NON-LOCAL APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS IN LINEAR ELASTICITY AND APPLICATION TO STOCHASTIC HOMOGENISATION

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Abstract. We analyse the Γ-convergence of general non-local convolution type functionals with varying densities depending on the space variable and on the symmetrized gradient. The limit is a local free-discontinuity functional, where the bulk term can be completely characterized in terms of an asymptotic cell formula. From that, we can deduce an homogenisation result in the stochastic setting.

Keywords: Non-local approximation, variational fracture, free discontinuity problems, functions of bounded deformation, Γ-convergence, deterministic and stochastic homogenisation.

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1. Introduction

This paper is focused on the approximation of brittle fracture energies for linearly elastic materials, by means of *non-local functionals* defined on Sobolev spaces. The asymptotic behavior of these functionals will simultaneously show the emergence both of effective energies for the elastic deformation (which may be, e.g., the output of homogenization), and of Griffith-type surface energies accounting for crack formation. In turn, this result can be further generalized to the setting of stochastic homogenization with fracture.

Precisely our results will extend the range of application of the recent papers [18, 23] while also providing some relevant technical improvement. We briefly comment on these previous contributions, in order to introduce our results. There, an approach originally devised by Braides and Dal Maso [7] for the approximation of the Mumford-Shah functional has been generalized to the linearly elastic setting. Namely, it was shown that, for a given bounded increasing function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ the energies

$$\frac{1}{\varepsilon_k}\int_U f\Big(\varepsilon_k W(e(u)) * \rho_k(x)\Big)\,\mathrm{d}x\,, \tag{1.1}$$

Γ-converge, in the $L^1(U)$-topology, to the functional

$$\alpha\int_U W(e(u))\,\mathrm{d}x + \beta\int_{J_u}\phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{1.2}$$

with $\alpha = f'(0)$ and $\beta = \lim_{t\to+\infty} f(t)$. Above, ρ_k are rescaled convolution kernels with unit mass and compact convex symmetrical support S, ϕ_ρ is (twice) the support function of S (see (2.4) for its precise definition), $W(e(u))$ is a convex elastic energy with superlinear p growth depending on the symmetrized gradient $e(u)$ of a vector-valued displacement u, whose jump set is denoted by J_u. Notice that the effective domains of the approximations and of the limit are different. Actually (1.1) is finite on the Sobolev space $W^{1,p}(U;\mathbb{R}^n)$, while the energy space of (1.2) is the one of generalized functions with bounded deformation $GSBD^p(U)$, introduced in [14].

We stress that the above results allowed one for a general (convex) bulk energy. The proof strategy cannot rely, at least when estimating the bulk part, on any slicing procedure. The latter is instead successful in the particular case $W(\xi) = |\xi|^p$, considered for instance in [21]. We also remark that the results of [18, 23] were obtained under an additional structural assumption on the kernels ρ_k, which have to be radial with respect to the norm induced by S. In the particular case considered in [21], this restriction was instead not needed.

A natural extension of the aforementioned models allows one to include an explicit dependence on k and on the space variable of the energy density. This amounts to consider functionals of the form

$$\frac{1}{\varepsilon_k}\int_U f\Big(\varepsilon_k W_k(\cdot, e(u)) * \rho_k(x)\Big)\,\mathrm{d}x\,, \tag{1.3}$$

whose limit behaviour is the object of the present paper. Functionals of the form (1.3) can be used to approximate (1.2) with some gain in the ease of minimization, for a proper choice of W_k. Actually, this more general setting is also suitable for further applications, if one thinks about the mechanical counterpart of the model. Indeed energy densities of type $W_k(y, M)$, where y is the position in the reference configuration, are customary when dealing with heterogeneous material with some microstructures. The prototypical example is the case of *homogenisation*, that is, when $W_k(y, M) = W(\frac{y}{\delta_k}, M)$ with $\delta_k \searrow 0$. Taking this point of view amounts to regard (1.3) as a nonlocal linearly elastic model, with a truncated potential $\frac{1}{\varepsilon_k} f(\varepsilon_k\cdot)$ accounting for the cost of breaking the elastic bonds on regions of size ε_k. In such a case, one is interested in deriving an effective asymptotic model for (1.3).

The main result of our paper is contained in Theorem 2.1. There we show that the functionals in (1.3) Γ-converge to a limit energy of the form

$$\alpha\int_U W(x, e(u))\,\mathrm{d}x + \beta\int_{J_u}\phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{1.4}$$

Above, the limit bulk density W can be characterised in terms of cell formula (see (2.7)-(2.8)). Remarkably, that coincides exactly with the asymptotic formula that one would obtain by considering the limit behaviour of the local energies $\int_U W_k(x, e(u))\,\mathrm{d}x$ in the Sobolev space $W^{1,p}(U)$. Hence, a decoupling effect between bulk and surface contribution occurs, since the volume energy only depends on f through its derivative at the origin. A similar effect has been observed in [11] where the analogue of (1.3) for energies depending on the full deformation gradient was taken into account. On the one hand, the possibility of using smooth truncations (a tool which is not available in $GSBD$) allowed the author there to replace f by a sequence f_k and to derive more general surface energies in the limit. On the other hand, the precise characterisation of the volume energy density was obtained at the expense of an additional technical condition on the W_k's (the so-called *stable* γ-convergence). It actually turns out, as an output of our proof strategy, that this extra assumption can be dropped (see Appendix A). Thus, our results also permit some improvement in the previous literature about non-local approximation of free-discontinuity problems.

We now come to the description of our proof technique. The most difficult point is the lower bound for the bulk contribution. This is done in Proposition 5.1, by means of a localisation and blow-up procedure which contains some elements of novelty in the non-local setting. More precisely we consider the blow-up of sequences with equi-bounded energies at a Lebesgue point for the limit energy. A crucial task is to gain a uniform control on the L^p norm of the symmetrized gradients of the blow-up functions up to sets with vanishing perimeter. This allows us to apply [19, Lemma 5.1] (which relies on the Korn-type inequality of [8]): we can substitute, with almost no change in the energy, the above mentioned sequence with a more regular one bounded in $W^{1,p}$. Exploiting the properties of f we are then reconducted to analyse the limit behaviour on small squares of a local energy in $W^{1,p}$, which can be estimated from below via a cell formula.

An optimal estimate from below for the surface term can be obtained by means of a slicing procedure (Proposition 5.3). As for the Γ-limsup inequality it can be achieved by a direct construction for a class of competitors with regular jump set, which are dense in energy. Here we use the classical approximation result of [9, 12].

We underline that even in the case of (1.1) (i.e., with W not depending on k) we have some technical improvement in comparison with the result of [18, 23]. First of all we do not need anymore to assume the kernels ρ_k to be radially symmetric. Secondly our Γ-convergence argument is carried out with respect to the convergence in measure instead of the L^1 convergence. This is (almost) the natural one for sequences with equibounded energy (see Theorem 2.1-(ii)). It can be indeed shown that such sequences are compact in the measure convergence up to an exceptional set U^∞, where their modulus diverges. However, this set can be easily made empty by adding a penalisation term in the energy (see the statement of Theorem 3.5 and Remark 2.2).

Eventually we complement our analysis with a *stochastic homogenisation* result Theorem 7.4. Namely we consider functionals of type (1.3) with *stationary* random integrands

$$W_k(\omega, y, M) = W\Big(\omega, \frac{y}{\delta_k}, M\Big), \tag{1.5}$$

where ω belongs to the sample space Ω of a probability space $(\Omega, \mathcal{T}, P)$ and $\delta_k \searrow 0$. Following the approach proposed by [15] (which relies on the Subadditive Ergodic Theorem in [1]) we show that, almost surely, such functionals Γ-converge to a free-discontinuity functional of the form (1.4) where the bulk energy density is independent of the space variable. A similar result was obtained in [3] in the context of elliptic approximation of free-discontinuity functionals.

Plan of the paper. The paper is structured as follows. After fixing the notation, in Section 2, we introduce the problem, discuss the assumptions and state our main results. Section 3 is devoted to recalling preliminary results which are useful for the analysis. The proof of Theorem 2.1 is carried out through the Sections 4–6, dealing with compactness, lower, and upper bound, respectively. In Section 7 we prove a stochastic homogenisation result Theorem 7.4. Eventually in the Appendix we briefly comment on the result of [11, Theorem 3.2], highlighting that the assumptions made there can actually be weakened. A complete statement is given for the readers' convenience in Theorem A.1.

2. Setting of the problem and main results

2.1. **Notation.** We start by collecting the notation adopted throughout the paper.

(a) $n \geq 2$ is a fixed integer and $p > 1$ is a fixed real number;
(b) $\mathbb{M}^{n\times n}$ denotes the space of $n \times n$ real matrices; $\mathbb{M}^{n\times n}_{\rm sym}$ and $\mathbb{M}^{n\times n}_{\rm skew}$ denote the spaces of symmetric and skew-symmetric matrices respectively;
(c) for a subset $A \subset \mathbb{R}^n$ $\partial^* A$ denotes the essential boundary of A;
(d) $\mathcal{L}^n$ and and $\mathcal{H}^{n-1}$ denote the Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$, respectively;
(e) for every $A \subset \mathbb{R}^n$ let χ_A denote the characteristic function of the set A;
(f) U denotes an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary;

(g) we denote by $\mathcal{A}(U)$ and $\mathcal{A}$ the collection of all open and bounded subsets of U and $\mathbb{R}^n$ respectively;
(h) If $A, B \in \mathcal{A}(U)$ (or $\mathcal{A}$) by $A \subset\subset B$ we mean that A is relatively compact in B;
(i) Q and Q' denote the open unit cube in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$ respectively with sides parallel to the coordinate axis, centred at the origin; for $x \in \mathbb{R}^n$ (respectively $x \in \mathbb{R}^{n-1}$) and $r > 0$ we set $Q_r(x) := rQ + x$ (respectively $Q'_r(x) := rQ' + x$);
(j) for every $\xi \in \mathbb{S}^{n-1}$ let R_ξ denote an orthogonal $(n \times n)$-matrix such that $R_\xi e_n = \xi$;
(k) for $x \in \mathbb{R}^n$, $r > 0$, and $\xi \in \mathbb{S}^{n-1}$, we define $Q^\xi_r(x) := R_\xi Q_r(x)$.
(l) for a given topological space X, $\mathcal{B}(X)$ denotes the Borel σ- algebra on X. If $X = \mathbb{R}^d$, with $d \in \mathbb{N}$, $d \geq 1$ we simply write $\mathcal{B}^d$ in place of $\mathcal{B}(\mathbb{R}^d)$. For $d = 1$ we write $\mathcal{B}$;
(m) we denote by $L^0(U;\mathbb{R}^n)$ the space of measurable functions;
(n) for $a, b \in \mathbb{R}^n$ the symbol $a \otimes b$ denotes the tensor product between a and b, while $a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$.

Throughout the paper C denotes a strictly positive constant which may vary from line to line and within the same expression.

2.2. **(G)SBV and (G)SBD functions.** We will work with the functional spaces $(G)SBV^p(U;\mathbb{R}^n)$ and $(G)SBD^p(U)$ for which we will recall the main properties and refer the reader to [2, 14] for a complete exposition of the subject. We say that $u \in L^1(U;\mathbb{R}^n)$ belongs to the space of *special functions with bounded variation*, i.e., $u \in SBV(U;\mathbb{R}^n)$, if its distributional gradient is a finite $\mathbb{M}^{n\times n}$-valued Radon measure without Cantor part, that is,

$$Du = \nabla u \mathcal{L}^n + [u] \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u\,,$$

where ∇u is the approximate gradient, J_u is the approximate jump set, $[u] = u^+ - u^-$ the jump opening and ν_u the unit normal to J_u. A function $u \in L^0(U;\mathbb{R}^n)$ belongs to the space of *generalised special functions with bounded variation*, i.e., $u \in GSBV(U;\mathbb{R}^n)$, if for any $\varphi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ with support of $\nabla\varphi$ compact it holds $\varphi \circ u \in SBV_{\rm loc}(U;\mathbb{R}^n)$.

We say that $u \in L^1(U;\mathbb{R}^n)$ belongs to the space of *special functions with bounded deformation*, and we write $u \in SBD(U)$, if its symmetrized distributional gradient is a finite $\mathbb{M}^{n\times n}_{\rm sym}$-valued Radon measure without Cantor part, that is,

$$Eu = \frac{Du + (Du)^T}{2} = e(u)\mathcal{L}^n + [u] \odot \nu_u \mathcal{H}^{n-1} \llcorner J_u\,,$$

where $e(u)$ is the approximate symmetric gradient with respect to the Lebesgue measure. On the contrary the space of *generalised special functions with bounded deformation*, $GSBD(U)$, cannot be defined analogously to the space $GSBV(U;\mathbb{R}^n)$ as if $u \in SBD(U)$ and φ is as above, then in general $\varphi \circ u \notin SBD(U)$. To overcome this issue, Dal Maso in [14] proposed a definition of this space by relying on a slicing argument which we describe in the following.
For $\xi \in \mathbb{R}^n \setminus \{0\}$ we let $\Pi^\xi := \{y \in \mathbb{R}^n : \langle \xi, y\rangle = 0\}$; for any $y \in \Pi^\xi$ and $A \in \mathcal{B}(U)$ we set

$$A_{\xi,y} := \{t \in \mathbb{R} : y + t\xi \in A\}\,.$$

Given $u : U \to \mathbb{R}^n$ we define $u^{\xi,y} : U_{\xi,y} \to \mathbb{R}$ by

$$u^{\xi,y}(t) := \langle u(y + t\xi), \xi\rangle\,.$$

If $u^{\xi,y} \in SBV(U_{\xi,y};\mathbb{R})$ we set

$$J^1_{u^{\xi,y}} := \{t \in J_{u^{\xi,y}} : |[u^{\xi,y}](t)| \geq 1\}\,.$$

We then say that $u \in L^0(U;\mathbb{R}^n)$ belongs to the space of *generalised special functions with bounded deformation*, and we write $u \in GSBD(U)$, if there exists a bounded Radon measure λ on U such that $u^{\xi,y} \in SBV_{\rm loc}(U_{\xi,y})$ for all $\nu \in \mathbb{S}^{n-1}$ and all $y \in \Pi^\xi$ and

$$\int_{\Pi^\xi} \Big(|Du^{\xi,y}|(A_{\xi,y} \setminus J^1_{u^{\xi,y}}) + \mathcal{H}^0(A_{\xi,y} \cap J^1_{u^{\xi,y}})\Big)\,\mathrm{d}\mathcal{H}^{n-1}(t) \leq \lambda(A)\,,$$

for all $A \in \mathcal{B}(U)$. Eventually we set

$$GSBV^p(U) := \{u \in (G)SBV(U;\mathbb{R}^n)\colon \nabla u \in L^p(U;\mathbb{M}^{n\times n}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\}\,;$$

and

$$GSBD^p(U) := \{u \in (G)SBD(U)\colon e(u) \in L^p(U;\mathbb{M}^{n\times n}_{\rm sym}) \text{ and } \mathcal{H}^{n-1}(J_u) < +\infty\}\,,$$

where ∇u and $e(u)$ are well defined also in $GSBV(U;\mathbb{R}^n)$ and $GSBD(U)$ respectively.

2.3. **Setting of the problem.** Let $1 < p < +\infty$; let c_1, c_2 be given positive constants such that $0 < c_1 \leq c_2 < +\infty$. Let $\mathcal{W} := \mathcal{W}(p, c_1, c_2)$ be the collection of all functions $W\colon \mathbb{R}^n \times \mathbb{M}^{n\times n} \to \mathbb{R}$ satisfying the following conditions:

$(W1)$ W is a Carathéodory function on $\mathbb{R}^n \times \mathbb{M}^{n\times n}$;

$(W2)$ $W(x,0) = 0$ for every $x \in \mathbb{R}^n$;

$(W3)$ for every $x \in \mathbb{R}^n$, $M \in \mathbb{M}^{n\times n}$ and $S \in \mathbb{M}^{n\times n}_{\rm skew}$

$$W(x, M+S) = W(x,M)\,;$$

$(W4)$ for every $x \in \mathbb{R}^n$ and every $M \in \mathbb{M}^{n\times n}$

$$c_1|M+M^T|^p \leq W(x,M) \leq c_2(|M+M^T|^p + 1)\,.$$

Let $f\colon [0,+\infty) \to [0,+\infty)$ be a concave[1] increasing function such that there exist $\alpha, \beta > 0$ with

$$\lim_{t\to 0^+} \frac{f(t)}{t} = \alpha\,, \quad \lim_{t\to+\infty} f(t) = \beta\,. \tag{2.1}$$

Note that for such f it holds

$$f(t) \leq \hat\alpha t \quad \forall \hat\alpha > \alpha\,; \tag{2.2}$$

moreover by [23, Lemma 2.10] there exist $(\alpha_i)_{i\in\mathbb{N}}$, $(\beta_i)_{i\in\mathbb{N}}$ sequences of positive numbers with $\sup_i \alpha_i = \alpha$, $\sup_i \beta_i = \beta$ such that

$$f(t) \geq f_i(t) := \alpha_i t \wedge \beta_i \quad \forall i \in \mathbb{N}\,,\ t \in \mathbb{R}\,. \tag{2.3}$$

Let $\rho \in L^\infty(\mathbb{R}^n;[0,+\infty))$ be a lower semi-continuous convolution kernel with $\int_{\mathbb{R}^n} \rho\,\mathrm{d}x = 1$ and $S := \{\rho > 0\}$ bounded, convex, symmetrical and with $0 \in S$. We denote by $|\cdot|_S$ the norm induced by S, namely,

$$|x|_S := \inf\{\lambda > 0\colon x \in \lambda S\}\,.$$

Under the above assumptions, $|\cdot|_S$ is a norm and $S = \{|x|_S < 1\}$. Then for any bounded set $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we let

$$\mathrm{d}_S(x,K) := \inf_{y\in K} |x-y|_S\,.$$

For any Borel set E and any $r > 0$ we denote by E_r and E_{-r} respectively the sets

$$E_r := \{x \in \mathbb{R}^n\colon\ \mathrm{d}_S(x,E) < r\}\,, \quad E_{-r} := \{x \in \mathbb{R}^n\colon\ \mathrm{d}_S(x,E^c) > r\}\,.$$

Finally we let $\phi_\rho\colon \mathbb{R}^n \to [0,+\infty)$ be given by

$$\phi_\rho(\nu) := 2\sup_{y\in S} |y\cdot\nu|\,. \tag{2.4}$$

[1]the need for this assumption is in deriving (2.2) which is used for the proof of Proposition 6.1. If W is not depending on k, as in [23], it can be weakened to mere lower semi-continuity.

For $\delta > 0$ we set $\rho_\delta(x) := \frac{1}{\delta^n}\rho(\frac{x}{\delta})$, $S_\delta(x) := x + \delta S$.

For $k \in \mathbb{N}$ let $(W_k) \subset \mathcal{W}$, let (ε_k) be a decreasing sequence of strictly positive real numbers converging to zero, as $k \to +\infty$ and let $\rho_k := \rho_{\varepsilon_k}$. We consider the family of functionals $F_k \colon L^0(U;\mathbb{R}^n) \to [0,+\infty]$ defined as

$$F_k(u) := \begin{cases} \dfrac{1}{\varepsilon_k}\displaystyle\int_U f\Big(\varepsilon_k W_k(\cdot, e(u)) * \rho_k(x)\Big)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathbb{R}^n)\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.5}$$

Here and henceforth, it remains understood that each $u \in W^{1,p}(U;\mathbb{R}^n)$ is extended to a fixed neighborhood of U to have a well-defined functional. The Γ-limit, as we will see, is independent of the considered extension. Let $x \in \mathbb{R}^n$, $M \in \mathbb{M}^{n\times n}$, $A \in \mathcal{A}$ and $u \in W^{1,p}(A;\mathbb{R}^n)$ be fixed. Set $u_M(y) := My$. We then define the minimisation problem

$$\mathbf{m}_k(u_M, A) := \inf\left\{\int_A W_k(x, e(v))\,\mathrm{d}x \colon v \in W^{1,p}(A;\mathbb{R}^n),\ v = u_M \text{ near } \partial A\right\}\,, \tag{2.6}$$

and the cell formulas

$$W'(x,M) := \limsup_{r\searrow 0^+}\liminf_{k\to+\infty}\frac{\mathbf{m}_k(u_M, Q_r(x))}{r^n}\,, \tag{2.7}$$

$$W''(x,M) := \limsup_{r\searrow 0^+}\limsup_{k\to+\infty}\frac{\mathbf{m}_k(u_M, Q_r(x))}{r^n}\,. \tag{2.8}$$

Notice that W' and W'' depend on the given sequence of k and are to be modified accordingly if one takes subsequences. This will be highlighted in the statement of our main result.

2.4. **Main Results.** In this Section we state our main results. The first one is a Γ-convergence theorem for the energies F_k.

Theorem 2.1 (Γ-convergence of F_k)**.** *Let F_k be as in* (2.5)*. Then the following hold:*

(*i*) *There exists a subsequence, not relabelled, such that for every $x \in U$ and every $M \in \mathbb{M}^{n\times n}$, and for W', W'' as in* (2.7) *and* (2.8) *(calculated for the given subsequence), one has*

$$W'(x,M) = W''(x,M) := W(x,M)\,. \tag{2.9}$$

and it holds $W(x,M) = W(x,\mathrm{sym}(M))$. Moreover, F_k Γ-converges with respect to the convergence in measure to the functional $F \colon L^0(U;\mathbb{R}^n) \to [0,+\infty]$ given by

$$F(u) := \begin{cases} \alpha\displaystyle\int_U W(x, e(u))\,\mathrm{d}x + \beta\int_{J_u}\phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in GSBD^p(U)\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{2.10}$$

with ϕ_ρ as in (2.4)*;*

(*ii*) *Let $(u_k) \subset L^0(U;\mathbb{R}^n)$ be such that $\sup_k F_k(u_k) < +\infty$. Set $U^\infty := \{x \in U \colon |u_k(x)| \to +\infty\}$. Then there exists $u \in GSBD^p(U)$ such that, up to subsequence, it holds $u_k \to u$ in measure on $U \setminus U^\infty$. If in addition*

$$\sup_{k\in\mathbb{N}}\int_U \psi(|u_k|)\,\mathrm{d}x < +\infty\,,$$

for some $\psi \colon [0,+\infty) \to [0,+\infty)$, continuous, increasing with $\lim_{s\to+\infty}\psi(s) = +\infty$, then $U^\infty = \emptyset$, so that $|u|$ is finite a.e., and $u_k \to u$ in measure on U.

Remark 2.2. The addition of a penalty term of the form $\int_U \psi(|u|)\,\mathrm{d}x$ to the energy enforces then compactness in measure, while causing no troubles in the Γ-convergence analysis. Indeed, such a term is clearly lower semicontinuous, hence the corresponding lower bound follows immediately. As for the upper bound, if one takes ψ as in Theorem 3.4, the argument of Proposition 6.1 can be readily adapted also in presence of such an additional term. As this is not the core of the argument, we will neglect lower order terms in our statements and proofs, directly assuming that convergence in measure holds everywhere. The technical details left to prove the upper bound are summarized in Remark 6.2 for the readers convenience.

The proof of Theorem 2.1 is divided into three main steps contained respectively in sections 4, 5 and 6. As a consequence of Theorem 2.1 and the Urysohn property of Γ-convergence [13, Proposition 8.3] we deduce the following corollary.

Corollary 2.3. *Let $(W_k) \subset \mathcal{W}$ and let F_k be the functionals as in* (2.5)*. Let W', W'' be as in* (2.7) *and* (2.8)*, respectively. Assume that*

$$W'(x,M) = W''(x,M) =: W(x,M)\,, \quad \textit{for a.e. } x \in \mathbb{R}^n \textit{ and for every } M \in \mathbb{M}^{n\times n}\,,$$

for some Borel function $W\colon \mathbb{R}^n \times \mathbb{M}^{n\times n} \to [0,+\infty)$. Let F defined as in (2.10) *accordingly. Then the functionals F_k Γ-converge with respect to the convergence in measure to F. Moreover*

$$W(x,M) = W(x,\mathrm{sym}(M)) = W'(x,M) = W''(x,M)\,,$$

for every $x \in U$ and every $M \in \mathbb{M}^{n\times n}$.

We now state a homogenisation theorem without assuming any spatial periodicity of the energy densities W_k. We start by introducing some notation. We fix $W \in \mathcal{W}$ and set

$$\mathbf{m}(u_M, A) := \inf\left\{\int_A W(x,e(v))\,\mathrm{d}x\colon v \in W^{1,p}(A;\mathbb{R}^n),\ v = u_M \text{ near } \partial A\right\}, \tag{2.11}$$

for all $A \in \mathcal{A}$ and all $M \in \mathbb{M}^{n\times n}$ Let also $(W_k) \subset \mathcal{W}$ be given by

$$W_k(x,M) := W\Big(\frac{x}{\delta_k}, M\Big)\,, \tag{2.12}$$

with $\delta_k \searrow 0$ when $k \to +\infty$.

Theorem 2.4 (Deterministic homogenisation)**.** *Let Let $W \in \mathcal{W}$ and let $\mathbf{m}(u_M, Q_t(tx))$ be as in* (2.11) *with $A = Q_t(tx)$. Assume that for every $x \in \mathbb{R}^n$, $M \in \mathbb{M}^{n\times n}$ the following limit*

$$\lim_{t\to+\infty} \frac{\mathbf{m}(u_M, Q_t(tx))}{t^n} =: W_{\mathrm{hom}}(M)\,, \tag{2.13}$$

exists and is independent of x. Then the functionals F_k defined in (2.5) *with W_k as in* (2.12) *Γ-converge with respect to the convergence in measure to the functional $F_{\mathrm{hom}}\colon L^0(U;\mathbb{R}^n) \to [0,+\infty]$ given by*

$$F_{\mathrm{hom}}(u) := \begin{cases} \alpha \displaystyle\int_U W_{\mathrm{hom}}(e(u))\,\mathrm{d}x + \beta \int_{J_u} \phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \textit{if } u \in GSBD^p(U)\,,\\ +\infty & \textit{otherwise}\,, \end{cases} \tag{2.14}$$

with ϕ_ρ as in (2.4)*. Moreover $W_{\mathrm{hom}}(M) = W_{\mathrm{hom}}(\mathrm{sym}(M))$ for all $M \in \mathbb{M}^{n\times n}$.*

Proof. Let W', W'' be respectively as in (2.7) and (2.8). By Corollary 2.3 it is sufficient to show that

$$W_{\mathrm{hom}}(M) = W'(x,M) = W''(x,M)\,, \tag{2.15}$$

for all $x \in \mathbb{R}^n$ and $M \in \mathbb{M}^{n\times n}$. We fix $x \in \mathbb{R}^n$, $M \in \mathbb{M}^{n\times n}$, $r > 0$ and $k \in \mathbb{N}$. For any $u \in W^{1,p}(Q_r(x);\mathbb{R}^n)$ with $u = u_M$ near $\partial Q_r(x)$ we let $u_k \in W^{1,p}(Q_{\frac{r}{\delta_k}}(\frac{x}{\delta_k});\mathbb{R}^n)$ be given by

$u_k(y) := \frac{1}{\delta_k} u(\delta_k y)$. Then clearly $u_k = u_M$ near $\partial Q_{\frac{r}{\delta_k}}(\frac{x}{\delta_k})$. Moreover by performing the change of variable $\hat{y} = \frac{y}{\delta_k}$ we find

$$\int_{Q_r(x)} W\Big(\frac{y}{\delta_k}, e(u)\Big)\,\mathrm{d}y = \delta_k^n \int_{Q_{\frac{r}{\delta_k}}(\frac{x}{\delta_k})} W(y, e(u_k))\,\mathrm{d}y\,.$$

Hence in particular

$$\mathbf{m}_k(u_M, Q_r(x)) = \delta_k^n \mathbf{m}(u_M, Q_{\frac{r}{\delta_k}}(\tfrac{x}{\delta_k})) = \frac{r^n}{t_k^n}\mathbf{m}(u_M, Q_{t_k}(t_k \tfrac{x}{r}))\,,$$

with $t_k := \frac{r}{\delta_k}$. Eventually passing to the limit as $k \to +\infty$ by (2.13) we deduce

$$\lim_{k\to+\infty} \frac{\mathbf{m}_k(u_M, Q_r(x))}{r^n} = \lim_{k\to+\infty} \frac{\mathbf{m}(u_M, Q_{t_k}(t_k\frac{x}{r}))}{t_k^n} = W_{\mathrm{hom}}(M)\,.$$

□

3. Some preliminary results

In this section we collect some useful results that will be employed throughout the paper. We start by recalling a Γ-convergence result for the bulk energies defined in (3.1) (Theorem 3.1) and a Γ-convergence result for one-dimensional non-local energies (Theorem 3.3). To follow we recall a density and a compactness result (cf., Theorem 3.4 and Theorem 3.5). We conclude this section with a series of technical lemmas (cf. Lemmas 3.6, 3.7, 3.8 and Corollary 3.9).

We consider the family of functionals $E_k\colon L^0(\mathbb{R}^n;\mathbb{R}^n)\times\mathcal{A}\to[0,+\infty]$ given by

$$E_k(u,A) := \begin{cases} \displaystyle\int_A W_k(x, e(u))\,\mathrm{d}x & \text{if } u \in W^{1,p}(A;\mathbb{R}^n)\,,\\ +\infty & \text{otherwise}\,. \end{cases} \tag{3.1}$$

Theorem 3.1 (Γ-convergence of E_k)**.** *Let E_k be as in* (3.1)*. Then there exists a subsequence, not relabelled, such that for every $A\in\mathcal{A}$ the functionals $E_k(\cdot,A)$ Γ-converge, with respect to the convergence in measure, to the functional $E(\cdot,A)$ with $E\colon L^0(\mathbb{R}^n;\mathbb{R}^n)\times\mathcal{A}\to[0,+\infty]$ given by*

$$E(u,A) = \begin{cases} \displaystyle\int_A W(x, e(u))\,\mathrm{d}x & \textit{if } u \in W^{1,p}(A;\mathbb{R}^n)\,,\\ +\infty & \textit{otherwise}\,, \end{cases} \tag{3.2}$$

where for every $x\in\mathbb{R}^n$ and every $M\in\mathbb{M}^{n\times n}$

$$W(x,M) = W(x,\mathrm{sym}(M)) = W'(x,M) = W''(x,M)\,, \tag{3.3}$$

with W', W'' as in (2.7) *and* (2.8) *for the given subsequence. The same Γ-convergence holds with respect to the $L^p_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ convergence.*

Observe that the above Theorem yields in particular a subsequence for which (2.9) holds. From now on, it remains understood that such a subsequence has been fixed, without relabelling. The proof of Theorem (3.1) is rather standard and follows by the localisation method (see e.g., [13, Sections 18,19]) and by suitably adapting the integral representation result in [4, Theorem 2] to our setting with the help of Korn-Poincaré inequality. For this reason we omit the proof here and we refer the reader to [19, Proposition 3.13] for more details. We only highlight that the result holds also for non regular open bounded subsets of $\mathbb{R}^n$. Since this may not be immediately clear from the statement given in [19, Proposition 3.13], we discuss this point in the remark below.

Remark 3.2. Let A be *any* open bounded subset of $\mathbb{R}^n$ and $u \in W^{1,p}(A;\mathbb{R}^n)$. We show that there exists a sequence $(u_k) \subset W^{1,p}(A;\mathbb{R}^n)$ such that $u_k \to u$ in $L^p(A;\mathbb{R}^n)$ and $E_k(u_k, A) \to E(u, A)$. With the use of Korn-Poincaré inequality, it is clear that this can be done if A is an extension domain. In the general case, consider smooth relatively compact subsets $A' \subset\subset A'' \subset\subset A$, and fix $\eta > 0$. We find a sequence $(v_k) \subset W^{1,p}(A'';\mathbb{R}^n)$ such that $E_k(v_k, A'') \to E(u, A'')$. By the liminf inequality, this also gives $E_k(v_k, A'' \setminus A') \to E(u, A'' \setminus A')$. With $(W4)$ we have

$$\limsup_{k\to+\infty} \int_{A''\setminus A'} |e(v_k)|^p \,\mathrm{d}x \le \frac{c_2}{c_1} \int_{A''\setminus A'} (1 + |e(u)|^p) \,\mathrm{d}x$$

Then, considering a cut-off φ between A' and A'' we set $u_k := \varphi v_k + (1-\varphi)u$. Clearly $u_k \to u$ in $L^p(A;\mathbb{R}^n)$. Furthermore, by $(W4)$ one has

$$\begin{aligned}
\limsup_{k\to+\infty} E_k(u_k, A) &= \limsup_{k\to+\infty} E_k(v_k, A') + E_k(u_k, A \setminus A') \\
&\le \limsup_{k\to+\infty} E_k(v_k, A'') + c_2 \Bigg[\int_{A\setminus A'} (1 + |e(u)|^p) \,\mathrm{d}x \\
&+ \int_{A''\setminus A'} (1 + |\nabla\phi \odot (v_k - u)|^p + |e(v_k)|^p) \,\mathrm{d}x\Bigg] \\
&\le E(u, A'') + c_2 \left(1 + \frac{c_2}{c_1}\right) \int_{A\setminus A'} (1 + |e(u)|^p) \,\mathrm{d}x \le E(u, A) + \eta \,,
\end{aligned}$$

provided $\mathcal{L}^n(A \setminus A')$ is sufficiently small. The limsup inequality, which is the only relevant one, follows by a diagonal argument.

We recall now the following one-dimensional result for non-local energies given in [5, Theorem 3.30].

Theorem 3.3 (Γ-convergence in 1d)**.** *Let $I \subset \mathbb{R}$ be a bounded interval. Let $f\colon [0,+\infty) \mapsto [0,+\infty)$ be a lower semi-continuos function satisfying* (2.1) *for some $\alpha, \beta > 0$. Consider the family of functionals $G_k\colon L^0(I) \to [0,+\infty]$ defined by*

$$G_k(w) := \frac{1}{\varepsilon_k} \int_I f\Big(\frac{1}{2} \int_{x-\varepsilon_k}^{x+\varepsilon_k} |\dot w(y)|^p \,\mathrm{d}y\Big) \,\mathrm{d}x \,,$$

if $u \in W^{1,p}(I)$ and $+\infty$ otherwise. Then G_k Γ-converge with respect to the convergence in measure, to the functional $G\colon L^0(I) \to [0,+\infty]$ given by

$$G(w) := \alpha \int_I |\dot w|^p \,\mathrm{d}x + 2\beta \#(J_w) \,,$$

if $w \in SBV(I)$ and $+\infty$ otherwise.

We next recall an approximation result [9, Theorem 1.1] and a compactness result in $GSBD^p$ in [10] (which generalises [14, Theorem 11.3]). To this aim we denote by $\mathcal{W}^\infty_{\mathrm{pw}}(U;\mathbb{R}^n) \subset GSBD^p(U)$ the space of "*piecewise smooth*" SBV-functions, that is,

$$\begin{aligned}
\mathcal{W}^\infty_{\mathrm{pw}}(U;\mathbb{R}^n) := \Big\{ u \in GSBD^p(U)\colon\ & u \in SBV(U;\mathbb{R}^n) \cap W^{m,\infty}(U \setminus \overline{J_u};\mathbb{R}^n),\ \forall m \in \mathbb{N}, \\
& \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0,\ \overline{J_u} = \cup_{i=1}^k K_i \subset\subset U \\
& \text{with } K_i \text{ connected (n-1)-rectifiable set}, \forall\, 1 \le i \le k \Big\}
\end{aligned} \tag{3.4}$$

Theorem 3.4 (Density in $GSBD^p$)**.** *Let ϕ be a norm on $\mathbb{R}^n$. Let $u \in GSBD^p(U)$. Then there exists a sequence $(u_j) \subset \mathcal{W}^\infty_{\mathrm{pw}}(U;\mathbb{R}^n)$ such that*

(i) $u_j \to u$ *in measure on* U ;
(ii) $e(u_j) \to e(u)$ *in* $L^p(U;\mathbb{M}^{n\times n}_{\rm sym})$;
(iii) $\lim_{j\to\infty}\int_{J_{u_j}} \phi(\nu_{u_j})\,\mathrm{d}\mathcal{H}^{n-1} = \int_{J_u}\phi(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$.

Moreover, if

$$\int_U \psi(|u|)\,\mathrm{d}x < +\infty\,,$$

for some $\psi\colon [0,+\infty) \to [0,+\infty)$, *continuous, increasing with*

$$\psi(0)=0\,,\ \psi(s+t) \le C(\psi(s)+\psi(t))\,,\ \psi(s) \le C(1+s^p)\,,\ \lim_{s\to+\infty}\psi(s) = +\infty\,;$$

then

$$\lim_{j\to+\infty}\int_U \psi(|u_j-u|)\,\mathrm{d}x = 0\,.$$

We notice that the approximating class considered above fulfils the additional requirement of having a jump set compactly contained in U. This is possible, as shown in [16, Theorem C].

Theorem 3.5 (Compactness in $GSBD^p$)**.** *Let* $(u_j) \subset GSBD^p(U)$ *be a sequence satisfying*

$$\sup_{j\in\mathbb{N}}\Big(\|e(u_j)\|_{L^p(U)} + \mathcal{H}^{n-1}(J_{u_j})\Big) < +\infty\,.$$

Then there exist a subsequence, still denoted by (u_j), *and* $u \in GSBD^p(U)$ *with the following properties:*

(i) *the set* $U^\infty := \{x \in U\colon |u_j| \to +\infty\}$ *has finite perimeter;*
(ii) $u_j \to u$ *in measure on* $U \setminus U^\infty$ *and* $u = 0$ *on* U^∞ ;
(iii) $e(u_j) \rightharpoonup e(u)$ *in* $L^p(U\setminus U^\infty;\mathbb{M}^{n\times n}_{\rm sym})$;
(iv) $\liminf\limits_{j\to+\infty}\mathcal{H}^{n-1}(J_{u_j}) \ge \mathcal{H}^{n-1}(J_u\cap(U\setminus U^\infty)) + \mathcal{H}^{n-1}(U\cap\partial^*U^\infty) \ge \mathcal{H}^{n-1}(J_u\cup(U\cap\partial^*U^\infty))$.

In the statement above, the last semicontinuity property is stated in a stronger form than in the original paper, but is also proved there (see [10, Formula (3.25)]). In the rest of this section we give some technical Lemmas.

Lemma 3.6. *Let* $g_j\colon \mathbb{R}^n \to [0,+\infty)$ *be a sequence of equi-integrable functions. Let* $E_j \subset \mathbb{R}^n$ *be such that* $\mathcal{L}^n(E_j) \to 0$ *and let* $\delta_j \searrow 0$ *as* $j \to +\infty$. *Then* $(g_j\chi_{E_j}) * \rho_{\delta_j} \to 0$ *strongly in* $L^1(\mathbb{R}^n)$.

Proof. By properties of convolution it holds that

$$\|g_j\chi_{E_j} * \rho_{\delta_j}\|_{L^1(\mathbb{R}^n)} \le \|g_j\chi_{E_j}\|_{L^1(\mathbb{R}^n)}.$$

By equi-integrability we have that for every $\varepsilon > 0$ there is $J \in \mathbb{N}$ such that for every $j \ge J$

$$\|g_j\chi_{E_j}\|_{L^1(\mathbb{R}^n)} = \int_{E_j} g_j\,\mathrm{d}x \le \varepsilon,$$

from which the thesis follows. □

Lemma 3.7. *Let* A' *be an open bounded subset of* $\mathbb{R}^n$. *Let* $g_j\colon A' \to [0,+\infty)$ *be a sequence of equi-integrable functions. Let* $\delta_j \searrow 0$ *as* $j \to +\infty$. *Then for every* $A \subset\subset A'$ *there holds*

$$\liminf_{j\to+\infty}\int_A g_j * \rho_{\delta_j}\,\mathrm{d}x \ge \liminf_{j\to+\infty}\int_A g_j\,\mathrm{d}x\,.$$

Proof. We consider the sequence of positive measures $\nu_j := g_j * \rho_{\delta_j}\mathcal{L}^n \llcorner A$. Since A' is bounded g_j turn out to be equi-bounded in $L^1(A')$, hence we get

$$\nu_j(A) = \int_A g_j * \rho_{\delta_j}\,\mathrm{d}x \le \int_{A'} g_j\,\mathrm{d}x \le C\,.$$

Therefore there exist a positive measure $\nu \in \mathcal{M}_b(A)$, a function $g \in L^1(A')$, and a not-relabelled subsequence such that $\nu_j \overset{*}{\rightharpoonup} \nu$ weakly $*$ in $\mathcal{M}_b(A)$ and $g_j \to g$ weakly in $L^1(A')$. It remains to show that $\nu = g\mathcal{L}^n \llcorner A$, indeed this would imply

$$\liminf_{j\to+\infty} \int_A g_j * \rho_{\delta_j}\,\mathrm{d}x = \liminf_{j\to+\infty} \nu_j(A) \ge \nu(A) = \int_A g\,\mathrm{d}x = \liminf_{j\to+\infty}\int_A g_j\,\mathrm{d}x\,,$$

and we could conclude. Let $\varphi \in C^\infty_c(A)$ and let $A \subset\subset A'' \subset\subset A'$ be fixed. By Fubini's theorem we have

$$\begin{aligned}\int_A \varphi\,\mathrm{d}\nu = \lim_{j\to+\infty}\int_A \varphi\,\mathrm{d}\nu_j &= \lim_{j\to+\infty}\int_A \varphi(g_j * \rho_{\delta_j})\,\mathrm{d}x\\ &= \lim_{j\to+\infty}\int_{A''}(\varphi * \hat\rho_{\delta_j})g_j\,\mathrm{d}x = \int_{A''}\varphi g\,\mathrm{d}x = \int_A \varphi g\,\mathrm{d}x,\end{aligned}$$

where $\hat\rho_{\delta_j}(x) := \rho_{\delta_j}(-x)$ and the last equality follows since $g_j \rightharpoonup g$ weakly in $L^1(A')$ and $\varphi * \hat\rho_{\delta_j}(x) \to \varphi$ strongly in $L^\infty(A')$. Thus we deduce $\nu = g\mathcal{L}^n \llcorner A$ and the proof is concluded. $\square$

Lemma 3.8. *Let $A \subset \mathbb{R}^{n-1}$. Let $(u_k) \subset L^1(A)$ be a sequence converging to u in $L^1(A)$. Let $A' \subset\subset A$ and let $w_k\colon A' \times Q' \to \mathbb{R}$ be given by $w_k(x, y) := u_k(x + \varepsilon_k y)$. Then w_k converges to u in $L^1(A' \times Q')$.*

Proof. By Frechet-Kolmogoroff's Theorem, for every $\eta > 0$ there is $h \in \mathbb{N}$ such that for all $k \ge h$ and $y \in Q'$ there holds

$$\int_{A'} |u_k(x + \varepsilon_k y) - u(x)|\,\mathrm{d}x \le \eta\,.$$

This together with Fubini's theorem yield

$$\int_{A'\times Q'} |w_k(x, y) - u(x)|\,\mathrm{d}x\,\mathrm{d}y \le \int_{Q'}\int_{A'} |u_k(x + \varepsilon_k y) - u(x)|\,\mathrm{d}x\,\mathrm{d}y \le \eta\,,$$

for all $k \ge h$. Eventually by letting $\eta \to 0$ we conclude. $\square$

Corollary 3.9. *Let $A \subset \mathbb{R}^{n-1}$. Let $(u_k) \subset L^0(A)$ be a sequence converging to u in measure. Let $A' \subset\subset A$ and let $w_k\colon A' \times Q' \to \mathbb{R}$ be given by $w_k(x, y) := u_k(x + \varepsilon_k y)$. Then w_k converges to u in measure.*

Proof. Since $\arctan(u_k)$ converges to $\arctan(u)$ in $L^1(A)$ by Lemma 3.8 we have that $\arctan(w_k)$ converges to $\arctan(u)$ in $L^1(A' \times Q')$. Hence w_k converges to u in measure. $\square$

4. Compactness

In this section we prove point (ii) of Theorem 2.1.

Proposition 4.1 (Compactness)**.** *Let F_k be as in (2.5). Let $(u_k) \subset L^0(U;\mathbb{R}^n)$ be such that $\sup_k F_k(u_k) < +\infty$. Then there exist $\bar u_k \in GSBV^p(U;\mathbb{R}^n)$ and $u \in GSBD^p(U)$ such that $\bar u_k - u_k \to 0$ in measure on U and, up to a subsequence, it holds*

$$\begin{gathered}\bar u_k \to u \quad \textit{in measure on } U \setminus U^\infty\,,\\ e(\bar u_k) \rightharpoonup e(u) \quad \textit{in } L^p_{\mathrm{loc}}(U \setminus U^\infty;\mathbb{M}^{n\times n}_{\mathrm{sym}})\,,\\ \liminf_{k\to+\infty}\mathcal{H}^{n-1}(J_{\bar u_k}) \ge \mathcal{H}^{n-1}(J_u \cup (\partial^* U^\infty \cap U))\,,\end{gathered}$$

where $U^\infty := \{x \in U : |u_k(x)| \to +\infty\}$*. If in addition*

$$\sup_{k\in\mathbb{N}} \int_U \psi(|u_k|)\,\mathrm{d}x < +\infty\,,$$

for some $\psi\colon [0,+\infty) \to [0,+\infty)$*, continuous, increasing with* $\lim_{s\to+\infty}\psi(s) = +\infty$*, then* $U^\infty = \emptyset$*, and all implications hold on* U*.*

Proof. The proof is inspired by that of [23, Proposition 4.1]. Let (u_k) be as in the statement and let $U' \subset\subset U$ be fixed. We will prove the following claim: there exist $(\bar u_k) \subset GSBV^p(U;\mathbb{R}^n)$ and $c_0 > 0$ (independent of k) such that

$$\bar u_k - u_k \to 0 \text{ in measure on } U\,, \tag{4.1}$$

$$\liminf_{k\to+\infty} F_k(u_k) \geq c_0 \limsup_{k\to+\infty}\left(\int_{U'} |e(\bar u_k)|^p\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{\bar u_k})\right). \tag{4.2}$$

Now, if (4.2) holds, we fix a sequence $U_i \nearrow U$ and apply Theorem 3.5 to each U_i. With a diagonal argument we deduce the existence of $u \in GSBD(U)$ with $u = 0$ on U^∞ such that, up to a subsequence,

$$\bar u_k \to u \text{ in measure on } U \setminus U^\infty\,,$$

$$e(\bar u_k) \rightharpoonup e(u) \text{ in } L^p_{\mathrm{loc}}(U\setminus U^\infty;\mathbb{M}^{n\times n}_{\mathrm{sym}})\,.$$

We remark that since the constant c_0 is independent of k, we indeed have $u \in GSBD(U)$ by the very same argument of [10, Formula (3.33)], with the minor difference that the Radon measure λ in the definition of $GSBD$ is first defined as local weak*-limit, but turns eventually out to be uniformly bounded by (4.2). We also get $e(u) \in L^p(U)$, since c_0 is independent of k. Concerning the remaining inequality, for fixed i we set $U_i^\infty := U^\infty \cap U_i$, then by Theorem 3.5 (iv) we have

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_{\bar u_k}) \geq \liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_{\bar u_k}\cap U_i) \geq \mathcal{H}^{n-1}(J_u \cap (U_i\setminus U^\infty)) + \mathcal{H}^{n-1}(U_i \cap \partial^* {U_i}^\infty)\,. \tag{4.3}$$

Observing that $U_j^\infty \nearrow U^\infty$ by lower semicontinuity of the relative perimeter we have

$$\liminf_{j\to+\infty} \mathcal{H}^{n-1}(U_i \cap \partial^* {U_j}^\infty) \geq \mathcal{H}^{n-1}(U_i\cap\partial^* U^\infty) \quad \forall i\,.$$

Letting $i \to +\infty$, by monotonicity we get

$$\lim_{i\to+\infty} \mathcal{H}^{n-1}(J_u\cap(U_i\setminus U^\infty)) = \mathcal{H}^{n-1}(J_u \cap (U\setminus U^\infty))\,, \tag{4.4}$$

and being $\mathcal{H}^{n-1}(U_i \cap \partial^* U_i^\infty) = \mathcal{H}^{n-1}(U_i\cap\partial^* U_j^\infty)$ for all $j \geq i$ it holds

$$\liminf_{i\to+\infty} \mathcal{H}^{n-1}(U_i\cap\partial^*{U_i}^\infty) \geq \liminf_{i\to+\infty}\liminf_{j\to+\infty} \mathcal{H}^{n-1}(U_i\cap\partial^*{U_j}^\infty) \geq \mathcal{H}^{n-1}(U\cap\partial^* U^\infty)\,. \tag{4.5}$$

Eventually combining (4.3) with (4.4) and (4.5)

$$\liminf_{k\to+\infty} \mathcal{H}^{n-1}(J_{\bar u_k}) \geq \mathcal{H}^{n-1}(J_u\cap(U\setminus U^\infty)) + \mathcal{H}^{n-1}(U\cap\partial^* U^\infty)) \geq \mathcal{H}^{n-1}(J_u \cup (\partial^* U^\infty \cap U))\,.$$

This also gives $u \in GSBD^p(U)$. As the remaining part of the statement follows directly from Chebycheff inequality and Fatou's lemma, we are only left to prove the claim.

For fixed $i \in \mathbb{N}$ let $f_i(t) = \alpha_i t \wedge \beta_i$ be as in (2.3). Choose $\eta > 0$ such that $Q_{2\eta}(0) \subset\subset S$ and let

$$m_\eta := \min_{x\in\overline{Q}_{2\eta}(0)} \rho(x) > 0 \quad\text{and}\quad f_i^{2\eta}(t) := f_i(m_\eta \eta^n t) = \alpha_i m_\eta \eta^n t \wedge \beta_i\,. \tag{4.6}$$

Then we have

$$\begin{aligned} F_k(u_k) &\geq \frac{1}{\varepsilon_k}\int_U f_i\Big(\varepsilon_k W_k(\cdot, e(u_k)) * \rho_k(x)\Big)\,\mathrm{d}x \\ &\geq \frac{1}{\varepsilon_k}\int_U f_i^{2\eta}\Big(\varepsilon_k \fint_{Q_{2\eta\varepsilon_k}(x)} W_k(y, e(u_k))\,\mathrm{d}y\Big)\,\mathrm{d}x\,. \end{aligned} \tag{4.7}$$

We set

$$A_k^1 := \left\{ x \in U \colon \varepsilon_k \fint_{Q_{\eta\varepsilon_k}(x)} W_k(y, e(u_k))\,\mathrm{d}y \geq \frac{\beta_i}{\alpha_i m_\eta \eta^n} \right\},$$

$$A_k^2 := \left\{ x \in U \colon \ \mathrm{dist}(x, A_k^1) \leq \eta\varepsilon_k \right\}.$$

Note that

$$A_k^1 \subset A_k^2 \subset \left\{ x \in U \colon \varepsilon_k \fint_{Q_{2\eta\varepsilon_k}(x)} W_k(y, e(u_k))\,\mathrm{d}y \geq \frac{\beta_i}{\alpha_i m_\eta (2\eta)^n} \right\}. \tag{4.8}$$

Indeed if $x \in A_k^2$ there is $z \in A_k^1$ with $Q_{\eta\varepsilon_k}(z) \subset Q_{2\eta\varepsilon_k}(x)$ and therefore

$$\varepsilon_k \fint_{Q_{2\eta\varepsilon_k}(x)} W_k(y, e(u_k))\,\mathrm{d}y \geq \frac{\varepsilon_k}{2^n} \fint_{Q_{\eta\varepsilon_k}(z)} W_k(y, e(u_k))\,\mathrm{d}y \geq \frac{\beta_i}{\alpha_i m_\eta (2\eta)^n}\,.$$

By combining together (4.7) and (4.8) we find

$$F_k(u_k) \geq \frac{\beta_i}{\varepsilon_k}\mathcal{L}^n(A_k^2)\,. \tag{4.9}$$

By the coarea formula (see e.g., [17, Theorem 3.14]) and the mean value theorem there exists $t_k \in (0, \eta\varepsilon_k)$ such that the set $A_k^3 := \{\mathrm{dist}(\cdot, A_k^1) \leq t_k\} \subset A_k^2$ satisfies

$$\mathcal{L}^n(A_k^2) \geq \eta\varepsilon_k \mathcal{H}^{n-1}(\partial A_k^3)\,. \tag{4.10}$$

Let now

$$\bar{u}_k(x) := \begin{cases} 0 & \text{if } x \in A_k^3\,, \\ u_k & \text{otherwise in } U\,. \end{cases}$$

By construction $\bar{u}_k \in GSBV^p(U;\mathbb{R}^n)$. By (4.9) and the fact that $A_k^3 \subset A_k^2$ we have $\mathcal{L}^n(A_k^3) \to 0$ as $k \to +\infty$ from which (4.1) follows. On the other hand as $J_{\bar{u}_k} \subseteq \partial A_k^3$ (4.9) and (4.10) yield

$$F_k(u_k) \geq \eta\beta_i \mathcal{H}^{n-1}(J_{\bar{u}_k})\,. \tag{4.11}$$

We next show that there exists $K(n) \geq 1$ such that for every $x \in U$

$$c_1\varepsilon_k \fint_{Q_{\eta\varepsilon_k}(x)} |e(\bar{u}_k(y))|^p\,\mathrm{d}y \leq K\frac{\beta_i}{\alpha_i m_\eta \eta^n}\,. \tag{4.12}$$

By $(W4)$ we have

$$W_k(x, e(u_k(x))) \geq c_1|e(u_k(x))|^p \geq c_1|e(\bar{u}_k(x))|^p \quad \text{for a.e. } x \in U\,. \tag{4.13}$$

Now if $x \in U \setminus A_k^3$, then $x \notin A_k^1$ and

$$c_1\varepsilon_k \fint_{Q_{\eta\varepsilon_k}(x)} |e(\bar{u}_k(y))|^p\,\mathrm{d}y \leq \varepsilon_k \fint_{Q_{\eta\varepsilon_k}(x)} W_k(y, e(u_k(y)))\,\mathrm{d}y \leq \frac{\beta_i}{\alpha_i m_\eta \eta^n}\,.$$

Assume instead that $x \in A_k^3$. Observe that $\bar{u}_k = 0$ in $Q_{\eta\varepsilon_k}(x) \cap A_k^3$, so that

$$\int_{Q_{\eta\varepsilon_k}(x)} |e(\bar{u}_k(y))|^p\,\mathrm{d}y = \int_{Q_{\eta\varepsilon_k}(x)\setminus A_k^3} |e(\bar{u}_k(y))|^p\,\mathrm{d}y\,.$$

Furthemore, we can cover $Q_{\eta\varepsilon_k}(x)\cap(Q\setminus A_k^3)$ with a finite number $K(n)\geq 1$ of balls of radius $\eta\varepsilon_k$ and centres $x_1,...,x_K\in U\setminus A_k^3$ (see e.g. [23, Remark 2.8]). Hence, we find

$$\begin{aligned} c_1\varepsilon_k \fint_{Q_{\eta\varepsilon_k}(x)} |e(\bar u_k(y))|^p\,\mathrm{d}y &\leq c_1\varepsilon_k\sum_{i=1}^K \fint_{Q_{\eta\varepsilon_k}(x_i)} |e(\bar u_k(y))|^p\,\mathrm{d}y \\ &\leq \varepsilon_k\sum_{i=1}^K \fint_{Q_{\eta\varepsilon_k}(x_i)} W_k(y,e(u_k(y)))\,\mathrm{d}y \leq K\frac{\beta_i}{\alpha_i m_\eta \eta^n}\,, \end{aligned}$$

and (4.12) follows. Finally by (4.7) (with η in place of 2η), the monotonicity of f_i^η, (4.13) we infer

$$\begin{aligned} F_k(u_k) &\geq \frac{1}{\varepsilon_k}\int_U f_i^\eta\Big(\varepsilon_k\fint_{Q_{\eta\varepsilon_k}(x)} c_1|e(\bar u_k(y))|^p\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\geq c_1\frac{\alpha_i m_\eta\eta^n}{K}\int_U\fint_{Q_{\eta\varepsilon_k}(x)} |e(\bar u_k(y))|^p\,\mathrm{d}y\,\mathrm{d}x\,, \end{aligned} \tag{4.14}$$

where the last inequality follows from (4.12) and the fact that $f_i^\eta(t)\geq\frac{\alpha_i m_\eta\eta^n}{K}t$ when $t\leq K\frac{\beta_i}{m_\eta\eta^n\alpha_i}$. Moreover by using in order the change of variable $y=x-\eta\varepsilon_k z$, Fubini's theorem, and the change of variable $\hat x = x-\eta\varepsilon_k z$ (for k large enough), we find

$$\begin{aligned} \int_U\fint_{Q_{\eta\varepsilon_k}(x)} |e(\bar u_k(y))|^p\,\mathrm{d}y\,\mathrm{d}x &= \int_Q\int_U |e(\bar u_k(x-\eta\varepsilon_k z))|^p\,\mathrm{d}x\,\mathrm{d}z \\ &\geq \int_{U'} |e(\bar u_k(x))|^p\,\mathrm{d}x\,. \end{aligned} \tag{4.15}$$

Eventually gathering together (4.10), (4.14), and (4.15), we deduce (4.2) with

$$c_0 := \frac12\Big(\frac{c_1\alpha_i m_\eta\eta^n}{K}\wedge\eta\beta_i\Big)\,.$$

□

5. Lower bound

In this section we prove the lower bound. To this purpose it is convenient to localise the functionals F_k, namely we set

$$F_k(u,A) := \frac{1}{\varepsilon_k}\int_A f\Big(\varepsilon_k W_k(\cdot,e(u))*\rho_k(x)\Big)\,\mathrm{d}x\,,\quad \text{for } u\in W^{1,p}(U)\,,\ A\subset U\,. \tag{5.1}$$

Proposition 5.1 (Lower bound: bulk contribution)**.** *Let $(u_k)\subset L^0(U;\mathbb{R}^n)$ be a sequence that converges in measure to $u\in L^0(U;\mathbb{R}^n)$. Assume moreover that $F_k(u_k)\leq C$ and that* (2.9) *holds. Then, for W as* (2.9)

$$\liminf_{k\to+\infty} F_k(u_k,A)\geq\alpha\int_A W(x,e(u))\,\mathrm{d}x\quad\forall A\in\mathcal{A}(U)\,,$$

Proof. Let (u_k) and u be as in the statement. By Proposition 4.1 $u\in GSBD^p(U)$. For every $k\in\mathbb{N}$ let μ_k be the Radon measure on $(U,\mathcal{B}(U))$ given by

$$\mu_k(A) := F_k(u_k,A)\,,\quad\forall A\in\mathcal{B}(U)\,. \tag{5.2}$$

As $\mu_k(A)\leq C$, by [2, Theorem 1.59] we deduce the existence of a subsequence, not relabelled, and of a Radon measure μ on $(A,\mathcal{B}(A))$ such that

$$\mu_k \overset{*}{\rightharpoonup}\mu \quad\text{and}\quad \liminf_{k\to+\infty}\mu_k(A)\geq\mu(A)\,. \tag{5.3}$$

By Radon-Nikodym's Theorem (in the version of [2, Theorem 1.28]) there exist two measures μ^a, μ^s with $\mu^a \ll \mathcal{L}^n$ and $\mu^s \perp \mathcal{L}^n$, and a function $h \in L^1(A)$ such that $\mu = \mu^a + \mu^s$ and $\mu^a = h\mathcal{L}^n$. This together with (5.3) imply that

$$\liminf_{k\to+\infty} F_k(u_k, A) \geq \int_A h(x)\,\mathrm{d}x\,.$$

Hence to conclude we need to show that

$$h(x) \geq \alpha W(x, e(u(x))) \quad \text{ for a.e. } x \in U\,. \tag{5.4}$$

with W as in (3.3) For $i \in \mathbb{N}$ fixed let $f_i(t) = \alpha_i t \wedge \beta_i$ be as in (2.3). Then it is enough to show that

$$h(x) \geq \alpha_i W(x, e(u(x))) \quad \text{ for a.e. } x \in U\,, \tag{5.5}$$

We divide the proof of (5.5) into four steps.

Step 1: In this step we show that for a.e. $x_0 \in U$ there exists a sequence $(k_j, r_j) \to (+\infty, 0)$ as $j \to +\infty$ such that setting $\delta_j := \frac{\varepsilon_{k_j}}{r_j}$,

$$u_k^r(v) := \frac{u_k(x_0 + rv) - u_k(x_0)}{r} \quad \text{ and } \quad W_k^r(x, M) := W(x_0 + rx, M)\,, \tag{5.6}$$

there hold

$$h(x_0) \geq \lim_{j\to+\infty} \frac{1}{r_j}\frac{1}{\delta_j}\int_Q f_i\Big(r_j\delta_j W_{k_j}^{r_j}(\cdot, e(u_{k_j}^{r_j})) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x\,, \tag{5.7}$$

and

$$u_{k_j}^{r_j} \to \nabla u(x_0)(\cdot) \quad \text{in measure on } Q\,, \tag{5.8}$$

together with

$$\lim_{j\to+\infty} \frac{\mathbf{m}_{k_j}(u_{\nabla u(x_0)}, Q_{r_j}(x_0))}{r_j^n} = W(x_0, e(u(x_0))) \tag{5.9}$$

By Besicovitch differentiation theorem and [8, Corollary 5.2] we have that for a.e. $x_0 \in U$ the following hold:

$$h(x_0) = \lim_{r\searrow 0^+} \frac{\mu(\overline{Q}_r(x_0))}{|Q_r(x_0)|}\,, \tag{5.10}$$

$$\lim_{r\searrow 0^+} \frac{1}{r^n}\mathcal{L}^n\left(\left\{y \in Q_r(x_0)\colon \frac{|u(y) - u(x_0) - \nabla u(x_0)(y - x_0)|}{|y - x_0|} > \delta\right\}\right) = 0 \quad \forall \delta > 0\,. \tag{5.11}$$

We fix $x_0 \in \Omega$ for which (5.10) and (5.11) hold. By [2, Proposition 1.62] we have

$$\mu(\overline{Q}_r(x_0)) \geq \limsup_{k\to+\infty} \mu_k(\overline{Q}_r(x_0))\,,$$

for every $r > 0$, which together with (5.10) yield

$$h(x_0) \geq \limsup_{r\searrow 0^+}\limsup_{k\to+\infty} \frac{\mu_k(\overline{Q}_r(x_0))}{|Q_r(x_0)|}\,. \tag{5.12}$$

Moreover from (5.2) and the change of variable $x = x_0 + rx'$ we get

$$\begin{aligned}\mu_k(\overline{Q}_r(x_0)) &= \frac{1}{\varepsilon_k}\int_{\overline{Q}_r(x_0)} f\Big(\varepsilon_k W_k(\cdot, e(u_k)) * \rho_k(x)\Big)\,\mathrm{d}x \\ &= \frac{r^n}{\varepsilon_k}\int_{\overline{Q}} f\Big(\varepsilon_k W_k(\cdot, e(u_k)) * \rho_k(x_0 + rx)\Big)\,\mathrm{d}x\,.\end{aligned} \tag{5.13}$$

From (5.6) and the change of variable $y = ry'$ we may deduce that

$$W_k(\cdot, e(u_k)) * \rho_k(x_0 + rx) = W_k^r(\cdot, e(u_k^r)) * \rho_{\frac{\varepsilon_k}{r}}(x)\,. \tag{5.14}$$

Gathering together (5.12), (5.13), (5.14) and using (2.3) we obtain

$$h(x_0) \geq \limsup_{r \searrow 0^+} \limsup_{k \to +\infty} \frac{1}{r} \frac{r}{\varepsilon_k} \int_Q f_i\Big(r \frac{\varepsilon_k}{r} W_k^r(\cdot, e(u_k^r)) * \rho_{\frac{\varepsilon_k}{r}}(x)\Big)\,\mathrm{d}x\,. \tag{5.15}$$

Now, from (5.11) and the fact that u_k converges to u in measure we can deduce that

$$\lim_{r \to 0} \lim_{k \to +\infty} \mathcal{L}^n\left(\{v \in Q \colon |u_k^r(v) - \nabla u(x_0)(v)| > \delta\}\right) = 0 \quad \forall \delta > 0\,.$$

If we fix a diagonal subsequence $(k_j, r_j) \to (+\infty, 0)$ as $j \to +\infty$ for which (5.8) and (5.9) hold, from (5.15) we also get (5.7) for $\delta_j := \frac{\varepsilon_{k_j}}{r_j}$ (up to taking a further subsequence to have a limit in place of a limsup). With this, the proof of step 1 is concluded.

Step 2: In this step we show that for any $0 < \zeta < 1$ and a.e. $x_0 \in U$ there exist $(\bar{u}_j) \subset GSBV^p(Q;\mathbb{R}^n)$ and $c_0 > 0$ independent of j such that

$$\lim_{j \to +\infty} \mathcal{L}^n\{\bar{u}_j \neq u_{k_j}^{r_j}\} = 0; \tag{5.16}$$

$$\bar{u}_j \to \nabla u(x_0)(\cdot) \quad \text{in measure on } Q\,; \tag{5.17}$$

$$\mathcal{H}^{n-1}(J_{\bar{u}_j} \cap Q) \to 0\,; \tag{5.18}$$

$$\int_{Q_{1-\zeta}(0)} |e(\bar{u}_j)|^p\,\mathrm{d}x \leq c_0\,. \tag{5.19}$$

By step 1 we have that $u_{k_j}^{r_j}$ converges in measure to $\nabla u(x_0)(\cdot)$ in Q as $j \to +\infty$ and for j large enough it satisfies

$$\frac{1}{r_j} \frac{1}{\delta_j} \int_Q f_i\Big(r_j \delta_j W_{k_j}^{r_j}(\cdot, e(u_{k_j}^{r_j})) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \leq C\,. \tag{5.20}$$

Next we fix $\eta > 0$ such that $Q_{2\eta}(0) \subset\subset S$ and let m_η and f_i^η be as in (4.6). Then we get

$$\int_Q f_i\Big(r_j \delta_j W_{k_j}^{r_j}(\cdot, e(u_{k_j}^{r_j})) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \geq \int_Q f_i^{2\eta}\Big(r_j \delta_j \fint_{Q_{2\eta\delta_j}(x)} W_{k_j}^{r_j}(y, e(u_{k_j}^{r_j}))\,\mathrm{d}y\Big)\,\mathrm{d}x\,. \tag{5.21}$$

We define the sets

$$A_j^1 := \left\{x \in Q \colon r_j \delta_j \fint_{Q_{\eta\delta_j}(x)} W_{k_j}^{r_j}(y, e(u_{k_j}^{r_j}))\,\mathrm{d}y \geq \frac{\beta_i}{\alpha_i m_\eta \eta^n}\right\},$$

$$A_j^2 := \left\{x \in Q \colon \operatorname{dist}(x, A_j^1) \leq \eta \delta_j\right\}.$$

Then arguing as in the proof of Proposition 4.1 we find that

$$A_j^1 \subset A_j^2 \subset \left\{x \in Q \colon r_j \delta_j \fint_{Q_{2\eta\delta_j}(x)} W_{k_j}^{r_j}(y, e(u_{k_j}^{r_j}))\,\mathrm{d}y \geq \frac{\beta_i}{\alpha_i m_\eta (2\eta)^n}\right\}. \tag{5.22}$$

(5.22) together with (5.20) and (5.21) imply that (for j large enough)

$$C \geq \frac{\beta_i}{r_j \delta_j} \mathcal{L}^n(A_j^2) = \frac{\beta_i}{\varepsilon_j} \mathcal{L}^n(A_j^2)\,. \tag{5.23}$$

By the coarea formula and the mean value theorem we can find $t_j \in (0, \eta\delta_j)$ such that setting $A_j^3 := \{\operatorname{dist}(\cdot, A_j^1) \leq t_j\} \subset A_j^2$

$$\mathcal{L}^n(A_j^2) \geq \eta \delta_j \mathcal{H}^{n-1}(\partial A_j^3)\,. \tag{5.24}$$

We finally define

$$\bar{u}_j(x) := \begin{cases} 0 & \text{if } x \in A^3_j, \\ u^{r_j}_{k_j} & \text{otherwise in } Q. \end{cases}$$

Recall that, by definition, $\frac{\varepsilon_j}{\delta_j} \to 0$. With this, as a consequence of (5.23) and (5.24) we have that both $\mathcal{L}^n(A^3_j)$ and $\mathcal{H}^{n-1}(\partial A^3_j) = \mathcal{H}^{n-1}(J_{\bar{u}_j})$ converge to 0 as $j \to +\infty$. Hence $\bar{u}_j \subset GSBV^p(Q;\mathbb{R}^n)$ and $\bar{u}_j - u^{r_j}_{k_j} \to 0$ in measure on Q which combined with (5.8) yield $\bar{u}_j \to \nabla u(x_0)(\cdot)$ in measure on Q. It remains to show (5.19). To this aim notice that arguing exactly as in the proof of Proposition 4.1 one can find $K(n) \geq 1$ such that for every $x \in \overline{Q}$

$$c_1 r_j \delta_j \fint_{Q_{\eta\delta_j}(x)} |e(\bar{u}_j(y))|^p \,\mathrm{d}y \leq K \frac{\beta_i}{\alpha_i m_\eta \eta^n}\,. \tag{5.25}$$

Next from (5.21) (with η in place of 2η) and the monotonicity of f^η_i we infer

$$\begin{aligned} \frac{1}{r_j\delta_j} \int_Q f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(u^{r_j}_{k_j})) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x &\geq \frac{1}{r_j\delta_j} \int_Q f^\eta_i\Big(c_1 r_j\delta_j \fint_{Q_{\eta\delta_j}(x)} |e(\bar{u}_j(y))|^p\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\geq c_1 \frac{\alpha_i m_\eta \eta^n}{K} \int_Q \fint_{Q_{\eta\delta_j}(x)} |e(\bar{u}_j(y))|^p \,\mathrm{d}y\,\mathrm{d}x \end{aligned} \tag{5.26}$$

where the last inequality follows from (5.25) and fact that $f^\eta_i(t) \geq \frac{\alpha_i m_\eta \eta^n}{K} t$ when $t \leq K\frac{\beta_i}{m_\eta \eta^n \alpha_i}$. Finally, for a fixed $0 < \zeta < 1$, arguing exactly as for (4.15) we get

$$\int_Q \fint_{Q_{\eta\delta_j}(x)} |e(\bar{u}_j(y))|^p \,\mathrm{d}y\,\mathrm{d}x \geq \int_{Q_{1-\zeta}(0)} |e(\bar{u}_j(x))|^p\,\mathrm{d}x \tag{5.27}$$

when j is sufficiently large. Eventually gathering together (5.20), (5.26), and (5.27), we deduce (5.19) with $c_0 := \frac{CK}{c_1 \alpha_i m_\eta \eta^n}$.

Step 3: In this step show that for a.e. $x_0 \in U$ there exists a sequence $(w_j) \subset W^{1,p}(Q;\mathbb{R}^n)$ such that:

$$(|\nabla w_j|^p) \quad \text{is equi-integrable;} \tag{5.28}$$

$$\lim_{j\to+\infty} \mathcal{L}^n(\{w_j \neq \bar{u}_j\} = 0\,; \tag{5.29}$$

$$\lim_{j\to+\infty} \|w_j - \nabla u(x_0)(\cdot)\|_{L^p(Q)} = 0\,; \tag{5.30}$$

$$h(x_0) \geq \liminf_{j\to+\infty} \frac{1}{r_j\delta_j} \int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \quad \forall i \in \mathbb{N}\,. \tag{5.31}$$

From step 2 we can apply [19, Lemma 5.1] to the sequence $\bar{u}_j$ and get the existence of $(w_j) \subset W^{1,p}(Q;\mathbb{R}^n)$ that satisfies (5.28)–(5.30). Moreover recalling $(W4)$ and the equi-integrability of $(|\nabla w_j|^p)$ we have that $W^{r_j}_{k_j}(x, e(w_j))$ is equi-integrable as well, while from the inclusion

$$E_j := \Big\{e(w_j) \neq e(u^{r_j}_{k_j})\Big\} \subset \{w_j \neq u^{r_j}_{k_j}\} \subset \{w_j \neq \bar{u}_j\} \cup \{\bar{u}_j \neq u^{r_j}_{k_j}\}\,,$$

it follows that $\mathcal{L}^n\left(\Big\{e(w_j) \neq e(u^{r_j}_{k_j})\Big\}\right) \to 0$. Thus, we can apply Lemma 3.6 with $g_j = W^{r_j}_{k_j}(x, e(w_j))$, and $E_j = \Big\{e(w_j) \neq e(u^{r_j}_{k_j})\Big\}$, and deduce that

$$\int_{Q_{1-\zeta}(0)} (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{E_j}) * \rho_{\delta_j}(x)\,\mathrm{d}x \to 0\,. \tag{5.32}$$

We also remark that by the definition of E_j we have

$$\begin{aligned} W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j} &= (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{E_j}^c) * \rho_{\delta_j} + (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{E_j}) * \rho_{\delta_j} \\ &\leq W^{r_j}_{k_j}(\cdot, e(u^{r_j}_{k_j})) * \rho_{\delta_j} + (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{E_j}) * \rho_{\delta_j} \end{aligned}$$

By monotonicity of f_i and since $f_i(t) \leq \alpha_i t$ we obtain the following estimate

$$\begin{aligned} &\frac{1}{r_j\delta_j}\int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \\ &\qquad \leq \frac{1}{r_j\delta_j}\int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(u^{r_j}_{k_j})) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x + \alpha_i \int_{Q_{1-\zeta}(0)} (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{E_j}) * \rho_{\delta_j}(x)\,\mathrm{d}x\,. \end{aligned}$$

Passing to the limit as $j \to +\infty$ in the above inequality and using (5.7) and (5.32) we infer (5.31).
Step 4: In this step we show that for a.e. $x_0 \in U$

$$\liminf_{j\to+\infty} \frac{1}{r_j\delta_j}\int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \geq \alpha_i W(x_0, e(u(x_0))) \quad \forall i \in \mathbb{N}\,. \tag{5.33}$$

We define the following partition

$$B^1_j := \left\{x \in Q_{1-\zeta}(0)\colon r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x) \geq \frac{\beta_i}{\alpha_i}\right\}\,, \quad B^2_j := Q_{1-\zeta}(0) \setminus B^1_j\,.$$

Since $f_i(t) = \alpha_i t$ when $t \leq \frac{\beta_i}{\alpha_i}$, and $(W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{B^2_j}) * \rho_{\delta_j} \leq \frac{\beta_i}{\alpha_i}$ by definition of B^2_j and standard properties of the convolution, we have

$$\frac{1}{r_j\delta_j}\int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \geq \alpha_i \int_{Q_{1-\zeta}(0)} (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{B^2_j}) * \rho_{\delta_j}(x)\,\mathrm{d}x\,. \tag{5.34}$$

As for j large enough there holds $\mathcal{L}^n(B^1_j) \leq C r_j\delta_j \to 0$, Lemma 3.6 implies

$$\int_{Q_{1-\zeta}(0)} (W^{r_j}_{k_j}(\cdot, e(w_j))\chi_{B^1_j}) * \rho_{\delta_j}(x)\,\mathrm{d}x \to 0\,. \tag{5.35}$$

Now, taking the liminf as $j \to +\infty$ in (5.34), and adding the vanishing term in (5.35) to the right-hand side, we get

$$\liminf_{j\to+\infty} \frac{1}{r_j\delta_j}\int_{Q_{1-\zeta}(0)} f_i\Big(r_j\delta_j W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\Big)\,\mathrm{d}x \geq \liminf_{j\to+\infty} \alpha_i \int_{Q_{1-\zeta}(0)} W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\,\mathrm{d}x\,. \tag{5.36}$$

From this, applying Lemma 3.7 with $g_j = W^{r_j}_{k_j}(x, e(w_j))$ we have

$$\liminf_{j\to+\infty} \alpha_i \int_{Q_{1-\zeta}(0)} W^{r_j}_{k_j}(\cdot, e(w_j)) * \rho_{\delta_j}(x)\,\mathrm{d}x \geq \alpha_i \liminf_{j\to+\infty} \int_{Q_{1-\zeta}(0)} W_{k_j}(x_0 + r_j x, e(w_j))\,\mathrm{d}x\,. \tag{5.37}$$

Next we modify w_j so that it coincides with $\nabla u(x_0)(\cdot)$ on $\partial Q_{1-\zeta}(0)$ without essentially increasing the energy. This can be achieved by relying on the following Fundamental Estimate than can be proved with standard arguments: for given $\gamma > 0$, there exist $C(\gamma)$ and a sequence $(\overline{w}_j) \subset W^{1,p}(Q_{1-\zeta}(0);\mathbb{R}^d)$ with $\overline{w}_i = \nabla u(x_0)(\cdot)$ in a neighbourhood of $\partial Q_{1-\zeta}(0)$ such that

$$\begin{aligned} \int_{Q_{1-\zeta}(0)} W_{k_j}(x_0 + r_j x, e(\overline{w}_j))\,\mathrm{d}x &\leq (1+\gamma)\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0 + r_j x, e(w_j))\,\mathrm{d}x \\ &\quad + (1+\gamma)\int_{Q_{1-\zeta}(0)\setminus Q_{1-\zeta-\gamma}(0)} W_{k_j}(x_0 + r_j x, e(u(x_0)))\,\mathrm{d}x \\ &\quad + C(\gamma)\|w_j - \nabla u(x_0)(\cdot)\|^p_{L^p(Q_{1-\zeta}(0))} + \gamma\,. \end{aligned} \tag{5.38}$$

By (5.30) we know that w_j converges to $\nabla u(x_0)(\cdot)$ in $L^p(Q)$, moreover from ((W4)) there holds

$$\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0+r_jx, e(u(x_0)))\,\mathrm{d}x \le c_2(|e(u(x_0))|^p+1)\mathcal{L}^n(Q_{1-\zeta}(0)\setminus Q_{1-\zeta-\gamma}(0)) \le c_2(|e(u(x_0))|^p+1)n\gamma\,.$$

This fact and (5.38) imply that

$$\begin{aligned}&\liminf_{j\to+\infty}\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0+r_jx, e(w_j))\,\mathrm{d}x\\ &\quad\ge \frac{1}{1+\gamma}\liminf_{j\to+\infty}\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0+r_jx, e(\overline{w}_j))\,\mathrm{d}x - c_2(|e(u(x_0))|^p+1)n\gamma - \frac{\gamma}{1+\gamma}\,.\end{aligned}\tag{5.39}$$

We now set $\widetilde{w}_j(x) := r_j\overline{w}_j((x-x_0)/r_j)$, which is admissible for $\mathbf{m}_{k_j}(u_{\nabla u(x_0)}, Q_{(1-\zeta)r_j}(0))$ in (2.6). Hence, by a change of variable in (5.39) we obtain

$$\begin{aligned}\liminf_{j\to+\infty}\alpha_i\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0+r_jx, e(\overline{w}_j))\,\mathrm{d}x &\ge \liminf_{j\to+\infty}\frac{\alpha_i}{r_j^n}\int_{Q_{(1-\zeta)r_j}(x_0)} W_{k_j}(x, e(\widetilde{w}_j))\,\mathrm{d}x\\ &\ge \liminf_{j\to+\infty}\alpha_i\frac{\mathbf{m}_{k_j}(u_{\nabla u(x_0)}, Q_{(1-\zeta)r_j}(x_0))}{r_j^n}\\ &= (1-\zeta)\alpha_i W(x_0, e(u(x_0)))\,.\end{aligned}\tag{5.40}$$

Gathering together (5.39) and (5.40), with (5.9) we deduce

$$\liminf_{j\to+\infty}\alpha_i\int_{Q_{1-\zeta}(0)} W_{k_j}(x_0+r_jx, e(w_j))\,\mathrm{d}x \ge \frac{(1-\zeta)^n}{1+\gamma}\alpha_i W(x_0, e(u(x_0))) - C\left(\gamma+\frac{\gamma}{1+\gamma}\right)\,.$$

With this, (5.36), and (5.37), we eventually deduce (5.33) by arbitrariness of ζ and γ.
Conclusion: from step 3 and step 4 we deduce the validity of (5.5) and the proof is concluded. ☐

Remark 5.2. We observe en passant that Proposition 5.1 indeed holds also for a sequence of functionals

$$F_k(u,A) := \frac{1}{\varepsilon_k}\int_A f_k\Big(\varepsilon_k W_k(\cdot, e(u))*\rho_k(x)\Big)\,\mathrm{d}x\,,\quad \text{for } u\in W^{1,p}(U)\,,\ A\subset U\,,$$

provided the functions f_k satisfy an estimate of the form

$$f_k(t)\ge \alpha_k t\wedge\beta$$

for all $t\in[0,+\infty)$, where β is a uniform constant and $\alpha = \lim_{k\to+\infty}\alpha_k$.

Proposition 5.3 (Lower bound: surface contribution)**.** *Let $(u_k)\subset L^0(U;\mathbb{R}^n)$ be a sequence that converges to in measure to $u\in L^0(U;\mathbb{R}^n)$. Assume moreover that $F_k(u_k)\le C$. Then there holds*

$$\liminf_{k\to+\infty}F_k(u_k,A)\ge\beta\int_{J_u\cap A}\phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\quad\forall A\in\mathcal{A}(U)\,.$$

Proof. Let (u_k) and u be as in the statement, so that by Proposition 4.1 $u\in GSBD^p(U)$. Let $A\in\mathcal{A}(U)$ be fixed. We claim that it suffices to show that for any $\xi\in\mathbb{S}^{n-1}$ fixed there holds

$$\liminf_{k\to+\infty}F_k^\xi(u_k,A)\ge\beta\int_{J_u^\xi\cap A}\mu_\xi|\langle\nu_u,\xi\rangle|\,\mathrm{d}\mathcal{H}^{n-1}\,,\tag{5.41}$$

with

$$F_k^\xi(u_k,A) := \frac{1}{\varepsilon_k}\int_A f\Big(c_1\varepsilon_k|\langle e(u_k)\xi,\xi\rangle|^p*\rho_k(x)\Big)\,\mathrm{d}x\,,$$

$$J_u^\xi := \{x\in J_u\colon \langle u^+(x)-u^-(x),\xi\rangle\neq 0\}\quad\text{and}\quad \mu_\xi := \mathcal{H}^1(\{x\in S\colon x=t\xi\ \text{ for }\ t\in\mathbb{R}\})\,.$$

Indeed, assume for the moment (5.41) holds true. Then (($W4$)) gives

$$W_k(x, e(u_k)) \geq c_1|e(u_k)|^p \geq c_1|\langle (e(u_k))\xi, \xi\rangle|^p\,.$$

Since f is nondecreasing, the above implies

$$\liminf_{k\to+\infty} F_k(u_k, A) \geq \liminf_{k\to+\infty} F_k^\xi(u_k, A) \geq \beta \int_{J_u^\xi\cap A} \mu_\xi|\langle \nu_u, \xi\rangle|\,\mathrm{d}\mathcal{H}^{n-1} = \beta\int_{J_u\cap A} \varphi_\xi\,\mathrm{d}\mathcal{H}^{n-1}\,,$$

with $\varphi_\xi\colon J_u \to [0, +\infty]$ given by

$$\varphi_\xi(x) := \begin{cases} \mu_\xi|\langle \nu_u(x), \xi\rangle| & \text{if } x \in J_u^\xi\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Now let $(\xi_h) \subset \mathbb{S}^{n-1}$ be a dense subset, in this way by [6, Proposition 1.16] it holds

$$\liminf_{k\to+\infty} F_k(u_k, A) \geq \beta\int_{J_u\cap A} \sup_h \varphi_{\xi_h}\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

On the other hand by [21, Lemma 4.5] we have

$$\phi_\rho(\nu) = \sup_{\xi\in\mathbb{S}^{n-1}} \mu_\xi|\langle \nu, \xi\rangle|\,, \tag{5.42}$$

which in turn implies $\phi_\rho(\nu_u(x)) = \sup_h \varphi_{\xi_h}(x)$ and the thesis follows. It remains to show (5.41) for which we will argue by slicing. As the set S is convex for $\delta \in (0, 1)$ fixed we can find $r = r(\delta, S)$ such that the cylinder

$$C_\xi^{r,\delta} := R_\xi(Q_r'(0) \times \big(-\mu_\xi\delta/2, \mu_\xi\delta/2\big)) \subset\subset S\,,$$

where $R_\xi \in SO(n)$ is such that $R_\xi e_n = \xi$ (see (j)). Let now $m_\xi := \min_{x\in\overline{C}_\xi^{r,\delta}} \rho(x)$ and

$$C_{\xi,k}^{r,\delta}(x) := \varepsilon_k C_\xi^{r,\delta} + x\,.$$

Next for any $x \in A$ we denote by $\hat{x}_\xi$ the projection of x onto $\Pi^\xi := \{y \in \mathbb{R}^n\colon \langle y, \xi\rangle = 0\}$ and

$$I_\xi := \{\tau \in \mathbb{R}\colon \hat{x}_\xi + \tau\xi \in A\}\,.$$

Then we have

$$\begin{aligned} F_k^\xi(u_k, A) &\geq \frac{1}{\varepsilon_k}\int_A f\bigg(\frac{c_1 m_\xi}{\varepsilon_k^{n-1}}\int_{C_{\xi,k}^{r,\delta}(x)} |\langle e(u_k(z))\xi, \xi\rangle|^p\,\mathrm{d}z\bigg)\,\mathrm{d}x\\ &= \frac{1}{\varepsilon_k}\int_{\Pi^\xi}\int_{I_\xi} f\bigg(\frac{c_1 m_\xi}{\varepsilon_k^{n-1}}\int_{C_{\xi,k}^{r,\delta}(\hat{x}_\xi+\tau\xi)} |\langle e(u_k(z))\xi, \xi\rangle|^p\,\mathrm{d}z\bigg)\mathrm{d}\tau\,\mathrm{d}\mathcal{H}^{n-1}(\hat{x}_\xi)\,, \end{aligned} \tag{5.43}$$

where the last equality follows by Fubini's Theorem. Noticing that f is concave and using the change of variable $z = \hat{x}_\xi + s\xi + \varepsilon_k r z'$ with

$$z' \in Q_\xi' := R_\xi\big(Q' \times \{0\}\big) \quad \text{and} \quad s \in \bigg(\tau - \frac{\mu_\xi\delta\varepsilon_k}{2}, \tau + \frac{\mu_\xi\delta\varepsilon_k}{2}\bigg)\,,$$

together with Jensen's inequality yield

$$\begin{aligned} f\bigg(\frac{c_1 m_\xi}{\varepsilon_k^{n-1}}\int_{C_{\xi,k}^{r,\delta}(\hat{x}_\xi+\tau\xi)} |\langle e(u_k(z))\xi, \xi\rangle|^p\,\mathrm{d}z\bigg) &\geq \fint_{Q_\xi'} \widetilde{f}\bigg(\int_{\tau-\frac{\mu_\xi\delta\varepsilon_k}{2}}^{\tau+\frac{\mu_\xi\delta\varepsilon_k}{2}} |\langle e(u_k(\hat{x}_\xi + \varepsilon_k r z' + s\xi))\xi, \xi\rangle|^p\,\mathrm{d}s\bigg)\,\mathrm{d}z'\\ &= \fint_{Q_\xi'} \widetilde{f}\bigg(\int_{\tau-\frac{\mu_\xi\delta\varepsilon_k}{2}}^{\tau+\frac{\mu_\xi\delta\varepsilon_k}{2}} \Big|\frac{\partial}{\partial s} w_{\xi,k}(\hat{x}_\xi, z', s)\Big|^p\,\mathrm{d}s\bigg)\,\mathrm{d}z' \end{aligned} \tag{5.44}$$

with $\widetilde{f}(t) := f\big(\frac{c_1 m_\xi}{r^{n-1}} t\big)$ and $w_{\xi,k}(\hat{x}_\xi, z', s) := \langle u_k(\hat{x}_\xi + \varepsilon_k r z' + s\xi), \xi\rangle$. Observe now that applying Corollary 3.9 and Fubini's Theorem to the functions $w_{\xi,k}(\hat{x}_\xi, z', s)$ we have that, for a.e. $(\hat{x}_\xi, z') \in \Pi_\xi \times Q'_\xi$ the functions $s \mapsto w_{\xi,k}(\hat{x}_\xi, z', s)$ converge to the section $u^{\hat{x}_\xi}(s) := \langle u(\hat{x}_\xi + s\xi), \xi\rangle$ in measure on I_ξ. Further, gathering together (5.43) and (5.44), and exchanging the order of integration it holds

$$F_k^\xi(u_k, A) \geq \int_{\Pi^\xi} \fint_{Q'_\xi} \left(\frac{1}{\varepsilon_k} \int_{I_\xi} \widetilde{f}\bigg(\int_{\tau - \frac{\mu_\xi \delta \varepsilon_k}{2}}^{\tau + \frac{\mu_\xi \delta \varepsilon_k}{2}} |\dot{w}_{\xi,k}^{\hat{x}_\xi, z'}(s)|^p \, \mathrm{d}s\bigg) \mathrm{d}\tau\right) \mathrm{d}z' \, \mathrm{d}\mathcal{H}^{n-1}(\hat{x}_\xi)\,, \tag{5.45}$$

where the shortcut $w_{\xi,k}^{\hat{x}_\xi, z'}(s)$ denotes the function $s \mapsto w_{\xi,k}(\hat{x}_\xi, z', s)$ for fixed $(\hat{x}_\xi, z')$. By Theorem 3.3 we get

$$\liminf_{k\to+\infty} \frac{1}{\varepsilon_k} \int_{I_\xi} \widetilde{f}\bigg(\int_{\tau - \frac{\mu_\xi \delta \varepsilon_k}{2}}^{\tau + \frac{\mu_\xi \delta \varepsilon_k}{2}} |\dot{w}_{\xi,k}^{\hat{x}_\xi, z'}(s)|^p \, \mathrm{d}s\bigg) \mathrm{d}\tau \geq \beta\delta\mu_\xi \#(J_{u^{\hat{x}_\xi}} \cap I_\xi)\,. \tag{5.46}$$

Combining (5.45) with (5.46) we finally obtain

$$\liminf_{k\to+\infty} F_k^\xi(u_k, A) \geq \delta\beta \int_{\Pi^\xi} \mu_\xi \#(J_{u^{\hat{x}_\xi}} \cap I_\xi) \, \mathrm{d}\mathcal{H}^{n-1}(\hat{x}_\xi) = \delta\beta \int_{J_u^\xi \cap A} \mu_\xi |\langle \nu_u, \xi\rangle| \, \mathrm{d}\mathcal{H}^{n-1}\,.$$

Eventually by the arbitrariness of δ we deduce (5.41). □

With the help of Propositions 5.1 and 5.3 we can now prove the following lower bound.

Proposition 5.4 (Lower-bound)**.** *Let F_k and F be as in* (2.5) *and* (2.10) *respectively. Let $(u_k) \subset L^0(U;\mathbb{R}^n)$ and u be such that u_k converges to u in measure. Then there exists a subsequence, not relabelled, such that*

$$\liminf_{k\to+\infty} F_k(u_k) \geq F(u)\,.$$

Proof. The result can be obtained exactly as in [23, Proposition 5.4]. □

6. Upper bound

In this section we prove the upper bound.

Proposition 6.1. *Let F_k and F be as in* (2.5) *and* (2.10) *respectively. Then for each $u \in L^0(U;\mathbb{R}^n)$ there is $(u_k) \subset L^0(U;\mathbb{R}^n)$ that converges in measure to u and such that*

$$\limsup_{k\to\infty} F_k(u_k) \leq F(u)\,.$$

Proof. Without loss of generality we assume $F(u) < C$ so that $u \in GSBD^p(U)$. Moreover, since W has p-growth from above, by Theorem 3.4 we can assume that $u \in \mathcal{W}^\infty_{\mathrm{pw}}(U;\mathbb{R}^n)$ and that J_u is an essentially closed connected $(n-1)$-rectifiable set compactly contained in U, since the above subspace is dense in energy. We fix $U' \in \mathcal{A}$ with $U \subset\subset U'$ and consider an extension of u on U', not relabelled, such that $u \in \mathcal{W}^\infty_{\mathrm{pw}}(U';\mathbb{R}^n)$. Then by Theorem 3.1 and Remark 3.2 we can find $(v_k) \subset W^{1,p}(U' \setminus \overline{J_u};\mathbb{R}^n)$ such that v_k converges strongly to u in $L^p(U' \setminus J_u;\mathbb{R}^n)$ and

$$\lim_{k\to\infty} E_k(v_k, U' \setminus \overline{J_u}) = E(u, U' \setminus \overline{J_u}) = \int_{U'} W(x, e(u)) \, \mathrm{d}x\,. \tag{6.1}$$

where the last equality clearly holds as J_u is a null set. For every $h > 0$ we set

$$(J_u)_h := \{x \in U \colon \mathrm{d}_S(x, J_u) < h\}\,,$$

so that for h small enough $(J_u)_h \subset\subset U$. Fix now $0 < \delta_k << \varepsilon_k$ and take $\varphi_k \in C_c^\infty(U')$ a cutoff between $(J_u)_{\delta_k}$ and $(J_u)_{2\delta_k}$. Next define $(u_k) \subset W^{1,p}(U';\mathbb{R}^n)$ as

$$u_k := v_k(1 - \varphi_k) \to u \quad \text{strongly in } L^p(U' \setminus J_u;\mathbb{R}^n)\,,$$

and in particular $u_k \to u$ in measure on U'. Then using that $u_k = v_k$ in $U' \setminus (J_u)_{2\delta_k}$ we have

$$F_k(u_k) \leq F_k(v_k, U \setminus \overline{J_u}) + \beta \frac{\mathcal{L}^n((J_u)_{2\delta_k+\varepsilon_k})}{\varepsilon_k}\,. \tag{6.2}$$

Now invoking [20, Theorem 3.7] we have

$$\lim_{k\to\infty} \frac{\mathcal{L}^n((J_u)_{2\delta_k+\varepsilon_k})}{\varepsilon_k} = \int_{J_u} \phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{6.3}$$

Fix $\hat\alpha > \alpha$. With (2.2), the change of variable $y = x - \varepsilon_k z$, Fubini's theorem, and the change of variable $\hat x = x - \varepsilon_k z$ we have

$$\begin{aligned}
F_k(v_k, U \setminus \overline{J_u}) &\leq \hat\alpha \int_{U\setminus J_u}\int_{\mathbb{R}^n} W_k(y, e(v_k))\rho_k(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \hat\alpha \int_U \int_{\mathbb{R}^n} W_k\big(x-\varepsilon_k z, e(v_k(x-\varepsilon_k\cdot))\big)\rho(z)\,\mathrm{d}z\,\mathrm{d}x \\
&= \hat\alpha \int_{\mathbb{R}^n} \rho(z) \int_U W_k\big(x-\varepsilon_k z, e(v_k(x-\varepsilon_k\cdot))\big)\,\mathrm{d}x\,\mathrm{d}z \\
&\leq \hat\alpha \int_{U'} W_k(x, e(v_k))\,\mathrm{d}x = E_k(v_k, U' \setminus \overline{J_u})\,.
\end{aligned}$$

Hence passing to the limit in k in the above inequality and using (6.1) we get

$$\limsup_{k\to\infty} F_k(v_k, U \setminus \overline{J_u}) \leq \hat\alpha E(u, U' \setminus \overline{J_u}) = \hat\alpha \int_{U'} W(x, e(u))\,\mathrm{d}x\,, \tag{6.4}$$

for all $\hat\alpha > \alpha$. Finally gathering together (6.2)-(6.4) we obtain

$$\limsup_{k\to\infty} F_k(u_k) \leq \hat\alpha \int_{U'} W(x, e(u))\,\mathrm{d}x + \beta \int_{J_u} \phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

Eventually by the arbitrariness of U' and $\hat\alpha$ we conclude. □

Remark 6.2. If a lower order term $\int_U \psi(|u|)\,\mathrm{d}x$, is added to the energy, the density argument above can still be applied if ψ complies with the assumptions of Theorem 3.4. Also observe that within the same assumptions, $\int_U \psi(|u_k|)\,\mathrm{d}x$ is equiintegrable whenever (u_k) is converging in L^p. For u_k and u as in the proof above, this entails the convergence $\int_U \psi(|u_k|)\,\mathrm{d}x \to \int_U \psi(|u|)\,\mathrm{d}x$.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Theorem 3.1 provides a subsequence for which (2.9) holds. Point (i) follows by combining Propositions 5.4 and 6.1, while (ii) is a consequence of Proposition 4.1. □

7. Stochastic homogenisation

In this section we are concerned with the Γ-convergence analysis of the functionals F_k when W_k are random integrands of type

$$W_k(\omega, y, M) = W\Big(\omega, \frac{y}{\delta_k}, M\Big)\,,$$

with ω belonging to the sample space Ω of a complete probability space $(\Omega, \mathcal{T}, P)$ and $\delta_k \searrow 0$. In order to do that we first give some definitions.

Definition 7.1 (Group of P-preserving transformations)**.** A group of P-preserving transformations on $(\Omega, \mathcal{T}, P)$ is a family $(\tau_z)_{z\in\mathbb{Z}^n}$ of mappings $\tau_z \colon \Omega \to \Omega$ satisfying the following:

- (a) (measurability) τ_z is $\mathcal{T}$-measurable for every $z \in \mathbb{Z}^n$;
- (b) (invariance) $P(\tau_z(E)) = P(E)$, for every $E \in \mathcal{T}$ and every $z \in \mathbb{Z}^n$;

(c) (group property) $\tau_0 = \mathrm{id}_\Omega$ and $\tau_{z+z'} = \tau_z \circ \tau_{z'}$ for every $z, z' \in \mathbb{Z}^n$.

If in addition, every $(\tau_z)_{z\in\mathbb{Z}^n}$-invariant set (that is, every $E \in \mathcal{T}$ with $\tau_z(E) = E$ for every $z \in \mathbb{Z}^n$) has probability 0 or 1, then $(\tau_z)_{z\in\mathbb{Z}^n}$ is called ergodic.

Let $a := (a_1, \dots, a_n)$, $b := (b_1, \dots, b_n) \in \mathbb{Z}^n$ with $a_i < b_i$ for all $i \in \{1, \dots, n\}$; we define the n-dimensional interval

$$[a, b) := \{x \in \mathbb{Z}^n \colon a_i \le x_i < b_i \text{ for } i = 1, \dots, n\}$$

and we set

$$\mathcal{I}_n := \{[a, b) \colon a, b \in \mathbb{Z}^n\,,\ a_i < b_i \text{ for } i = 1, \dots, n\}\,.$$

Definition 7.2 (Subadditive process)**.** A discrete subadditive process with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of P-preserving transformations on $(\Omega, \mathcal{T}, P)$ is a function $\mu \colon \Omega \times \mathcal{I}_n \to \mathbb{R}$ satisfying the following:

(a) (measurability) for every $A \in \mathcal{I}_n$ the function $\omega \mapsto \mu(\omega, A)$ is $\mathcal{T}$-measurable;
(b) (covariance) for every $\omega \in \Omega$, $A \in \mathcal{I}_n$, and $z \in \mathbb{Z}^n$ we have $\mu(\omega, A + z) = \mu(\tau_z(\omega), A)$;
(c) (subadditivity) for every $A \in \mathcal{I}_n$ and for every finite family $(A_i)_{i\in I} \subset \mathcal{I}_n$ of pairwise disjoint sets such that $A = \cup_{i\in I} A_i$, we have

$$\mu(\omega, A) \le \sum_{i\in I} \mu(\omega, A_i) \quad \text{for every } \omega \in \Omega\,;$$

(d) (boundedness) there exists $c > 0$ such that $0 \le \mu(\omega, A) \le c\mathcal{L}^n(A)$ for every $\omega \in \Omega$ and $A \in \mathcal{I}_n$.

Definition 7.3 (Stationarity)**.** Let $(\tau_z)_{z\in\mathbb{Z}^n}$ be a group of P-preserving transformations on $(\Omega, \mathcal{T}, P)$. We say that $W \colon \Omega \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to [0, +\infty)$ is *stationary* with respect to $(\tau_z)_{z\in\mathbb{Z}^n}$ if

$$W(\omega, x + z, M) = W(\tau_z(\omega), x, M)$$

for every $\omega \in \Omega$, $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$ and $M \in \mathbb{M}^{n\times n}$. Moreover we say that a stationary random integrand W is *ergodic* if $(\tau_z)_{z\in\mathbb{Z}^n}$ is ergodic.

For our purposes we consider random integrands $W \colon \Omega \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to [0, +\infty)$ satisfying the following assumptions:

(w1) W is $(\mathcal{T} \otimes \mathcal{B}^n \otimes \mathcal{B}^{n\times n})$-measurable;
(w2) $W(\omega, \cdot, \cdot) \in \mathcal{W}$ for every $\omega \in \Omega$;
(w3) the map $M \mapsto W(\omega, x, M)$ is lower semicontinuous for every $\omega \in \Omega$ and every $x \in \mathbb{R}^n$.

Let W be a random integrand satisfying (w1)-(w3) and $\delta_k \searrow 0$. We consider the family of functionals $F_k(\omega) \colon L^0(U; \mathbb{R}^n) \to [0, +\infty]$ defined as

$$F_k(\omega)(u) := \frac{1}{\varepsilon_k} \int_U f\Big(\varepsilon_k W\Big(\omega, \frac{\cdot}{\delta_k}, e(u)\Big) * \rho_k(x)\Big)\,\mathrm{d}x\,, \tag{7.1}$$

if $u \in W^{1,p}(U; \mathbb{R}^n)$, and extended to $+\infty$ otherwise. Let also for $\omega \in \Omega$ and $A \in \mathcal{A}$

$$\mathbf{m}_\omega(u_M, A) := \inf\Big\{\int_A W(\omega, x, e(v))\,\mathrm{d}x \colon v \in W^{1,p}(A; \mathbb{R}^n),\ v = u_M \text{ near } \partial A\Big\}\,. \tag{7.2}$$

We now state the main theorem of this section.

Theorem 7.4 (Stochastic homogenisation)**.** *Let W be a random integrand satisfying (w1)-(w3). Assume moreover W is stationary with respect to a group $(\tau_z)_{z\in\mathbb{Z}^n}$ of P-preserving transformations* *on $(\Omega, \mathcal{T}, P)$. For every $\omega \in \Omega$ let $F_k(\omega)$ be as in (7.1) and $\mathbf{m}_\omega$ be as in (7.2). Then there exists $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ such that for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$, $M \in \mathbb{M}^{n\times n}$ the limit*

$$\lim_{t\to+\infty} \frac{\mathbf{m}_\omega(u_M, Q_t(tx))}{t^n} = \lim_{t\to+\infty} \frac{\mathbf{m}_\omega(u_M, Q_t(0))}{t^n} =: W_{\mathrm{hom}}(\omega, M) \tag{7.3}$$

exists and is independent of x. The function $W_{\mathrm{hom}} \colon \Omega \times \mathbb{M}^{n\times n} \to [0, +\infty)$ is $(\mathcal{T} \otimes \mathcal{B}^{n\times n})$-measurable. Moreover, for every $\omega \in \Omega'$ the functionals $F_k(\omega)$ Γ-converge in measure to the functional $F_{\mathrm{hom}}(\omega) \colon L^0(U; \mathbb{R}^n) \to [0, +\infty]$ given by

$$F_{\mathrm{hom}}(\omega)(u) := \begin{cases} \alpha \displaystyle\int_U W_{\mathrm{hom}}(\omega, e(u))\,\mathrm{d}x + \beta \int_{J_u} \phi_\rho(\nu_u)\,\mathrm{d}\mathcal{H}^{n-1} & \text{if } u \in GSBD^p(U)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

If, in addition, W is ergodic, then W_{hom} is independent of ω and

$$W_{\mathrm{hom}}(M) = \lim_{t\to+\infty} \frac{1}{t^n} \int_\Omega \mathbf{m}_\omega(u_M, Q_t(0))\,\mathrm{d}P(\omega)\,, \tag{7.4}$$

and thus F_{hom} is deterministic.

The proof of Theorem 7.4 is quite standard and can be achieved as in [15] (see also [22]). For this reason here we only detail the main adaptations.

Proposition 7.5. *Let W be a stationary random integrand satisfying $(w1)$-$(w3)$ and let $\mathbf{m}_\omega$ be as in (7.2). Then for every $M \in \mathbb{M}^{n\times n}$ the function $\mu_M \colon \Omega \times \mathcal{I}_n \to \mathbb{R}$ given by $\mu_M(\omega, A) := \mathbf{m}_\omega(u_M, A)$ defines a subadditive process on $(\Omega, \mathcal{T}, P)$. Moreover*

$$0 \le \mu_M(\omega, A) \le c_2(|M + M^T|^p + 1)\mathcal{L}^n(A)\,, \tag{7.5}$$

for P-a.e. $\omega \in \Omega$ and for every $A \in \mathcal{I}_n$.

Proof. Let $M \in \mathbb{M}^{n\times n}$ be fixed. Then we need to show that μ_M satisfies properties (a)–(d). The proof of froperties (b)–(d) and of (7.5) are standard and therefore we omit it here. It then remains to prove (a). Let $A \in \mathcal{I}_n$ be fixed. For $N \in \mathbb{N}$ let

$$W^N(\omega, x, M) := \inf_{\xi \in \mathbb{M}^{n\times n}} \{W(\omega, x, \xi) + N|\xi - M|\}$$

be the Moreau-Yosida regularisation of $M \mapsto W(\omega, x, M)$ which is N-Lipschitz. Let also

$$F^N(\omega) \colon W^{1,p}(A) \to [0, +\infty)\,,$$

be defined as

$$F^N(\omega)(u) := \int_A W^N(\omega, x, e(u))\,\mathrm{d}x\,.$$

Arguing as in the proof of [22, Lemma C.1.] it can be shown that $(\omega, u) \mapsto F^N(\omega)(u)$ is $\mathcal{T} \otimes \mathcal{B}(W^{1,p}(A))$-measurable. By $(w3)$ $W^N \nearrow W$ pointwise, and in particular $F^N(\omega)(u)$ converges to $\int_A W(\omega, x, e(u))\,\mathrm{d}x$ pointwise. As a consequence $(\omega, u) \mapsto \int_A W(\omega, x, e(u))\,\mathrm{d}x$ is also $\mathcal{T} \otimes \mathcal{B}(W^{1,p}(A))$-measurable. Now we note that $F(\omega)(u_M) < +\infty$. This together with $(w3)$ and [22, Lemma C.2.] imply that $\omega \mapsto \mu_M(u_M, A)$ is $\mathcal{T}$-measurable. □

The proof of Theorem 7.4 follows by Proposition 7.5 and the Subadditive Ergodic Theorem [1, Theorem 2.4] arguing as in [15].

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Appendix A. A remark on the non-local approximation of free-discontinuity problems in $GSBV$

This Appendix is devoted to the statement of a Γ-convergence Theorem for non-local functionals depending on the full deformation gradient ∇u. The result we are going to state has actually been proved in [11, Theorem 3.2], under an additional technical assumption, the so called *stable* γ-*convergence* of the functionals

$$\widetilde{E}_k(u,A) := \begin{cases} \displaystyle\int_A W_k(x,\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A;\mathbb{R}^n)\,, \\ +\infty & \text{otherwise}\ . \end{cases} \tag{A.1}$$

This assumption, stated in [11, Definition 7.2] is stronger than simple Γ-convergence, and introduces a limitation to the class of functionals to which the theorem applies, although relevant examples fulfilling this condition can be readily provided (see [11, Examples 7.3-7.5]). Actually, the inspection of the proof of Proposition 5.1, which can be clearly adapted to the $GSBV$ setting, shows that it is not needed. For the reader's convenience we give a precise statement of the result, after recalling the structural assumptions on the non-local approximation energies under which it is formulated.

The functions W_k are assumed to satisfy $(W1)$–$(W2)$, together with

$(W4')$ for every $x \in \mathbb{R}^n$ and every $M \in \mathbb{M}^{n\times n}$

$$c_1|M|^p \le W_k(x,M) \le c_2(|M|^p+1)\,.$$

We will denote with $\widetilde{E}$ the Γ-limit with respect to the convergence in measure of the functionals $\widetilde{E}_k$ in (A.1), given by (see [13, Theorem 20.4])

$$\widetilde{E}(u,A) := \begin{cases} \displaystyle\int_A W(x,\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A;\mathbb{R}^n)\,, \\ +\infty & \text{otherwise}\ . \end{cases}$$

where, for every $x \in u$ and every $M \in \mathbb{M}^{n\times n}$

$$W(x,M) = W'(x,M) = W''(x,M)\,. \tag{A.2}$$

Above, W' and W'' are defined in (2.7), and (2.8), respectively, provided that E_k is replaced by $\widetilde{E}_k$. We then consider the non-local functionals

$$\widetilde{F}_k(u) := \begin{cases} \displaystyle\frac{1}{\varepsilon_k}\int_U f_k\Big(\varepsilon_k W_k(\cdot,\nabla u) * \rho_k(x)\Big)\,\mathrm{d}x & \text{if } u \in W^{1,p}(U;\mathbb{R}^n)\,, \\ +\infty & \text{otherwise}\ . \end{cases} \tag{A.3}$$

where ρ_k are as in Section 2.3, while $f_k\colon [0,+\infty) \to [0,+\infty)$ are concave and satisfy

$$a_1 t \wedge b_1 \le f_k(t) \le b_2 \tag{A.4}$$

for suitable uniform constants a_1, b_1, $b_2 > 0$. We then have the following theorem.

Theorem A.1. *Assume (W1), (W2), and (W4′). Consider a sequence of concave functions* f_k *as in* (A.4) *and convolution kernels* ρ_k *as in Section 2.3. Let the functionals* $\widetilde{F}_k$ *be given by* (A.3). *Finally, assume that*

$$\alpha_k t \wedge b_1 \leq f_k(t) \leq b_2 \quad \textit{with} \quad \lim_{k\to+\infty} \alpha_k - f_k'(0) = 0\,. \tag{A.5}$$

Then $\widetilde{F}_k$ Γ*-converge, with respect to the convergence in measure, to a functional of the form*

$$\alpha \int_U W(x, \nabla u)\,\mathrm{d}x + \int_{J_u} \varphi(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}$$

where W *is given by* (A.2), $\alpha = \liminf f_k'(0)$, *and* φ *is a suitable Carathéodory integrand.*

Proof. By [11, Theorem 3.1] we have that the Γ-limit of $\widetilde{F}_k$ is an integral functional of the form

$$\int_U W_\infty(x, \nabla u)\,\mathrm{d}x + \int_{J_u} \varphi(x, [u], \nu_u)\,\mathrm{d}\mathcal{H}^{n-1}\,.$$

For W' and W'' as in (2.7), and (2.8), respectively, one has only to show that $W_\infty \leq \alpha W''$ and $W_\infty \geq \alpha W'$. The first inequality is actually already proved in [11, Proposition 7.1]. As for the second, notice under assumption (A.5) and taking into account Remark 5.2, it can be recovered by exactly following the argument of Proposition 5.1, provided one is willing to replace each occurrence of $e(u)$ with ∇u. □

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