Overdetermined boundary value problems for the ∞-Laplacian

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Abstract: We consider overdetermined boundary value problems for the ∞-Laplacian in a domain Ω of $\mathbb{R}^n$ and discuss what kind of implications on the geometry of Ω the existence of a solution may have. The classical ∞-Laplacian, the normalized or game-theoretic ∞-Laplacian and the limit of the p-Laplacian as $p \to \infty$ are considered and provide different answers, even if we restrict our domains to those that have only web-functions as solutions.

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1 Motivation

Suppose that $\Omega \subset \mathbb{R}^n$ is connected and bounded, with boundary at least of class C^1, and that $u \in C^1(\overline{\Omega})$ is a positive solution of the overdetermined boundary value problem

$$-\Delta_p u_p := -\operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = 1 \quad \text{in } \Omega, \tag{1.1}$$

$$u_p = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$-\frac{\partial u_p}{\partial \nu} = a \quad \text{on } \partial\Omega, \tag{1.3}$$

where $p \in (1, \infty)$ and a is a positive constant. Does this have consequences on the geometry of Ω? This question was answered in 1971 for $p = 2$ by Serrin [17] and Weinberger [18], and for general p in 1987 by Garofalo and Lewis [6]. See also Farina and Kawohl [5] for related results. In both cases the domain Ω must be a ball of fixed radius related to a. This result leads us to the question: what happens if the p-Laplacian is replaced by the infinity Laplacian?

The answer depends on how we define the ∞-Laplacian and the notion of solution. In case of equation (1.1) and $p = 2$ Serrin and Weinberger had classical $C^2(\Omega)$ solutions in mind, while for general $p \in (1, \infty)$ the solutions were weak in the sense that

$$\int_\Omega |\nabla u_p|^{p-2} \nabla u_p \nabla v \; dx = \int_\Omega v \; dx \qquad \text{for every } v \in W_0^{1,p}(\Omega).$$

2 The classical ∞-Laplacian

The classical ∞-Laplacian operator is usually defined as $\Delta_\infty u := \langle D^2 u Du, Du \rangle$, with Du denoting the gradient and $D^2 u$ the Hessian matrix of u. For functions in C^2 the second directional derivative in direction ν is given by $\langle D^2 u \; \nu, \nu \rangle$. If ν denotes the direction $-Du/|Du|$ of steepest descent of u, the equation $-\Delta_\infty u = 1$ can be rewritten as

$$-u_{\nu\nu} |u_\nu|^2 = 1, \tag{2.1}$$

and if Ω should happen to be a ball of radius R centered at zero, $u(x)$ is necessarily a radial function. In fact, then

$$u(r) = \frac{3^{4/3}}{4} (R^{4/3} - r^{4/3}) \qquad \text{and } u_r(R) = -(3R)^{1/3}$$

imply that R must be equal to $a^3/3$ to match both boundary conditions. Notice that this function is exactly of class $C^{1,1/4}$, which is the conjectured optimal regularity for ∞-harmonic functions v, that is for functions satisfying $\Delta_\infty v = 0$.

Therefore we cannot expect classical solutions. Since the equation is not in divergence form, we cannot expect a notion of weak solution either. Instead we define a *viscosity solution* u of the equation

$$F(Du, D^2 u) := -\langle D^2 u Du, Du \rangle - 1 = 0$$

as a continuous function which is both a viscosity sub- and viscosity supersolution. A viscosity subsolution has the property that $F(D\varphi, D^2\varphi)(x) \le 0$ whenever φ is a C^2-function such that $\varphi - u$ has a local minimum at x. A viscosity supersolution has the property that $F(D\psi, D^2\psi)(x) \ge 0$ whenever ψ is a C^2-function such that $\psi - u$ has a local maximum at x, see for instance [2]. In our autonomous case we may also assume that φ touches u from above at x if we check the definition of subsolutions, and that ψ touches from below at x if we check supersolutions.

Let us see that the explicit radial function $c - kr^{4/3}$, with $k = 3^{4/3}/4$ is a viscosity solution of $F(Du, D^2u) = 0$ at $x = 0$. If φ is a smooth function touching u from above, then $\nabla\varphi(0) = 0$, so $\varphi_\nu = 0$ and $F(D\varphi, D^2\varphi) = -1$, which is less or equal to zero, as required for subsolutions. For supersolutions the set of test functions ψ that touch u from below in the origin is empty, so that the condition for a supersolution is trivially satified. Effects like this happen quite often when viscosity solutions are not smooth. Checking the property of sub- or supersolution is somehow easier in points where the solutions loose smoothness.

Now suppose that Ω is not necessarily a ball, but a more general smooth domain.

Remark 2.1 From every point x_0 on $\partial\Omega$ we can follow the line of steepest ascent, parametrized as $x(t)$ by solving the initial value problem

$$x(0) = x_0, \qquad \frac{dx_i}{dt} = u_{x_i} \quad \text{for small but positive } t. \tag{2.2}$$

A simple calculation shows, assuming that u is locally of class C^2,

$$\frac{d}{dt}\left(\left|\frac{dx}{dt}\right|^2\right) = 2u_{x_i}u_{x_ix_j}u_{x_j} = -2, \tag{2.3}$$

so that upon integration from 0 to t

$$\left|\frac{dx}{dt}\right|^2 = |\nabla u(x(t)|^2 = a^2 - 2t. \tag{2.4}$$

Note that this works until t reaches $a^2/2$, at which time $\nabla u = 0$. Subsequently we get the estimate

$$|x(t) - x_0| = \left|\int_0^t x_t(s)\ ds\right| \leq \frac{1}{3}\left(a^3 - (a^2 - 2t)^{3/2}\right) \leq \frac{a^3}{3}. \tag{2.5}$$

This shows that our trajectories can never reach a distance greater than $a^3/3$ from the boundary of Ω and that any critical point of u that can be approached this way has at most distance $a^3/3$ from $\partial\Omega$.

Notice that the radial solution on a ball is a web-function in the sense of [3], i.e. a function, whose value depends only on the distance to $\partial\Omega$. From now on we assume that a solution of (2.1) (1.2) (1.3) happens to be a webfunction for a general domain as well. This may be justified via the Cauchy-Kowalewski Theorem or by using the remark above, but we could not give a precise proof. Under this assumption we can interpret equation (2.1) as an ordinary differential equation for a function $u(d)$ that depends only on the distance $d = d(x, \partial\Omega)$ to the boundary, with initial condition (1.2) and (1.3) at $d = 0$. Then we arrive after the first integration at $u_\nu^3(d) - a^3 = -3d$ or $-u_\nu = (a^3 - 3d)^{1/3}$ and after a second integration at

$$u(d) = \int_0^d (a^3 - 3t)^{1/3}\ dt = \frac{1}{4}\left[a^4 - \left(a^3 - 3d\right)^{4/3}\right].$$

Clearly the integrations are only justifiable for sufficiently small d and as long as d is locally of class $C^{1,1}$. When $d = a^3/3$, the gradient of u vanishes and we have reached the peak on our way uphill from the boundary. This shows that Ω has an inradius of exactly $a^3/3$. Incidentally, the points in

$$M(\Omega) := \{y \in \Omega \mid d(y, \partial\Omega) = \max_{x\in\Omega} d(x, \partial\Omega)\}$$

belong to the *ridge* of Ω or *cut locus* of $\partial\Omega$, which is defined as follows. Let G be the largest open subset of Ω such that every point x in G has a unique closest point on $\partial\Omega$. Then we call

$$\mathcal{R}(\Omega) := \Omega \setminus G$$

the ridge $\mathcal{R}(\Omega)$. In G, the distance $d(x, \partial\Omega)$ to the boundary is at least of class C^1, and also smooth, i.e., of class C^2 or $C^{k,\alpha}$ with $k \geq 2$ and $\alpha \in (0, 1)$ provided $\partial\Omega$ is of the same class, see [4, 11]. It is remarkable that even for a convex plane domain the ridge can have positive measure, see pages 10 and 11 in [14]. Simple examples such as an ellipse or a rectangle show that in general $M(\Omega)$ is a genuine subset of the ridge, but there are many domains with the property $M(\Omega) = \mathcal{R}(\Omega)$.

Examples of such domains are for instance a stadium domain (convex hull of two balls of same radius and different center), an annulus, or plane domains which are generated as follows. Let γ be a compact $C^{1,1}$ curve with curvature not exceeding K in modulus and $\Omega = U_b(\gamma) = \{x \in \mathbb{R}^2 \mid d(x, \gamma) < b\}$ with $b < 1/K$. Then $M(\Omega) = \mathcal{R}(\Omega)$, see Figure 1.

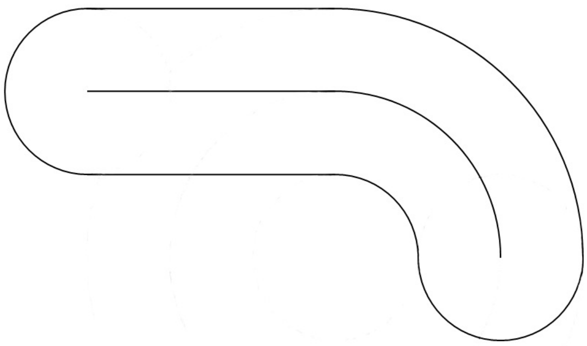

Figure 1: A domain satisfying $M(\Omega) = \mathcal{R}(\Omega)$

Theorem 2.2 *Suppose that $\partial\Omega$ is of class C^2. Then a webfunction $u \in C^1(\overline{\Omega})$ is a viscosity solution of* (2.1) (1.2) (1.3) *if and only if $M(\Omega) = \mathcal{R}(\Omega)$ and every $x \in \partial\Omega$ has distance $a^3/3$ to $\mathcal{R}(\Omega)$.*

Proof. In fact, if $M(\Omega) = \mathcal{R}(\Omega)$, then the function

$$u(x) = \frac{1}{4}\left[a^4 - \left(a^3 - 3d(x, \partial\Omega)\right)^{4/3}\right]$$

is well defined and differentiable everywhere in Ω. Moreover, according to [4], it is of class $C^2(\Omega \setminus \mathcal{R}(\Omega))$ and solves (2.1) in $\Omega \setminus \mathcal{R}(\Omega)$ in the classical (and a fortiori in the viscosity) sense. Finally on $M(\Omega) = \mathcal{R}(\Omega)$ we can argue as in the radial case to see that u is a viscosity solution there as well. This shows that the geometric constraint $M(\Omega) = \mathcal{R}(\Omega)$ is sufficient for the existence of solutions to (2.1) (1.2) (1.3).

To prove necessity, suppose that $M(\Omega)$ is a genuine subset of $\mathcal{R}(\Omega)$, so that there exists a $z \in \mathcal{R}(\Omega) \setminus M(\Omega)$. But then $d(z, \partial\Omega) < a^3/3$ and $d(z, \partial\Omega)$ has a kink in the sense that some directional derivative of d, and subsequently of u, is discontinuous at z. This is incompatible with being a viscosity solution, because one can then find an admissible test function $\varphi \in C^2(\Omega)$ for which $F(D\varphi, D^2\varphi)$ fails to satisfy the proper inequality. To be precise, suppose that Ω is essentially a rectangle (with rounded corners to make it smooth) or an ellipse. Then z lies on a line segment and $d(x, \partial\Omega)$ is concave near z and has one-sided nonzero derivatives in direction η orthogonal to the ridge in z. But then one can choose a C^2 function φ, touching u from above in z such that $\nabla\varphi(z) \neq 0$ points in direction η and $\varphi_{\eta\eta}(z) < -K$, where K is an arbitrarily large number. Therefore $F(D\varphi, D^2\varphi)(z) > 0$, which contradicts the requirement for subsolutions. There is a similar reasoning using supersolutions, if Ω is essentially L-shaped and u is convex and nondifferentiable on parts of its ridge. □

3 The normalized or game-theoretic ∞-Laplacian

Recently the following operator has received considerable attention (see for instance [15, 16, 9, 12, 13, 20]) in the PDE community

$$\Delta_\infty^N u = \langle D^2 u Du, Du\rangle |Du|^{-2}.$$

Here $u(x)$ denotes the (unique) running costs in a differential game called "tug of war", see [20]. Let us therefore study the differential equation

$$-u_{\nu\nu} = 1 \qquad \text{in } \Omega \tag{3.1}$$

under boundary conditions (1.2) and (1.3). A simple integration shows that certainly for a ball of radius $R = a$ this overdetermined problem has the explicit solution $u(r) = (a^2 - r^2)/2$, provided we can live with the ambiguity that ν is not properly defined at the origin. Fortunately the notion of viscosity solution allows us to do so. A viscosity solution u of

$$G(Du, D^2u)(x) := -\frac{\langle D^2 u Du, Du\rangle}{|Du|^2}(x) - 1 = 0 \qquad \text{in } \Omega \tag{3.2}$$

is a viscosity subsolution of $G_*(Du, D^2u) = 0$ and a viscosity supersolution of $G^*(Du, D^2u) = 0$. Here G_* and G^* are the upper and lower semicontinuous envelopes of G, see Remark 6.3 in [2]. Thus $u \in C(\Omega)$ is a *viscosity subsolution* of (3.1) or (3.2), if for every $x \in \Omega$ and every smooth test function φ, that touches u from above (only) in x, the following relations hold:

$$\begin{cases} G(\nabla\varphi(x), D^2\varphi(x)) \le 0 & \text{when } \nabla\varphi(x) \neq 0, \\ \quad -\Lambda(D^2\varphi(x)) - 1 \le 0 & \text{when } \nabla\varphi(x) = 0. \end{cases} \tag{3.3}$$

In a similar fashion *viscosity supersolutions* $u \in C(\Omega)$ of (3.1) are characterized by the fact that

$$\begin{cases} G(\nabla\psi(x), D^2\psi(x)) \ge 0 & \text{when } \nabla\psi(x) \neq 0, \\ \quad -\lambda(D^2\psi(x)) - 1 \ge 0 & \text{when } \nabla\psi(x) = 0. \end{cases} \tag{3.4}$$

for every smooth test function ψ that touches u from below (only) in x. Here $\Lambda(X)$ and $\lambda(X)$ denote the maximal and minimal (nonnegative) eigenvalue of the symmetric matrix X.

For a more general Ω, if we interpret (3.1) again as an ODE and (1.2) and (1.3) as initial data on $\partial\Omega$, then an integration like in the previous section along lines of steepest ascent of u leads to the local representation

$$u(x) = \frac{d(x, \partial\Omega)}{2}\big(2a - d(x, \partial\Omega)\big) \quad \text{in } \Omega \setminus \mathcal{R}(\Omega).$$

Theorem 3.1 *Suppose that $\partial\Omega$ is of class C^2. Then a webfunction $u \in C^1(\overline{\Omega})$ is a viscosity solution of* (3.1) (1.2) (1.3) *if and only if $M(\Omega) = \mathcal{R}(\Omega)$ and every $x \in \partial\Omega$ has distance a to $\mathcal{R}(\Omega)$.*

The proof parallels the one of Theorem 2.2 and is left to the reader.

Remark 3.2 Notice that annuli provide examples of domains (other than balls) for which a smooth solution of this problem (but not of Serrin's and Weinberger's original problem) exists.

4 The limit of u_p

It is well-known, that p-harmonic functions or viscosity solutions of $\Delta_p u = 0$ converge to the viscosity solution of $\Delta_\infty u = 0$ as $p \to \infty$. Therefore one is inclined to believe that solutions u_p of the inhomogeneous equation (1.1) should converge to those of (2.1). This is not the case, and in the present section we investigate this limit. For Ω a ball in $\mathbb{R}^n$ the solutions of (1.1), (1.2) were explicitly calculated and shown to converge uniformly to $d(x, \partial\Omega)$ in [10]. Let us demonstrate that this behaviour happens for any connected domain, even for a nonsmooth one. First one has to note that u_p on Ω can be estimated in L^q for any $q \in [0,\infty]$ by the corresponding solution U_p on a ball Ω^* of same volume as Ω, so that the u_p are uniformly bounded in $L^\infty(\Omega)$ as $p \to \infty$. Furthermore u_p minimizes the functional

$$J_p(v) = \int_\Omega \left[\frac{1}{p}|\nabla v(x)|^p - v(x)\right] dx \qquad \text{on } W_0^{1,p}(\Omega).$$

In particular

$$J_p(u_p(x)) \leq J_p(d(x,\partial\Omega)) = \frac{1}{p}|\Omega| - \int_\Omega d(x,\partial\Omega)\,dx,$$

the right hand of which is negative for sufficiently large p. Thus

$$\int_\Omega |\nabla u_p|^p\,dx \leq p\int_\Omega u_p\,dx,$$

or for $p > q$ and q large enough

$$\int_\Omega |\nabla u_p|^q\,dx \leq \left(\int_\Omega |\nabla u_p|^p\,dx\right)^{q/p} |\Omega|^{1-q/p} \leq \left(p\int_\Omega u_p\,dx\right)^{q/p} |\Omega|^{1-q/p}.$$

But this implies $||\nabla u_p||_q \leq p^{1/p}||u_p||_\infty^{1/p}|\Omega|^{1/q}$, so that the family $\{u_p\}_{p\to\infty}$ is uniformly bounded in every $W^{1,q}(\Omega)$ and converges uniformly to some limit u_∞ with Lipschitz constant 1.

Therefore $|\nabla u_\infty| \leq 1$ a.e. in Ω, and this implies not only that $u_\infty(x) \leq d(x,\partial\Omega)$ in Ω, but it (almost) proves the first half of our following result.

Theorem 4.1 *The limit u_∞ is a viscosity solution of the eikonal equation $|Du(x)| - 1 = 0$ in Ω under the Dirichlet boundary condition $u = 0$ on $\partial\Omega$.*

Remark 4.2 Since this Hamilton-Jacobi equation has a unique viscosity solution, see e.g. [2], we obtain $u_\infty := d(x,\partial\Omega)$ as a Lipschitz solution for a highly overdetermined boundary value problem. It satisfies not only $|Du| - 1 = 0$ in Ω but also $-\Delta_\infty u_\infty = 0$ in $\Omega \setminus \mathcal{R}(\Omega)$, and not only $u = 0$ on $\partial\Omega$ but also $-\frac{\partial u}{\partial \nu} = 1$ on differentiable parts of $\partial\Omega$.

Remark 4.3 Notice that the statement $M(\Omega) = \mathcal{R}(\Omega)$ is conspicuously missing in Theorem 4.1. Under the additional assumption $M(\Omega) = \mathcal{R}(\Omega)$, however, the function u_∞ is moreover (up to multiplication by a constant) the unique eigenfunction for the ∞–Laplacian operator, i.e. it satisfies in addition

$$\min\left\{-\langle D^2u_\infty(x)Du_\infty(x), Du_\infty(x)\rangle,\ -|Du(x)| + \Lambda_\infty u(x)\right\} = 0 \qquad \text{in } \Omega$$

in the viscosity sense, see [7, 19]. Here Λ_∞ is the inverse of the inradius of Ω. Without this assumption, as demonstrated in [8] there is nonuniqueness of this eigenfunction.

Proof of Theorem 4.1. Let us first realize that $|Du_\infty| \le 1$ a.e. in Ω implies $|Du_\infty| - 1 \le 0$ in the viscosity sense. Otherwise there would be a function $\varphi \in C^2$ touching u from above in some x_0 such that $|Du(x_0)| \ge 1+\gamma$, with $\gamma > 0$, and $|Du(x) \ge 1 + \gamma/2$ in a neighbourhood $B_\varepsilon(x_0)$. But then $u(x_0) - u(x) \ge \varphi(x_0) - \varphi(x) \ge (1+\gamma/2)|x_0 - x|$ for a suitable $x \in B_\varepsilon(x_0)$. This contradicts the fact that u_∞ has Lipschitz constant 1.

To show the reverse inequality, it is instructive to follow ideas in [7, 1] and to identify the limiting equation. Suppose that φ is a C^2-function such that $\varphi - u_\infty$ has a local minimum at $x_0 \in \Omega$. Then without loss of generality we may assume that $\varphi - u_\infty \ge \delta > 0$ on $\partial B_\varepsilon(x_0) \subset \Omega$. Moreover, for p large enough, $\varphi - u_p$ has a local minimum at some $x_p \in B_\varepsilon(x_0)$ and $x_p \to x_0$ as $p \to \infty$. Since u_p is a viscosity subsolution of (1.1)

$$-|Du|^{p-2}\left(tr(D^2u) + (p-2)\frac{\langle D^2uDu, Du\rangle}{|Du|^2}\right) - 1 = 0 \quad \text{in } \Omega, \tag{4.1}$$

it follows

$$-|D\varphi(x_p)|^{p-2}\left(tr(D^2\varphi(x_p)) + (p-2)\frac{\langle D^2\varphi(x_p)D\varphi(x_p), D\varphi(x_p)\rangle}{|D\varphi(x_p)|^2}\right) \le 1.$$

Now either $|D\varphi(x_0)| \le 1$ or otherwise there exists a positive constant γ independent of p, such that $|D\varphi(x_p)| > 1+\gamma$ for large p. Upon division of the last inequality by $(p-2)|D\varphi(x_p)|^{p-4}$ one sees that in this case the first term on the left and the right hand side in

$$-\tfrac{1}{p-2}|D\varphi(x_p)|^2 trD^2\varphi(x_p) - \langle D^2\varphi(x_p)D\varphi(x_p), D\varphi(x_p)\rangle \le \tfrac{1}{p-2}|D\varphi(x_p)|^{4-p}$$

converge to zero as $p \to \infty$, so that $-\langle D^2\varphi(x_0)D\varphi(x_0), D\varphi(x_0)\rangle \le 0$. This proves that u_∞ is a viscosity subsolution of

$$\min\{\ |Du| - 1, -\langle D^2uDu, Du\rangle\ \} = 0 \quad \text{in } \Omega. \tag{4.2}$$

A similar reasoning holds for supersolutions. Since u_p is a viscosity supersolution of (4.1), we have

$$-|D\psi(x_p)|^{p-2}\left(tr(D^2\psi(x_p)) + (p-2)\frac{\langle D^2\psi(x_p)D\psi(x_p), D\psi(x_p)\rangle}{|D\psi(x_p)|^2}\right) \ge 1$$

for testfunctions $\psi \in C^2$ such that $u - \psi$ has a local maximum at x_0 and $u_p - \psi$ has a local maximum at x_p. This time we can rule out that $D\psi(x_p) = 0$, otherwise the last inequality cannot hold. Arguing as before, the inequality

$$-\tfrac{1}{p-2}|D\psi(x_p)|^2 trD^2\psi(x_p) - \langle D^2\psi(x_p)D\psi(x_p), D\psi(x_p)\rangle \ge \tfrac{1}{p-2}|D\psi(x_p)|^{4-p}$$

follows and leads to $|D\psi(x_0)| \ge 1$, because else the right hand side would explode for $p \to \infty$, as well as to $-\langle D^2\psi(x_0)D\psi(x_0), D\psi(x_0)\rangle \ge 0$. This shows that u_∞ is also a viscosity supersolution of (4.2). In particular u_∞ satisfies $|Du| \geq 1$ in the viscosity sense, and this completes the proof of Theorem 4.1. □

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