ON GENERALIZED NONPARAMETRIC MINIMAL HYPERFURFACES IN HIGH DIMENSION

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Abstract. Nonparametric g-surfaces in Euclidean space have recently been characterized by Bildhauer-Fuchs in terms of closure of a 1-form associated to the so called asymptotic normal. This 1-form can be written by means of the pull-back of a canonical vector-valued 1-form through a suitable map depending on the asymptotic normal, that in the minimal surfaces case agrees with the Gauss graph map. We show that a similar characterization holds true for g-hypersurfaces of any high dimension N, but this time in terms of a canonical vector valued form of degree $N-1$. In the minimal hypersurfaces case, we finally discuss the lack of a relationship between the previous result and existence of good parameterizations, when N is greater than two.

Introduction

We deal with critical points of the functional

$$\mathcal{F}_g(u) := \int_{B^N} g(|\nabla u|)\, d\mathcal{L}^N\,, \qquad u \in C^2(B^N, \mathbb{R})$$

on smooth real valued functions u defined in the unit ball B^N in $\mathbb{R}^N$, in any dimension $N \geq 2$.

The isotropic functional is given by integration with respect to Lebesgue measure $\mathcal{L}^N$ of a non-negative and smooth integrand $g : [0, +\infty) \to \mathbb{R}$ acting on the modulus of the gradient ∇u.

The associated Euler-Lagrange equation reads as

$$\operatorname{div}\big(\Xi(|\nabla u|)\nabla u\big) = 0\,, \quad \Xi(t) := \frac{g'(t)}{t} \tag{0.1}$$

provided that $\Xi(t)$ and $\Xi'(t)$ are bounded functions in $[0, +\infty)$, see (2.5).

If a smooth function u satisfies equation (0.1), the *graph* $\mathcal{G}_u$ is commonly said to be a *g-hypersurface* in $\mathbb{R}^{N+1}$.

2010 *Mathematics Subject Classification.* 49Q15; 53A10; 58E15.

Key words and phrases. generalized surfaces, canonical forms, parameterizations.

I wish to thank M. Bildhauer for several interesting discussions. The research of D.M. was partially supported by the GNAMPA of INDAM.

In this paper, we show in any dimension $N \geq 2$ that the validity of equation (0.1) is equivalent to the closure of a suitable $\mathbb{R}^{N+1}$-valued $(N-1)$-form in B^N. This differential form is essentially obtained through the *pull-back* of a *canonical* vector valued differential form by means of a natural extension of the *asymptotic normal* introduced by Bildhauer-Fuchs [3] in dimension $N = 2$.

More precisely, denoting respectively by $\mathbb{R}_x^{N+1}$ and $\mathbb{R}_y^{N+1}$ the ambient spaces where the graph $\mathcal{G}_u$ and the *g-normal* $\widetilde{\nu}_u$ to u live, our Main Result involves a map depending on both the graph map and g-normal,

$$\widetilde{\Phi}_u : B^N \to \mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1}$$

see (1.1), (1.2), and (1.4).

Notice that in the model case when $g(t) = \sqrt{1+t^2}$, so that $\mathcal{F}_g(u)$ is the *area functional*, we have $\Xi(t) = (1+t^2)^{-1/2}$ and (0.1) reduces to the *nonparametric minimal hypersurfaces equation*:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0\,.$$

Moreover, in that case the g-normal reduces to the unit normal ν_u to $\mathcal{G}_u$

$$\nu_u := \frac{1}{\sqrt{1+|\nabla u|^2}}\,\big(-\nabla u, 1\big) \tag{0.2}$$

and finally $\widetilde{\Phi}_u$ agrees with the *Gauss graph map*

$$\Phi_u(\widetilde{x}) := \big((\widetilde{x}, u(\widetilde{x})), \nu_u(\widetilde{x})\big)\,, \quad \widetilde{x} \in B^N\,. \tag{0.3}$$

Furthermore, we denote by $\widetilde{\Phi}_u^{\#}\omega$ the pull-back through the map $\widetilde{\Phi}_u$ of a differential form ω in $\mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1}$, by d the *exterior derivative* operator, and by $\Omega^{(N)}$ the (naturally oriented) volume N-form in $\mathbb{R}^N$, see (1.5). We finally remark that for vector valued forms, both pull-back and exterior derivative are defined componentwise.

Referring to Sec. 1 for further notation and details, we are now in position to state the Main Result of this paper, that holds true in any dimension.

Theorem 0.1. *Let $N \geq 2$ integer. There exists a canonical $\mathbb{R}^{N+1}$-valued $(N-1)$-form $\bar{\omega}^{(N-1)}$ in $\mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1}$ such that for any smooth function $u \in C^2(B^N, \mathbb{R})$*

$$\mathrm{d}\widetilde{\Phi}_u^{\#}\bar{\omega}^{(N-1)} = \operatorname{div}\Big(\Xi(|\nabla u|)\,\nabla u\Big)\,(-\nabla u,\, 1) \wedge \Omega^{(N)}\,.$$

Therefore, the graph $\mathcal{G}_u$ is a g-hypersurface in $\mathbb{R}_x^{N+1}$ if and only if $\widetilde{\Phi}_u^{\#}\bar{\omega}^{(N-1)}$ is a closed $\mathbb{R}^{N+1}$-valued $(N-1)$-form in B^N.

We refer to Theorems 2.1 or 4.1 for a more precise statement in dimension $N = 2$ or $N \geq 3$, and to equations (2.1), (4.5), (4.6) for the explicit expression of the canonical form $\bar{\omega}^{(N-1)}$ in dimension $N = 2, 3, 4$, respectively.

In low dimension $N = 2$, compare equation (2.3) below, our Main Result was essentially obtained in [3], where the authors extended a classical property concerning minimal surfaces in $\mathbb{R}^3$. This crucial property, which yields to existence of isothermal parameters, was written in terms of differential forms by Dierkes-Hildebrandt-Sauvigny in Sec. 2.2 of their treatise [5].

The role of the Gauss graph map (0.3) in the analysis of functionals depending on curvatures of codimension one surfaces, goes back to the excellent work by Anzellotti-Serapioni-Tamanini [2], see also [4]. The Gauss graph map is a main tool also in [9], where a relaxed curvature energy for nonparametric surfaces in $\mathbb{R}^3$ is analyzed, and more recently in [8], where elastic thin shells without through-the-thickness shear are depicted as Gauss graphs of parametric surfaces.

We finally present the plan of the paper. Notation is fixed in Sec. 1, whereas Theorem 0.1 in low dimension $N = 2$ is proved in Sec. 2. In Sec. 3, we then collect some known results concerning (asymptotic) conformal parameterizations, showing how they can be obtained from our Main Result in low dimension $N = 2$. Theorem 0.1 in high dimension $N \geq 3$ is proved in Sec. 4. Finally, in Sec. 5 we discuss the reason why in high dimension $N \geq 3$ our Main Result does not lead to existence of "good parameterizations", compared to the two-dimensional case treated by Bildhauer-Fuchs [3].

1. Notation

We set $x = (\widetilde{x}, x_{N+1}) \in \mathbb{R}^{N+1}_x$, where $\widetilde{x} := (x_1, \dots, x_N)$, so that the graph of a function $u \in C^2(B^N, \mathbb{R})$ is the nonparametric hypersurface

$$\mathcal{G}_u := \{x \in \mathbb{R}^{N+1}_x \mid x_{N+1} = u(\widetilde{x})\} .$$

We also denote by $f_{,i}$ the partial derivative of a smooth function $f : B^N \to \mathbb{R}$ in the i-th coordinate direction, so that the gradient of u reads as $\nabla u = (u_{,1}, \dots, u_{,N})$, and by $f_{,ij}$ the second order partial derivatives

$$f_{,ij} := \partial_{x_i} \partial_{x_j} f = \partial_{x_j} \partial_{x_i} f \,, \quad i, j = 1, \dots, N .$$

Extending to high dimension $N \geq 3$ the definition of asymptotic normal introduced in [3] in case $N = 2$, for a given integrand g as in the introduction, we call *g-normal* to the graph $\mathcal{G}_u$ at $(\widetilde{x}, u(\widetilde{x}))$ the $(N + 1)$-vector

$$\widetilde{\nu}_u(\widetilde{x}) := \big(\widetilde{\nu}_u^1(\widetilde{x}), \dots, \widetilde{\nu}_u^N(\widetilde{x}), \widetilde{\nu}_u^{N+1}(\widetilde{x})\big)$$

with first N components defined by

$$\widetilde{\nu}_u^j := -\Xi(|\nabla u|)\, u_{,j} \,, \quad j = 1, \dots, N \tag{1.1}$$

where $\Xi(t)$ is given by (0.1), and last component

$$\widetilde{\nu}_u^{N+1} := \Xi(|\nabla u|) + \vartheta(|\nabla u|) \,, \quad \vartheta(t) := g(t) - t g'(t) - \Xi(t) . \tag{1.2}$$

Therefore, in the minimal hypersurfaces case $g(t) = \sqrt{1 + t^2}$, we get

$$\Xi(t) = \frac{1}{\sqrt{1 + t^2}} \,, \quad \vartheta(t) \equiv 0 \,, \quad \widetilde{\nu}_u = \nu_u \tag{1.3}$$

where ν_u is the unit normal to $\mathcal{G}_u$, see (0.2).

Denoting by $y = (y_1, \dots, y_N, y_{N+1})$ the coordinates in the vector space $\mathbb{R}^{N+1}_y$ where the g-normal lives, we correspondingly introduce the map

$$\widetilde{\Phi}_u : B^N \to \mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$$

defined in terms of the g-normal (1.1)–(1.2) by

$$\widetilde{\Phi}_u(\widetilde{x}) := \big((\widetilde{x}, u(\widetilde{x})), \widetilde{\nu}_u(\widetilde{x})\big) . \tag{1.4}$$

Moreover, $(\mathrm{d}x^1, \dots, \mathrm{d}x^N, \mathrm{d}x^{N+1})$ and $(\mathrm{d}y^1, \dots, \mathrm{d}y^N, \mathrm{d}y^{N+1})$ denote the dual bases of covectors in $\mathbb{R}^{N+1}_x$ and $\mathbb{R}^{N+1}_y$, respectively, where d is the exterior derivative operator. Therefore, the volume N-form in the domain $\mathbb{R}^N$ that appears in Theorem 0.1 is:

$$\Omega^{(N)} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \tag{1.5}$$

whereas the differential of e.g. the function u and the j-th component of $\widetilde{\nu}_u$ become the 1-forms:

$$\mathrm{d}u = \sum_{i=1}^N u_{,i}\, \mathrm{d}x^i , \quad \mathrm{d}\widetilde{\nu}^j_u = \sum_{i=1}^N \widetilde{\nu}^j_{u,i}\, \mathrm{d}x^i , \quad j = 1, \dots, N+1 .$$

We also denote by $\widetilde{\Phi}_u^{\#}\omega$ the pull-back through the map $\widetilde{\Phi}_u$ of a differential form ω in $\mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$, and recall that for vector valued forms, both pull-back and exterior derivative are defined componentwise. For further details on differential forms we refer e.g. to Sec. 2.2.2 of the treatise [6].

Remark 1.1. We finally point out that the nonparametric hypersurface $\mathcal{G}_u$ is the image of B^N through the graph map $X(\widetilde{x}) := (\widetilde{x}, u(\widetilde{x}))$, and hence it is naturally equipped with the metric $\mathfrak{g}_{ij} := \partial_i X \bullet \partial_j X = \delta_{ij} + u_{,i} u_{,j}$, for $i, j = 1, \dots N$, where $\bullet$ is the scalar product in $\mathbb{R}^{N+1}_x$ and δ_{ij} is Kronecker symbol, so that

$$\mathfrak{g} := \det(\mathfrak{g}_{ij}) = 1 + |\nabla u|^2 .$$

Denoting by $(\mathfrak{g}^{ij})$ the inverse to the metric tensor $(\mathfrak{g}_{ij})$, we also have

$$\mathfrak{g}^{ii} = \mathfrak{g}^{-1} \cdot (1 + |\nabla u|^2 - {u_{,i}}^2) , \qquad \mathfrak{g}^{ij} = -\mathfrak{g}^{-1} \cdot u_{,i} u_{,j} \quad \text{if} \quad i \neq j .$$

2. The surface case

In this section, we prove Theorem 0.1 in low dimension $N = 2$. Namely, in Theorem 2.1 we recover a result that goes back to [3, Thm. 1.2].

For this purpose, we introduce the $\mathbb{R}^3$-valued 1-form $\bar{\omega}^{(1)}$ in $\mathbb{R}^3_x \times \mathbb{R}^3_y$

$$\bar{\omega}^{(1)} := \begin{pmatrix} -y_2\, \mathrm{d}x^3 + y_3\, \mathrm{d}x^2 \\ -y_3\, \mathrm{d}x^1 + y_1\, \mathrm{d}x^3 \\ -y_1\, \mathrm{d}x^2 + y_2\, \mathrm{d}x^1 \end{pmatrix} \tag{2.1}$$

(where from now on we denote vector-valued forms as column vectors) and observe that the $\mathbb{R}^3$-valued 1-form in B^2 given by the pull-back of $\bar\omega^{(1)}$ through the map $\widetilde{\Phi}_u$ from (1.4) becomes:

(2.2)
$$\widetilde{\Phi}_u^{\#}\bar\omega^{(1)} = \begin{pmatrix} \Xi(|\nabla u|)\,u_{,1}u_{,2}\,\mathrm{d}x^1 + \big(\Xi(|\nabla u|)\,(1+{u_{,2}}^2) + \vartheta(|\nabla u|)\big)\,\mathrm{d}x^2 \\ -\big(\Xi(|\nabla u|)\,(1+{u_{,1}}^2) + \vartheta(|\nabla u|)\big)\,\mathrm{d}x^1 - \Xi(|\nabla u|)\,u_{,1}u_{,2}\,\mathrm{d}x^2 \\ \Xi(|\nabla u|)\,u_{,1}\,\mathrm{d}x^2 - \Xi(|\nabla u|)\,u_{,2}\,\mathrm{d}x^1 \end{pmatrix}.$$

In particular, one recovers the notation from [3] in terms of vector product $\times$ in $\mathbb{R}^3$. In fact, denoting by v^T the transpose of a line vector $v \in \mathbb{R}^3$, after an identification of $\mathbb{R}^3_y$ with $\mathbb{R}^3_x$ we have:

$$\widetilde{\Phi}_u^{\#}\bar\omega^{(1)} = -(\widetilde{\nu}_u \times \mathrm{d}X)^T\,, \quad X(x_1,x_2) := (x_1, x_2, u(x_1,x_2))\,. \tag{2.3}$$

In the model case $g(t) = \sqrt{1+t^2}$, so that equations (1.3) hold, and hence $\widetilde{\Phi}_u$ agrees with the Gauss graph map (0.3), it is readily checked that

$$\mathrm{d}\Phi_u^{\#}\bar\omega^{(1)} = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(-u_{,1},\,-u_{,2},\,1)^T\,\mathrm{d}x^1\wedge\mathrm{d}x^2$$

so that $\Phi_u^{\#}\bar\omega^{(1)}$ is a closed 1-form in B^2 if and only if the graph $\mathcal{G}_u$ is a nonparametric minimal surface in $\mathbb{R}^3$.

Theorem 2.1. *Let $N = 2$ and let $\widetilde{\Phi}_u$ be given by* (1.4)*, with g-normal defined by* (1.1) *and* (1.2) *for some integrand g as in the introduction. Then, for any smooth function $u \in C^2(B^2,\mathbb{R})$, we have*

$$\mathrm{d}\widetilde{\Phi}_u^{\#}\bar\omega^{(1)} = \mathrm{div}\Big[\Xi(|\nabla u|)\,\nabla u\Big]\,(-u_{,1},\,-u_{,2},\,1)^T\,\mathrm{d}x^1\wedge\mathrm{d}x^2$$

where the function $\Xi(t)$ is given by (0.1) *and the canonical 1-form $\bar\omega^{(1)}$ by* (2.1)*. Therefore, the graph $\mathcal{G}_u$ is a g-surface in $\mathbb{R}^3$ if and only if $\widetilde{\Phi}_u^{\#}\bar\omega^{(1)}$ is a closed $\mathbb{R}^3$-valued 1-form in B^2.*

Proof. We first observe that by (2.2) we can write the differential

(2.4)
$$\mathrm{d}\widetilde{\Phi}_u^{\#}\bar\omega^{(1)} = \begin{pmatrix} \Big[\mathrm{div}\big(\Xi(|\nabla u|)\,({u_{,2}}^2,\,-u_{,1}u_{,2})\big) + \partial_{x_1}(\Xi+\vartheta)(|\nabla u|)\Big]\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ \Big[\mathrm{div}\big(\Xi(|\nabla u|)\,(-u_{,1}u_{,2},\,{u_{,1}}^2)\big) + \partial_{x_2}(\Xi+\vartheta)(|\nabla u|)\Big]\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ \mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \end{pmatrix}.$$

Recalling (1.2), we get

$$\begin{aligned} \Xi(t) = \frac{g'(t)}{t}\,, \quad \Xi'(t) = \frac{g''(t)\,t - g'(t)}{t^2}\,, \\ (\Xi+\vartheta)'(t) = -t\,g''(t) \quad \forall\, t>0 \end{aligned} \tag{2.5}$$

so that for $i = 1,2$ we infer:

$$\begin{aligned} \partial_{x_i}\Xi(|\nabla u|) = &\ \frac{g''(t)\,t - g'(t)}{t^3}\,u_{,\alpha}u_{,\alpha i} \\ \partial_{x_i}(\Xi+\vartheta)(|\nabla u|) = &\ -g''(t)\,u_{,\alpha}u_{,\alpha i} \end{aligned} \tag{2.6}$$

where (here and in the sequel) in the right-hand side we have set $t = |\nabla u|$, and the summation on repeated indices $\alpha = 1, 2$ is adopted.

Denoting by Δu the Laplacean of u and by $\bullet$ the scalar product in $\mathbb{R}^2$, we have:

$$\begin{aligned}
\operatorname{div}\big(\Xi(|\nabla u|)\,\nabla u\big) &= \nabla(\Xi(|\nabla u|)\bullet\nabla u + \Xi(|\nabla u|)\,\Delta u \\
&= \frac{g''(t)t - g'(t)}{t^3}\,\big((u_{,1}u_{,11} + u_{,2}u_{,12})\,u_{,1} + (u_{,1}u_{,12} + u_{,2}u_{,22})\,u_{,2}\big) \\
&\quad + \frac{g'(t)}{t}\,\big(u_{,11} + u_{,22}\big) \\
&= \frac{g''(t)}{t^2}\,\big({u_{,1}}^2 u_{,11} + {u_{,2}}^2 u_{,22} + 2\,u_{,1}u_{,2}u_{,12}\big) \\
&\quad + \frac{g'(t)}{t^3}\,\big({u_{,2}}^2 u_{,11} + {u_{,1}}^2 u_{,22} - 2\,u_{,1}u_{,2}u_{,12}\big)\,.
\end{aligned}$$

Moreover, as to e.g. the second line in equation (2.4), we compute:

$$\begin{aligned}
-\big[&\operatorname{div}\big(\Xi(|\nabla u|)\,(-u_{,1}u_{,2},\, {u_{,1}}^2)\big) + \partial_{x_2}(\Xi + \vartheta)(|\nabla u|)\big] \\
&= \frac{g''(t)t - g'(t)}{t^3}\,\big((u_{,1}u_{,11} + u_{,2}u_{,12})\,u_{,1}u_{,2} - (u_{,1}u_{,12} + u_{,2}u_{,22})\,{u_{,1}}^2\big) \\
&\quad + \frac{g'(t)}{t}\,\big(u_{,2}u_{,11} - u_{,1}u_{,12}\big) + g''(t)\,\big(u_{,1}u_{,12} + u_{,2}u_{,22}\big) \\
&= \frac{g''(t)}{t^2}\,\big({u_{,1}}^2 u_{,2}u_{,11} + {u_{,2}}^3 u_{,22} + 2\,u_{,1}{u_{,2}}^2 u_{,12}\big) \\
&\quad + \frac{g'(t)}{t^3}\,\big({u_{,2}}^3 u_{,11} + {u_{,1}}^2 u_{,2}u_{,22} - 2\,u_{,1}{u_{,2}}^2 u_{,12}\big) \\
&= u_{,2}\operatorname{div}\big(\Xi(|\nabla u|)\,\nabla u\big)\,.
\end{aligned}$$

Finally, concerning the first line in equation (2.4), we similarly obtain

$$\operatorname{div}\big(\Xi(|\nabla u|)\,({u_{,2}}^2,\, -u_{,1}u_{,2})\big) + \partial_{x_1}(\Xi + \vartheta)(|\nabla u|) = -u_{,1}\operatorname{div}\big(\Xi(|\nabla u|)\,\nabla u\big)$$

and hence the assertion readily follows. □

Remark 2.2. In the model case when $g(t) = \sqrt{1 + t^2}$, on account of Remark 1.1, equation (2.4) becomes:

$$\mathrm{d}\Phi_u^{\#}\bar\omega^{(1)} = \begin{pmatrix} \operatorname{div}\big(\mathfrak{g}^{-1/2}(1 + {u_{,2}}^2,\, -u_{,1}u_{,2})\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ \operatorname{div}\big(\mathfrak{g}^{-1/2}(-u_{,1}u_{,2},\, 1 + {u_{,1}}^2)\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ \operatorname{div}\big(\mathfrak{g}^{-1/2}\nabla u\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2 \end{pmatrix}. \tag{2.7}$$

Therefore, denoting by $A \in C^2(B^2, \mathbb{R}^{2\times 2})$ the symmetric tensor valued function with components by

$$A^i_j := \mathfrak{g}^{1/2}\,\mathfrak{g}^{ij} \tag{2.8}$$

for $i, j = 1, 2$, in the previous proof we have just checked that

$$-\operatorname{div} A = (\nabla u)^T \operatorname{div}(\mathfrak{g}^{-1/2}\nabla u) \tag{2.9}$$

on B^2, where divergence is computed along the raw components.

3. (Asymptotic) conformal parameterizations

In this section, we apply Theorem 2.1 to find existence of "good parameterizations" of nonparametric g-surfaces. For completeness, we also recall how isothermal parameters are obtained in the minimal surfaces case.

Using an argument similar to the one exploited by Bildhauer-Fuchs in [3], we obtain the following

Corollary 3.1. *Let $N=2$ and let $u\in C^2(B^2,\mathbb{R})$ satisfy the Euler-Lagrange equation* (0.1)*. Then, there exists a smooth vector field $\widetilde{F}:B^2\to\mathbb{R}^2$ such that for each $\widetilde{x}\in B^2$*

$$\nabla\widetilde{F}=\begin{pmatrix}\Xi(|\nabla u|)\,(1+{u_{,1}}^2)+\vartheta(|\nabla u|) & \Xi(|\nabla u|)\,u_{,1}u_{,2}\\ \Xi(|\nabla u|)\,u_{,1}u_{,2} & \Xi(|\nabla u|)\,(1+{u_{,2}}^2)+\vartheta(|\nabla u|)\end{pmatrix}. \tag{3.1}$$

Conversely, the existence of a smooth vector field satisfying (3.1) *implies the validity of Euler-Lagrange equation* (0.1)*.*

Proof. Consider the couple of 1-forms

$$\begin{aligned}\widetilde{\omega}^1 &:= \Big(\Xi(|\nabla u|)\,(1+{u_{,1}}^2)+\vartheta(|\nabla u|)\Big)\,\mathrm{d}x^1+\Xi(|\nabla u|)\,u_{,1}u_{,2}\,\mathrm{d}x^2\\ \widetilde{\omega}^2 &:= \Xi(|\nabla u|)\,u_{,1}u_{,2}\,\mathrm{d}x^1+\Big(\Xi(|\nabla u|)\,(1+{u_{,2}}^2)+\vartheta(|\nabla u|)\Big)\,\mathrm{d}x^2\,.\end{aligned} \tag{3.2}$$

In Theorem 2.1, we have seen that their differentials satisfy equations

$$\begin{aligned}\mathrm{d}\widetilde{\omega}^1 &= u_{,2}\cdot\mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\\ \mathrm{d}\widetilde{\omega}^2 &= -u_{,1}\cdot\mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big)\,\mathrm{d}x^1\wedge\mathrm{d}x^2\,.\end{aligned}$$

Therefore, B^2 being simply-connected, both $\widetilde{\omega}^1$ and $\widetilde{\omega}^2$ are exact 1-forms in B^2 if and only if the function u is a solution to equation (0.1). In that case, it then suffices to choose $\widetilde{F}=(\widetilde{F}^1,\,\widetilde{F}^2)$, where $\widetilde{F}^i\in C^2(B^2,\mathbb{R})$ satisfies $\mathrm{d}\widetilde{F}^i=\widetilde{\omega}^i$, for $i=1,2$. □

In the minimal surfaces case, one then readily obtains the classical existence result of a conformal parameterization for the graph map $X(\widetilde{x})=(\widetilde{x},u(\widetilde{x}))$, compare e.g. [5, Sec. 2.3].

Proposition 3.2. *If $\mathcal{G}_u$ is a nonparametric minimal surface in $\mathbb{R}^3$, and $\widetilde{F}$ is given by Corollary* 3.1 *in correspondence to $g(t)=\sqrt{1+t^2}$, then the vector field*

$$\Lambda(\widetilde{x}):=\widetilde{x}+\widetilde{F}(\widetilde{x}) \tag{3.3}$$

defines a smooth diffeomorphism $z=\Lambda(\widetilde{x})$ from B^2 onto its image, a smooth domain $\widehat{\Omega}$ of $\mathbb{R}^2$, and the parameterization

$$\widehat{X}(z):=\big(\Lambda^{-1}(z),u(\Lambda^{-1}(z))\big)\,,\qquad z=(z_1,z_2)\in\widehat{\Omega} \tag{3.4}$$

of the graph map is conformal. Precisely, at any point $z\in\widehat{\Omega}$

$$\partial_{z_i}\widehat{X}\bullet\partial_{z_j}\widehat{X}=\delta_{ij}\,U^2\,,\qquad i,j=1,2 \tag{3.5}$$

with conformal factor $U^2(z) := f(\mathfrak{g}\,(\Lambda^{-1}(z)))$, *where*

$$f(\mathfrak{g}\,) = \frac{\mathfrak{g}}{2\mathfrak{g}^{\,1/2} + (1+\mathfrak{g}\,)}\,, \quad \mathfrak{g}\, = 1 + |\nabla u|^2\,.$$

Proof. When $g(t) = \sqrt{1+t^2}$, the differentials of the 1-forms $\widetilde{\omega}^i$ in (3.2) satisfy equations:

$$\begin{aligned} \mathrm{d}\big(\mathfrak{g}^{\,-1/2}(1+u_{,1}{}^2)\,\mathrm{d}x^1 + \mathfrak{g}^{\,-1/2}\,u_{,1}u_{,2}\,\mathrm{d}x^2\big) &= u_{,2}\cdot \mathrm{div}\big(\mathfrak{g}^{\,-1/2}\nabla u\big)\,\mathrm{d}x^1\wedge \mathrm{d}x^2 \\ \mathrm{d}\big(\mathfrak{g}^{\,-1/2}\,u_{,1}u_{,2}\,\mathrm{d}x^1 + \mathfrak{g}^{\,-1/2}(1+u_{,2}{}^2)\,\mathrm{d}x^2\big) &= -u_{,1}\cdot \mathrm{div}\big(\mathfrak{g}^{\,-1/2}\nabla u\big)\,\mathrm{d}x^1\wedge \mathrm{d}x^2 \end{aligned}$$

and hence we obtain a smooth vector field $\widetilde{F} : B^2 \to \mathbb{R}^2$ such that

(3.6)
$$\nabla \widetilde{F} = \big(\mathfrak{g}^{\,-1/2}\mathfrak{g}_{\,ij}\big) = \begin{pmatrix} \mathfrak{g}^{\,-1/2}\,(1+u_{,1}{}^2) & \mathfrak{g}^{\,-1/2}\,u_{,1}u_{,2} \\ \mathfrak{g}^{\,-1/2}\,u_{,1}u_{,2} & \mathfrak{g}^{\,-1/2}\,(1+u_{,2}{}^2) \end{pmatrix} \quad \text{on } B^2$$

see (3.1). With this choice, definition (3.3) gives a smooth diffeomorphism onto its image (cf. e.g. [3, Prop. 5.1]) and on account of (2.8) we obtain

$$\det \nabla\Lambda = 1 + \mathrm{tr}\,A + \det A = 1 + \mathfrak{g}^{\,-1/2}(2+|\nabla u|^2) + 1 = 2 + \mathfrak{g}^{\,-1/2}(1+\mathfrak{g}\,)$$

$$\nabla\Lambda^{-1} = \frac{1}{\det\nabla\Lambda}\begin{pmatrix} 1+\mathfrak{g}^{\,-1/2}(1+u_{,2}{}^2) & -\mathfrak{g}^{\,-1/2}u_{,1}u_{,2} \\ -\mathfrak{g}^{\,-1/2}u_{,1}u_{,2} & 1+\mathfrak{g}^{\,-1/2}(1+u_{,1}{}^2) \end{pmatrix} =: \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}$$

so that the parameterization $\widehat{X}$ in (3.4) satisfies

$$\nabla\widehat{X} = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \\ \alpha\,u_{,1} + \gamma\,u_{,2} & \gamma\,u_{,1} + \beta\,u_{,2} \end{pmatrix}.$$

Therefore, the conformality relations (3.5) hold, with conformal factor

$$U^2 = \frac{2\mathfrak{g}^{\,1/2} + (1+\mathfrak{g}\,)}{\big(2+\mathfrak{g}^{\,-1/2}(1+\mathfrak{g}\,)\big)^2} = \frac{\mathfrak{g}}{2\mathfrak{g}^{\,1/2} + (1+\mathfrak{g}\,)}$$

where $\mathfrak{g}$ is computed at $\widetilde{x} = \Lambda^{-1}(z) \in B^2$. Further details are omitted. □

We recall that the first general existence proof for the nonparametric Plateau problem was given by A. Haar [7] in 1927, whereas analyticity of minimizers was firstly achieved by T. Radó. The starting point of the classical proof is the following exactness criterion for 1-forms in $\mathbb{R}^2$ with continuous coefficients:

Lemma (Haar) *Let* $\Omega \subset \mathbb{R}^2$ *be a simply connected, bounded, open set, and let* $u,\, v \in C^0(\bar\Omega)$ *such that*

$$\int_\Omega (u\,\zeta_{,1} + v\,\zeta_{,2})\,d\mathcal{L}^2 = 0 \qquad \forall\,\zeta \in C^1_0(\Omega)\,.$$

Then, the 1*-form* $\omega := u\,\mathrm{d}x^2 - v\,\mathrm{d}x^1$ *is exact in* Ω.

Referring to the mimeographed notes [1] for further details on the classical approach, we only point out that Haar's lemma yields to existence of isothermal parameters, but it only works in dimension $N = 2$. In some sense, that is the reason why in high dimension $N \geq 3$ our Main Result does not lead to existence of "good parameterizations", see Sec. 5 below.

Finally, we recall that the previous argument was essentially exploited in [3] for g-surfaces $\mathcal{G}_u$, provided that g is of class C^2, with $g'(0) = 0$, $g''(t) > 0$ for all $t > 0$, that for some real numbers $a, A > 0$, $b, B \geq 0$,

$$at - b \leq g(t) \leq At + B \quad \text{for all } t \geq 0$$

and finally that

$$\int_0^{+\infty} t\, g''(t)\, dt < \infty\,.$$

With these assumptions, in fact, in [3, Thm. 1.3] it is shown that the vector field from (3.3) is a smooth diffeomorphism $z = \Lambda(\widetilde{x})$ onto its image, and that equation (3.4) defines a so called *asymptotic conformal parameterization* of the g-surface $\mathcal{G}_u$.

4. The high dimension case

In this section, we prove Theorem 0.1 in high dimension $N \geq 3$. It is restated in Theorem 4.1 below.

For this purpose, we come back to Remark 2.2. Following the notation from Remark 1.1, we denote again by $A \in C^2(B^N, \mathbb{R}^{N\times N})$ the symmetric tensor valued function with components as in (2.8), for $i, j = 1, \ldots, N$, and observe that formula (2.9) continues to hold. Therefore, we wish to find a canonical $\mathbb{R}^{N+1}$-valued $(N-1)$-form $\bar{\omega}^{(N-1)}$, that in components reads as

$$\bar{\omega}^{(N-1)} = \big(\omega_1^{(N-1)}, \omega_2^{(N-1)}, \ldots, \omega_N^{(N-1)}, \omega_{N+1}^{(N-1)}\big)^T$$

in such a way that according to equation (2.7) one has

$$\text{(4.1)} \qquad \mathrm{d}\Phi_u^{\#}\bar{\omega}^{(N-1)} = \begin{pmatrix} \mathrm{div}(A_1^1, \ldots, A_N^1)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \\ \mathrm{div}(A_1^2, \ldots, A_N^2)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \\ \vdots \\ \mathrm{div}(A_1^N, \ldots, A_N^N)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \\ \mathrm{div}(\mathfrak{g}^{-1/2}\nabla u)\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \end{pmatrix}.$$

Clearly, the last component of $\bar{\omega}^{(N-1)}$ is given by

$$\text{(4.2)} \qquad \omega_{N+1}^{(N-1)} := -\sum_{j=1}^{N} (-1)^{j-1} y_j\, \widehat{dx^j}$$

where for $j = 1, \ldots, N$ we denote by $\widehat{\mathrm{d}x^j}$ the $(N-1)$-covector in $\mathbb{R}^N$ obtained by deleting $\mathrm{d}x^j$ from the ordered N-covector $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N$, i.e.,

$$\widehat{\mathrm{d}x^j} := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{j-1} \wedge \mathrm{d}x^{j+1} \wedge \cdots \wedge \mathrm{d}x^N$$

so that

$$\text{(4.3)} \qquad (-1)^{j-1}\mathrm{d}x^j \wedge \widehat{\mathrm{d}x^j} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N\,.$$

In fact, recalling (1.1), (1.2), and (1.4), by (4.2) we compute the pull-back

$$\widetilde{\Phi}_u^{\#}\omega_{N+1}^{(N-1)} = \sum_{j=1}^{N}(-1)^{j-1}\Xi(|\nabla u|)\, u_{,j}\, \widehat{\mathrm{d}x^j}$$

so that by (4.3) we get:

$$\mathrm{d}\widetilde{\Phi}_u^{\#}\omega_{N+1}^{(N-1)} = \mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^N\,. \tag{4.4}$$

When $N=3$, we define the four components of $\bar{\omega}^{(2)}$ as follows:

$$\begin{cases} \omega_1^{(2)} := & y_2\,\mathrm{d}x^3\wedge\mathrm{d}x^4 + y_3\,\mathrm{d}x^4\wedge\mathrm{d}x^2 + y_4\,\mathrm{d}x^2\wedge\mathrm{d}x^3 \\ \omega_2^{(2)} := & -\big(y_3\,\mathrm{d}x^4\wedge\mathrm{d}x^1 + y_4\,\mathrm{d}x^1\wedge\mathrm{d}x^3 + y_1\,\mathrm{d}x^3\wedge\mathrm{d}x^4\big) \\ \omega_3^{(2)} := & y_4\,\mathrm{d}x^1\wedge\mathrm{d}x^2 + y_1\,\mathrm{d}x^2\wedge\mathrm{d}x^4 + y_2\,\mathrm{d}x^4\wedge\mathrm{d}x^1 \\ \omega_4^{(2)} := & -\big(y_1\mathrm{d}x^2\wedge\mathrm{d}x^3 + y_2\mathrm{d}x^3\wedge\mathrm{d}x^1 + y_3\mathrm{d}x^1\wedge\mathrm{d}x^2\big) \end{cases} \tag{4.5}$$

and when $N=4$, instead, the five components of $\bar{\omega}^{(4)}$ are:

$$\begin{cases} \bar{\omega}_1^{(3)} := & -y_2\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5 + y_3\mathrm{d}x^4\wedge\mathrm{d}x^5\wedge\mathrm{d}x^2 \\ & -y_4\mathrm{d}x^5\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + y_5\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4 \\ \bar{\omega}_2^{(3)} := & -y_3\mathrm{d}x^4\wedge\mathrm{d}x^5\wedge\mathrm{d}x^1 + y_4\mathrm{d}x^5\wedge\mathrm{d}x^1\wedge\mathrm{d}x^3 \\ & -y_5\mathrm{d}x^1\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4 + y_1\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5 \\ \bar{\omega}_3^{(3)} := & -y_4\mathrm{d}x^5\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + y_5\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^4 \\ & -y_1\mathrm{d}x^2\wedge\mathrm{d}x^4\wedge\mathrm{d}x^5 + y_2\mathrm{d}x^4\wedge\mathrm{d}x^5\wedge\mathrm{d}x^1 \\ \omega_4^{(3)} := & -y_5\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 + y_1\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^5 \\ & -y_2\mathrm{d}x^3\wedge\mathrm{d}x^5\wedge\mathrm{d}x^1 + y_5\mathrm{d}x^5\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 \\ \omega_5^{(3)} := & -y_1\mathrm{d}x^2\wedge\mathrm{d}x^3\wedge\mathrm{d}x^4 + y_2\mathrm{d}x^3\wedge\mathrm{d}x^4\wedge\mathrm{d}x^1 \\ & -y_3\mathrm{d}x^4\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2 + y_4\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\,. \end{cases} \tag{4.6}$$

With this notation, in fact, it can be checked that equation (4.1) holds true for $N=3,4$. Notice moreover that the 3-form $\bar{\omega}^{(3)}$ has a similar structure to the one of the 1-form $\bar{\omega}^{(1)}$ we defined in (2.1) when $N=2$.

For $N\geq 5$, we have to define $\bar{\omega}^{(N-1)}$ in such a way that equation (4.1) continues to hold. Therefore, for $N\geq 5$ odd, the structure of $\bar{\omega}^{(N-1)}$ is similar to the one of case $N=3$ in (4.5), whereas for $N\geq 6$ even, its structure is similar to the one of case $N=4$ in (4.6). Their explicit expression can be obtained starting from the expression in cases $N=3$ or $N=4$, and by distinguishing between $N\geq 5$ odd or even.

More precisely, for $i=1,\ldots,N+1$, the i-th component of $\bar{\omega}^{(N-1)}$ is made of N terms, each one involving a coefficient y_{j_1} and $N-1$ differentials $\mathrm{d}x^{j_2}\wedge\cdots\wedge\mathrm{d}x^{j_N}$, where the N indices j_k, for $k=1,\ldots,N$, are defined in an increasing and cyclical way by means of the ordered multi-index which complements i in $(1,\ldots,N+1)$. The main feature is that when N is odd, compare (4.5), a constant sign ± 1 appears, depending on the parity of the index i, whereas when N is even, compare (4.6), alternating signs appear.

Since we did not find a satisfactory synthetic notation, for $N\geq 5$ the explicit expression of $\bar{\omega}^{(N-1)}$ is omitted, for the sake of brevity.

We are now in position to prove the Main Result of this paper:

Theorem 4.1. *Let $N \geq 3$ and let $\widetilde{\Phi}_u$ be given by* (1.4)*, with g-normal defined by* (1.1) *and* (1.2) *for some integrand g as in the introduction. Moreover, let $\bar{\omega}^{(N-1)}$ denote the canonical $\mathbb{R}^{N+1}$-valued $(N-1)$-form defined as above (see* (4.5) *and* (4.6) *for $N = 3, 4$, respectively). Then, for any smooth function $u \in C^2(B^N, \mathbb{R})$*

$$\mathrm{d}\widetilde{\Phi}_u^{\#}\bar{\omega}^{(N-1)} = \mathrm{div}\Big[\Xi(|\nabla u|)\,\nabla u\Big]\,(-\nabla u,\,1)^T\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \tag{4.7}$$

where the function $\Xi(t)$ is given by (0.1)*. Therefore, the graph $\mathcal{G}_u$ is a g-hypersurface in $\mathbb{R}^{N+1}$ if and only if $\widetilde{\Phi}_u^{\#}\bar{\omega}^{(N-1)}$ is a closed $\mathbb{R}^{N+1}$-valued $(N-1)$-form in B^N.*

Proof. Let $\mathfrak{a} \in C^1(B^N, \mathbb{R}^{N\times N})$ be the symmetric tensor-valued field associated to a given function $u \in C^2(B^N, \mathbb{R})$ and with components

$$\mathfrak{a}^{\,ij} := \delta_{ij}|\nabla u|^2 - u_{,i}\,u_{,j}\,, \quad i, j = 1, \ldots, N\,. \tag{4.8}$$

Also, denote by $\mathfrak{a}^{\,i}$ the i-th raw vector field of $\mathfrak{a}$, namely:

$$\mathfrak{a}^{\,i} := \big(\mathfrak{a}^{\,i1}, \ldots, \mathfrak{a}^{\,iN}\big)\,, \quad i = 1, \ldots, N\,. \tag{4.9}$$

According to Remark 1.1, we point out that the inverse $(\mathfrak{g}^{\,ij})$ of the metric tensor $(\mathfrak{g}_{\,ij})$ of the nonparametric hypersurface $\mathcal{G}_u$ satisfies

$$\mathfrak{g}^{\,ij} = \mathfrak{g}^{\,-1}\,(\delta_{ij} + \mathfrak{a}^{\,ij}) \quad \forall\, i, j = 1, \ldots, N\,.$$

In particular, definition (2.8) can be equivalently written as

$$A^i_j := \mathfrak{g}^{\,-1/2}\,(\delta_{ij} + \mathfrak{a}^{\,ij})\,, \quad i, j = 1, \ldots, N\,. \tag{4.10}$$

With this notation, and recalling that $\omega_i^{(N-1)}$ denotes the i-th component of the canonical form $\bar{\omega}^{(N-1)}$, we have already obtained that the last component satisfies equation (4.4). On account of formulas (1.1), (1.2), and (1.4), it then suffices to check the validity for $i = 1, \ldots, N$ of equations

$$\mathrm{d}\widetilde{\Phi}_u^{\#}\omega_i^{(N-1)} = \big[\mathrm{div}\big(\Xi(|\nabla u|)\,\mathfrak{a}^{\,i}\big) + \partial_{x_i}\big((\Xi+\vartheta)(|\nabla u|)\big)\big]\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^N \tag{4.11}$$

and then of equations

$$\mathrm{div}\big(\Xi(|\nabla u|)\,\mathfrak{a}^{\,i}\big) + \partial_{x_i}\big((\Xi + \vartheta)(|\nabla u|)\big) = -u_{,i} \cdot \mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big) \tag{4.12}$$

in any dimension $N \geq 2$. In fact, equation (4.7) readily follows from (4.4), (4.11), and (4.12).

Notice that on account of (4.10), when $g(t) = \sqrt{1+t^2}$ equation (4.11) becomes the i-th line of formula (4.1), whereas in accordance with (2.9) for the case $N = 2$, equation (4.12) reads as

$$\mathrm{div} A^i = -u_{,i} \cdot \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)\,, \quad i = 1, \ldots, N\,.$$

The rest of the proof is then divided in three steps. Firstly, we write more explicitly the expression in the right-hand side of equation (4.4). Secondly, according to the notation from (4.8), we show that for $i = 1, \dots, N$

$$\widetilde{\Phi}_u^{\#}\omega_i^{(N-1)} = \sum_{j=1}^{N}(-1)^{j-1}\Xi(|\nabla u|)\,\mathfrak{a}^{ij}\,\widehat{\mathrm{d}x^j} + (-1)^{i-1}(\Xi+\vartheta)(|\nabla u|)\,\widehat{\mathrm{d}x^i} \tag{4.13}$$

so that on account of (4.9) we readily obtain the validity of equations (4.11), by differentiation. Finally, we show that formulas (4.12) hold true for every $i = 1, \dots, N$.

We shall give the details of the proof of formulas (4.13) and (4.12) for $i = 1$ and in dimension $N = 3$. When $N \geq 4$ or $i \geq 2$, the previous formulas are checked in a similar way, by essentially distinguishing when N is odd or even. Therefore, the proof in these other cases will be omitted, for the sake of brevity. Finally, we recall that when $N = 2$ formulas (4.13) and (4.12) have been proved in Theorem 2.1. Therefore, we follow the same strategy.

Step 1: we write explicitly the expression of $\mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big)$. To this purpose, recalling formulas (2.5), equations (2.6) hold for each $i = 1, \dots, N$, where again we shall denote $t = |\nabla u|$, and the summation on repeated indices $\alpha, \beta = 1, \dots, N$ is adopted. Therefore, denoting by Δu the Laplacean of u and by $\bullet$ the scalar product in $\mathbb{R}^N$, in any dimension $N \geq 2$ we have:

$$\begin{aligned} \mathrm{div}\big(\Xi(|\nabla u|)\,\nabla u\big) &= \nabla(\Xi(|\nabla u|) \bullet \nabla u + \Xi(|\nabla u|)\,\Delta u \\ &= \frac{g''(t)t - g'(t)}{t^3}\,u_{,\alpha}u_{,\beta}u_{,\alpha\beta} + \frac{g'(t)}{t}\,u_{,\alpha\alpha} \\ &= \frac{g''(t)}{t^2}\,u_{,\alpha}u_{,\beta}u_{,\alpha\beta} + \frac{g'(t)}{t^3}\,\sigma_{\alpha\beta}u_{,\alpha}u_{,\beta}u_{,\alpha\beta} \end{aligned} \tag{4.14}$$

where in the last addendum we have set

$$\sigma_{\alpha\beta} := \begin{cases} +1 & \text{if } \alpha = \beta \\ -1 & \text{if } \alpha \neq \beta \end{cases} \qquad \alpha, \beta = 1, \dots, N\,.$$

Step 2: we prove formula (4.13) for $N = 3$ and $i = 1$. By using the first line in definition (4.5), we compute the pull-back

$$\begin{aligned} \widetilde{\Phi}_u^{\#}\bar{\omega}_1^{(2)} &= \widetilde{\nu}_u^2\,\mathrm{d}x^3 \wedge \mathrm{d}u + \widetilde{\nu}_u^3\,\mathrm{d}u \wedge \mathrm{d}x^2 + \widetilde{\nu}_u^4\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= -\Xi(t)\,u_{,2}\,\mathrm{d}x^3 \wedge \mathrm{d}u - \Xi(t)\,u_{,3}\,\mathrm{d}u \wedge \mathrm{d}x^2 + (\Xi+\vartheta)(t)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \Xi(t)\,({u_{,2}}^2 + {u_{,3}}^2)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \Xi(t)\,u_{,1}u_{,2}\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &\quad -\Xi(t)\,u_{,1}u_{,3}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + (\Xi+\vartheta)(t)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \end{aligned}$$

that on account of definition (4.8) agrees with the right-hand side of formula (4.13), when $N = 3$ and $i = 1$.

Step 3: we prove formula (4.12) for $N = 3$ and $i = 1$. Since by (4.8)–(4.9)

$$\begin{aligned} \mathfrak{a}^1 &= ({u_{,2}}^2 + {u_{,3}}^2,\ -u_{,1}u_{,2},\ -u_{,1}u_{,3}) \\ \mathrm{div}\,\mathfrak{a}^1 &= u_{,2}u_{,12} + u_{,3}u_{,13} - u_{,1}(u_{,22} + u_{,33}) \end{aligned}$$

using again equations (2.6) we compute:

$$
\begin{aligned}
&\operatorname{div}\big(\Xi(|\nabla u|)\,\mathfrak{a}^{1}\big) + \partial_{x_1}\big((\Xi+\vartheta)(|\nabla u|)\big)\\
&\quad = \frac{g''(t)t - g'(t)}{t^3}\Big((u_{,1}u_{,11} + u_{,2}u_{,12} + u_{,3}u_{,13})\,({u_{,2}}^2 + {u_{,3}}^2)\\
&\qquad\quad -(u_{,1}u_{,12} + u_{,2}u_{,22} + u_{,3}u_{,23})\,u_{,1}u_{,2}\\
&\qquad\quad -(u_{,1}u_{,13} + u_{,2}u_{,23} + u_{,3}u_{,33})\,u_{,1}u_{,3}\Big)\\
&\qquad +\frac{g'(t)}{t}\big(u_{,2}u_{,12} + u_{,3}u_{,13} - u_{,1}(u_{,22} + u_{,33})\big)\\
&\qquad -g''(t)\big(u_{,1}u_{,11} + u_{,2}u_{,12} + u_{,3}u_{,13}\big)\\
&\quad = -u_{,1}\cdot\frac{g''(t)}{t^2}\big({u_{,1}}^2u_{,11} + {u_{,2}}^2u_{,22} + {u_{,3}}^2u_{,33}\\
&\qquad\quad +2\,(u_{,1}u_{,2}u_{,12} + u_{,1}u_{,3}u_{,13} + u_{,2}u_{,3}u_{,23})\big)\\
&\qquad -u_{,1}\cdot\frac{g'(t)}{t^3}\big({u_{,1}}^2u_{,11} + {u_{,2}}^2u_{,22} + {u_{,3}}^2u_{,33}\\
&\qquad\quad -2\,(u_{,1}u_{,2}u_{,12} + u_{,1}u_{,3}u_{,13} + u_{,2}u_{,3}u_{,23})\big)\,.
\end{aligned}
$$

Therefore, since when $N = 3$ equation (4.14) becomes:

$$
\begin{aligned}
\operatorname{div}\big(\Xi(|\nabla u|)\,\nabla u\big) = \;& \frac{g''(t)}{t^2}\big({u_{,1}}^2u_{,11} + {u_{,2}}^2u_{,22} + {u_{,3}}^2u_{,33}\\
&\quad +2\,(u_{,1}u_{,2}u_{,12} + u_{,1}u_{,3}u_{,13} + u_{,2}u_{,3}u_{,23})\big)\\
& +\frac{g'(t)}{t^3}\big({u_{,1}}^2u_{,11} + {u_{,2}}^2u_{,22} + {u_{,3}}^2u_{,33}\\
&\quad -2\,(u_{,1}u_{,2}u_{,12} + u_{,1}u_{,3}u_{,13} + u_{,2}u_{,3}u_{,23})\big)
\end{aligned}
$$

formula (4.12) holds true for $N = 3$ and $i = 1$, as required. □

5. On good parameterizations of g-hypersurfaces

In this section, we discuss the lack of validity of a similar argument to the one in Corollary 3.1, in high dimension $N \geq 3$.

Namely, one might ask if it exists a smooth vector field $\widetilde{F} : B^N \to \mathbb{R}^N$ such that a property similar to (3.1) holds true, for g-hyperfusfaces $\mathcal{G}_u$. Recall that in the particular case of nonparametric minimal surfaces in $\mathbb{R}^3$, condition (3.1) becomes (3.6).

When e.g. $N = 3$, according to the notation (4.8)–(4.9), in Theorem 4.1 we have shown that if equation $\operatorname{div}\big(\mathfrak{g}^{-1/2}\nabla u\big) = 0$ holds, then the 2-forms

$$
\begin{aligned}
\omega^1 &:= \mathfrak{g}^{-1/2}\Big((1+\mathfrak{a}^{11})\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + \mathfrak{a}^{12}\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + \mathfrak{a}^{13}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)\\
\omega^2 &:= \mathfrak{g}^{-1/2}\Big(\mathfrak{a}^{21}\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + (1+\mathfrak{a}^{22})\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + \mathfrak{a}^{13}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)\\
\omega^3 &:= \mathfrak{g}^{-1/2}\Big(\mathfrak{a}^{31}\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + \mathfrak{a}^{32}\,\mathrm{d}x^3\wedge\mathrm{d}x^1 + (1+\mathfrak{a}^{13})\,\mathrm{d}x^1\wedge\mathrm{d}x^2\Big)
\end{aligned}
$$

are closed, whence exact in B^3. Therefore, there exist three smooth 1-forms η^i in B^3 such that $\mathrm{d}\eta^i = \omega^i$ for $i = 1, 2, 3$.

Such a property is clearly equivalent to the existence of three smooth vector fields $\Psi^i : B^3 \to \mathbb{R}^3$ such that $\operatorname{curl} \Psi^i = f_i$ for $i = 1, 2, 3$, where

$$\begin{aligned} f_1 &:= \mathfrak{g}^{-1/2} \left(1 + u_{,2}^2 + u_{,3}^2,\, -u_{,1}u_{,2},\, -u_{,1}u_{,3}\right) \\ f_2 &:= \mathfrak{g}^{-1/2} \left(-u_{,1}u_{,2},\, 1 + u_{,3}^2 + u_{,1}^2,\, -u_{,2}\, u_{,3}\right) \\ f_3 &:= \mathfrak{g}^{-1/2} \left(-u_{,1}u_{,3},\, -u_{,2}u_{,3},\, 1 + u_{,1}^2 + u_{,2}^2\right). \end{aligned}$$

On the other hand, on account of (4.7) and (4.10), when $N = 3$ we have seen that the tensor-valued field

$$A := \mathfrak{g}^{-1/2} \begin{pmatrix} 1 + u_{,2}^2 + u_{,3}^2 & -u_{,1}u_{,2} & -u_{,1}u_{,3} \\ -u_{,1}u_{,2} & 1 + u_{,3}^2 + u_{,1}^2 & -u_{,2}u_{,3} \\ -u_{,1}u_{,3} & -u_{,2}u_{,3} & 1 + u_{,1}^2 + u_{,2}^2 \end{pmatrix} \tag{5.1}$$

satisfies $\operatorname{div} A = 0$, where divergence is computed along the raw vector fields A^i, compare (3.6). However, given a tensor-valued field $\widetilde{A} \in C^1(B^3, \mathbb{R}^{3\times 3})$ depending on u, the existence of a vector field $F : B^3 \to \mathbb{R}^3$ such that

$$\nabla F = \widetilde{A} \quad \text{on } B^3$$

implies the necessary condition $\operatorname{curl} \widetilde{A} = 0$, where curl is again computed along the raw vector fields $\widetilde{A}^i$. Such a curl-free condition should be obtained as a consequence of the validity of equation $\operatorname{div}\left(\mathfrak{g}^{-1/2}\nabla u\right) = 0$, and of course this is not the case for $\widetilde{A} = A$ in (5.1). In a similar way, in any high dimension $N \geq 3$ it is not clear how to obtain a suitable tensor-valued field $\widetilde{A} \in C^1(B^N, \mathbb{R}^{N\times N})$ depending on u that agrees with the gradient of a smooth vector field $F \in C^2(B^N, \mathbb{R}^N)$, by exploiting Theorem 4.1 for minimal hypersurfaces.

In fact, if a function $u \in C^2(B^N, \mathbb{R})$ satisfies the Euler-Lagrange equation (0.1), by Theorem 4.1 we infer the existence of a $\mathbb{R}^{N+1}$-valued $(N-2)$-form $\eta^{(N-2)}$ in B^N such that

$$\mathrm{d}\eta^{(N-2)} = \widetilde{\Phi}_u^{\#}\bar{\omega}^{(N-1)}$$

and hence it is only in low dimension $N = 2$ that one may proceed as in Corollary 3.1, by working with the first two components of the smooth function $\eta^{(0)} \in C^1(B^2, \mathbb{R}^3)$.

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