FRACTIONAL HARDY-RELLICH INEQUALITIES VIA A POHOZAEV IDENTITY

NICOLA DE NITTI AND SIDY MOCTAR DJITTE

Abstract. We prove a fractional Hardy-Rellich inequality with an explicit constant in bounded domains of class C^α with $\alpha > \max\{1, 2s\}$. The strategy of the proof generalizes an approach pioneered by E. Mitidieri (*Mat. Zametki*, 2000) by relying on a fractional Pohozaev identity.

1. Introduction

In [35], G. H. Hardy proved that, if $p > 1$ and f is a non-negative function in $L^p(0, \infty)$, then f is integrable over the interval $(0, x)$ for every $x > 0$ and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^p \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,\mathrm{d}x$$

holds; or, letting $u(x) = \int_0^x f(t)\,\mathrm{d}t$,

$$\int_0^\infty \frac{u(x)^p}{x^p}\,\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,\mathrm{d}x.$$

The constant $(p/(p-1))^p$ was proved to be sharp by Landau in [41]. On a bounded interval, e.g. $\Omega = (0, 1) \subset \mathbb{R}$, Hardy also proved that

$$\int_0^\infty \frac{u(x)^p}{\mathrm{d}^p_{(0,1)}(x)}\,\mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(x)|^p\,\mathrm{d}x,$$

where $\mathrm{d}_{(0,1)}(x) = \min\{x, 1-x\}$. From these beginnings, many Hardy-type inequalities have been proven and have become fundamental tools in several branches of analysis. We refer to the surveys [39, 4, 17, 54] for further information and historical context.

The classical N-dimensional generalization of the Hardy inequality states that, for $N > 1$, $1 \le p < \infty$, with $p \ne N$, and for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, it holds

$$\int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \leqslant \left(\frac{p}{|N-p|}\right)^p \int_{\mathbb{R}^N} |\nabla u(x)|^p\,\mathrm{d}x.$$

More precisely, u may belong to $W^{1,p}\left(\mathbb{R}^N\right)$ when $1 \le p < N$ and $W^{1,p}\left(\mathbb{R}^N \setminus \{0\}\right)$ when $N < p < \infty$. Here the constant $(p/|N-p|)^p$ is sharp and is not attained in these Sobolev spaces. If $p = 1$, equality holds for any symmetric decreasing function.

On a domain $\Omega \subset \mathbb{R}^N$ with nonempty boundary and $1 \le p < \infty$, it holds

$$c\int_\Omega \frac{|u(x)|^p}{\mathrm{d}^p_\Omega(x)}\,\mathrm{d}x \le \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x,$$

where $\mathrm{d}_\Omega(x) := \min\{|x-y| : y \notin \Omega\}$ is the distance of x from $\partial\Omega$. We refer to, e.g., [53, 43, 5, 44, 11, 19, 20, 33, 43, 45, 46, 50] for further results, including on the best value of the constant $c_{N,p}$ involved. The problem posed on nilpotent groups has also attracted much attention recently (see, e.g., [18, 2, 59, 3, 61]).

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A related inequality is due to Rellich (see [55, 56]):

$$\frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^4}\,\mathrm{d}x \le \int_{\mathbb{R}^N}|\Delta u(x)|^2\,\mathrm{d}x,$$

for $N\in\mathbb{N}\setminus\{2\}$ and $u\in C_0^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$; for $N=2$, the inequality holds (with constant equal to 1) for $u\in C_0^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$ satisfying (in polar coordinates)

$$\int_0^{+\infty} u(t,\theta)\cos\theta\,\mathrm{d}\theta = \int_0^{+\infty} u(t,\theta)\sin\theta\,\mathrm{d}\theta = 0$$

We point to [4, Chapter 6], [27, Chapter 7], and [20, 13, 26] for further information on Rellich-type inequalities. Among several generalizations, we point out that

$$c_{p,\theta}^p\int_\Omega\frac{|u(x)|^p}{|x|^{\theta+2}}\,\mathrm{d}x \le \int_\Omega\frac{|\Delta u(x)|^p}{|x|^{\theta+2-2p}}\,\mathrm{d}x, \tag{1.1}$$

holds for $u\in C_0^\infty(\Omega)$, $p>1$ and $N>\theta+2$, with $\theta\in\mathbb{R}$ (as proved, e.g., in [50]); the sharp constant is given by

$$c_{p,\theta} = \frac{(N-2-\theta)[(p-1)(N-2)+\theta]}{p^2}.$$

In the present contribution, we shall focus on establishing a counterpart of (1.1) where the Laplace operator is replaced by the *fractional Laplacian operator* (see [21, 1]): namely,

$$\int_\Omega\frac{|u|^p(x)}{|x|^{\theta+2s}}\,\mathrm{d}x \lesssim \int_\Omega\frac{|(-\Delta)^s u(x)|^p}{|x|^{\theta+2s-2sp}}\,\mathrm{d}x. \tag{1.2}$$

Several Hardy-type inequalities are already available in the fractional setting. A starting point, let us recall the following sharp one proved in [31, Theorem 1.1] (see also [47, Theorem 2] and [48]): Let $N\in\mathbb{N}$, with $N\ge 1$, and $0<s<1$. Then for all $u\in\dot{W}^{s,p}\left(\mathbb{R}^N\right)$ in case $1\le p<N/s$, and for all $u\in\dot{W}^{s,p}\left(\mathbb{R}^N\setminus\{0\}\right)$ in case $p>N/s$,

$$\mathcal{C}_{N,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}\,\mathrm{d}x \le \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.3}$$

with

$$\mathcal{C}_{N,s,p} := 2\int_0^1 r^{ps-1}\left|1-r^{(N-ps)/p}\right|^p\Phi_{N,sp}(r)\,\mathrm{d}r \tag{1.4}$$

and

$$\begin{aligned}
\Phi_{N,s,p}(r) &:= \operatorname{vol}(\mathbb{S}^{N-2})\int_{-1}^1\frac{\left(1-t^2\right)^{\frac{N-3}{2}}}{\left(1-2rt+r^2\right)^{\frac{N+ps}{2}}}\,\mathrm{d}t, && N\ge 2,\\
\Phi_{1,s,p}(r) &:= \left(\frac{1}{(1-r)^{1+ps}}+\frac{1}{(1+r)^{1+ps}}\right), && N=1.
\end{aligned}$$

The constant $\mathcal{C}_{N,s,p}$ is optimal[1]. If $p=1$, equality holds if and only if u is proportional to a symmetric decreasing function. If $p>1$, the inequality is strict for any function $0\not\equiv u\in\dot{W}^{s,p}\left(\mathbb{R}^N\right)$ or $\dot{W}^{s,p}\left(\mathbb{R}^N\setminus\{0\}\right)$, respectively. Here the homogeneous Sobolev spaces $\dot{W}_p^s\left(\mathbb{R}^N\right)$ and $\dot{W}_p^s\left(\mathbb{R}^N\setminus\{0\}\right)$ are defined as the completion of $C_0^\infty\left(\mathbb{R}^N\right)$ for $1\le p<N/s$ and $C_0^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$ for $p>N/s$, respectively, with respect to the Gagliardo seminorm which is defined in the left-hand side of (1.3).

[1]See also [31, Eq (3.5) and Eq. (3.6)] for equivalent expressions of $\mathcal{C}_{N,s,p}$. Moreover, from [31, Eq. (1.6)], if $p=2$, then the constant is given more explicitly by

$$\mathcal{C}_{N,s,2} = 2\pi^{N/2}\frac{\Gamma((N+2s)/4)^2}{\Gamma((N-2s)/4)^2}\frac{|\Gamma(-s)|}{\Gamma((N+2s)/2)}.$$

Previously in [37], Herbst proved the inequality

$$\tilde{\mathcal{C}}^p_{N,s,p}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x|^{ps}}\,\mathrm{d}x \le \|(-\Delta)^{s/2}u\|^p_{L^p(\mathbb{R}^N)} \tag{1.5}$$

for $1<p<\infty, s>0, N>ps$ and $u\in C_0^\infty\left(\mathbb{R}^N\right)$, with the optimal constant

$$\tilde{\mathcal{C}}_{N,s,p} = 2^{-s}\frac{\Gamma\left(\frac{N(p-1)}{2p}\right)\Gamma\left(\frac{N-ps}{2p}\right)}{\Gamma\left(\frac{N}{2p}\right)\Gamma\left(\frac{N(p-1)+ps}{2p}\right)}.$$

We refer also to [30, 9, 62, 6, 38, 42, 27] for related results. In particular, the problem of classifying and proving the nondegeneracy of minimizers for the fractional Hardy-Sobolev inequality was studied in [51]. We remark that, for $p=2$, the right-hand side of (1.5) is proportional to the one in (1.3) because of [21, Proposition 3.6] whereas, for $p\neq 2$ and $0<s<1$, it is not; on the other hand, as pointed out in [31], from [60, Chapter V], it follows that a one-sided inequality holds depending on whether $1<p<2$ or $p>2$.

Some versions of the fractional Hardy inequality have been obtained in [15, Theorem 1.4], [25, Theorem 1], [36, Theorem 6.1], and [28, Theorem 1.3]. The inequality on convex sets has been specifically studied in [10, 7]. Hardy-type inequalities for fractional relativistic operators have been proved in [57] and, for fractional powers of a discrete Laplacian, in [14]. We refer also to [17] for a short survey of some of these developments. The higher-order version of the Hardy inequality in $\mathbb{R}^N$ is contained in [52, 63] and a survey of some fractional counterpart of the Rellich inequality are also available in the recent monographs [4, 27]. Namely, from [27, Theorem 7.14],

$$E_{k,N,p,ps}G(ps,k,p)\int_\Omega\frac{|u(x)|^p}{M_{k+s}(x)^{ps+kp}}\,\mathrm{d}x \le S_{k,p'}\sum_{|\alpha|=k}\int_\Omega\int_\Omega\frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

holds for $\alpha\in\mathbb{N}_0^N$, $k\in\mathbb{N}$, $1<p<\infty$, $1/p<s<1$, and all $u\in C_0^\infty(\Omega)$, where M_{k+s} is a mean distance function from the boundary of Ω defined in [27, Eq. (7.4.6)], the constants $S_{k,p'}$, $E_{k,N,p,ps}$, and $G(ms,k,p)$ are defined in [27, Eq. (7.4.3), Eq. (7.4.8), Eq. (7.4.10)], and the following bound holds for $p=2$ (as shown in [27, Eq. (7.6.3)]):

$$\frac{1}{2}c_{N,s}\sum_{|\alpha|=k}\int_\Omega\int_\Omega\frac{|D^\alpha u(x)-D^\alpha u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \le \int_{\mathbb{R}^N}|(-\Delta)^{\frac{s+k}{2}}u(x)|^2\,\mathrm{d}x.$$

However, to the best of our knowledge, no fractional analogue of (1.1) of the type (1.2) is available in the literature. The main aim of this paper is to prove it (see Theorem 2.1 below for the precise statement) by generalizing the strategy pioneered by Mitidieri in [50, Section 3]. In contrast to the classical case, this approach does not seem to yield sharp constants in the fractional context. On the other hand, we believe that it is of interest in itself for its simplicity. The key ingredient of the proof is the fractional Pohozaev identity of [23, 58] (see the next paragraphs for a more detailed outline of the proof and the main technical difficulties). As a byproduct, we deduce some versions of the fractional Hardy inequality (1.3).

1.1. **Outline and strategy.** Let us outline the strategy employed in [50, Section 3] to prove that, for any function $u\in C_0^\infty(\Omega)$,

$$c^p_{p,\theta}\int_\Omega\frac{|u(x)|^p}{|x|^{\theta+2}}\,\mathrm{d}x \le \int_\Omega\frac{|\Delta u(x)|^p}{|x|^{\theta+2-2p}}\,\mathrm{d}x, \tag{1.6}$$

for $p>1$ and $N>\theta+2$, with $\theta\in\mathbb{R}$. The key observation is that (1.6) can be deduced from an identity of Rellich-Pohozaev type by suitably choosing an auxiliary function.

More precisely, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $u, v \in C^2(\bar{\Omega})$. Then, from [49, Corollary 2.1], we have the following Rellich-Pohozaev identity:

$$\begin{aligned}&\int_\Omega \Big(\Delta u(x\cdot\nabla v) + \Delta v(x\cdot \nabla u)\Big)\,\mathrm{d}x\\ &\qquad = (N-2)\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\partial\Omega}\Big(\partial_\nu u(x\cdot\nabla v) + \partial_\nu v(x\cdot\nabla u) - (\nabla u\cdot\nabla v)(x\cdot\nu)\Big)\,\mathrm{d}\sigma,\end{aligned}\tag{1.7}$$

where ν denotes the outward-pointing normal unit vector and ∂_ν denotes the external normal derivative at the point $x \in \partial\Omega$. By suitably choosing the function v and applying Hölder's inequality, (1.7) implies (1.6). For completeness, we reproduce the computation below.

Let $u \in C_0^\infty(\Omega)$ and, to begin with, let us additionally assume $u > 0$. Plugging u^p (for $p > 1$) into (1.7) yields

$$\begin{aligned}&\int_\Omega u^{p-1}\Delta u(x\cdot\nabla v)\,\mathrm{d}x + (p-1)\int_\Omega u^{p-2}|\nabla u|^2(x\cdot\nabla v)\,\mathrm{d}x + \int_\Omega u^{p-1}(x\cdot\nabla u)\Delta v\,\mathrm{d}x\\ &\qquad = (N-2)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\,\mathrm{d}x.\end{aligned}\tag{1.8}$$

We choose $v = v_\varepsilon := (|x|^\theta + \varepsilon)^{-1}$, for $\varepsilon > 0$ and $\theta \in \mathbb{R}$, and compute

$$\begin{aligned}\nabla\left(\frac{1}{|x|^\theta+\varepsilon}\right) &= -\theta\frac{x\,|x|^{\theta-2}}{(|x|^\theta+\varepsilon)^2},\\ \Delta\left(\frac{1}{|x|^\theta+\varepsilon}\right) &= -(N-1)\theta\frac{|x|^{2\theta-2}}{(|x|^\theta+\varepsilon)^3} + 2\theta^2\frac{|x|^{2\theta-2}}{(|x|^\theta+\varepsilon)^3} - \theta(\theta-1)\frac{|x|^{\theta-2}}{(|x|^\theta+\varepsilon)^2}.\end{aligned}$$

Plugging these into (1.8) and letting $\varepsilon \to 0^+$ (here we use $N > \theta + 2$), we get

$$\begin{aligned}&\int_\Omega \frac{u^{p-1}\Delta u}{|x|^\theta}\,\mathrm{d}x + (p-1)\int_\Omega \frac{u^{p-2}|\nabla u|^2}{|x|^\theta}\,\mathrm{d}x\\ &= \underbrace{[(N-2)-(N-1)+2\theta-(\theta-1)]}_{=\theta}\int_\Omega \frac{u^{p-1}x\cdot\nabla u}{|x|^{\theta+2}}\,\mathrm{d}x.\end{aligned}\tag{1.9}$$

The divergence theorem yields

$$\begin{aligned}\int_\Omega \frac{u^{p-1}x\cdot\nabla u}{|x|^{\theta+2}}\,\mathrm{d}x &= -\frac{1}{p}\int_\Omega u^p\mathrm{div}\Big(\frac{x}{|x|^{\theta+2}}\Big)\,\mathrm{d}x\\ &= -\frac{N-2-\theta}{p}\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x.\end{aligned}\tag{1.10}$$

By using Cauchy-Schwarz and Hölder's inequalities, we have

$$\begin{aligned}-\frac{N-2-\theta}{p}\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x = \int_\Omega \frac{u^{p-1}x\cdot\nabla u}{|x|^{\theta+2}}\,\mathrm{d}x &\le \int_\Omega \frac{|\nabla u|\,|x|\,|u|^{p-1}}{|x|^{\theta+2}}\,\mathrm{d}x\\ &\le \left(\int_\Omega \frac{|\nabla u|^2|u|^{p-2}}{|x|^\theta}\,\mathrm{d}x\right)^{\frac12}\left(\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x\right)^{\frac12}\end{aligned}$$

which gives

$$\int_\Omega \frac{|\nabla u|^2|u|^{p-2}}{|x|^\theta}\,\mathrm{d}x \ge \left(\frac{N-2-\theta}{p}\right)^2\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x.\tag{1.11}$$

Plugging (1.10)–(1.11) into (1.9), we have

$$\left[\frac{N-2-\theta}{p^2}\Big((p-1)(N-2)+\theta\Big)\right]\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x \le -\int_\Omega \frac{u^{p-1}\Delta u}{|x|^\theta}\,\mathrm{d}x.\tag{1.12}$$

Applying Hölder's inequality with conjugate exponents p and $p/(p-1)$, we estimate

$$\begin{aligned}\int_\Omega \frac{u^{p-1}\Delta u}{|x|^\theta}\,\mathrm{d}x &= \int_\Omega \frac{u^{p-1}}{|x|^\alpha}\,\frac{\Delta u}{|x|^{\theta-\alpha}}\,\mathrm{d}x \\ &\le \left(\int_\Omega \frac{u^p}{|x|^{\frac{\alpha p}{p-1}}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{|\Delta u|^p}{|x|^{(\theta-\alpha)p}}\,\mathrm{d}x\right)^{\frac{1}{p}},\end{aligned}$$

for $\alpha \in \mathbb{R}$. Plugging this – with $\alpha = (\theta+2)\frac{(p-1)}{p}$ – into (1.12), we obtain

$$\left[\frac{N-2-\theta}{p^2}\Big((p-1)(N-2)+\theta\Big)\right]\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x \le \left(\int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_\Omega \frac{|\Delta u|^p}{|x|^{\theta+2-p}}\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

That is,

$$c_{p,\theta}^p \int_\Omega \frac{u^p}{|x|^{\theta+2}}\,\mathrm{d}x \le \int_\Omega \frac{|\Delta u|^p}{|x|^{\theta+2-2p}}\,\mathrm{d}x,$$

where

$$c_{p,\theta} = \frac{(N-2-\theta)[(p-1)(N-2)+\theta]}{p^2}.$$

To remove the extra assumption $u > 0$, an approximation argument is needed: we use $u_\mu := (u^2+\mu^2)^{\frac{1}{2}} - \mu$ (with $\mu > 0$), follow the steps above, and finally let $\mu \to 0^+$.

Our aim here is to extend this method to the fractional setting, our starting point being the fractional Pohozaev identity proved in [23, Theorem 1.3]. Several technical difficulties ensue. The main one is that functions of the form $v_\varepsilon(\cdot) := \frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}$ are not admissible in the identity [23, Theorem 1.3]. To overcome this issue, we use an approximation technique. We approximate v_ε by $\zeta_k v_\varepsilon$ where ζ_k is a suitable cut-off function supported in Ω (see Lemma A.1). Secondly, we need to compute $\lim_{\varepsilon\to 0^+}(-\Delta)^s\left((\varepsilon^2+|\cdot|^2)^{-\frac{\theta}{2}}\right)(x)$, which is done in Lemma A.2 below. This computation might be well-known, but we present it here for the sake of completeness. Finally, we need to estimate $(-\Delta)^s u^p$, for which we rely on Cordoba-Cordoba's inequality (see [12, Theorem 1.1]). However, as a drawback, we do not expect the constants involved to be sharp as opposed to the classical case.

The paper is organized as follows. In Section 2, we state our main theorems and present the needed preliminary notions. The proofs are developed in Section 3. The main technical lemmas used in the arguments are collected in Appendix A.

2. Generalized fractional Hardy-type inequalities

For $0 < s < 1$, the *fractional Laplacian operator* $(-\Delta)^s u$ is defined, for any $u \in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, through the singular integral

$$(-\Delta)^s u(x) := c_{N,s}\,\mathrm{p.v.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y, \tag{2.1}$$

where

$$c_{N,s} := \frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{N/2}\Gamma(1-s)} \tag{2.2}$$

(see [32, Proposition 5.6]). It can also be defined weakly for any $u \in H^s(\mathbb{R}^N)$ by letting

$$\big\langle(-\Delta)^s u, v\big\rangle := \frac{c_{N,s}}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y =: \mathcal{E}_s(u,v).$$

Here, $H^s(\mathbb{R}^N)$ is the subspace of those L^2-functions u for which the seminorm $\mathcal{E}_s(u,u)$ is finite. Throughout this manuscript, we denote by $\mathcal{H}_0^s(\Omega)$ the subset of L^2-functions belonging to $H^s(\mathbb{R}^N)$ and such that $u \equiv 0$ in $\mathbb{R}^N\setminus\Omega$. We recall that, by [29, Theorem 2], if Ω has a continuous boundary, then $\mathcal{H}_0^s(\Omega)$ coincides with the closure of $C_0^\infty(\Omega)$ with respect to the seminorm $\mathcal{E}_s(\cdot,\cdot)$ (see also [34, Theorem 1.4.2.1]).

Our main results read as follows.

Theorem 2.1 (Generalized fractional Hardy-type inequality in bounded domains)**.** *Let $s \in (0,1)$ and $\theta \geq 0$ be given. Let Ω be a bounded open set of class C^α, with $\alpha > \max\{1, 2s\}$. Let $N \in \mathbb{N}$ with $N > \theta + 2s$. Then, for all $u \in C_0^\alpha(\Omega)$, we have*

$$\left[\frac{b_{N,s,\theta}}{p}\right]^p \int_\Omega \frac{|u|^p}{|x|^{\theta+2s}}\,\mathrm{d}x \leq \int_\Omega \frac{|(-\Delta)^s u|^p}{|x|^{\theta+2s-2sp}}\,\mathrm{d}x, \tag{2.3}$$

for $p > 1$ and

$$b_{N,s,\theta} \int_\Omega \frac{|u|}{|x|^{\theta+2s}}\,\mathrm{d}x \leq \int_\Omega \frac{\operatorname{sign}(u)(-\Delta)^s u}{|x|^\theta}\,\mathrm{d}x, \tag{2.4}$$

(corresponding to the case $p = 1$), where

$$\begin{aligned} b_{N,s,\theta} &= c_{N,s} \int_0^1 r^{2s-1}(1-r^\theta)(1-r^{N-2s-\theta})\psi(r)\,\mathrm{d}r, \quad c_{N,s} = \frac{s4^s\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \\ \psi(r) &= 2\operatorname{vol}(\mathbb{S}^{N-2}) \int_{-1}^1 \frac{(1-h^2)^{\frac{N-3}{2}}}{\left(1+r^2-2rh\right)^{(N+2s)/2}}\,\mathrm{d}h. \end{aligned}$$

Taking $\theta = 2sp - 2s$ in (2.3) above gives the following version of a result by Herbst [37] (see also [28, Theorem 1.3] and [62] for related results).

Corollary 2.2. *(Improved fractional Hardy inequality) Let $s \in (0,1)$, $\frac{N}{p} > 2s$ and $p > 1$. Let Ω be a bounded open set of $\mathbb{R}^N$ of class C^α with $\alpha > \max\{1, 2s\}$. Then for all $u \in C_0^\alpha(\Omega)$, we have*

$$\left[\frac{b_{N,s,2s(p-1)}}{p}\right]^p \int_\Omega \frac{|u|^p}{|x|^{2sp}}\,\mathrm{d}x \leq \int_\Omega |(-\Delta)^s u|^p\,\mathrm{d}x \tag{2.5}$$

In particular, for $p = 2$ we deduce, for all $u \in \mathcal{H}_0^s(\Omega)$,

$$\begin{aligned} \left[\frac{b_{N,s/2,s}}{2}\right]^2 \int_\Omega \frac{u(x)^2}{|x|^{2s}}\,\mathrm{d}x &\leq \int_\Omega |(-\Delta)^{s/2} u|^2\,\mathrm{d}x \\ &\leq \frac{c_{N,s}}{2}\left[\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^N\setminus\Omega}\left(\int_\Omega \frac{u(y)}{|x-y|^{N+s}}\,\mathrm{d}y\right)^2\mathrm{d}x\right]. \end{aligned} \tag{2.6}$$

In fact, we have the following more general identity.

Theorem 2.3. *Let $N \in \mathbb{N}$ with $N > \theta + 2s$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class C^α with $\alpha > \max\{1, 2s\}$. Let $s \in (0,1)$. For any $X \in C^{0,1}\left(\mathbb{R}^N, \mathbb{R}^N\right)$, we define the nonlocal operator*

$$\left[\mathscr{L}_{\mathcal{K}_Y} u\right](x) := \text{p.v.} \int_{\mathbb{R}^N} (u(x)-u(y))\mathcal{K}_Y(x,y)\,\mathrm{d}y \tag{2.7}$$

where

$$\mathcal{K}_Y(x,y) = \frac{c_{N,s}}{2}\left[\operatorname{div} Y(x) + \operatorname{div} Y(y) - (N+2s)\frac{(Y(x)-Y(y))\cdot(x-y)}{|x-y|^2}\right]|x-y|^{-N-2s}. \tag{2.8}$$

Let $\theta \geq 0$ and $u \in C_0^\alpha(\Omega)$. Then, we have

$$\begin{aligned} &- b_{N,s,\theta} \int_\Omega \frac{u^2(x)}{|x|^{\theta+2s}} \operatorname{div} Y(x)\,\mathrm{d}x + b_{N,s,\theta}(\theta+2s) \int_\Omega u^2(x) \frac{x\cdot Y(x)}{|x|^{\theta+2s+2}}\,\mathrm{d}x \\ &= \theta \int_\Omega \frac{x\cdot Y(x)}{|x|^{\theta+2}}(-\Delta)^s u^2\,\mathrm{d}x - \int_\Omega \frac{\mathscr{L}_{\mathcal{K}_Y} u^2}{|x|^\theta}\,\mathrm{d}x. \end{aligned} \tag{2.9}$$

Remark 2.4. Let us collect a few remarks on the main results.

(1) Taking the limit when $s \to 1^-$ in (2.3), we recover the inequality (1.6) announced in [50] with the constant $\left[\frac{2\theta}{p}\frac{\Gamma(\frac{N-\theta}{2})}{\Gamma(\frac{N-\theta-2}{2})}\right]^p$.

Indeed, from the fact that

$$\lim_{\varepsilon\to 0^+}(-\Delta)^s\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x)=(-\Delta)^s\left[\frac{1}{|\cdot|^\theta}\right](x), \quad \forall x\in\mathbb{R}^N\setminus\{0\}$$

(which can be seen by computing $(-\Delta)^s(|\cdot|^{-\theta})(x)$ as in Lemma A.2) and the identity

$$(-\Delta)^s\left[\frac{1}{|\cdot|^\theta}\right](x)=2^{2s}\frac{\Gamma\left(\frac{N-\theta}{2}\right)\Gamma(\frac{2s+\theta}{2})}{\Gamma\left(\frac{N-\theta-2s}{2}\right)\Gamma\left(\frac{\theta}{2}\right)}\;|x|^{-(\theta+2s)}, \qquad \forall x\neq 0, \quad N>\theta>-2s, \tag{2.10}$$

(which is contained in [40, Table 1]), we deduce that

$$b_{N,s,\theta}=2^{2s}\frac{\Gamma\left(\frac{N-\theta}{2}\right)\Gamma(\frac{2s+\theta}{2})}{\Gamma\left(\frac{N-\theta-2s}{2}\right)\Gamma\left(\frac{\theta}{2}\right)}\longrightarrow\frac{2\theta\Gamma(\frac{N-\theta}{2})}{\Gamma\left(\frac{N-\theta-2}{2}\right)} \quad \text{as} \quad s\to 1^-.$$

(2) If we replace Ω by $\mathbb{R}^N$, using the Pohozaev identity

$$\int_{\mathbb{R}^N} x\cdot\nabla u(-\Delta)^s u\,\mathrm{d}x=(2s-N)\int_{\mathbb{R}^N}u(-\Delta)^s u\,\mathrm{d}x \quad \text{for all } u\in C_0^\infty(\mathbb{R}^N),$$

we prove by a similar argument (much simpler in fact) that

$$\left[\frac{b_{N,s,\theta}}{p}\right]^p\int_{\mathbb{R}^N}\frac{|u|^p}{|x|^{\theta+2s}}\,\mathrm{d}x\le\int_{\mathbb{R}^N}\frac{|(-\Delta)^s u|^p}{|x|^{\theta+2s-2sp}}\,\mathrm{d}x$$

for all $\theta>-2s,\ p>1$ and for all $u\in C_0^\infty\left(\mathbb{R}^N\right)$.

(3) When $\theta=0$, the constant $b_{N,s,0}$ vanishes. Equation (2.5) gives in particular that

$$\int_\Omega \operatorname{sign}(u)(-\Delta)^s u\,\mathrm{d}x\ge 0, \quad \text{for all } u\in C_0^\alpha(\Omega). \tag{2.11}$$

Note that an estimate like (2.11) follows also by using a Kato-type inequality and the symmetry of the fractional Laplace operator (see [22]):

$$\int_{\mathbb{R}^N}\operatorname{sign}(u)\,(-\Delta)^s u\,\mathrm{d}x\ge\int_{\mathbb{R}^N}(-\Delta)^s|u|\,\mathrm{d}x=0 \qquad \text{for all } u\in C_0^\infty(\mathbb{R}^N).$$

(4) We believe that more general Hardy-type inequalities may be obtained by choosing suitably the vector field Y in (2.9). Indeed, note that the estimate (2.6) follows from (2.9) by taking $Y\equiv \mathrm{id}_{\mathbb{R}^N}$.

Remark 2.5. Following closely the proof, we can also obtain the following version of the fractional Hardy inequality (1.3):

$$\begin{aligned}
\mathcal{C}_{N,s,2}\int_\Omega\frac{w^2(x)}{|x|^{2s}}\,\mathrm{d}x &\le \frac{2}{c_{N,s}}\int_\Omega\frac{w}{|x|^{\frac{N-2s}{4}}}(-\Delta)^s(|\cdot|^{\frac{N-2s}{4}}w)\,\mathrm{d}x\\
&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(|x|^{-\frac{N-2s}{4}}w(x)-|y|^{-\frac{N-2s}{4}}w(y))(|x|^{\frac{N-2s}{4}}w(x)-|y|^{\frac{N-2s}{4}}w(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y-\iint_{\Omega\times\Omega}\frac{(|x|^{\frac{N-2s}{4}}-|y|^{\frac{N-2s}{4}})(w(x)v(y)-w(y)v(x))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

for all $w\in C_0^\alpha(\Omega\setminus\{0\})$. Here, $v(x):=\frac{w(x)}{|x|^{\frac{N-2s}{4}}}$ and $\mathcal{C}_{N,s,2}$ is the sharp constant given in (1.4). This follows by choosing $\theta=\frac{N-2s}{2}$ and $u=|x|^{\frac{N-2s}{4}}w$ in (3.11) below (indeed, note that $b_{N,s,\frac{N-2s}{2}}=\mathcal{C}_{N,s,2}\times c_{N,s}$).

3. Proof of the main theorems

In this section, we prove our main results. We start with Theorem 2.1.

Proof of Theorem 2.1. We recall the following approximated Pohozaev identity from [23, Lemma 2.1]. Let Ω be a bounded open set and let $u \in C_0^\alpha(\Omega)$ with $\alpha > \max\{1, 2s\}$. Let $Y : \mathbb{R}^N \to \mathbb{R}^N$ be a globally Lipschitz vector field. Then, denoting

$$\mathcal{E}_Y(u,u) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))^2 \mathcal{K}_Y(x,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{3.1}$$

where

$$\mathcal{K}_Y(x,y) = \frac{c_{N,s}}{2}\left[\operatorname{div} Y(x) + \operatorname{div} Y(y) - (N+2s)\frac{(Y(x)-Y(y))\cdot(x-y)}{|x-y|^2}\right]|x-y|^{-N-2s}, \tag{3.2}$$

we have

$$\mathcal{E}_Y(u,u) = -2\int_\Omega Y\cdot\nabla u(-\Delta)^s u\,\mathrm{d}x, \quad \text{for all } u \in C_0^\alpha(\Omega). \tag{3.3}$$

Taking $Y \equiv \mathrm{id}_{\mathbb{R}^N}$, $u := u + tv$, $t > 0$ in (3.3), and differentiating at $t = 0$ implies, in particular, that

$$\begin{aligned} &\int_\Omega x\cdot\nabla u(-\Delta)^s v\,\mathrm{d}x + \int_\Omega x\cdot\nabla v(-\Delta)^s u\,\mathrm{d}x \\ &= -(N-2s)\frac{c_{N,s}}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{3.4}$$

for all $u, v \in C_0^\alpha(\Omega)$ with $\alpha > \max\{1, 2s\}$.

Step 1. *Proof for non-negative functions* $u \in C_0^\alpha(\Omega)$. We start by considering a function $u \in C_0^\alpha(\Omega)$ satisfying $u \geq 0$. In (3.4), we replace u by u^p and v by $v_k = \frac{1-\rho_k}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}} := \frac{\zeta_k}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}$ where ρ_k is defined as in Lemma A.1. By the same lemma, we know that v_k is admissible in (3.4). With this substitution, from (3.4), we deduce

$$\begin{aligned} &\underbrace{\int_\Omega x\cdot\nabla u^p(-\Delta)^s\left(\frac{\zeta_k}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\right)\mathrm{d}x}_{=:I_k} + \int_\Omega \zeta_k x\cdot\nabla\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\right)(-\Delta)^s u^p\,\mathrm{d}x \\ &\quad + \underbrace{\int_\Omega \frac{x\cdot\nabla\zeta_k}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}(-\Delta)^s u^p\,\mathrm{d}x}_{=:J_k} \\ &= -(N-2s)\int_\Omega \frac{\zeta_k(-\Delta)^s u^p}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\,\mathrm{d}x. \end{aligned} \tag{3.5}$$

Recalling the product rule for the fractional Laplacian – i.e.

$$(-\Delta)^s(uv) = u(-\Delta)^s v + v(-\Delta)^s u - I_s(u,v),$$

where

$$I_s(u,v)(x) := c_{N,s}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

which holds for functions u and v such that $(-\Delta)^s u$ and $(-\Delta)^s v$ exist and

$$\int_{\mathbb{R}^N}\frac{|(u(x)-u(y))(v(x)-v(y))|}{|x-y|^{N+2s}}\,\mathrm{d}y < \infty$$

(see [8, Proposition 1.5 and Remark 1.6]) – we compute

$$I_k := \int_\Omega x\cdot\nabla u^p(-\Delta)^s\left(\frac{\zeta_k}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\right)\mathrm{d}x$$

$$= \int_{\Omega} x \cdot \nabla u^p \left[\zeta_k (-\Delta)^s \left(\frac{1}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}} \right) + \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}} (-\Delta)^s \zeta_k - I_s \left(\zeta_k, \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}} \right) \right] \mathrm{d}x$$
$$:= I_k^1 + I_k^2 + I_k^3. \tag{3.6}$$

By continuity, we have

$$\lim_{k \to +\infty} I_k^1 = \int_{\Omega} x \cdot \nabla u^p (-\Delta)^s \left(\frac{1}{(\varepsilon^2 + |\cdot|^2)^{\frac{\theta}{2}}} \right) \mathrm{d}x. \tag{3.7}$$

On the other hand, by Lemma A.1, we deduce

$$\lim_{k \to +\infty} I_k^2 = 0 = \lim_{k \to +\infty} I_k^3. \tag{3.8}$$

To deal with J_k, we consider the global vector field $Y_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto Y_\varepsilon(x) := \frac{x}{(\varepsilon^2 + |x|^2)^{\theta/2}}$. If $\theta \geq 0$, then we can compute that $\|Y_\varepsilon\|_{C^1(\mathbb{R}^N, \mathbb{R}^N)} \leq C$ for some $C = C(\varepsilon, \theta) > 0$ and therefore $Y_\varepsilon \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$. Applying (3.3) to $U(t, u^p, \zeta_k) := u^p + t\zeta_k$, $t > 0$ with $Y = Y_\varepsilon$ and differentiating with respect to t yields

$$\begin{aligned} J_k :&= \int_{\Omega} \frac{x \cdot \nabla \zeta_k}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}} (-\Delta)^s u^p \, \mathrm{d}x \\ &= \int_{\Omega} Y_\varepsilon \cdot \nabla \zeta_k (-\Delta)^s u^p \, \mathrm{d}x \\ &= -\int_{\Omega} Y_\varepsilon \cdot \nabla u^p (-\Delta)^s \zeta_k \, \mathrm{d}x - \int_{\mathbb{R}^{2N}} (\zeta_k(x) - \zeta_k(y))(u^p(x) - u^p(y)) \mathcal{K}_{Y_\varepsilon}(x, y) \, \mathrm{d}x \, \mathrm{d}y, \end{aligned}$$

where $\mathcal{K}_{Y_\varepsilon}(\cdot, \cdot)$ is defined as in (3.2). Since $\mathcal{K}_{Y_\varepsilon}(\cdot, \cdot)$ is symmetric, we may write

$$\int_{\mathbb{R}^{2N}} (\zeta_k(x) - \zeta_k(y))(u^p(x) - u^p(y)) \mathcal{K}_{Y_\varepsilon}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\Omega} u^p [\mathcal{L}_{\mathcal{K}_{Y_\varepsilon}}](\zeta_k) \, \mathrm{d}x$$

with

$$[\mathcal{L}_{\mathcal{K}_{Y_\varepsilon}}](w)(x) := 2 \, \mathrm{p.v.} \int_{\mathbb{R}^N} (w(x) - w(y)) \mathcal{K}_{Y_\varepsilon}(x, y) \, \mathrm{d}y.$$

Putting everything together, we end up with

$$J_k = -\int_{\Omega} Y_\varepsilon \cdot \nabla u^p (-\Delta)^s \zeta_k \, \mathrm{d}x - \int_{\Omega} u^p [\mathcal{L}_{\mathcal{K}_{Y_\varepsilon}}](\zeta_k) \, \mathrm{d}x =: J_k^1 + J_k^2. \tag{3.9}$$

From (3.8), we know that J_k^1 converges to zero as $k \to +\infty$. Next, arguing as in the proof of (A.2) from Lemma A.1 and using the fact that

$$\left|\mathcal{K}_{Y_\varepsilon}(x, y)\right| \leq C(\varepsilon, \theta, N, s) |x - y|^{-N-2s} \quad \text{for all } x, y \in \mathbb{R}^N,\ x \neq y,$$

we deduce that J_k^2 converges to zero as well. In conclusion, we have that

$$\lim_{k \to \infty} J_k = \lim_{k \to \infty} \int_{\Omega} \frac{x \cdot \nabla \zeta_k}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}} (-\Delta)^s u^p \, \mathrm{d}x = 0. \tag{3.10}$$

Putting the computations in (3.5), (3.6), (3.7), (3.8), and (3.10) together; using Lemma A.2 and passing to the limit $k \to +\infty$ and $\varepsilon \to 0$, we deduce

$$b_{N,s,\theta} \int_{\Omega} x \cdot \nabla u^p \frac{\mathrm{d}x}{|x|^{\theta + 2s}} - \theta \int_{\Omega} \frac{(-\Delta)^s u^p}{|x|^\theta} \, \mathrm{d}x = -(N - 2s) \int_{\Omega} \frac{(-\Delta)^s u^p}{|x|^\theta} \, \mathrm{d}x$$

for all $u \in C_0^\alpha(\Omega)$ with $\alpha > \max\{1, 2s\}$. Using the divergence theorem in the first term then yields

$$-b_{N,s,\theta}(N - 2s - \theta) \int_{\Omega} \frac{u^p}{|x|^{\theta + 2s}} \, \mathrm{d}x - \theta \int_{\Omega} \frac{(-\Delta)^s u^p}{|x|^\theta} \, \mathrm{d}x = -(N - 2s) \int_{\Omega} \frac{(-\Delta)^s u^p}{|x|^\theta} \, \mathrm{d}x. \tag{3.11}$$

To conclude the argument, let us consider three cases separately: $p = 1$, $p \in [2, +\infty)$, and $p \in (1, 2)$.

Case (a). For $p=1$, the inequality (3.11) reads

$$-b_{N,s,\theta}(N-2s-\theta)\int_\Omega \frac{u}{|x|^{\theta+2s}}\,\mathrm{d}x-\theta\int_\Omega \frac{(-\Delta)^s u}{|x|^\theta}\,\mathrm{d}x=-(N-2s)\int_\Omega \frac{(-\Delta)^s u}{|x|^\theta}\,\mathrm{d}x,$$

from which the conclusion follows.

Case (b). For $p\geq 2$, we estimate (3.11) using first Cordoba-Cordoba's inequality (see [12, Theorem 1.1] or [16, Theorem 1.1]) with $\varphi(t)=t^p\in C^2(\mathbb{R}_+)$ and then Hölder's inequality:

$$\begin{aligned}
b_{N,s,\theta}\int_\Omega \frac{u^p}{|x|^{\theta+2s}}\,\mathrm{d}x&=\int_\Omega \frac{(-\Delta)^s u^p}{|x|^\theta}\,\mathrm{d}x\\
&\leq p\int_\Omega \frac{u^{p-1}(-\Delta)^s u}{|x|^\theta}\,\mathrm{d}x=\int_\Omega \frac{u^{p-1}}{|x|^\alpha}\,\frac{(-\Delta)^s u}{|x|^{\theta-\alpha}}\,\mathrm{d}x\\
&\leq p\left(\int_\Omega \frac{u^p}{|x|^{\frac{\alpha p}{p-1}}}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_\Omega \frac{|(-\Delta)^s u|^p}{|x|^{(\theta-\alpha)p}}\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

Plugging this with $\alpha=(\theta+2s)\frac{(p-1)}{p}$ finishes the prove of the claim for $u\in C_0^\alpha(\Omega)$ and $u\geq 0$.

Case (c). For $p\in(1,2)$, we cannot apply Cordoba-Cordoba's inequality directly. Instead, we start over by replacing u^p by its convex approximation $\varphi_t(u)=(t^2+u^2)^{\frac{p}{2}}-t^p$ (with $t>0$) in the Pohozaev identity. Following the same steps as above leads to

$$-b_{N,s,\theta}(N-2s-\theta)\int_\Omega \frac{\varphi_t(u)}{|x|^{\theta+2s}}\,\mathrm{d}x-\theta\int_\Omega \frac{(-\Delta)^s\varphi_t(u)}{|x|^\theta}\,\mathrm{d}x=-(N-2s)\int_\Omega \frac{(-\Delta)^s\varphi_t(u)}{|x|^\theta}\,\mathrm{d}x.$$

Then, we compute

$$\begin{aligned}
b_{N,s,\theta}\int_\Omega \frac{\varphi_t(u)}{|x|^{\theta+2s}}\,\mathrm{d}x&\leq\int_\Omega \frac{(-\Delta)^s\varphi_t(u)}{|x|^\theta}\,\mathrm{d}x\\
&\leq\int_\Omega \frac{\varphi_t'(u)(-\Delta)^s u}{|x|^\theta}\,\mathrm{d}x\leq p\int_\Omega \frac{u^{p-1}}{|x|^\alpha}\,\frac{|(-\Delta)^s u|}{|x|^{\theta-\alpha}}\,\mathrm{d}x\\
&\leq p\left(\int_\Omega \frac{u^p}{|x|^{\frac{\alpha p}{p-1}}}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_\Omega \frac{|(-\Delta)^s u|^p}{|x|^{(\theta-\alpha)p}}\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

The conclusion follows by letting $\alpha=(\theta+2s)\frac{(p-1)}{p}$ and by using Fatou's lemma.

Step 2. *General case $u\in C_0^\alpha(\Omega)$.* Let us consider a function $u\in C_0^\alpha(\Omega)$ and define, for $\mu>0$, the approximation $u_\mu:=(u^2+\mu^2)^{\frac12}-\mu$. Since $u_\mu\in C_0^\alpha(\Omega)$ and $u_\mu\geq 0$, we can follow the computations of Step 1. We conclude by noticing that $\left|u(u^2+\mu^2)^{-\frac12}\right|\leq 1$ and passing to the limit as $\mu\to 0^+$ thanks to Fatou's lemma. □

Next, we prove Corollary 2.2.

Proof of Corollary 2.2. We only need to prove that (2.6) holds for all $u\in\mathcal{H}_0^s(\Omega)$. Let $u\in\mathcal{H}_0^s(\Omega)$ and $u_n\in C_0^\infty(\Omega)$ such that $u_n\to u$ in $\mathcal{H}_0^s(\Omega)$. Since

$$\left[\frac{b_{N,s/2,s}}{2}\right]^2\int_\Omega \frac{u_n(x)^2}{|x|^{2s}}\,\mathrm{d}x\leq\int_\Omega |(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x\leq\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2\,\mathrm{d}x=[u_n]^2_{\mathcal{H}_0^s(\Omega)},$$

it follows that

$$\int_\Omega \frac{u_n(x)^2}{|x|^{2s}}\,\mathrm{d}x\to\int_\Omega \frac{u(x)^2}{|x|^{2s}}\,\mathrm{d}x\quad\text{as}\quad n\to\infty. \tag{3.12}$$

On the other hand, applying Hölder's inequality and using the fact that $(a+b)^2\leq 2(a^2+b^2)$ yields

$$\begin{aligned}
\int_\Omega\left||(-\Delta)^{s/2}u_n|^2-|(-\Delta)^{s/2}u|^2\right|\mathrm{d}x&\leq\int_\Omega\left|(-\Delta)^{s/2}(u_n-u)\right|\left|(-\Delta)^{s/2}u_n+(-\Delta)^{s/2}u\right|\mathrm{d}x\\
&\leq\sqrt{2}\Big(\int_\Omega\left|(-\Delta)^{s/2}(u_n-u)\right|^2\mathrm{d}x\Big)^{1/2}\Big([u_n]^2_{\mathcal{H}_0^s(\Omega)}+[u]^2_{\mathcal{H}_0^s(\Omega)}\Big)^{1/2}
\end{aligned}$$

$$\leq C\big[u_n - u\big]_{\mathcal{H}^s_0(\Omega)}\Big(1 + [u]^2_{\mathcal{H}^s_0(\Omega)}\Big)^{1/2} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Thanks to (3.12)–(3.13), we conclude by passing to the limit in

$$\left[\frac{b_{N,s/2,s}}{2}\right]^2 \int_\Omega \frac{u_n(x)^2}{|x|^{2s}}\,\mathrm{d}x \leq \int_\Omega |(-\Delta)^{s/2} u_n|^2\,\mathrm{d}x.$$

□

Finally, we give a proof of Theorem 2.3.

Proof of Theorem 2.3. The proof follows the idea of the proof of Theorem 2.1 except that in here we use the following more general identity from [23, Lemma 2.1]:

$$\begin{aligned}\int_\Omega Y\cdot\nabla u(-\Delta)^s w\,\mathrm{d}x + \int_\Omega Y\cdot\nabla w(-\Delta)^s u\,\mathrm{d}x &= -\int_{\mathbb{R}^{2N}} (w(x)-w(y))(u(x)-u(y))\mathcal{K}_Y(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &= -\int_\Omega u\,\mathcal{L}_{\mathcal{K}_Y}(w)\,\mathrm{d}x \end{aligned} \tag{3.14}$$

for all $u, w \in C^\alpha_0(\Omega)$ ($\alpha > \max\{1, 2s\}$) and all $Y \in C^{0,1}(\mathbb{R}^N, \mathbb{R}^N)$ where $\mathcal{L}_{\mathcal{K}_Y}$ and $\mathcal{K}_Y$ are respectively given by (2.7) and (2.8) . □

Appendix A. Technical lemmas

In this appendix, we collect the technical lemmas that have been used in the proof of the main results. First, we estimate the fractional Laplacian of a suitably constructed cut-off function involving the distance to the boundary of Ω.

Lemma A.1. *Let Ω be a bounded open set of class C^α with $\alpha > \max\{1, 2s\}$. Let $\rho \in C^\infty(\mathbb{R})$ with $\rho \equiv 1$ in $(-\infty, 1]$, $\rho \equiv 0$ in $[2, +\infty)$ and define $\rho_k(\cdot) = \rho(k\delta_\Omega(\cdot))$, where $\delta_\Omega : \mathbb{R}^N \to \mathbb{R}$ is a $C^\alpha(\mathbb{R}^N)$ function which coincides with the signed distance function near the boundary $\partial\Omega$ (note that, since the boundary is C^α, the sign distance function is C^α as well). Moreover, we assume that δ_Ω is positive in Ω and negative in $\mathbb{R}^N \setminus \Omega$. Let $\zeta_k := 1 - \rho_k$. Then, we have*

$$\frac{\zeta_k}{(\varepsilon^2 + |\cdot|^2)^{\frac{\theta}{2}}} \in \mathcal{H}^s_0(\Omega) \qquad \textit{and} \qquad \frac{\zeta_k}{(\varepsilon^2 + |\cdot|^2)^{\frac{\theta}{2}}} \in C^\alpha_0(\Omega). \tag{A.1}$$

Moreover, for any $u \in C^\alpha_0(\Omega)$, $u \geq 0$, we have

$$\int_\Omega x\cdot\nabla u^p \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}}\big[(-\Delta)^s\zeta_k\big](x)\,\mathrm{d}x \to 0 \qquad \textit{as} \quad k \to \infty, \tag{A.2}$$

$$\int_\Omega x\cdot\nabla u^p I_s\left(\zeta_k, \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{\theta}{2}}}\right)\mathrm{d}x \to 0 \qquad \textit{as} \quad k \to +\infty, \tag{A.3}$$

where

$$I_s(v, w)(\cdot) = \frac{c_{N,s}}{2}\int_{\mathbb{R}^N} \frac{(v(\cdot) - v(\cdot + y))(w(\cdot) - w(\cdot + y))}{|y|^{N+2s}}\,\mathrm{d}y.$$

Proof. We start by proving (A.2) and (A.3). Let $\Omega'' \Subset \Omega' \subset \Omega$ so that $\mathrm{supp}(u) \subset \Omega''$ and let $x \in \Omega''$. Since $\delta_\Omega(x) \geq c > 0$, then for k sufficiently large we have $\rho(k\delta_\Omega(x)) = 0$. Therefore,

$$\begin{aligned}\big[(-\Delta)^s\zeta_k\big](x) &= -c_{N,s}\,\mathrm{p.v.}\int_{\mathbb{R}^N} \frac{\rho_k(x) - \rho_k(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\\ &= c_{N,s}\int_{\Omega_{2/k}} \frac{\rho(k\delta_\Omega(y))}{|x-y|^{N+2s}}\,\mathrm{d}y\\ &\leq \bar{c}(\Omega', \Omega'', N, s)\,\mathrm{vol}(\Omega_{2/k}).\end{aligned}$$

It follows that

$$\begin{aligned}
&\left|\int_\Omega x\cdot\nabla u^p \frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\left[(-\Delta)^s\zeta_k\right](x)\,\mathrm{d}x\right| \\
&\qquad \le \bar c(\Omega',\Omega'')\mathrm{vol}(\Omega_{2/k})\int_\Omega \left|x\cdot\nabla u^p\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}\right|\mathrm{d}x \to 0 \quad \text{as} \quad k\to\infty,
\end{aligned}$$

which gives (A.2).

Similarly, with $x\in\Omega''$, we have

$$\begin{aligned}
\left|I_s\left(\zeta_k,\frac{1}{(\varepsilon+|x|^2)^{\frac{\theta}{2}}}\right)(x)\right| &= \frac{c_{N,s}}{2}\left|\int_{\Omega_{2/k}}\frac{-\rho(k\delta_\Omega(x))\big((\varepsilon^2+|x|^2)^{-\frac{\theta}{2}}-(\varepsilon^2+|y|^2)^{-\frac{\theta}{2}}\big)}{|x-y|^{N+2s}}\,\mathrm{d}y\right| \\
&\le \bar c(\Omega',\Omega'',N,s,\varepsilon)\,\mathrm{vol}(\Omega_{2/k}),
\end{aligned}$$

from which (A.3) follows. To see that $\frac{\zeta_k}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\in\mathcal{H}^s_0(\Omega)$, first observe that $\frac{\zeta_k}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\equiv 0$ in $\mathbb{R}^N\setminus\Omega$. Next let $R_\Omega>1$ so that $\Omega\Subset B_{R_\Omega}$ and write

$$\begin{aligned}
\frac{2}{c_{N,s}}\left[\frac{\zeta_k}{\varepsilon^2+|\cdot|}\right]^2_{H^s(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(\zeta_k(x)(\varepsilon^2+|x|^2)^{-\frac{\theta}{2}}-\zeta_k(y)(\varepsilon^2+|y|^2)^{-\frac{\theta}{2}}\big)^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{B_{R_\Omega}}\int_{B_{R_\Omega}}\frac{\big(\zeta_k(x)(\varepsilon^2+|x|^2)^{-\frac{\theta}{2}}-\zeta_k(y)(\varepsilon^2+|y|^2)^{-\frac{\theta}{2}}\big)^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_\Omega\frac{\zeta_k^2(x)}{(\varepsilon^2+|x|^2)^\theta}\int_{\mathbb{R}^N\setminus B_{R_\Omega}}\frac{\mathrm{d}y}{|x-y|^{N+2s}}\,\mathrm{d}x \\
&\le 2\int_{B_{R_\Omega}}\int_{B_{R_\Omega}}\frac{\big(\rho_k(x)-\rho_k(y)\big)^2(\varepsilon^2+|x|^2)^{-\theta}}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\int_{B_{R_\Omega}}\int_{B_{R_\Omega}}\frac{\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}}-\frac{1}{(\varepsilon^2+|y|^2)^{\frac{\theta}{2}}}\right)^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + c\int_\Omega\frac{\zeta_k^2(x)}{(\varepsilon^2+|x|^2)^{\theta/2}}\,\mathrm{d}x\int_{\mathbb{R}^N\setminus B_{R_\Omega}}\frac{\mathrm{d}y}{1+|y|^{N+2s}} \\
&\le \left(\frac{2}{\varepsilon^{2\theta}}\|\nabla\rho_k\|_{L^\infty(\mathbb{R})}+2\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right]_{C^1_{loc}(\mathbb{R}^N)}\right)\int_{B_{R_\Omega}}\int_{B_{R_\Omega}}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{N+2s-2}} \\
&\quad + c\int_\Omega\frac{\zeta_k^2(x)}{(\varepsilon^2+|x|^2)^{\theta/2}}\,\mathrm{d}x\int_{\mathbb{R}^N\setminus B_{R_\Omega}}\frac{\mathrm{d}y}{1+|y|^{N+2s}} \\
&< \infty.
\end{aligned}$$

□

Finally, we shall give an explicit computation of the quantity

$$\lim_{\varepsilon\to0^+}(-\Delta)^s\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x).$$

Lemma A.2. *Let $\varepsilon>0$ and $\theta>-2s$. Then, we have*

$$\lim_{\varepsilon\to0^+}(-\Delta)^s\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x) = |x|^{-(\theta+2s)}b_{N,s,\theta}, \quad \textit{for all } x\in\mathbb{R}^N\setminus\{0\}. \tag{A.4}$$

The convergence also holds locally uniformly in $\mathbb{R}^N \setminus \{0\}$. Here, we introduced the following notation:

$$b_{N,s,\theta} = c_{N,s} \int_0^1 r^{2s-1}(1-r^\theta)(1-r^{N-2s-\theta})\psi(r)\,\mathrm{d}r, \tag{A.5}$$

$$c_{N,s} = \frac{s4^s\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{A.6}$$

$$\psi(t) = 2\,\mathrm{vol}(\mathbb{S}^{N-2}) \int_{-1}^1 \frac{(1-h^2)^{\frac{N-3}{2}}}{\big(1+r^2-2rh\big)}\,\mathrm{d}h. \tag{A.7}$$

Proof. For the sake of brevity, we let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\begin{aligned}\frac{1}{\mathrm{vol}(\mathbb{S}^{N-2})}\psi(r) :&= \int_0^\pi \frac{\sin^{N-2}(\alpha_1)}{\big(1+r^2-2r\cos(\alpha_1)\big)^{(N+2s)/2}}\,\mathrm{d}\alpha_1 \\ &= 2\int_{-1}^1 \frac{(1-h^2)^{\frac{N-3}{2}}}{\big(1+r^2-2rh\big)^{(N+2s)/2}}\,\mathrm{d}h.\end{aligned}$$

Let $x \in \mathbb{R}^N \setminus \{0\}$. By definition and passing into polar coordinates, we compute

$$\begin{aligned}
&(-\Delta)^s\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x) = \text{ p.v.} \int_{\mathbb{R}^N}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+|y|^2)^{\frac{\theta}{2}}}\right)\frac{\mathrm{d}y}{|x-y|^{N+2s}} \\
&= c_{N,s}\,\text{p.v.}\int_0^\infty r^{N-1}\left((\varepsilon^2+|x|^2)^{-\frac{\theta}{2}} - (\varepsilon^2+r^2)^{-\frac{\theta}{2}}\right)\int_{\mathbb{S}^{N-1}}\frac{\mathrm{d}y}{(|x|^2+r^2-2r|x|y_1)^{\frac{N+2s}{2}}}\,\mathrm{d}r \\
&= c_{N,s}|x|^{-2s}\,\text{p.v.}\int_0^\infty r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right)\int_{\mathbb{S}^{N-1}}\frac{\mathrm{d}y}{(1+r^2-2ry_N)^{\frac{N+2s}{2}}}\,\mathrm{d}r \\
&= 2c_{N,s}|x|^{-2s}\,\text{p.v.}\int_0^\infty r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right) \\
&\qquad\qquad \times \mathrm{vol}(\mathbb{S}^{N-2})\int_0^\pi \frac{\sin^{N-2}(\alpha_1)}{\big(1+r^2-2r\cos(\alpha_1)\big)^{(N+2s)/2}}\,\mathrm{d}\alpha_1 \\
&= c_{N,s}|x|^{-2s}\,\text{p.v.}\int_0^\infty r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right)\psi(r)\,\mathrm{d}r \\
&= c_{N,s}|x|^{-2s}\left(\text{p.v.}\int_0^1 \cdots\,\mathrm{d}r \ + \text{p.v.}\int_1^\infty \cdots\,\mathrm{d}r\right) \\
&= c_{N,s}|x|^{-2s}\,\text{p.v.}\int_0^1 \psi(r)\left(r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right)\right. \\
&\qquad\qquad\qquad \left. + r^{2s-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+\frac{|x|^2}{r^2})^{\frac{\theta}{2}}}\right)\right)\mathrm{d}r.
\end{aligned} \tag{A.8}$$

Let us prove that the integral above is finite. To this aim, we distinguish two cases.

Case (a): $r \in (0,\gamma]$ for some $\gamma \in (\frac{6-\sqrt{32}}{2}, 1)$. In this case, we use the bound

$$\begin{aligned}&\left|r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right) + r^{2s-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+\frac{|x|^2}{r^2})^{\frac{\theta}{2}}}\right)\right| \\ &\le \frac{r^{N-1}}{|x|^\theta}\left(1+\frac{1}{r^\theta}\right) + \frac{r^{2s-1}}{|x|^\theta}(1+r^\theta).\end{aligned} \tag{A.9}$$

Case (b): $r \in [\gamma, 1)$. Here, we use the power series expansion

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\, x^n := \sum_{n=0}^{\infty} c_{\alpha,n}\, x^n \qquad (\text{for } \ |x|<1)$$

to get, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
&\left| r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right) + r^{2s-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+\frac{|x|^2}{r^2})^{\frac{\theta}{2}}}\right)\right| \\
&= \frac{1}{|x|^\theta}\left|\sum_{k=0}^{\infty} c_{-\frac{\theta}{2},k}\big(1-r^{\theta+2k}\big)\big(r^{2s-1}-r^{N-1-\theta-2k}\big)\frac{\varepsilon^{2k}}{|x|^{2k}}\right| \\
&\le \frac{1}{|x|^\theta}\left((1-r^\theta)\big|r^{2s-1}-r^{N-1-\theta}\big| + \sum_{k=1}^{\infty}\big|c_{-\frac{\theta}{2},k}\big|\big|r^{2s-1+2k}-r^{N-1-\theta}\big|\frac{\varepsilon^{2k}}{(r|x|)^{2k}}\right) \\
&\le \frac{1}{|x|^\theta}\left((1-r^\theta)\big|r^{2s-1}-r^{N-1-\theta}\big| + C\sum_{k=1}^{\infty}\frac{\varepsilon^{2k}}{(r|x|)^{2k}}\right) \\
&\le \frac{C}{|x|^\theta}(1-r^\theta)\big|r^{2s-1}-r^{N-1-\theta}\big| \qquad (A.10)
\end{aligned}$$

for some $C = C(\theta) > 0$. In the last line, we used the fact that the power series in the line before converges uniformly (in r) to zero and hence is controlled by $(1-r^\theta)|r^{2s-1}-r^{N-1-\theta}|$.

Moreover, we have

$$\begin{aligned}
\int_0^1 \Bigg(&\chi_{(0,\gamma]}(r)\left(r^{N-1}\left(1+\frac{1}{r^\theta}\right) + r^{2s-1}(1+r^\theta)\right)\psi(r) \\
&+ \chi_{[\gamma,1)}(r)\frac{|1-r^\theta|}{(1-r)^{1+2s}} r^{2s-1}|1-r^{N-\theta-2s}|(1-r)^{1+2s}\psi(r)\Bigg)\,\mathrm{d}r < \infty. \qquad (A.11)
\end{aligned}$$

Indeed, since $\psi(r)$ is bounded near zero,

$$\int_0^1 \chi_{(0,\gamma]}(r)\left(r^{N-1}\left(1+\frac{1}{r^\theta}\right) + r^{2s-1}(1+r^\theta)\right)\psi(r)\,\mathrm{d}r < \infty \qquad (A.12)$$

provided that $\theta > -2s$. On the other hand, since

$$(1-r)^{1+2s}\psi(r) = \frac{4}{(2\sqrt{r})^{\frac{N-1}{2}}}\left(\frac{1-r}{2\sqrt{r}}\right)^{1+2s}\int_0^1 \frac{(h(1-h))^{\frac{N-3}{2}}}{\left((\frac{1-r}{2\sqrt{r}})^2+h\right)^{\frac{N+2s}{2}}}\,\mathrm{d}h,$$

which is $C^{s+\kappa(s)}([\gamma,1])$ (for some κ depending on s) by the choice of $\gamma \in (\frac{6-\sqrt{32}}{2}, 1)$ and [24, Lemma 2.1], we have that

$$\int_0^1 \chi_{[\gamma,1)}(r)\frac{|1-r^\theta|}{(1-r)^{1+2s}} r^{2s-1}|1-r^{N-\theta-2s}|(1-r)^{1+2s}\psi(r)\,\mathrm{d}r < \infty. \qquad (A.13)$$

The claim (A.11) then follows from (A.12) and (A.13).

This implies, in view of (A.9) and (A.10), that the integral in (A.8) converges. Consequently,

$$\begin{aligned}
&(-\Delta)^s\left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x) \\
&= \frac{c_{N,s}}{|x|^{2s}}\int_0^1 \psi(r)\left(r^{N-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}}\right) + r^{2s-1}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{1}{(\varepsilon^2+\frac{|x|^2}{r^2})^{\frac{\theta}{2}}}\right)\right)\mathrm{d}r.
\end{aligned}$$

In view of (A.9), (A.10), and (A.11), by Lebesgue's dominated convergence theorem, we have that

$$\begin{aligned}
&\frac{|x|^{2s}}{c_{N,s}} \lim_{\varepsilon\to 0^+} (-\Delta)^s \left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x) = \\
&\lim_{\varepsilon\to 0^+} \int_0^1 \psi(r)\left(\frac{r^{N-1}}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{r^{N-1}}{(\varepsilon^2+r^2|x|^2)^{\frac{\theta}{2}}} + \frac{r^{2s-1}}{(\varepsilon^2+|x|^2)^{\frac{\theta}{2}}} - \frac{r^{2s-1}}{(\varepsilon^2+\frac{|x|^2}{r^2})^{\frac{\theta}{2}}}\right)\mathrm{d}r \\
&= \frac{1}{|x|^\theta}\int_0^1 r^{2s-1}(1-r^\theta)(1-r^{N-2s-\theta})\psi(r)\,\mathrm{d}r.
\end{aligned}$$

In other words, we conclude that

$$\lim_{\varepsilon\to 0^+} (-\Delta)^s \left[\frac{1}{(\varepsilon^2+|\cdot|^2)^{\frac{\theta}{2}}}\right](x) = |x|^{-(\theta+2s)} c_{N,s}\int_0^1 r^{2s-1}(1-r^\theta)(1-r^{N-2s-\theta})\psi(r)\,\mathrm{d}r.$$

□

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References

[1] N. Abatangelo and E. Valdinoci. Getting acquainted with the fractional Laplacian. In *Contemporary research in elliptic PDEs and related topics*, volume 33 of *Springer INdAM Ser.*, pages 1–105. Springer, Cham, 2019.

[2] Adimurthi and A. Mallick. A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(3):917–949, 2018.

[3] L. Ambrosio, A. Pinamonti, and G. Speight. Weighted Sobolev spaces on metric measure spaces. *J. Reine Angew. Math.*, 746:39–65, 2019.

[4] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy's inequality*. Universitext. Springer, Cham, 2015.

[5] G. Barbatis, S. Filippas, and A. Tertikas. A unified approach to improved L^p Hardy inequalities with best constants. *Trans. Amer. Math. Soc.*, 356(6):2169–2196, 2004.

[6] W. Beckner. Pitt's inequality and the uncertainty principle. *Proc. Amer. Math. Soc.*, 123(6):1897–1905, 1995.

[7] F. Bianchi, L. Brasco, and A. C. Zagati. On the sharp Hardy inequality in Sobolev-Slobodeckiĭ spaces. *ArXiv:2209.03012*, 2022.

[8] U. Biccari, M. Warma, and E. Zuazua. Local elliptic regularity for the Dirichlet fractional Laplacian. *Adv. Nonlinear Stud.*, 17(2):387–409, 2017.

[9] K. Bogdan, B. o. Dyda, and P. Kim. Hardy inequalities and non-explosion results for semigroups. *Potential Anal.*, 44(2):229–247, 2016.

[10] L. Brasco and E. Cinti. On fractional Hardy inequalities in convex sets. *Discrete Contin. Dyn. Syst.*, 38(8):4019–4040, 2018.

[11] H. Brezis and M. Marcus. Hardy's inequalities revisited. volume 25, pages 217–237 (1998). 1997. Dedicated to Ennio De Giorgi.

[12] L. A. Caffarelli and Y. Sire. On some pointwise inequalities involving nonlocal operators. In *Harmonic analysis, partial differential equations and applications*, Appl. Numer. Harmon. Anal., pages 1–18. Birkhäuser/Springer, Cham, 2017.

[13] B. Cassano, L. Cossetti, and L. Fanelli. Improved Hardy-Rellich inequalities. *Commun. Pure Appl. Anal.*, 21(3):867–889, 2022.

[14] O. Ciaurri and L. Roncal. Hardy's inequality for the fractional powers of a discrete Laplacian. *J. Anal.*, 26(2):211–225, 2018.

[15] E. Cinti and F. Ferrari. Geometric inequalities for fractional Laplace operators and applications. *NoDEA Nonlinear Differential Equations Appl.*, 22(6):1699–1714, 2015.

[16] A. Córdoba and D. Córdoba. A pointwise estimate for fractionary derivatives with applications to partial differential equations. *Proc. Natl. Acad. Sci. USA*, 100(26):15316–15317, 2003.

[17] A. Cotsiolis and N. Labropoulos. On the Hardy-Sobolev inequalities. In *Differential and integral inequalities*, pages 265–287. Cham: Springer, 2019.

[18] L. D'Ambrosio. Hardy-type inequalities related to degenerate elliptic differential operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(3):451–486, 2005.

[19] E. B. Davies. The Hardy constant. *Quart. J. Math. Oxford Ser. (2)*, 46(184):417–431, 1995.

[20] E. B. Davies and A. M. Hinz. Explicit constants for Rellich inequalities in $L_p(\Omega)$. *Math. Z.*, 227(3):511–523, 1998.

[21] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[22] J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez. The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach. *Nonlinear Anal.*, 177(part A):325–360, 2018.

[23] S. M. Djitte, M. M. Fall, and T. Weth. A generalized fractional Pohozaev identity and applications. *ArXiv:2112.10653*, 2021.

[24] S. M. Djitte and S. Jarohs. Nonradiality of second fractional eigenfunctions of thin annuli. *Communications on Pure and Applied Analysis, https://doi.org/10.1007/s00526-021-02094-3*, 2022.

[25] B. Dyda and A. V. Vähäkangas. A framework for fractional Hardy inequalities. *Ann. Acad. Sci. Fenn. Math.*, 39(2):675–689, 2014.

[26] D. E. Edmunds and W. D. Evans. The Rellich inequality. *Rev. Mat. Complut.*, 29(3):511–530, 2016.

[27] D. E. Edmunds and W. D. Evans. *Fractional Sobolev spaces and inequalities*, volume 230 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2023.

[28] M. M. Fall. Semilinear elliptic equations for the fractional Laplacian with Hardy potential. *Nonlinear Anal.*, 193:111311, 29, 2020.

[29] A. Fiscella, R. Servadei, and E. Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.

[30] R. L. Frank, E. H. Lieb, and R. Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.*, 21(4):925–950, 2008.

[31] R. L. Frank and R. Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255(12):3407–3430, 2008.

[32] N. Garofalo. Fractional thoughts. In *New developments in the analysis of nonlocal operators*, volume 723 of *Contemp. Math.*, pages 1–135. Amer. Math. Soc., [Providence], RI, 2019.

[33] F. Gazzola, H.-C. Grunau, and E. Mitidieri. Hardy inequalities with optimal constants and remainder terms. *Trans. Amer. Math. Soc.*, 356(6):2149–2168, 2004.

[34] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[35] G. H. Hardy. Note on a theorem of Hilbert. *Math. Z.*, 6(3-4):314–317, 1920.

[36] H. P. Heinig, A. Kufner, and L.-E. Persson. On some fractional order Hardy inequalities. *J. Inequal. Appl.*, 1(1):25–46, 1997.

[37] I. W. Herbst. Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys.*, 53(3):285–294, 1977.

[38] V. F. Kovalenko, M. A. Perelmuter, and Y. A. Semenov. Schrödinger operators with $L^{1/2}_W(\mathbf{R}^l)$-potentials. *J. Math. Phys.*, 22(5):1033–1044, 1981.

[39] A. Kufner, L. Maligranda, and L.-E. Persson. The prehistory of the Hardy inequality. *Amer. Math. Monthly*, 113(8):715–732, 2006.

[40] M. Kwaśnicki. Fractional Laplace operator and its properties. In *Handbook of fractional calculus with applications. Vol. 1*, pages 159–193. De Gruyter, Berlin, 2019.

[41] E. Landau. A note on a theorem concerning series of positive terms. Extract from a letter of Prof. E. Landau to Prof. I. Schur (communicated by G. H. Hardy). *J. Lond. Math. Soc.*, 1:38–39, 1926.

[42] M. Loss and C. Sloane. Hardy inequalities for fractional integrals on general domains. *J. Funct. Anal.*, 259(6):1369–1379, 2010.

[43] M. Marcus, V. J. Mizel, and Y. Pinchover. On the best constant for Hardy's inequality in $\mathbf{R^n}$. *Trans. Amer. Math. Soc.*, 350(8):3237–3255, 1998.

[44] M. Marcus and I. Shafrir. An eigenvalue problem related to Hardy's L^p inequality. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 29(3):581–604, 2000.

[45] T. Matskewich and P. E. Sobolevskii. The best possible constant in generalized Hardy's inequality for convex domain in $\mathbf{R}^n$. *Nonlinear Anal.*, 28(9):1601–1610, 1997.

[46] T. Matskewich and P. E. Sobolevskii. The sharp constant in Hardy's inequality for complement of bounded domain. *Nonlinear Anal.*, 33(2):105–120, 1998.

[47] V. Mazya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[48] V. Mazya and T. Shaposhnikova. Erratum to: "On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces" [J. Funct. Anal. **195** (2002), no. 2, 230–238; MR1940355 (2003j:46051)]. *J. Funct. Anal.*, 201(1):298–300, 2003.
[49] E. Mitidieri. A Rellich type identity and applications. *Comm. Partial Differential Equations*, 18(1-2):125–151, 1993.
[50] E. Mitidieri. A simple approach to Hardy inequalities. *Mat. Zametki*, 67(4):563–572, 2000.
[51] R. Musina and A. I. Nazarov. Complete classification and nondegeneracy of minimizers for the fractional Hardy-Sobolev inequality, and applications. *J. Differ. Equations*, 280:292–314, 2021.
[52] R. Musina and A. I. Nazarov. A note on higher order fractional Hardy-Sobolev inequalities. *Nonlinear Anal.*, 203:Paper No. 112168, 3, 2021.
[53] J. Nečas. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 16:305–326, 1962.
[54] B. Opic and A. Kufner. *Hardy-type inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.
[55] F. Rellich. Halbbeschränkte Differentialoperatoren höherer Ordnung. In *Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III*, pages 243–250. Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.
[56] F. Rellich. *Perturbation theory of eigenvalue problems*. Gordon and Breach Science Publishers, New York-London-Paris, 1969. Assisted by J. Berkowitz, With a preface by Jacob T. Schwartz.
[57] L. Roncal. Hardy type inequalities for the fractional relativistic operator. *Math. Eng.*, 4(3):Paper No. 018, 16, 2022.
[58] X. Ros-Oton and J. Serra. The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.*, 213(2):587–628, 2014.
[59] M. Ruzhansky and D. Suragan. On horizontal Hardy, Rellich, Caffarelli-Kohn-Nirenberg and p-sub-Laplacian inequalities on stratified groups. *J. Differential Equations*, 262(3):1799–1821, 2017.
[60] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[61] F. Vigneron. A simple proof of the Hardy inequality on Carnot groups and for some hypoelliptic families of vector fields. *Tunis. J. Math.*, 2(4):851–880, 2020.
[62] D. Yafaev. Sharp constants in the Hardy-Rellich inequalities. *J. Funct. Anal.*, 168(1):121–144, 1999.
[63] J. Yang. Fractional Sobolev-Hardy inequality in $\mathbb{R}^N$. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 119:179–185, 2015.

(N. De Nitti) Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Data Science, Chair for Dynamics, Control and Numerics (Alexander von Humboldt Professorship), Cauerstr. 11, 91058 Erlangen, Germany.

Email address: nicola.de.nitti@fau.de

(S. M. Djitte) Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Data Science, Chair for Dynamics, Control and Numerics (Alexander von Humboldt Professorship), Cauerstr. 11, 91058 Erlangen, Germany.

African Institute for Mathematical Sciences in Senegal (AIMS Senegal), KM 2, Route de Joal, B.P. 14 18. Mbour, Sénégal.

Email address: sidy.m.djitte@aims-senegal.org