HOMOGENIZATION OF HIGH-CONTRAST MEDIA IN FINITE-STRAIN ELASTOPLASTICITY

ELISA DAVOLI, CHIARA GAVIOLI, AND VALERIO PAGLIARI

Abstract. This work is devoted to the analysis of the interplay between internal variables and high-contrast microstructure in inelastic solids. As a concrete case-study, by means of variational techniques, we derive a macroscopic description for an elastoplastic medium. Specifically, we consider a composite obtained by filling the voids of a periodically perforated stiff matrix by soft inclusions. We study the Γ-convergence of the related energy functionals as the periodicity tends to zero. The main challenge is posed by the lack of coercivity brought about by the degeneracy of the material properties in the soft part. We prove that the Γ-limit, which we compute with respect to a suitable notion of convergence, is the sum of the contributions resulting from each of the two components separately. Eventually, convergence of the energy minimizing configurations is obtained.

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Contents

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1. Introduction

The present paper is concerned with the variational analysis of some integral functionals that model the stored energy of materials governed by finite-strain elastoplasticity with hardening. Our goal is to derive, by means of Γ-convergence, the effective macroscopic energy of a special class of heterogeneous materials, those with a so called high-contrast microstructure. The interest in such media stems from the experimental observation of an infinite number of band gaps in their mechanical behavior. In other words, high-contrast materials exhibit infinitely many interval of frequencies in which wave propagation is not allowed. This, in turn, makes them extremely interesting for possible cloaking applications. Some recent ones in civil engineering, e.g. in seismic waves cloaking, and in the modeling of advanced sensor and actuator devices call for advancements in the mathematical modeling of those classes of high-contrast materials that have not been fully studied yet, like the ones we consider here.

The mathematical literature on high-contrast materials is vast. To keep our presentation concise, we only point out that, besides results for stratified elastoplastic composites [14, 15, 22, 25], the only additional available contributions in the inelastic setting concern the study of brittle fracture problems [5, 6, 42]. For the modeling of nonlinear elastic high-contrast composites we single out the works [10, 13].

When undertaking the analysis of high-contrast media beyond the elastic purview, hurdles are posed by the mathematical treatment of possible internal variables and dissipative effects, as well as by their interplay with the high-contrast microstructure. In this paper we initiate such task by focusing on the case-study of finite elastoplasticity (see, e.g., [37]). At this first stage we neglect both the difficulties due to possible lack of coercivity for the dissipative effects and those associated with time evolution. Thus, we focus here on a static model for a single time-step with a global regularization on the gradient of the plastic strain, and leave the analysis of different regimes and the passage to the limit in the quasistatic evolutions for future investigations.

The present study grounds on a previous result that we obtained in [24], where we addressed the static homogenization of elastoplastic microstructures in the large strain regime. As in that work, our starting point is the description of the medium at the microscopic level. We let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with Lipschitz boundary, and we suppose it to be the reference configuration of an elastoplastic body that exhibits the following microstructure: denoting by $\varepsilon > 0$ the microscale, we suppose that a stiff perforated matrix Ω^1_ε sits in Ω and that its pores are filled by soft inclusions, which form the set Ω^0_ε (see Figure 2). Let us denote by $\mathsf{SL}(3)$ the group of 3×3 real matrices with determinant equal to 1. When the matrix and the inclusions exhibit the same plastic-hardening H, the functionals encoding the stored energy associated with the deformation $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ and the plastic strain $P \in W^{1,q}(\Omega;\mathsf{SL}(3))$ read

$$\begin{aligned} \mathcal{J}_\varepsilon(y,P) \coloneqq \int_{\Omega^0_\varepsilon} W^0_\varepsilon\left(\varepsilon \nabla y(x) P^{-1}(x)\right) \mathrm{d}x + \int_{\Omega^1_\varepsilon} W^1\left(\nabla y(x) P^{-1}(x)\right) \mathrm{d}x \\ + \int_\Omega H(P(x))\, \mathrm{d}x + \int_\Omega |\nabla P(x)|^q \, \mathrm{d}x, \end{aligned} \tag{1.1}$$

where $\{W^0_\varepsilon\}_{\varepsilon>0}$ and W^1 are, respectively, the elastic energy densities of the inclusions and of the matrix.

Let us briefly comment on some modeling choices underlying position (1.1). The factor ε multiplying the argument of W^0_ε encodes the high-contrast between the two components, and it results in a loss of coercivity in the problem. From a modeling perspective, this heuristically means that very large deformations of the inclusions are allowed or, in other words, that the inclusions are very soft – whence the expression *high-contrast* to describe the difference between the phases.

As for the hardening term, note that also additional hardening variables have been taken into account in the literature, see [38, 39] for a modeling overview. Here, to the purpose of putting the high-contrast behavior to the foreground, we give up full generality and restrict ourselves to the case in which only a hardening dependence on the plastic strain is given. A discussion on alternative modeling choices is also presented in Remark 2.3.

Our main result describes the asymptotics of the functionals $\mathcal{J}_\varepsilon$, and it is presented in Theorem 2.7. The precise mathematical framework of our analysis is described in Section 2, where further details on the definitions and on the roles of the terms in $\mathcal{J}_\varepsilon$ may be found.

We work under the classical assumption that the elastic behavior of our sample Ω is independent of preexistent plastic distortions. Then, the deformation gradient ∇y associated with any deformation $y\colon \Omega \to \mathbb{R}^3$ of the body decomposes into an elastic strain and a plastic one. In the framework of linearized elastoplasticity the decomposition would take an additive form. In the case at stake, that of finite plasticity [34, 36, 39, 38], the existence of an intermediate configuration determined by purely plastic distortions is instead assumed. It is then supposed that elastic deformations are applied on such intermediate configuration. Mathematically, these hypotheses amount to a multiplicative decomposition of the gradient of any deformation $y \in W^{1,2}(\Omega;\mathbb{R}^3)$:

$$\nabla y(x) = F_{\mathrm{el}}(x)P(x) \quad \text{for a. e. } x \in \Omega,$$

for a suitable *elastic strain* $F_{\mathrm{el}} \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and a *plastic strain* $P \in L^2(\Omega;\mathrm{SL}(3))$. On one hand, such multiplicative structure has recently found an atomistic validation in the framework of crystal plasticity by means of a discrete-to-continuum analysis [18, 19]. On the other hand, alternative models for finite plasticity have been proposed. However, since a discussion of fine modeling issues goes beyond the scopes of our work, we do not dwell here on a comparison of the various modeling theories. We refer the reader interested on this point to, e.g., [23, 31, 32, 40].

We finally comment on the regularizing term in ∇P in the energy (1.1). As mentioned before, at this stage we assume it to provide coercivity of the energy with respect to the plastic-strain variables on the whole set Ω. From a modeling point of view, we note that this regularization is common in engineering models, for it prevents the formation of microstructures, see [7, 11]. Alternative higher order regularizations are discussed in [27].

Let us conclude our introduction with a few words on the proofs. A delicate point is choosing a convergence that ensures effective compactness properties. Indeed, the fact that the energy contributions in the soft inclusions are evaluated in terms of $\varepsilon\nabla y$ leads to a loss of coercivity for which compactness in classical weak Sobolev topologies is prevented. On the other hand, arguing with strong two-scale convergence of the gradients, as in [13] does not guarantee convergence of minimizers of $\mathcal{J}_\varepsilon$ to minimizers of the limiting functional. To cope with this difficulty, we adapt the approach in [26] and introduce an ad hoc notion of convergence for deformations, to which we refer as *convergence in the sense of extensions*. Roughly speaking, a sequence of deformations converges in the sense of extensions if it is bounded in L^2 and can be extended in $W^{1,2}$ in such a way that the extensions are weakly compact in the Sobolev sense, cf. Definition 2.4 and Remarks 2.5 and 2.6 for the precise definition and some basic properties. For the plastic strains, we argue instead with the weak convergence in $W^{1,q}$. This choice is motivated by the fact that sequences of deformations and plastic strains with uniformly bounded energies are precompact with respect to the above topology. Thus our Γ-convergence analysis directly entails convergence of minimizers. We observe that this result easily extends to functionals which take into account also plastic dissipation. On this point we refer to Section 6.

The strategy relies on extension results on perforated domains, on two-scale convergence and periodic unfolding techniques, as well as on equiintegrability arguments to control the behavior of the microstructure close to the boundary of the set Ω. A key-step is a splitting procedure that allows to treat the soft and the stiff parts separately.

Outline. The setup of our analysis and the main result, Theorem 2.7, are presented in Section 2. Section 3 contains some preliminary useful facts. In Section 4 we discuss the equicoercivity of the energy functionals under consideration and the splitting procedure. The asymptotic behavior of the soft inclusions is characterized in Section 5. The ground is then laid for the proof of the Theorem 2.7, which is contained in Section 6 together with a variant including plastic dissipation and a comparison with an aforementioned result from [13].

2. Mathematical setting and results

Hereafter, Ω is an open, bounded, and connected set with Lipschitz boundary in $\mathbb{R}^3$. Working in the 3-dimensional space is not essential, and our analysis can be easily adapted to the setting of $\mathbb{R}^d$ with $d = 2$ or $d > 3$. Real-valued 3×3 and $3 \times 3 \times 3$ tensors are denoted by $\mathbb{R}^{3\times 3}$ and by $\mathbb{R}^{3\times 3\times 3}$, respectively. We adopt the symbol I for the identity matrix. With $|\cdot|$ we denote indiscriminately the Euclidean norms in $\mathbb{R}^3$, $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{3\times 3\times 3}$. To deal with plastic strains, we recall the classical notation

$$\mathsf{SL}(3) \coloneqq \{F \in \mathbb{R}^{3\times 3} : \det F = 1\}.$$

If $A \subset \mathbb{R}^3$ is a measurable set, we will denote by $\mathcal{L}^3(A)$ its three-dimensional Lebesgue measure.

A fundamental role in our study is played by the following notion of variational convergence, see the monograph [21] for a thorough treatment:

Definition 2.1. Let X be a set endowed with a notion of convergence. We say that the family $\{\mathcal{G}_\varepsilon\}$, with $\mathcal{G}_\varepsilon \colon X \to [-\infty, +\infty]$, Γ-*converges* as $\varepsilon \to 0$ to $\mathcal{G} \colon X \to [-\infty, +\infty]$ if for all $x \in X$ and all infinitesimal sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$ the following holds:

(1) for every sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ such that $x_k \to x$, we have

$$\mathcal{G}(x) \le \liminf_{k\to+\infty} \mathcal{G}_{\varepsilon_k}(x_k);$$

(2) there exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ such that $x_k \to x$ and

$$\limsup_{k\to+\infty} \mathcal{G}_{\varepsilon_k}(x_k) \le \mathcal{G}(x).$$

When X is equipped with a topology τ, we write e.g. $\Gamma(\tau)$-convergence to stress what the underlying convergence for sequences in X is. In what follows, for notational convenience, we indicate the dependence on ε_k by means of the subscript k alone, e.g. $\mathcal{J}_k \coloneqq \mathcal{J}_{\varepsilon_k}$.

Our aim is to study elastoplastic media with high-contrast periodic microstructure in the case of soft inclusions inserted in a perforated stiff matrix. To describe the geometry in precise terms, let $Q \coloneqq [0,1)^3$ be the periodicity cell, and let $Q^0 \subset Q$ be an open set such that $Q^1 \coloneqq Q \setminus \overline{Q^0}$ is connected and has a Lipschitz boundary (see Figure 1). The set Ω, which represents the region of space occupied by the composite, is then subdivided by means of the sets

$$\Omega^0_\varepsilon \coloneqq \bigcup_{t\in T_\varepsilon} \varepsilon(t + Q^0), \quad \text{with } T_\varepsilon \coloneqq \{t \in \mathbb{Z}^3 : \varepsilon(t + \overline{Q^0}) \subset \Omega\}, \tag{2.1}$$

$$\Omega^1_\varepsilon \coloneqq \Omega \setminus \overline{\Omega^0_\varepsilon}, \tag{2.2}$$

FIGURE 1. The periodicity cell Q and its partition into the soft inclusion Q^0 (white) and the stiff matrix Q^1 (gray).

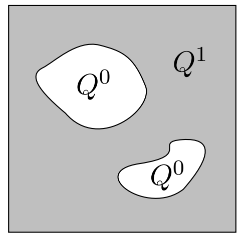
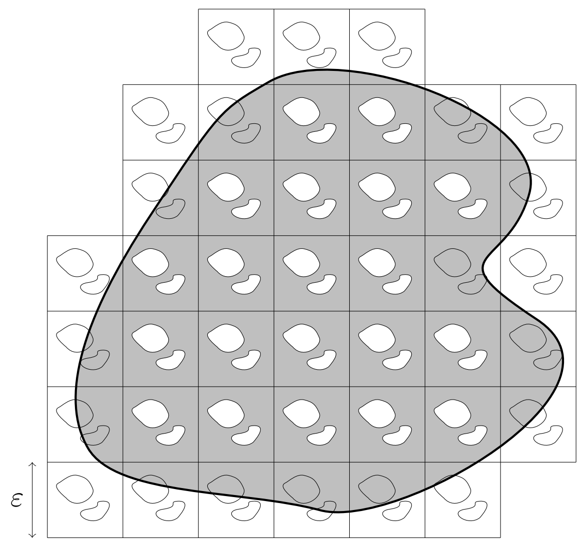


FIGURE 2. The microstructure of the composite in Ω. The soft inclusions that form Ω_ε^0 correspond to the white holes, while the grey region represents the matrix Ω_ε^1.



which stand respectively for the collection of the inclusions and for the matrix (see Figure 2). We also define the Q-periodic set

$$E^1 \coloneqq \bigcup_{t\in\mathbb{Z}^3} (t + Q^1), \tag{2.3}$$

where we say that a set $E \subset \mathbb{R}^3$ is Q-periodic if $E + t = E$ for all $t \in \mathbb{Z}^3$. Note that the set Ω_ε^1 is connected and Lipschitz, because (2.1) ensures that the inclusions are well separated from $\partial\Omega$. Our assumptions allow for some flexibility on the geometry of the inclusions, which could for instance form interconnected fibers (see Figure 3).

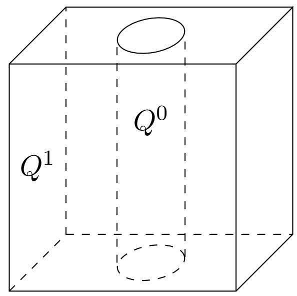
Our Γ-convergence result deals with the asymptotic behavior, as ε tends to 0, of the family $\{\mathcal{J}_\varepsilon\}$ defined by (1.1). Before stating the result, we collect the hypotheses we use in the following lines.

The elastic energy density of the stiff matrix $W^1 \colon \mathbb{R}^{3\times3} \to [0, +\infty]$ satisfies the following:

E1: It is 2-coercive and has at most quadratic growth, i.e., there exist $0 < c_1 \le c_2$ such that for all $F \in \mathbb{R}^{3\times3}$

$$c_1|F|^2 \le W^1(F) \le c_2\left(|F|^2 + 1\right).$$

FIGURE 3. In the 3-dimensional space, interconnected soft fibers do not disconnect the matrix. A simple case is depicted here: the cylindrical perforation Q^0 runs through the periodicity cell and its complement Q^1 is connected.



E2: It is 2-Lipschitz: there exists $c_3 > 0$ such that for all $F_1, F_2 \in \mathbb{R}^{3\times3}$

$$|W^1(F_1) - W^1(F_2)| \le c_3 (1 + |F_1| + |F_2|) |F_1 - F_2|.$$

The assumptions on the soft densities $W^0_\varepsilon \colon \mathbb{R}^{3\times3} \to [0, +\infty]$ are analogous:

E3: There exist $0 < c_1 \le c_2$ such that for all $F \in \mathbb{R}^{3\times3}$, and all $\varepsilon > 0$,

$$c_1|F|^2 \le W^0_\varepsilon(F) \le c_2 \left(|F|^2 + 1\right).$$

E4: There exists $c_3 > 0$ such that for all $F_1, F_2 \in \mathbb{R}^{3\times3}$, and all $\varepsilon > 0$,

$$\left|W^0_\varepsilon(F_1) - W^0_\varepsilon(F_2)\right| \le c_3 (1 + |F_1| + |F_2|) |F_1 - F_2|.$$

E5: There exists $W^0 \colon \mathbb{R}^{3\times3} \to [0, +\infty]$ such that for all $F \in \mathbb{R}^{3\times3}$

$$\lim_{\varepsilon\to0} W^0_\varepsilon(F) = W^0(F).$$

Remark 2.2. The function W^0 possesses the same growth and regularity properties of W^0_ε.

Our assumptions rule out non-impenetrability constraints at the level of the energy. A blow up of the energy on matrices with non-positive determinant is desirable from a modeling point of view, but it is at the same time very hard to be handled with in the context of homogenization. Frame indifference is instead compatible with our hypotheses and we point out that, up to a normalization, we can require all energy densities to vanish on the identity.

We list next the assumptions on the hardening $H \colon \mathbb{R}^{3\times3} \to [0, +\infty]$.

H1: Assume that a *Finsler structure* on $\mathsf{SL}(3)$ is assigned. $H(F)$ is finite if and only if $F \in K$, where $K \subset \mathsf{SL}(3)$ is a geodesically convex, compact neighborhood of I.

H2: The restriction of H to K is Lipschitz continuous.

The requirement that K is geodesically convex with respect to the Finsler structure assigned on $\mathsf{SL}(3)$ is the crucial ingredient to invoke [24, Theorem 2.2], which in our context is employed to capture the asymptotic behaviour of the stiff matrix, see Theorem 3.8. We refer to [24] for a discussion on the role of the Finsler geometry for the homogenization of elastoplastic media, and to Subsection 3.5 for a summary of the tools from that theory that we need here. In particular, the existence of a set K complying with H1 is settled in Corollary 3.11 below.

Requirement H1 prescribes that the effective domain of H coincides with a compact set K containing I. It follows then there exists $c_K > 0$ such that

$$|F| + |F^{-1}| \le c_K \quad \text{for every } F \in K, \tag{2.4}$$

because $\mathsf{SL}(3)$ is by definition well separated from 0. As a consequence, plastic strains with finite hardening are uniformly bounded in L^∞, and, in particular, we infer that for any $F \in K$ and $G \in \mathbb{R}^{3\times 3}$

$$|G| = \left|GF^{-1}F\right| \leq c_K \left|GF^{-1}\right|. \tag{2.5}$$

Remark 2.3. Note that in principle it would be reasonable to suppose that the soft and the stiff components feature different hardening behaviors. For instance, it could be imposed that the soft hardening is evaluated on an ε-rescaling of the plastic stress, thus replicating the structure of the elastic contribution. As the only available tool to deal with periodic homogenization at finite strains is [24, Theorem 2.2], we leave such scenarios for possible future investigation and we restrain ourselves to a simpler setting, namely we choose to model both hardening terms by a single function satisfying H1 and H2. We point out that under these assumptions making a distinction between $H^i = H^i(P)$, $i = 0, 1$ would not require any substantial change in our approach, therefore we dispense with it. Qualitatively, keeping the soft hardening contribution of order 1 amounts to the situation in which, for small ε, elastic deformations of a much larger magnitude than the plastic ones are allowed.

We can now state the homogenization result for high-contrast elastoplastic media. Since we want our analysis to yield convergence of minima and minimizers of $\mathcal{J}_\varepsilon$ to the ones of the limiting energy, we need to introduce a convergence that is compliant with the degeneracy of the soft inclusions. For shortness, we refer to it as *convergence in the sense of extensions*, even though the name is not at all standard.

Definition 2.4. Let $\{\varepsilon_k\}$ be an infinitesimal sequence. We say that $\{y_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ converges to $y \in W^{1,2}(\Omega; \mathbb{R}^3)$ in the sense of extensions with respect to the scales ε_k if the following hold:

(1) $\{y_k\}$ is bounded in $L^2(\Omega; \mathbb{R}^3)$;
(2) there exists a sequence $\{\tilde{y}_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ such that $y_k = \tilde{y}_k$ in $\Omega^1_k := \Omega^1_{\varepsilon_k}$ and $\tilde{y}_k \rightharpoonup y$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$.

Remark 2.5. Let $\tilde{y}_k = \tilde{y}'_k$ a.e. in Ω^1_k. Let as well $\tilde{y}_k \rightharpoonup y$ and $\tilde{y}'_k \rightharpoonup y'$ weakly in $W^{1,2}(\Omega; \mathbb{R}^3)$. Then, recalling (2.2)–(2.3) and observing that $\Omega \cap \varepsilon_k E^1 \subset \Omega^1_k$,

$$0 = \lim_{k\to+\infty} \int_{\Omega^1_k} |\tilde{y}_k - \tilde{y}'_k| \,\mathrm{d}x \geq \lim_{k\to+\infty} \int_\Omega \chi_{\varepsilon_k E^1}(x) |\tilde{y}_k - \tilde{y}'_k| \,\mathrm{d}x = c \int_\Omega |y - y'| \,\mathrm{d}x,$$

for a constant $c > 0$. From this, we conclude that $y = y'$ a.e. in Ω. In particular, if the limit in the sense of extensions exists, then it is unique.

Remark 2.6. By the definition of Ω^1_k, there exists a tubular neighborhood O of $\partial\Omega$ such that $\Omega^1_k \cap O \equiv \Omega \cap O$. Therefore, if y and $\tilde{y}$ coincide in Ω^1_k, their traces on $\partial\Omega$ are also equal.

The asymptotic behavior of the family $\{\mathcal{J}_\varepsilon\}$ with respect to the notion of convergence that we have just introduced is described in the next theorem:

Theorem 2.7. *Let $\{W^1\}$ and $\{W^0_\varepsilon\}$ satisfy E1–E5, and let H satisfy H1–H2. For all $y \in L^2(\Omega; \mathbb{R}^3)$ and $P \in L^q(\Omega; \mathsf{SL}(3))$ there exists*

$$\mathcal{J}(y, P) := \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(y, P),$$

where the underlying convergences are the one in the sense of extensions and the uniform one, respectively for the first and for the second argument. The Γ-limit is characterized as follows:

$$\mathcal{J}(y, P) = \mathcal{J}^0(0, P) + \mathcal{J}^1(y, P),$$

where

$$\mathcal{J}^0(y,P) := \begin{cases} \mathcal{L}^3(Q^0)\displaystyle\int_\Omega \Big[\mathcal{Q}'W^0(\nabla y(x), P^{-1}(x)) + H(P(x))\Big]\,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \text{if } y = 0 \text{ and } P \in W^{1,q}(\Omega;K), \\ +\infty \qquad\qquad\qquad\quad \text{otherwise in } L^2(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathsf{SL}(3)), \end{cases} \tag{2.6}$$

and

$$\mathcal{J}^1(y,P) := \begin{cases} \displaystyle\int_\Omega \Big[\widetilde{W}^1_{\mathrm{hom}}(\nabla y(x), P(x)) + \mathcal{L}^3(Q^1)H(P(x)) + |\nabla P(x)|^q\Big]\,\mathrm{d}x \\ \qquad\qquad\qquad\qquad \text{if } (y,P) \in W^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;K), \\ +\infty \qquad\qquad\qquad\quad \text{otherwise in } L^2(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathsf{SL}(3)). \end{cases} \tag{2.7}$$

Here, for $F, G \in \mathbb{R}^{3\times 3}$*,*

$$\mathcal{Q}'W^0(F,G) := \inf\left\{\int_Q W^0\Big((F+\nabla v(z))G\Big)\,\mathrm{d}z : v \in W^{1,2}_0(Q;\mathbb{R}^3)\right\}, \tag{2.8}$$

while

$$\widetilde{W}^1_{\mathrm{hom}}(F,G) := \lim_{\lambda\to+\infty}\frac{1}{\lambda^3}\inf\left\{\int_{(0,\lambda)^3\cap E^1} W^1\Big((F+\nabla y(x))G^{-1}\Big)\,\mathrm{d}x : y \in W^{1,2}_0((0,\lambda)^3;\mathbb{R}^3)\right\}.$$

The formula defining $\mathcal{Q}'W^0$ provides a variant of the classical quasiconvex envelope of W^0. We refer to Section 5 for further discussion on this point.

Remark 2.8. In principle, it cannot be excluded that some nontrivial energy densities W^0_ε do not contribute to the elastic homogenized energy, in the sense that, when finite, for the corresponding $\mathcal{J}^0$ we have

$$\mathcal{J}^0(0,P) = \mathcal{L}^3(Q^0)\int_\Omega H(P(x))\,\mathrm{d}x.$$

As an instance of this phenomenon, we consider the following example. For any $F \in \mathbb{R}^{3\times 3}$, we let $W^0_\varepsilon(F) = W^0(F) := |F|^2$. Conditions E3–E5 are satisfied by definition. Since for any fixed $G \in \mathbb{R}^{3\times 3}$ the function $F \mapsto W^0_G(F) := W^0(FG)$ is convex, it is, in particular, also quasiconvex. Hence, $\mathcal{Q}'W^0(0,G) = W^0(0,G) = W^0(0) = 0$.

As a byproduct of our asymptotic analysis, we are in a position to infer convergence of the minimum problems associated with the energy functionals and of the related (quasi) minimizers.

Corollary 2.9. *Let the same assumptions and notation of Theorem 2.7 hold, and let* $\{(y_k, P_k)\} \subset W^{1,2}_0(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;\mathsf{SL}(3))$ *be a sequence of almost minimizers, that is,*

$$\lim_{k\to+\infty}\Big(\mathcal{J}_k(y_k,P_k) - \inf \mathcal{J}_k(y,P)\Big) = 0,$$

where the infimum is taken over $W^{1,2}_0(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;\mathsf{SL}(3))$*. Then, there exists a minimizer* $(y,P) \in W^{1,2}_0(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;\mathsf{SL}(3))$ *of* $\mathcal{J}$ *such that, up to subsequences,* $y_k \to y$ *in the sense of extensions and* $P_k \to P$ *uniformly. Moreover,*

$$\inf \mathcal{J}_k \to \min \mathcal{J}.$$

The proof of Theorem 2.7 consists of three steps. First, we study the compactness properties of sequences $\{(y_\varepsilon, P_\varepsilon)\}$ satisfying $\sup_\varepsilon \mathcal{J}_\varepsilon(y_\varepsilon, P_\varepsilon) \le C$, and characterize their limits. Second, we show that the two components of the material can be studied independently. Finally, we perform the analysis of each single component. In view of this approach, it is useful to introduce the functionals that account for the two different contributions, namely

$$\mathcal{E}^0_\varepsilon(y,P) \coloneqq \int_\Omega \chi^0_\varepsilon(x)\Big[W^0_\varepsilon\left(\varepsilon\nabla y(x)P^{-1}(x)\right) + H(P(x))\Big]\mathrm{d}x, \tag{2.9}$$

$$\mathcal{E}^1_\varepsilon(y,P) \coloneqq \int_\Omega \chi^1_\varepsilon(x)\Big[W^1\left(x, \nabla y(x)P^{-1}(x)\right) + H(P(x))\Big]\mathrm{d}x, \tag{2.10}$$

where, for $i = 0, 1$, $\chi^i_\varepsilon(x)$ denotes the characteristic function of Ω^i_ε, i.e. $\chi^i_\varepsilon(x) = 1$ if $x \in \Omega^i_\varepsilon$ and $\chi^i_\varepsilon(x) = 0$ otherwise. We also decompose the functional $\mathcal{J}_\varepsilon$ accordingly:

$$\mathcal{J}_\varepsilon = \mathcal{J}^0_\varepsilon + \mathcal{J}^1_\varepsilon,$$

with

$$\mathcal{J}^0_\varepsilon(y,P) \coloneqq \begin{cases} \mathcal{E}^0_\varepsilon(y,P) & \text{if } (y,P) \in W^{1,2}_0(\Omega^0_\varepsilon;\mathbb{R}^3) \times W^{1,q}(\Omega;K), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3)), \end{cases} \tag{2.11}$$

$$\mathcal{J}^1_\varepsilon(y,P) \coloneqq \begin{cases} \mathcal{E}^1_\varepsilon(y,P) + \|\nabla P\|^q_{L^q(\Omega;\mathbb{R}^{3\times3\times3})} & \text{if } (y,P) \in W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;K), \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3)). \end{cases} \tag{2.12}$$

In contrast to $\mathcal{J}^1_\varepsilon(y,P)$, whose asymptotic behavior is derived from [24, Theorem 2.2], the soft part requires a dedicated treatment. This happens already in the setting of nonlinear elasticity (see [13]). Recall the topology τ in (3.5). We obtain the following:

Proposition 2.10. *Let $(v,P) \in W^{1,2}_0(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;\mathsf{SL}(3))$. For an infinitesimal sequence $\{\varepsilon_k\}$, consider $\mathcal{J}^0_k$ and $\mathcal{J}^0$ as in (2.11) and (2.6), respectively.*

(1) For every sequence $\{(v_k,P_k)\} \subset W^{1,2}_0(\Omega^0_k;\mathbb{R}^3) \times W^{1,q}(\Omega;\mathsf{SL}(3))$ such that $(\varepsilon_k v_k, P_k) \xrightarrow{\tau} (v,P)$ we have

$$\mathcal{J}^0(v,P) \le \liminf_{k\to+\infty} \mathcal{J}^0_k(v_k,P_k),$$

provided that $\{v_k\}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$ and that $\{\varepsilon_k\nabla v_k\}$ is 2-equiintegrable.

(2) There exists a sequence $\{v_k\} \subset W^{1,2}_0(\Omega^0_k;\mathbb{R}^3)$ such that $\varepsilon_k v_k \to v$ in $L^2(\Omega;\mathbb{R}^3)$ and that

$$\limsup_{k\to+\infty} \mathcal{J}^0_k(v_k,P_k) \le \mathcal{J}^0(v,P),$$

provided $P_k \to P$ uniformly.

In the statement above, the space $W^{1,2}_0(\Omega^0_\varepsilon;\mathbb{R}^3)$ is regarded for each ε as a subset of $W^{1,2}(\Omega;\mathbb{R}^3)$ by extending its elements to 0 on Ω^1_ε.

Remark 2.11. Let $\Omega \subset \mathbb{R}^3$ be bounded Lipschitz domain and, for $p > 1$, let us consider the local integral functionals on $W^{1,p}(\Omega;\mathbb{R}^3)$

$$v \mapsto \int_\Omega W_k(\nabla v)\,\mathrm{d}x.$$

If the energy densities $\{W_k\}$ satisfy standard p-growth conditions, as a consequence of Rellich-Kondrachov theorem, the Γ-limits with respect to the strong L^p-convergence and with respect to the weak $W^{1,p}$-convergence coincide (if they exist).

For the sequence of functionals

$$v \mapsto \int_\Omega W_k(\varepsilon_k \nabla v)\,\mathrm{d}x, \tag{2.13}$$

again under standard growth conditions for $\{W_k\}$, the analysis is more delicate. The natural bound that follows from the p-coercivity is $\|\varepsilon_k \nabla v_k\|_{L^p} \le C$, and it suggests the use of weak two-scale convergence (see Subsection 3.3). However, this estimate alone is not enough to deduce convergence of the sequence $\{v_k\}$: a further control on the ε-difference quotients is required to guarantee that a two-scale variant of Rellich-Kondrachov theorem holds (see [44, Theorem 4.4]).

In other words, in our degenerate setting, compactness of sequences of gradients, say $\{\varepsilon_k \nabla v_k\}$, does not bring compactness of $\{v_k\}$. This explains why in Proposition 2.10 we need to require a bound also for $\|v_k\|_{L^2}$ in order to establish the lower limit inequality.

We note incidentally that, by means of Lemma 3.6(4) below, it can be shown that the Γ-limit of the functionals (2.13) with respect to the strong two-scale convergence in L^p of $\{v_k\}$ is the same as the one computed by combining the latter convergence and the weak two-scale convergence of $\{\varepsilon_k \nabla v_k\}$. Those are not suitable choices for our goals, though, because, as we commented above, they do not match the natural compactness of the problem. This explains why in [13], where strong two-scale convergence is considered, the asymptotic behavior of minimum problems is not immediately determined by the Γ-convergence (see [13, Sec. 10]). We also refer to the Appendix for a comparison between our findings and the ones in [13].

3. Preliminaries

We gather in this section the technical tools to be employed in the sequel.

3.1. A decomposition lemma.

In our analysis of heterogeneous media it will be often desirable to disregard the energy contributions arising from the region close to $\partial\Omega$, for the composite fails to be periodic there (recall positions (2.1)–(2.2)). To this aim, it is natural to resort to p-equiintegrability arguments, because such boundary strip has small measure. We recall that a family $\mathcal{C} \subset L^p(\Omega;\mathbb{R}^3)$ is said to be p-equiintegrable if for all $\delta > 0$ there exists $m > 0$ such that

$$\sup_{u\in\mathcal{C}} \int_E |u|^p\,\mathrm{d}x < \delta \quad \text{whenever } E \subset \Omega \text{ satisfies } \mathcal{L}^3(E) < m.$$

The ensuing lemma grants that for any bounded sequence in L^p we can always find another one which is p-equiintegrable and "does not differ too much" from the given one.

Lemma 3.1 (Theorem 2.20 in [3]; see also Lemma 1.2 in [29])**.** *Let Ω be as in Section 2. For any sequence $\{v_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $v_k \rightharpoonup v$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ there exist a subsequence $\{k_j\}$ and a sequence $\{u_j\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying the following:*

(1) $u_j \rightharpoonup v$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$;
(2) $u_j = v$ in a neighborhood of $\partial\Omega$;
(3) $\{\nabla u_j\}$ is 2-equiintegrable;
(4) $\lim_{j\to+\infty} \mathcal{L}^3(\{x \in \Omega : v_{k_j}(x) \neq u_j(x)\}) = 0$.

Point (4) yields $\lim_{j\to+\infty} \mathcal{L}^3(\{\nabla v_{k_j} \neq \nabla u_j\}) = 0$, because by standard properties of Sobolev functions (see e.g. [30, Lemma 7.7]) the inclusion $\{v_{k_j} \neq u_j\} \supseteq \{\nabla v_{k_j} \neq \nabla u_j\}$ holds true.

3.2. **A couple of tools to deal with periodic heterogeneous media.** The periodic geometry of the composite calls for an extension result for Sobolev maps on perforated domains. Since the perforations of the matrix are well detached from the boundary, by applying [9, Lemma B.7] the following can be proved:

Lemma 3.2 (Lemma 8 in [13])**.** *Let Ω be open and bounded, and let Ω^1_ε be as in Section 1. There exists a linear and continuous extension operator*

$$\mathsf{T}_\varepsilon \colon W^{1,2}(\Omega^1_\varepsilon;\mathbb{R}^3) \to W^{1,2}(\Omega;\mathbb{R}^3)$$

such that for all $y \in W^{1,2}(\Omega^1_\varepsilon;\mathbb{R}^3)$

$$\begin{gathered}\mathsf{T}_\varepsilon y = y \quad a.\,e.\ in\ \Omega^1_\varepsilon,\\ \|\mathsf{T}_\varepsilon y\|_{L^2(\Omega;\mathbb{R}^3)} \le c\,\|y\|_{L^2(\Omega^1_\varepsilon;\mathbb{R}^3)},\\ \|\nabla(\mathsf{T}_\varepsilon y)\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \le c\,\|\nabla y\|_{L^2(\Omega^1_\varepsilon;\mathbb{R}^{3\times3})},\end{gathered}$$

where c is independent of ε and Ω.

Remark 3.3. Even though the lemma above is a classical result, it is worth clarifying the way we employ it.

In the sequel, we always work with sequences which are already defined on the whole Ω. When we apply Lemma 3.2 to such a sequence, say $\{y_\varepsilon\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$, it is tacitly understood that the functions that are extended are the restrictions $y_\varepsilon \llcorner \Omega^1_\varepsilon$. So, in a sense, the process modifies y_ε on the region occupied by the soft inclusions rather than extending it. Note that the modification is a true one, because T_ε cannot be the identity. The two crucial points for our analysis are that

(1) if $\{y_\varepsilon \llcorner \Omega^1_\varepsilon\}$ and $\{\nabla y_\varepsilon \llcorner \Omega^1_\varepsilon\}$ are bounded in L^2, then $\{\mathsf{T}_\varepsilon y_\varepsilon\}$ is bounded in $W^{1,2}(\Omega;\mathbb{R}^3)$;
(2) if $\{y_\varepsilon\}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$ and $\{\nabla y_\varepsilon\}$ is a 2-equiintegrable sequence, then $\{\nabla(\mathsf{T}_\varepsilon y_\varepsilon)\}$ is 2-equiintegrable as well.

The second point follows from the construction of T_ε, which is modeled on the proof of [9, Lemma B.8] by patching together the extensions from $W^{1,2}(Q^1;\mathbb{R}^3)$ to $W^{1,2}(Q;\mathbb{R}^3)$ given by [9, Lemma B.7] via partitions of unity (this is also the reason why the constant c above depends only on Q^1 and Q). The extensions in [9, Lemma B.7] preserve equiintegrability, because they rely on the classical reflection procedure.

The first application of the extension lemma is the following Poincaré inequality on periodic heterogeneous media (cf. formula (4.5) in [2] where, however, the proof is not provided).

Proposition 3.4. *Let Ω, Ω^0_ε and Ω^1_ε be as in Section 1. There exists a constant c independent of ε, and such that for every $y \in W^{1,2}_0(\Omega;\mathbb{R}^3)$*

$$\|y\|_{L^2(\Omega;\mathbb{R}^3)} \le c\left(\varepsilon\|\nabla y\|_{L^2(\Omega^0_\varepsilon;\mathbb{R}^{3\times3})} + \|\nabla y\|_{L^2(\Omega^1_\varepsilon;\mathbb{R}^{3\times3})}\right).$$

Proof. For ε fixed, we use the extension operator T_ε from Lemma 3.2 to obtain

$$\begin{aligned}\|y\|_{L^2} &\le \|y-\mathsf{T}_\varepsilon y\|_{L^2} + \|\mathsf{T}_\varepsilon y\|_{L^2}\\ &= \|y-\mathsf{T}_\varepsilon y\|_{L^2(\Omega^0_\varepsilon)} + \|\mathsf{T}_\varepsilon y\|_{L^2}.\end{aligned} \tag{3.1}$$

Observe that $\mathsf{T}_\varepsilon y \in W^{1,2}_0(\Omega;\mathbb{R}^3)$, as $\mathsf{T}_\varepsilon y = y$ a. e. in Ω^1_ε and there exists a tubular neighborhood O of $\partial\Omega$ such that $\Omega^1_\varepsilon \cap O \equiv \Omega \cap O$. Then, by the standard Poincaré's inequality,

$$\|\mathsf{T}_\varepsilon y\|_{L^2} \le c\|\nabla(\mathsf{T}_\varepsilon y)\|_{L^2} \le c\|\nabla y\|_{L^2(\Omega^1_\varepsilon)}. \tag{3.2}$$

Observe that $y - \mathsf{T}_\varepsilon y \in W_0^{1,2}(\Omega_\varepsilon^0;\mathbb{R}^3)$ as well. In view of the periodic structure of Ω_ε^0 and of Poincaré inequality on each cube, we infer

$$\begin{aligned}\|y - \mathsf{T}_\varepsilon y\|^2_{L^2(\Omega_\varepsilon^0)} &= \sum_{t\in T_\varepsilon} \|y - \mathsf{T}_\varepsilon y\|^2_{L^2(\varepsilon(t+D_0))}\\ &= \sum_{t\in T_\varepsilon} \varepsilon^3 \int_{D_0} |y(\varepsilon(t+z)) - \mathsf{T}_\varepsilon y(\varepsilon(t+z))|^2\,\mathrm{d}z\\ &\le c\sum_{t\in T_\varepsilon} \varepsilon^5 \int_{D_0} |\nabla(y - \mathsf{T}_\varepsilon y)(\varepsilon(t+z))|^2\,\mathrm{d}z\\ &= c\varepsilon^2 \|\nabla(y - \mathsf{T}_\varepsilon y)\|^2_{L^2(\Omega_\varepsilon^0)},\end{aligned}$$

where c depends only on D_0. By applying again Lemma 3.2 we find

$$\|y - \mathsf{T}_\varepsilon y\|_{L^2(\Omega_\varepsilon^0)} \le c\left(\varepsilon\|\nabla y\|_{L^2(\Omega_\varepsilon^0)} + \|\nabla y\|_{L^2(\Omega_\varepsilon^1)}\right).$$

This, together with (3.1) and (3.2), yields the result. □

3.3. **Two-scale convergence and the unfolding method.** From a mathematical perspective, the high-contrast structure of the functional $\mathcal{J}_\varepsilon$ results in the absence of uniform bounds in L^2 for sequences with equibounded energy; indeed, only bounds on $\{\varepsilon\nabla y_\varepsilon P_\varepsilon^{-1}\}$ are available. Such degenerate bounds are conveniently dealt with by means of two-scale convergence [2, 41], whose definition we recall next. Hereafter, the subscript per denotes spaces of Q-periodic functions, e.g.

$$W^{1,2}_{\mathrm{per}}(\mathbb{R}^3) := \{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3) : u(x+t) = u(x) \text{ a.e. for all } t \in \mathbb{Z}^3\}.$$

Definition 3.5. Let $\{\varepsilon_k\} \subset (0,+\infty)$ be infinitesimal. A sequence $\{y_k\} \subset L^2(\Omega;\mathbb{R}^3)$ *weakly two-scale converges in* L^2 to a function $y \in L^2(\Omega; L^2_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$ if for every $v \in L^2(\Omega; C_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$

$$\lim_{k\to+\infty} \int_\Omega y_k(x)\cdot v\left(x, \frac{x}{\varepsilon_k}\right)\mathrm{d}x = \int_\Omega\int_Q y(x,z)\cdot v(x,z)\,\mathrm{d}z\,\mathrm{d}x.$$

A sequence $\{y_k\} \subset L^2(\Omega;\mathbb{R}^3)$ *strongly two-scale converges in* L^2 to $y \in L^2(\Omega; L^2_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$ if $y_k \overset{2}{\rightharpoonup} y$ in L^2 and $\|y_k\|_{L^2(\Omega;\mathbb{R}^3)} \to \|y\|_{L^2(\Omega\times Q;\mathbb{R}^3)}$. We use the notations $y_k \overset{2}{\rightharpoonup} y$ and $y_k \overset{2}{\to} y$ for the weak and strong two-scale convergence, respectively.

Recalling that for $i = 0, 1$ $\chi_k^i(x) = 1$ if $x \in \Omega_k^i$ and $\chi_k^i(x) = 0$ otherwise, an example of strong two-scale convergence is provided by the sequences $\{\chi_k^0\}$ and $\{\chi_k^1\}$. Indeed,

$$\chi_k^i \overset{2}{\to} \chi^i \quad \text{strongly two-scale in } L^2, \tag{3.3}$$

where $\chi^i(x,z) := \chi_{Q^i}(z)$ for all $(x,z) \in \Omega\times Q$.

We collect in the next lemma some basic properties of two-scale convergence which we will resort to in the following. Proofs and more details can be found in [2, 43, 44].

Lemma 3.6. *Let $\{\varepsilon_k\} \subset (0,+\infty)$ be infinitesimal and consider $\{y_k\} \subset L^2(\Omega;\mathbb{R}^3)$.*

(1) If $\{y_k\}$ is weakly two-scale convergent, then it is bounded in $L^2(\Omega;\mathbb{R}^3)$; conversely, if $\{y_k\}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$, then it admits a weakly two-scale convergent subsequence.

(2) If $y_k \overset{2}{\rightharpoonup} y$ weakly two-scale in L^2, then $y_k \rightharpoonup \int_Q y(\,\cdot\,,z)\,\mathrm{d}z$ weakly in $L^2(\Omega;\mathbb{R}^3)$.

(3) If $y_k \overset{2}{\rightharpoonup} y$ weakly two-scale in L^2 and if $\{u_k\} \subset L^2(\Omega;\mathbb{R}^3)$ converges to u strongly two-scale in L^2, then $y_k u_k \overset{2}{\rightharpoonup} yu$ weakly two-scale in L^2.

(4) Suppose that $\{y_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *and that* $\{y_k\}$ *and* $\{\varepsilon_k \nabla y_k\}$ *are bounded in* L^2*. Then, there exists* $y \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$ *such that, up to subsequences,* $y_k \xrightharpoonup{2} y$ *and* $\varepsilon_k \nabla y_k \xrightharpoonup{2} \nabla_z y$ *weakly two-scale in* L^2*.*

Two-scale convergence in L^2 can be related to L^2 convergence by means of *unfolding operator*, which, for $\varepsilon > 0$, is the map $\mathsf{S}_\varepsilon \colon L^2(\Omega) \to L^2(\mathbb{R}^3; L^2_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$ defined as

$$\mathsf{S}_\varepsilon y(x,z) := \hat{y}\Big(\varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon z\Big), \tag{3.4}$$

where $\hat{y}$ denotes the extension of y by 0 outside Ω.

Lemma 3.7. *If* $\{y_\varepsilon\} \subset L^2(\Omega;\mathbb{R}^3)$ *is bounded, the following hold:*

(1) $y_\varepsilon \xrightharpoonup{2} y$ *weakly two-scale in* L^2 *if and only if* $\mathsf{S}_\varepsilon y_\varepsilon \rightharpoonup y$ *weakly in* $L^2(\mathbb{R}^3 \times Q;\mathbb{R}^3)$*;*

(2) $y_\varepsilon \xrightarrow{2} y$ *strongly two-scale in* L^2 *if and only if* $\mathsf{S}_\varepsilon y_\varepsilon \to y$ *strongly in* $L^2(\mathbb{R}^3 \times Q;\mathbb{R}^3)$*.*

In addition, if $\{y_\varepsilon\}$ *is* 2*-equiintegrable, the family of unfoldings* $\{\mathsf{S}_\varepsilon y_\varepsilon\}$ *is as well* 2*-equiintegrable on* $\mathbb{R}^3 \times Q$*. Lastly, if* $y \in W^{1,2}(\Omega;\mathbb{R}^3)$*, then*

$$\mathsf{S}_\varepsilon(\varepsilon \nabla y)(x,z) = \nabla_z(\mathsf{S}_\varepsilon y)(x,z).$$

For a proof of Lemma 3.7 and for further reading on the unfolding operator we refer to [43, 44, 16, 17].

3.4. **Homogenization of connected media in finite plasticity.** We present a variant of [24, Theorem 2.2] that is instrumental in dealing with the analysis of the stiff matrix. Its proof is an adaptation of the one in [24], the most substantial difference being the use of [9, Theorem 19.1] instead of [9, Theorem 14.5].

We work in the space $W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;\mathsf{SL}(3))$ endowed with the topology τ characterized by

$$(y_k, P_k) \xrightarrow{\tau} (y,P) \quad \text{if and only if} \quad \begin{cases} y_k \to y & \text{strongly in } L^2(\Omega;\mathbb{R}^3), \\ P_k \to P & \text{uniformly.} \end{cases} \tag{3.5}$$

Theorem 3.8. *Let* E *be an open and connected set that is* Q*-periodic and that has Lipschitz boundary. For every* $(y,P) \in W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;K)$*, let*

$$\widetilde{W}(x,F) := \chi_E(x) W^1(F), \qquad \widetilde{H}(x,P) := \chi_E(x) H(P),$$

and define

$$\mathcal{F}_\varepsilon(y,P) := \int_\Omega \widetilde{W}\left(\frac{x}{\varepsilon}, \nabla y(x) P^{-1}(x)\right) \mathrm{d}x + \int_\Omega \widetilde{H}\left(\frac{x}{\varepsilon}, P(x)\right) \mathrm{d}x + \int_\Omega |\nabla P(x)|^q \,\mathrm{d}x, \tag{3.6}$$

which we extend by setting

$$\mathcal{F}_\varepsilon(y,P) = +\infty \quad \text{on } \big[L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3))\big] \setminus \big[W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;K)\big].$$

If W^1 *and* H *satisfy E1–E2 and H1–H2, respectively, then for all* $(y,P) \in L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3))$ *the* Γ*-limit*

$$\mathcal{F}(y,P) := \Gamma(\tau)\text{-}\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(y,P)$$

exists and we have that

$$\mathcal{F}(y,P) = \begin{cases} \displaystyle\int_\Omega \Big(\widetilde{W}_{\mathrm{hom}}(\nabla y(x), P(x)) + \widetilde{H}_{\mathrm{hom}}(P(x)) + |\nabla P(x)|^q\Big) \,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad \text{if } (y,P) \in W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;K), \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise in } L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3)), \end{cases}$$

where $\widetilde{W}_{\mathrm{hom}}\colon \mathbb{R}^{3\times 3}\times K \to [0,+\infty)$ *and* $\widetilde{H}_{\mathrm{hom}}\colon K\to[0,+\infty)$ *are defined as*

$$\widetilde{W}_{\mathrm{hom}}(F,G) \coloneqq \lim_{\lambda\to+\infty}\frac{1}{\lambda^3}\inf\left\{\int_{(0,\lambda)^3}\widetilde{W}(x,(F+\nabla y(x))G^{-1})\,\mathrm{d}x : y\in W^{1,2}_0((0,\lambda)^3;\mathbb{R}^3)\right\},$$

$$\widetilde{H}_{\mathrm{hom}}(F)\coloneqq\int_Q \widetilde{H}(z,F)\,\mathrm{d}z.$$

We observe that the theorem above is similar in spirit to homogenization results for perforated domains. The case at stake is however different, in that later we will deal with functions defined on the nonperforated domain Ω. This makes the analysis simpler because it spares us the need of extending $\mathsf{SL}(3)$-valued Sobolev maps.

Thanks to Lemma 3.1, we are able to refine the choice of recovery sequences for $\mathcal{F}$. This will come in handy in the proof of Corollary 2.9.

Corollary 3.9. *Under the same assumptions of Theorem 3.8, for any* $(y,P)\in W^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;K)$ *there exists a recovery sequence* (y_k,P_k) *for* $\mathcal{F}(y,P)$ *satisfying the following:*

- *(1)* $y_k \rightharpoonup y$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^3)$*;*
- *(2)* $y_k = y$ *in a neighborhood of* $\partial\Omega$*;*
- *(3)* $\{\nabla y_k\}$ *is* 2*-equiintegrable.*

Proof. Let $\{(w_k,P_k)\}$ be a recovery sequence for $\mathcal{F}(y,P)$ as provided by Theorem 3.8. We apply Lemma 3.1 to $\{w_k\}$. We deduce the existence of sequences $\{k_j\}$ and $\{u_j\}\subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that the sequence defined by

$$y_k \coloneqq \begin{cases} u_j & \text{if } k=k_j \text{ for some } j\in\mathbb{N},\\ y & \text{otherwise}\end{cases}$$

satisfies properties (1)–(3) and $(y_k,P_k)\xrightarrow{\tau}(y,P)$. Moreover

$$\lim_{j\to+\infty}\mathcal{L}^3(N_j)=0,$$

where $N_j\coloneqq\{x\in\Omega : w_{k_j}(x)\neq u_j(x)\}$.

We are left to prove that $\{(y_k,P_k)\}$ satisfies the upper limit inequality. Loosely speaking, this is a consequence of the fact that passing to a 2-equiintegrable sequence "does not increase the energy". Upon passing to a subsequence, which we do not relabel, we can assume that $\{\mathcal{F}_k(y_k,P_k)\}$ is convergent. We provisionally focus just on the elastic and hardening parts of the energy $\mathcal{F}_{k_j}$. It holds

$$\begin{aligned}
&\int_\Omega\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla w_{k_j}P_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x\\
&\quad=\int_{N_j}\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla w_{k_j}P_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x\\
&\qquad+\int_{\Omega\setminus N_j}\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla u_jP_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x\\
&\quad\geq\int_{\Omega\setminus N_j}\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla u_jP_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x,
\end{aligned}$$

so that

$$\begin{aligned}\limsup_{j\to+\infty}&\int_\Omega\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla w_{k_j}P_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x\\&\geq\limsup_{j\to+\infty}\int_{\Omega\setminus N_j}\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla u_jP_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x\\&=\limsup_{j\to+\infty}\int_{\Omega}\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla u_jP_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x,\end{aligned}$$

where the equality follows from the growth condition E1 and from the 2-equiintegrability of $\{\nabla u_j\}$ (recall that $\sup_{k\in\mathbb{N}}\|P_k^{-1}\|_\infty\leq C$), together with the boundedness of H. Therefore, coming back to the full functional $\mathcal{F}_{k_j}$,

$$\begin{aligned}&\lim_{j\to+\infty}\mathcal{F}_{k_j}(w_{k_j},P_{k_j})\\&\geq\limsup_{j\to+\infty}\int_\Omega\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla w_{k_j}P_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x+\liminf_{j\to+\infty}\int_\Omega|\nabla P_{k_j}|^q\,\mathrm{d}x\\&\geq\limsup_{j\to+\infty}\int_\Omega\left[W\left(\frac{x}{\varepsilon_{k_j}},\nabla u_jP_{k_j}^{-1}\right)+H\left(\frac{x}{\varepsilon_{k_j}},P_{k_j}\right)\right]\mathrm{d}x+\liminf_{j\to+\infty}\int_\Omega|\nabla P_{k_j}|^q\,\mathrm{d}x\\&\geq\lim_{j\to+\infty}\mathcal{F}_{k_j}(u_j,P_{k_j}).\end{aligned}\tag{3.7}$$

Recalling that $\{(w_k,P_k)\}$ is a recovery sequence, we find

$$\lim_{k\to+\infty}\mathcal{F}_k(y_k,P_k)=\lim_{j\to+\infty}\mathcal{F}_{k_j}(u_j,P_{k_j})\leq\lim_{j\to+\infty}\mathcal{F}_{k_j}(w_{k_j},P_{k_j})=\mathcal{F}(y,P),$$

which in turn yields that $\{(y_k,P_k)\}$ is also a recovery sequence. □

3.5. **Finsler structure on $\mathsf{SL}(3)$.** In order to apply the results on homogenization of elastoplastic media in [24] we endow $\mathsf{SL}(3)$ with a Finsler structure. In doing so, we follow [38], whose approach is based on the notion of *plastic dissipation*. Such line of thought links the geometry of $\mathsf{SL}(3)$ to the physics of the system under consideration, and allows to conveniently include dissipation effects in the model, see Subsection 6.3.

We start from the observation that $\mathsf{SL}(3)$ is a smooth manifold with respect to the topology induced by the inclusion in $\mathbb{R}^{3\times3}$. For every $F\in\mathsf{SL}(3)$ the tangent space at F is characterized as

$$\mathrm{T}_F\mathsf{SL}(3)=F\mathsf{sl}(3)\coloneqq\{FM\in\mathbb{R}^{3\times3}:\mathrm{tr}M=0\},$$

and, in particular, $\mathrm{T}_I\mathsf{SL}(3)$ coincides with $\mathsf{sl}(3)\coloneqq\{M\in\mathbb{R}^{3\times3}:\mathrm{tr}M=0\}$. To the purpose of endowing $\mathsf{SL}(3)$ with a Finsler structure, we first consider a C^2 function $\Delta_I\colon\mathsf{sl}(3)\to[0,+\infty)$, on which we make the following assumptions:

D1: It is positively 1-homogeneous: $\Delta_I(cM)=c\Delta_I(M)$ for all $c\geq0$ and $M\in\mathsf{sl}(3)$;
D2: It is 1-coercive and has at most linear growth: there exist $0<c_4\leq c_5$ such that for all $M\in\mathsf{sl}(3)$

$$c_4|M|\leq\Delta_I(M)\leq c_5|M|.$$

D3: It is strictly convex.

Note that we consider more restrictive regularity assumptions than the ones in [38], because we appeal to results of differential geometry, where smoothness is customarily required. The drawback of this choice is that in our analysis we cannot encompass some models, such as single crystal plasticity. However, on the positive side, our assumptions cover Von Mises plasticity, see [33, 38].

Let $\mathsf{TSL}(3)$ denote the tangent bundle to $\mathsf{SL}(3)$. We can "translate" Δ_I to the tangent spaces other than $\mathsf{sl}(3)$ by setting

$$\begin{array}{rccc}\Delta\colon & \mathsf{TSL}(3) & \to & [0,+\infty) \\ & (F,M) & \mapsto & \Delta_I(F^{-1}M).\end{array} \tag{3.8}$$

Then, it can be proved that $(\mathsf{SL}(3),\Delta)$ is a C^2 Finsler manifold. For an introduction to Finsler geometry we refer to the monograph [4].

Next, we introduce the family $\mathcal{C}(F_0,F_1)$ of piecewise C^2 curves $\Phi\colon[0,1]\to\mathsf{SL}(3)$ such that $\Phi(0)=F_0$ and $\Phi(1)=F_1$. We set

$$D(F_0,F_1)\coloneqq\inf\left\{\int_0^1\Delta\big(\Phi(t),\dot\Phi(t)\big)\mathrm{d}t:\Phi\in\mathcal{C}(F_0,F_1)\right\}, \tag{3.9}$$

where $\dot\Phi$ is the velocity of the curve. The function D provides a non-symmetric distance on $\mathsf{SL}(3)$: it is positive, attains 0 if and only if it is evaluated on the diagonal of $\mathsf{SL}(3)\times\mathsf{SL}(3)$, and satisfies the triangular inequality; in general, however, $D(F_0,F_1)\neq D(F_1,F_0)$.

By the direct method of the calculus of variations (cf. [38, Theorem 5.1]) it can be proved that for every $F_0,F_1\in\mathsf{SL}(3)$ there exists a curve $\Phi\in C^{1,1}([0,1];\mathsf{SL}(3))$ such that $\Phi(0)=F_0$, $\Phi(1)=F_1$ and

$$D(F_0,F_1)=\int_0^1\Delta\big(\Phi(t),\dot\Phi(t)\big)\,\mathrm{d}t. \tag{3.10}$$

We call such Φ a shortest path between F_0 and F_1. We need the following local uniqueness result for shortest paths, which wraps up the content of [4, Exercises 6.3.3].

Proposition 3.10. *For any point F in the Finsler manifold $\mathsf{SL}(3)$ there exists a relatively compact neighborhood U of F such that for any $F_0,F_1\in U$ there exists a unique shortest path Φ joining F_0 and F_1, and such path depends smoothly on its endpoints F_0 and F_1.*

From Proposition 3.10 we deduce the existence of a set K as in H1, but we first need to recall some terminology from differential geometry. A *geodesic* between F_0 and F_1 is a path that is a critical point of the length functional under variations that do not alter the endpoints. When for any couple of points in a given subset S of a Finsler manifold there is a unique shortest path contained in S joining those points, we say that S is *geodesically convex.*

Corollary 3.11. *Assume that a C^2 Finsler structure on $\mathsf{SL}(3)$ is assigned. Then, there exists a geodesically convex, compact neighborhood of I.*

Proof. Owing to Proposition 3.10, there exists a relatively compact neighborhood U of $I\in\mathsf{SL}(3)$ such that for any $F_0,F_1\in U$ there is a unique shortest path Φ joining F_0 and F_1. Thanks to a Finsler variant of a theorem by Whitehead [4, Exercise 6.4.3], there is an open neighborhood V of I that is compactly contained in U and geodesically convex. Let us set $K\coloneqq\bar V$. Since $K\subset U$, there is a unique shortest path Φ from F_0 to F_1 for any $F_0,F_1\in K$. The fact that K is geodesically convex as well may be proved by the same argument that proves that the closure of a convex set is still convex. □

4. Compactness and splitting

From now on we turn to the analysis of the high-contrast energy in (1.1). We investigate in this section the compactness properties of sequences with equibounded energy. We will see that, as a consequence of the behavior of the hardening functional H, we can reduce the problem to the case of pure elasticity addressed by K. Cherdantsev & M. Cherednichenko [13], and we adapt their approach.

Lemma 4.1 (Compactness)**.** *Let $\{\varepsilon_k\}$ be an infinitesimal sequence. We suppose that $\{(y_k, P_k)\}_{k\in\mathbb{N}} \subset W^{1,2}(\Omega;\mathbb{R}^3) \times W^{1,q}(\Omega;\mathsf{SL}(3))$ satisfies*

$$\|y_k\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \qquad \mathcal{J}_k(y_k, P_k) \leq C$$

for some $C \geq 0$, uniformly in k. Let us denote by $\tilde{y}_k$ the extension of y_k in the sense of Remark 3.3 above. Then, there exist subsequences of $\{\varepsilon_k\}$, $\{y_k\}$, and $\{P_k\}$, which we do not relabel, as well as $y \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$, $y^1 \in W^{1,2}(\Omega;\mathbb{R}^3)$, $v \in L^2(\Omega; W^{1,2}_0(Q^0;\mathbb{R}^3))$, and $P \in W^{1,q}(\Omega;\mathsf{SL}(3))$ such that the following hold:

$$y(x,z) = y^1(x) + v(x,z) \quad \text{for a. e. } (x,z) \in \Omega \times Q, \tag{4.1}$$

$$y_k \xrightharpoonup{2} y \quad \text{weakly two-scale in } L^2, \tag{4.2}$$

$$\varepsilon_k \nabla y_k \xrightharpoonup{2} \nabla_z v \quad \text{weakly two-scale in } L^2, \tag{4.3}$$

$$\tilde{y}_k \rightharpoonup y^1 \quad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3), \tag{4.4}$$

$$P_k \to P, \quad P_k^{-1} \to P^{-1} \quad \text{weakly in } W^{1,q}(\Omega;\mathsf{SL}(3)) \text{ and uniformly in } C(\bar{\Omega};\mathsf{SL}(3)),$$
$$\nabla \tilde{y}_k P_k^{-1} \rightharpoonup \nabla y^1 P^{-1} \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{4.5}$$

Proof. From the definition of $\mathcal{J}_k$, for all $k \in \mathbb{N}$

$$\|\nabla P_k\|_{L^q} \leq C. \tag{4.6}$$

Besides, for all k, hypothesis E3, the definition of H and the bound (2.4) imply

$$\left\|\varepsilon_k \chi_k^0 \nabla y_k P_k^{-1}\right\|_{L^2} + \left\|\chi_k^1 \nabla y_k P_k^{-1}\right\|_{L^2} \leq C, \tag{4.7}$$

$$\|P_k\|_{L^\infty} + \left\|P_k^{-1}\right\|_{L^\infty} \leq C. \tag{4.8}$$

Thanks to (2.5), from the first estimate we deduce

$$\left\|\varepsilon_k \chi_k^0 \nabla y_k\right\|_{L^2} + \left\|\chi_k^1 \nabla y_k\right\|_{L^2} \leq C, \tag{4.9}$$

which is precisely formula (21) in [13]. Thus, for what concerns the sequence of deformations, the same bounds as the purely elastic case are retrieved. While referring to [13] for details, here we limit ourselves to sketch how (4.9) entails two-scale compactness.

The boundedness of $\{y_k\}$ in L^2 and Lemma 3.6(4) yield the existence of a function $y \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$ such that, up to subsequences, (4.2) holds and

$$\varepsilon_k \nabla y_k \xrightharpoonup{2} \nabla_z y \quad \text{weakly two-scale in } L^2. \tag{4.10}$$

Thanks to (3.3) and Lemma 3.6(3), we also infer that

$$\chi_k^1 y_k \xrightharpoonup{2} \chi^1 y, \quad \varepsilon_k \chi_k^1 \nabla y_k \xrightharpoonup{2} \chi^1 \nabla_z y \quad \text{weakly two-scale in } L^2.$$

Moreover, there exist $y^1 \in W^{1,2}(\Omega;\mathbb{R}^3)$ and $v \in L^2(\Omega; W^{1,2}_0(Q^0;\mathbb{R}^3))$ such that the decomposition (4.1) and the convergence (4.4) hold. By combining (4.1) and (4.10), (4.3) follows.

We now turn to the sequence of plastic strains. By (4.6) and (4.8), we see that $\{P_k\}$ is bounded in $W^{1,q}(\Omega;\mathsf{SL}(3))$. Since $q > 3$, Morrey's embedding yields the uniform convergence of (a subsequence of) $\{P_k\}$ to some $P \in W^{1,q}(\Omega;\mathsf{SL}(3))$. Therefore, by definition of the inverse matrix

$$P_k^{-1} = \frac{(\operatorname{cof} P_k)^T}{\det P_k} = (\operatorname{cof} P_k)^T,$$

we also deduce that $P_k^{-1} \to P^{-1}$ uniformly.

Finally, we observe that, thanks to (4.4) and the uniform convergence of $\{P_k^{-1}\}$, (4.5) is also inferred. □

It is well-known that Γ-limits are not additive. In our case, however, we are able to show that the asymptotic behavior of the functionals $\mathcal{J}_\varepsilon$ is given exactly by the sum of the Γ-limits of the soft and of the stiff contributions. Such splitting will enable us to treat the Γ-limits of $\mathcal{J}_\varepsilon^0$ and of $\mathcal{J}_\varepsilon^1$ separately. We premise a simple lemma, which deals with the hardening part of the energy. We recall that, for $i = 0, 1$, χ_k^i is the characteristic function of Ω_k^i.

Lemma 4.2. *Under assumptions H1–H2, for any sequence $\{P_k\} \subset W^{1,q}(\Omega; K)$ converging uniformly to $P \in W^{1,q}(\Omega; K)$ it holds*

$$\lim_{k\to+\infty} \int_\Omega \chi_k^i(x) H(P_k(x))\,\mathrm{d}x = \mathcal{L}^3(Q^i) \int_\Omega H(P(x))\,\mathrm{d}x \qquad \text{for } i = 0, 1.$$

Proof. Let us focus on the case $i = 0$ first. We set

$$E^0 \coloneqq \bigcup_{t\in\mathbb{Z}^3} (t + Q^0) = \mathbb{R}^3 \setminus \overline{E^1}, \qquad \hat{\Omega}_k^0 \coloneqq \bigcup_{t\in\hat{T}_k} \varepsilon_k(t + Q^0),$$

where

$$\hat{T}_k \coloneqq \{t \in \mathbb{Z}^3 : \varepsilon_k(t + Q) \subset \Omega\} \subset T_k. \tag{4.11}$$

By definition of Ω_k^0 (see (2.1)), we have

$$\varepsilon_k E^0 \setminus \Omega_k^0 \subset \varepsilon_k E^0 \setminus \hat{\Omega}_k^0.$$

Note that $\Omega \cap (\varepsilon_k E^0 \setminus \hat{\Omega}_k^0)$ is contained in the strip $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \sqrt{3}\varepsilon_k\}$. Since $\{H(P_k)\}$ is uniformly bounded by H1 and H2, we see that

$$\begin{aligned}
&\lim_{k\to+\infty} \int_\Omega \chi_k^0(x) H(P_k(x))\,\mathrm{d}x \\
&\quad= \lim_{k\to+\infty} \int_\Omega \chi_{\varepsilon_k E^0}(x) H(P_k(x))\,\mathrm{d}x - \lim_{k\to+\infty} \int_\Omega (\chi_{\varepsilon_k E^0}(x) - \chi_k^0(x)) H(P_k(x))\,\mathrm{d}x \\
&\quad= \lim_{k\to+\infty} \int_\Omega \chi_{\varepsilon_k E^0}(x) H(P_k(x))\,\mathrm{d}x.
\end{aligned}$$

Then, by the Lipschitz continuity of H on its domain,

$$\begin{aligned}
\lim_{k\to+\infty} \int_\Omega \chi_{\varepsilon_k E^0}(x) H(P_k(x))\,\mathrm{d}x &= \lim_{k\to+\infty} \int_\Omega \chi_{\varepsilon_k E^0}(x) H(P(x))\,\mathrm{d}x \\
&= \mathcal{L}^3(Q^0) \int_\Omega H(P(x))\,\mathrm{d}x.
\end{aligned}$$

The case $i = 1$ follows from the previous one by the identities $\chi_k^1 = \chi_\Omega - \chi_k^0$ and $\mathcal{L}^3(Q^1) = 1 - \mathcal{L}^3(Q^0)$. □

The splitting process is explained by the ensuing proposition.

Proposition 4.3 (Splitting)**.** *Let $\{\varepsilon_k\}$ be an infinitesimal sequence, and let $\{(y_k, P_k)\}_{k\in\mathbb{N}} \subset W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; \mathsf{SL}(3))$ be a sequence satisfying*

$$\|y_k\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \quad \mathcal{J}_k(y_k, P_k) \leq C$$

for some $C \geq 0$, uniformly in k. Let $\tilde{y}_k$ be the extension of y_k in the sense of Remark 3.3, and let $v \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3))$ be as in Lemma 4.1. Then, defining $v_k := y_k - \tilde{y}_k$, the following hold:

$$\{v_k\} \subset W_0^{1,2}(\Omega_k^0;\mathbb{R}^3),$$
$$\|v_k\|_{L^2(\Omega;\mathbb{R}^3)} \leq C,$$
$$\varepsilon_k \nabla v_k \xrightharpoonup{2} \nabla_z v \quad \textit{weakly two-scale in } L^2, \tag{4.12}$$
$$\liminf_{k\to+\infty} \mathcal{J}_k^0(v_k, P_k) + \liminf_{k\to+\infty} \mathcal{J}_k^1(\tilde{y}_k, P_k) \leq \liminf_{k\to+\infty} \mathcal{J}_k(y_k, P_k), \tag{4.13}$$
$$\limsup_{k\to+\infty} \mathcal{J}_k(y_k, P_k) \leq \limsup_{k\to+\infty} \mathcal{J}_k^0(v_k, P_k) + \limsup_{k\to+\infty} \mathcal{J}_k^1(\tilde{y}_k, P_k). \tag{4.14}$$

Moreover, in (4.13), $\{v_k\}$ may be replaced with another sequence $\{w_k\} \subset W_0^{1,2}(\Omega_k^0;\mathbb{R}^3)$ such that $\{\varepsilon_k \nabla w_k\}$ is 2-equiintegrable and $\varepsilon_k \nabla w_k \rightharpoonup 0$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$.

Proof. We first prove that (4.12) – (4.14) hold for the sequence $\{v_k\}$. Afterwards, we will show how to recover the equiintegrability for the sequence of gradients.

We split the functional $\mathcal{J}_k$ evaluated on (y_k, P_k) as follows:

$$\begin{aligned}\mathcal{J}_k(y_k, P_k) &= \mathcal{J}_k^0(y_k, P_k) + \mathcal{J}_k^1(y_k, P_k)\\ &= \mathcal{J}_k^0(v_k, P_k) + \mathcal{J}_k^1(\tilde{y}_k, P_k) + \mathcal{R}_k,\end{aligned} \tag{4.15}$$

where $\mathcal{J}_k^0$ and $\mathcal{J}_k^1$ are as in (2.11) and (2.12), and

$$\begin{aligned}\mathcal{R}_k &:= \mathcal{J}_k^0(y_k, P_k) - \mathcal{J}_k^0(v_k, P_k)\\ &= \int_\Omega \chi_k^0 \left[W_\varepsilon^0\left(\varepsilon_k \nabla y_k P_k^{-1}\right) - W_\varepsilon^0\left(\varepsilon_k \nabla v_k P_k^{-1}\right)\right] \mathrm{d}x.\end{aligned}$$

We next show that $\mathcal{R}_k$ is asymptotically negligible.

Hypothesis E4 yields

$$|\mathcal{R}_k| \leq c_3 \int_\Omega \chi_k^0 \left(1 + \left|\varepsilon_k \nabla y_k P_k^{-1}\right| + \left|\varepsilon_k \nabla v_k P_k^{-1}\right|\right) \left|\varepsilon_k \nabla \tilde{y}_k P_k^{-1}\right| \mathrm{d}x. \tag{4.16}$$

Since $\{(y_k, P_k)\}$ is equibounded in energy, the sequences $\{\varepsilon_k \chi_k^0 \nabla y_k P_k^{-1}\}$, $\{\chi_k^1 \nabla y_k P_k^{-1}\}$, and $\{P_k^{-1}\}$ are bounded in suitable Lebesgue spaces (see (4.7) and (4.8)). By the properties of the extension operator T_ε in Lemma 3.2, we deduce that

$$\int_\Omega \left|\nabla \tilde{y}_k P_k^{-1}\right|^2 \mathrm{d}x \leq c \int_\Omega |\nabla \tilde{y}_k|^2 \,\mathrm{d}x \leq c \int_\Omega \left|\chi_k^1 \nabla y_k\right|^2 \mathrm{d}x \leq c \int_\Omega \left|\chi_k^1 \nabla y_k P_k^{-1}\right|^2 \mathrm{d}x \leq C$$

(recall estimate (2.5)). So, thanks to (4.3), we deduce that

$$\varepsilon_k \nabla v_k = \varepsilon_k \nabla y_k - \varepsilon_k \nabla \tilde{y}_k \xrightharpoonup{2} \nabla_z v \quad \text{weakly two-scale in } L^2,$$

In particular, by Lemma 3.6(1), $\{\varepsilon_k \chi_k^0 \nabla v_k P_k^{-1}\}$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$. By applying Hölder's inequality to the right-hand side of (4.16), we then find $\mathcal{R}_k = O(\varepsilon_k)$. Owing to (4.15) we conclude that (4.13) and (4.14) hold.

To complete the proof, we are only left to establish the existence of the sequence $\{w_k\}$. Upon extraction of a subsequence, which we do not relabel, we may assume that in (4.13) the lower limit involving $\mathcal{J}_k^0$ is a limit. From the equiboundedness of the energy, by arguing as in the lines before (4.9), we get

$$\|\varepsilon_k \nabla y_k\|_{L^2} \leq C, \qquad \|\chi_k^1 \nabla y_k\|_{L^2} \leq C. \tag{4.17}$$

Then, (4.3) holds and, by Lemma 3.6(2), we obtain

$$\varepsilon_k \nabla y_k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}).$$

Lemma 3.1 applied to the sequence $\{\varepsilon_k \nabla y_k\}$ yields two sequences, $\{k_j\}$ and $\{u_j\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$, such that $\{\varepsilon_{k_j}\nabla u_j\}$ is 2-equiintegrable,

$$\begin{gathered}\varepsilon_{k_j}\nabla u_j \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}), \\ \lim_{j\to+\infty} \mathcal{L}^3(N_j) = 0, \quad \text{with } N_j \coloneqq \{x\in\Omega : y_{k_j}(x) \neq u_j(x)\}.\end{gathered} \tag{4.18}$$

Besides, we have

$$\varepsilon_{k_j}\chi^1_{k_j}\nabla u_j \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{4.19}$$

Indeed, it holds

$$\begin{aligned}\|\varepsilon_{k_j}\chi^1_{k_j}\nabla u_j\|_{L^2} &= \|\varepsilon_{k_j}\chi^1_{k_j}\nabla u_j\|_{L^2(N_j)} + \|\varepsilon_{k_j}\chi^1_{k_j}\nabla y_{k_j}\|_{L^2(\Omega\setminus N_j)} \\ &\le \|\varepsilon_{k_j}\nabla u_j\|_{L^2(N_j)} + \varepsilon_{k_j}\|\chi^1_{k_j}\nabla y_{k_j}\|_{L^2},\end{aligned}$$

and the conclusion follows by the 2-equiintegrability of $\{\varepsilon_{k_j}\nabla u_j\}$ and from (4.17).

We now define $\tilde{u}_j \coloneqq \mathsf{T}_{k_j}u_j$, with T_{k_j} as in Lemma 3.2. From Remark 3.3 it follows that $\{\varepsilon_{k_j}\nabla\tilde{u}_j\}$ is 2-equiintegrable as well. Thus, the sequence defined by

$$w_k \coloneqq \begin{cases} u_j - \tilde{u}_j & \text{if } k = k_j \text{ for some } j\in\mathbb{N}, \\ 0 & \text{otherwise}\end{cases}$$

has the properties that $w_k \in W^{1,2}_0(\Omega^0_k;\mathbb{R}^3)$ and $\{\varepsilon_k\nabla w_k\}$ is 2-equiintegrable. Moreover,

$$\varepsilon_k\nabla w_k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}).$$

To see this, we write

$$\varepsilon_{k_j}\nabla w_{k_j} = \varepsilon_{k_j}\nabla u_j - \varepsilon_{k_j}\nabla\tilde{u}_j.$$

The first term converges to 0 weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$, as stated in (4.18). Additionally, Lemma 3.2 entails

$$\|\varepsilon_{k_j}\nabla\tilde{u}_j\|_{L^2} \le c\|\varepsilon_{k_j}\chi^1_{k_j}\nabla u_j\|_{L^2},$$

and the weak convergence of $\{\varepsilon_k\nabla w_k\}$ follows from (4.19).

We are now ready to prove the validity of (4.13) when $\{\varepsilon_k\nabla v_k\}$ is replaced by the 2-equiintegrable sequence $\{\varepsilon_k\nabla w_k\}$. By the definition of the sequence at stake, we have

$$\varepsilon_{k_j}(\nabla v_{k_j} - \nabla w_{k_j}) = \varepsilon_{k_j}(\nabla y_{k_j} - \nabla u_j) - \varepsilon_{k_j}(\nabla\tilde{y}_{k_j} - \nabla\tilde{u}_j) \quad \text{a. e. in } \Omega. \tag{4.20}$$

Lemma 3.2 yields

$$\begin{aligned}\varepsilon_{k_j}\|\nabla\tilde{y}_{k_j} - \nabla\tilde{u}_j\|_{L^2} &= \varepsilon_{k_j}\|\nabla(\mathsf{T}_{k_j}(y_{k_j} - u_j))\|_{L^2} \\ &\le c\varepsilon_{k_j}\|\chi^1_{k_j}\nabla(y_{k_j} - u_j)\|_{L^2} \\ &= c\varepsilon_{k_j}\|\chi^1_{k_j}(\nabla y_{k_j} - \nabla u_j)\|_{L^2(N_j)} \\ &\le c\left(\varepsilon_{k_j}\|\chi^1_{k_j}\nabla y_{k_j}\|_{L^2} + \|\varepsilon_{k_j}\nabla u_j\|_{L^2(N_j)}\right).\end{aligned}$$

Thus, (4.17) and the 2-equiintegrability of $\{\varepsilon_{k_j}\nabla u_j\}$ entail

$$\varepsilon_{k_j}\left(\nabla\tilde{y}_{k_j} - \nabla\tilde{u}_j\right) \to 0 \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{4.21}$$

Therefore, using (4.20) and the fact that the densities $W^0_{k_j}$ are bounded from below, we have

$$\begin{aligned}
&\int_\Omega \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla v_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x\\
&= \int_{N_j} \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla v_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x\\
&\quad + \int_{\Omega\setminus N_j} \chi^0_{k_j}(x) W^0_{k_j}\Big((\varepsilon_{k_j}\nabla w_{k_j}(x) - \varepsilon_{k_j}(\nabla\tilde{y}_{k_j}(x) - \nabla\tilde{u}_j(x)))P^{-1}_{k_j}(x)\Big)\,\mathrm{d}x\\
&\quad - \int_{\Omega\setminus N_j} \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla w_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x + \int_{\Omega\setminus N_j} \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla w_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x\\
&\geq -c\left(\int_{\Omega\setminus N_j} |\varepsilon_{k_j}(\nabla\tilde{y}_{k_j}(x) - \nabla\tilde{u}_j(x))|^2\,\mathrm{d}x\right)^{1/2} + \int_{\Omega\setminus N_j} \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla w_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x,
\end{aligned}$$

where the Lipschitz regularity E4 and Hölder's inequality were employed to derive the last bound (recall that $\sup_{k\in\mathbb{N}}\|P_k^{-1}\|_\infty \leq C$). We now take the limit in the inequality above. According to Lemma 4.2, the hardening term has a limit. Therefore, also the elastic contribution is convergent, and it satisfies

$$\lim_{k\to+\infty} \mathcal{J}^0_k(v_k, P_k) = \lim_{j\to+\infty}\int_\Omega \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla v_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x + \mathcal{L}^3(Q^0)\int_\Omega H(P(x))\,\mathrm{d}x.$$

The strong converge (4.21) implies

$$\begin{aligned}
&\lim_{j\to+\infty}\int_\Omega \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla v_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x\\
&\quad\geq \liminf_{j\to+\infty}\int_{\Omega\setminus N_j} \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla w_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x\\
&\quad= \liminf_{j\to+\infty}\int_\Omega \chi^0_{k_j}(x) W^0_{k_j}(\varepsilon_{k_j}\nabla w_{k_j}(x)P^{-1}_{k_j}(x))\,\mathrm{d}x,
\end{aligned}$$

where the equality follows from the growth condition E3 and from the equiintegrability of $\{\varepsilon_{k_j}\nabla w_{k_j}\}$. We thereby infer

$$\liminf_{k\to+\infty} \mathcal{J}^0_k(w_k, P_k) \leq \liminf_{j\to+\infty} \mathcal{J}^0_{k_j}(w_{k_j}, P_{k_j}) \leq \lim_{k\to+\infty} \mathcal{J}^0_k(v_k, P_k),$$

and this concludes the proof. □

5. Γ-limit of the soft component

We devote this section to the study of the asymptotics of the functional $\mathcal{J}^0_\varepsilon$ in (2.11), which encodes the energy of the inclusions. After some observations on the limiting functional $\mathcal{J}^0$ in (2.6), in the second and third subsections we deal respectively with the lower and with the upper limit inequality for the elastic part of the energy. The other contributions will be taken into account in Subsection 5.4, where we prove Proposition 2.10.

5.1. **The limiting functional.** The definition of $\mathcal{Q}'W^0$ in (2.8), which encodes the limiting elastic contribution of the soft inclusions, may be regarded as a variant of the well known Dacorogna's formula for the quasiconvex envelope [20, Theorem 6.9]. As such, the infimum in (2.8) does not depend on Q, and we may rewrite $\mathcal{Q}'W^0$ as follows:

$$\mathcal{Q}'W^0(F,G) = \inf\left\{ \fint_{Q^0} W^0\Big((F + \nabla v(z))G\Big)\,\mathrm{d}z : v \in W^{1,2}_0(Q^0;\mathbb{R}^3)\right\}. \tag{5.1}$$

Note that here quasiconvexification occurs just with respect to the first argument, since a very strong convergence is considered for the second one. The fact that different variables in a problem may call for different relaxation procedures has been already observed. As an example, we mention the concept of cross-quasiconvexity introduced by Le Dret & Raoult [35] to deal with dimension reduction problems in elasticity.

For the sake of completeness, we explicitly mention some basic properties of $\mathcal{Q}'W^0$.

Lemma 5.1. *Let $W^0\colon \mathbb{R}^{3\times 3}\to\mathbb{R}$, and assume there exist $0<c_1\le c_2$ such that for all $F\in\mathbb{R}^{3\times 3}$*

$$c_1|F|^2 \le W^0(F) \le c_2\left(|F|^2+1\right).$$

Let $\mathcal{Q}'W^0$ be as in (2.8).

(1) For all $F,G\in\mathbb{R}^{3\times 3}$

$$c_1|FG|^2 \le \mathcal{Q}'W^0(F,G) \le c_2\left(|FG|^2+1\right),$$

and for all $G\in\mathbb{R}^{3\times 3}$ there exists $c\coloneqq c(G)>0$ such that for all $F_1,F_2\in\mathbb{R}^{3\times 3}$

$$\left|\mathcal{Q}'W^0(F_1,G)-\mathcal{Q}'W^0(F_2,G)\right| \le c\,(1+|F_1|+|F_2|)\,|F_1-F_2|.$$

Suppose further that there exists $c_3>0$ such that for all $F_1,F_2\in\mathbb{R}^{3\times 3}$

$$\left|W^0(F_1)-W^0(F_2)\right| \le c_3\,(1+|F_1|+|F_2|)\,|F_1-F_2|. \tag{5.2}$$

(2) Then, $\mathcal{Q}'W^0(F,\,\cdot\,)$ is continuous for all $F\in\mathbb{R}^{3\times 3}$.

(3) If $\{P_k\}\subset W^{1,q}(\Omega;\mathsf{SL}(3))$ converges weakly to $P\in W^{1,q}(\Omega;\mathsf{SL}(3))$, then for any $V\in L^2(\Omega;\mathbb{R}^{3\times 3})$

$$\lim_{k\to+\infty}\int_\Omega \mathcal{Q}'W^0(V(x),P_k^{-1}(x))\,\mathrm{d}x = \int_\Omega \mathcal{Q}'W^0(V(x),P^{-1}(x))\,\mathrm{d}x.$$

Proof. The growth conditions on $\mathcal{Q}'W^0$ are an immediate consequence of the ones on W^0 and of the definition of $\mathcal{Q}'W^0$.

For what concerns the 2-Lipschitz property, let us set $W^0_G(F)\coloneqq W^0(FG)$ for any fixed $G\in\mathbb{R}^{3\times 3}$. Then, $\mathcal{Q}'W^0(\,\cdot\,,G)$ coincides with the quasiconvex envelope of W^0_G. By [20, Remark 5.3(iii)] it follows that $\mathcal{Q}'W^0(\,\cdot\,,G)$ is separately convex, and hence, in view of the growth assumptions on W^0, the proof of item (1) is concluded by [20, Proposition 2.32].

As for point (2), let $G_k\to G$ in $\mathbb{R}^{3\times 3}$. In view of (5.2), for every $\delta>0$ there exists $c_\delta>0$ such that

$$\mathcal{Q}'W^0(F,G_k)-\mathcal{Q}'W^0(F,G)\le c_\delta|G_k-G|+\delta.$$

Similarly, for any $k\in\mathbb{N}$ there exists $v_k\in W^{1,p}_0(Q;\mathbb{R}^{3\times 3})$ such that

$$\begin{aligned}\mathcal{Q}'W^0&(F,G_k)-\mathcal{Q}'W^0(F,G)\\ &\ge -c_3|G_k-G|\int_Q (1+|(F+\nabla v_k)G_k|+|(F+\nabla v_k)G|)|F+\nabla v_k|\,\mathrm{d}x-\frac{1}{k}.\end{aligned}$$

Thanks to the coercivity of the integrand, it follows that $\{\nabla v_k\}$ is bounded in L^2, whence

$$\mathcal{Q}'W^0(F,G_k)-\mathcal{Q}'W^0(F,G)\ge -c\,|G_k-G|-\frac{1}{k}$$

for a constant c independent of k. The continuity of $\mathcal{Q}'W^0(F,\,\cdot\,)$ is then proved by letting first $k\to+\infty$ and then $\delta\to 0$.

Eventually, taking into account points (1) and (2), as well as the compact embedding of $W^{1,q}(\Omega)$ into $C(\bar{\Omega})$, we can employ the dominated convergence theorem to obtain the continuity property in (3). ☐

We now exhibit an alternative expression for the soft limiting elastic energy, which is to be exploited in the proof of Proposition 5.7.

Lemma 5.2. *For every couple $(V,P) \in L^2(\Omega;\mathbb{R}^{3\times 3}) \times W^{1,q}(\Omega;\mathsf{SL}(3))$ we have*

$$\begin{aligned}\int_\Omega \mathcal{Q}' W^0(V(x), P^{-1}(x))\,\mathrm{d}x \\ = \inf\left\{ \int_\Omega \fint_{Q^0} W^0\Big((V(x) + \nabla_z w(x,z))P^{-1}(x)\Big)\,\mathrm{d}z : w \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3))\right\}.\end{aligned} \tag{5.3}$$

The identity above rests on a measurable selection criterion that we recall next.

Lemma 5.3 (Lemma 3.10 in [28]). *Let S be a multifunction defined on the measurable space X and taking values in the collection of subsets of the separable metric space Y. If $S(x)$ is nonempty and open in Y for every $x \in X$, and if the set $\{ x \in X : y \in S(x) \}$ is measurable for every $y \in Y$, then S admits a measurable selection, that is, there exists a measurable function $s\colon X \to Y$ such that $s(x) \in S(x)$ for all $x \in X$.*

The previous lemma is a variant of [12, Theorem III.6], and we refer to that monograph for a comprehensive treatment of measurable selection principles.

Proof of Lemma 5.2. The argument follows the one proposed in [28, Corollary 3.2].

Let us fix $w \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3))$, so that, for almost every $x \in \Omega$, $w(x,\cdot) \in W_0^{1,2}(Q^0;\mathbb{R}^3)$. Hence, according to (5.1), we have

$$\mathcal{Q}' W^0(V(x), P^{-1}(x)) \le \fint_{Q^0} W^0\Big((V(x) + \nabla_z w(x,z))P^{-1}(x)\Big)\,\mathrm{d}z \qquad \text{for a.e. } x \in \Omega,$$

whence, after integration over Ω, we deduce that in (5.3) the left-hand side is smaller that the righ-hand one.

In order to establish the opposite inequality, we first observe that, by (5.1), for every $x \in \Omega$ and every $\delta > 0$ there exists $v_{x,\delta} \in W_0^{1,2}(Q^0;\mathbb{R}^3)$ such that

$$\fint_{Q^0} W^0\Big((V(x) + \nabla v_{x,\delta}(z))P^{-1}(x)\Big)\,\mathrm{d}z - \mathcal{Q}' W^0(V(x), P^{-1}(x)) < \delta. \tag{5.4}$$

We introduce the multifunction S defined for $x \in \Omega$ by

$$S(x) \coloneqq \Big\{ v \in W_0^{1,2}(Q^0;\mathbb{R}^3) : (5.4) \text{ holds for } v_{x,\delta} = v \Big\}.$$

We show that it admits a measurable selection. To this purpose, observe that, as a consequence of the growth assumptions on W^0 and of the dominated convergence theorem, $S(x)$ is a nonempty, open subset of $W_0^{1,2}(Q^0;\mathbb{R}^3)$ for every $x \in \Omega$. Second, for every $v \in W_0^{1,2}(Q^0;\mathbb{R}^3)$ the set $\{x \in \Omega : v \in S(x)\}$ is measurable, because it is the sublevel set of a measurable function.

Thanks to Lemma 5.3, for every $\delta > 0$ we retrieve a measurable function $w_\delta\colon \Omega \to W_0^{1,2}(Q^0;\mathbb{R}^3)$ that satisfies

$$\int_\Omega \fint_{Q^0} W^0\Big((V(x) + \nabla_z w_\delta(x,z))P^{-1}(x)\Big)\,\mathrm{d}z\,\mathrm{d}x \le \int_\Omega \mathcal{Q}' W^0(V(x), P^{-1}(x)) + O(\delta).$$

In particular, by the growth conditions on W^0, w_δ must belong to $L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3))$. Therefore, since δ is arbitrary, we conclude that the left-hand side in (5.3) bounds from above the right-hand one. □

5.2. **Lower bound for the elastic energy.** The goal of this subsection is to prove the ensuing:

Proposition 5.4. *Let $\{W_k^0\}_k$ satisfy assumptions E3–E5, and let $P \in W^{1,q}(\Omega; \mathsf{SL}(3))$. For every sequence $\{(v_k, P_k)\} \subset W_0^{1,2}(\Omega_k^0;\mathbb{R}^3) \times W^{1,q}(\Omega;\mathsf{SL}(3))$ such that $\{\varepsilon_k \nabla v_k\}$ is 2-equiintegrable and $P_k \to P$ uniformly, it holds*

$$\mathcal{L}^3(Q^0)\int_\Omega \mathcal{Q}'W^0(0, P^{-1}(x))\,\mathrm{d}x \le \liminf_{k\to+\infty}\int_\Omega \chi_k^0(x) W_k^0(\varepsilon_k\nabla v_k(x)P_k^{-1}(x))\,\mathrm{d}x. \tag{5.5}$$

At a first glance, it may look bizarre that no convergence for the sequence $\{\varepsilon_k\nabla v_k\}$ is prescribed. The statement becomes clearer once we recall that if $\mathcal{Q}f$ is the quasiconvex envelope of $f\colon \mathbb{R}^{3\times3}\to\mathbb{R}$, then

$$\mathcal{Q}f(0) \le \fint_\Omega f(\nabla v(x))\,\mathrm{d}x$$

for any $v \in W_0^{1,\infty}(\Omega;\mathbb{R}^3)$.

In order to establish (5.5), it is convenient to unfold the elastic energy.

Lemma 5.5. *Let $\{W_k^0\}_k$ satisfy assumptions E3–E5. For any $(v,P) \in W^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;\mathsf{SL}(3))$ it holds*

$$\int_\Omega \chi_k^0(x) W_k^0\left(\varepsilon_k \nabla v(x) P^{-1}(x)\right)\,\mathrm{d}x = \sum_{t\in T_k}\int_{\varepsilon_k(t+Q)}\int_{Q^0} W_k^0\left(\nabla_z\hat{v}(x,z)\hat{P}^{-1}(x,z)\right)\mathrm{d}z\,\mathrm{d}x \tag{5.6}$$

where $\hat{v} := \mathsf{S}_k v$, $\hat{P} := \mathsf{S}_k P$ and $\mathsf{S}_k := \mathsf{S}_{\varepsilon_k}$ is the unfolding operator introduced in Lemma 3.7.

Proof. According to the definition of Ω_k^0 in (2.1), the left-hand side of (5.6) equals

$$\varepsilon_k^3\sum_{t\in T_k}\int_{Q^0} W_k^0\Big(\varepsilon_k\nabla v(\varepsilon_k(t+z))P^{-1}(\varepsilon_k(t+z))\Big)\,\mathrm{d}z.$$

We use the unfolding operator to rewrite this quantity as a double integral. Recalling Lemma 3.7, we firstly observe that for every $t\in T_k$ and $z \in Q^0$ we have the identities

$$\mathsf{S}_k(\varepsilon_k\nabla v)(\varepsilon_k t, z) = \varepsilon_k\nabla v(\varepsilon_k(t+z)), \quad \mathsf{S}_k P^{-1}(\varepsilon_k t, z) = P^{-1}(\varepsilon_k(t+z)).$$

Then, we also have

$$\mathsf{S}_k(\varepsilon_k\nabla v) = \nabla_z(\mathsf{S}_k v) = \nabla_z\hat{v}, \quad \mathsf{S}_k P^{-1} = (\mathsf{S}_k P)^{-1} = \hat{P}^{-1}.$$

We obtain

$$\begin{aligned}
&\int_\Omega \chi_k^0(x)W_k^0(\varepsilon_k\nabla v(x)P^{-1}(x))\,\mathrm{d}x\\
&\quad= \varepsilon_k^3\sum_{t\in T_k}\int_{Q^0} W_k^0(\mathsf{S}_k(\varepsilon_k\nabla v)(\varepsilon_k t, z)\mathsf{S}_k(P^{-1})(\varepsilon_k t, z))\,\mathrm{d}z\\
&\quad= \sum_{t\in T_k}\int_{\varepsilon_k(t+Q)}\int_{Q^0} W_k^0\left(\nabla_z\hat{v}\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor, z\right)\hat{P}^{-1}\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor, z\right)\right)\mathrm{d}z\,\mathrm{d}x,
\end{aligned}$$

because $\lfloor x/\varepsilon_k\rfloor = t$ for all $x\in\varepsilon_k(t+Q)$. Since, in general, it holds

$$\mathsf{S}_k u\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor, z\right) = u\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor + \varepsilon_k z\right) = \mathsf{S}_k u(x,z),$$

(5.6) follows. □

A crucial ingredient in the proof of Proposition 5.4 is a sort of lower semicontinuity result for the elastic contribution to the energy.

Lemma 5.6. *Let $\{W_k^0\}_k$ satisfy assumptions E3–E5. Let also $\{w_k\} \subset L^2(\Omega; W_0^{1,2}(Q_0;\mathbb{R}^3))$ be such that $\{\nabla_z w_k\}$ is 2-equiintegrable. Then, for all $P \in W^{1,q}(\Omega; \mathsf{SL}(3))$,*

$$\mathcal{L}^3(Q^0)\int_\Omega \mathcal{Q}' W^0(0, P^{-1}(x))\,\mathrm{d}x \le \liminf_{k\to+\infty}\int_\Omega\int_{Q^0} W_k^0(\nabla_z w_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x,$$

whenever $P_k \to P$ uniformly.

Proof. From (5.1) it follows that for all $k \in \mathbb{N}$

$$\mathcal{L}^3(Q^0)\int_\Omega \mathcal{Q}' W^0(0, P_k^{-1}(x))\,\mathrm{d}x \le \int_\Omega\int_{Q^0} W^0(\nabla_z w_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x. \tag{5.7}$$

Next, relying on the pointwise convergence of $\{W_k^0\}$ to W^0, we adapt the argument in the proof of [21, Theorem 5.14] to pass from W^0 to W_k^0 on the right-hand side (see also [26, Lemma 5.2] for a similar result in the context of $\mathscr{A}$-quasiconvexity). Fix $\delta > 0$. If $\{\nabla_z w_k\}$ is 2-equiintegrable, then so is $\{\nabla_z w_k P_k^{-1}\}$. Therefore, since the 2-growth assumptions on $\{W_k^0\}$ transfer to the pointwise limit W^0, there exists $r > 0$ such that

$$\sup_{k\in\mathbb{N}}\int_{\{(x,z)\in\Omega\times Q^0 : |\nabla_z w_k(x,z)P_k^{-1}(x)|>r\}} W^0(\nabla_z w_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x \le \delta. \tag{5.8}$$

Owing to assumption E4 and Remark 2.2, we can find $\rho > 0$ such that for every $F, G \in \mathbb{R}^{3\times 3}$ contained in the open ball $B(0,\rho)$

$$\sup_{k\in\mathbb{N}} |W_k^0(F) - W_k^0(G)| + |W^0(F) - W^0(G)| \le \delta. \tag{5.9}$$

Let now $F_1, \dots, F_n \in B(0,r)$ be such that

$$\overline{B(0,r)} \subset \bigcup_{i=1}^n B(F_i,\rho). \tag{5.10}$$

Due to the pointwise convergence of W_k^0 to W^0, for any such F_i there exist $\bar{k}_i \in \mathbb{N}$ such that $|W_k^0(F_i) - W^0(F_i)| \le \delta$ if $k > \bar{k}_i$. Letting $\bar{k} := \max\{\bar{k}_1, \dots, \bar{k}_n\}$, it follows that for any $i = 1, \dots, n$

$$|W_k^0(F_i) - W^0(F_i)| \le \delta \quad \text{if } k > \bar{k}. \tag{5.11}$$

By (5.10), for every $G \in \overline{B(0,r)}$ there exists $i \in \{1,\dots,n\}$ such that $G \in B(F_i,\rho)$. For this particular i, the combination of the triangle inequality, (5.9) and (5.11) yields

$$|W_k^0(G) - W^0(G)| \le |W_k^0(G) - W_k^0(F_i)| + |W_k^0(F_i) - W^0(F_i)| + |W^0(G) - W^0(F_i)| \le 3\delta, \tag{5.12}$$

for every $G \in \overline{B(0,r)}$ and every $k > \bar{k}$.

Thanks to Lemma 5.1(3) and (5.7) we deduce

$$
\begin{aligned}
&\mathcal{L}^3(Q^0)\int_\Omega \mathcal{Q}'W^0(0,P^{-1}(x))\,\mathrm{d}x\\
&\quad= \mathcal{L}^3(Q^0)\lim_{k\to+\infty}\int_\Omega \mathcal{Q}'W^0(0,P_k^{-1}(x))\,\mathrm{d}x\\
&\quad\leq \liminf_{k\to+\infty}\int_\Omega\int_{Q^0} W^0(\nabla_z w_k(x,z)P^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x\\
&\quad\leq \liminf_{k\to+\infty}\int_{\{(x,z)\in\Omega\times Q^0:|\nabla_z w_k(x,z)P_k^{-1}(x)|\leq r\}} W^0(\nabla_z w_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x+\delta\\
&\quad\leq \liminf_{k\to+\infty}\int_\Omega\int_{Q^0} W_k^0(\nabla_z w_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x+3\delta\mathcal{L}^6(\Omega\times Q^0)+\delta,
\end{aligned}
$$

where the second inequality is due to (5.8), and the last one to (5.12). The arbitrariness of $\delta>0$ yields the conclusion. □

We are now ready to prove the lower bound for the elastic contribution of the soft part.

Proof of Proposition 5.4. Let $\hat{v}_k \coloneqq \mathsf{S}_k v_k$. In view of the 2-equiintegrability of the sequence $\{\varepsilon_k\nabla v_k\}$ and of Lemma 3.7, $\{\nabla_z\hat{v}_k\}$ is 2-equiintegrable as well. Hence it is *a fortiori* bounded in L^2. From Lemma 5.5, restricting the summation in (5.6) to the set of translations in (4.11), we deduce

$$
\liminf_{k\to+\infty}\int_\Omega \chi_k^0(x)W_k^0(\varepsilon_k\nabla v_k(x)P_k^{-1}(x))\,\mathrm{d}x \quad\geq\quad \liminf_{k\to+\infty}\int_{\Omega_k^Q}\int_{Q^0} W_k^0(\nabla_z\hat{v}_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x,
$$

where

$$
\Omega_k^Q \coloneqq \bigcup_{t\in\hat{T}_k}\varepsilon_k(t+Q). \tag{5.13}
$$

We rewrite the right-hand side of the previous inequality as the difference between the integrals

$$
\begin{aligned}
I_k' &\coloneqq \int_\Omega\int_{Q^0} W_k^0(\nabla_z\hat{v}_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x,\\
I_k'' &\coloneqq \int_{\Omega\setminus\Omega_k^Q}\int_{Q^0} W_k^0(\nabla_z\hat{v}_k(x,z)P_k^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}
$$

Being $\{\nabla_z\hat{v}_k\}$ 2-equiintegrable, the sequence $\{\nabla_z\hat{v}_kP_k^{-1}\}$ is still 2-equiintegrable. Thus, by the growth condition E3, we obtain

$$
\lim_{k\to+\infty} I_k''=0.
$$

Taking into account Lemma 5.6 we conclude

$$
\liminf_{k\to+\infty}\int_\Omega \chi_k^0(x)W_k^0(\varepsilon_k\nabla v_k(x)P_k^{-1}(x))\,\mathrm{d}x \geq \liminf_{k\to+\infty} I_k' \geq \mathcal{L}^3(Q^0)\int_\Omega \mathcal{Q}'W^0(0,P^{-1}(x))\,\mathrm{d}x.
$$

□

5.3. **Upper bound for the elastic energy.** In this subsection we address the proof of Γ-upper limit inequality for the elastic contribution of the soft component. Differently from the previous subsection, in order to establish the desired inequality we perform an analysis that is genuinely two-scale, in the sense that we interpret 0 as the average with respect to the periodic variable of the two-scale limit of the sequence $\{\varepsilon_k \nabla v_k\}$.

Proposition 5.7. *Let $\{W_k^0\}_k$ satisfy assumptions E3–E5, and let $P \in W^{1,q}(\Omega; \mathsf{SL}(3))$. For all $\delta > 0$ there exists a sequence $\{v_k\} \subset W_0^{1,2}(\Omega_k^0; \mathbb{R}^3)$ such that $\varepsilon_k \nabla v_k \rightharpoonup 0$ weakly in $L^2(\Omega; \mathbb{R}^{3\times 3})$ and that*

$$\limsup_{k\to+\infty} \int_\Omega \chi_k^0(x) W_k^0 \left(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)\right) \mathrm{d}x < \mathcal{L}^3(Q^0) \int_\Omega \mathcal{Q}' W^0(0, P^{-1}(x))\, \mathrm{d}x + \delta, \tag{5.14}$$

whenever $P_k \to P$ uniformly.

We begin with a lemma that provides a strong two-scale approximation of any sufficiently regular function. The result has already appeared in [13], where, however, the proof is just sketched. In order to keep the exposition self-contained, we include it in the Appendix, where we also compare our result with the one in [13].

Lemma 5.8. *Let $w \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3)$. Then, there exists a sequence $\{v_k\} \subset L^2(\Omega; \mathbb{R}^3)$ such that, letting $\hat{v}_k := \mathsf{S}_k v_k$, it holds*

$$\nabla_z \hat{v}_k \to \nabla_z w \quad \textit{strongly in } L^2(\Omega \times Q; \mathbb{R}^{3\times 3}). \tag{5.15}$$

We are now ready to prove the Γ-limsup inequality for the soft inclusions functional.

Proof of Proposition 5.7. According to Lemma 5.2, for every $\delta > 0$ there exists $w_\delta \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3))$ satisfying

$$\int_\Omega \int_{Q^0} W^0(\nabla_z w_\delta(x,z) P^{-1}(x))\, \mathrm{d}z\, \mathrm{d}x < \mathcal{L}(Q^0) \int_\Omega \mathcal{Q}' W^0(0, P^{-1}(x))\, \mathrm{d}x + \delta \tag{5.16}$$

We would like to apply Lemma 5.8 which, however, requires $w_\delta \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3)$. We therefore establish the inequality first for a sufficiently regular w_δ, and we then extend the result by a density argument.

<u>CASE 1: w_δ REGULAR</u>

Let $w_\delta \in L^2(\Omega; W_0^{1,2}(Q^0; \mathbb{R}^3)) \cap C^2(\Omega \times Q^0; \mathbb{R}^3)$. We consider the recovery sequence $\{v_k\}$ coming from Lemma 5.8. Lemmas 3.7 and 3.6(2) yield $\varepsilon_k \nabla v_k \rightharpoonup 0$ weakly in $L^2(\Omega; \mathbb{R}^{3\times 3})$. Assumption E4 and Hölder's inequality entail

$$\begin{aligned} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} \left| W_k^0 \left(\nabla_z \hat{v}_k(x,z) P_k^{-1}(x)\right) - W_k^0 \left(\nabla_z w_\delta(x,z) P_k^{-1}(x)\right)\right| \mathrm{d}z\, \mathrm{d}x \\ \leq c \sum_{t\in T_k} \left(\int_{\varepsilon_k(t+Q)} \int_{Q^0} |\nabla_z \hat{v}_k(x,z) - \nabla_z w_\delta(x,z)|^2\, \mathrm{d}z\, \mathrm{d}x \right)^{1/2}, \end{aligned}$$

where the constant c bounds $\|P_k^{-1}\|_{L^\infty}$. Thanks to the strong convergence of $\{\nabla_z \hat{v}_k\}$, we obtain that the term above is infinitesimal when $k \to +\infty$. From Lemma 5.5 we then deduce

$$
\begin{aligned}
\limsup_{k\to+\infty} \int_\Omega \chi_k^0(x) W_k^0 \left(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)\right) \mathrm{d}x \\
&= \limsup_{k\to+\infty} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} W_k^0 \left(\nabla_z w_\delta(x,z) P_k^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x \\
&= \limsup_{k\to+\infty} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} W_k^0 \left(\nabla_z w_\delta(x,z) P^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x \\
&= \limsup_{k\to+\infty} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} W^0 \left(\nabla_z w_\delta(x,z) P^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x,
\end{aligned}
$$

where the second identity follows from E4 and the last one from E5. Note also that, by absolute continuity of the Lebesgue integral,

$$
\begin{aligned}
\limsup_{k\to+\infty} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} W^0 \left(\nabla_z w_\delta(x,z) P^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x \\
&= \int_\Omega \int_{Q^0} W^0 \left(\nabla_z w_\delta(x,z) P^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x.
\end{aligned}
$$

Therefore, by combining the equalities that we have just found with (5.16), we achieve the conclusion in the case under consideration.

CASE 2: w_δ GENERIC

Let now $w_\delta \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3))$. By mollification, we retrieve a function $\tilde{w}_\delta \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3)) \cap C^2(\Omega \times Q^0;\mathbb{R}^3)$ such that

$$
\int_\Omega \int_{Q^0} W^0(\nabla_z \tilde{w}_\delta(x,z) P^{-1}(x))\, \mathrm{d}z \leq \int_\Omega \int_{Q^0} W^0(\nabla_z w_\delta(x,z) P^{-1}(x))\, \mathrm{d}z + \delta.
$$

To achieve the conclusion, it only suffices to repeat the argument in Case 1 for $\tilde{w}_\delta$. □

5.4. **Proof of Proposition 2.10.** We are eventually in a position to reap the fruits of the previous subsections.

Proof of Proposition 2.10. Let us start with the lower limit inequality. If the lower limit of $\mathcal{J}_k^0(v_k, P_k)$ is not finite, there is nothing to prove. Otherwise, recalling Lemma 3.6(4), we deduce that $\varepsilon_k \nabla v_k \overset{2}{\rightharpoonup} \nabla_z \tilde{v}$ weakly two-scale in L^2 for some $\tilde{v} \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$. In particular, by Lemma 3.6(2), it must be

$$
\nabla v(x) = \int_Q \nabla_z \tilde{v}(x,z)\, \mathrm{d}z = 0 \quad \text{for a. e. } x \in \Omega,
$$

whence, being Ω connected, must v be identically zero. Statement (1) in Proposition 2.10 then follows by combining Proposition 5.4 and Lemma 4.2.

We now turn to the upper bound. The only nontrivial case corresponds to $v = 0$. Proposition 5.7 provides for all $\delta > 0$ a sequence $\{v_k\} \subset W_0^{1,2}(\Omega_k^0;\mathbb{R}^3)$ such that $\varepsilon_k \nabla v_k \rightharpoonup 0 = \nabla v$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and (5.14) holds. By the Rellich-Kondrachov theorem in $W_0^{1,2}(\Omega;\mathbb{R}^3)$, it follows that $\varepsilon_k v_k \to 0$ strongly in $L^2(\Omega;\mathbb{R}^3)$ (up to subsequences). We employ again Lemma 4.2 to deduce that

$$
\limsup_{k\to+\infty} \mathcal{J}_k^0(v_k, P_k) < \mathcal{J}^0(v, P) + \delta.
$$

This inequality is actually equivalent to the desired one (cf. [8, Section 1.2]), and the proof is therefore concluded. □

6. Conclusions and a variant

We devote this final section to the proof of the homogenization result for high-contrast composites and to the discussion of a variant of the problem featuring plastic dissipation.

6.1. **Proof of Theorem 2.7 and convergence of minimum problems.** As we outlined before, the proof of Theorem 2.7 is achieved by combining the splitting procedure in Proposition 4.3 with Theorem 3.8 and Proposition 2.10, which account for the asymptotics of the stiff and the soft components, respectively. Once the homogenization theorem is on hand, the convergence of the minimum problems and of their minimizers will follow thanks to the compactness result in Lemma 4.1.

Proof of Theorem 2.7. Let $\{\varepsilon_k\}$ be an infinitesimal sequence and let us fix $y \in L^2(\Omega;\mathbb{R}^3)$ and $P \in L^q(\Omega;\mathsf{SL}(3))$. We separate the proof of the lower and of the upper limit inequalities.

Lower bound

We consider a sequence $\{(y_k,P_k)\} \subset L^2(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathsf{SL}(3))$ such that $y_k \to y$ in the sense of extensions and that $P_k \to P$ uniformly. The only case to discuss is the one in which the lower limit of $\mathcal{J}_k(y_k,P_k)$ is finite, and we may thus assume that $\{\mathcal{J}_k(y_k,P_k)\}$ is bounded. Keeping in force the notation of Definition 2.4, we let $\{\tilde{y}_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence such that $y_k = \tilde{y}_k$ in Ω^1_k and $\tilde{y}_k \rightharpoonup y$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. In the light of (4.4) and Remark 2.5, we may without loss of generality assume that $\tilde{y}_k \coloneqq \mathsf{T}_k y_k$, with T_k as in Lemma 3.2.

We now apply Proposition 4.3, which yields $\{v_k\} \subset W^{1,2}_0(\Omega^0_k;\mathbb{R}^3)$ satisfying (4.13) and such that $\{v_k\}$ is bounded in L^2, $\{\varepsilon_k \nabla v_k\}$ is 2-equiintegrable and $\varepsilon_k v_k \to 0$ strongly in L^2. In particular, $(\varepsilon_k v_k, P_k) \xrightarrow{\tau} (0,P)$ and Proposition 2.10 yields

$$\mathcal{J}^0(0,P) \leq \liminf_{k\to+\infty} \mathcal{J}^0_k(v_k,P_k).$$

At this stage, recalling (4.13), the proof of the lower bound is concluded as soon as we show that

$$\mathcal{J}^1(y,P) \leq \liminf_{k\to+\infty} \mathcal{J}^1_k(\tilde{y}_k,P_k) = \liminf_{k\to+\infty} \mathcal{J}^1_k(y_k,P_k) \tag{6.1}$$

with $\mathcal{J}^1(y,P)$ given by (2.7). This is what we prove next.

Let us set

$$\widehat{W}^1(x,F) \coloneqq \chi_{E^1}(x) W^1(F), \qquad \widehat{H}(x,P) \coloneqq \chi_{E^1}(x) H(P),$$
$$\widehat{\mathcal{J}}^1_k(y,P) \coloneqq \int_\Omega \left[\widehat{W}^1\left(\frac{x}{\varepsilon_k}, \nabla \tilde{y} P^{-1}\right) + \widehat{H}\left(\frac{x}{\varepsilon_k}, P\right) + |\nabla P|^q \right] \mathrm{d}x. \tag{6.2}$$

It holds

$$\liminf_{k\to+\infty} \widehat{\mathcal{J}}^1_k(\tilde{y}_k,P_k) \leq \liminf_{k\to+\infty} \mathcal{J}^1_k(\tilde{y}_k,P_k).$$

Since $(\tilde{y}_k,P_k) \xrightarrow{\tau} (y,P)$, by applying Theorem 3.8 to the left-hand side of the previous inequality, (6.1) is deduced.

Upper bound

If $P \notin W^{1,q}(\Omega;K)$ there is nothing to prove; let us then assume that $P \in W^{1,q}(\Omega;K)$.

As we have already observed, $\{\widehat{\mathcal{J}}_k^1\}$ satisfies the requirements of Theorem 3.8. In view of Corollary 3.9, for any $(y, P) \in W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ there exists a sequence $\{(u_k, P_k)\} \subset W^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ such that $\{\nabla u_k\}$ is 2-equiintegrable, $(u_k, P_k) \xrightarrow{\tau} (y, P)$, and

$$\limsup_{k\to+\infty} \widehat{\mathcal{J}}_k^1(u_k, P_k) \le \mathcal{J}^1(y, P).$$

Note that

$$\begin{aligned} 0 &\le \mathcal{J}_k^1(u_k, P_k) - \widehat{\mathcal{J}}_k^1(u_k, P_k) \\ &= \int_\Omega \big(\chi_k^1(x) - \chi_{\varepsilon_k E^1}(x)\big)\big(W^1(\nabla u_k P_k^{-1}) + H(P_k)\big)\,\mathrm{d}x \\ &\le c \int_\Omega \big(\chi_k^1(x) - \chi_{\varepsilon_k E^1}(x)\big)\big(|\nabla u_k|^2 + 1\big)\,\mathrm{d}x \end{aligned}$$

for all $k \in \mathbb{N}$. Thanks to the 2-equiintegrability of $\{\nabla u_k\}$, we deduce

$$\limsup_{k\to+\infty} \mathcal{J}_k^1(u_k, P_k) = \limsup_{k\to+\infty} \widehat{\mathcal{J}}_k^1(u_k, P_k) \le \mathcal{J}^1(y, P). \tag{6.3}$$

We focus now on the soft part. Proposition 2.10 grants the existence of a sequence $\{v_k\} \subset W_0^{1,2}(\Omega_k^0; \mathbb{R}^3)$ such that $\varepsilon_k v_k \to 0$ strongly in $L^2(\Omega; \mathbb{R}^{3\times 3})$ and that

$$\limsup_{k\to+\infty} \mathcal{J}_k^0(v_k, P_k) \le \mathcal{J}^0(0, P), \tag{6.4}$$

where $\{P_k\}$ is as in (6.3). Notice that if $y_k \coloneqq u_k + v_k$, then $\{\mathcal{J}_k(y_k, P_k)\}$ is bounded and $\{y_k\}$ converges to y in the sense of extensions. Letting $\tilde{y}_k \coloneqq \mathsf{T}_k y_k$, thanks to (4.14) we conclude the proof of the upper limit inequality:

$$\begin{aligned} \limsup_{k\to+\infty} \mathcal{J}_k(y_k, P_k) &\le \limsup_{k\to+\infty} \mathcal{J}_k^0(y_k - \tilde{y}_k, P_k) + \limsup_{k\to+\infty} \mathcal{J}_k^1(\tilde{y}_k, P_k) \\ &= \limsup_{k\to+\infty} \mathcal{J}_k^0(v_k, P_k) + \limsup_{k\to+\infty} \mathcal{J}_k^1(u_k, P_k) \\ &\le \mathcal{J}(y, P). \end{aligned}$$

In the previous lines, the equality is a consequence of the facts that $\{\nabla u_k\}$ and $\{\nabla \tilde{y}_k\}$ are bounded and that $u_k = y_k$ on Ω_k^1, whereas the last bound accounts for (6.3) and (6.4). ☐

Finally, we are only left to establish the convergence of the minimum problems associated with the energy functionals $\mathcal{J}_\varepsilon$. What we need is an adaptation of the Γ-convergence statement that we have just proved so as to make it comply with Dirichlet boundary conditions. To this aim, as it is customary (see e.g. [9, Proposition 11.7]), we could employ the fundamental estimate derived in [24] on the functionals $\{\widehat{\mathcal{J}}_k^1\}$ in (6.2); indeed, boundary data concern only the stiff part, cf. Remark 2.6. In the light of Corollary 3.9 we can adopt an alternative strategy.

Proof of Corollary 2.9. Since $\{(y_k, P_k)\}$ is a sequence of almost-minimizers, there exists C such that $\mathcal{J}_k(y_k, P_k) \le C$. The 2-growth condition from below, together with Lemma 3.4, provides a bound on $\|y_k\|_{L^2}$. By Proposition 4.1, there exists $(y, P) \in W_0^{1,2}(\Omega; \mathbb{R}^3) \times W^{1,q}(\Omega; K)$ such that, up to subsequences, $y_k \to y$ in the sense of extensions and $P_k \to P$ uniformly. Theorem 2.7 ensures that

$$\mathcal{J}(y, P) \le \liminf_{k\to+\infty} \mathcal{J}_k(y_k, P_k).$$

We now prove the existence of a recovery sequence meeting the boundary conditions. As suggested by Remark 2.6, we focus on the stiff part. Let us consider again the functional $\widehat{\mathcal{J}}_k^1$ in (6.2). Since the sequence $\{\widehat{\mathcal{J}}_k^1\}$ falls within the scopes of Theorem 3.8, for any $(\widehat{y}, \widehat{P}) \in W_0^{1,2}(\Omega; \mathbb{R}^3) \times$

$W^{1,q}(\Omega;K)$ Corollary 3.9 provides a sequence $\{(u_k,\widehat{P}_k)\}\subset W_0^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;K)$ such that $\{\nabla u_k\}$ is 2-equiintegrable, $(u_k,\widehat{P}_k)\xrightarrow{\mathcal{T}}(\widehat{y},\widehat{P})$ and

$$\limsup_{k\to+\infty}\widehat{\mathcal{J}}_k^1(u_k,\widehat{P}_k)\leq\mathcal{J}^1(y,P).$$

By reasoning as in the proof of the upper bound in Theorem 2.7 we retrieve a sequence $\{\widehat{y}_k,\widehat{P}_k\}\in W_0^{1,2}(\Omega;\mathbb{R}^3)\times W^{1,q}(\Omega;K)$ such that $\widehat{y}_k\to\widehat{y}$ in the sense of extensions, $\widehat{P}_k\to\widehat{P}$ uniformly and

$$\limsup_{k\to+\infty}\mathcal{J}_k(\widehat{y}_k,\widehat{P}_k)\leq\mathcal{J}(\widehat{y},\widehat{P}),$$

whence

$$\limsup_{k\to+\infty}(\inf\mathcal{J}_k)\leq\inf\mathcal{J}.$$

Recalling that $\{(y_k,P_k)\}$ is a sequence of almost minimizers, we conclude

$$\inf\mathcal{J}\leq\mathcal{J}(y,P)\leq\liminf_{k\to+\infty}\mathcal{J}_k(y_k,P_k)=\liminf_{k\to+\infty}\inf\mathcal{J}_k\leq\inf\mathcal{J},$$

as desired. □

6.2. **A non degenerate upper bound for the soft component.** We proved in Section 5 that the limiting behavior of the soft inclusions is described by a degenerate functional. However, under our assumptions, a non-degenerate upper bound may still be established, as we prove in the remainder. The argument follows [13], where Cherdantsev & Cherednichenko derived the effective energy of high-contrast nonlinear elastic materials. Differently from us, the Γ-limit that they retrieve keeps track of both the macro- and the microscopic variable, and this roots in the choice of a stronger notion of convergence. The drawback of such an approach is the lack of compactness for sequences with equibounded energy. It was shown in [26, Example 2.12] that, when weaker topologies are considered, the quasiconvex envelope does not provide the correct limiting energy density for the Γ-lower limit.

We start by proving a more detailed version of Lemma 5.8.

Lemma 6.1 (cf. Lemma 22 in [13])**.** *Let $w\in L^2(\Omega;W_0^{1,2}(Q^0;\mathbb{R}^3))\cap C^2(\Omega\times Q^0;\mathbb{R}^3)$. Then, there exists a sequence $\{w_k\}\subset L^2(\Omega;W_{\mathrm{per}}^{1,2}(\mathbb{R}^3;\mathbb{R}^3))$ such that $\nabla_z w_k\to\nabla_z w$ strongly in $L^2(\Omega\times Q;\mathbb{R}^{3\times3})$. Besides, setting for $x\in\Omega$*

$$v_k(x):=w_k\left(x,\frac{x}{\varepsilon_k}\right),\tag{6.5}$$

$\{v_k\}$ converges strongly two-scale to w in L^2 and (5.15) holds.

Proof. We extend w by setting it equal to 0 on $Q\backslash Q^0$, so as to obtain a function in $L^2(\Omega;W_{\mathrm{per}}^{1,2}(\mathbb{R}^3;\mathbb{R}^3))$ which, by a slight abuse of notation, we denote again by w.

Keeping in mind the definition of Ω_k^Q (see (5.13)), for $(\bar{x},\bar{z})\in\Omega\times\mathbb{R}^3$ we define $w_k(\bar{x},\bar{z})$ in terms of the averages of $w(\,\cdot\,,\bar{z})$ on the cubes that form Ω_k^Q:

$$w_k(\bar{x},\bar{z}):=\begin{cases}\displaystyle\fint_{\varepsilon_k(t+Q)}w(x,\bar{z})\,\mathrm{d}x & \text{if } \bar{x}\in\varepsilon_k(t+Q) \text{ for some } t\in\hat{T}_k,\\ 0 & \text{for any other } \bar{x}\in\Omega.\end{cases}\tag{6.6}$$

By definition, $w_k(\,\cdot\,,z)$ is piecewise constant for all $z\in\bar{Q}$. Moreover, for almost every $x\in\Omega$, $w_k(x,\,\cdot\,)$ is Q-periodic as well as weakly differentiable, and $\nabla_z w_k\to\nabla_z w$ strongly in $L^2(\Omega\times$

$Q;\mathbb{R}^{3\times3})$. Indeed, from (6.6) and Jensen's inequality, we have that

$$\begin{aligned}
&\int_\Omega \int_Q |\nabla_z w_k(x,z) - \nabla_z w(x,z)|^2 \,\mathrm{d}z\,\mathrm{d}x \\
&\quad = \int_{\Omega_k^Q} \int_Q |\nabla_z w_k(x,z) - \nabla_z w(x,z)|^2 \,\mathrm{d}z\,\mathrm{d}x + \int_{\Omega\setminus\Omega_k^Q} \int_Q |\nabla_z w(x,z)|^2 \,\mathrm{d}z\,\mathrm{d}x \\
&\quad = \sum_{t\in\hat{T}_k} \int_{\varepsilon_k(t+Q)} \int_Q |\nabla_z w_k(x,z) - \nabla_z w(x,z)|^2 \,\mathrm{d}z\,\mathrm{d}x + o(1) \\
&\quad \le \sum_{t\in\hat{T}_k} \int_{\varepsilon_k(t+Q)} \int_Q \fint_{\varepsilon_k(t+Q)} |\nabla_z w(\xi,z) - \nabla_z w(x,z)|^2 \,\mathrm{d}\xi\,\mathrm{d}z\,\mathrm{d}x + o(1),
\end{aligned}$$

and the last term is infinitesimal for $k\to+\infty$ (recall that $w\in C^2$ and the mean value theorem applies).

We now turn to the functions v_k given by (6.5). First of all, we point out that, thanks to the regularity of w, v_k is measurable and vanishes on Ω_k^1. Besides, it belongs to $W_0^{1,2}(\Omega_k^0;\mathbb{R}^3)$. Secondly, we show that $\{v_k\}$ converges weakly two-scale to w in L^2. To this aim, let us fix $\phi\in C(\bar{\Omega}; C_{\mathrm{per}}(\mathbb{R}^3;\mathbb{R}^3))$. We find

$$\begin{aligned}
\int_\Omega v_k(x)\cdot\phi\left(x,\frac{x}{\varepsilon_k}\right)\mathrm{d}x &= \int_{\Omega_k^0} w_k\left(x,\frac{x}{\varepsilon_k}\right)\cdot\phi\left(x,\frac{x}{\varepsilon_k}\right)\mathrm{d}x \\
&= \sum_{t\in T_k}\int_{\varepsilon_k(t+Q^0)} w_k\left(x,\frac{x}{\varepsilon_k}\right)\cdot\phi\left(x,\frac{x}{\varepsilon_k}\right)\mathrm{d}x \\
&= \varepsilon_k^3\sum_{t\in T_k}\int_{Q^0} w_k(\varepsilon_k(t+z),z)\cdot\phi(\varepsilon_k(t+z),z)\,\mathrm{d}z \\
&= \sum_{t\in\hat{T}_k}\int_{Q^0}\int_{\varepsilon_k(t+Q)} w(x,z)\cdot\phi(\varepsilon_k(t+z),z)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_{\Omega_k^Q}\int_{Q^0} w(x,z)\cdot\phi_k(x,z)\,\mathrm{d}z\,\mathrm{d}x,
\end{aligned}$$

where $\phi_k(x,z) \coloneqq \phi(\varepsilon_k(t+z),z)$ if $x\in\varepsilon_k(t+Q)$ with $t\in\hat{T}_k$. By the dominated convergence theorem, we infer

$$\lim_{k\to+\infty}\int_\Omega v_k(x)\cdot\phi\left(x,\frac{x}{\varepsilon_k}\right)\mathrm{d}x = \int_\Omega\int_{Q^0} w(x,z)\cdot\phi(x,z)\,\mathrm{d}z\,\mathrm{d}x,$$

that is, $v_k \xrightharpoonup{2} w$ weakly two-scale in L^2 (recall that $w(x,z)=0$ if $z\in Q^1$).

In order to prove that strong two-scale convergence actually holds, we study the limiting behavior of the L^2 norm of $\{v_k\}$. On one hand, the weak two-scale convergence yields

$$\|w\|_{L^2(\Omega\times Q)} \le \liminf_{k\to+\infty}\|v_k\|_{L^2(\Omega)}. \tag{6.7}$$

On the other hand, from the properties of $\{w_k\}$ and a change of variables we have the identities

$$\begin{aligned}
\int_\Omega |v_k(x)|^2\,\mathrm{d}x &= \int_{\Omega_k^0}\left|w_k\left(x,\frac{x}{\varepsilon_k}\right)\right|^2\mathrm{d}x = \sum_{t\in T_k}\int_{\varepsilon_k(t+Q^0)}\left|w_k\left(x,\frac{x}{\varepsilon_k}\right)\right|^2\mathrm{d}x \\
&= \sum_{t\in T_k}\varepsilon_k^3\int_{Q^0}|w_k(\varepsilon_k(t+z),z)|^2\,\mathrm{d}z = \sum_{t\in\hat{T}_k}\varepsilon_k^3\int_{Q^0}\left|\fint_{\varepsilon_k(t+Q)} w(x,z)\,\mathrm{d}x\right|^2\mathrm{d}z.
\end{aligned}$$

Thanks to Jensen's inequality we deduce

$$\int_\Omega |v_k(x)|^2 \,\mathrm{d}x \le \sum_{t\in\hat{T}_k} \varepsilon_k^3 \int_{Q^0} \fint_{\varepsilon_k(t+Q)} |w(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z = \int_{Q^0}\int_{\Omega_k^Q} |w(x,z)|^2 \,\mathrm{d}x\,\mathrm{d}z.$$

This, combined with (6.7), ensures that

$$\lim_{k\to+\infty} \|v_k\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega\times Q)}.$$

In view of Definition 3.5 the conclusion is achieved.

Finally, the strong convergence (5.15) follows by observing that, if $x \in \varepsilon_k(t+Q)$, it holds

$$\nabla_z \hat{v}_k(x,z) = \nabla_z w_k(\varepsilon_k(t+z), z).$$

□

We are now in a position to prove a non-degenerate Γ-upper limit inequality that is the counterpart of the one in Proposition 5.7 under the current stronger convergence assumptions.

Proposition 6.2. *Let $\{W_k^0\}_k$ satisfy assumptions E3–E5. For any $(w,P) \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3)) \times W^{1,q}(\Omega;\mathsf{SL}(3))$. there exists a sequence $\{v_k\} \subset W_0^{1,2}(\Omega_k^0;\mathbb{R}^3)$ such that:*

(1) $v_k \xrightarrow{2} w$ strongly two-scale in L^2;
(2) $\varepsilon_k \nabla v_k \xrightharpoonup{2} \nabla_z w$ weakly two-scale in L^2;
(3) whenever $P_k \to P$ uniformly, it holds

$$\limsup_{k\to+\infty} \int_\Omega \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x)) \,\mathrm{d}x \quad \le \quad \int_\Omega \int_{Q^0} \mathcal{Q}' W^0(\nabla_z w(x,z), P^{-1}(x)) \,\mathrm{d}z\,\mathrm{d}x,$$

where $\mathcal{Q}'W^0$ is given by (2.8).

The conclusion is not a straightforward consequence of Lemma 6.1, because along the sequence $\{v_k\}$ in (6.5) we would not end up with the correct limiting energy density. Therefore, the actual recovery sequence is obtained by adding a "correction" to v_k.

Proof of Proposition 6.2. The proof consists of several steps. At first, to circumvent measurability issues, it is convenient to consider a sufficiently regular w. Under such assumption, we are able to construct a recovery sequence of the form $v_k = \tilde{v}_k + \tilde{w}_k$, where $\{\tilde{v}_k\}$ is provided by Lemma 6.1 and $\{\tilde{w}_k\}$ allows to pass from the densities W_k^0 to $\mathcal{Q}'W_k^0$. The definition of $\tilde{w}_k$ is given in Step 1, while Step 2 deals with the upper limit inequality in the regular case. The general statement is eventually retrieved by approximation.

STEP 1: CONSTRUCTION OF $\tilde{w}_k$ FOR A REGULAR w

Let us assume that $w \in L^2(\Omega; W_0^{1,2}(Q^0;\mathbb{R}^3)) \cap C^2(\Omega\times Q^0;\mathbb{R}^3)$. We consider a cover of Q^0 made of cubes whose edge length is ε_k. We set $\hat{\Sigma}_k := \{\, s\in\mathbb{Z}^3 : \varepsilon_k(s+Q) \subset \bar{Q}^0 \,\}$ and, for all $(t,s) \in \hat{T}_k \times \hat{\Sigma}_k$, we introduce the averages

$$A_k(t,s) := \fint_{\varepsilon_k(t+Q)} \fint_{\varepsilon_k(s+Q)} \nabla_z w(x,z) \,\mathrm{d}z\,\mathrm{d}x \tag{6.8}$$

and the piecewise constant functions

$$A_k(x,z) := \begin{cases} A_k(t,s) & \text{if } (x,z) \in \varepsilon_k(t+Q)\times\varepsilon_k(s+Q), (t,s)\in\hat{T}_k\times\hat{\Sigma}_k, \\ 0 & \text{otherwise.} \end{cases}$$

We record here for later use that, by means of Lebesgue differentiation and dominated convergence theorems, it follows

$$\begin{aligned}
&\lim_{k\to+\infty}\|A_k-\nabla_z w\|^2_{L^2(\Omega\times Q)}\\
&\quad=\lim_{k\to+\infty}\sum_{t\in\hat T_k}\sum_{s\in\hat\Sigma_k}\int_{\varepsilon_k(t+Q)}\int_{\varepsilon_k(s+Q)}|A_k(t,s)-\nabla_z w(x,z)|^2\,\mathrm{d}z\,\mathrm{d}x \qquad (6.9)\\
&\quad=0.
\end{aligned}$$

By the definition of $\mathcal{Q}'W^0_k$, for all $k\in\mathbb{N}$ there exists $\psi_k\in W^{1,2}_0(Q;\mathbb{R}^3)$ such that

$$\int_Q \chi^0(z)W^0_k\Big((A_k(t,s)+\nabla\psi_k(z))P_k^{-1}(x)\Big)\,\mathrm{d}z\le \mathcal{Q}'W^0_k(A_k(t,s),P^{-1}(x))+\frac{1}{k}. \qquad (6.10)$$

Note that, due to the smoothness of w, the averages A_k are bounded uniformly in k, t and s. In the light of Lemma 5.1, the values $\mathcal{Q}'W^0_k(A_k(t,s),P^{-1}(x))$ are uniformly bounded as well. Therefore, by combining (6.10) with assumption E3, we deduce that $\{\psi_k\}$ is bounded in $W^{1,2}_0(Q;\mathbb{R}^3)$.

A change of variables in (6.10) yields

$$\begin{aligned}
&\int_{\varepsilon_k(s+Q)}\chi^0\left(\frac{z}{\varepsilon_k}-s\right)W^0_k\left(\left(A_k(t,s)+\nabla\psi_k\left(\frac{z}{\varepsilon_k}-s\right)\right)P^{-1}(x)\right)\mathrm{d}z\\
&\qquad\qquad\le\varepsilon_k^3\left(\mathcal{Q}'W^0_k(A_k(t,s),P^{-1}(x))+\frac{1}{k}\right), \qquad (6.11)
\end{aligned}$$

and that suggests us to introduce the functions

$$\tilde\psi_k(x,z):=\begin{cases}\varepsilon_k\psi_k\left(\dfrac{z}{\varepsilon_k}-s\right) & \text{if } (x,z)\in\varepsilon_k(t+Q)\times\varepsilon_k(s+Q),\ (t,s)\in\hat T_k\times\hat\Sigma_k,\\ 0 & \text{otherwise.}\end{cases}$$

Note that, for each k and $x\in\Omega$, $\tilde\psi_k(x,\cdot\,)$ admits a weak derivative with respect to z; thus, by summing over $(t,s)\in\hat T_k\times\hat\Sigma_k$, from (6.11) we may write

$$\begin{aligned}
&\sum_{(t,s)\in\hat T_k\times\hat\Sigma_k}\int_{\varepsilon_k(t+Q)}\int_{\varepsilon_k(s+Q)}\chi^0\left(\frac{z}{\varepsilon_k}-s\right)W^0_k\Big((A_k(x,z)+\nabla_z\tilde\psi_k(x,z))P^{-1}(x)\Big)\,\mathrm{d}z\,\mathrm{d}x\\
&\qquad\le\sum_{(t,s)\in\hat T_k\times\hat\Sigma_k}\int_{\varepsilon_k(t+Q)}\varepsilon_k^3\left(\mathcal{Q}'W^0_k(A_k(t,s),P_k^{-1}(x))+\frac{1}{k}\right)\mathrm{d}x. \qquad (6.12)
\end{aligned}$$

We also observe that, since $\{\psi_k\}$ is bounded, $\tilde\psi_k\to0$ strongly in $L^2(\Omega\times Q;\mathbb{R}^3)$. Then, given that $\{\nabla_z\tilde\psi\}$ is bounded $L^2(\Omega\times Q;\mathbb{R}^{3\times3})$, it must converge weakly in L^2 to 0. It follows that, if w_k is as in Lemma 6.1 and if $(x,z)\in\varepsilon_k(t+Q)\times\varepsilon_k(s+Q)$ with $(t,s)\in\hat T_k\times\hat\Sigma_k$,

$$\nabla_z(w_k+\tilde\psi_k)\rightharpoonup\nabla_z w\quad\text{weakly in } L^2(\Omega\times Q;\mathbb{R}^{3\times3}). \qquad (6.13)$$

We further notice that

$$\begin{aligned}
\tilde w_k(x)&:=\tilde\psi_k\left(x,\frac{x}{\varepsilon_k}\right)\\
&=\sum_{(t,s)\in\hat T_k\times\hat\Sigma_k}\varepsilon_k\psi_k\left(\frac{x}{\varepsilon_k^2}-s\right)\chi_{\varepsilon_k(t+Q)}(x)\chi_{\varepsilon_k(s+Q)}\left(\frac{x}{\varepsilon_k}\right)
\end{aligned}$$

is a measurable function. A quick application of the definition of weak derivative proves also that $\tilde w_k$ belongs to $W^{1,2}_0(\Omega^0_k;\mathbb{R}^3)$.

STEP 2: w REGULAR

We now turn to the proof of the limsup inequality along the sequence $\{v_k\}$ defined as

$$v_k := \tilde{v}_k + \tilde{w}_k, \tag{6.14}$$

where

$$\tilde{v}_k(x) := w_k\Big(x, \frac{x}{\varepsilon_k}\Big)$$

with w_k as in Lemma 6.1, and where $\{\tilde{w}_k\}$ was introduced in Step 1. We have

$$\hat{v}_k(x,z) := \mathsf{S}_k v_k(x,z) = w_k\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor + \varepsilon_k z, z\right) + \tilde{\psi}_k\left(\varepsilon_k\left\lfloor\frac{x}{\varepsilon_k}\right\rfloor + \varepsilon_k z, z\right),$$

so that if $(x,z) \in \varepsilon_k(t+Q) \times \varepsilon_k(s+Q)$

$$\nabla_z \hat{v}_k(x,z) = \nabla_z w_k(\varepsilon_k(t+z), z) + \nabla\psi_k\left(\frac{z}{\varepsilon_k} - s\right). \tag{6.15}$$

Taking into account (6.13), (6.15) and Lemma 3.7(1), it follows that

$$\varepsilon_k \nabla v_k \xrightharpoonup{2} \nabla_z w \quad \text{weakly two-scale in } L^2.$$

Recalling Lemma 5.5, we have that

$$\begin{aligned}
&\limsup_{k\to+\infty} \int_\Omega \chi_k^0(x) W_k^0(\varepsilon_k \nabla v_k(x) P_k^{-1}(x))\, \mathrm{d}x \\
&\quad= \limsup_{k\to+\infty} \sum_{t\in T_k} \int_{\varepsilon_k(t+Q)} \int_{Q^0} W_k^0\left(\nabla_z \hat{v}_k(x,z) P_k^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x \\
&\quad= \limsup_{k\to+\infty} I_k,
\end{aligned}$$

where

$$I_k := \sum_{(t,s)\in\hat{T}_k\times\hat{\Sigma}_k} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} W_k^0(\nabla_z \hat{v}_k(x,z) P_k^{-1}(x))\, \mathrm{d}z\, \mathrm{d}x.$$

Indeed, $\hat{v}_k$ vanishes if $x \in \Omega\setminus\Omega_k^Q$ or if $z \in Q^0 \setminus \cup\{\varepsilon_k(s+Q) : s \in \hat{\Sigma}_k\}$, and the sequence $\{W_k^0(0)\}$ is bounded by virtue of E3. Therefore, since the measure of $\Omega\setminus\Omega_k^Q$ and of $Q^0\setminus\cup\{\varepsilon_k(s+Q) : s \in \hat{\Sigma}_k\}$ vanish for $k \to +\infty$, the second equality holds.

Being the value of $\nabla_z \hat{v}_k(x,z)$ expressed by formula (6.15), we introduce

$$I_k' := \sum_{t,s} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} W_k^0\left(\left(A_k(t,s) + \nabla\psi_k\left(\frac{z}{\varepsilon_k} - s\right)\right) P_k^{-1}(x)\right) \mathrm{d}z\, \mathrm{d}x,$$

where the summation runs over $\hat{T}_k \times \hat{\Sigma}_k$. By exploiting assumption E4 and Hölder's inequality, we obtain the estimate

$$|I_k - I_k'| \le c \sum_{t,s} \int_{\varepsilon_k(t+Q)} \int_{\varepsilon_k(s+Q)} \left|\Big(\nabla_z w_k(\varepsilon_k(t+z), z) - A_k(t,s)\Big) P_k^{-1}(x)\right|^2 \mathrm{d}z\, \mathrm{d}x.$$

In view of Lemma 6.1 and (6.9) we deduce

$$\lim_{k\to+\infty} |I_k - I_k'| = 0. \tag{6.16}$$

Next, let us set

$$I_k'' := \int_{\Omega_k^Q} \int_{Q^0} \mathcal{Q}' W_k^0(A_k(x,z), P_k^{-1}(x))\, \mathrm{d}z\, \mathrm{d}x.$$

According to (6.12), the difference between the integrands of I'_k and I''_k is of order k^{-1}:

$$\lim_{k\to+\infty} |I'_k - I''_k| = 0. \tag{6.17}$$

Finally, we compare I''_k and the limiting functional. We have

$$\begin{aligned}
&\left| I''_k - \int_\Omega \int_{Q^0} \mathcal{Q}'W^0(\nabla_z w(x,z), P^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x \right| \\
&\leq \int_{\Omega_k^Q} \int_{Q^0} \left| \mathcal{Q}'W_k^0(A_k(x,z), P_k^{-1}(x)) - \mathcal{Q}'W_k^0(\nabla_z w(x,z), P_k^{-1}(x)) \right| \mathrm{d}z\,\mathrm{d}x \\
&\quad + \int_{\Omega_k^Q} \int_{Q^0} \left| \mathcal{Q}'W_k^0(\nabla_z w(x,z), P_k^{-1}(x)) - \mathcal{Q}'W_k^0(\nabla_z w(x,z), P^{-1}(x)) \right| \mathrm{d}z\,\mathrm{d}x \\
&\quad + \int_{\Omega_k^Q} \int_{Q^0} \left| \mathcal{Q}'W_k^0(\nabla_z w(x,z), P^{-1}(x)) - \mathcal{Q}'W^0(\nabla_z w(x,z), P^{-1}(x)) \right| \mathrm{d}z\,\mathrm{d}x \\
&\quad + \int_{\Omega\setminus\Omega_k^Q} \int_{Q^0} \mathcal{Q}'W^0(\nabla_z w(x,z), P^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

All the terms on the right-hand side vanish as $k\to+\infty$. Indeed, by using the Lipschitz continuity of $\mathcal{Q}'W_k^0$ (see Lemma 5.1(1)) and the uniform bound on $\{P_k\}$, the first summand is controlled by the norm of $A_k - \nabla_z v$, which, according to (6.9), is infinitesimal. For what concerns the second term, Lemma 5.1(2) and the uniform convergence of $\{P_k\}$ imply that the integrand is infinitesimal for $k\to+\infty$. The third quantity vanishes because $\{\mathcal{Q}'W_k^0\}$ pointwise converges to $\mathcal{Q}'W^0$ (recall that they are just variants of the quasiconvex envelopes). Lastly, the fourth summand is negligible since $\mathcal{L}^3(\Omega\setminus\Omega_k^Q)$ tends to 0.

On the whole, taking into account (6.16) and (6.17), we conclude

$$\lim_{k\to+\infty} I_k = \int_\Omega \int_{Q^0} \mathcal{Q}'W^0(\nabla_z w(x,z), P^{-1}(x))\,\mathrm{d}z\,\mathrm{d}x.$$

STEP 3: w GENERIC

The argument follows the one of Case 2 in the proof of Proposition 5.7.

□

6.3. **A variant with plastic dissipation.** With a view to applying Theorem 2.7 to time-dependent problems, it is useful to modify the functionals $\mathcal{J}_\varepsilon$ by adding a term that encodes the plastic dissipation mechanism of the system. Precisely, we take into account the non-symmetric distance $D\colon \mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\to[0,+\infty]$ in (3.9) and we define the dissipation between $P_0, P_1\colon \Omega\to\mathsf{SL}(3)$ as

$$\mathcal{D}(P_0;P_1) := \int_\Omega D(P_0,P_1)\,\mathrm{d}x.$$

From a physical viewpoint, if $P_0, P_1\colon\Omega\to\mathsf{SL}(3)$ are admissible plastic strains, $\mathcal{D}(P_0,P_1)$ is interpreted as the minimum amount of energy that is dissipated when the system moves from a plastic configuration to another. Then, assuming that $\bar P\in W^{1,q}(\Omega;\mathsf{SL}(3))$ represents a pre-existent plastic strain of the body, we set

$$\mathcal{J}_\varepsilon^{\mathrm{diss}}(y,P) := \mathcal{E}_\varepsilon(y,P) + \mathcal{D}(\bar P;P) + \|\nabla P\|^q_{L^q(\Omega;\mathbb{R}^{3\times3\times3})}. \tag{6.18}$$

In the same spirit of (2.9) and (2.10), we distinguish between the dissipation of the inclusions and the one of the matrix, respectively

$$\mathcal{D}^0_\varepsilon(\bar{P};P) \coloneqq \int_\Omega \chi^0_\varepsilon(x) D(\bar{P},P)\,\mathrm{d}x, \qquad \mathcal{D}^1_\varepsilon(\bar{P};P) \coloneqq \int_\Omega \chi^1_\varepsilon(x) D(\bar{P},P)\,\mathrm{d}x.$$

For what concerns the compactness of sequences with equibounded energy, we notice that the presence of the dissipation $\mathcal{D}$ does not affect Lemma 4.1: the same conclusions hold if the bound on $\mathcal{J}_k(y_k,P_k)$ is replaced by a bound on $\mathcal{J}^{\mathrm{diss}}_k(y_k,P_k)$.

Also our Γ-convergence results easily extend to the family $\{\mathcal{J}^{\mathrm{diss}}_\varepsilon\}$. The dissipation is indeed a continuous perturbation:

Lemma 6.3. *Let $P,\bar{P} \in C(\overline{\Omega};K)$ be given. If $\{P_k\} \subset C(\overline{\Omega};K)$ converges uniformly to P, then*

$$\lim_{k\to+\infty} \mathcal{D}^i_k(\bar{P};P_k) = \mathcal{L}^3(Q^i)\mathcal{D}(\bar{P};P) \qquad \textit{for } i=0,1.$$

Proof. We firstly observe that if $P_k \to P$ pointwise, then

$$D\big(P_k(x),P(x)\big) \to 0, \qquad D\big(P(x),P_k(x)\big) \to 0. \tag{6.19}$$

To see this, let γ be such that for all $(t,F,G) \in [0,1]\times \mathsf{SL}(3)\times\mathsf{SL}(3)$, $\gamma(t,F,G)$ is the evaluation at t of the unique minimizing geodesic connecting F and G, cf. Corollary 3.11. Then, by (3.9) and the definition of γ,

$$\begin{aligned} D\big(P_k(x),P(x)\big) &= \int_0^1 \Delta\Big(\gamma\big(t,P_k(x),P(x)\big),\dot{\gamma}\big(t,P_k(x),P(x)\big)\Big)\mathrm{d}t \\ &\le c\int_0^1 |\dot{\gamma}\big(t,P_k(x),P(x)\big)|\,\mathrm{d}t, \end{aligned}$$

where the inequality follows from the definition of Δ in (3.8) and (2.4). Since $\dot{\gamma}$ is continuous and bounded, by dominated convergence we deduce that the last term vanishes as $k\to+\infty$. In a similar fashion, we show that $D(P,P_k)\to 0$ as well.

As second step, we notice that

$$D\big(\bar{P}(x),P_k(x)\big) \to D\big(\bar{P}(x),P(x)\big). \tag{6.20}$$

Indeed, the triangular inequality yields

$$D\big(\bar{P}(x),P(x)\big) - D\big(P_k(x),P(x)\big) \le D\big(\bar{P}(x),P_k(x)\big) \le D\big(\bar{P}(x),P(x)\big) + D\big(P(x),P_k(x)\big),$$

and the assertion follows as a consequence of (6.19).

Finally, we observe that (6.20) grants that

$$\lim_{k\to+\infty} \mathcal{D}^i_k(\bar{P};P_k) = \lim_{k\to+\infty}\int_\Omega \chi^i_k(x) D\big(\bar{P}(x),P(x)\big)\mathrm{d}x,$$

and the conclusion is achieved by arguing as in Lemma 4.2. ☐

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E. Davoli, C. Gavioli, V. Pagliari.

Institute of Analysis and Scientific Computing, TU Wien,

Wiedner Hauptstraße 8-10, 1040 Vienna, Austria.

E-mails: elisa.davoli@tuwien.ac.at, chiara.gavioli@tuwien.ac.at, valerio.pagliari@tuwien.ac.at.