FULL DOUBLE HÖLDER REGULARITY OF THE PRESSURE IN BOUNDED DOMAINS

LUIGI DE ROSA, MICKAËL LATOCCA, AND GIORGIO STEFANI

Abstract. We consider Hölder continuous weak solutions $u \in C^\gamma(\Omega)$, $u \cdot n|_{\partial\Omega} = 0$, of the incompressible Euler equations on a bounded and simply connected domain $\Omega \subset \mathbb{R}^d$. If Ω is of class $C^{2,\delta}$, for some $\delta > 0$, then the corresponding pressure satisfies $p \in C^{2\gamma}_*(\Omega)$ in the case $\gamma \in (0, \frac{1}{2}]$, where $C^{2\gamma}_*$ is the Hölder–Zygmund space, which coincides with the usual Hölder space for $\gamma < \frac{1}{2}$. This result, together with our previous one in [10] covering the case $\gamma \in (\frac{1}{2}, 1)$, yields the full double regularity of the pressure on bounded and sufficiently regular domains. The interior regularity comes from the corresponding $C^{2\gamma}_*$ estimate for the pressure on the whole space $\mathbb{R}^d$, which in particular extends and improves the known double regularity results (in the absence of a boundary) in the borderline case $\gamma = \frac{1}{2}$. The boundary regularity features the use of local normal geodesic coordinates, pseudodifferential calculus and a fine Littlewood–Paley analysis of the modified equation in the new coordinate system.

1. Introduction

Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class C^2. The time evolution in Ω of an incompressible inviscid fluid is described by the *Euler equations*

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p &= 0 \quad \text{in } \Omega \times (0,T) \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T) \\ u \cdot n &= 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where $u\colon \Omega \times (0,T) \to \mathbb{R}^d$ and $p\colon \Omega \times (0,T) \to \mathbb{R}$ are the *velocity* of the fluid and its *hydrodynamic pressure*, respectively, and $n\colon \partial\Omega \to \mathbb{R}^d$ is the outward unit normal to $\partial\Omega$. The boundary condition $u(\cdot,t) \cdot n = 0$ on $\partial\Omega$ is the usual *no-flow condition*, which prohibits the fluid to cross the boundary of the container Ω.

1.1. The pressure equation.

The pressure equation. Forgetting about the time dependence of the unknowns and only focusing on the spatial one, straightforward computations yield that the pressure p solves the following elliptic Neumann boundary value problem

$$\begin{cases} -\Delta p &= \operatorname{div}\operatorname{div}(u \otimes u) \quad \text{in } \Omega \\ \partial_n p &= u \otimes u : \nabla n \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Date: January 16, 2023.

2010 *Mathematics Subject Classification.* 76B03, 35D30, 35J15, 35J25.

Key words and phrases. Incompressible fluids, hydrodynamic pressure, boundary Schauder's estimates, Littlewood–Paley analysis.

Acknowledgements. The first-named author has been partially funded by the SNF grant FLUTURA: Fluids, Turbulence, Advection No. 212573. The second-named author thanks Nicolas Burq for an enlightening discussion leading to the choice of geodesic normal coordinates. The third-named author is member of the Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), is partially supported by the INdAM–GNAMPA 2022 Project *Analisi geometrica in strutture subriemanniane*, codice CUP_E55F22000270001, and has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 945655).

Indeed, the interior elliptic equation is obtained by taking the divergence of the first equation in (1.1), while the boundary condition follows by scalar multiplying the same equation by the unit normal $n\colon \partial\Omega \to \mathbb{R}^d$ and noticing that

$$\begin{aligned}\partial_n p = \nabla p \cdot n &= -\operatorname{div}(u\otimes u)\cdot n = -\partial_i(u^i u^j)n^j = -u^i\partial_i(u^j)n^j \\ &= -u_i\partial_i(u^j n^j) + u^i u^j \partial_i n^j = -u\cdot\nabla(u\cdot n) + u\otimes u : \nabla n = u\otimes u : \nabla n.\end{aligned} \tag{1.3}$$

Here we used that $\partial\Omega$ is a level set of the scalar function $u\cdot n$, so $\nabla(u\cdot n)|_{\partial\Omega}$ is parallel to n. In the previous chain of equalities, we implicitly assumed that the normal, which always satisfies $n \in C^1(\partial\Omega)$ on any C^2 domain, is extended to a neighbourhood of $\partial\Omega$ in order to compute its gradient. Clearly, the pressure in (1.2) is always determined up to a constant and its uniqueness can be restored by imposing $\int_\Omega p = 0$.

To run the computations in (1.3), we have used $u \in C^1(\overline{\Omega})$. In order to deal with $u \in C^\gamma(\Omega)$, for $\gamma < 1$, we need to interpret (1.2) in the weak sense, that is, we consider a scalar function $p \in C^0(\overline{\Omega})$ such that

$$-\int_\Omega p\,\Delta\varphi\,dx + \int_{\partial\Omega} p\,\partial_n\varphi\,dx = \int_\Omega u\otimes u : H\varphi\,dx, \qquad \text{for all } \varphi \in C^2(\overline{\Omega}), \tag{1.4}$$

where $H\varphi$ denotes the Hessian matrix of the scalar function φ. As usual, (1.4) is obtained by multiplying the first equation in (1.2) by a test function φ and then integrating by parts. Such weak formulation makes sense for every $u, p \in C^0(\overline{\Omega})$ and it is actually the equation satisfied by any bounded and uniformly continuous couple (u, p) weakly solving (1.1). For the rigorous derivation, we refer the reader to [10].

1.2. **Previous results.** In the whole discussion below, we will always assume that the incompressible vector field u is γ-Hölder regular and, when considering a bounded domain Ω, it is also tangent to the boundary.

Looking at the equation $-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u)$ and forgetting the technicalities that might arise from the presence of the boundary, by standard Shauder's estimates we get that p is exactly as γ-Hölder regular as u, for every $\gamma \in (0,1)$. However, it has been recently noted that the quadratic structure of the right-hand side $\operatorname{div}\operatorname{div}(u\otimes u)$, together with the divergence-free condition $\operatorname{div} u = 0$, allows to increase the Hölder regularity of the pressure up to 2γ.

At the best of our knowledge, such double regularity has been observed for the first time by L. Silvestre [20] when $\Omega = \mathbb{R}^d$, and then in [6, 15] by different proofs which in particular generalize the double regularity to the periodic setting $\Omega = \mathbb{T}^d$. More precisely, such results show that, when Ω is either the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$, the pressure enjoys

$$p \in \begin{cases} C^{2\gamma} & \text{if } 0 < \gamma < \frac{1}{2} \\ C^{1,2\gamma-1} & \text{if } \frac{1}{2} < \gamma < 1, \end{cases} \tag{1.5}$$

as soon as $u \in C^\gamma$. Soon after, P. Constantin [8] proved that $p \in \mathrm{Lip}_{\log}(\mathbb{R}^d)$ in the borderline case $\gamma = \frac{1}{2}$ by relying on some useful new local formulas for the pressure on $\mathbb{R}^d$. Let us emphasize that the case $\gamma = \frac{1}{2}$ is naturally borderline for the regularity (1.5), because of the well-known failure of Schauder's estimates in Hölder spaces with integer exponents. We refer the reader to [7], where the double pressure regularity on $\mathbb{T}^d$ has been generalized to any Sobolev or Besov space.

Remarkably, the double Hölder regularity of the pressure on $\mathbb{R}^d$ or $\mathbb{T}^d$ (together with several other fine regularity estimates along the flow of u) has been used by P. Isett in [15] to prove the smoothness of trajectories of Euler flows when the velocity is strictly less than Lipschitz regular in the spatial variable, thus in a regime in which trajectories are not necessarily unique.

Very recently, the double regularity of the pressure has been also crucially used in [9] to establish rigorous intermittency-type results in the framework of fully developed turbulence. An extensive description of the relevance of intermittency phenomena in the mathematical theory of turbulent flows can be found in the monograph [12]. Moreover, on a bounded domain $\Omega \subset \mathbb{R}^d$, where the presence of the boundary adds highly non-trivial complications, the Hölder regularity of the pressure plays a fundamental role in the study of anomalous dissipation, see [2, 4, 18, 19] and the references therein. For these reasons, the study of the regularity of the pressure up to the boundary is of crucial importance, from both the mathematical and the physical point of view, and our result might be seen as a first step towards the extension of [9, 15] to bounded domains.

As usual when considering boundaries, the above analysis becomes more delicate when $\Omega \subset \mathbb{R}^d$ is a bounded and simply connected domain. Indeed, even the more standard *single* regularity $p \in C^\gamma(\Omega)$ is not a straightforward consequence of Schauder's estimates up to the boundary. Such regularity was first established by C. Bardos and E. Titi in [3] for 2-dimensional domains of class C^2 by relying on the global geodesic coordinates at the price of modifying the natural Neumann boundary condition.

In our previous work [10], we extended Bardos–Titi's result to any dimension without introducing any different boundary condition. Actually, we have been able to also partially doubling the pressure regularity by proving that, in any dimension $d \geq 2$, if $\Omega \subset \mathbb{R}^d$ is a simply connected open set of class $C^{2,\delta}$ for some $\delta > 0$, then the pressure enjoys

$$p \in \begin{cases} C^\gamma(\Omega) & \text{if } 0 < \gamma < \frac{1}{2} \text{ and } \delta > 0 \\ C^{1,2\gamma-1}(\Omega) & \text{if } \frac{1}{2} < \gamma < 1 \text{ and } \delta = 2\gamma - 1. \end{cases} \tag{1.6}$$

In the case $\gamma < \frac{1}{2}$, we exploited the explicit representation formula for p via the Green–Neumann kernel on Ω, while, for $\gamma \in (\frac{1}{2}, 1)$, we relied on the already known double regularity (1.5) on $\mathbb{R}^d$ by suitably extending the vector field u to the whole space. Note that, when $\gamma > \frac{1}{2}$, the requirement $\delta = 2\gamma - 1$ in (1.6) is necessary, since otherwise the boundary condition $\partial_n p = u \otimes u : \nabla n \in C^{\min\{\delta,\gamma\}}(\partial\Omega)$ would be incompatible with $\nabla p \in C^{2\gamma-1}(\Omega)$. Moreover, once the single γ-Hölder regularity has been established, by the abstract interpolation argument developed in [7], an almost double regularity $p \in C^{2\gamma-\varepsilon}(\Omega)$, for any $\varepsilon > 0$ arbitrarily small, follows directly as soon as $\partial\Omega \in C^{3,\delta}$ for some $\delta > 0$, see [10, Theorem 1.3].

In this paper, we complete the picture initiated in [10] by establishing the double Hölder regularity (up to the boundary) of the pressure in the whole range $\gamma \in (0, \frac{1}{2}]$ in any bounded, simply connected and sufficiently regular domain $\Omega \subset \mathbb{R}^d$, including the borderline case $\gamma = \frac{1}{2}$, in which we prove that the pressure belongs to the Hölder–Zygmund space. Together with [10], this gives the full double regularity for every $\gamma \in (0, 1)$, in a sufficiently regular simply connected bounded domain $\Omega \subset \mathbb{R}^d$. Our main result is rigorously stated below.

1.3. **New main result.** We let $C^1_*(\Omega)$ be the usual Hölder–Zygmund space, see Section 2.2, and in particular (2.7), for its precise definition. We prove the following

Theorem 1.1. *Let $\gamma \in (0, \frac{1}{2}]$ and let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$ for some $\delta > 0$. If $u \in C^\gamma(\Omega)$ is a divergence-free vector field such that $u \cdot n|_{\partial\Omega} = 0$, then there exists a unique zero-average solution $p \in C^0(\overline{\Omega})$ of (1.2) such that*

$$\|p\|_{C^{2\gamma}(\Omega)} \leq C\|u\|^2_{C^\gamma(\Omega)} \quad \text{for } \gamma < \frac{1}{2} \tag{1.7}$$

and

$$\|p\|_{C^1_*(\Omega)} \leq C\|u\|^2_{C^{\frac{1}{2}}(\Omega)} \quad \text{for } \gamma = \frac{1}{2}, \tag{1.8}$$

where $C > 0$ is a constant depending on Ω and γ.

Similarly to [15], the proof of Theorem 1.1 is based on the Littlewood–Paley analysis in the frequency space, that we introduce in Section 2.2. Moreover, to achieve the above result, we prove the interior and the local boundary regularity estimates separately, see Theorem 3.1 and Theorem 4.1, respectively.

The interior regularity comes as a consequence of the more general Theorem 3.2, providing the double pressure regularity on the whole $\mathbb{R}^d$ for $u \in C^\gamma_c(\mathbb{R}^d)$. In the case $\gamma < \frac{1}{2}$, this was indeed already known (as discussed above in (1.5)). The main novelty here is for the borderline value $\gamma = \frac{1}{2}$, for which we achieve $p \in C^1_*(\mathbb{R}^d)$, thus improving the $\mathrm{Lip}_{\log}$ regularity proved in [8] (see Remark 2.4). The main reason for such a sharper regularity in the borderline case lies in the flexibility of the Littlewood–Paley analysis when dealing with estimates in the borderline Hölder–Zygmund space.

To deal with the boundary regularity, we pass to the *normal geodesic coordinate* system (see Proposition 2.1) in a neighbourhood of $\partial\Omega$. This new coordinate frame, more precisely the new local induced metric, allows to suitably extend the datum and the unknown to the whole space. Once on $\mathbb{R}^d$, we analyze the corresponding transformed equation by means of the pseudodifferential formalism. The quantitative regularity estimates in $C^{2\gamma}_*$ are then obtained via Littlewood–Paley analysis. The present strategy is more robust than the one adopted in our previous work [10], and seems flexible enough to be helpful also in less regular settings, such as Sobolev or Besov. We refer to Section 6 for a discussion on possible extensions of Theorem 1.1.

The assumption $\partial\Omega \in C^{2,\delta}$, for some $\delta > 0$, is needed in order to work with the regular approximation of the velocity given by Lemma 2.13. This, in turn, ensures the existence of solutions of the regularized problem via classical Elliptic Theory. Thus, for rougher domains with $\partial\Omega \in C^2$ only, we believe that the double regularity property might become more technical, even though still likely true.

The corresponding result for solutions p which are not necessarily average-free directly follows.

Corollary 1.2. *Let $\gamma \in (0, \frac{1}{2}]$ and let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$, for some $\delta > 0$. If $u \in C^\gamma(\Omega)$ is a divergence-free vector field such that $u \cdot n|_{\partial\Omega} = 0$, then every weak solution $p \in C^0(\overline{\Omega})$ of (1.2) is unique up to constants and satisfies*

$$\left\| p - \int_\Omega p(x)\, dx \right\|_{C^{2\gamma}(\Omega)} \leq C \|u\|^2_{C^\gamma(\Omega)} \quad \text{for } \gamma < \frac{1}{2}$$

and

$$\left\| p - \int_\Omega p(x)\, dx \right\|_{C^1_*(\Omega)} \leq C \|u\|^2_{C^{\frac{1}{2}}(\Omega)} \quad \text{for } \gamma = \frac{1}{2},$$

where $C > 0$ is a constant depending on the Ω and γ.

1.4. **Organization of the paper.** In Section 2 we discuss all the main technical ingredients that are needed to run our proof: we introduce the local geodesic coordinates, the Littlewood–Paley analysis, the definition of the Hölder–Zygmund spaces and the pseudodiffrential formalism. In Section 3 we prove the interior boundary regularity. Section 4 and Section 5 are devoted to the boundary analysis: in Section 4 we transform the equation, locally at the boundary, in the new coordinate system, while in Section 5 we conclude the proof providing the quantitative stability estimate for the equation in the new coordinate system. We conclude with Section 6 in which we discuss further extensions.

2. Preliminaries

In this section we prove or simply recall results that we will need in our proof.

2.1. **Geodesic coordinates.** Let $\Omega \subset \mathbb{R}^d$ be a domain of class $C^{2,\delta}$ for some $\delta > 0$, endowed with the constant Euclidean metric $g^{ij} = g_{ij} = \delta_{ij}$ for all $i, j \in \{1, \dots, d\}$, where $g_{ij} = \langle e_i, e_j \rangle$ is the scalar product between the basis vectors e_i and e_j.

Let $x_0 \in \partial\Omega$ and fix a small neighborhood U of x_0 in $\mathbb{R}^d$. Let $U_0 \coloneqq U \cap \partial\Omega$, which is a neighborhood of x_0 in $\partial\Omega$, and consider a local coordinate chart for $\partial\Omega$ at x_0, *i.e.*, a couple (U_0, φ), where $\varphi\colon U_0 \to V_0 =: \varphi(U_0) \subset \mathbb{R}^{d-1}$ is a $C^{2,\delta}$-diffeomorphism. We write $\varphi(x) = (\theta^1(x), \dots, \theta^{d-1}(x))$ and call the θ^i's the *local coordinates* on $\partial\Omega$ around x_0. Up to reducing the neighborhood U, one can assume that U is a tubular neighborhood of $\partial\Omega$, that is, $U = \{x + tn(x) : 0 \le t \le c,\ x \in \Sigma \subset \partial\Omega\}$ for some $c > 0$, some portion of the boundary $\Sigma \subset \partial\Omega$ and where $n(x)$ stands for the inward pointing unit normal to $\partial\Omega$ at x.

We can express the Euclidean metric g in any new coordinate system y^i (and where we let $\frac{\partial}{\partial y^i}$ be the associated dual vector field) using the following change of variable formula for all $x \in \mathbb{R}^d$

$$g_{ij}(x) = \left\langle \frac{\partial x}{\partial y^i}, \frac{\partial x}{\partial y^j} \right\rangle = \sum_{k=1}^{d} \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j}.$$

In our proof we will use the so-called *normal geodesic coordinates* which are defined as follows. For any point $x \in U$, let us denote $\pi(x)$ the orthogonal projection (for the Euclidean metric) of x on $\partial\Omega$ and write $r(x) = \operatorname{dist}(x, \partial\Omega)$. We also let $(\theta^i(x))_{2 \le i \le d}$ be the coordinates of $\pi(x)$ on $\partial\Omega$. Our new coordinates are $y^1 = r$ and $y^j = \theta^j(x)$, $j \ge 2$, which define local coordinates in U.

Proposition 2.1. *With the above notation, the following statements hold true.*

(1) There exists a symmetric matrix $g^{\theta\theta} = [g^{\theta^i\theta^j}]_{2 \le i,j \le d}$ with $C^{1,\delta}$ coefficients such that the inverse metric g in the new coordinates $(y^i)_{1 \le i \le d}$ reads as

$$g = \begin{bmatrix} 1 & 0 \\ 0 & g^{\theta\theta} \end{bmatrix}.$$

(2) There exists $c > 0$ such that $\xi^T g(r,\theta)\xi \ge c|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and all $(r, \theta) \in U$.

Proof. Let us prove the two claims separately.

Proof of (1). We are going to prove that $\bar{g} = [g_{ij}]_{1 \le i,j \le d}$ in the coordinates y^i assumes the block-diagonal form, therefore the inverse $g = [g^{ij}]_{1 \le i,j \le d}$ will have the same form. Note that by definition of n, π and r, there holds $x = r(x)n(\pi(x)) + \pi(x)$ for any $x \in U$.

We start by proving that $g_{11} = 1$. With the notation $y^1 = r$ and $(y^2, \dots, y^d) = \theta$, we have to prove $\left|\frac{\partial x(r,\theta)}{\partial r}\right|^2 = 1$ for all $(r, \theta) \in V$. Observe that, for all h small enough (in order to remain in the tubular neighborhood), there holds

$$x(r + h, \theta) - x(r, \theta) = hn(\pi(r, \theta)),$$

therefore $\frac{\partial x(r,\theta)}{\partial r} = n(\pi(r, \theta))$, which has norm 1.

Let us now prove that, for all $i \in \{2, \dots, d\}$, there holds $g_{1i} = 0$, which is equivalent to $\left\langle \frac{\partial x(r,\theta)}{\partial \theta^i}, n(\pi(r, \theta)) \right\rangle = 0$. Let $(e_i)_{2 \le i \le d}$ be the Euclidean basis of $\mathbb{R}^{d-1}$. Let $k \ge 2$ and $\eta = (0, \dots, 0, \eta_k, 0, \dots, 0)$. We can therefore write

$$x(r, \theta + \eta) = n(r, \theta + \eta) + \pi(r, \theta + \eta) = n(r, \theta + \eta) + \varphi^{-1}((\theta^i + \delta_{ik}\eta_k)e_i),$$

so that, using Taylor's formula at order one, we find

$$x(r, \theta + \eta) - x(r, \theta) = \eta_k \partial_{\theta^k} n(r, \theta) + \eta_k \partial_k(\varphi^{-1})(\theta^i e_i) + \mathcal{O}(\eta_k^2).$$

Differentiating the equality $|n(r, \theta)|^2 = 1$ in θ^k, we see that $\partial_{\theta^k} n(r, \theta)$ is orthogonal to $n(r, \theta)$. Finally, we observe that $\partial_k(\varphi^{-1})(\theta^i e_i)$ is also orthogonal to $n(r, \theta)$, therefore completing the proof of the claim. Note that $g_{\theta\theta}$ is a matrix whose coefficients are functions of $\nabla x(r,\theta)$, therefore are $C^{1,\delta}$ as well as their inverse, *i.e.*, $g^{\theta\theta} = (g_{\theta\theta})^{-1} \in C^{1,\delta}$.

Proof of (2). Let us observe that, since g is block-diagonal, it is enough to prove that

$$\eta^T g^{\theta\theta}(r,\theta)\eta \geq c|\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^{d-1}.$$

Letting J denote the Jacobian of $\varphi\colon U \to V \subset \mathbb{R}^{d-1}$, we have $g^{\theta\theta} = J^T J$, so that we only need to explain why $\eta^T g^{\theta\theta}(r,\theta)\eta = |J\eta|^2 \geq c > 0$ when $|\eta| = 1$. But the latter property readily follows since J is invertible at every $(r,\theta) \in V$ and the unitary sphere is a compact set. □

We can write the Laplacian in the new coordinates as

$$-\Delta p = -(\det(g))^{-1/2}\partial_i\left((\det(g))^{1/2} g^{ij}\partial_j p\right) = -\frac{1}{G}\partial_i(G g^{ij}\partial_j p), \tag{2.1}$$

where $G(r,\theta) \coloneqq \sqrt{\det g(r,\theta)}$ and ∂_i denote the derivatives in the new coordinates r,θ. Moreover, in the new local coordinates system, the boundary condition $u\cdot n = 0$ on $\partial\Omega$ reads as $u^r(0,\theta) = 0$ for all θ and, similarly, the boundary condition $\partial_n p = 0$ on $\partial\Omega$ is equivalent to $\partial_r p(0,\theta) = 0$. Finally, for a vector field $F = F^i e^i$ there holds

$$\operatorname{div} F = (\det g)^{-1/2}\partial_i\left(F^i(\det g)^{1/2}\right), \tag{2.2}$$

from which we get

$$\operatorname{div}\operatorname{div}(u\otimes u) = (\det g)^{-1/2}\partial^2_{ij}\left((\det g)^{1/2}u^i u^j\right) = \frac{1}{G}\partial^2_{ij}\left(G u^i u^j\right). \tag{2.3}$$

2.2. **Littlewood–Paley analysis, Hölder and Hölder–Zygmund spaces.** We introduce a smooth Littlewood–Paley partition of the unity

$$1 = \sum_{N\in 2^{\mathbb{N}}} \mathbf{P}_N(\xi)$$

as follows. Let φ be some smooth bump function which is non-negative, radially symmetric, supported on $\{|\xi| \leq 2\}$ and such that $\varphi(\xi) = 1$ for all $|\xi| \leq 1$. We let $\mathbf{P}_1(\xi) \coloneqq \varphi(\xi)$ and $\mathbf{P}_N(\xi) = \varphi(N^{-1}\xi) - \varphi(2N^{-1}\xi)$ for all $N \geq 2$. Therefore $\mathbf{P}_N$ is supported on $|\xi| \sim N$. Let $u \in \mathcal{S}'$ be a tempered distribution. For all dyadic integers $N \in 2^{\mathbb{N}}$ we define

$$u_N \coloneqq \mathbf{P}_N u \coloneqq \mathcal{F}^{-1}(\mathbf{P}_N) * u,$$

which is just $\hat{u}_N(\xi) = \mathbf{P}_N(\xi)\hat{u}(\xi)$ on the Fourier side. Because the Fourier transform of u_N is compactly supported, u_N is smooth. Also, due to the partition of unity property, there holds

$$u = \sum_{N\in 2^{\mathbb{N}}} u_N.$$

Let us also denote

$$u_{\leq N} \coloneqq \sum_{M\leq N} u_M \quad \text{and} \quad u_{\geq N} \coloneqq \sum_{M\geq N} u_M,$$

where the summation is on dyadic $M \in 2^{\mathbb{N}}$. In this article, when a summation is taken on capitalized letters, it is always assumed that the summation is on dyadic integers only. For a much more extensive presentation, we refer the reader to [1].

Using the frequency localization of u_N and Young's inequality, one can infer the following very useful estimates that we shall constantly use in Section 5.

Theorem 2.2 (Bernstein's Theorem [21, Appendix A])**.** *Let $s \in \mathbb{R}$ and $p,q \in [1,\infty]$ such that $p \leq q$. Then, the following hold:*

(i) $\|u_N\|_{L^q} \lesssim N^{d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_N\|_{L^p}$;

(ii) $\||\nabla|^s u_N\|_{L^p} \sim N^s \|u_N\|_{L^p}$ *if* $N \geq 2$ *and* $\||\nabla|^s u_1\|_{L^p} \lesssim \|u_1\|_{L^\infty}$*, where* $|\nabla|^s$ *is the Fourier multiplier* $|\xi|^s$*;*

(iii) $\|u_{\leq N}\|_{L^p} \lesssim \|u\|_{L^p}$ *and* $\|u_{\geq N}\|_{L^p} \lesssim \|u\|_{L^p}$*.*

In the latter inequalities, the implicit constants depend on d*,* p*,* q *and* s *but not on* N*.*

The definition of the smooth truncation $\mathbf{P}_N$ implies that $\mathbf{P}_N\mathbf{P}_M = 0$, unless $\frac{1}{4}M \leq N \leq 4M$, which we abbreviate as $N \sim M$. We also write $N \gg M$ when $N \geq 8M$.

If $N \geq M$, then $u_N v_M$ has frequency in $\{N - M \leq |\xi| \leq N + M\}$. In the case $N \sim M$, this means that $u_N v_M$ is frequency supported in $\{|\xi| \lesssim N\}$, so that $\mathbf{P}_K(u_N v_M) = 0$ unless $K \lesssim N$. The other interesting case is when $N \gg M$, in which case $u_N v_M$ is frequency localized in $\{|\xi| \sim N\}$, so that $\mathbf{P}_K(u_N v_M) = 0$ unless $K \sim N$. These considerations will be frequently used in Section 5. We shall also repeatedly use the following simple facts about dyadic sums

$$\sum_{M<N} 1 \lesssim \log(N), \qquad \sum_{M<N} M^s \lesssim N^s \quad \text{and} \quad \sum_{M>N} M^{-s} \lesssim N^{-s},$$

for any $s > 0$, where the implicit constants in the last two estimates might also depend on s.

We now recall some basic facts about Hölder–Zygmund spaces. For a more detailed account, we refer to [22]. For $s \in \mathbb{R}$, we define

$$\|u\|_{C^s_*} := \sup_{N \in 2^{\mathbb{N}}} N^s \|u_N\|_{L^\infty}, \tag{2.4}$$

which is nothing but the Besov norm $\|u\|_{B^s_{\infty,\infty}}$. It is well known that $C^s(\mathbb{R}^d) = C^s_*(\mathbb{R}^d)$ whenever $s \in \mathbb{R}^+ \setminus \mathbb{N}$, where $C^s(\mathbb{R}^d)$ denotes the usual space of s-Hölder continuous functions, while $C^s(\mathbb{R}^d) \subset C^s_*(\mathbb{R}^d)$ with strict inclusion if $s \in \mathbb{N}$, see [22, Appendix A] for instance.

Let us also recall the following useful facts about the borderline case $s = 1$.

Proposition 2.3. *The space* $C^1_*(\mathbb{R}^d)$ *is the classical Zygmund space, and the norm* (2.4) *is equivalent to*

$$\|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{x \in \mathbb{R}^d} \sup_{h \in \mathbb{R}^d,\, h \neq 0} \frac{|u(x+h) + u(x-h) - 2u(x)|}{|h|}. \tag{2.5}$$

Moreover, $C^1_*(\mathbb{R}^d) \subset \mathrm{Lip}_{\log}(\mathbb{R}^d)$ *with continuous embedding, where* $\mathrm{Lip}_{\log}(\mathbb{R}^d)$ *denotes the space of functions such that*

$$\|u\|_{\mathrm{Lip}_{\log}(\mathbb{R}^d)} := \|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|\,(1 + |\log|x-y||)} < \infty. \tag{2.6}$$

Proof. For (2.5), we refer to [22, Appendix A]. To prove the embedding, define $w(h) := (u(x+h) - u(x))/|h|$ for $h \neq 0$. Since $u \in C^1_*(\mathbb{R}^d)$, we have

$$\left| w(h) - w\left(\frac{h}{2}\right) \right| = \frac{\left| u(x+h) + u(x) - 2u\left(x + \frac{h}{2}\right) \right|}{|h|} \leq C \|u\|_{C^1_*(\mathbb{R}^d)},$$

from which we deduce

$$|w(2^{k+1}h) - w(2^k h)| \leq C \|u\|_{C^1_*(\mathbb{R}^d)}, \quad \forall k \geq 0.$$

The constant $C > 0$ may vary from line to line in the next computations, but it is important that it will be always independent on h and k.

Choose $k_0 \in \mathbb{N}$ such that $e^{k_0}|h| \sim 1$ and add the previous inequality for all $0 \leq k < k_0$, getting

$$|w(2^{k_0}h) - w(h)| \leq k_0 C \|u\|_{C^1_*(\mathbb{R}^d)} \leq C \|u\|_{C^1_*(\mathbb{R}^d)} \left|\log|h|\right|.$$

Thus we finally achieve

$$\begin{aligned}|w(h)| &\le |w(2^{k_0}h)| + |w(2^{k_0}h) - w(h)| \le 2^{-k_0}|h|^{-1}\|u\|_{L^\infty(\mathbb{R}^d)} + C\|u\|_{C^1_*(\mathbb{R}^d)}\,|\log|h|| \\ &\le C\left(\|u\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{C^1_*(\mathbb{R}^d)}\right)(1+|\log|h||)\,,\end{aligned}$$

which, in terms of u, reads as

$$|u(x+h)-u(x)| \le C\left(\|u\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{C^1_*(\mathbb{R}^d)}\right)|h|\,(1+|\log|h||)$$

yielding the claimed embedding. □

We can use the equivalent norm (2.5) to naturally define the Zygmund space C^1_* on any bounded domain Ω by setting

$$\|u\|_{C^1_*(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{\substack{x,\, x+h,\, x-h\,\in\,\Omega \\ h\neq 0}} \frac{|u(x+h)+u(x-h)-2u(x)|}{|h|}. \tag{2.7}$$

Also the $\mathrm{Lip}_{\log}$ norm (2.6) extends to any bounded domain by setting

$$\|u\|_{\mathrm{Lip}_{\log}(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|\,(1+|\log|x-y||)} < \infty. \tag{2.8}$$

Remark 2.4. Note that the function $u\colon[-1,1]\to\mathbb{R}$ given by $u(x) = -|x|\log|x|$, satisfies $u \in \mathrm{Lip}_{\log}([-1,1]) \setminus C^1_*([-1,1])$, proving that $C^1_*([-1,1]) \subsetneq \mathrm{Lip}_{\log}([-1,1])$.

Finally, as it has been first observed by J. M. Bony [5], we recall that the product of a distribution in $C^r_*(\mathbb{R}^d)$ for $r<0$ with a function in $C^s_*(\mathbb{R}^d)$ defines a distribution if $s+r>0$, see [14, Lemma 2.1] for instance.

Lemma 2.5 (Product of distributions)**.** *Let $s, r \in \mathbb{R}$ be such that $r<0<s$ and $r+s>0$. If $u \in C^r_*$ and $v \in C^s_*$, then $uv \in C^r_*$, with*

$$\|uv\|_{C^r_*} \lesssim \|u\|_{C^r_*}\|v\|_{C^s_*}. \tag{2.9}$$

Notice that defining the product of two distributions is not trivial, and indeed it involves a *Bony decomposition* together with *paradifferential* calculus. However, we remark that we are going to use Lemma 2.5 on products of bounded and continuous functions, thus in this case uv is trivially defined and the estimate (2.9) becomes an easy exercise in Littlewood–Paley analysis.

2.3. **Pseudodifferential operators and symbols.** In what follows we introduce the pseudodifferential formalism from [24]. We will repeatedly use the notation $\langle\xi\rangle := \sqrt{1+|\xi|^2}$.

Definition 2.6 (Classes of symbols)**.** Let $m\in\mathbb{R}$ and $\delta\in[0,1]$. A symbol $a \in \mathcal{C}^\infty_{x,\xi}$ is in $S^m_{1,\delta}$ if

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \le C(\alpha,\beta)\langle\xi\rangle^{m-|\alpha|+\delta|\beta|}.$$

We let $S^m := S^m_{1,0}$ be the space of classical symbols of order m.

When dealing with limited regularity in the variable x, one has the following generalization.

Definition 2.7 (Class $C^s_* S^m_{1,\delta}$)**.** A symbol a with regularity C^s_* in x is in the class $C^s_* S^m_{1,\delta}$ when

$$\left\|\partial^\alpha_\xi a(\cdot,\xi)\right\|_{C^s_*} \le C(s)\langle\xi\rangle^{m-|\alpha|+s\delta}.$$

Definition 2.8 (Quantization of a symbol a)**.** Let $a \in C^r_* S^m_{1,\delta}$. The *quantization* of a is the operator $\mathrm{Op}(a) : \mathcal{S}\to\mathcal{S}$ defined for every $u\in\mathcal{S}$ by

$$\mathrm{Op}(a)u(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i\xi\cdot(x-y)} a(x,\xi)u(y)\,\mathrm{d}\xi\,\mathrm{d}y = \int_{\mathbb{R}^d} K(x,y)u(y)\,\mathrm{d}y,$$

where $K(x,y) = \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)} a(x,\xi)\,\mathrm{d}\xi$.

This quantization is such that for example when $a(x,\xi) = ib(x)\xi_k$ then

$$\mathrm{Op}(a)u(x) = b(x)\partial_k u(x).$$

Other quantizations exist, but this one (which is the classical one) fits to our problem.

It is well-known that operators in the class $\mathrm{Op}(S^m)$ enjoy good continuity bounds in $W^{s,p}$ or $\mathcal{C}^s_*$ spaces, as known from the more general Calderón–Vaillancourt Theorem.

Theorem 2.9 ([24, Chapter 13, Proposition 8.6])**.** *Let $m \in \mathbb{R}$, $\delta \in [0,1)$ and $a \in S^m_{1,\delta}$. Then, for any $s \in \mathbb{R}$, there holds*

$$\mathrm{Op}(a) : C^{s+m}_* \to C^s_*.$$

Operators associated to symbols in the class $S^m_{1,\delta}$ enjoy nice composition properties.

Theorem 2.10 ([23, Chapter 7, Proposition 3.3])**.** *Let $A = \mathrm{Op}(a)$ and $B = \mathrm{Op}(b)$ two pseudodifferential operators with symbols $a \in S^m_{1,\delta}$ and $b \in S^n_{1,\delta}$, for some $n, m \in \mathbb{R}$ and $\delta \in [0,1)$. Then, the composition $C := A \circ B$ is a pseudodifferential operator $C = \mathrm{Op}(c)$, where $c \in S^{m+n}_{1,\delta}$. Moreover, for any integer $N \geq 0$, it holds*

$$c(x,\xi) - \sum_{|\alpha|\leq N} \frac{i^{|\alpha|}}{\alpha!} \partial^\alpha_\xi a(x,\xi) \partial^\alpha_x b(x,\xi) \in S^{m+n-(1-\delta)(N+1)}_{1,\delta}.$$

We will need to invert elliptic operators. In our context we will use the following definition of elliptic operators.

Definition 2.11 (Elliptic symbol on a compact set)**.** We say that a symbol $a \in S^m_{1,\delta}$ is elliptic on a compact set $K \subset \mathbb{R}^d$ if there exists $R > 0$ and $c > 0$ such that, for all $|\xi| \geq R$ and all $x \in K$, there holds

$$|a(x,\xi)| \geq c\langle\xi\rangle^m.$$

With this definition, an elliptic operator can be (locally) inverted modulo smooth functions.

Theorem 2.12 (Elliptic operators inversion, [22, Section 0.4])**.** *Let $m > 0$ and $\delta \in [0,1)$ and let $A \in \mathrm{Op}(S^m_{1,\delta})$ be an elliptic operator on a compact set K. Then, there exist $B \in \mathrm{Op}(S^{-m}_{1,\delta})$ and $C \in \mathrm{Op}(S^{-2(1-\delta)}_{1,\delta})$ such that*

$$B \circ A = \mathrm{Op}(\chi(x)) + C,$$

for some localization function χ supported in K

Proof. We provide a proof for the sake of completeness. Note that this result is a variant of the usual proof of inversion of elliptic operators modulo smooth functions [23, Chapter 7.4].

We seek an operator $B = \mathrm{Op}(b)$, where $b = b_{-m} + b_{-m-(1-\delta)}$ for some $b_{-m-(1-\delta)} \in S^{-m-(1-\delta)}_{1,\delta}$ to be found. The first term b_{-m} is actually easy to identify. In order to write it, let us introduce some cutoff function χ supported in K. Also, let ψ be a cutoff function which takes values 1 for $|\xi| \geq 2R$ and which is 0 for $|\xi| \leq R$. Let us set

$$b_{-m}(x,\xi) = \frac{\chi(x)\psi(\xi)}{a(x,\xi)},$$

which, thanks to our assumptions on a and the chain rule, is a symbol in $S^{-m}_{1,\delta}$. Observe that $b_{-m}(x,\xi)a(x,\xi) = \chi(x)\psi(\xi)$, so Theorem 2.10 applied for $N = 1$ to $\mathrm{Op}(b_{-m})$ and $\mathrm{Op}(a)$ gives

$$\mathrm{Op}(b_{-m}) \circ \mathrm{Op}(a) = \mathrm{Op}(\chi(x)\psi(\xi)) + \mathrm{Op}(i\nabla_\xi b_{-m} \cdot \nabla_x a) + C_0,$$

where $C_0 \in \mathrm{Op}\left(S_{1,\delta}^{-2(1-\delta)}\right)$.

Now observe that, again by applying Theorem 2.10 to $\mathrm{Op}(b_{-m-(1-\delta)}) \circ \mathrm{Op}(a)$, there exists $C_1 \in \mathrm{Op}\left(S_{1,\delta}^{-2(1-\delta)}\right)$ such that

$$\begin{aligned}\mathrm{Op}(b) \circ \mathrm{Op}(a) &= \mathrm{Op}(\chi(x)\psi(\xi)) + \mathrm{Op}(i\nabla_\xi b_{-m} \cdot \nabla_x a) + C_0 + \mathrm{Op}(b_{-m-(1-\delta)}a) + C_1 \\ &= \mathrm{Op}(\chi(x)\psi(\xi)) + C_0 + C_1,\end{aligned}$$

provided that we take $b_{-m-(1-\delta)} := -\frac{i}{a}\nabla_\xi b_{-m} \cdot \nabla_x a$, which is a symbol in $S_{1,\delta}^{-m-(1-\delta)}$.

Let us finally observe that $\chi(x)\psi(\xi) = \chi(x) + \chi(x)(\psi(\xi) - 1)$ and that $c_2 := \chi(x)(\psi(\xi) - 1)$ belongs to any $S_{1,\delta}^{-M}$ class ($M > 0$). We can thus write

$$\mathrm{Op}(b) \circ \mathrm{Op}(a) = \mathrm{Op}(\chi(x)) + C,$$

where $C = C_0 + C_1 + \mathrm{Op}(c_2)$ as wanted. □

2.4. **Reduction to smooth functions.** Since it will be very convenient to work with a $C^1(\overline{\Omega})$ vector field u, and thus to justify all our computations in the classical sense, let us consider u_ε, the smooth regularization of u given by [10, Lemma 2.1 and Remark 2.3], that we recall here

Lemma 2.13 (Velocity approximation). *Let $d \geq 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$, for some $\delta > 0$. Let $\gamma \in (0,1)$ and let $u \in C^\gamma(\Omega)$ be such that $\operatorname{div} u = 0$ and $u \cdot n|_{\partial\Omega} = 0$. Then, there exists a family $(u_\varepsilon)_{\varepsilon>0} \subset C^\infty(\Omega) \cap C^{1,\delta}(\Omega)$ such that $u_\varepsilon \to u$ in $C^0(\overline{\Omega})$ as $\varepsilon \to 0^+$, $\operatorname{div} u_\varepsilon = 0$ and $u_\varepsilon \cdot n|_{\partial\Omega} = 0$ for all $\varepsilon > 0$, and*

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{C^\gamma(\Omega)} \leq C\|u\|_{C^\gamma(\Omega)}$$

for some constant $C > 0$.

The proof of Theorem 1.1 relies on the following

Theorem 2.14. *Let $\gamma \in (0, \frac{1}{2}]$ and $\delta > 0$. Let $\Omega \subset \mathbb{R}^d$ be a simply connected bounded domain in the class $C^{2,\delta}$. Let $u \in C^{1,\delta}(\Omega) \cap C^\infty(\Omega)$ be a divergence-free vector field such that $u \cdot n|_{\partial\Omega} = 0$. Then, letting $\beta = \min\{\gamma, \delta\}$, there exists a unique zero-average solution $p \in C^{1,\beta}(\Omega)$ to (1.2) such that*

$$\|p\|_{C_*^{2\gamma}(\Omega)} \leq C\left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)}\right), \tag{2.10}$$

for some constant $C > 0$ which depending on Ω and γ only.

Thanks to the above result, the proof of Theorem 1.1 follows by standard tools in analysis. We give the details for the reader's convenience.

Proof of Theorem 1.1. We divide the proof in two steps: we first eliminate the term $\|p\|_{L^\infty(\Omega)}$ from the right-hand side of (2.10), and then we show how such a quantitative continuity estimate leads to the existence of solutions to (1.2) for any $u \in C^\gamma(\Omega)$ not necessarily smooth.

Step 1. We prove that, if $u \in C^{1,\delta}(\Omega) \cap C^\infty(\Omega)$, then the unique zero-average solution p of (1.2) (which always exists thanks to Theorem 2.14) enjoys

$$\|p\|_{C_*^{2\gamma}(\Omega)} \leq C\|u\|^2_{C^\gamma(\Omega)}. \tag{2.11}$$

Indeed, suppose that (2.11) does not hold. Then, for all $k \in \mathbb{N}$, we can find a divergence-free vector field u_k and a scalar function p_k solving

$$\begin{cases} -\Delta p_k = \operatorname{div}\operatorname{div}(u_k \otimes u_k) & \text{in } \Omega \\ \partial_n p_k = u_k \otimes u_k : \nabla n & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega p_k = 0$ and

$$1 = \|p_k\|_{C_*^{2\gamma}(\Omega)} \geq k\|u_k\|^2_{C^\gamma(\Omega)}.$$

In particular, $u_k \to 0$ in $C^\gamma(\Omega)$, which, by passing to the limit in the weak formulation (1.4), implies that the uniform limit of $(p_k)_k$, say q, (which we can always suppose to exist by Arzelà–Ascoli and up to a subsequence) solves

$$\begin{cases} -\Delta q &=& 0 & \text{in } \Omega \\ \partial_n q &=& 0 & \text{on } \partial\Omega. \end{cases}$$

Being q average-free, we get $q \equiv 0$, which contradicts (2.10), since

$$1 = \|p_k\|_{C^{2\gamma}_*(\Omega)} \le C \left(\|u_k\|^2_{C^\gamma(\Omega)} + \|p_k\|_{L^\infty(\Omega)} \right) \to 0.$$

Thus, the validity of (2.11) follows.

Step 2. Let u be as in the statement of Theorem 1.1. We wish to prove the existence of a unique solution p such that (1.7) holds true. Let u_ε be the regular approximation given by Lemma 2.13 and let p^ε the unique zero-average solution (which exists by Theorem 2.14) of

$$\begin{cases} -\Delta p^\varepsilon &=& \operatorname{div}\operatorname{div}(u_\varepsilon \otimes u_\varepsilon) & \text{in } \Omega \\ \partial_n p^\varepsilon &=& u_\varepsilon \otimes u_\varepsilon : \nabla n & \text{on } \partial\Omega. \end{cases}$$

By (2.11) and Lemma 2.13, we get

$$\|p^\varepsilon\|_{C^{2\gamma}_*(\Omega)} \le C\|u_\varepsilon\|^2_{C^\gamma(\Omega)} \le C\|u\|^2_{C^\gamma(\Omega)}.$$

Thus $(p^\varepsilon)_\varepsilon$ is a bounded sequence in $C^{2\gamma}_*(\Omega)$. In particular, by Arzelà—Ascoli Theorem, up to a (non-relabeled) subsequence, we can suppose that there exists $p \in C^{2\gamma}_*(\Omega)$ such that

$$\int_\Omega p\,dx = 0, \quad p^\varepsilon \to p \text{ in } C^0(\overline{\Omega}) \quad \text{and} \quad \|p\|_{C^{2\gamma}_*(\Omega)} \le C\|u\|^2_{C^\gamma(\Omega)}.$$

Since $u_\varepsilon \to u$ in $C^0(\overline{\Omega})$, by the weak formulation (1.4) we deduce that p (weakly) solves (1.2). To prove the uniqueness of p, suppose p_1 and p_2 are two zero-average solutions of (1.2). Then

$$\begin{cases} -\Delta(p_1 - p_2) &=& 0 & \text{in } \Omega \\ \partial_n(p_1 - p_2) &=& 0 & \text{on } \partial\Omega, \end{cases}$$

which, together with the constraint $\int_\Omega p_1 = \int_\Omega p_2 = 0$, gives $p_1 = p_2$. □

Thus, from now on, since the main goal is to prove Theorem 2.14, we always work with $u, p \in C^{1,\delta}(\Omega) \cap C^\infty(\Omega)$. Moreover, note that the continuity estimate (2.10) is the only thing that needs to be proven, since the existence (and the uniqueness) of the solution p is a direct consequence of standard Elliptic Theory (see [13, 25] for instance). Indeed, since $\partial\Omega \in C^{2,\delta}$ implies $\nabla n \in C^\delta(\partial\Omega)$, one has

$$\operatorname{div}\operatorname{div}(u \otimes u) = \partial_i u^j \partial_j u^i \in C^\delta(\Omega), \quad u \otimes u : \nabla n \in C^{\min\{\delta,\gamma\}}(\partial\Omega),$$

together with the compatibility condition

$$\begin{aligned} \int_\Omega \operatorname{div}\operatorname{div}(u \otimes u) &= \int_{\partial\Omega} \partial_j (u^i u^j) n^i = \int_{\partial\Omega} u^j \partial_j u^i n^i = \int_{\partial\Omega} u^j \partial_j (u \cdot n) - \int_{\partial\Omega} u^i u^j \partial_j n^i \\ &= -\int_{\partial\Omega} u \otimes u : \nabla n, \end{aligned}$$

which ensures the solvability of (1.2). Thus, from now on, we can forget about the existence and uniqueness statement and only focus on the estimate (2.10). This, in turn, is a consequence of Theorem 3.1 and Theorem 4.1 proved in Sections 3 and 4, respectively.

3. Interior regularity

In this section we wish to prove the following

Theorem 3.1 (Interior regularity). *Let $\gamma \in (0, \frac{1}{2}]$ and let $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$, for some $\delta > 0$. Let $u \in C^{1,\delta}(\Omega) \cap C^\infty(\Omega)$ be a divergence-free vector field such that $u \cdot n|_{\partial\Omega} = 0$. Then, for every $\widetilde{\Omega} \Subset \Omega$, the unique zero-average solution $p \in C^{1,\beta}(\Omega)$ of (1.2), where $\beta = \min\{\delta, \gamma\}$, enjoys*

$$\|p\|_{C^{2\gamma}_*(\widetilde{\Omega})} \le C \left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)} \right), \tag{3.1}$$

for some constant $C > 0$ depending on Ω, $\widetilde{\Omega}$ and γ only.

Being the previous theorem a purely interior regularity result, in the case $\gamma < \frac{1}{2}$ it does not contain anything new with respect to the double regularity established in [6, 15] in the whole space, since one can simply localize the equation (1.2) strictly inside Ω and then extending it to $\mathbb{R}^d$. However, since Theorem 3.1 additionally provides the new estimate in the borderline case $\gamma = \frac{1}{2}$ (which was left open in [6, 15]), we are going to give a detailed proof in the whole range $\gamma \le \frac{1}{2}$, noticing that the borderline case $\gamma = \frac{1}{2}$ does not require any different argument with respect to case $\gamma < \frac{1}{2}$.

Proof. Let u be as in the statement. Extend it to the whole space (see for instance [16, Section 5]), getting a divergence-free vector field $\tilde{u} \in C^1_c(\mathbb{R}^d)$ such that

$$\|\tilde{u}\|_{C^\gamma(\mathbb{R}^d)} \le C\|u\|_{C^\gamma(\mathbb{R}^d)}. \tag{3.2}$$

Define q to be the unique bounded solution, decaying at infinity, of

$$-\Delta q = \operatorname{div}\operatorname{div}(\tilde{u} \otimes \tilde{u}) \quad \text{in } \mathbb{R}^d.$$

Then $p - q$ is harmonic in Ω and thus we can estimate it, for every $\gamma \le \frac{1}{2}$ and every $\widetilde{\Omega} \Subset \Omega$, as

$$\|p - q\|_{C^{2\gamma}_*(\widetilde{\Omega})} \le C\|p - q\|_{C^1(\widetilde{\Omega})} \le C\|p - q\|_{L^\infty(\Omega)} \le C \left(\|p\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\mathbb{R}^d)} \right),$$

from which we infer that

$$\|p\|_{C^{2\gamma}_*(\widetilde{\Omega})} \le \|p - q\|_{C^{2\gamma}_*(\widetilde{\Omega})} + \|q\|_{C^{2\gamma}_*(\widetilde{\Omega})} \le C \left(\|p\|_{L^\infty(\Omega)} + \|q\|_{C^{2\gamma}_*(\mathbb{R}^d)} \right).$$

Moreover, by Theorem 3.2 below, and also using (3.2), we have

$$\|q\|_{C^{2\gamma}_*(\mathbb{R}^d)} \le C\|\tilde{u}\|^2_{C^\gamma(\mathbb{R}^d)} \le C\|u\|^2_{C^\gamma(\mathbb{R}^d)}$$

for some constant $C > 0$ depending on γ. This concludes the proof. □

The following theorem provides the full double regularity of the pressure in the whole space, for all $2\gamma \le 1$, thus both extending the results [6, 15] to the borderline case $\gamma = \frac{1}{2}$ and also improving to $C^1_*(\mathbb{R}^d)$ the $\mathrm{Lip}_{\log}(\mathbb{R}^d)$ regularity obtained in [8]. By straightforward modifications, the very same result holds on the periodic setting of the torus $\mathbb{T}^d$.

Theorem 3.2 (Double regularity on $\mathbb{R}^d$). *Let $\gamma \in (0, \frac{1}{2}]$ and $v \in C^\gamma_c(\mathbb{R}^d)$ be a compactly supported divergence-free vector field. Then, there exists a unique solution decaying at infinity of*

$$-\Delta q = \operatorname{div}\operatorname{div}(v \otimes v) \quad \text{in } \mathbb{R}^d$$

satisfying

$$\|q\|_{C^{2\gamma}_*(\mathbb{R}^d)} \le C\|v\|^2_{C^\gamma(\mathbb{R}^d)}. \tag{3.3}$$

Proof. Uniqueness is trivial. To prove the existence, one can simply regularize the vector filed v as $v_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ and then looking for the corresponding solution q^ε. By smoothness of the right-hand side, there exists $q^\varepsilon \in C^\infty(\mathbb{R}^d)$ solving $-\Delta q^\varepsilon = \operatorname{div}\operatorname{div}(v_\varepsilon \otimes v_\varepsilon)$. Since, for all $\varepsilon > 0$,

$$\|v_\varepsilon\|_{C^\gamma(\mathbb{R}^d)} \le \|v\|_{C^\gamma(\mathbb{R}^d)},$$

it is then enough to prove the continuity estimate (3.3) for q^ε and v_ε, from which one can then obtain a solution q by Ascoli-Arzelà Theorem, up to a subsequence. To lighten the notation we simply write q and v in place of q^ε and v_ε.

The global double regularity (3.3) follows by a Littlewood–Paley analysis. We believe that the following computations also give a clearer intuition on why one should expect that the very same double regularity can still be derived in our more general case of (4.7). We refer the reader to [15], where the Littlewood–Paley formalism has been indeed used to prove the double pressure regularity for the first time, for every $0 < 2\gamma < 1$.

Let us write $q = \mathbf{P}_1 q + \mathbf{P}_{>1} q$, where $\mathbf{P}_1$ stands for the first Littlewood–Paley smooth truncation on frequencies $|\xi| \le 1$. We refer to Section 2.2 for the notation and the basic facts on Littlewood–Paley calculus. The reason for distinguishing between high and low frequencies is that the high-frequency part of the Laplacian is invertible. More precisely, recall that $\mathbf{P}_{>1}(\xi) = 1 - \chi(\xi)$ for some smooth function compactly function χ which equals 1 in a neighborhood of 0. The symbol

$$a(\xi) = -\frac{1-\chi(\xi)}{|\xi|^2}$$

belongs to the class S^{-2} and therefore $A_{-2} := \mathrm{Op}(a(\xi)) \in \mathrm{Op}(S^{-2})$ is a good candidate for the inverse of $-\Delta$ on high frequencies. We indeed have $A_{-2} \circ (-\Delta) = \mathbf{P}_{>1}$, this equality being exact because the two operators are Fourier multipliers. Therefore, we can write

$$\mathbf{P}_{>1} q = A_{-2} \partial^2_{ij}(v^i v^j).$$

Let us use a Littlewood–Paley decomposition for v as

$$v^i = \sum_{M \in 2^{\mathbb{N}}} v^i_M,$$

so that we can write

$$v^i v^j = \sum_{M \sim K} v^i_M v^j_K + \sum_{M \ll K} (v^i_M v^j_K + v^i_K v^j_M).$$

Since the v^i_N are smooth and divergence-free (note that the Littlewood–Paley truncation $\mathbf{P}_N$ commutes with the divergence, as they are both Fourier multipliers), we have

$$\partial^2_{ij}(v^i_K v^j_M) = \partial_j v^i_K \partial_i v^j_M,$$

which we use on the $\sum_{M \ll K}$ terms to obtain

$$\mathbf{P}_{>1} q = \sum_{M \sim K} A_{-2} \partial^2_{ij}(v^i_M v^j_K) + \sum_{M \ll K} A_{-2} \left(\partial_j v^i_K \partial_i v^j_M + \partial_j v^i_M \partial_i v^j_K \right).$$

Finally, since $\mathbf{P}_N$ is a Fourier multiplier, and so commutes with A_{-2} and ∂^2_{ij}, for $N \ge 2$ we have

$$q_N = \sum_{M \sim K} A_{-2} \partial^2_{ij} \mathbf{P}_N (v^i_M v^j_K) + \sum_{M \ll K} A_{-2} \mathbf{P}_N \left(\partial_j v^i_K \partial_i v^j_M + \partial_j v^i_M \partial_i v^j_K \right) =: I + J.$$

Estimation of I. First, let us remark that $v^i_M v^j_K$ is frequency supported in $|\xi| \le \max\{M, K\}$, therefore $\mathbf{P}_N(v^i_M v^j_K) = 0$ unless $\max\{M, K\} \gtrsim N$. We thus have

$$I = \sum_{K \sim M \gtrsim N} A_{-2} \partial^2_{ij} \mathbf{P}_N (v^i_M v^j_K).$$

Now, $A_{-2}\partial^2_{ij} = \mathrm{Op}\left(-\frac{(1-\chi(\xi))\xi_i\xi_j}{|\xi|^2}\right) \in \mathrm{Op}(S^0)$ is a $C^0_* \to C^0_*$ continuous 0-order operator. Thus,

$$\|I\|_{C^0_*} \lesssim \sum_{M\sim K\gtrsim N} \left\|\mathbf{P}_N(v^i_M v^j_K)\right\|_{C^0_*} \lesssim \sum_{M\sim K\gtrsim N} \left\|v^i_M v^j_K\right\|_{L^\infty},$$

where we used the definition of the C^0_* norm in the last inequality. We now use that $v \in C^\gamma_*(\mathbb{R}^d)$, so that $\|v^j_K\|_{L^\infty} \lesssim K^{-\gamma}\|v\|_{C^\gamma_*}$, and therefore

$$\|I\|_{C^0_*} \lesssim \sum_{M\sim K\gtrsim N} M^{-\gamma}K^{-\gamma}\|v\|^2_{C^\gamma_*} \lesssim \sum_{M\gtrsim N} M^{-2\gamma}\|v\|^2_{C^\gamma_*} \lesssim N^{-2\gamma}\|v\|^2_{C^\gamma_*}.$$

In particular, being I frequency localized at N, and using the definition of the C^0_* norm, the previous estimate implies

$$\|I\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{-2\gamma}\|v\|^2_{C^\gamma_*}. \tag{3.4}$$

Notice that, in the estimate of this term, we have been able to double the regularity, because the frequencies of the two v's were similar, therefore responsible for the addition of regularities.

Estimation of J. Note that $\partial_j v^i_K \partial_i v^j_M$ is supported in $\{|\xi| \sim K\}$ because $M \ll K$, and therefore $\mathbf{P}_N(\partial_j v^i_K \partial_i v^j_M) = 0$ unless $N \sim K$. We combine this remark and the continuity of $A_{-2}\colon C^{-2}_* \to C^0_*$ (because A_{-2} is a pseudodifferential operator of order -2) to obtain

$$\begin{aligned}\|J\|_{C^0_*} &\lesssim \sum_{M\ll K\sim N} \left\|\mathbf{P}_N\left(\partial_j v^i_K \partial_i v^j_M + \partial_j v^i_M \partial_i v^j_K\right)\right\|_{C^{-2}_*} \\ &\lesssim N^{-2} \sum_{M\ll K\sim N} \left\|\partial_j v^i_K \partial_i v^j_M + \partial_j v^i_M \partial_i v^j_K\right\|_{L^\infty}.\end{aligned}$$

By Bernstein's inequality (see (ii) in Theorem 2.2) and the fact that $v \in C^\gamma_*(\mathbb{R}^d)$, we see that $\|\partial_i v^j_K\|_{L^\infty(\mathbb{R}^d)} \lesssim K^{1-\gamma}\|v\|_{C^\gamma_*(\mathbb{R}^d)}$ so that

$$\|J\|_{C^0_*(\mathbb{R}^d)} \lesssim N^{-2} \sum_{M\ll K\sim N} M^{1-\gamma}K^{1-\gamma}\|v\|^2_{C^\gamma_*(\mathbb{R}^d)} \lesssim N^{-2\gamma}\|v\|^2_{C^\gamma_*(\mathbb{R}^d)}.$$

In particular, as before, this implies

$$\|J\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{-2\gamma}\|v\|^2_{C^\gamma_*(\mathbb{R}^d)}. \tag{3.5}$$

Notice that, in the above estimate, we gained double regularity because the double divergence has been rewritten as a product of gradients. Without this rewriting, since $K \gg M$, all derivatives should fall on the function localized at $\sim K$, but with the rewriting we transfer one derivative to the lowest frequency M, and this derivative is estimated by M (instead of K), hence the gain.

Combining (3.4) and (3.5) we finally obtain that, for all $N \geq 2$, there holds

$$N^{2\gamma}\|q_N\|_{L^\infty} \lesssim \|v\|^2_{C^\gamma_*}. \tag{3.6}$$

Recall that $q = \mathbf{P}_1 q + \mathbf{P}_{>1} q$, so we need to estimate $\mathbf{P}_1 q$. By Sobolev embedding, if $s > \frac{d}{2}$, then

$$\begin{aligned}\|\mathbf{P}_1 q\|^2_{L^\infty(\mathbb{R}^d)} &\lesssim \||\nabla|^s \mathbf{P}_1 q\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left|\widehat{\mathbf{P}_1 q}(\xi)\right|^2 |\xi|^{2s}\chi_{|\xi|\leq 1}(\xi)\, d\xi \\ &\lesssim \int_{\mathbb{R}^d} \left|\widehat{\mathbf{P}_1 q}(\xi)\right|^2 |\xi|^4\chi_{|\xi|\leq 1}(\xi)\, d\xi \lesssim \int_{\mathbb{R}^d} \left|\widehat{v^i v^j}(\xi)\right|^2 |\xi|^4\chi_{|\xi|\leq 1}(\xi)\, d\xi \\ &\lesssim \|v^i v^j\|^2_{L^2(\mathbb{R}^d)} \lesssim \|v\|^4_{C^0(\mathbb{R}^d)},\end{aligned}$$

where in the fourth inequality we have used that $-|\xi|^2 q(\xi) = \xi_i\xi_j\widehat{v^i v^j}(\xi)$, for every $\xi \in \mathbb{R}^d$. This, together with (3.6), gives (3.3). The proof is thus complete. ☐

4. Boundary regularity, part 1: extending to the whole space

The aim of this section, and of the subsequent Section 5, is to provide a proof of the following

Theorem 4.1 (Local boundary regularity)**.** *Let $\gamma \in (0, \frac{1}{2}]$, $\delta > 0$ and $\Omega \subset \mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$. Let $u \in C^{1,\delta}(\Omega) \cap C^\infty(\Omega)$ be a divergence-free vector field such that $u \cdot n|_{\partial\Omega} = 0$ and let $x_0 \in \partial\Omega$. Then, there exists a ball $B_{R_0}(x_0)$ such that the unique zero-average solution $p \in C^{1,\beta}(\Omega)$ of (1.2), where $\beta = \min\{\delta, \gamma\}$, enjoys*

$$\|p\|_{C^{2\gamma}_*(\Omega \cap B_{R_0}(x_0))} \leq C \left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)} \right) \tag{4.1}$$

for some constant $C > 0$ depending on γ, R_0 and Ω only.

Note that, by compactness, we can always cover $\partial\Omega$ with a finite number of balls. This immediately implies that, by patching all the local estimates (4.1),

$$\|p\|_{C^{2\gamma}_*(\Omega \setminus \Omega_{\tilde{R}})} \leq C \left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)} \right),$$

for a (possibly larger) constant $C > 0$ and some $\tilde{R} > 0$, where we defined

$$\Omega_{\tilde{R}} := \left\{ x \in \Omega \, : \, \text{dist}\,(x, \partial\Omega) \geq \tilde{R} \right\}.$$

This, together with the interior regularity result of Theorem 3.1, proves Theorem 2.14, from which we already deduced Theorem 1.1.

Thus, from now on, we only focus on (4.1). To prove such a quantitative local estimate, we use the normal geodesic coordinates introduced in Proposition 2.1, getting a modified equation in the new variables. We then extend such equation to the whole space $\mathbb{R}^d$ and apply pseudodifferential calculus, together with Littlewood–Paley analysis, in order to prove the desired regularity. In this Section 4 we focus on the first step, *i.e.*, changing coordinates and extending to the whole space. Then, in Section 5, we prove the quantitative estimate (4.1).

4.1. Removing the boundary datum.

Here we prove that the boundary datum is not the main obstacle for proving the double regularity of the pressure, but it only represents a compatibility condition in order to guarantee that (1.2) admits a solution. Removing the boundary datum is then useful in order to pass to the normal geodesic coordinates, avoiding some extra (tedious) terms at the boundary.

Note that, in the result below, u does not need to be divergence-free or tangent to $\partial\Omega$.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^d$ be of class $C^{2,\delta}$, for some $\delta > 0$. Let $\gamma > 0$ and $u \in C^\gamma(\Omega)$ be any vector field. Denote $\beta = \min\{\delta, \gamma\}$ and $A = \frac{1}{|\Omega|} \int_{\partial\Omega} u \otimes u : \nabla n$. Then, there exists a unique zero-average $\psi \in C^{1,\beta}(\Omega)$ such that*

$$\begin{cases} \Delta\psi = A & \text{in } \Omega \\ \partial_n \psi = u \otimes u : \nabla n & \text{on } \partial\Omega, \end{cases}$$

and, moreover, there exists a constant $C > 0$ such that

$$\|\psi\|_{C^{1,\beta}(\Omega)} \leq C \|u\|^2_{C^\gamma(\Omega)}. \tag{4.2}$$

Proof. Clearly, $A \in C^\infty(\overline{\Omega})$ and $u \otimes u : \nabla n \in C^\beta(\partial\Omega)$. Moreover, the compatibilty condition

$$\int_{\partial\Omega} u \otimes u : \nabla n = \int_{\partial\Omega} \partial_n \psi = \int_\Omega \Delta\psi = |\Omega| A$$

holds true. Thus, by standard elliptic arguments (see [25, Theorem 1.2] for instance), we infer that there exists a unique $\psi \in C^{1,\beta}(\Omega)$ such that $\int_\Omega \psi = 0$ and

$$\|\psi\|_{C^{1,\beta}(\Omega)} \leq C \left(|A| + \|u \otimes u : \nabla n\|_{C^\beta(\partial\Omega)} + \|\psi\|_{L^\infty(\Omega)} \right).$$

Since

$$|A| \le C\|u\|^2_{C^0(\overline{\Omega})} \qquad \text{and} \qquad \|u\otimes u : \nabla n\|_{C^\beta(\partial\Omega)} \le C\|u\|^2_{C^\gamma(\Omega)},$$

for some constant $C>0$ depending on Ω only, we achieve

$$\|\psi\|_{C^{1,\beta}(\Omega)} \le C\left(\|u\|^2_{C^\gamma(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\right).$$

The desired estimate (4.2) then follows by eliminating the term $\|\psi\|_{L^\infty(\Omega)}$ from the right-hand side of the previous estimate. This can be done by exploiting the very same contradiction argument already used in the proof of Theorem 1.1 above. We omit the details. □

Up to changing p with $p-\psi$, we are left with proving double regularity to solutions of

$$\begin{cases} -\Delta p &= \operatorname{div}\operatorname{div}(u\otimes u) + A & \text{in } \Omega \\ \partial_n p &= 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

for some constant $A\in\mathbb{R}$ guaranteeing the compatibility condition for solving (4.3). More precisely, Theorem 4.1 is a consequence of the following

Theorem 4.3. *Let $\gamma\in(0,\frac{1}{2}]$, $\delta>0$ and $\Omega\subset\mathbb{R}^d$ be a bounded and simply connected domain of class $C^{2,\delta}$. Let $u\in C^{1,\delta}(\Omega)\cap C^\infty(\Omega)$ be some divergence-free vector field such that $u\cdot n|_{\partial\Omega}=0$ and let $x_0\in\partial\Omega$. Then there exists a ball $B_{R_0}(x_0)$ such that the unique zero-average solution $p\in C^{2,\beta}(\Omega)$ of (4.3) with $|\Omega|A=\int_{\partial\Omega} u\otimes u:\nabla n$, where $\beta=\min\{\delta,\gamma\}$, enjoys*

$$\|p\|_{C^{2\gamma}_*(\Omega\cap B_{R_0}(x_0))} \le C\left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)}\right). \tag{4.4}$$

for some constant $C>0$ depending on γ, R_0 and Ω only.

Note that, since $u\in C^1(\overline{\Omega})\cap C^\infty(\Omega)$ is divergence-free and tangent to the boundary (which in particular gives that $\partial\Omega$ is a level set of the scalar function $u\cdot n$), we have

$$\begin{aligned}\int_\Omega \operatorname{div}\operatorname{div}(u\otimes u) &= \int_{\partial\Omega}\partial_j(u^iu^j)n^i = \int_{\partial\Omega} u^j\partial_j u^i n^i = \int_{\partial\Omega} u^j\partial_j(u\cdot n) - \int_{\partial\Omega} u^iu^j\partial_j n^i \\ &= -\int_{\partial\Omega} u\otimes u:\nabla n = -|\Omega|A.\end{aligned}$$

Therefore, the compatibility condition in the Neumann boundary value problem (4.3) is satisfied and the existence of a solution, say $p\in C^{2,\beta}(\Omega)$, unique up to constants, follows by standard Elliptic Theory. Thus we only have to prove the estimate (4.4).

Even if straightforward, we give a detailed proof on how the previous result implies Theorem 4.1 for the reader's convenience.

Proof of Theorem 4.1. Let u be as in the statement of Theorem 4.1. Let $A=\frac{1}{|\Omega|}\int_{\partial\Omega} u\otimes u:\nabla n$ and let ψ the corresponding unique solution of

$$\begin{cases} \Delta\psi &= A & \text{in } \Omega \\ \partial_n\psi &= u\otimes u:\nabla n & \text{on } \partial\Omega \end{cases}$$

given by Lemma 4.2, with $\int_\Omega \psi = 0$.

Let q be the unique zero-average solution of (4.3) given by Theorem 4.3. Then $p=q+\psi$ solves (1.2) and it is also average-free (since both q and ψ are), which in particular implies its uniqueness. Moreover, since $2\gamma\le 1$, by (4.2) and (4.4) we get

$$\begin{aligned}\|p\|_{C^{2\gamma}_*(\Omega\cap B_{R_0}(x_0))} &\le \|q\|_{C^{2\gamma}_*(\Omega\cap B_{R_0}(x_0))} + \|\psi\|_{C^{2\gamma}_*(\Omega\cap B_{R_0}(x_0))} \\ &\le C\left(\|u\|^2_{C^\gamma(\Omega)} + \|q\|_{L^\infty(\Omega)} + \|\psi\|_{C^1(\overline{\Omega})}\right)\end{aligned}$$

$$\leq C\left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)} + \|\psi\|_{C^1(\overline{\Omega})}\right)$$
$$\leq C\left(\|u\|^2_{C^\gamma(\Omega)} + \|p\|_{L^\infty(\Omega)}\right),$$

where in the third inequality we also used $\|q\|_{L^\infty(\Omega)} \leq \|p\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)}$. □

Thus the rest of the paper is devoted to the proof of Theorem 4.3, that is, as already pointed out, to the proof of the estimate (4.4).

4.2. **Straightening of the boundary.** We now reduce the proof of (4.4) to the flat-boundary case by straightening the boundary using the results of Section 2.1.

Let U be a neighborhood of $x_0 \in \partial\Omega$. We can assume U to be small enough so that, in $U \cap \Omega$, we have local geodesic coordinates $(r, \theta^2, \dots, \theta^d) = (r, \theta) \in (-r_0, r_0) \times \Theta$ such that the metric g^{ij} in these coordinates satisfies the properties of Proposition 2.1. Therefore (4.3) (denoting the solution by p, thus slightly abusing notation) now reads as

$$-\frac{1}{G}\partial_i\left(Gg^{ij}\partial_j p\right) = \frac{1}{G}\partial^2_{ij}\left(Gu^i u^j\right) + A \quad \text{in } \Omega,$$

with the boundary condition

$$\partial_r p(0, \theta) = 0 \quad \text{for all } \theta \in \Theta,$$

where $G = G(r, \theta) = \sqrt{\det g(r, \theta)} \in C^{1,\delta}$. Pay attention: here u^i, u^j refer to u^r and u^{θ^i}, as well as the partial derivatives ∂_i stands for ∂_r and ∂_{θ^i}. Moreover, the condition $u \cdot n = 0$ on $U \cap \partial\Omega$ reads $u^r(0, \theta) = 0$. Multiplying the equation in the interior by G leads to

$$\begin{cases} -\partial_i\left(Gg^{ij}\partial_j p\right) &= \partial^2_{ij}\left(Gu^i u^j\right) + GA \quad \text{in } \Omega \cap U \\ \partial_r p(0, \theta) &= 0 \quad \text{for all } \theta \in \Theta. \end{cases} \tag{4.5}$$

Remark 4.4. Note that the function GA plays a purely compatibility role only. Indeed

$$\|GA\|_{C^1(\Omega\cap U)} \leq |A|\|G\|_{C^1(\Omega\cap U)} \leq C(\Omega)\|u\|^2_{C^0(\overline{\Omega})},$$

which is a term that can be easily incorporated in all the estimates in Section 5 below, where we prove the double regularity of the 'modified' pressure equation in the local geodesic coordinates. Thus, from now on, we can forget about such a very regular term and assume that $A = 0$.

4.3. **Extension for negative r.** Now we extend the functions for $r < 0$ as follows: for any $r > 0$ we let $\tilde{g}(-r, \theta) := g(r, \theta)$, $\tilde{u}^r(-r, \theta) := -u^r(r, \theta)$, $\tilde{u}^\theta(-r, \theta) := u^\theta(r, \theta)$ and $\tilde{p}(-r, \theta) := p(r, \theta)$.

Remark 4.5. Note that we evenly extend g and u^θ because we have no information about their value at $r = 0$. On the other hand, we exploit the fact that $u^r(0, \theta) = 0$ by extending u^r oddly. Similarly, since $\partial_r p(0, \theta) = 0$, we want $\partial_r p$ to be extended oddly, so p should be extended evenly.

Thanks to the boundary conditions satisfied by u, we have the following

Lemma 4.6 (Regularity of the extensions). *Outside $\{r = 0\}$ all the functions above are C^∞. Moreover, if Ω is a $C^{2,\delta}$ domain, for some $\delta > 0$, we have the following global regularities:*

(1) $\tilde{p}, \tilde{u} \in \mathrm{Lip}_{r,\theta}$ and $\partial_r \tilde{u}^r \in C^0_{r,\theta}$;
(2) $\tilde{g}^{\theta\theta} \in \mathrm{Lip}_{r,\theta}$;
(3) $\partial_{\theta^i}\tilde{u}^r$, $\partial_{\theta^i}\tilde{u}^\theta$ and $\partial_{\theta^i}\tilde{p}$ are $C^0_{r,\theta}$ functions;
(4) $\partial_{\theta^k}\tilde{g}^{\theta^i\theta^j} \in C^\delta_{r,\theta}$, for all k, i, j.

Proof. Since $u^r(0, \theta) = 0$ and $u \in C^1(\overline{\Omega})$, then clearly $\tilde{u} \in \mathrm{Lip}_{r,\theta}$ and $\partial_r \tilde{u}^r \in C^0_{r,\theta}$. The same reasoning applies to $\tilde{p}$, giving (1). Since Ω is a $C^{2,\delta}$ domain, $g \in C^{1,\delta}$, thus all the θ components of its even extension satisfy $\tilde{g}^{\theta\theta} \in \mathrm{Lip}_{r,\theta}$, which gives (2). Moreover, since we are evenly extending g

in the r variable only, it is also clear that $\tilde{g}^{\theta^i\theta^j}(r,\cdot) \in C^{1,\delta}_\theta$ for every fixed $r \in \mathbb{R}$ and $\partial_{\theta^k}\tilde{g}^{\theta^i\theta^j}(\cdot,\theta) \in C^\delta_r$ if θ is fixed. In particular, this gives (4). For the very same reason, also (3) holds true, giving the C^1 regularity in the θ variables. □

Remark 4.7. It turns out that (4) will not be used in our analysis, as the Lipschitz regularity of $\tilde{g}$ is enough. One should however keep in mind that what (4) shows is that the purely Lipschitz regularity of $\tilde{g}$ is only given by the normal direction, as tangential directions are smoother. This might be indeed useful in other contexts.

Similarly, $\tilde{G} \coloneqq \sqrt{\det \tilde{g}}$ is even in r and globally Lipschitz. Therefore, we have the following

Lemma 4.8 (Divergence of the extension)**.** *Such extension of u preserves the incompressibility, i.e.,* $\operatorname{div}\tilde{u} = 0$*, where the divergence is taken in the $\tilde{g}$ metric.*

Proof. First, let us compute the divergence at (r,θ) for $r > 0$. In this case, there holds $\operatorname{div}\tilde{u} = \frac{1}{\tilde{G}}\partial_i(\tilde{G}\tilde{u}^i) = \operatorname{div} u(r,\theta) = 0$. Similarly, for $r < 0$, the parity properties of u give the same result. Note that this computation is not enough. Indeed, one needs to prove that there is no jump across $\{r = 0\}$, but this is easily checked via Lemma 4.6, since we are taking derivatives of Lipschitz functions, thus (by Rademacher's Theorem) not producing any distributional jump. □

Similarly, we need to make sure that $-\partial_i(\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p})$ produces no distributional jumps, *i.e.*, the distribution $-\partial_i(\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p})$ agrees with its evaluation outside $\{r = 0\}$. A standard calculus exercise shows that it is enough to have $\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p} \in C^0_{r,\theta}$, which we prove in the following

Lemma 4.9 (Laplacian of the extension)**.** *For all $i = 1,\dots,d$ it holds $\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p} \in C^0_{r,\theta}$. Thus the distribution $-\partial_i(\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p})$ coincides with its values outside $\{r = 0\}$.*

Proof. Clearly $\tilde{G}\tilde{g}^{ij} \in C^0_{r,\theta}$ by Lemma 4.6, thus it is enough to check that $\partial_j\tilde{p} \in C^0_{r,\theta}$ for all $j = 1,\dots,d$. By Lemma 4.6 we have $\partial_{\theta^j}\tilde{p} \in C^0_{r,\theta}$, for all $j \neq 1$. Moreover, since $\partial_r\tilde{p}$ is odd in r and $\partial_r p(0,\theta) = 0$, we conclude that $\partial_r\tilde{p} \in C^0_{r,\theta}$. □

We just need to exclude the possible jumps for the double divergence term.

Lemma 4.10 (Double divergence of the extension)**.** *The distributional derivative of order two of $\partial_{ij}(\tilde{G}\tilde{u}^i\tilde{u}^j)$ agrees with the function given by the pointwise derivatives computed outside $\{r = 0\}$, and this holds for all $(r,\theta) \in (-r_0,r_0)\times\Theta$.*

Proof. First, note that $\tilde{G}\tilde{u}^i\tilde{u}^j$ is a Lipschitz function, therefore has no jump across $\{r = 0\}$. Thus, the distributional derivative $\partial_j(\tilde{G}\tilde{u}^i\tilde{u}^j)$ agrees with its pointwise computation

$$\partial_j(\tilde{G}\tilde{u}^i\tilde{u}^j) = \partial_j(\tilde{G}\tilde{u}^j)\tilde{u}^i + \tilde{G}\tilde{u}^j\partial_j\tilde{u}^i = \tilde{G}\tilde{u}^j\partial_j\tilde{u}^i,$$

where we used the divergence free condition $\partial_j(\tilde{G}\tilde{u}^j) = 0$. As before, we now only need to make sure that $\tilde{G}\tilde{u}^j\partial_j\tilde{u}^i \in C^0_{r,\theta}$ for all $i = 1,\dots,d$. Now observe that, thanks to Lemma 4.6, the functions $\partial_j\tilde{u}^i$ are continuous when $j \geq 2$ (or, equivalently, all the derivatives that are not in the r direction), so that when $(i,j) \in \{1,\dots,d\}\times\{2,\dots,d\}$ the function $\tilde{G}\tilde{u}^j\partial_j\tilde{u}^i$ is continuous and has no jumps. Finally, let us deal with the case $j = 1$. Since $\tilde{u}^1$ is uniformly continuous with $\tilde{u}^1(0,\theta) = 0$ and $\partial_1\tilde{u}^i$ is bounded and also continuous on $\{r \neq 0\}$ (in the case $i = 1$ even $\partial_1\tilde{u}^1$ is continuous by Lemma 4.6), it follows that the function $\tilde{G}\tilde{u}^1\partial_1\tilde{u}^i$ is continuous. Therefore there is no jump of $\tilde{G}\tilde{u}^1\partial_1\tilde{u}^i$ across $\{r = 0\}$ for all $i = 1,\dots,d$. This proves our claim. □

Remark 4.11. Lemma 4.10 uses the straightening of the boundary with geodesic coordinates in an essential way. Indeed, the terms $\tilde{u}^1\partial_1\tilde{u}^i$ for $i = 1,\dots,d$ do not produce distributional jumps across $\{r = 0\}$ because $\tilde{u}^1(0,\theta) = 0$. Also the terms $\tilde{u}^j\partial_j\tilde{u}^i$ for $(i,j) \in \{1,\dots,d\}\times\{2,\dots,d\}$ are fine thanks to the additional regularity in the tangential direction given by Lemma 4.6.

Combining the above lemmas (and keeping in mind that, in virtue of Remark 4.4, we can neglect the constant A) we infer that, on $U = (-r_0, r_0) \times \Theta$, the functions $\tilde{u}$ and $\tilde{p}$ satisfy

$$-\partial_i \left(\tilde{g}^{ij}\tilde{G}\partial_j\tilde{p}\right) = \partial^2_{ij}\left(\tilde{G}\tilde{u}^i\tilde{u}^j\right) \quad \text{in } U. \tag{4.6}$$

4.4. **Extension to the whole space.** Our goal is now to extend the solution $\tilde{p}$ and the right-hand side datum $\tilde{u}$ of (4.6) to $\mathbb{R}^d$. In order to do so, let ψ be a (non-negative, smooth) localization function supported in $(-r_0, r_0) \times \Theta$ and which equals 1 on $U_0 := (-r_0/2, r_0/2) \times \Theta/2$.

Let $\bar{p} := \psi\tilde{p}$, $\bar{u} := \psi\tilde{u}$ defined as functions on $\mathbb{R}^d$. Also introduce $\bar{G} = \psi\tilde{G}$ and $\bar{g}^{ij} = \psi\tilde{g}^{ij}$. Also, let $\tilde{\psi}$ be another localization function such that $\operatorname{supp}\tilde{\psi} \subset \{\psi \equiv 1\}$ and a third one $\tilde{\tilde{\psi}}$ supported in $\{\tilde{\psi} \equiv 1\}$. In particular, observe that $\tilde{\psi}\partial_i\psi = 0$. Therefore, we can compute

$$\tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\bar{G}\bar{g}^{ij}\partial_j\bar{p}\right) = \tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\tilde{G}\tilde{g}^{ij}(\partial_j\psi\tilde{p} + \psi\partial_j\tilde{p})\right),$$

where we used that $\tilde{\psi}\psi = \tilde{\psi}$. Now, because $\tilde{\psi}\partial_j\psi = 0$, we obtain

$$\tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\bar{G}\bar{g}^{ij}\partial_j\bar{p}\right) = \tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p}\right),$$

so that, finally, using $\tilde{\tilde{\psi}}\tilde{\psi} = \tilde{\tilde{\psi}}$ and $\tilde{\tilde{\psi}}\partial_j\tilde{\psi} = 0$, we obtain

$$\tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\bar{G}\bar{g}^{ij}\partial_j\bar{p}\right) = \tilde{\tilde{\psi}}\partial_i\left(\tilde{G}\tilde{g}^{ij}\partial_j\tilde{p}\right).$$

Similarly, one can verify that

$$\tilde{\tilde{\psi}}\partial^2_{ij}\left(\tilde{\psi}\bar{G}\bar{u}^i\bar{u}^j\right) = \tilde{\tilde{\psi}}\partial^2_{ij}\left(\tilde{G}\tilde{u}^i\tilde{u}^j\right),$$

so that, in conclusion, the equation satisfied by $\bar{u}$ and $\bar{p}$ is

$$-\tilde{\tilde{\psi}}\partial_i\left(\tilde{\psi}\bar{G}\bar{g}^{ij}\partial_j\bar{p}\right) = \tilde{\tilde{\psi}}\partial^2_{ij}\left(\tilde{\psi}\bar{G}\bar{u}^i\bar{u}^j\right). \tag{4.7}$$

5. Boundary regularity, part 2: Hölder regularity in the full space

We will rewrite (4.7) as a pseudodifferential equation. First, let us change notation a little bit by writing g^{ij} instead of $\tilde{\psi}\bar{G}\bar{g}^{ij}$, G instead of $\tilde{\psi}\bar{G}$, q instead of $\bar{p}$, $\tilde{u}^i$ instead of $\bar{u}^i$ and ψ instead of $\tilde{\tilde{\psi}}$. We aim to prove our last result, namely, the $C^{2\gamma}_*$ estimate on (4.7). More precisely, in this last section we shall prove that

$$\|q\|_{C^{2\gamma}_*(\mathbb{R}^d)} \le C\left(\|G\tilde{u}\|^2_{C^\gamma(\mathbb{R}^d)} + \|q\|_{L^\infty(\mathbb{R}^d)}\right). \tag{5.1}$$

Note that, since the change of variables is bi-Lipschitz, the previous estimate for q automatically translates into (4.4) for a certain ball $B_{R_0}(x_0)$, since we have chosen the cutoff function ψ such that $\psi \equiv 1$ in an open neighborhood of x_0.

For sake of clarity, in our new notation, the equation is

$$-\psi\partial_i\left(g^{ij}\partial_j q\right) = \psi\partial^2_{ij}\left(G\tilde{u}^i\tilde{u}^j\right). \tag{5.2}$$

The operator $E_{1,i} := g^{ij}\partial_j$ is a pseudodifferential operator, $E_{1,i} = \mathrm{Op}(e_{1,i})$, where $e_{1,i} := e_{1,i}(r,\theta,\xi) \in C^1_*S^1$ is the symbol defined by

$$e_{1,i}(r,\theta,\xi) := g^{ij}(r,\theta)\xi^j.$$

Let $\delta \in (0,1)$. We define the *sharp* part of $e_{1,i}$ as

$$e^\sharp_{1,i}(r,\theta,\xi) := \sum_{K \le M^\delta} g^{ij}_K(r,\theta)\mathbf{P}_M(\xi)\xi^j, \tag{5.3}$$

where $g_K^{ij} := \mathbf{P}_K g^{ij}$ and the summation runs on all $M, K \in 2^{\mathbb{N}}$ such that $K \le M^\delta$. We also define the *flat* part of the symbol $e_{1,i}$ as

$$e^\flat_{1-\delta,i} := e_{1,i} - e^\sharp_{1,i}.$$

Note that $e^\sharp_{1,i}$ is a symbol of order 1. More precisely, we have the following

Lemma 5.1. *For all $i = 1, \dots, d$, we have $e^\sharp_{1,i} \in S^1_{1,\delta}$.*

Proof. To prove that $e^\sharp_{1,i} \in S^1_{1,\delta}$, we need to show the convergence of the series in (5.3) in all C^k_x spaces (for any non-negative k), with the right bounds. Note that, for any K, g_K^{ij} is a smooth function, and recall that $\mathbf{P}_M(\xi)$ is the Littlewood–Paley partition introduced in Section 2.2. Let α, β be two multi-indices, with $|\beta| = k$ and set $x = (r, \theta)$. By the triangle inequality, we have

$$|\partial_\xi^\alpha \partial_x^\beta e^\sharp_{1,i}(x,\xi)| \le \sum_{K \le M^\delta} \left|\partial_x^\beta g_K^{ij}(x)\right| \left|\partial_\xi^\alpha (\mathbf{P}_M(\xi)\xi^j)\right|.$$

Now observe that, by Theorem 2.2, there holds $|\partial_x^\beta g_K^{ij}(x)| \lesssim K^{|\beta|}\|g^{ij}\|_{L^\infty} \lesssim K^{|\beta|}$. By direct computations, as soon as $M > 1$, one also has the bound

$$|\partial_\xi^\alpha(\mathbf{P}_M(\xi)\xi^j)| \lesssim \langle\xi\rangle^{1-|\alpha|}\chi(M^{-1}\xi),$$

where χ is some compactly supported function in the annulus $\{M/4 \le |\xi| \le 4M\}$. Therefore,

$$|\partial_\xi^\alpha \partial_x^\beta e^\sharp_{1,i}(x,\xi)| \lesssim \sum_{K \le M^\delta} K^{|\beta|}\langle\xi\rangle^{1-|\alpha|}\chi(M^{-1}\xi).$$

After a summation in K and M, we hence get

$$|\partial_\xi^\alpha \partial_x^\beta e^\sharp_{1,i}(x,\xi)| \lesssim \sum_M M^{\delta|\beta|}\langle\xi\rangle^{1-|\alpha|}\chi(M^{-1}\xi) \lesssim \langle\xi\rangle^{1-|\alpha|+\delta|\beta|}$$

and the conclusion follows. □

Remark 5.2. With similar computations, one gets $e^\flat_{1-\delta,i} \in C^1_* S^{1-\delta}_{1,\delta}$, therefore justifying its indices. However, since of no use in our proof, we leave the proof of this property to the reader.

Thus, equation (5.2) translates into

$$-\psi\partial_i\left(E^\sharp_{1,i}(q)\right) = \psi\partial_{ij}\left(G\tilde{u}^i\tilde{u}^j\right) + \psi\partial_i\left(E^\flat_{1-\delta,i}(q)\right).$$

Our next observation is that the principal part of the operator $\psi\partial_i\left(E^\sharp_{1,i}\,\cdot\right)$ is elliptic. In order to see it, we apply Theorem 2.10 with $N = 0$, and get $\psi\partial_i\left(E^\sharp_{1,i}\,\cdot\right) = \mathrm{Op}(c_i)$ where

$$c_i(r,\theta,\xi) = \psi e^\sharp_{1,i}(r,\theta,\xi)\xi^i + h(r,\theta,\xi),$$

and where $h \in S^{1+\delta}_{1,\delta}$. Since $1 + \delta < 2$, the operator $R_{1+\delta} := \mathrm{Op}(h)$ can be considered as a remainder term. Thus, we can further rewrite the equation as

$$-E^\sharp_2 q = \psi\partial_{ij}\left(G\tilde{u}^i\tilde{u}^j\right) + \psi\partial_i\left(E^\flat_{1-\delta,i}q\right) + R_{1+\delta}(q), \tag{5.4}$$

where $E^\sharp_2 := \mathrm{Op}(e^\sharp_2)$ is the second-order operator associated to the symbol

$$e^\sharp_2 := \psi(r,\theta)\sum_{K \le M^\delta} g_K^{ij}(r,\theta)\mathbf{P}_M(\xi)\xi^i\xi^j.$$

Lemma 5.3. *The symbol $e^\sharp_2 \in S^2_{1,\delta}$ is elliptic.*

Proof. Since $e^\sharp_{1,i} \in S^1_{1,\delta}$ and $\xi^i \in S^1_{1,\delta}$, we immediately see that $e^\sharp_2 \in S^2_{1,\delta}$. To prove the ellipticity of $e^\sharp_2$, let us first observe that the symbol

$$e_2(r,\theta,\xi) := \psi(r,\theta) g^{ij}(r,\theta)\xi^i\xi^j$$

is elliptic. Indeed, by Proposition 2.1, we have $g^{ij}(x)\xi^i\xi^j \geq c|\xi|^2$, for all $|\xi| \geq R$ and $x \in U$, for some constant $c > 0$. Since $\operatorname{Spt}\psi \subset U$, the latter inequality holds for all x in $U_0 := \{\psi \equiv 1\}$, hence implying that e_2 is elliptic on U_0 with constant c.

In order to prove that $e^\sharp_2$ is elliptic, we only need to prove that, for M_0 large enough and $|\xi| \geq M_0$ there holds

$$\sum_{K \leq M^\delta} g^{ij}_K(x)\mathbf{P}_M(\xi)\xi^i\xi^j \geq \frac{c}{2}|\xi|^2, \tag{5.5}$$

where the summation runs on both K and M.

To obtain (5.5), let us bound

$$\left| \sum_{K > M^\delta} g^{ij}_K(x)\mathbf{P}_M(\xi)\xi^i\xi^j \right| \leq \sum_{K > M^\delta} \left|g^{ij}_K(x)\right| \left|\mathbf{P}_M(\xi)\right| \left|\xi^i\right| \left|\xi^j\right| \leq d|\xi|^2 \sum_{K : K > M_0^\delta} \|g_K\|_{L^\infty},$$

where we have also used the partition of unity property $\sum_M \mathbf{P}_M(\xi) = 1$ to get rid of the sum on M. Now observe that there holds

$$\sum_{K : K > M_0^\delta} \|g_K\|_{L^\infty} \leq C \sum_{K : K > M_0^\delta} K^{-1}\|g\|_{C^1_*} \leq 2CM_0^{-\delta}\|g\|_{C^1_*},$$

which can be made smaller than $\frac{c}{2d}$ for M_0 large enough. Therefore, we get (5.5). $\square$

Since $E^\sharp_2 = \operatorname{Op}(e^\sharp_2)$ is elliptic on U_0, we can invert it by using Theorem 2.12. Hence there exist $E^\sharp_{-2} \in \operatorname{Op}(S^{-2}_{1,\delta})$ and $R_{-2(1-\delta)} \in \operatorname{Op}(S^{-2(1-\delta)}_{1,\delta})$ such that

$$E^\sharp_{-2} \circ E^\sharp_2 = \operatorname{Op}(\chi) + R_{-2(1-\delta)}, \tag{5.6}$$

where χ is some compactly supported function in U_0 such that $\chi \equiv 1$ in the open set U_1 containing the point x_0. First, since $2\gamma \leq 1$ and $\delta < \frac{1}{2}$, we have $2\gamma - 2(1-\delta) < 0$ and therefore, thanks to Theorem 2.9, we have the continuity property for $R_{-2(1-\delta)}$, which reads as

$$\|R_{-2(1-\delta)}(q)\|_{C^{2\gamma}_*} \lesssim \|q\|_{C^0_*} \lesssim \|q\|_{L^\infty}. \tag{5.7}$$

We are going to perform a last change of variable, by introducing $v^i := G\tilde{u}^i$ and $a := \frac{1}{G} \in C^1_*$, so that $\psi\partial^2_{ij}(G\tilde{u}^i\tilde{u}^j) = \psi\partial^2_{ij}(av^iv^j)$. Note that the divergence-free condition now reads as

$$\partial_i v^i = 0. \tag{5.8}$$

Observe that (5.1) can be rephrased as

$$\|q\|_{C^{2\gamma}_*} \leq C\left(\|v\|^2_{C^\gamma_*} + \|q\|_{L^\infty}\right). \tag{5.9}$$

In order to obtain (5.9), let us apply $-E^\sharp_{-2}$ to (5.4). Hence we can write

$$\chi q = \underbrace{-E^\sharp_{-2}\left(\psi\partial^2_{ij}(av^iv^j)\right)}_{A} \underbrace{- E^\sharp_{-2}\left(\psi\partial_i\left(E^\flat_{1-\delta,i}q\right)\right)}_{B} \underbrace{- E^\sharp_{-2} \circ R_{1+\delta}(q) + R_{-2(1-\delta)}(q)}_{R},$$

so we just need to estimate the right-hand side. The term A is the main contribution, whereas the terms B and R should be thought as remainder terms. We postpone the treatment of A and B in Section 5.1 and Section 5.2.

Let us consider the term R. Using the continuity property $C_*^{2\gamma-2} \to C_*^{2\gamma}$ of $E_{-2}^\sharp \in \mathrm{Op}(S_{1,\delta}^{-2})$ given by Theorem 2.9, together with the continuity property (5.7), we can estimate

$$\begin{aligned}\|R\|_{C_*^{2\gamma}} &\lesssim \|R_{1+\delta}(q)\|_{C_*^{2\gamma-2}} + \|R_{-2(1-\delta)}(q)\|_{C_*^{2\gamma}} \\ &\lesssim \|q\|_{C_*^{2\gamma-1+\delta}} + \|q\|_{L^\infty} \lesssim \|q\|_{C_*^{2\gamma-\frac{\delta}{2}}},\end{aligned}$$

where in the third inequality we have used that $2\gamma \le 1$ and that $4\gamma > 3\delta$ (this last property can be indeed assumed without any loss of generality, by choosing $\delta > 0$ sufficiently small). Now, by interpolation and Young's inequality, we can estimate

$$C\|q\|_{C_*^{2\gamma-\frac{\delta}{2}}} \le C\|q\|_{C_*^{2\gamma}}^{1-\frac{\delta}{4\gamma}} \|q\|_{L^\infty}^{\frac{\delta}{4\gamma}} \le \varepsilon\|q\|_{C_*^{2\gamma}} + C(\varepsilon)\|q\|_{L^\infty},$$

for some (possibly large) constant $C(\varepsilon) > 0$. Therefore (5.9) reduces to

$$\|A\|_{C_*^{2\gamma}} \lesssim \|v\|_{C^\gamma}^2 \tag{5.10}$$

and

$$\|B\|_{C_*^{2\gamma}} \lesssim \|q\|_{C_*^{2\gamma-\frac{\delta}{2}}} \le \varepsilon\|q\|_{C_*^{2\gamma}} + C(\varepsilon)\|q\|_{L^\infty}. \tag{5.11}$$

The combination of these inequalities indeed imply a bound of the form

$$\|\chi q\|_{C_*^{2\gamma}} \le 2\varepsilon\|q\|_{C_*^{2\gamma}} + C(\varepsilon)(\|q\|_{L^\infty} + \|v\|_{C_*^\gamma}^2)$$

(for all $\varepsilon > 0$ arbitrarily small), which itself implies (5.9). Indeed, the estimate holds in $\{\chi \equiv 1\}$, and can be obtained (with the same proof) locally around any point in U_0. Thus the $\|q\|_{C_*^{2\gamma}}$ norm on the right-hand side can be absorbed in the left-hand side by using standard techniques in elliptic PDEs, see [11, Proof of Theorem 2.16] or [13, Theorem 6.2] for instance.

The goal of the following subsections is to prove the two last estimates (5.10) and (5.11).

5.1. **Term A.** Let us write

$$a = \sum_{L \in 2^{\mathbb{N}}} a_L \quad \text{and} \quad v^i = \sum_{K \in 2^{\mathbb{N}}} v_K^i,$$

so that

$$A = E_{-2}^\sharp \left(\sum_{L,K,M \ge 1} \psi \partial_{ij}^2 \left(a_L v_K^i v_M^j \right) \right).$$

At this stage, our goal is to apply the same strategy of Section 3. To do so, we use the double-divergence form when the frequencies of v are similar, $K \sim M$. This allows us to split the derivatives and gain the double regularity. Also, when $L \ge \max\{K, M\}$, the derivatives ∂_{ij}^2 apply to a, which is smoother than v, therefore the double-divergence form also gives us the double regularity. When L is not the largest frequency and one of the frequencies among K, M is significantly larger than the other one, the double-divergence form could not achieve double regularity. Therefore, in this case, we rely on the divergence-free structure of v, which allows to rewrite the double-divergence as some product of terms of order 1. Indeed, we have

Lemma 5.4. *There holds*

$$\partial_{ij}^2(a_L v_K^i v_M^j) = \partial_j(\partial_i a_L v_K^i v_M^j) + \partial_j a_L v_K^i \partial_i v_M^j + a_L \partial_j v_K^i \partial_i v_M^j.$$

Proof. Note that $\mathbf{P}_K$ commutes with ∂_i, as they are both Fourier multipliers. Therefore, since $\partial_i v^i = 0$ by (5.8), we obtain $\partial_i v_K^i = 0$. Using this, we can write

$$\begin{aligned}\partial_{ij}^2(a_L v_K^i v_M^j) &= \partial_j(\partial_i a_L v_K^i v_M^j) + \partial_j(a_L v_K^i \partial_i v_M^j) \\ &= \partial_j(\partial_i a_L v_K^i v_M^j) + \partial_j a_L v_K^i \partial_i v_M^j + a_L \partial_j v_K^i \partial_i v_M^j,\end{aligned}$$

where in the last line we used again $\partial_j v^j_M = 0$. □

Thanks to the above discussion, we write $A = A_1 + A_2$, with

$$A_1 := E^\sharp_{-2}\left(\sum_{\substack{K\sim M\\ L\ll\max\{K,M\}}} + \sum_{L\geq\frac18\max\{K,M\}}\right)\psi\partial^2_{ij}(a_L v^i_K v^j_M) =: A_{11} + A_{12}$$

and

$$\begin{aligned}A_2 &:= E^\sharp_{-2}\sum_{\substack{K\ll M \text{ or } M\ll K\\ L\ll\max\{K,M\}}}\psi\partial^2_{ij}(a_L v^i_K v^j_M)\\ &= E^\sharp_{-2}\psi\sum_{\substack{K\ll M \text{ or } M\ll K\\ L\ll\max\{K,M\}}}\Big(\partial_j(\partial_i a_L v^i_K v^j_M) + \partial_j a_L v^i_K\partial_i v^j_M + a_L\partial_j v^i_K\partial_i v^j_M\Big)\\ &=: A_{21} + A_{22} + A_{23},\end{aligned}$$

where we also used Lemma 5.4. We are left with estimating A_{11}, A_{12}, A_{21}, A_{22} and A_{23}.

As a preliminary remark, let us observe that, by composition of pseudodifferential operators with smooth symbols (see Theorem 2.10), we recognize that $E^\sharp_{-2}\psi\partial^2_{ij} \in \mathrm{Op}(S^0_{1,\delta})$, therefore continuous $C^{2\gamma}_* \to C^{2\gamma}_*$, thanks to Theorem 2.9.

Estimating A_{11}. By continuity of $E^\sharp_{-2}\psi\partial^2_{ij}$, we start by estimating

$$\|A_{11}\|_{C^{2\gamma}_*} \lesssim \left\|\sum_{\substack{K\sim M\\ L\ll\max\{K,M\}}} a_L v^i_K v^j_M\right\|_{C^{2\gamma}_*}.$$

As the sum is symmetrical in i and j, we can assume $M \geq K$, therefore $K \in \{\frac{M}{2}, M\}$. As v^j_M and v^i_K are frequency supported at similar values $M \sim K$, $v^i_K v^j_M$ is frequency supported in $\{|\xi| \lesssim M\}$. Since $L < M$, then $a_L v^i_K v^j_M$ is also frequency supported in $\{|\xi| \lesssim M\}$. Therefore,

$$\mathbf{P}_N\left(\sum_{\substack{K\sim M\\ L\ll\max\{K,M\}}} a_L v^i_K v^j_M\right) = \sum_{\substack{K\sim M\geq N\\ L\ll M}}\mathbf{P}_N(a_L v^i_K v^j_M),$$

so that

$$\|A_{11}\|_{C^{2\gamma}_*} \lesssim \sup_{N\geq1} N^{2\gamma}\sum_{\substack{M\geq N\\ L\ll M}}\sum_{K=\frac{M}{2},M}\left\|a_L v^i_K v^j_M\right\|_{L^\infty}.$$

Using $a \in C^1_*$ and $v \in C^\gamma_*$, we see that

$$\left\|a_L v^i_K v^j_M\right\|_{L^\infty} \lesssim L^{-1}(KM)^{-\gamma}\|v\|^2_{C^\gamma_*}\|a\|_{C^1_*} \lesssim L^{-1}K^{-2\gamma}\|v\|^2_{C^\gamma_*},$$

where we used that $K \sim M$. Therefore, we finally arrive at

$$\begin{aligned}\|A_{11}\|_{C^{2\gamma}_*} &\lesssim \sup_{N\geq1}\sum_{\substack{M\geq N\\ L\ll M}}\sum_{K=\frac{M}{2},M} N^{2\gamma}L^{-1}M^{-2\gamma}\|v\|^2_{C^\gamma_*} \lesssim \sup_{N\geq1}\sum_{\substack{M\geq N\\ L\geq1}} N^{2\gamma}L^{-1}M^{-2\gamma}\|v\|^2_{C^\gamma_*}\\ &\lesssim \sup_{N\geq1} N^{2\gamma}\sum_{M\geq N} M^{-2\gamma}\|v\|^2_{C^\gamma_*} \lesssim \|v\|^2_{C^\gamma_*},\end{aligned}$$

proving that A_{11} satisfies (5.10).

Estimating A_{12}. By continuity of $E^\sharp_{-2}\psi\partial^2_{ij}$, we start by estimating

$$\|A_{12}\|_{C^{2\gamma}_*} \lesssim \left\| \sum_{L\geq\frac{1}{8}\max\{K,M\}} a_L v^i_K v^j_M \right\|_{C^{2\gamma}_*}.$$

By symmetry of K and M, we can assume $M \geq K$. The frequency localization gives

$$\begin{aligned}\mathbf{P}_N\left(\sum_{L\geq\frac{1}{8}\max\{K,M\}} a_L v^i_K v^j_M\right) &= \mathbf{P}_N\left(\sum_{L\gg M\geq K} a_L v^i_K v^j_M\right) + \mathbf{P}_N\left(\sum_{L\sim M\geq K} a_L v^i_K v^j_M\right)\\ &= \sum_{\substack{L\sim N\\ K\leq M\ll L}} \mathbf{P}_N(a_L v^i_K v^j_M) + \sum_{\substack{N\lesssim L\sim M\\ K\leq M}} \mathbf{P}_N(a_L v^i_K v^j_M),\end{aligned}$$

so that

$$\begin{aligned}\|A_{12}\|_{C^{2\gamma}_*} &\lesssim \sup_{N\geq1} N^{2\gamma}\left(\sum_{\substack{L\sim N\\ K\leq M\ll L}} L^{-1}M^{-\gamma}K^{-\gamma} + \sum_{\substack{N\lesssim L\sim M\\ K\leq M}} L^{-1-\gamma}K^{-\gamma}\right)\|v\|^2_{C^\gamma_*}\\ &\lesssim \sup_{N\geq1} N^{2\gamma}\left(N^{-1}\sum_{M\geq1} M^{-\gamma}\sum_{K\geq1} K^{-\gamma} + \sum_{L\gtrsim N} L^{-(1+\gamma)}\sum_{K\geq1} K^{-\gamma}\right)\|v\|^2_{C^\gamma_*}\\ &\lesssim \sup_{N\geq1}\left(N^{2\gamma-1}+N^{\gamma-1}\right)\|v\|^2_{C^\gamma_*} \lesssim \|v\|^2_{C^\gamma_*},\end{aligned}$$

since $2\gamma-1\leq 0$. This proves that A_{12} satisfies (5.10).

Estimating A_{21}. To estimate A_{21}, we use the $C^{2\gamma-1}_* \to C^{2\gamma}_*$ continuity of $E^\sharp_{-2}\psi\partial_j \in \mathrm{Op}(S^{-1}_{1,\delta})$. Since K and M play a symmetrical role, let us assume that $K \ll M$ (which means $K < \frac{M}{2}$), so

$$\|A_{21}\|_{C^{2\gamma}_*} \lesssim \left\| \sum_{\substack{K\ll M\\ L\ll M}} \partial_i a_L v^i_K v^j_M \right\|_{C^{2\gamma-1}_*}.$$

Now observe that

$$\mathbf{P}_N\left(\sum_{\substack{K\ll M\\ L\ll M}} \partial_i a_L v^i_K v^j_M\right) = \sum_{\substack{M\sim N\\ K,L\ll M}} \mathbf{P}_N(\partial_i a_L v^i_K v^j_M),$$

and therefore

$$\|A_{21}\|_{C^{2\gamma}_*} \lesssim \sup_{N\geq1} N^{2\gamma-1}\sum_{\substack{M\sim N\\ K,L\ll M}} \left\|\partial_i a_L v^i_K v^j_M\right\|_{L^\infty}.$$

Since $a\in \mathrm{Lip}$, we have $\|\partial_j a_L\|_{L^\infty} \lesssim \|\partial_j a\|_{L^\infty} \lesssim 1$, see Theorem 2.2. Being $v\in C^\gamma_*$, there holds

$$\left\|\partial_i a_L v^i_K v^j_M\right\|_{L^\infty} \lesssim (KM)^{-\gamma}\|v\|^2_{C^\gamma_*},$$

and we can bound

$$\|A_{21}\|_{C^{2\gamma}_*} \lesssim \sup_{N\geq1} N^{2\gamma-1}\|v\|^2_{C^\gamma_*}\sum_{\substack{M\sim N\\ K,L\ll M}} (KM)^{-\gamma} \lesssim \sup_{N\geq1} N^{\gamma-1}\log(N)\|v\|^2_{C^\gamma_*} \lesssim \|v\|^2_{C^\gamma},$$

because $\gamma-1<0$.

Estimating A_{22}. To estimate A_{22}, we use the $C_*^{2\gamma-2}\to C_*^{2\gamma}$ continuity of $E_{-2}^\sharp\psi\in \mathrm{Op}(S^{-2}_{1,\delta})$, so

$$\|A_{22}\|_{C_*^{2\gamma}}\lesssim\Big\|\sum_{\substack{K\ll M\\ L\ll M}}\partial_j a_L v_K^i\partial_i v_M^j\Big\|_{C_*^{2\gamma-2}}+\Big\|\sum_{\substack{M\ll K\\ L\ll K}}\partial_j a_L v_K^i\partial_i v_M^j\Big\|_{C_*^{2\gamma-2}}.$$

For the first term, we can write

$$\begin{aligned}\Big\|\sum_{\substack{K\ll M\\ L\ll M}}\partial_j a_L v_K^i\partial_i v_M^j\Big\|_{C_*^{2\gamma-2}}&\lesssim\sup_{N\geq 1}N^{2\gamma-2}\|v\|^2_{C_*^\gamma}\sum_{K,L\ll M\sim N}K^{-\gamma}M^{1-\gamma}\\&\lesssim\sup_{N\geq1}N^{\gamma-1}\log(N)\|v\|^2_{C_*^\gamma}\lesssim\|v\|^2_{C_*^\gamma},\end{aligned}$$

because $\gamma-1<0$. To estimate the second term, we observe that

$$\begin{aligned}\Big\|\sum_{\substack{M\ll K\\ L\ll K}}\partial_j a_L v_K^i\partial_i v_M^j\Big\|_{C_*^{2\gamma-2}}&\lesssim\sup_{N\geq 1}N^{2\gamma-2}\|v\|^2_{C_*^\gamma}\sum_{\substack{M\ll K\sim N\\ L\ll K}}\|\partial_j a_L\|_{L^\infty}K^{-\gamma}M^{1-\gamma}\\&\lesssim\sup_{N\geq1}N^{-1}\log N\|v\|^2_{C_*^\gamma}\lesssim\|v\|^2_{C_*^\gamma},\end{aligned}$$

where we used that $\|\partial_j a_L\|_{L^\infty}\lesssim\|\partial_j a\|_{L^\infty}\lesssim 1$, see Theorem 2.2.

Estimating A_{23}. We start by using the $C_*^{2\gamma-2}\to C_*^{2\gamma}$ continuity of $E_{-2}^\sharp$. Since again K and M play a symmetrical role, we also assume that $K\leq M$. Hence, we can write

$$\|A_{23}\|_{C_*^{2\gamma}}\lesssim\Big\|\sum_{\substack{K\ll M\\ L\ll M}}a_L\partial_j v_K^i\partial_i v_M^j\Big\|_{C_*^{2\gamma-2}},$$

Therefore, by taking into account the frequency localization, this gives

$$\|A_{23}\|_{C_*^{2\gamma}}\lesssim\sup_{N\geq1}N^{2\gamma-2}\|v\|^2_{C_*^\gamma}\sum_{\substack{K\ll M\sim N\\ L\ll M}}L^{-1}K^{1-\gamma}M^{1-\gamma}\lesssim\|v\|^2_{C_*^\gamma},$$

where we used that $1-\gamma>0$.

5.2. Term B.

We start by estimating

$$\left\|E_{-2}^\sharp\left(\psi\partial_i\left(E^\flat_{1-\delta,i}(q)\right)\right)\right\|_{C_*^{2\gamma}}\lesssim\sum_{i=1}^d\left\|E^\flat_{1-\delta,i}(q)\right\|_{C_*^{2\gamma-1}}.$$

By definition of $E^\flat_{1-\delta,i}$, we can write

$$E^\flat_{1-\delta,i}(q)=\sum_{K>M^\delta}g_K^{ij}\partial_j q_M=\sum_{M^\delta<K\ll M}g_K^{ij}\partial_j q_M+\sum_{K\sim M}g_K^{ij}\partial_j q_M+\sum_{K\gg M}g_K^{ij}\partial_j q_M,$$

so that, by frequency localization of the above terms, there holds

$$\mathbf{P}_N E^\flat_{1-\delta,i}(q)=\sum_{\substack{M^\delta<K\ll M\\ M\sim N}}\mathbf{P}_N(g_K^{ij}\partial_j q_M)+\sum_{K\sim M\geq N}\mathbf{P}_N(g_K^{ij}\partial_j q_M)+\sum_{M\ll K\sim N}\mathbf{P}_N(g_K^{ij}\partial_j q_M).$$

Thus, by using $\|q_M\|_{L^\infty} \lesssim M^{-2\gamma+\frac{\delta}{2}}\|q\|_{C_*^{2\gamma-\frac{\delta}{2}}}$ and $\|g_K^{ij}\|_{L^\infty} \lesssim K^{-1}$ (recall that the extended metric is Lipschitz), we obtain

$$\begin{aligned}\|E^\flat_{1-\delta,i}(q)\|_{C_*^{2\gamma-1}} &\lesssim \sup_{N\geq 1} N^{2\gamma-1}\|q\|_{C_*^{2\gamma-\frac{\delta}{2}}}\left(\sum_{\substack{M^\delta<K\ll M\\ M\sim N}} + \sum_{K\sim M\geq N} + \sum_{M\ll K\sim N}\right) K^{-1}M^{1-2\gamma+\frac{\delta}{2}}\\ &\lesssim \sup_{N\geq 1}\left(N^{-\frac{\delta}{2}} + N^{-1+\frac{\delta}{2}} + N^{-1+\frac{\delta}{2}}\right)\|q\|_{C_*^{2\gamma-\frac{\delta}{2}}} \lesssim \|q\|_{C_*^{2\gamma-\frac{\delta}{2}}}.\end{aligned}$$

Here we used $\sum_{M^\delta<K\ll M} K^{-1} \leq \sum_{K>M^\delta} K^{-1} \lesssim M^{-\delta}$ and, in the second summation, we also assumed $-2\gamma + \frac{\delta}{2} < 0$ (this can be clearly ensured by choosing $\delta > 0$ sufficiently small).

Remark 5.5. As mentioned in Remark 5.2, $E^\flat_{1-\delta} \in \mathrm{Op}(C^1_* S^{1-\delta}_{1,\delta})$ by similar computations. Therefore, the estimate of the term B can be handled using a generalization of Theorem 2.9 to non-smooth symbols, see [24, Chapter 13, Proposition 9.10]. However, we preferred to provide a proof adapted to our operator in order to keep the proof as self-contained as possible.

6. Final comments and extensions

Let us now conclude our paper with some comments on possible extensions of our results.

6.1. Rougher domains. Essentially, the content of Theorem 1.1 is the validity of the double regularity estimate (1.7) on domains which are a little more regular than C^2. Maybe, the pseudodifferential approach used in this article might achieve the same regularity on less regular domains. In the following discussion, we will forget about the approximation procedure (which, in our case, works on C^{2^+} domains only).

Let Ω be a $C^{1,\alpha}$ domain. Since $u\otimes u \in C^\gamma$ and $\nabla n \in C^{\alpha-1}$, it is enough that $\gamma+\alpha-1>0$ to ensure the well-posedness of the boundary condition as a distribution (see Lemma 2.5). Thus, in order to have $\partial_n p \in C^{2\gamma-1}$, together with its compatibility with the boundary condition, we also need $2\gamma-1 \leq \min\{\alpha-1,\gamma\} = \alpha-1$. The combination of these two conditions give the conjecture for the optimal domain regularity, see Figure 1.

We believe that the pseudodifferential part of the proof should still apply to this situation. Indeed, the regularity of $a \in C^\alpha$ in Section 5.1 and in Section 5.2 is enough for the argument. Thus, the main difficulty would be to extend the good approximation procedure of Lemma 2.13.

6.2. Extension to Besov velocities. Another interesting extension of Theorem 1.1 would be to investigate the double regularity in Besov space, *i.e.*, the validity of the estimate

$$\|p\|_{B^{2\gamma}_{r,\infty}} \lesssim \|u\|^2_{B^{\gamma}_{2r,\infty}}, \tag{6.1}$$

for some values of $r \in [1,\infty)$, as $r=\infty$ is exactly the case of Theorem 1.1. Indeed, from the pioneering work of A. Kolmogorov [17], the Besov classes seem the right setting in which to embed the local structure of turbulent flows. The estimate (6.1) has been indeed recently proved in [7] in the absence of the boundary, *i.e.*, on $\mathbb{T}^d$ and on $\mathbb{R}^d$.

First, let us mention that as in the previous observation, the pseudodifferential part will work similarly, with the slight difference that, instead of using continuity estimates of pseudodifferential operators in C^s_* spaces, one should use the continuity estimates in Besov spaces, *i.e.*, Theorem 2.9 in Besov classes, which follows by interpolation between L^r and $W^{s,r}$.

Discarding the difficulty of approximation results such as Lemma 2.13, a new problem arises, that is, the traces on the boundary. Indeed, at a formal level, the trace of $u \in B^\gamma_{2r,\infty}$ at the boundary is only in $B^{\gamma-\frac{1}{2r}}_{2r,\infty}$. Therefore, for smooth enough domains (say C^3, for example), and

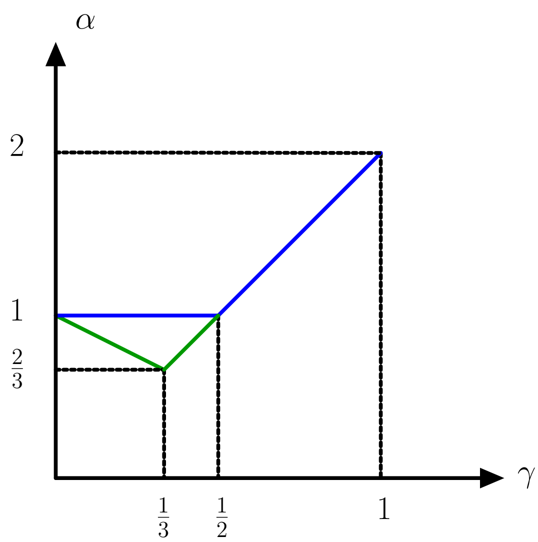

FIGURE 1. Double Hölder regularity on $C^{1,\alpha}$ domains: blue lines represent what is proven in Theorem 1.1 for $\gamma \leq \frac{1}{2}$, as well as in [10] when $\gamma > \frac{1}{2}$. Green lines represent the conjectured optimal domain regularity.

for $\gamma \geq \frac{1}{2r}$ (to let $u \otimes u$ be well defined), there holds $u \otimes u : \nabla n \in B^{\gamma-\frac{1}{2r}}_{r,\infty}$. This is in fact compatible with $\partial_n p \in B^{2\gamma-1-\frac{1}{r}}_{r,\infty}$ when $\gamma \leq 1 + \frac{1}{2r}$, which is the case. So, for example, in the case $r = 3$, which is the relevant case in the K41 theory for fully developed turbulence (see [12] for an extensive description), one should be able to obtain (6.1) for all $\gamma > \frac{1}{6}$, thus including $\gamma = \frac{1}{3}$. This is in fact the exponent which plays a pivotal role in turbulence theory relating to the Kolmogorov $\frac{4}{5}$-law: an exact result relating the energy dissipation of a solution to its third order (signed) structure function.

REFERENCES

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften (343), Springer Berlin, Heidelberg, 2011.

[2] C. Bardos and E. S. Titi, *Onsager's conjecture for the incompressible Euler equations in bounded domains*, Arch. Ration. Mech. Anal. **228** (2018), no. 1, 197–207.

[3] ———, *$C^{0,\alpha}$ boundary regularity for the pressure in weak solutions of the 2d Euler equations*, Phil. Trans. R. Soc. A. **380** (2022).

[4] C. Bardos, E. S. Titi, and E. Wiedemann, *Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit*, Comm. Math. Phys. **370** (2019), no. 1, 291–310.

[5] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 2, 209–246 (French).

[6] M. Colombo and L. De Rosa, *Regularity in time of Hölder solutions of Euler and hypodissipative Navier-Stokes equations*, SIAM J. Math. Anal. **52** (2020), no. 1, 221–238.

[7] M. Colombo, L. De Rosa, and L. Forcella, *Regularity results for rough solutions of the incompressible Euler equations via interpolation methods*, Nonlinearity **33** (2020), no. 9, 4818–4836.

[8] P. Constantin, *Local formulas for hydrodynamic pressure and their applications*, Uspekhi Mat. Nauk **69** (2014), no. 3(417), 3–26 (Russian, with Russian summary); English transl., Russian Math. Surveys **69** (2014), no. 3, 395–418.

[9] L. De Rosa and P. Isett, *Intermittency and lower dimensional dissipation in incompressible fluids: quantifying Landau* (2022). Preprint available at arXiv:2212.08176.

[10] L. De Rosa, M. Latocca, and G. Stefani, *On Double Hölder Regularity of the Hydrodynamic Pressure in Bounded Domains*, Calc. Var. Partial Differ. Equ. (2022), to appear. Preprint available at arXiv:2205.00929.

[11] X. Fernandez-Real and X. Ros-Oton, *Regularity Theory for Elliptic PDE*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, 2023.
[12] U. Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995.
[13] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[14] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75.
[15] P. Isett, *Regularity in time along the coarse scale flow for the incompressible Euler equations*, Trans. Amer. Math. Soc. (2023), to appear. Preprint available at arXiv:1307.0565.
[16] T. Kato, M. Mitrea, G. Ponce, and M. Taylor, *Extension and representation of divergence-free vector fields on bounded domains*, Math. Res. Lett. **7** (2000), no. 5-6, 643–650.
[17] A. Kolmogoroff, *The local structure of turbulence in incompressible viscous fluid for very large Reynold's numbers*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30** (1941), 301–305.
[18] J. C. Robinson, J. L. Rodrigo, and J. W. D. Skipper, *Energy conservation for the Euler equations on* $\mathbb{T}^2\times\mathbb{R}_+$ *for weak solutions defined without reference to the pressure*, Asymptot. Anal. **110** (2018), no. 3-4, 185–202.
[19] J. C. Robinson, J. L. Rodrigo, and J. W. D. Skipper, *Energy conservation in the 3D Euler equation on* $\mathbb{T}^2\times\mathbb{R}_+$, Partial differential equations in fluid mechanics, London Math. Soc. Lecture Note Ser., vol. 452, Cambridge Univ. Press, Cambridge, 2018, pp. 224–251.
[20] L. Silvestre, *A non obvious estimate for the pressure* (2011), unpublished note, available at `http://math.uchicago.edu/~luis/preprints/pressureestimate.pdf`.
[21] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS Regional Conference Series in Mathematics (106), American Mathematical Society, 2006.
[22] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Progress in Mathematics (100), Birkhäuser Boston, MA, 1991.
[23] ———, *Partial Differential Equations II: Qualitative Studies of Linear Equations*, Applied Mathematical Sciences, Springer New York, 2010.
[24] ———, *Partial differential equations III. Nonlinear equations*, 2nd ed., Applied Mathematical Sciences, vol. 117, Springer, New York, 2011.
[25] S. Vita, *Boundary regularity estimates in Hölder spaces with variable exponent*, Calc. Var. Partial Differential Equations **61** (2022), no. 5, Paper No. 166, 31.
[26] M. Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics Volume: 138, American Mathematical Society, 2012.

(L. De Rosa) Department Mathematik und Informatik, Universität Basel, Spiegelgasse 1, 4051 Basel, Switzerland
Email address: `luigi.derosa@unibas.ch`

(M. Latocca) Department of Mathematics, University of Maryland, College Park, MD 20742, USA
Email address: `mickael.latocca@umd.edu`

(G. Stefani) Scuola Internazionale Superiore di Studi Avanzati (SISSA), via Bonomea 265, 34136 Trieste (TS), Italy
Email address: `gstefani@sissa.it` *or* `giorgio.stefani.math@gmail.com`