LAGRANGIAN STABILITY FOR A SYSTEM OF NON-LOCAL CONTINUITY EQUATIONS UNDER OSGOOD CONDITION

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Abstract. We extend known existence and uniqueness results of weak measure solutions for systems of non-local continuity equations beyond the usual Lipschitz regularity. Existence of weak measure solutions holds for uniformly continuous vector fields and convolution kernels, while uniqueness follows from a Lagrangian stability estimate under an additional Osgood condition.

1. Introduction

1.1. **Statement of the problem.** For fixed $T \in (0, +\infty)$ and $k, d \in \mathbb{N}$, we consider the system of non-local continuity equations

$$\begin{cases} \partial_t \varrho^i + \operatorname{div}\left(\varrho^i\, V^i(t, x, \varrho * \eta^i)\right) = 0, & t \in (0,T),\ x \in \mathbb{R}^d, \\ \varrho^i(0) = \bar\varrho^i, & i = 1, \dots, k, \end{cases} \tag{1.1}$$

where the unknown $\varrho = (\varrho^1, \dots, \varrho^k) \in L^\infty([0,T]; \mathcal{M}^+(\mathbb{R}^d)^k)$ is a time-dependent k-vector of non-negative Borel measures on $\mathbb{R}^d$, the initial datum $\bar\varrho = (\bar\varrho^1, \dots, \bar\varrho^k) \in \mathcal{M}^+(\mathbb{R}^d)^k$ is a k-vector of non-negative Borel measures,

$$V = (V^1, \dots, V^k) \in L^\infty([0,T]; C_b(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)^k) \tag{1.2}$$

is a uniformly-in-time bounded continuous k-vector field and

$$\eta^i = (\eta^{i,1}, \dots, \eta^{i,k}) \in L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^k)) \tag{1.3}$$

is a uniformly-in-time bounded continuous k-vector of convolution kernels, with the convolution $\varrho * \eta^i = (\varrho^1 * \eta^{i,1}, \dots, \varrho^k * \eta^{i,k})$ occurring in the space variable only. In the entire paper, we frequently consider the 1-norm (i.e., the sum of the absolute values of the entries) on both vectors and matrices. In particular, $|\varrho| = |\varrho^1| + \dots + |\varrho^k|$ and thus $\|\varrho\|_{\mathcal{M}} = \|\varrho^1\|_{\mathcal{M}} + \dots + \|\varrho^k\|_{\mathcal{M}}$ for all $\varrho \in \mathcal{M}(\mathbb{R}^d)$. When considering other norms, constants depending on d and/or k may be dropped without notice.

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Solutions of the system (1.1) are understood in the usual distributional sense, which is well-set thanks to (1.2) and (1.3).

Definition 1.1 (Weak solution)**.** We say that $\varrho \in L^\infty([0,T];\mathcal{M}^+(\mathbb{R}^d)^k)$ is a *weak solution* of the system (1.1) starting from the initial datum $\bar\varrho \in \mathcal{M}^+(\mathbb{R}^d)^k$ if

$$\int_0^T \int_{\mathbb{R}^d} \Big(\partial_t\varphi + V^i(t,x,\varrho * \eta^i)\cdot\nabla\varphi\Big)\,\mathrm{d}\varrho^i(t,x)\,\mathrm{d}t + \int_{\mathbb{R}^d}\varphi(0,x)\,\mathrm{d}\bar\varrho^i(x) = 0 \tag{1.4}$$

for each $i = 1,\dots,k$ and any $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^d)$.

Any weak solution in the sense of Definition 1.1 admits a weakly continuous representative in duality with the space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at infinity, see [2, Lem. 8.1.2] and [1, 14]. So, from now on, we restrict our attention to weakly-continuous weak solutions $\varrho \in C([0,T];\mathcal{M}^+(\mathbb{R}^d)^k - w^*)$ only.

The system (1.1) is used in several physical situations—for instance, pedestrian traffic, sedimentation models and supply chains—to describe the time evolution of the density of a vectorial quantity (e.g., pedestrians or particles), possibly concentrating in some points or along hypersurfaces. Far from being complete, we refer the reader for example to [4, 10–13, 16, 18, 21, 24, 25] for a panoramic on the related literature. Because of the physical relevance of the system (1.1), here we deal with non-negative solutions only.

The system (1.1) can be also interpreted in the sense of the Control Theory. Indeed, the convolution kernel η can be viewed as a *non-local* control for the (non-linear) PDE in (1.1). Therefore, assuming V is fixed for simplicity, any stability result for the solutions of the system (1.1) in terms of the convolution kernel η yields a continuous dependence of the curve $t \mapsto \varrho_t[\eta]$ solving (1.1) in terms of the control η.

The well-posedness of the system (1.1) was established in [14], provided that V and η are bounded and Lipschitz continuous uniformly in time, namely,

$$V \in L^\infty([0,T];\mathrm{Lip}_b(\mathbb{R}^d\times\mathbb{R}^k;\mathbb{R}^d)^k) \quad \text{and} \quad \eta \in L^\infty([0,T];\mathrm{Lip}_b(\mathbb{R}^d;\mathbb{R}^k)^k). \tag{1.5}$$

The crucial ingredient of [14] is a stability estimate (in terms of the 1-Wasserstein distance between two solutions, see [14, Prop. 4.2]) which, in turn, allows to obtain existence and uniqueness of the solution of (1.1) via a fix point argument. The idea of exploiting the Lipschitz regularity to gain stability of trajectories has been later applied to several other related problems, see [5, 7, 9, 17, 23] and the references therein for instance.

1.2. **Main results.** The aim of the present note is to prove the well-posedness of the system (1.1) under less restrictive assumptions than (1.5), that is, to extend the existence and uniqueness result of [14] beyond the Lipschitz regularity. Our interest is motivated by some recent works [1, 3, 6, 15, 19, 20] dealing with non-Lipschitz velocity fields.

Our first main result deals with the existence of weak solutions of the system (1.1), in the spirit of the celebrated Peano's Theorem. To this aim, we consider the following structural hypotheses (where *modulus of continuity* means a non-decreasing concave function vanishing continuously at zero):

(V) The vector field $V \in L^\infty([0,T];C_b(\mathbb{R}^d\times\mathbb{R}^k;\mathbb{R}^d)^k)$ satisfies

$$\operatorname*{ess\,sup}_{t\in[0,T]} |V(t,x,u) - V(t,y,v)| \le \omega_V(|x-y| + |u-v|) \quad \forall x,y\in\mathbb{R}^d,\ u,v\in\mathbb{R}^k, \tag{1.6}$$

where $\omega_V\colon [0,+\infty) \to [0,+\infty)$ is a modulus of continuity.

(η) For each $i = 1, \dots, k$, the convolution kernel $\eta^i \in L^\infty([0,T]; C_0(\mathbb{R}^d; \mathbb{R}^k))$ satisfies

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\eta^i(t,x) - \eta^i(t,y)| \le \omega_\eta(|x-y|) \quad \forall x, y \in \mathbb{R}^d, \tag{1.7}$$

where $\omega_\eta \colon [0,+\infty) \to [0,+\infty)$ is a modulus of continuity.

Theorem 1.2 (Existence)**.** *If* (V) *and* (η) *hold, then the system* (1.1) *admits a weak solution starting from any given initial datum in* $\mathcal{M}^+(\mathbb{R}^d)^k$.

To prove Theorem 1.2, we first consider the smoothed functions V_ε and η_ε and obtain a weak solution ϱ_ε of the corresponding system (1.1) for all $\varepsilon > 0$ in virtue of the main result of [14]. Then, we pass to the limit as $\varepsilon \to 0^+$ showing that ϱ_ε (weakly) converges to a weak solution of the system (1.1). The needed a priori compactness is achieved via an Aubin–Lion-type Lemma which is inspired by [15, Th. A.1].

In order to achieve uniqueness of weak solutions of the system (1.1), we need to impose a further *Osgood condition* on the composition of the two moduli of continuity of V and η:

(O) for each $\lambda > 0$, it holds

$$\int_{0+} \frac{\mathrm{d}r}{\omega_V(r + \lambda\,\omega_\eta(r))} = +\infty.$$

For example, condition (O) is satisfied as soon as $\omega_V \circ \omega_\eta$ is a log-linear function, such as $r|\log r|$, $r \log|\log r|$ and similar, with $r > 0$ sufficiently small.

Our uniqueness result deals with *Lagrangian* weak solutions of the system (1.1).

Definition 1.3 (Lagrangian weak solution)**.** A weak solution $\varrho \in L^\infty([0,T]; \mathcal{M}^+(\mathbb{R}^d)^k)$ of the system (1.1) starting from the initial datum $\bar\varrho \in \mathcal{M}^+(\mathbb{R}^d)^k$ is *Lagrangian* if

$$\varrho^i(t,\cdot) = X^i(t,\cdot)_\# \bar\varrho^i, \quad i = 1, \dots, k,$$

where $X^i \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the (classical) solution of the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X^i(t,x) &= V^i\Big(t, X^i(t,x), \varrho * \eta^i(t, X^i(t,x))\Big), & t \in (0,T),\ x \in \mathbb{R}^d, \\ X^i(0,x) &= x, & x \in \mathbb{R}^d. \end{cases} \tag{1.8}$$

Thanks to Proposition 1.4 below, the Osgood condition in (O) ensures the well-posedness of the ODE in (1.8).

Proposition 1.4 (Associated vector field)**.** *Let assumptions* (V) *and* (η) *be in force. If* $\varrho \in C([0,T]; \mathcal{M}^+(\mathbb{R}^d)^k\text{–}w^*)$ *is a weak solution of the system* (1.1) *starting from the initial datum* $\bar\varrho \in \mathcal{M}^+(\mathbb{R}^d)^k$, *then the vector field*

$$b^i_{V,\eta,\varrho}(t,x) = V^i\Big(t, x, \varrho * \eta^i(t,x)\Big), \quad t \in [0,T],\ x \in \mathbb{R}^d,\ i = 1, \dots, k, \tag{1.9}$$

appearing in (1.8) *satisfies* $b \in L^\infty([0,T]; C_b(\mathbb{R}^d; \mathbb{R}^d)^k)$ *with*

$$\operatorname*{ess\,sup}_{t\in[0,T]} |b_{V,\eta,\varrho}(t,x) - b_{V,\eta,\varrho}(t,y)| \lesssim \omega_V\Big(|x-y| + \|\bar\varrho\|_{\mathcal{M}}\,\omega_\eta(|x-y|)\Big) \quad \forall x, y \in \mathbb{R}^d.$$

With the above notation, our main uniqueness result reads as follows.

Theorem 1.5 (Uniqueness)**.** *If* (V), (η) *and* (O) *hold, then* (1.1) *admits a unique Lagrangian weak solution starting from any given initial datum in* $\mathcal{M}^+(\mathbb{R}^d)^k$.

The word “Lagrangian” in Theorem 1.5 can be dropped, since any weak solution of the system (1.1) is in fact Lagrangian because of [1, Th. 1] (also see [8]) and of Proposition 1.4. However, this regularity result is not at all elementary, so we prefer to state Theorem 1.5 for Lagrangian solutions only in order to emphasize what is possible to achieve just relying on our elementary approach.

The strategy of [14] exploits the linearity of ω_η in an essential way. Indeed, the authors need the Lipschitz continuity of η in order to recover the 1-Wasserstein distance between two weak solutions of (1.1) in terms of its dual Kantorovich–Rubinstein formulation (see [14, Lem. 4.1]). We do not know if the strategy of [14] can be adapted to deal with a more general modulus of continuity ω_η.

To overcome this issue, we adopt a different point of view, which is inspired by the elementary uniqueness result achieved in the recent work [15]. Instead of controlling the 1-Wasserstein distance between two weak solutions of the system (1.1), we exploit their Lagrangian property to quantitatively estimate the difference between the two associated ODE flows, thus providing a *Lagrangian* stability of weak solutions from which Theorem 1.5 immediately follows.

Theorem 1.6 (Lagrangian stability). *Let $V, U \in L^\infty([0,T]; C_b(\mathbb{R}^d \times \mathbb{R}^k; \mathbb{R}^d)^k)$ satisfy* (1.6) *with the same modulus of continuity ω_V and let $\eta, \nu \in L^\infty([0,T]; C_0(\mathbb{R}^d; \mathbb{R}^k)^k)$ satisfy* (1.7) *with the same same modulus of continuity ω_η. Let $\varrho, \sigma \in C([0,T]; \mathcal{M}^+(\mathbb{R}^d)^k - w^*)$ be two weak solutions of the system* (1.1) *starting from the initial data $\bar\varrho, \bar\sigma \in \mathcal{M}^+(\mathbb{R}^d)^k$, with vector fields V, U and convolution kernels η, ν, respectively. Assume that ϱ, σ are Lagrangian, namely, $\varrho = X(t,\cdot)_\# \bar\varrho$ and $\sigma = Y(t,\cdot)_\# \bar\sigma$ for $t \in [0,T]$, where X, Y are the flows solving the corresponding ODEs in* (1.8). *Then, there exists a modulus of continuity $\Omega\colon [0,+\infty) \to [0,+\infty)$, only depending on*

$$T,\ \|\bar\varrho\|_{\mathcal{M}},\ \|\bar\sigma\|_{\mathcal{M}},\ \|\eta\|_{L^\infty(C)},\ \|\nu\|_{L^\infty(C)},\ \omega_V,\ \omega_\eta,$$

such that

$$\sup_{t\in[0,T]} \|X(t,\cdot) - Y(t,\cdot)\|_{L^\infty} \le \Omega\left(\|\bar\varrho - \bar\sigma\|_{\mathcal{M}} + \|V - U\|_{L^\infty(C)} + \|\nu - \eta\|_{L^\infty(C)}\right). \tag{1.10}$$

The modulus of continuity Ω in Theorem 1.6 can be explicitly computed as soon as one can invert the integral function

$$G_{V,\eta,\lambda}(r) = \int_{r_0}^{r} \frac{\mathrm{d}s}{\omega_V(s + \lambda\, \omega_\eta(s))}, \quad r \ge 0,\ r_0 > 0, \tag{1.11}$$

naturally brought by the Osgood condition assumed in (O). In fact, the stability estimate (1.10) follows by simply differentiating a localized integral distance between the flows with respect to the time variable, and then applying the classical Bihari–LaSalle inequality (see [22, Th. 2.3.1] for instance) with Osgood modulus of continuity $r \to \omega_V(r + \lambda\, \omega_\eta(r))$, for some specific parameter $\lambda > 0$ depending on $\|\bar\varrho\|_{\mathcal{M}}$ and $\|\bar\sigma\|_{\mathcal{M}}$.

Theorem 1.6 clearly rephrases as a stability result of the flow of the ODE in (1.8). From the point of view of Control Theory, the stability estimate in (1.10) yields a continuous dependence of the (Lagrangian) solutions of the system (1.1), i.e., of the flows induced by the corresponding ODE in (1.8), in terms of the (non-local) control given by the convolution kernel, as well as of the velocity vector field and of the initial datum.

2. Proofs

2.1. **Existence of weak solutions.** To prove Theorem 1.2, we need some preliminary results. We begin with an Aubin–Lions-type Lemma, which is inspired by [15, Th. A.1].

Lemma 2.1 (Compactness)**.** *Let* $(\varrho_n)_{n\in\mathbb{N}} \subset C([0,T];\mathcal{M}(\mathbb{R}^d)-w^*)$ *be such that*

$$\sup_{n\in\mathbb{N}} \|\varrho_n\|_{L^\infty(\mathcal{M})} < +\infty. \tag{2.1}$$

Assume that, for each $\varphi \in C_c^\infty(\mathbb{R}^d)$*, the functions* $F_n[\varphi]\colon [0,T] \to \mathbb{R}$*, given by*

$$F_n[\varphi](t) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\varrho_n(t,\cdot), \quad t \in [0,T],$$

are uniformly equicontinuous on $[0,T]$*, that is,*

$$\forall\varepsilon > 0\ \exists\delta > 0\ :\ s,t \in [0,T],\ |s-t| < \delta \implies \sup_{n\in\mathbb{N}} |F_n[\varphi](s) - F_n[\varphi](t)| < \varepsilon. \tag{2.2}$$

Then, there exist a subsequence $(\varrho_{n_k})_{k\in\mathbb{N}}$ *and* $\varrho \in C([0,T],\mathcal{M}(\mathbb{R}^d)-w^*)$ *such that*

$$\lim_{k\to+\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\varrho_{n_k}(t,\cdot) - \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\varrho(t,\cdot) \right| = 0 \tag{2.3}$$

for all $\varphi \in C_0(\mathbb{R}^d)$*.*

Proof. Let $\mathscr{D} \subset C_c(\mathbb{R}^d)$ be a countable and dense set in $C_0(\mathbb{R}^d)$. In virtue of (2.1) and (2.2), for each $\varphi \in \mathscr{D}$ the sequence $(F_n[\varphi])_{n\in\mathbb{N}}$ is equibounded and equicontinuous on $[0,T]$. By Ascoli–Arzelà Theorem and a standard diagonal argument, we can find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that, for each $\varphi \in \mathscr{D}$, the sequence $(F_{n_k}[\varphi])_{k\in\mathbb{N}}$ is uniformly convergent to some $F[\varphi] \in C([0,T])$, with

$$\|F[\varphi]\|_{L^\infty([0,T])} \le \|\varphi\|_{L^\infty} \sup_{n\in\mathbb{N}} \|\varrho_n\|_{L^\infty(\mathcal{M})}. \tag{2.4}$$

By construction, the function $\varphi \mapsto F[\varphi](t)$ is a continuous linear functional on $\mathscr{D}$ for each $t \in [0,T]$. Thus, for each fixed $t \in [0,T]$, we can extend the map $\varphi \mapsto F[\varphi](t)$ to a linear and continuous functional on $C_0(\mathbb{R}^d)$ for which we keep the same notation. A plain approximation argument readily proves that, for each $\varphi \in C_0(\mathbb{R}^d)$, the map $t \mapsto F[\varphi](t)$ is continuous on $[0,T]$ and satisfies (2.4). By Riesz's Representation Theorem, for each $t \in [0,T]$ there exists a finite Borel measure $\varrho(t,\cdot) \in \mathcal{M}(\mathbb{R}^d)$ such that

$$F[\varphi](t) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\varrho(t,\cdot) \quad \text{for all } \varphi \in C_0(\mathbb{R}^d),$$

so that $\varrho \in C([0,T];\mathcal{M}(\mathbb{R}^d)-w^*)$. Finally, in virtue of (2.1) and (2.4), for $\varphi \in C_0(\mathbb{R}^d)$ and $\psi \in \mathscr{D}$, we can estimate

$$\sup_{t\in[0,T]} |F_{n_k}[\varphi](t) - F[\varphi](t)| \le \sup_{t\in[0,T]} |F_{n_k}[\psi](t) - F[\psi](t)| + 2\,\|\psi-\varphi\|_{L^\infty} \sup_{n\in\mathbb{N}} \|\varrho_n\|_{L^\infty(\mathcal{M})}$$

and the desired (2.3) readily follows. □

In order to exploit Lemma 2.1, we need the following mass preservation property for weak solutions of the system (1.1).

Lemma 2.2 (Mass preservation). *Let V and η be as in* (1.2) *and* (1.3)*, respectively. If $\varrho \in C([0,T];\mathcal{M}^+(\mathbb{R}^d)^k\!-\!w^*)$ is a weak solution of the system* (1.1) *starting from the initial datum $\bar\varrho \in \mathcal{M}^+(\mathbb{R}^d)^k$, then*

$$\|\varrho^i(t,\cdot)\|_{\mathcal{M}} = \|\bar\varrho^i\|_{\mathcal{M}} \tag{2.5}$$

for $t \in [0,T]$ and $i = 1,\dots,k$.

Proof. Let $i \in \{1,\dots,k\}$ be fixed. By applying (1.4) to the test function $\varphi(t,x) = \alpha(t)\,\beta(x)$, $(t,x) \in [0,T]\times\mathbb{R}^d$, where $\alpha \in C^\infty_c([0,T))$ and $\beta \in C^\infty_c(\mathbb{R}^d)$, we get

$$\int_0^T \int_{\mathbb{R}^d} \Big(\alpha'\beta + \alpha\, V^i(t,x,\varrho*\eta^i)\cdot\nabla\beta\Big)\,\mathrm{d}\varrho^i(t,\cdot)\,\mathrm{d}t + \alpha(0)\int_{\mathbb{R}^d}\beta(x)\,\mathrm{d}\bar\varrho^i = 0.$$

Since $\alpha \in C^\infty_c([0,T))$ is arbitrary and $\varrho \in C([0,T];\mathcal{M}^+(\mathbb{R}^d)^k\!-\!w^*)$, we infer that

$$t \mapsto \int_{\mathbb{R}^d}\beta\,\mathrm{d}\varrho^i(t,\cdot) \in AC^{1,1}([0,T];\mathbb{R}) \tag{2.6}$$

with

$$\int_{\mathbb{R}^d}\beta\,\mathrm{d}\varrho^i(t,\cdot) = \int_{\mathbb{R}^d}\beta\,\mathrm{d}\bar\varrho^i + \int_0^t\int_{\mathbb{R}^d} V^i(s,\cdot,\varrho*\eta^i)\cdot\nabla\beta\,\mathrm{d}\varrho^i(s,\cdot)\,\mathrm{d}s \tag{2.7}$$

for all $t\in[0,T]$. Now let $t\in[0,T]$ be fixed. We let $(\beta_R)_{R>0}\subset C^\infty_c(\mathbb{R}^d)$ be such that

$$\beta_R \ge 0, \quad \operatorname{supp}\beta_R \subset B_{2R}, \quad \beta_R = 1 \text{ on } B_R, \quad \|\nabla\beta_R\|_{L^\infty} \le \frac{2}{R}$$

for all $R>0$. By the Monotone Convergence Theorem, we infer that

$$\lim_{R\to+\infty}\int_{\mathbb{R}^d}\beta_R\,\mathrm{d}\varrho^i(t,\cdot) = \|\varrho^i(t,\cdot)\|_{\mathcal{M}}$$

as well as

$$\lim_{R\to+\infty}\int_{\mathbb{R}^d}\beta_R\,\mathrm{d}\bar\varrho^i = \|\bar\varrho^i\|_{\mathcal{M}}.$$

Since

$$\left|\int_0^t\int_{\mathbb{R}^d} V^i(s,\cdot,\varrho*\eta^i)\cdot\nabla\beta_R\,\mathrm{d}\varrho^i(s,\cdot)\,\mathrm{d}s\right| \le \frac{2}{R}\,\|\varrho^i\|_{L^\infty(\mathcal{M})}\,\|V^i\|_{L^\infty(C)}$$

for all $R>0$, we get (2.5) by applying (2.7) to β_R and passing to the limit as $R\to+\infty$. □

We are ready to prove our existence result.

Proof of Theorem 1.2. Let $(\ell_\varepsilon)_{\varepsilon>0}\subset C^\infty_c(\mathbb{R}^{d+k})$ and $(\jmath_\varepsilon)_{\varepsilon>0}\in C^\infty_c(\mathbb{R}^d)$ be two families of standard non-negative mollifiers and set

$$V^{i,j}_\varepsilon(t,\cdot) = V^{i,j}(t,\cdot)*\ell_\varepsilon, \quad \eta^{i,j}_\varepsilon = \eta^{i,j}(t,\cdot)*\jmath_\varepsilon,$$

where in both cases the (component-wise) convolution occur in the spatial variables only. Since V_ε and η_ε clearly satisfy the Lipschitz property (1.5) for each $\varepsilon>0$, by [14, Th. 1.1] there exists a weak solution

$$\varrho_\varepsilon \in C([0,T],\mathcal{M}^+(\mathbb{R}^d)^k\!-\!w^*)$$

of the system (1.1) starting from the initial datum $\bar\varrho\in\mathcal{M}^+(\mathbb{R}^d)^k$, so that

$$\int_0^T\int_{\mathbb{R}^d}\Big(\partial_t\varphi + V^i_\varepsilon(t,\cdot,\varrho_\varepsilon*\eta^i_\varepsilon)\cdot\nabla\varphi\Big)\,\mathrm{d}\varrho^i_\varepsilon(t,\cdot)\,\mathrm{d}t + \int_{\mathbb{R}^d}\varphi(0,\cdot)\,\mathrm{d}\bar\varrho^i = 0 \tag{2.8}$$

for each $i=1,\dots,k$ and $\varepsilon>0$ and $\varphi\in C^\infty_c([0,T)\times\mathbb{R}^d)$.

Now let $i \in \{1, \dots, k\}$ be fixed. We claim that (any sequence in) the family $(\varrho^i_\varepsilon)_{\varepsilon>0}$ satisfies the assumptions (2.1) and (2.2) of Lemma 2.1. Indeed, from Lemma 2.2 we get

$$\|\varrho^i_\varepsilon(t,\cdot)\|_{\mathcal{M}} = \|\bar{\varrho}^i\|_{\mathcal{M}} \tag{2.9}$$

for all $t \in [0,T]$ and $\varepsilon > 0$, from which (2.1) immediately follows. To prove (2.2), we simply argue as in the proof of Lemma 2.2. Recalling (2.6) and (2.7), we easily recognize that the time derivative of the function

$$F_\varepsilon[\beta](t) = \int_{\mathbb{R}^d} \beta(\cdot)\,\mathrm{d}\varrho^i_\varepsilon(t,\cdot), \quad t \in [0,T], \tag{2.10}$$

is bounded by

$$\left| \int_{\mathbb{R}^d} V^i_\varepsilon(t,x,\varrho_\varepsilon * \eta^i_\varepsilon) \cdot \nabla\beta \,\mathrm{d}\varrho^i_\varepsilon(t,x) \right| \le \|V^i\|_{L^\infty(C)}\, \|\nabla\beta\|_{L^\infty}\, \|\bar{\varrho}^i\|_{\mathcal{M}}$$

for a.e. $t \in [0,T]$ and for each $\varepsilon > 0$. In particular, the family $(F_\varepsilon[\beta])_{\varepsilon>0}$ in (2.10) is equi-Lipschitz and thus satisfies (2.2). Therefore, by Lemma 2.1, we find a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}} \subset C([0,T];\mathcal{M}^+(\mathbb{R}^d)^k - w^*)$ and $\varrho \in C([0,T];\mathcal{M}^+(\mathbb{R}^d)^k - w^*)$ such that

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \left| \int_{\mathbb{R}^d} \beta\,\mathrm{d}\varrho_{\varepsilon_n}(t,\cdot) - \int_{\mathbb{R}^d} \beta\,\mathrm{d}\varrho(t,\cdot) \right| = 0 \tag{2.11}$$

for all $\beta \in C_0(\mathbb{R}^d)$.

To conclude, we just need to prove that ϱ is a weak solution of (1.1) starting from the initial datum $\bar{\varrho}$. We do so by passing to the limit in (2.8) along $(\varepsilon_n)_{n\in\mathbb{N}}$ as $n \to +\infty$ for each given $\varphi \in C^\infty_c([0,+\infty) \times \mathbb{R}^d)$. Indeed, on the one side, since

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \partial_t\varphi\,\mathrm{d}\varrho^i_{\varepsilon_n}(t,\cdot) = \int_{\mathbb{R}^d} \partial_t\varphi\,\mathrm{d}\varrho^i(t,\cdot)$$

because of (2.11) and

$$\left| \int_{\mathbb{R}^d} \partial_t\varphi\,\mathrm{d}\varrho^i_{\varepsilon_n}(t,\cdot) \right| \le \|\partial_t\varphi\|_{L^\infty}\, \|\bar{\varrho}^i\|_{\mathcal{M}}$$

because of (2.9), for all $t \in [0,T]$, by the Dominated Convergence Theorem we infer that

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^d} \partial_t\varphi\,\mathrm{d}\varrho^i_{\varepsilon_n}(t,\cdot)\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \partial_t\varphi\,\mathrm{d}\varrho^i(t,\cdot)\,\mathrm{d}t. \tag{2.12}$$

On the other side, since $\eta^i(t,\cdot) \in C_0(\mathbb{R}^d)$ in virtue of the assumption (η), we have that $\eta^i_{\varepsilon_n}(t,\cdot) \to \eta^i(t,\cdot)$ in $C_0(\mathbb{R}^d)$ as $n \to +\infty$, so that

$$\begin{aligned} \lim_{n\to+\infty} \Big(\varrho_{\varepsilon_n}(t,\cdot) * \eta^i_{\varepsilon_n}(t,\cdot)\Big)(x) &= \lim_{n\to+\infty} \int_{\mathbb{R}^d} \eta^i_{\varepsilon_n}(t,x-y)\,\mathrm{d}\varrho_{\varepsilon_n}(t,y) \\ &= \int_{\mathbb{R}^d} \eta^i(t,x-y)\,\mathrm{d}\varrho(t,y) = \Big(\varrho(t,\cdot) * \eta^i(t,\cdot)\Big)(x) \end{aligned} \tag{2.13}$$

for each $x \in \mathbb{R}^d$ and all $t \in [0,T]$ as a weak-strong convergent pair, due to (2.11). Moreover, again in virtue of (2.9) and (η), we can estimate

$$\|\varrho_{\varepsilon_n}(t,\cdot) * \eta^i_{\varepsilon_n}(t,\cdot)\| \le \|\bar{\varrho}^i\|_{\mathcal{M}}\, \|\eta^i\|_{L^\infty(C)}$$

and

$$\begin{aligned} &\Big| \Big(\varrho_{\varepsilon_n}(t,\cdot) * \eta^i_{\varepsilon_n}(t,\cdot)\Big)(x) - \Big(\varrho_{\varepsilon_n}(t,\cdot) * \eta^i_{\varepsilon_n}(t,\cdot)\Big)(y) \Big| \\ &\qquad \le \int_{\mathbb{R}^d} \Big|\eta^i_{\varepsilon_n}(t,x-\cdot) - \eta^i_{\varepsilon_n}(t,y-\cdot)\Big|\,\mathrm{d}\varrho_{\varepsilon_n}(t,\cdot) \le \omega_\eta(|x-y|)\, \|\bar{\varrho}^i\|_{\mathcal{M}} \end{aligned}$$

for all $n \in \mathbb{N}$ and $t \in [0,T]$. By Arzelà–Ascoli's Theorem, we thus get that the pointwise convergence in (2.13) must be uniform on compact sets in $\mathbb{R}^d$, uniformly in $t \in [0,T]$. An analogous argument relying on the assumption (V) proves that also $V^i_{\varepsilon_n}(t,\cdot) \to V^i(t,\cdot)$ as $n \to +\infty$ uniformly on compact sets in $\mathbb{R}^d$, uniformly in $t \in [0,T]$. Again by (2.11), by weak-strong convergence and by the Dominated Convergence Theorem, we hence get

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^d} V^i_{\varepsilon_n}(t,\cdot,\varrho_{\varepsilon_n} * \eta^i_{\varepsilon_n}) \cdot \nabla\varphi \, \mathrm{d}\varrho^i_{\varepsilon_n}(t,\cdot)\, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} V^i(t,\cdot,\varrho * \eta^i) \cdot \nabla\varphi \, \mathrm{d}\varrho^i(t,\cdot)\, \mathrm{d}t. \tag{2.14}$$

Thus, the conclusion follows by combining (2.12) with (2.14). □

2.2. **Lagrangian stability.** We deal with the Lagrangian stability of weak solutions. We begin with the proof of Proposition 1.4.

Proof of Proposition 1.4. Let $t \in [0,T]$ be fixed. Given $x, y \in \mathbb{R}^d$ and $i \in \{1,\dots,k\}$, in virtue of assumption (η) and of Lemma 2.2, we can estimate

$$\begin{aligned}
|\varrho * \eta^i(t,x) - \varrho * \eta^i(t,y)| &\le \sum_{j=1}^k \int_{\mathbb{R}^d} |\eta^{i,j}(t,x-z) - \eta^{i,j}(t,y-z)| \, \mathrm{d}\varrho^j(t,z) \\
&\le \sum_{j=1}^k \int_{\mathbb{R}^d} \omega_\eta(|x-y|) \, \mathrm{d}\varrho^k(t,z) = \|\varrho(t,\cdot)\|_{\mathcal{M}} \, \omega_\eta(|x-y|) \\
&= \|\bar\varrho\|_{\mathcal{M}} \, \omega_\eta(|x-y|).
\end{aligned}$$

Thus, thanks to assumption (V), we get that

$$\begin{aligned}
\Big|V^i\Big(t,x,\varrho * \eta^i(t,x)\Big) - V^i\Big(t,y,\varrho * \eta^i(t,y)\Big)\Big| &\le \omega_V\Big(|x-y| + |\varrho * \eta^i(t,x) - \varrho * \eta^i(t,y)|\Big) \\
&\le \omega_V\Big(|x-y| + \|\bar\varrho\|_{\mathcal{M}} \, \omega_\eta(|x-y|)\Big)
\end{aligned}$$

and the conclusion immediately follows. □

We conclude our paper with the proof of Theorem 1.6.

Proof of Theorem 1.6. Let V, U, η, ν, $\bar\varrho, \bar\sigma$, X, Y and ϱ, σ be as in the statement. Fix $\zeta \in C(\mathbb{R}^d)$ with $\zeta \ge 0$ and $\int_{\mathbb{R}^d} \zeta(x)\, \mathrm{d}x = 1$. Letting $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ be defined by $\mu = |\bar\varrho| + |\bar\sigma| + \zeta \mathscr{L}^d$, we consider the quantity

$$Q_\zeta(t) = \sum_{i=1}^k \fint_{\mathbb{R}^d} |X^i(t,\cdot) - Y^i(t,\cdot)| \, \mathrm{d}\mu$$

for all $t \in [0,T]$. Note that $t \mapsto Q_\zeta(t)$ is well defined and Lipschitz, with $Q_\zeta(0) = 0$ and

$$|Q_\zeta(s) - Q_\zeta(t)| \le k \left(\|U\|_{L^\infty(C)} + \|V\|_{L^\infty(C)}\right) |s-t|$$

for all $s, t \in [0,T]$. Therefore, for a.e. $t \in [0,T]$, we can write

$$\begin{aligned}
Q'_\zeta(t) &\le \sum_{i=1}^k \fint_{\mathbb{R}^d} |V^i(t,X^i,\varrho * \eta^i(t,X^i)) - U^i(t,Y^i,\sigma * \nu^i(t,Y^i))| \, \mathrm{d}\mu \\
&\le \sum_{i=1}^k (1)_i + (2)_i + (3)_i + (4)_i,
\end{aligned}$$

where (dropping the variables of X and Y for notational convenience)

$$\begin{aligned}
(1)_i &= \fint_{\mathbb{R}^d} |V^i(t, X^i, \varrho * \eta^i(t, X^i)) - V^i(t, Y^i, \varrho * \eta^i(t, Y^i))| \, \mathrm{d}\mu, \\
(2)_i &= \fint_{\mathbb{R}^d} |V^i(t, Y^i, \varrho * \eta^i(t, Y^i)) - V^i(t, Y^i, \sigma * \eta^i(t, Y^i))| \, \mathrm{d}\mu, \\
(3)_i &= \fint_{\mathbb{R}^d} |V^i(t, Y^i, \sigma * \eta^i(t, Y^i)) - V^i(t, Y^i, \sigma * \nu^i(t, Y^i))| \, \mathrm{d}\mu, \\
(4)_i &= \fint_{\mathbb{R}^d} |V^i(t, Y^i, \sigma * \nu^i(t, Y^i)) - U^i(t, Y^i, \sigma * \nu^i(t, Y^i))| \, \mathrm{d}\mu.
\end{aligned}$$

We now estimate each term separately at a given $t \in [0, T]$. By Proposition 1.4 and Jensen's inequality, we can easily estimate the first term as

$$\begin{aligned}
(1)_i &\le \fint_{\mathbb{R}^d} \omega_V\Big(|X^i - Y^i| + \|\bar\varrho\|_{\mathcal{M}} \, \omega_\eta(|X^i - Y^i|)\Big) \, \mathrm{d}\mu \\
&\le \omega_V \left(\fint_{\mathbb{R}^d} |X^i - Y^i| \, \mathrm{d}\mu + \|\bar\varrho\|_{\mathcal{M}} \, \omega_\eta \left(\fint_{\mathbb{R}^d} |X^i - Y^i| \, \mathrm{d}\mu \right) \right) \\
&\le \omega_V\Big(Q_\zeta(t) + \|\mu\|_{\mathcal{M}} \, \omega_\eta(Q_\zeta(t))\Big).
\end{aligned}$$

Concerning the second term, since

$$\begin{aligned}
|(\varrho - \sigma) * \eta^i(t, x)| &= \left| \int_{\mathbb{R}^d} \eta^i(t, x - y) \, \mathrm{d}(X_\# \bar\varrho(y) - Y_\# \bar\sigma(y)) \right| \\
&\le \sum_{j=1}^k \int_{\mathbb{R}^d} |\eta^{i,j}(t, x - X^j) - \eta^{i,j}(t, x - Y^j)| \, \mathrm{d}\bar\varrho^j + \int_{\mathbb{R}^d} |\eta^{i,j}(t, x - Y^j)| \, \mathrm{d}|\bar\varrho^j - \bar\sigma^j| \\
&\le \sum_{j=1}^k \int_{\mathbb{R}^d} \omega_\eta(|X^j - Y^j|) \, \mathrm{d}\bar\varrho^j + \|\eta\|_{L^\infty(C)} \|\bar\varrho^j - \bar\sigma^j\|_{\mathcal{M}} \\
&\le \int_{\mathbb{R}^d} \omega_\eta \left(\sum_{j=1}^k |X^j - Y^j| \right) \mathrm{d}|\bar\varrho| + \|\eta\|_{L^\infty(C)} \|\bar\varrho - \bar\sigma\|_{\mathcal{M}}
\end{aligned}$$

for all $x \in \mathbb{R}^d$, again by Jensen's inequality we get

$$\begin{aligned}
(2)_i &\le \fint_{\mathbb{R}^d} \omega_V\Big(|(\varrho - \sigma) * \eta^i(t, Y^i)|\Big) \, \mathrm{d}\mu \\
&\le \omega_V \left(\int_{\mathbb{R}^d} \omega_\eta \left(\sum_{i=1}^k |X^i - Y^i| \right) \mathrm{d}|\bar\varrho| + \|\eta\|_{L^\infty(C)} \|\bar\varrho - \bar\sigma\|_{\mathcal{M}} \right) \\
&\le \omega_V \left(\|\mu\|_{\mathcal{M}} \, \omega_\eta(Q_\zeta(t)) + \|\eta\|_{L^\infty(C)} \|\bar\varrho - \bar\sigma\|_{\mathcal{M}} \right) \\
&\le \omega_V\Big(Q_\zeta(t) + \|\mu\|_{\mathcal{M}} \, \omega_\eta(Q_\zeta(t))\Big) + \omega_V \left(\|\eta\|_{L^\infty(C)} \|\bar\varrho - \bar\sigma\|_{\mathcal{M}} \right).
\end{aligned}$$

The last two terms can be trivially estimated as

$$\begin{aligned}
(3)_i &\le \omega_V\Big(\|\sigma\|_{L^\infty(\mathcal{M})} \, \|\eta - \nu\|_{L^\infty(C)}\Big) = \omega_V\Big(\|\bar\sigma\|_{\mathcal{M}} \, \|\eta - \nu\|_{L^\infty(C)}\Big) \\
&\le \omega_V\Big(\|\mu\|_{\mathcal{M}} \, \|\eta - \nu\|_{L^\infty(C)}\Big)
\end{aligned}$$

thanks to Lemma 2.2, and

$$(4)_i \le \|V - U\|_{L^\infty(C)}.$$

Putting everything altogether, we conclude that

$$Q'_\zeta(t) \lesssim \omega_V\Big(Q_\zeta(t) + \lambda\,\omega_\eta(Q_\zeta(t))\Big) + M,$$

where $\lambda = \|\bar{\varrho}\|_{\mathcal{M}} + \|\bar{\sigma}\|_{\mathcal{M}} + 1$ and

$$M = \omega_V\left(\|\eta\|_{L^\infty(C)}\|\bar{\varrho} - \bar{\sigma}\|_{\mathcal{M}}\Big) + \omega_V\Big(\lambda\,\|\eta - \nu\|_{L^\infty(C))}\right) + \|V - U\|_{L^\infty(C)}.$$

At this point, we just need to recall the Osgood condition assumed in (O) and the integral function in (1.11). Indeed, by the classical Bihari–LaSalle inequality (see [22, Th. 2.3.1] for instance), we find a modulus of continuity $\Omega\colon [0,+\infty) \to [0,+\infty)$, only depending on

$$T,\ \|\bar{\varrho}\|_{\mathcal{M}},\ \|\bar{\sigma}\|_{\mathcal{M}},\ \|\eta\|_{L^\infty(C)},\ \|\nu\|_{L^\infty(C)},\ \omega_V,\ \omega_\eta,$$

such that

$$\sup_{t\in[0,T]} Q_\zeta(t) \le \Omega\left(\|\bar{\varrho} - \bar{\sigma}\|_{\mathcal{M}} + \|V - U\|_{L^\infty(C)} + \|\nu - \eta\|_{L^\infty(C)}\right). \tag{2.15}$$

We remark that Ω is independent of ζ, as long as we choose $\zeta \ge 0$ and $\|\zeta\|_{L^1} = 1$. To conclude, we choose a family $(\zeta_{x_0,\varepsilon})_{\varepsilon>0}$ of standard mollifiers around $x_0 \in \mathbb{R}^d$. Since the flows $X(t,\cdot), Y(t,\cdot)$ are continuous maps, we deduce that

$$\lim_{\varepsilon\to 0^+} Q_{\zeta_{x_0,\varepsilon}}(t) = |X(t,x_0) - Y(t,x_0)|. \tag{2.16}$$

Thus, (1.10) follows from (2.15) and (2.16) and the proof is complete. □

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