A NEW PROOF OF THE RIEMANNIAN PENROSE INEQUALITY

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Abstract. We give an overview of our recent new proof of the Riemannian Penrose inequality in the case of a single black hole. The proof is based on a new monotonicity formula, holding along the level sets of the p–capacitary potential of the connected boundary of an asymptotically flat 3–manifold, with nonnegative scalar curvature.

1. The Riemannian Penrose Inequality

The *Penrose conjecture* is a longstanding open problem in mathematical relativity. In the case of an isolated gravitational system containing a single black hole and obeying the *dominant energy condition* (see [19, Chapter 7]), it says that the "mass" of an asymptotically flat initial datum for the *Einstein field equations* is at least as large as the one of a reference space–like Schwarzschild manifold, whose horizon boundary has the same area as the horizon of the given initial datum. Here, the relevant concept of mass is the *ADM mass*, a global invariant introduced by Arnowitt, Deser and Misner in the late fifties (see [3]), which is supposed to measure the total amount of mass contained in an asymptotically flat initial datum. We recall that in the PDE's formulation of general relativity, the Einstein field equations are interpreted as a hyperbolic system whose solutions are four–dimensional space–time Lorentzian manifolds. Then, an asymptotically flat initial datum is a triple (M, g, h), where (M, g) is a three–dimensional asymptotically flat Riemannian manifold and h is a symmetric $(2, 0)$–tensor field, representing the second fundamental form of M inside his (globally) hyperbolic space–time development. In this framework, the Penrose conjecture affirms that

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{|\partial M|}{16\pi}}\,, \tag{1.1}$$

where $|\partial M|$ denotes the area of the horizon boundary of M.

Penrose viewed it as a test result for the validity of his *final–state conjecture* (see [18, Section 5] and [8, Section 1]) which, roughly speaking, says that the space–time evolution of a generic asymptotically flat initial datum eventually gets closer and closer to a stationary Kerr solution (see [25]) in the long run. If such a far reaching conjecture would be correct, then the *Penrose inequality* (1.1) would also be true. Indeed, by *Hawking's area theorem*, the area of the horizon is nondecreasing along the evolution of the initial datum, whereas the *Trautman–Bondi mass* (see [11]) – another concept of mass, which coincides with the ADM mass on the initial datum – is nonincreasing. On the other hand, the inequality

$$m_{\mathrm{TB}} \geq \sqrt{\frac{\text{Horizon Area}}{16\pi}} \tag{1.2}$$

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can be seen to hold on any Kerr solution, hence at the asymptotic final–stage of the evolution, provided the final–state conjecture is correct. Combining the latter fact with the opposite monotonicities of mass and area, it turns out that the inequality (1.2) must hold *a fortiori* on the initial datum, where it coincides with the Penrose inequality (1.1). This shows that the Penrose inequality is a necessary condition for the validity of the final–state conjecture.

It is worth mentioning that the Penrose conjecture is still an open problem in its full generality. Remarkably, its validity has has been established within the class of *time–symmetric* initial data, i.e. for initial data whose second fundamental form is vanishing inside their globally hyperbolic development. Since time–symmetric initial data are often called *Riemannian* initial data in the physics literature, inequality (1.1) is often referred to as the *Riemannian Penrose inequality* (RPI), in this context.

In dimension three, proofs of the RPI were obtained by G. Huisken and T. Ilmanen in 1997, for the case of a single black hole [17] and by H. Bray in 1999, for the general case of multiple black holes [7]. The proof of Huisken and Ilmanen is based on the monotonicity of a certain quantity, called *Hawking mass*, along a family of surfaces moving by (a weak form of) the *inverse mean curvature flow*. Bray's argument uses instead a flow of metrics, called *conformal flow*, that deforms the original metric to a Schwarzschild one – representing a spherically symmetric black hole in the vacuum – in a way that the total mass of the space does not increase and the total area of the horizons does not decrease. Therefore, these two approaches use two different geometric flows, while our proof relies on a monotonicity formula holding along the level sets of appropriate p–capacitary potentials. Although our argument cannot apparently be generalized to the case of multiple black holes and does not allow us, so far, to characterize the equality case, its brevity and simplicity distinguish it from the previous proofs of the Riemannian Penrose inequality. Actually, we think that our proof is elementary enough to be understood by graduate students with a standard training in PDE's and Riemannian geometry.

We mention that several monotone quantities, involving appropriate harmonic functions [2, 20, 21, 22] or suitable p–harmonic functions [1, 9, 15], have been discovered thanks to the groundbreaking work of Stern [23]. They have been used to obtain comparison results (concerning the area of the level sets of the considered harmonic or p–harmonic functions, the bottom spectrum, the ADM mass and the capacity). All these monotone quantities are compared in [15], except those in [22], which are the only ones modeled on the Schwarzschild manifolds.

In order to state precisely the result, we recall some definitions.

Definition 1.1. A complete 3–dimensional Riemannian manifold (M, g), with or without boundary and with one single end, is said to be *asymptotically flat with decay rate* τ, if the following conditions are satisfied:

(1) there exists a compact set $K \subseteq M$ such that the end $E = M \setminus K$ is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$ centered at the origin, through a so called *asymptotically flat coordinate chart* $(E, (x^1, x^2, x^3))$;
(2) in such a chart, the metric tensor can be expressed as
$$g \,=\, g_{ij}\, dx^i \otimes dx^j \,=\, (\delta_{ij} + \gamma_{ij})\, dx^i \otimes dx^j \,,$$
with
$$\sum_{i,j} |x|^\tau \, |\gamma_{ij}| + \sum_{i,j,k} |x|^{1+\tau} \, |\partial_k \gamma_{ij}| + \sum_{i,j,k,\ell} |x|^{2+\tau} \, |\partial_\ell \partial_k \gamma_{ij}| \,=\, O(1)\,,$$

as $|x| \to +\infty$ (here, δ is the Kronecker delta function).

According to the physicists Arnowitt, Deser and Misner who first introduced it in [3], the *ADM mass* of an asymptotically flat Riemannian 3–manifold is defined as follows

$$m_{\mathrm{ADM}} \,=\, \lim_{r \to +\infty} \frac{1}{16\pi} \int\limits_{\{|x|=r\}} (\partial_j g_{ij} - \partial_i g_{jj})\, \frac{x^i}{|x|}\, d\overline{\sigma}\,,$$

where $d\overline{\sigma}$ is the area element on the sphere $\{|x| = r\}$ with respect to the Euclidean metric. It can then be shown that if the scalar curvature of (M, g) is nonnegative and the decay rate τ is strictly larger than $1/2$, the ADM mass is a well defined geometric invariant, i.e., the above limit exists (possibly equal to $+\infty$) and its value does not depend on the particular asymptotically flat coordinate chart in which the limit is computed (see [4] and [10]).

We can then state the Riemannian Penrose inequality for single black hole initial data.

Theorem 1.2 (Riemannian Penrose inequality for single black hole initial data)**.** *Let (M, g) be a 3–dimensional, complete, connected, noncompact Riemannian manifold with a smooth, compact, connected boundary and one single end. Assume that:*

(1) the metric g has nonnegative scalar curvature $\mathrm{R} \geq 0$*;*
(2) (M, g) is asymptotically flat with decay rate $\tau = 1$;
(3) ∂M is the unique closed minimal surface in (M, g).

Then, the ADM mass satisfies

$$m_{\mathrm{ADM}} \,\geq\, \sqrt{\frac{|\partial M|}{16\pi}}\,, \tag{1.3}$$

where $|\partial M|$ denotes the area of ∂M.

We are going to present here the proof with some technical simplifications, skipping details that are fully addressed in [1]. We also mention, for completeness, that another proof of Riemannian Penrose inequality under the optimal decay assumptions was been showed in [6], through the use of the *isoperimetric mass* introduced by Huisken in [16].

2. A proof via nonlinear potential theory

The main idea of the proof is to consider, for every $1 < p < 3$, the elliptic problem

$$\begin{cases} \Delta_p u = 0 & \text{in } M \\ \quad u = 0 & \text{on } \partial M \\ \quad u \to 1 & \text{at } \infty \end{cases} \tag{2.1}$$

where $\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2} \nabla u\big)$ is the p–Laplacian operator of (M, g) (in the whole paper, ∇ denotes the Levi–Civita connection of (M, g)). This problem admits a unique (weak) solution $u_p \in W^{1,p}_{\mathrm{loc}}(M)$ in $\mathscr{C}^{1,\beta}$ on any bounded open set, for some $\beta > 0$ (hence, a *p–harmonic function*). It takes values in $[0, 1)$ and it is a proper function from M to $[0, 1)$. Furthermore, u_p is smooth at the points where the gradient does not vanish and attains smoothly the datum on the boundary, as zero is a regular value, by the Hopf lemma (see [5, Section 2]).

We then recall that, for any $1 \leq p < 3$, the p–capacity of ∂M is defined as

$$\mathrm{Cap}_p(\partial M) \,=\, \inf \left\{ \int\limits_M |\nabla v|^p \, d\mu \,:\, v \in \mathscr{C}^\infty_c(M),\, v = 1 \text{ on } \partial M \right\}$$

and, when $1 < p < 3$, it is related to u_p through the following identities

$$\mathrm{Cap}_p(\partial M) \;=\; \int\limits_M |\nabla u_p|^p \, d\mu \;=\; \int\limits_{\{u_p = t\}} |\nabla u_p|^{p-1} \, d\sigma \,, \tag{2.2}$$

for every regular value t of u_p, see [5, Section 2]. We set

$$c_p \;=\; \left(\frac{\mathrm{Cap}_p(\partial M)}{4\pi}\right)^{\frac{1}{p-1}} , \tag{2.3}$$

moreover, whenever there is no possibility of misunderstanding, we will drop the subscript p and we will simply denote with u the solution of problem (2.1).

With these notations, we introduce the vector field X as

$$X = \frac{c_p^{\frac{p-1}{3-p}}}{\left[\frac{3-p}{p-1}\,(1-u)\right]^{\frac{p-1}{3-p}}} \left\{ \frac{|\nabla u|^{p-2}\nabla u}{c_p^{p-1}} + \frac{\nabla|\nabla u| - \frac{\Delta u}{|\nabla u|}\nabla u}{\frac{3-p}{p-1}\,(1-u)} + \frac{|\nabla u|\nabla u}{\left[\frac{3-p}{p-1}\,(1-u)\right]^2} \right\} ,$$

which is well defined and smooth away from the critical points of u. Then, we consider the function

$$F_p(t) \;=\; \int\limits_{\{u=\alpha_p(t)\}} \left\langle X, \frac{\nabla u}{|\nabla u|} \right\rangle d\sigma \;=\; 4\pi t \;-\; \frac{t^{\frac{2}{p-1}}}{c_p} \int\limits_{\{u=\alpha_p(t)\}} |\nabla u| \, \mathrm{H} \, d\sigma \;+\; \frac{t^{\frac{5-p}{p-1}}}{c_p^2} \int\limits_{\{u=\alpha_p(t)\}} |\nabla u|^2 \, d\sigma \,, \tag{2.4}$$

where

$$\alpha_p(t) = 1 - \left(\frac{t_p}{t}\right)^{\frac{3-p}{p-1}} , \qquad \text{with} \qquad t_p = \left(\frac{p-1}{3-p}\, c_p\right)^{\frac{p-1}{3-p}} , \tag{2.5}$$

and the variable t ranges in $[\,t_p, +\infty)$. The function F_p is then well defined whenever $\alpha_p(t)$ is a regular value of u and the second equality follows by a straightforward computation, taking into account the equalities (2.2), (2.3) and the expression

$$\mathrm{H} \;=\; -(p-1)\, \frac{\nabla\nabla u(\nabla u, \nabla u)}{|\nabla u|^3}$$

for the mean curvature H of a regular level set of u (with respect to the unit normal $\nabla u/|\nabla u|$) which is obtained by making explicit the equation $\Delta_p u = 0$.

The first step of the proof is to show that the function F_p is monotone nondecreasing. We are going to prove this under the favorable hypothesis that the function u has no critical points, indeed, this fact has two relevant consequences: first, all the level sets of u are connected closed surfaces, being all diffeomorphic to $\{u = 0\} = \partial M$ which is a connected closed surface and second, all the previous quantities are well defined and smooth everywhere. In the general case, without such assumption, one has to deal with the possible lack of these two properties, making the proof of the monotonicity of F_p technically much more complicated (we refer the reader to [1]).

Step 1. Monotonicity of F_p. With the help of the Bochner formula and the twice contracted Gauss equation, the divergence of X can be expressed as

$$\operatorname{div} X = \frac{c_p^{\frac{p-1}{3-p}} |\nabla u|}{\left[\frac{3-p}{p-1}(1-u)\right]^{\frac{p-1}{3-p}+1}} \left\{ \frac{|\nabla^\Sigma |\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{5-p}{p-1}\left(\frac{|\nabla u|}{\frac{3-p}{p-1}(1-u)} - \frac{\mathrm{H}}{2}\right)^2 \right.$$

$$\left. + \frac{|\nabla u|^{p-1}}{c_p^{p-1}} - \frac{\mathrm{R}^\Sigma}{2} \right\},$$

where $\mathrm{R}^\Sigma(q)$, ∇^Σ and $\mathring{\mathrm{h}}(q)$ represent the scalar curvature, the Levi–Civita connection and the trace–free second fundamental form of the level set $\Sigma = \{u = u(q)\}$ passing through the point $q \in M$. Then, since all the values in the range of u are regular, the monotonicity of F_p can be deduced by means of the divergence theorem and the coarea formula,

$$\begin{aligned} F_p(t) - F_p(s) &= \int_{\{u=\alpha_p(t)\}} \left\langle X, \frac{\nabla u}{|\nabla u|} \right\rangle d\sigma - \int_{\{u=\alpha_p(s)\}} \left\langle X, \frac{\nabla u}{|\nabla u|} \right\rangle d\sigma \\ &= \int_{\{\alpha_p(s)<u<\alpha_p(t)\}} \operatorname{div} X \, d\mu = \int_{(\alpha_p(s),\alpha_p(t))} d\tau \int_{\{u=\tau\}} \frac{\operatorname{div} X}{|\nabla u|} \, d\sigma \\ &= \int_s^t d\tau \int_{\{u=\alpha_p(\tau)\}} \left[\frac{|\nabla^\Sigma |\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{5-p}{p-1}\left(\frac{|\nabla u|}{\frac{3-p}{p-1}(1-u)} - \frac{\mathrm{H}}{2}\right)^2 \right] d\sigma \\ &\quad + \int_s^t d\tau \int_{\{u=\alpha_p(\tau)\}} \left(\frac{|\nabla u|^{p-1}}{c_p^{p-1}} - \frac{\mathrm{R}^\Sigma}{2} \right) d\sigma \, . \end{aligned}$$

Now, recalling that $\mathrm{R}^\Sigma/2$ is equal to the Gaussian curvature G of the surface Σ and formulas (2.2), (2.3), the last integral is equal to

$$\int_s^t \left(4\pi - \int_{\{u=\alpha_p(\tau)\}} \mathrm{G} \, d\sigma \right) d\tau = \int_s^t \left[4\pi - 2\pi\chi\big(\{u = \alpha_p(\tau)\}\big) \right] d\tau \geq 0 \, ,$$

where the equality is given by the Gauss–Bonnet theorem. Here, $\chi\big(\{u = \alpha_p(\tau)\}\big)$ is the Euler characteristic of the level set $\{u = \alpha_p(\tau)\}$. When $|\nabla u| > 0$, all the level sets are diffeomorphic to each other, in particular they are all diffeomorphic to the boundary ∂M, which is a *connected* closed surface. Hence $4\pi - 2\pi\chi(\{u = \alpha_p(\tau)\}) \geq 0$, for every τ, and the conclusion then follows as $\mathrm{R} \geq 0$, by assumption. □

Remark 2.1. As we said the monotonicity part is formal, rigorous only if the critical points of the function u are not present, being the vector field X well defined and smooth only away from these points, hence we cannot in general apply the divergence theorem. Nevertheless, through a suitable sequence of cut–off functions, we can always limit from below the difference $F_p(t) - F_p(s)$ with the integral on the open set $\{\alpha_p(s) < u < \alpha_p(t)\}$ of the extension of $\operatorname{div} X$ which is zero on such critical points and subsequently apply the coarea formula as above. Actually, the main difference between the case without critical points and the general one is that there is no control on the set of the critical values of u, which a priori could even have positive Lebesgue measure. Indeed, being the p–harmonic functions in general only of class $\mathscr{C}^{1,\beta}$, Sard theorem cannot be applied. This clearly could affect the regularity of "too many" surfaces–level sets of the function u, not allowing the application of the Gauss–Bonnet theorem as we did above. In order to overcome these difficulties, the idea is to "locally" approximate the p–harmonic function u with the solutions $u^{\varepsilon,T}$ of the following perturbed problem (inspired by the works of Di Benedetto [12, 13] and similarly done in [9])

$$\begin{cases} \operatorname{div}\left(|\nabla v|_\varepsilon^{p-2}\nabla v\right) = 0 & \text{in } \{0 \leq u \leq T\} \\ v = 0 & \text{on } \partial M \\ v = T & \text{on } \{u = T\} \end{cases}$$

where $|\nabla v|_\varepsilon = \sqrt{|\nabla v|^2 + \varepsilon^2}$ and T is a fairly large regular value of u. The functions $u^{\varepsilon,T}$ are smooth (so Sard theorem can be applied) and $\mathscr{C}^k_{\mathrm{loc}}$–converge to the function u outside $\{|\nabla u| = 0\}$, for every $k \in \mathbb{N}$, as $\varepsilon \to 0$, then, with the same line as before we can consider analogous functions F_p^ε, pointwise converging to F_p, as $\varepsilon \to 0$, which are "almost" nondecreasing, up to an "error term" going to zero as $\varepsilon \to 0$. Hence, sending $\varepsilon \to 0$, we obtain the monotonicity of the original function F_p. For full details, we refer to [1, Section 1].

The monotonicity of F_p clearly leads to the inequality

$$F_p(t_p) \ \leq \ \lim_{t \to +\infty} F_p(t)\,, \tag{2.6}$$

where the limit of F_p at infinity is well defined, by the "good" asymptotic behavior at infinity of the function u, described by the expansion

$$u \ = \ 1 - \frac{p-1}{3-p}\,\frac{c_p}{|x|^{\frac{3-p}{p-1}}} + o_2\big(|x|^{-\frac{3-p}{p-1}}\big) \tag{2.7}$$

(see [5, Theorem 3.1]).

Step 2. The limit of F_p at infinity is bounded above by $8\pi m_{\mathrm{ADM}}$. By formula (2.4) we have

$$\lim_{t \to +\infty} F_p(t) \ = \ \lim_{t \to +\infty} \frac{4\pi \ - \ \frac{t^{\frac{3-p}{p-1}}}{c_p} \displaystyle\int_{\{u=\alpha_p(t)\}} |\nabla u|\,\mathrm{H}\,d\sigma \ + \ \frac{t^{\frac{2(3-p)}{p-1}}}{c_p^2} \displaystyle\int_{\{u=\alpha_p(t)\}} |\nabla u|^2\,d\sigma}{1/t}\,,$$

where the right hand limit is an indeterminate form $0/0$, by the expansion (2.7). Hence, applying the generalized version of de l'Hôpital's theorem in [24, Theorem II], we obtain the estimate

$$
\begin{aligned}
\lim_{t\to+\infty} F_p(t) &\leq \limsup_{t\to+\infty} \frac{\dfrac{d}{dt}\left[4\pi - \dfrac{t^{\frac{3-p}{p-1}}}{c_p}\displaystyle\int_{\{u=\alpha_p(t)\}} |\nabla u|\,\mathrm{H}\,d\sigma + \dfrac{t^{\frac{2(3-p)}{p-1}}}{c_p^2}\displaystyle\int_{\{u=\alpha_p(t)\}} |\nabla u|^2\,d\sigma\right]}{-1/t^2} \\
&= \limsup_{t\to+\infty} \Bigg\{ - t \int_{\{u=\alpha_p(t)\}} \left[\frac{|\nabla^\Sigma |\nabla u||^2}{|\nabla u|^2} + \frac{\mathrm{R}}{2} + \frac{|\mathring{\mathrm{h}}|^2}{2} + \frac{3-p}{2(p-1)}\left(\frac{2|\nabla u|}{\frac{3-p}{p-1}(1-u)} - \mathrm{H}\right)^2\right] d\sigma \\
&\qquad\qquad + t \int_{\{u=\alpha_p(t)\}} \frac{\mathrm{R}^\Sigma}{2}\,d\sigma - t\int_{\{u=\alpha_p(t)\}} \frac{\mathrm{H}^2}{4}\,d\sigma \Bigg\} \\
&\leq \limsup_{t\to+\infty} \; t \int_{\{u=\alpha_p(t)\}} \frac{\mathrm{R}^\Sigma}{2}\,d\sigma - t\int_{\{u=\alpha_p(t)\}} \frac{\mathrm{H}^2}{4}\,d\sigma \\
&= \limsup_{t\to+\infty} \; \frac{t}{4}\left(16\pi - \int_{\{u=\alpha_p(t)\}} \mathrm{H}^2\,d\sigma\right),
\end{aligned}
$$

where the last identity follows by the Gauss–Bonnet theorem, as the level sets are all diffeomorphic to a 2–dimensional sphere, for t large enough. Then, the way to deal with the term

$$
M_p(t) = \frac{t}{4}\left(16\pi - \int_{\{u=\alpha_p(t)\}} \mathrm{H}^2\,d\sigma\right),
$$

is to compare it with the analogous quantity computed with respect to the Euclidean background metric. By means of the knowledge of the behavior of u at infinity, formula (2.7), one gets

$$
M_p(t) = \frac{t}{4}\left(16\pi - \int_{\Sigma_t} \overline{\mathrm{H}}^2\,d\overline{\sigma}\right) + \frac{1}{2}\int_{\Sigma_t} \overline{\mathrm{div}}_{\Sigma_t} Y^\top\,d\overline{\sigma} + \frac{1}{2}\int_{\Sigma_t} (\partial_j g_{ij} - \partial_i g_{jj})\overline{\nu}^i\,d\overline{\sigma} + o(1)\,,
$$

where the bar denotes the quantities in the Euclidean metric, Y is the vector field defined by $(g_{ij}-\delta_{ij})\overline{\nu}^i\partial_j$ and $Y^\top$ denotes its tangential component, namely $Y^\top = Y - \overline{g}(Y,\overline{\nu})\overline{\nu}$.
The first term in the right hand side of the equality is nonpositive by the *Euclidean Willmore inequality* (see [26]), the second vanishes by the divergence theorem and it is well known that the third one tends to $8\pi m_{\mathrm{ADM}}$, as $t\to+\infty$ (see [4, Proposition 4.1]). Thus,

$$
\lim_{t\to+\infty} F_p(t) \leq \limsup_{t\to+\infty} M_p(t) \leq 8\pi m_{\mathrm{ADM}}\,.
$$

$\square$

Step 3. Proof of the Riemannian Penrose inequality – Theorem 1.2. By inequality (2.6) and the above estimate, recalling the expression (2.4) of F_p, one gets

$$
8\pi m_{\mathrm{ADM}} \geq F_p(t_p) = 4\pi t_p - \frac{t_p^{\frac{2}{p-1}}}{c_p}\int_{\partial M} |\nabla u|\,\mathrm{H}\,d\sigma + \frac{t_p^{\frac{5-p}{p-1}}}{c_p^2}\int_{\partial M} |\nabla u|^2\,d\sigma \geq 4\pi t_p\,,
$$

as $\{u = \alpha_p(t_p)\} = \{u = 0\} = \partial M$ is a minimal surface, hence $\mathrm{H} = 0$. Making explicit c_p and t_p with formulas (2.3) and (2.5), we obtain the following inequality involving the p–capacity of ∂M,

$$m_{\mathrm{ADM}} \,\geq\, \frac{1}{2}\left(\frac{p-1}{3-p}\right)^{\frac{p-1}{3-p}}\left(\frac{\mathrm{Cap}_p(\partial M)}{4\pi}\right)^{\frac{1}{3-p}}.$$

Then, the Riemannian Penrose inequality (1.3) follows by sending $p \to 1^+$, as there holds

$$\lim_{p\to 1^+} \mathrm{Cap}_p(\partial M) \,=\, |\partial M|$$

by [14, Theorem 1.2]. □

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References

1. V. Agostiniani, C. Mantegazza, L. Mazzieri, and F. Oronzio, *Riemannian Penrose inequality via nonlinear potential theory*, ArXiv Preprint Server – arXiv:2205.11642v2, 2022.
2. V. Agostiniani, L. Mazzieri, and F. Oronzio, *A Green's function proof of the positive mass theorem*, ArXiv Preprint Server – arXiv:2108.08402, 2021.
3. R. Arnowitt, S. Deser, and C. W. Misner, *Dynamical structure and definition of energy in general relativity*, Phys. Rev. **116** (1959), 1322–1330.
4. R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), 661–693.
5. L. Benatti, M. Fogagnolo, and L. Mazzieri, *The asymptotic behaviour of p–capacitary potentials in asymptotically conical manifolds*, Math. Ann. (2022).
6. ———, *On the isoperimetric Riemannian Penrose inequality*, ArXiv Preprint Server – arXiv:2212.10215v3, 2023.
7. H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Diff. Geom. **59** (2001), 177–267.
8. H. L. Bray and P. T. Chruściel, *The Penrose inequality*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 39–70.
9. P.-Y. Chan, J. Chu, M.-C. Lee, and T.-Y. Tsang, *Monotonicity of the p–Green functions*, ArXiv Preprint Server –arXiv:2202.13832, 2022.
10. P. T. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological properties and global structure of space–time (Erice, 1985), NATO Adv. Sci. Inst. Ser. B Phys., vol. 138, Plenum Press, 1986, pp. 49–59.
11. P. T. Chruściel, J. Jezierski, and M. A. H MacCallum, *Uniqueness of the Trautman–Bondi mass*, Phys. Rev. D (3) **58** (1998), no. 8, 084001, 16.
12. E. Di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.
13. ———, *Interior and boundary regularity for a class of free boundary problems*, Free boundary problems: theory and applications, Vol. I, II (Montecatini, 1981), Res. Notes in Math., vol. 78, Pitman, Boston, Mass.–London, 1983, pp. 383–396.
14. M. Fogagnolo and L. Mazzieri, *Minimising hulls, p–capacity and isoperimetric inequality on complete Riemannian manifolds*, J. Funct. Anal. **283** (2022), Paper No. 109638.
15. S. Hirsch, P. Miao, and L.-F Tam, *Monotone quantities of p–harmonic functions and their applications*, ArXiv Preprint Server –arXiv:2211.06939v1, 2022.

16. G. Huisken, *An isoperimetric concept for the mass in general relativity*, Oberwolfach Reports 3.3 (2006), 1898–1899.
17. G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Diff. Geom. **59** (2001), 353–437.
18. K. Landsman, *Singularities, black holes, and cosmic censorship: a tribute to Roger Penrose*, Found. Phys. **51** (2021), no. 2, Paper No. 42, 38.
19. D. A. Lee, *Geometric relativity*, Graduate Studies in Mathematics, vol. 201, AMS, 2019.
20. P. Miao, *Mass, capacitary functions, and the mass–to–capacity ratio*, ArXiv Preprint Server – arXiv:2207.03467v4, 2022.
21. O. Munteanu and J. Wang, *Comparison theorems for* 3*D dimensional manifolds with scalar curvature bound*, Int. Math. Res. Notices (2023), no. 3, 2215–2242.
22. F. Oronzio, *ADM mass, area and capacity in asymptotically flat* 3*–manifolds with nonnegative scalar curvature*, ArXiv Preprint Server – arXiv:2208.06688, 2022.
23. D. Stern, *Scalar curvature and harmonic maps to* $\mathbb{S}^1$, J. Diff. Geom. **122** (2022), no. 2, 259–269.
24. A. E. Taylor, *L'Hospital's rule*, Amer. Math. Monthly **59** (1952), 20–24.
25. Wikipedia, *Kerr metric*, https://en.wikipedia.org/wiki/Kerr_metric, 2023.
26. T. Willmore, *Total curvature in Riemannian geometry*, Wiley, 1982.

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