omega]\!])$. We use [31, 26.18] to compute

$$\begin{aligned} \partial S =& F_\sharp(\partial(I \times [\![\Omega]\!])) = F_\sharp(\partial I \times [\![\Omega]\!] - I \times \partial [\![\Omega]\!]) \\ =& F_\sharp(\delta_1 \times [\![\Omega]\!]) - F_\sharp(\delta_0 \times [\![\Omega]\!]) - F_\sharp(I \times \partial[\![\Omega]\!]) \\ =& (\bar{\mathbf{u}})_\sharp [\![\Omega]\!] - (\mathbf{u})_\sharp [\![\Omega]\!] - F_\sharp(I \times \partial [\![\Omega]\!]) \\ =& [\![gr(\bar{u})]\!] - [\![gr(u)]\!] - F_\sharp(I \times \partial[\![\Omega]\!]) = [\![gr(\bar{u})]\!] - [\![gr(u)]\!], \end{aligned}$$

where the last equality is due to the fact that $\bar{\mathbf{u}} = \mathbf{u}$ on $\partial\Omega$. Hence, by the homotopy formula, see [15, §4.1.14], we can estimate

$$\begin{aligned} \mathbb{F}([\![gr(\bar{u})]\!] - [\![gr(u)]\!]) \leq \mathbb{M}(S) \leq& \|\bar{\mathbf{u}} - \mathbf{u}\|_\infty \sup_{x\in\Omega} (|D\mathbf{u}(x)| + |D\bar{\mathbf{u}}(x)|)^{m-1} \mathbb{M}([\![\Omega]\!]) \\ \leq& C\|\bar{\mathbf{u}} - \mathbf{u}\|_{C^{3,\alpha}} = C\|\bar{u} - u\|_{C^{\ell,\alpha}}. \end{aligned} \tag{2.6}$$

Therefore, since $\mathbf{\Phi}$ is of class $C^{h,\beta}$, we infer

$$\mathbb{F}(\bar{b} - b) = \mathbb{F}(\mathbf{\Phi}_\sharp([\![gr(\bar{u})]\!] - [\![gr(u)]\!])) \leq \mathbb{M}(\mathbf{\Phi}_\sharp S) \leq C\|\bar{u} - u\|_{C^{\ell,\alpha}} = C\, d(\bar{b}, b),$$

where the last identity follows from the definition of the distance d. □

Lemma 2.2. *For every $b \in \mathcal{B}_\eta$ there exists a current $S \in \mathscr{I}_m(\mathcal{M})$ such that $[\![\Gamma]\!] - b = \partial S$. In particular all the elements of $\mathcal{B}_\eta$ are in the same homology class.*

Proof. For every connected component of Γ, we consider the corresponding $U_i, \mathbf{\Phi}_i, \Omega_i, f_i$, defined in the introduction. We now argue as in the proof of Lemma 2.1, replacing $\bar{b}$ with $[\![\Gamma]\!]$ to define a current $S_i \in \mathscr{I}_m(\mathbb{R}^{m+n})$ such that $\partial S_i = [\![gr(f_i)]\!] - [\![gr(u_i)]\!]$. The current $S := \sum_{i=1}^{N} (\mathbf{\Phi}_i)_\sharp(S_i)$ satisfies the requirement. □

3 THE TYPICAL $C^{\ell,\alpha}$ GRAPH AVOIDS $C^{h,\beta}$ SUBMANIFOLDS

The proof of Theorem 1.1 is obtained combining the boundary regularity result of [9] and the following property of the typical map $u \in X_\varepsilon(P)$, see (1.2). For every open set $V \subset U \subset \mathbb{R}^{n+m}$ such that $gr(u \llcorner \Omega') \cap V \neq \emptyset$ and, for every m-dimensional submanifold $\mathcal{N}$ of class $C^{h,\beta}$ in $\mathbb{R}^{m+n}$ with $\partial\mathcal{N} \cap V = \emptyset$ it holds $gr(u) \cap V \not\subset \mathcal{N}$.

For the sake of generality, in this section we prove this result for $u : \mathbb{R}^{m-k} \to \mathbb{R}^{n+k}$, for every $k < m$. For the purpose of this paper, this generalization is unnecessary, however, we include it since it does not require any additional effort.

In the following let $n, m \geq 1$, and $0 \leq k < m$. Throughout this section we will denote $\{e_1, \ldots, e_{m+n}\}$ the standard basis of $\mathbb{R}^{m+n}$. Let Ω be a fixed open bounded set in $\mathbb{R}^{m-k} = \langle e_1, \ldots, e_{m-k} \rangle$. We further fix $h \in \mathbb{N} \setminus \{0\}$, $\ell \in \mathbb{N}$, $\alpha, \beta \in [0,1]$ and $\gamma \in [0,2] \setminus \{1\}$ so that $\ell + \alpha < \ell + \gamma < h + \beta$, a function $f : \Omega \to \mathbb{R}^{n+k}$ of class $C^{\ell,\alpha}$ and an open set Ω' compactly contained in Ω. For fixed $\varepsilon > 0$, we let

$$X_\varepsilon := \{u \in C^{\ell,\alpha}(\Omega, \mathbb{R}^{n+k}) : f - u \equiv 0 \text{ on } \Omega \setminus \Omega',\ \|f - u\|_{C^{\ell,\alpha}} \leq \varepsilon\},$$

where we denoted

$$\|u\|_{C^{\ell,\alpha}} = \|u\|_{C^\ell} + [D^\ell u]_\alpha := \|u\|_\infty + \sum_{j=1}^{\ell} \|D^j u\|_\infty + \sup_{x \neq y \in \Omega} \frac{|D^\ell u(x) - D^\ell u(y)|}{|x - y|^\alpha}.$$

we further endow X_ε with the norm $\|\cdot\|_{C^{h,\alpha}}$. We observe that the space $X_\varepsilon(P)$ defined in (1.2), fits this definition with $k = 1$ and $\ell \geq 3$.

We begin with the following observation.

Lemma 3.1. *The space $(X_\varepsilon, \|\cdot\|_{C^{\ell,\alpha}})$ is complete. In particular the space $(\mathcal{B}_\eta, d)$ is also complete.*

Proof. It suffices to show that X_ε is closed in $(C^{\ell,\alpha}, \|\cdot\|_{C^{\ell,\alpha}})$. Let u_n be a sequence of elements in X_ε and let $u \in C^{\ell,\alpha}$ be such that $\|u_n - u\|_{C^{\ell,\alpha}} \to 0$. Obviously $f - u \equiv 0$ on $\Omega \setminus \Omega'$ and $u \in C^{\ell,\alpha}$, hence $u \in X_\varepsilon$.

The fact that $\mathcal{B}_\eta$ is complete follows from the fact that Ψ is an isometry between the product space X defined in (1.5) endowed with the distance induced by the norm (1.6) and $(\mathcal{B}_\eta, d)$. □

We then introduce a subset of X_ε which roughly consists of those functions whose graph has small intersection with any submanifold of class $C^{h,\beta}$. We let $\pi_\Omega : \Omega \times \mathbb{R}^{n+k} \to \Omega$ be the orthogonal projection on the first $m - k$ coordinates of $\mathbb{R}^{m+n}$. For every open set $A \subset \Omega$ we denote

$$C_A := \{(z_1, z_2) \in \Omega \times \mathbb{R}^{n+k} : z_1 \in A\}$$

and we abbreviate $C(x, r) := C_{B(x,r)}$.

Definition 3.2. Let $\mathcal{A}$ be the set of those $w \in X_\varepsilon$ for which there exists an embedded m-dimensional manifold $\mathcal{N} \subset \mathbb{R}^{m+n}$ of class $C^{h,\beta}$ and an open set $O \subset C_{\Omega'}$ such that

$$\partial \mathcal{N} \cap O = \emptyset \qquad \text{and} \qquad \emptyset \neq \operatorname{gr}(w) \cap O \subset \mathcal{N}.$$

The aim of this section is to prove the following proposition.

Proposition 3.3. *The set $\mathcal{A}$ is of first category in X_ε, i.e. it is contained in a countable union of closed sets with empty interior.*

Thanks to Lemma 3.1 and Baire's theorem, Proposition 3.3 implies in particular that $X_\varepsilon \setminus \mathcal{A}$ is dense in X_ε. Our strategy to prove Proposition 3.3 uses the relation between topological properties of sets in the sense of Baire categories and the existence of a winning strategy for a suitable topological game. Let us quickly recall such general result.

Definition 3.4 (Banach-Mazur game)**.** Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be an arbitrary subset. The *Banach-Mazur game* associated to A is a game between two players, $P1$ and $P2$ with the following rules: $P1$ chooses arbitrarily an open set $\mathscr{U}_1 \subseteq X$; then $P2$ chooses an open set $\mathscr{V}_1 \subseteq \mathscr{U}_1$; then $P1$ chooses an open set $\mathscr{U}_2 \subseteq \mathscr{V}_1$ and so on. If the set $(\bigcap_{i\in\mathbb{N}} \mathscr{V}_i) \cap A$ is non-empty then $P1$ wins. Otherwise $P2$ wins.

The following proposition relates the Banach-Mazur game to the topology of the space on which it is played. We say that a set is of *first category* if it is contained in a countable union of closed subsets with empty interior. A set is *residual* if its complement is of first category. We say that a certain property holds for the *typical* element of X, if it holds for every element of a residual set.

Proposition 3.5. *Suppose the metric space X is complete. Then there exists a winning strategy for $P2$ if and only if A is of first category in X.*

Proof. The proof of this result is given in [29] only in the case of the real line. However the same argument works verbatim in any complete metric space. □

Definition 3.6. Let A be the set of those $w \in X_\varepsilon$ for which there exists a map $M : \Omega \times \langle e_{m-k+1}, \dots, e_m \rangle \to \mathbb{R}^n$ of class $C^{h,\beta}$ and an open set $W \subset \Omega'$ such that

$$\pi_\Omega(C_W \cap \mathrm{gr}(M) \cap \mathrm{gr}(w)) = W.$$

The main step for the proof of Proposition 3.3 is the following

Proposition 3.7. *The set A is of first category in X_ε.*

We postpone the proof to the end of the section and begin with the following elementary lemma.

Lemma 3.8. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a function of class $C^{h,\beta}$. Then, for any $x \in \mathbb{R}$ there exists a $t_0 > 0$ and a bounded function $r_{h+1}(t) : [-t_0, t_0] \to \mathbb{R}$ such that for every $t \in [-t_0, t_0]$*

$$g(x+t) = g(x) + dg(x)t + \ldots + \frac{d^h g(x)}{h!} t^h + r_{h+1}(t) t^{h+\beta}.$$

In addition $\|r_{h+1}(t)\|_\infty \le \frac{[d^h g(x)]_{C^\beta}}{h!}$.

Proof. The Taylor expansion of g yields

$$g(x+t) = g(x) + dg(x)t + \ldots + \frac{d^{h-1} g(x)}{(h-1)!} t^{h-1} + \frac{d^h g(x+\zeta_t)}{h!} t^h, \tag{3.1}$$

for some $\zeta_t \in [0,t]$. However, this shows that

$$g(x+t) - g(x) - dg(x)t - \ldots - \frac{d^{h-1}g(x)}{(h-1)!}t^{h-1} - \frac{d^h g(x)}{h!}t^h = \frac{d^h g(x+\zeta_t)t^h - d^h g(x)t^h}{h!}. \tag{3.2}$$

Define $r_{h+1}(t) := \frac{d^h g(x+\zeta_t) - d^h g(x)}{h! t^\beta}$ and note that $|r_{h+1}(t)| \leq \frac{|d^h g(x+\zeta_t) - d^h g(x)|}{h! \zeta_t^\beta} \leq \frac{[d^h g(x)]_{C^\beta}}{h!}$. □

The following proposition provides the main tool to find a winning strategy for the Banach-Mazur game associated to A, allowing us to prove Proposition 3.7. We denote the balls in X_ε by $\mathscr{B}(w,\rho) = \{u \in X_\varepsilon : \|u - w\|_{C^{\ell,\alpha}} < \rho\}$.

Proposition 3.9. *Let $\bar{w} \in X_\varepsilon$ be fixed and let $\bar{\rho} > 0$, $j \in \mathbb{N} \setminus \{0\}$. Then, for any $x \in \Omega'$ there exist $u \in X_\varepsilon$ and $\rho > 0$ such that*

(i) $\mathscr{B}(u,\rho) \subseteq \mathscr{B}(\bar{w},\bar{\rho})$;

(ii) for every $w \in \mathscr{B}(u,\rho)$ and $M : \Omega \times \langle e_{m-k+1}, \ldots, e_m \rangle \to \mathbb{R}^n$ of class $C^{h,\beta}$ with $\|M\|_{C^{h,\beta}} \leq j$ we have

$$\pi_\Omega(\mathrm{gr}(M) \cap \mathrm{gr}(w) \cap C(x,r)) \neq B(x,r),$$

where $r := \min\{1/j, \mathrm{dist}(x, \Omega \setminus \Omega')\}$.

Proof. Assume by contradiction that there is an $x \in \Omega'$ such that for every $u \in \mathscr{B}(\bar{w},\bar{\rho})$ there is an infinitesimal sequence $\rho_i \leq \bar{\rho} - \|u - \bar{w}\|_{C^{\ell,\alpha}}$ for which property (ii) fails.

Fix $0 < \delta < 1$ to be chosen later and let ψ_δ be the function on $\mathbb{R}^{m-k}$ defined by

$$\psi_\delta(z) := \delta^{1+\gamma}|x_1 - z_1|^{\ell+\gamma},$$

where z_i are the coordinates of $z \in \mathbb{R}^{m-k}$. Let r be as in (ii). Let $\eta : \Omega \to [0,1]$ be a smooth cutoff function such that $\eta \equiv 0$ on $\Omega \setminus B_r(x)$ and $\eta \equiv 1$ on $B_{r/2}(x)$. Observe that $\eta\psi_\delta \in C^{\ell,\alpha}$ and more precisely

$$\lim_{\delta \to 0} \delta^{-1}\|\eta\psi_\delta\|_{C^{\ell,\alpha}} = 0 \qquad \text{and} \qquad |\psi_\delta(x + te_1)| = \delta^{1+\gamma}|t|^{\ell+\gamma} \qquad \text{for any } |t| \leq r/2. \tag{3.3}$$

Throughout the rest of the proof, we let φ be a mollification kernel, that is a non-negative, radial smooth function supported on $B(0,1) \subseteq \mathbb{R}^{m-k}$ such that $\varphi \equiv 1$ on $B(0,1/2)$ and $\int \varphi = 1$. For any $\iota \in \mathbb{N} \setminus \{0\}$ we further let $\varphi_\iota(y) := \iota^{m-k}\varphi(\iota y)$. Denote $f_\delta := (1-\delta)\bar{w} + \delta f$ and define

$$v := \varphi_\iota * (\eta f_\delta) + (1-\eta) f_\delta.$$

Observe that for ι sufficiently large we have that $f - v \equiv 0$ on $\Omega \setminus \Omega'$. Moreover, the function v is smooth on $B(x, r/2)$. Denote

$$u := (v_1, \ldots, v_{n+k-1}, v_{n+k} + \eta\psi_\delta).$$

We can estimate

$$\begin{aligned} \|u - f\|_{C^{\ell,\alpha}} &\leq \|u - v\|_{C^{\ell,\alpha}} + \|v - f_\delta\|_{C^{\ell,\alpha}} + \|(1-\delta)(\bar{w} - f)\|_{C^{\ell,\alpha}} \\ &\leq \|\eta\psi_\delta\|_{C^{\ell,\alpha}} + \|\varphi_\iota * (\eta f_\delta) - \eta f_\delta\|_{C^{\ell,\alpha}} + (1-\delta)\varepsilon. \end{aligned}$$

Hence it follows from (3.3) that for δ sufficiently small and ι sufficiently large, $u \in X_\varepsilon$. Moreover for δ sufficiently small and ι sufficiently large we have $u \in \mathscr{B}(\bar{w}, \bar{\rho}/2)$. Indeed,

$$\begin{aligned}\|u - \bar{w}\|_{C^{\ell,\alpha}} &\le \|u - v\|_{C^{\ell,\alpha}} + \|v - f_\delta\|_{C^{\ell,\alpha}} + \|\delta(\bar{w} - f)\|_{C^{\ell,\alpha}} \\ &\le \|\eta\psi_\delta\|_{C^{\ell,\alpha}} + \|\varphi_\iota * (\eta f_\delta) - \eta f_\delta\|_{C^{\ell,\alpha}} + \delta\varepsilon.\end{aligned}$$

By assumption there is a sequence $\rho_i < \bar{\rho}/2$ with $\rho_i \to 0$, such that there exist $w^i \in \mathscr{B}(u, \rho_i)$ and $M^i : \Omega \times \langle e_{m-k+1}, \dots, e_m \rangle \to \mathbb{R}^n$ of class $C^{h,\beta}$ with $\|M^i\|_{C^{h,\beta}} \le j$ for which

$$\pi_\Omega(\mathrm{gr}(M^i) \cap \mathrm{gr}(w^i) \cap C(x,r)) = B(x,r).$$

This means that for any $y \in B(x,r)$ we find $y' \in \mathbb{R}^k$ such that

$$(y, y', M^i(y,y')) = (y, \underline{w}^i(y), \underline{\underline{w}}^i(y)) \in \mathbb{R}^{m-k} \times \mathbb{R}^k \times \mathbb{R}^n, \qquad \text{for any } i \in \mathbb{N},$$

where we denote $\underline{w}^i := (w^i_1, \dots, w^i_k)$ and $\underline{\underline{w}}^i := (w^i_{k+1}, \dots, w^i_{n+k})$. In particular, for $y = x_t := x + te_1$ with $t \in [0, r/2]$, we have

$$(x_t, \underline{w}^i(x_t), \underline{\underline{w}}^i(x_t)) = (x_t, \underline{w}^i(x_t), M^i(x_t, \underline{w}^i(x_t))),$$

and comparing the last components, we deduce that for every $t \in [0, r/2]$, we have

$$w^i_{n+k}(x_t) = M^i_n(x_t, \underline{w}^i(x_t)) \qquad \text{for all } i \in \mathbb{N}. \tag{3.4}$$

Thanks to Arzelà-Ascoli theorem, we can show that there exists $M \in C^{h,\beta}$ with $\|M\|_{C^{h,\beta}} \le j$ such that, up to subsequences, $\lim_{i\to\infty} \|M^i - M\|_{C^{h,\beta}} = 0$. Indeed, up to subsequences, one can find maps $\mathfrak{M}^i_l$ for $l = 0, \dots, h$ such that $D^l M^i$ converges uniformly to $\mathfrak{M}_l$ on $\overline{\Omega}$, the map $\mathfrak{M}_h$ is of class C^β and $\sum_{l=1}^N \|\mathfrak{M}_l\|_\infty + [\mathfrak{M}_h]_\beta \le j$. The fact that $D\mathfrak{M}_l = \mathfrak{M}_{l+1}$ for $l = 0, \dots, h-1$ is an elementary consequence of Lemma 3.8 and of the fact that $\|M^i\|_{C^{h,\beta}} \le j$.

In addition, since $\rho_i \to 0$ and $w^i \in \mathscr{B}(u, \rho_i)$, we also have that $\lim_{i\to\infty} \|w^i - u\|_{C^1} = 0$ and hence, by continuity of all the functions involved, the fact that $w^i \to u$ and $\underline{u} = \underline{v}$, (3.4) implies that

$$u_{n+k}(x_t) = M_n(x_t, \underline{u}(x_t)) = M_n(x_t, \underline{v}(x_t)), \tag{3.5}$$

for every $t \in [0, r/2]$. On the other hand, using Lemma 3.8 and the fact that M is of class $C^{h,\beta}$ and $\underline{u}$ is smooth, we find constants $c_0, \dots, c_h$ and a function $c_{h+1}(t)$ with $\|c_{h+1}\|_{L^\infty(0,r_\delta)} < C_{h+1}$ such that for every $t \in [0, r/2]$, it holds

$$M_n(x_t, \underline{v}(x_t)) = c_0 + c_1 t + \dots + c_h \frac{t^h}{h!} + c_{h+1}(t) t^{h+\beta}. \tag{3.6}$$

Observe that v is smooth, hence we can expand it

$$v_{n+k}(x_t) = v_{n+k}(x) + \partial_1 v_{n+k}(x) t + \dots + \frac{\partial_1^h v_{n+k}(x)}{h!} t^h + \frac{\partial_1^{h+1} v_{n+k}(\zeta)}{(h+1)!} t^{h+1}, \tag{3.7}$$

for some $\zeta \in [x, x + te_1]$.

Now we estimate the size of the added bump.

$$\eta(x_t)\psi_\delta(x_t) = (u_{n+k} - v_{n+k})(x_t) \overset{(3.5)}{=} M_n(x_t, \underline{v}(x_t)) - v_{n+k}(x_t). \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) we infer that for every $t \in [0, r/2]$ we have

$$\begin{aligned}\left|\sum_{\kappa=0}^{h}(c_\kappa - \partial_1^\kappa v_{n+k}(x))\frac{t^\kappa}{\kappa!} + c_{h+1}(t)t^{h+\beta} - \frac{\partial_1^{h+1}v_{n+k}(\zeta)}{(h+1)!}t^{h+1}\right| &= |M_n(x_t, \underline{v}(x_t)) - v_{n+k}(x_t)| \\ &\overset{(3.8)}{=} |\eta(x_t)\psi_\delta(x_t)| \overset{(3.3)}{=} \delta^{1+\gamma}t^{\ell+\gamma}.\end{aligned} \tag{3.9}$$

As $\gamma > 0$, we deduce that

$$c_\kappa = \partial_1^\kappa v_{n+k}(x) \qquad \text{for all} \quad 0 \le \kappa \le \ell + \gamma. \tag{3.10}$$

Moreover, we infer from (3.9) that for any $t \in [0, r/2]$, we have

$$\delta^{1+\gamma}t^{\ell+\gamma} \overset{(3.10)}{=} \left|\sum_{\kappa=\lfloor \ell+\gamma+1\rfloor}^{h}(c_\kappa - \partial_1^\kappa v_{n+k}(x))\frac{t^\kappa}{\kappa!} + c_{h+1}(t)t^{h+\beta} - \frac{\partial_1^{h+1}v_{n+k}(\zeta)}{(h+1)!}t^{h+1}\right|, \tag{3.11}$$

which is a contradiction to $\ell + \alpha < \ell + \gamma < h + \beta$. □

Proof of Proposition 3.7. Let us prove that $P2$ has a winning strategy for the Banach-Mazur game associated to A.

Let us assume that the players $P1$ and $P2$ have played already κ moves which are associated to open sets $\mathscr{U}_1, \dots, \mathscr{U}_\kappa$ and $\mathscr{V}_1, \dots, \mathscr{V}_\kappa$ chosen by $P1$ and $P2$ respectively in such a way that

$$\mathscr{V}_\kappa \subseteq \mathscr{U}_\kappa \subseteq \mathscr{V}_{\kappa-1} \subseteq \cdots \subseteq \mathscr{V}_1 \subseteq \mathscr{U}_1.$$

The $(\kappa+1)$th move for $P1$ is an open set $\mathscr{U}_{\kappa+1} \subseteq \mathscr{V}_\kappa$. Now we describe how to choose properly the set $\mathscr{V}_{\kappa+1}$.

Let us fix a dense sequence $\{x_\iota\}_{\iota\in\mathbb{N}}$ in Ω'. First $P2$ picks some $\bar{w} \in \mathscr{U}_{\kappa+1}$ and $\bar{\rho} > 0$ such that $\mathscr{B}(\bar{w}, \bar{\rho}) \subseteq \mathscr{U}_{\kappa+1}$. By Proposition 3.9 applied with these choices of $\bar{w}$ and $\bar{\rho}$ and with $x = x_{\kappa+1}$, $j = \kappa + 1$ we obtain $u^{\kappa+1} \in \mathscr{B}(\bar{w}, \bar{\rho})$ and $0 < \rho_{\kappa+1} < 1/(\kappa+1)$ such that

(i) $\mathscr{B}(u^{\kappa+1}, \rho_{\kappa+1}) \subseteq \mathscr{U}_{\kappa+1}$;

(ii) for every $w \in \mathscr{B}(u^{\kappa+1}, \rho_{\kappa+1})$ and $M : \Omega \times \langle e_{m-k+1}, \dots, e_m\rangle \to \mathbb{R}^n$ of class $C^{h,\beta}$ with $\|M\|_{C^{h,\beta}} \le \kappa + 1$ we have

$$\pi_\Omega(\mathrm{gr}(M) \cap \mathrm{gr}(w) \cap C(x, r)) \ne B(x, r),$$

where $r := \min\{1/(\kappa+1), \mathrm{dist}(x, \Omega \setminus \Omega')\}$.

Note that with the choice $\mathscr{V}_{\kappa+1} := \mathscr{B}(u^{\kappa+1}, \rho_{\kappa+1})$ there exists $w_\infty \in X_\varepsilon$ such that $\{w_\infty\} = \bigcap_{j\in\mathbb{N}} \mathscr{V}_j$. Let us show that $w_\infty \notin A$. Let $U \subseteq \Omega'$ be an open set and pick $\iota \in \mathbb{N}$ such that $B(x_\iota, 1/\iota)$ is compactly contained in U. Since $w_\infty \in \cap_{\kappa\in\mathbb{N}} \mathscr{V}_\kappa$, in particular $w_\infty \in \mathscr{V}_\iota$ and we can deduce that

$$\pi_\Omega(\mathrm{gr}(M) \cap \mathrm{gr}(w_\infty) \cap C(x_\iota, 1/\iota)) \neq B(x_\iota, 1/\iota),$$

for every $M : \Omega \times \langle e_{m-k+1}, \ldots, e_m \rangle \to \mathbb{R}^n$ with $\|M\|_{C^{h,\beta}} \leq \iota$. Thanks to the arbitrariness of U and since ι can be chosen arbitrarily large, we conclude that $w_\infty \notin A$. Hence this is a winning strategy for $P2$ and this concludes the proof. □

Proof of Proposition 3.3. Let $w \in \mathcal{A}$ and let U and $\mathcal{N}$ be as in Definition 3.2. Let $p \in \mathrm{gr}(w \llcorner \Omega') \cap U$. We claim that there exists a ball $B \subseteq U$ centred at p such that the manifold $\mathcal{N}$ inside the ball B coincides with the graph of a map $N : V \to V^\perp$ of class $C^{h,\beta}$, where V is an m-dimensional coordinate plane in $\mathbb{R}^{m+n} = \langle e_1, \ldots, e_{m+n} \rangle$. This is due to the implicit function theorem and to the fact that the tangent of $\mathcal{N}$ at p must be a graph with respect to one of the coordinate planes.

Furthermore, it is also clear that V must contain $\langle e_1, \ldots, e_{m-k} \rangle$. This is due to the fact that otherwise $\mathrm{gr}(w \llcorner \Omega')$ and $\mathrm{gr}(N)$ would be transversal. In particular $\Omega \subseteq V$. For any m-dimensional coordinate plane denote with A_V the subset of X_ε obtained replacing $\langle e_1, \ldots, e_m \rangle$ with V in Definition 3.6. Note that the above discussion implies that $\mathcal{A} \subseteq \cup_V A_V$, where the union is taken on the coordinate m-dimensional planes in $\mathbb{R}^{m+n}$, and thus $\mathcal{A}$ is of first category by Proposition 3.7. □

4 PROOF OF THE MAIN RESULTS

Given $\mathcal{M}, \Gamma$ as in Section 1 and $T \in \mathbf{AMC}(b)$ with $b = [\![\Gamma]\!]$, we recall that a point $p \in \Gamma$ is a *regular boundary point* for T if there exist a neighborhood W of p and a regular embedded m-dimensional submanifold $\Sigma \subset W \cap \mathcal{M}$ (without boundary in W) such that $\mathrm{supp}(T) \cap W \subset \Sigma$. The set of regular boundary points is denoted by $\mathrm{Reg}_\mathrm{b}(T)$ and its complement in Γ will be denoted by $\mathrm{Sing}_\mathrm{b}(T)$.

Let $p \in \mathrm{Reg}_\mathrm{b}(T)$. Up to restrictions of W so that $W \cap \Sigma$ is diffeomorphic to an m-dimensional ball, there exists a positive integer Q (called *multiplicity*) such that $T \llcorner W = Q[\![\Sigma^+]\!] + (Q-1)[\![\Sigma^-]\!]$, where Σ^+ and Σ^- are the two disjoint regular submanifolds of W divided by $\Gamma \cap W$ and with boundaries Γ and $-\Gamma$, respectively. We define the *density* of a regular boundary point p in $\Gamma \cap W$ as $\Theta(T, p) := Q - 1/2$. This definition is equivalent to the definition of density of T at every regular boundary point p as

$$\Theta(T, p) := \lim_{r\to 0} \frac{\|T\|(B(p, r))}{\omega_m r^m},$$

where the numerator and the denominator represent respectively the mass of the current in a ball of radius r and the m-dimensional volume of an m-dimensional ball of radius r. Regular boundary points where $Q = 1$ are called *one-sided* boundary points. Regular boundary points where $Q > 1$ are called *two-sided*. The main result of [9] is that, assuming $\mathcal{M}, \Gamma$ and T as above, $\mathrm{Reg}_\mathrm{b}(T)$ is (open and) dense in Γ.

Analogously, we say that $p \in \operatorname{supp}(T) \setminus \Gamma$ is an *interior regular point* if there is a positive radius $\bar{r} > 0$, a regular embedded submanifold $\Sigma \subset \mathcal{M}$ and a positive integer Q such that $T \llcorner B(x_0, \bar{r}) = Q[\![\Sigma]\!]$. The set of interior regular points, which is relatively open in $\operatorname{supp}(T) \setminus \Gamma$, is denoted by $\operatorname{Reg_i}(T)$. Its complement, *i.e.* $\operatorname{supp}(T) \setminus (\Gamma \cup \operatorname{Reg_i}(T))$, is denoted by $\operatorname{Sing_i}(T)$ and is called the *interior singular set* of T.

4.1 *Proof of Theorem 1.1*

Observe that for $N = 1$ the conclusion of Theorem 1.1 holds for *every* $b \in \mathcal{B}_\eta$, due to [9, Theorem 2.1]. For $N > 1$ and every $i = 1, \dots, N$ we consider the set $X_\eta(p_i)$ and we define the corresponding set $\mathcal{A}_i$ as in Definition 3.2. By Proposition 3.3 we have that $\mathcal{A}_i$ is a set of first category in $X_\eta(p_i)$ so that $\mathscr{R} := \prod_{i=1}^N (X_\eta(p_i) \setminus \mathcal{A}_i)$ is a residual set in $\mathbf{X}_\eta$, see (1.5). Since the map Ψ defined in (1.7) is an isometry, then $\Psi(\mathscr{R})$ is a residual set in $\mathcal{B}_\eta$, see (1.8). Moreover, for every $i = 1, \dots, N$ and for every $u \in X_\eta(p_i) \setminus \mathcal{A}_i$ the following property holds: for every open set $W \subset \mathbf{\Phi}_i(\Omega_i') \subset \mathcal{M}$ and for every m-dimensional submanifold $\mathcal{N} \subset \mathcal{M}$ of class $C^{h,\beta}$ such that $\partial\mathcal{N} \cap W = \emptyset$ we have

$$W \cap \mathbf{\Phi}_i(gr(u)) \not\subset \mathcal{N}. \tag{4.1}$$

Now consider $b \in \Psi(\mathscr{R})$ and assume by contradiction that there exists an area-minimizing integral current T with $\partial T = b$ which does not satisfy the conclusion of Theorem 1.1. By [9, Theorem 1.6 and Theorem 2.1], the open and dense set of regular boundary points of T contains at least a two-sided point p. By [9, Theorem 2.1] the dense set of regular points in the connected component of $\operatorname{supp}(\tilde{b})$ containing p consists of two-sided points. This contradicts (4.1) because for any two-sided point p, then $\operatorname{supp}(T)$ must be contained in a $C^{h,\beta}$ submanifold, locally around p. □

4.2 *Proof of Corollary 1.2*

By [32, Theorem 9.1], for every area-minimizing integral current T with $\partial T = b$ every point $P \in \operatorname{supp}(b)$ is regular. As in the proof of Theorem 1.1, for every $b \in \mathscr{R}$ there are no two-sided regular points, which implies that every point of b has density $1/2$. □

4.3 *Proof of Theorem 1.3*

Consider the subset of $\mathcal{B}_\eta$ of those boundaries admitting more than one minimizer:

$$NU := \{b \in \mathcal{B}_\eta : \text{ there exist } T^1, T^2 \in \mathbf{AMC}(b) \text{ such that } T^1 \neq T^2\}. \tag{4.2}$$

We aim to prove that NU is a set of first category in $\mathcal{B}_\eta$. The following lemma shows that it is sufficient to prove that $\mathcal{B}_\eta \setminus NU$ is dense. A similar strategy is adopted in [5].

Lemma 4.1. *There exists a constant $\eta_0 = \eta_0(\mathcal{M}) > 0$ such that if the parameter η in (1.4) is smaller than η_0 the following property holds: if the set $\mathcal{B}_\eta \setminus NU$ is dense in $(\mathcal{B}_\eta, d)$, then it is residual.*

Proof. For every $\mathbf{m} \in \mathbb{N} \setminus \{0\}$, consider the sets

$$NU_{\mathbf{m}} := \{b \in \mathcal{B}_\eta : \text{ there exist } T^1, T^2 \in \mathbf{AMC}(b) \text{ with } \mathbb{F}(T^2 - T^1) \geq \mathbf{m}^{-1}\}.$$

Since $NU_{\mathbf{m}} \subset NU$, then $(\mathcal{B}_\eta \setminus NU_{\mathbf{m}}) \supset (\mathcal{B}_\eta \setminus NU)$ and hence, by assumption, $\mathcal{B}_\eta \setminus NU_{\mathbf{m}}$ is dense in $\mathcal{B}_\eta$ for every $\mathbf{m}$. Therefore $NU_{\mathbf{m}}$ has empty interior in $\mathcal{B}_\eta$ for every $\mathbf{m}$. We conclude by proving that $NU_{\mathbf{m}}$ is closed for every $\mathbf{m}$.

Fix $\mathbf{m}$ and consider a sequence b_j of elements of $NU_{\mathbf{m}}$ and let b be such that $d(b_j, b) \to 0$. Since $\mathcal{B}_\eta$ is complete, see Lemma 3.1, we can assume $b \in \mathcal{B}_\eta$. By Lemma 2.1 we deduce that $\mathbb{F}(b_j - b) \to 0$. Observe that, denoting $u(b_j) = (u_1^j, \dots, u_N^j)$, we have

$$\mathbb{M}(b_j) \leq \mathbb{M}(b_0 \llcorner (\mathcal{M} \setminus \bigcup_{i=1}^N \mathbf{\Phi}_i(U_i))) + \sum_{i=1}^N \mathbb{M}((\mathbf{u}_i^j)_\sharp [\![\Omega_i]\!]) \leq C + \sum_{i=1}^N \mathrm{Lip}(\mathbf{u}_i^j)^{m-1} \mathscr{H}^{m-1}(\Omega_i), \tag{4.3}$$

where we recall that $\mathbf{u}_i^j$ are defined in (2.5). Therefore the masses of b_j are equibounded because

$$\mathrm{Lip}(\mathbf{u}_i^j) \leq \mathrm{Lip}(f_i) + \|u_i^j - f_i\|_{C^1} \leq \mathrm{Lip}(f_i) + \varepsilon_i.$$

For every $j \in \mathbb{N}$, take

$$T_j, \bar{T}_j \in \mathbf{AMC}(b_j) \quad \text{with} \quad \mathbb{F}(T_j - \bar{T}_j) \geq \mathbf{m}^{-1}.$$

Let $T \in \mathbf{AMC}(b)$ and observe that by [17, Lemma 3.4], if the parameter η defined in (1.4) is smaller than a constant η_0 depending only on $\mathcal{M}$, for every j there exists $S_j, \bar{S}_j \in \mathscr{I}_{m+1}(\mathcal{M})$ such that $T - T_j = \partial S_j$ and $\mathbb{M}(S_j) \leq C\,\mathbb{F}(T - T_j)$ and similarly $T - \bar{T}_j = \partial \bar{S}_j$ and $\mathbb{M}(\bar{S}_j) \leq C\,\mathbb{F}(T - \bar{T}_j)$. Therefore we can estimate

$$\mathbb{M}(T_j) \leq \mathbb{M}(T) + C\,\mathbb{F}(b - b_j), \qquad \mathbb{M}(\bar{T}_j) \leq \mathbb{M}(T) + C\,\mathbb{F}(b - b_j) \tag{4.4}$$

and so the masses of T_j and $\bar{T}_j$ are equibounded. Moreover the same argument used in [17, Lemma 3.4] implies that $\mathrm{supp}(T_j)$ and $\mathrm{supp}(\bar{T}_j)$ are contained in a fixed tubular neighbourhood of Γ and therefore they are all supported on a unique compact set $K \subset \mathcal{M}$. Combining (4.3) and (4.4) we deduce from the compactness theorem [15, §4.2.17] that there exist integral currents $T, \bar{T} \in \mathscr{I}_m(K)$, such that $\partial T = \partial \bar{T} = b$ and, up to subsequences, $\mathbb{F}(T_j - T) \to 0$, $\mathbb{F}(\bar{T}_j - \bar{T}) \to 0$. Clearly $\mathbb{F}(\bar{T} - T) \geq 1/\mathbf{m}$. By [31, Theorem 34.5], we have $T, \bar{T} \in \mathbf{AMC}(b)$, hence $b \in NU_{\mathbf{m}}$. □

Proposition 4.2. *The set $\mathcal{B}_\eta \setminus NU$ is dense in $\mathcal{B}_\eta$.*

Proof. Fix $0 < \mu < 1$. Let $b \in \mathcal{B}_\eta$ and take $(u_1, \dots, u_N) \in \mathbf{X}_\eta$ such that $b = \Psi(u_1, \dots, u_N)$, see (1.7). Consider

$$(w_1, \dots, w_N) := (1 - \mu)(u_1, \dots, u_N) - \mu(f_1, \dots, f_N)$$

and observe that $(w_1, \dots, w_N) \in \mathbf{X}_{(1-\mu)\eta}$. By Proposition 3.3, there is $(\tilde{w}_1, \dots, \tilde{w}_N) \in \mathbf{X}_{(1-\mu)\eta} \setminus \mathcal{A}$ with

$$\sum_{i=1,\dots,N} \|\tilde{w}_i - w_i\|_{C^{\ell,\alpha}} < \mu\eta/2. \tag{4.5}$$

Now define $b_\mu := \Psi(w_1, \ldots, w_N)$ and $\tilde{b} := \Psi(\tilde{w}_1, \ldots, \tilde{w}_N)$ and observe that by (4.5)

$$d(\tilde{b}, b) \le d(\tilde{b}, b_\mu) + d(b_\mu, b) \le \mu\eta/2 + \mu\eta = 3\mu\eta/2. \tag{4.6}$$

Moreover, for every $T \in \mathbf{AMC}(\tilde{b})$ there exists an open and dense subset of $\mathrm{supp}(\tilde{b})$ which points have density $1/2$ for T. Indeed, fix such a current T and observe that for every $i = 1, \ldots, N$ [9, Theorem 1.6] implies that $\mathbf{\Phi}_i(gr(\tilde{w}_i \llcorner \Omega_i'))$ contains at least one regular boundary point q_i for T. On the other hand, by the same argument used in the proof of Theorem 1.1, q_i cannot be two-sided and therefore it has density $1/2$. Hence [9, Theorem 2.1] implies that all points in the open dense set of regular boundary points for T in the connected component of $\mathrm{supp}(\tilde{b})$, which contains q_i have density $1/2$.

Now we construct a boundary $\hat{b}$ which is a local perturbation of $\tilde{b}$ around every q_i, with the property that $\mathbf{AMC}(\hat{b})$ is a singleton. From now on we we work in every Ω_i separately and drop the index i. Without loss of generality, and up to choosing a subset of U, we can assume that the diffeomorphism $\mathbf{\Phi} : U \to \mathbf{\Phi}(U) \subseteq \mathcal{M}$ is of the form

$$\mathbf{\Phi}(z) = (z, \Phi(z)) \in \mathbb{R}^d \qquad \text{for } z \in U \subset \mathbb{R}^{n+m},$$

with $\Phi : U \to \mathbb{R}^{d-m-n}$ of class $C^{h,\beta}$. Moreover up to rotation and translation, we can assume that

- $(0, \Phi(0)) = q$ and $D\Phi(0) = 0$,
- there exist $r > 0$, a open set $\Lambda \subset B(0,r) \subset \mathbb{R}^m$ containing the origin and a $C^{h,\beta}$ function $F : \Lambda \to \mathbb{R}^n$ with $F(0) = 0$ and $DF(0) = 0$ such that

$$\begin{aligned}&\mathrm{supp}(T) \cap C_\Lambda \cap B^d(q,r) = \\ &\left\{ (x', x_m, F(x', x_m), \Phi(x', x_m, F(x', x_m))) : x' \in \Omega', x_m > \tilde{w}_1(x') \text{ with } (x', x_m) \in \Lambda \right\},\end{aligned} \tag{4.7}$$

  where $B^d(q,r)$ denotes the d-dimensional ball, $C_\Lambda \subset \mathbb{R}^d$ the cylinder above Λ and $\tilde{w}_1$ the first component of $\tilde{w}$.

Now consider a non-zero, smooth bump function $\rho : \mathbb{R}^{m-1} \to [0, \infty)$ with $\mathrm{supp}(\rho) \subset \Lambda \cap \Omega'$ and

$$\|\rho\|_{C^{\ell,\alpha}} < \frac{\mu\eta}{4(1 + \|F\|_{C^{\ell,\alpha}})}.$$

Define $v : \Omega \to \mathbb{R}^{1+n}$ by

$$v(x') = (\tilde{w}_1(x') + \rho(x'), F(x', \tilde{w}_1(x') + \rho(x')))$$

and observe that denoting $\hat{b} := \Psi(v)$ we have

$$d(\tilde{b}, \hat{b}) < \mu\eta/2 \tag{4.8}$$

and that that $\mathbf{\Phi}(gr(v)) \subset \mathrm{supp}(T)$. Combining (4.8) and the fact that $\tilde{b} \in \mathcal{B}_{(1-\mu/2)\eta}$ we deduce that $\hat{b} \in \mathcal{B}_\eta$. Moreover by (4.6) and (4.8) we also have

$$d(b, \hat{b}) < 2\mu\eta. \tag{4.9}$$

Define the current

$$T' := T \llcorner \left\{ (x', x_m, F(x', x_m), y) \in \mathbb{R}^d : x' \in \Lambda \cap \Omega', \, \tilde{w}_1(x') < x_m < v_1(x') \right\}.$$

We repeat this construction in every connected component on b and call the resulting current T_i'. Consider the current

$$\hat{T} := T - \sum_{i=1}^{N} T_i'.$$

Since $T \in \mathbf{AMC}(\tilde{b})$, it follows that $\hat{T} \in \mathbf{AMC}(\hat{b})$. Indeed assuming by contradiction that $S \in \mathbf{AMC}(\hat{b})$ satisfies $\mathbb{M}(S) < \mathbb{M}(\hat{T})$, we obtain that $\partial(S + \sum_i T_i') = \tilde{b}$ and moreover, as $\mathrm{supp}(T_i')$ are disjoint and T_j' is the restriction of T to a set, we have

$$\mathbb{M}\Big(S + \sum_i T_i'\Big) \leq \mathbb{M}(S) + \sum_i \mathbb{M}(T_i') < \mathbb{M}(\hat{T}) + \sum_i \mathbb{M}(T_i') = \mathbb{M}(T),$$

which contradicts the minimality of T. We claim that $\hat{T}$ is the unique element of $\mathbf{AMC}(\hat{b})$. The validity of the claim concludes the proof due to (4.9) and the arbitrariness of μ.

We show the validity of the claim following [24]. Assume by contradiction that there exists a current $\hat{S} \in \mathbf{AMC}(\hat{b})$ with $\hat{S} \neq \hat{T}$. Define $S := \hat{S} + \sum_i T_i'$. By interior regularity, see [10, 11, 12], there exists a point $q^i \in \mathrm{Reg_i}(T) \cap \mathrm{Reg_i}(S) \cap \mathrm{supp}(\hat{b}) \setminus \mathrm{supp}(\tilde{b})$ in every connected component of $\mathrm{supp}(\hat{b}) \setminus \mathrm{supp}(\tilde{b})$. Since both T and S are smooth minimal surfaces in a neighbourhood of q^i which coincide on $\mathrm{supp}(T_i')$, the unique continuation principle of [25, Lemma 7.2] implies that there exists a neighbourhood of q^i where $\mathrm{supp}(S) = \mathrm{supp}(T)$. By [9, Theorem 2.1] we know that in every connected component of $\mathrm{supp}(\hat{T}) \setminus \mathrm{supp}(\partial\hat{T})$ and of $\mathrm{supp}(\hat{S}) \setminus \mathrm{supp}(\partial\hat{S})$ respectively, the sets of interior regular points $\mathrm{Reg_i}(\hat{T})$ and $\mathrm{Reg_i}(\hat{S})$ are connected. Moreover each of these connected components touch (at least) one of the q^i. We therefore conclude that $\mathrm{supp}(\hat{S}) = \mathrm{supp}(\hat{T})$.

Since the points of $\mathrm{supp}(\hat{b}) \cap B^d(q, r)$ are one-sided for $\hat{T}$, then the multiplicity (and the orientation) of $\hat{S}$ coincides with that of $\hat{T}$, which concludes the proof of the claim and of Proposition 4.2. □

Proof of Theorem 1.3. By Proposition 4.2, $\mathcal{B}_\eta \setminus NU$ is dense in $\mathcal{B}_\eta$ and by Lemma 4.1 it is also residual. □

Remark 4.3. The validity of Theorem 1.3 can be extended to the case in which $\mathcal{B}_\eta$ is replaced by the corresponding space of boundaries of class C^∞, where $\mathbf{X}_\eta$ is endowed with the classical metric inducing the smooth convergence. The argument for the proof differs only in the final steps. We begin the construction of the boundary $\hat{b}$ starting from $\tilde{b} := b_\mu$ and observe that $\tilde{b}$ could be an element of $\Psi(\mathcal{A})$, so that the regular boundary points of $T \in \mathbf{AMC}(\tilde{b})$ in a fixed connected component of $\mathrm{supp}(\tilde{b})$ might fail to have density $1/2$. Let us fix a point q in a connected component of $\mathrm{supp}(\tilde{b})$.

By [4, Theorem 3.2] the current T can be written as a sum of two area-minimizing currents $T_1 + T_2$, with

$$\mathbb{M}(T) = \mathbb{M}(T_1) + \mathbb{M}(T_2) \quad \text{and} \quad \mathbb{M}(\partial T) = \mathbb{M}(\partial T_1) + \mathbb{M}(\partial T_2),$$

where T_1 is multiplicity-one and $q \notin \operatorname{supp}(\partial T_2)$. Since T_1 has the same local structure which the current T has in the proof of Theorem 1.3, with same argument we can define a boundary $\hat{b}_1$ pushing the connected component of $\tilde{b}$ containing q inside the support of T_1 through a map φ and prove that the corresponding current $R := \hat{T}_1 + T_2$ is a minimizer for $\hat{b}_1$. Now consider any area-minimizing $S \neq R$ with $\partial S = \hat{b}_1$. Again by [4, Theorem 3.2] the current S can be written as a sum of two area-minimizing currents $S_1 + S_2$, with

$$\mathbb{M}(S) = \mathbb{M}(S_1) + \mathbb{M}(S_2) \quad \text{and} \quad \mathbb{M}(\partial S) = \mathbb{M}(\partial S_1) + \mathbb{M}(\partial S_2),$$

where S_1 is multiplicity-one and $\varphi(q) \notin \operatorname{supp}(\partial S_2)$. The unique continuation argument used in the proof of Theorem 1.3 guarantees that $\hat{T}_1 = S_1$.

Now define $\check{b}_1 := \hat{b}_1 - \partial T_2$, pick a connected component of the boundary ∂T_2 and iterate the procedure obtaining a new boundary $\check{b}_2$. Since such number of connected components is strictly decreasing along the iteration, there exists $M \in \mathbb{N}$ such that $\hat{b}_{M+1} = 0$. Obviously there is a unique area-minimizing current with boundary $\hat{b} := \check{b}_1 + \cdots + \check{b}_M$.

5 GENERIC UNIQUENESS WITH RESPECT TO THE FLAT NORM

In this section we aim to obtain a result in the spirit of Theorem 1.3, replacing the space $\mathcal{B}_\eta$ with a larger space of boundaries. On the other hand, the strong norm considered on $\mathcal{B}_\eta$ needs to be naturally substituted by a weaker one. In this section we work on the manifold $\mathcal{M} := \mathbb{R}^{m+n}$, with $m > 1$.

We fix an arbitrary $C > 0$, a compact, convex set $K \subset \mathbb{R}^{m+n}$ with nonempty interior and define

$$\mathcal{R}_C := \{b \in \mathscr{B}_{m-1}(K) \cap \mathscr{I}_{m-1}(K) : \mathbb{M}(b) \leq C\}. \tag{5.1}$$

We metrize $\mathcal{R}_C$ with the distance $d_\flat$ induced by the flat norm, see (2.3).

Lemma 5.1. *The space $(\mathcal{R}_C, d_\flat)$ is a nontrivial complete metric space.*

Proof. It is sufficient to prove that $\mathcal{R}_C$ is closed, then completeness follows from [15, §4.2.17]. Let b_j be a sequence of elements of $\mathcal{R}_C$ and let b be such that $\mathbb{F}(b_j - b) \to 0$. By the lower semicontinuity of the mass, we have $\mathbb{M}(b) \leq C$. For any $j \in \mathbb{N}$, let $T_j \in \mathbf{AMC}(b_j)$. By the isoperimetric inequality, see [15, §4.2.10], we have $\sup\{\mathbb{M}(T_j)\} < \infty$. By [15, §4.2.17], there exists $T \in \mathscr{I}_m(\mathbb{R}^{m+n})$ such that, up to (non relabeled) subsequences $\mathbb{F}(T_j - T) \to 0$. By the continuity of the boundary operator we have $\partial T = b$ and hence $b \in \mathcal{R}_C$. □

We state the main result of this section.

Theorem 5.2. *For the typical boundary $b \in \mathcal{R}_C$, the set $\mathbf{AMC}(b)$ is a singleton.*

In analogy with (4.2), we consider the following subset of $\mathcal{R}_C$:

$$\mathcal{NU}_C := \{b \in \mathcal{R}_C : \text{ there exist } T^1, T^2 \in \mathbf{AMC}(b) \text{ such that } T^1 \neq T^2\}.$$

The following lemma is the counterpart of Lemma 4.1 for the flat norm.

Lemma 5.3. *Assume that the set $\mathcal{R}_C \setminus \mathcal{NU}_C$ is dense in $\mathcal{R}_C$. Then it is residual.*

Proof. For $\mathbf{m} \in \mathbb{N} \setminus \{0\}$, consider the sets

$$\mathcal{NU}_C^{\mathbf{m}} := \{b \in \mathcal{R}_C : \text{ there exist } T^1, T^2 \in \mathbf{AMC}(b) \text{ with } \mathbb{F}(T^2 - T^1) \geq \mathbf{m}^{-1}\}.$$

It suffices to prove that $\mathcal{NU}_C^{\mathbf{m}}$ is closed for every $\mathbf{m}$. Consider a sequence b_j of elements of $\mathcal{NU}_C^{\mathbf{m}}$ and let $b \in \mathcal{R}_C$ be such that $\mathbb{F}(b_j - b) \to 0$. For every $j \in \mathbb{N}$, take $T_j^1, T_j^2 \in \mathbf{AMC}(b_j)$ with $\mathbb{F}(T_j^2 - T_j^1) \geq 1/\mathbf{m}$. As in the proof of Lemma 5.1, we deduce that there exist $T^1, T^2 \in \mathscr{I}_m(\mathbb{R}^{m+n})$ such that $\partial T^1 = \partial T^2 = b$ and, up to (non relabeled) subsequences, $\mathbb{F}(T_j^1 - T^1) \to 0$, $\mathbb{F}(T_j^2 - T^2) \to 0$ and $\mathbb{F}(T^2 - T^1) \geq 1/\mathbf{m}$. By [31, Theorem 34.5], we have $T_1, T_2 \in \mathbf{AMC}(b)$, hence $b \in \mathcal{NU}_C^{\mathbf{m}}$. □

To prove Theorem 5.2 we are left to show that the set of boundaries $b \in \mathcal{R}_C$ for which $\mathbf{AMC}(b)$ is a singleton is dense in the metric space $(\mathcal{R}_C, d_\flat)$. The proof can be roughly summarized as follows: firstly, we approximate $b \in \mathcal{R}_C$ with an integer polyhedral boundary $b_P \in \mathcal{R}_{C-\delta}$, for some $\delta > 0$, see Lemma 5.4. Then, we fix $S \in \mathbf{AMC}(b_P)$ and for every connected component of $\mathrm{Reg}_\mathrm{i}(S)$ there exists, by [10, 11, 12], an interior regular point x_i. We define the current $b' := \partial(S - S \llcorner \bigcup_i B(x_i, r_i))$ where r_i are suitably small radii, so that $b' \in \mathcal{R}_C$ and $\mathbb{F}(b - b')$ is small. An argument similar to that used in Proposition 4.2 proves that $\mathbf{AMC}(b')$ is a singleton.

Lemma 5.4. *For any $b \in \mathcal{R}_C$ and $\varepsilon > 0$ there exist a $\delta > 0$ and $b_P \in \mathcal{R}_{C-\delta} \cap \mathscr{P}_{m-1}(K)$ such that*

$$\mathbb{F}(b - b_P) \leq \varepsilon.$$

Proof. Without loss of generality and up to rescaling, we can assume $C = 1$. We consider a map $\phi : K \to K$ which is $(1 - \varepsilon/(4m))$-Lipschitz and $\|\mathrm{Id} - \phi\|_\infty < \varepsilon/2^m$. Consider $\bar{b} := \phi_\sharp b$. Applying the homotopy formula as in (2.6), we obtain that

$$\mathbb{F}(b - \bar{b}) \leq 2^{m-1} \|Id - \phi\|_\infty \leq \varepsilon/2 \tag{5.2}$$

and

$$\mathbb{M}(\bar{b}) \leq \left(1 - \frac{\varepsilon}{4m}\right)^{m-1} \mathbb{M}(b) \leq \left(1 - \frac{\varepsilon}{2}\right) \mathbb{M}(b).$$

Observe in particular that $\bar{b} \in \mathcal{R}_{C-\varepsilon/2}$ and moreover $\mathrm{supp}(\bar{b})$ is contained in the interior of K. We can thus apply [15, §4.2.21] to obtain an integer polyhedral current b_P such that

$$\mathbb{F}(b_P - \bar{b}) \leq \varepsilon/2, \quad \partial b_P = 0 \quad \text{and} \quad \mathbb{M}(b_P) \leq (1 + \varepsilon/2)\mathbb{M}(\bar{b}), \tag{5.3}$$

deducing from (5.2) and (5.3) that

$$\mathbb{F}(b_P - b) \leq \varepsilon \quad \text{and} \quad \mathbb{M}(b_P) \leq (1 - \varepsilon^2/4)\mathbb{M}(b).$$

In particular b_P satisfies the requirement of the lemma for $\delta = \varepsilon^2/4$.

□

Proof of Theorem 5.2. Fix $\varepsilon > 0$, $b \in \mathcal{R}_C$ and b_P as in Lemma 5.4 and consider $S \in \mathbf{AMC}(b_P)$. It is sufficient to prove the theorem assuming $\mathrm{Reg}_{\mathrm{i}}(S)$ is connected, indeed the same argument can be applied to each connected component of $\mathrm{Reg}_{\mathrm{i}}(S)$.

Let $x_0 \in \mathrm{Reg}_{\mathrm{i}}(S)$, so that there exists a positive radius $\bar{r} > 0$, a smooth embedded submanifold $\Sigma \subset \mathbb{R}^{m+n}$ and a positive integer Q such that $S \llcorner B(x_0, \bar{r}) = Q[\![\Sigma]\!]$. Fix some positive radius r such that $r < \bar{r}$ and define

$$S' := S - S \llcorner B(x_0, r) \quad \text{and} \quad b' := \partial S'.$$

Note that, since $b \in \mathcal{R}_{C-\delta}$, then for r sufficiently small $b' \in \mathcal{R}_C$. Note further that $S' \in \mathbf{AMC}(b')$, which can be proved by the same argument used in the proof of Proposition 4.2. Hence we only need to show that $\mathbf{AMC}(b') = \{S'\}$.

Suppose there exists $S'' \in \mathbf{AMC}(b')$ such that $S' \neq S''$ and denote $\hat{S} := S'' + S \llcorner B(x_0, r)$. Observe that since

$$\mathbb{M}(S) \leq \mathbb{M}(\hat{S}) \leq \mathbb{M}(S'') + \mathbb{M}(S \llcorner B(x_0, r)) \leq \mathbb{M}(S),$$

then $\hat{S} \in \mathbf{AMC}(b_P)$. By the minimality of S one immediately sees that $\mathrm{supp}(\hat{S}) \supset \mathrm{supp}(S) \cap B(x_0, r)$. By interior regularity, there exists $x_1 \in \partial B(x_0, r) \cap \mathrm{Reg}_{\mathrm{i}}(S) \cap \mathrm{Reg}_{\mathrm{i}}(\hat{S})$. For a sufficiently small radius ρ we can write

$$S \llcorner B(x_1, \rho) = Q_1[\![\Sigma_1]\!] \llcorner B(x_1, \rho) \quad \text{and} \quad \hat{S} \llcorner B(x_1, \rho) = Q_2[\![\Sigma_2]\!] \llcorner B(x_1, \rho).$$

By the same argument of Lemma 4.2, the two submanifolds Σ_1, Σ_2 must coincide locally around x_1. Since by [9, Theorem 2.1] $\mathrm{Reg}_{\mathrm{i}}(S')$ and $\mathrm{Reg}_{\mathrm{i}}(S'')$ are connected, unique continuation implies that $\mathrm{Reg}_{\mathrm{i}}(S') = \mathrm{Reg}_{\mathrm{i}}(S'')$. Since all points of $\partial B(x_0, r)$ have density $Q/2$, then the multiplicity (and the orientation) of S' coincides with that of S'', contradicting $S' \neq S''$. This proves that the set of boundaries $b \in \mathcal{R}_C$ for which $\mathbf{AMC}(b)$ is a singleton is dense in $(\mathcal{R}_C, d_\flat)$ and hence, by Lemma 5.3, we conclude the proof of Theorem 5.2. □

The preliminary approximation of Lemma 5.4 is motivated by the following remark.

Remark 5.5. Given an integral current $b \in \mathscr{B}_{m-1}(K) \cap \mathscr{I}_{m-1}(K)$ and $S \in \mathbf{AMC}(b)$, it is not possible to conclude $\mathrm{Reg}_{\mathrm{i}}(S) \neq \emptyset$, as the following example shows. Consider a sequence of positive real numbers r_j such that $\sum_j r_j < \infty$ and a sequence q_j that is dense in $B(0,1) \subset \mathbb{R}^2$. Denote $\rho_j := \min\{r_j, 1 - |q_j|\}$, and consider the balls $B(q_j, \rho_j)$ and the 2-dimensional current defined as

$$T := \sum_{j \in \mathbb{N}} [\![B(q_j, \rho_j)]\!].$$

Note T is well defined and has finite mass because $\sum_{j \in \mathbb{N}} \mathbb{M}([\![B(q_j, \rho_j)]\!]) < \infty$ and moreover

$$\mathbb{M}(\partial T) = \mathbb{M}\Big(\sum_{j \in \mathbb{N}} \partial [\![B(q_j, \rho_j)]\!]\Big) \leq \sum_{j \in \mathbb{N}} \mathbb{M}(\partial [\![B(q_j, \rho_j)]\!]). \tag{5.4}$$

Hence, by [31, Theorem 30.3], $T \in \mathscr{I}_2(\overline{B(0,1)})$. Moreover, since the intersection between two circumferences with different centers is $\mathscr{H}^1$-null, then the inequality in (5.4) is an equality and therefore $\partial B(q_j, \rho_j) \subset \mathrm{supp}(\partial T)$ for every j implying that $\mathrm{supp}(\partial T) = \overline{B(0,1)}$. Now take $S \in \mathbf{AMC}(\partial T)$. Since $\mathrm{supp}(S) \subset \mathrm{conv}(\mathrm{supp}(\partial T))$, we deduce that $\mathrm{Reg}_{\mathrm{i}}(S) \subset \mathrm{supp}(S) \setminus \mathrm{supp}(\partial T) = \emptyset$.

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