MINIMIZING MOVEMENTS FOR ANISOTROPIC AND INHOMOGENEOUS MEAN CURVATURE FLOWS

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Abstract. In this paper we address anisotropic and inhomogeneous mean curvature flows with forcing and mobility, and show that the minimizing movements scheme converges to level set/viscosity solutions and to distributional solutions *à la* Luckhaus-Sturzenhecker to such flows, the latter result holding in low dimension and conditionally to the convergence of the energies. By doing so we generalize recent works concerning the evolution by mean curvature by removing the hypothesis of translation invariance, which in the classical theory allows one to simplify many arguments.

1. Introduction

In this paper we deal with the anisotropic, inhomogeneous mean curvature flow with forcing and mobility. By inhomogeneous we mean that the flow is driven by surface tensions depending on the position in addition to the orientation of the surface. The evolution of sets $t \mapsto E_t \subseteq \mathbb{R}^N$ considered is (formally) governed by the law

$$V(x,t) = \psi(x, \nu_{E_t}(x)) \left(-H^{\phi}_{E_t}(x) + f(x,t) \right), \quad x \in \partial E_t, \ t \in (0,T), \tag{1}$$

where $V(x,t)$ is the (outer) normal velocity of the boundary ∂E_t at x, $\phi(x,p)$ is a given anisotropy representing the *surface tension*, H^ϕ is the *anisotropic mean curvature* of ∂E_t associated to ϕ, $\psi(x,p)$ is an anisotropy evaluated at the outer unit normal $\nu_{E_t}(x)$ to ∂E_t which represents a *velocity* modifier (also called the *mobility* term), and f is the *forcing* term. We will be mainly concerned with smooth anisotropies (and the regularity assumptions will be made precise later on): in this case, the curvature H^ϕ is the first variation of the anisotropic and inhomogeneous perimeter associated to the anisotropy ϕ (in short, $\phi-$perimeter) defined as

$$P_\phi(E) := \int_{\partial^* E} \phi(x, \nu_E(x)) \, \mathrm{d}\mathcal{H}^{N-1}(x) \tag{2}$$

for any set E of finite perimeter (where $\partial^* E$ denotes the reduced boundary of E) and, if E is sufficiently smooth, it takes the form

$$H^\phi_E(x) = \operatorname{div}(\nabla_p \phi(x, \nu_E(x))),$$

where with ∇_p we denote the gradient made with respect to the second variable. Note that evolution (1) can be red as the motion of sets in $\mathbb{R}^N$, when the latter is endowed with the Finsler metric induced by the anisotropy (see Remark 4.13). Equation (1) is relevant in Material Sciences, Crystal Growth, Image Segmentation, Geometry Processing and other fields see e.g. [1, 19, 28, 43, 44].

The mathematical literature for inhomogeneous mean curvature flows is not as extensive as in the homogeneous case, mainly due to the difficulties arising from the lack of translational invariance. Indeed, assuming that the evolution is invariant under translations allows to simplify many arguments used in the classical proofs of, for example, comparison results and estimates on the speed of evolution. In the homogeneous case the well-posedness theory is nowadays well established and quite satisfactory, both in the local and nonlocal case, and even in the much more challenging crystalline case (that is, when the anisotropy ϕ is piecewise affine) see [2, 3, 8, 12, 13, 14, 16, 27, 36, 38, 41] to cite a few. Concerning the inhomogeneous mean curvature flow, we cite [30, 31] where the short time existence of smooth solutions on manifolds is shown, and [26, 34], where the viscosity level set approach (introduced for the homogeneous evolution in [16, 23]) is extended, respectively, to the equation (1) and to the Riemannian setting.

In the present work we implement the minimizing movement approach à la Almgren-Taylor-Wang (in short, ATW scheme) [3] to prove existence via approximation of a level set solution to the generalized anisotropic and inhomogeneous motion (1). To carry on this scheme (which has only been sketched in [8], but lacks a formal proof) we gain insights from [14]. We also show that, under the additional hypothesis of convergence of the energies (4) and low dimension (14)(which are nowadays classical for this approach), the same approximate solutions provide in the limit a suitable notion of "BV-solutions", also termed distributional solutions, see [38, 41].

There are many more concepts of weak solution for the mean curvature flow. In particular, we cite the diffuse-interface approximation provided by the Allen-Cahn equation [22, 33, 29, 37] and the threshold dynamic scheme [40, 20] (see also the relative entropy methods of [36]). Other recent results concern the weak-strong uniqueness problem, which consists in proving that weak solutions coincide with the smooth ones as long as the latter exist. After classical works concerning viscosity solutions, a new definition of "BV-solution" (whose existence is proved via the Allen-Cahn approximation scheme) allows the authors in [29, 37] to prove weak-strong uniqueness for isotropic and anisotropic mean curvature flows. This result is based upon the so-called *optimal dissipation inequality* satisfied by their weak solution. In general, it is very difficult to say if the ATW scheme could satisfy such a property, mainly because of the "degeneracy" of the dissipation term in the incremental problem defined via the distance function. Even if all these results concern the translationally invariant case, a study of some of these properties in the inhomogeneous setting seems very interesting and challenging.

Other remarks on possible research directions are the following. To begin with, the new arguments which are used to compensate the lack of translation invariance are based on the locality of the anisotropic curvature H^ϕ associated with a smooth anisotropy ϕ. This implies that the proofs are not straightforwardly adaptable to the so-called "variational curvatures" considered in [14], which are non-local in nature. On the other hand, since the crystalline curvatures are highly nonlocal and degenerate operators (see e.g. [12, 10]), they do not fall in the theory constructed in the present work. In principle, it would be possible to follow the same perturbative study conducted in [12] in order to prove at least existence for an inhomogeneous and crystalline mean curvature flow. However, a satisfactory characterization of the limiting motion equation bearing a comparison principle is lacking so far.

This work can be seen as a first step towards constructing a general theory of motions driven by non-translationally invariant and possibly nonlocal curvatures, in the spirit of [14].

1.1. **Main results.**

Now briefly recall the minimizing movements procedure in order to state the main results of the paper. Given an initial bounded set E_0 and a parameter $h > 0$, we define the *discrete flow* $E_t^{(h)} := T_{h,t-h}E_{t-h}^{(h)}$ for any $t \geq h$ and $E_t^{(h)} = E_0$ for $t \in [0, h)$, where the functional $T_{h,t}$ is defined for $t \geq 0$ as follows: for any bounded set E we set $T_{h,t}E$ (or, sometimes, $T^-_{h,t}E$) as the minimal solution to the problem

$$\min\left\{P_\phi(F) + \int_F \left(\frac{\mathrm{sd}_E^\psi(x)}{h} + \fint_{[\frac{t}{h}]h}^{[\frac{t}{h}]h+h} f(x,s)\,\mathrm{d}s\right)\mathrm{d}\mathcal{H}^{N-1}(x)\ :\ F \text{ is measurable}\right\}, \tag{3}$$

where $\mathrm{sd}_E^\psi(x)$ is the signed geodesic distance between x and E induced by the anisotropy ψ (see (10) for the precise definition) and $[s] = \max\{n \leq s,\ n \in \mathbb{N}\cup\{0\}\}$ denotes the integer part of a non-negative real number $s \in [0, +\infty)$. We will then define $T^+_{h,t}E$ as the maximal solution to the problem above. Any L^1-limit point as $h \to 0$ of the family $\{E_t^{(h)}\}_{t\geq 0}$ will be called a *flat flow*. In the whole paper we will assume that

$$\text{(H0)}\qquad \begin{aligned}&\phi \in \mathscr{E} \text{ (see Definition 2.2) and } \psi \text{ is an anisotropy as in Definition 2.1,}\\ &\forall t \in [0,+\infty) \text{ it holds } f(\cdot,t) \in C^0(\mathbb{R}^N), \|f\|_{L^\infty(\mathbb{R}^N\times[0,+\infty))} < \infty.\end{aligned}$$

With more effort one could weaken the hypothesis and require $\int_0^t f(\cdot,s)\,\mathrm{d}s$ to be continuous (see [15]). For the sake of simplicity we will require the global-in-time boundedness. We prove existence and H ölder regularity for flat flows.

Theorem 1.1 (Existence of flat flows)**.** *Let E_0 be a bounded set of finite perimeter and ϕ, ψ, f satisfy* (H0)*. Fix $T > 0$. For any $h > 0$, let $\{E_t^{(h)}\}_{t\in[0,T)}$ be a discrete flow with initial datum E_0. Then, there exists a family of sets of finite perimeter $\{E_t\}_{t\in[0,T)}$ and a subsequence $h_k \searrow 0$ such that*

$$E_t^{(h)} \to E_t \quad \text{in } L^1,$$

for a.e. $t \in [0,T)$. Such flow satisfies the following regularity property: there exists a constant c, depending on T, such that for every $0 \le s \le t < T$,

$$|E_s \triangle E_t| \le c|t-s|^{1/2},$$
$$P_\phi(E_t) \le P_\phi(E_0) + c.$$

Subsequently, we will show that flat flow s are distributional solutions, as defined in [38]. We will require additional hypothesis: firstly, low dimension (14) (linked to the complete regularity of the $\phi-$perimeter minimizer, compare [38, 41]), moreover

$$\text{(H1)}\qquad \exists\, c_\psi > 0 \text{ s.t. } |\psi(x,v) - \psi(y,v)| \le c_\psi |x-y|, \quad \forall x,y \in \mathbb{R}^N, v \in S^{N-1},$$

$$\text{(H2)}\qquad f \in C^0(\mathbb{R}^N \times [0,\infty)]).$$

Theorem 1.2 (Existence of distributional solutions)**.** *Assume* (H0)*,* (H1)*,* (H2) *and* (14)*. For any $T > 0$, if*

$$\lim_{k\to\infty} \int_0^T P_\phi(E_t^{(h_k)}) = \int_0^T P_\phi(E_t), \tag{4}$$

then $\{E_t\}_{t\in[0,T]}$ is a distributional solution (1) *with initial datum E_0 in the following sense:*

(1) *for a.e. $t \in [0,T)$ he set E_t has weak $\phi-$curvature $H^\phi_{E_t}$ (see* (19) *for details) satisfying*

$$\int_0^T \int_{\partial^* E_t} |H^\phi_{E_t}|^2 < \infty;$$

(2) *there exist $v : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ with $\int_0^T \int_{\partial^* E_t} v^2 \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t < \infty$ and $v(\cdot,t)\big|_{\partial E_t} \in L^2(\partial E_t)$ for a.e. $t \in [0,T)$, such that*

$$-\int_0^T \int_{\partial^* E_t} v\eta \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t = \int_0^T \int_{\partial^* E_t} \left(H^\phi_{E_t} - f\right)\eta \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \tag{5}$$

$$\int_0^T \int_{E_t} \partial_t \eta \,\mathrm{d}x\,\mathrm{d}t + \int_{E_0} \eta(\cdot,0)\,\mathrm{d}x = -\int_0^T \int_{\partial^* E_t} \psi(\cdot,\nu_{E_t}) v\eta \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t, \tag{6}$$

for every $\eta \in C^1_c(\mathbb{R}^N \times [0,T))$.

The definitions $1), 2)$ extend to our case the definition of BV-solutions of [38] and the distributional solutions of [41]. We recall that hypothesis (4) ensures that the evolving sets avoid the so-called "fattening" phenomenon. It is known that this hypothesis is satisfied in the case of evolution of convex or mean-convex sets, see e.g. [10, 17, 24], but in general is not known under which general hypothesis it is valid. We also remark that the proof of the theorem above provides a detailed proof of [10, Theorem 3.2], which had only been sketched. Moreover, we bypass the use of a Bernstein-type result (which is usually employed) by a double blow-up technique.

In the second part of the work we will focus on the level set approach. Briefly, given an initial compact set E_0, we set u_0 such that $\{u_0 \geq 0\} = E_0$ and we look for a solution u in the viscosity sense (in a sense made precise in Definition 4.5) to

$$\begin{cases} \partial_t u + \psi(x, -\nabla u)\left(\mathrm{div}\nabla_p \phi(x, \nabla u(x)) - f(x,t)\right) = 0 \\ u(\cdot, t) = u_0. \end{cases} \tag{7}$$

Classical remarks ensure that any level set $\{u \geq s\}$ is evolving following the mean curvature flow (1). To prove existence for (7) we use an approximating procedure. For $h > 0$ and $t \in (0, +\infty)$ we set iteratively $u_h^{\pm}(\cdot, t) = u_0$ for $t \in [0, h)$ and for $t \geq h$

$$u_h^+(x,t) := \sup\left\{ s \in \mathbb{R} \ : \ x \in T^+_{h,t-h}\{u_h^+(\cdot, t-h) \geq s\} \right\}$$
$$u_h^-(x,t) := \sup\left\{ s \in \mathbb{R} \ : \ x \in T^-_{h,t-h}\{u_h^-(\cdot, t-h) > s\} \right\},$$

where the operator $T^{\pm}_{h,t}$ has been previously introduced. We remark that these are maps piecewise constant in time, since $T^{\pm}_{h,t} = T^{\pm}_{h,[t/h]h}$, which are only upper and lower semicontinuous in space respectively. Then, we will pass to the limit $h \to 0$ on the families $\{u_h^{\pm}\}_h$ to find functions u^+, u^- which are viscosity sub - and supersolution respectively of equation (7). Passing to the limit as $h \to 0$ in our case is not straightforward. The main issue is that we do not have an uniform estimate on the modulus of continuity of the functions u_h (compare [14]) and thus we can not pass to the (locally) uniform limit of the sequence. (More precisely, our best estimate contained in Lemma 4.8 decays too fast as $h \to 0$ to provide any useful information). Nonetheless, motivated by [6, 5, 7] we can define the half-relaxed limits

$$\begin{aligned} u^+(x,t) &:= \sup_{(x_h,t_h)\to(x,t)} \limsup_{h\to 0} u_h^+(x_h, t_h) \\ u^-(x,t) &:= \inf_{(x_h,t_h)\to(x,t)} \liminf_{h\to 0} u_h^-(x_h, t_h), \end{aligned} \tag{8}$$

and prove that the functions defined above are sub - and supersolutions, respectively, to (7). The main difficulty in this regard is that we need to work with just semicontinuous functions in space, as in the translationally invariant setting one can easily prove the uniform equicontinuity of the approximating sequence. We prove the following.

Theorem 1.3. *Assume* (H0), (H1) *and* $f \in C^0(\mathbb{R}^n \times [0, +\infty))$. *The function* u^+ *(respectively* u^-*) defined in* (8) *is a viscosity subsolution (respectively a viscosity supersolution) of* (7).

Thanks to the results of [16] we then prove that, under the additional hypothesis

$$\begin{aligned} &\nabla_x \nabla_p \phi(\cdot, p) \text{ and } \nabla^2_p \phi(\cdot, p) \text{ are Lipschitz, uniformly for } p \in S^{N-1} \\ &\nabla^2_p \phi^2(x, p) \text{ is uniformly elliptic in } p, \text{uniformly in } x \\ &\psi(\cdot, p) \text{ Lipschitz continuous, uniformly in } p \\ &f(\cdot, t) \text{ Lipschitz continuous, uniformly in } t, \end{aligned} \tag{H3}$$

the following uniqueness result holds.

Theorem 1.4. *Assume* (H0) *and* (H3). *If* u_0 *is a continuous function which is spatially constant outside a compact set, equation* (7) *with initial condition* u_0 *admits a unique continuous viscosity solution* u *given by* (8). *In particular,* $u^+ = u^- = u$ *is the unique continuous viscosity solution to* (7) *and* $u_h^{\pm} \to u$ *as* $h \to 0$*, locally uniformly.*

The previous result yields a proof of consistency between the level set approach and the minimizing movements one to study the evolution (1). We recall that it has been established for the classical mean curvature flow in [11], in the anisotropic but homogeneous case in [21] and in a very general nonlocal setting in [14].

2. Preliminaries

We start introducing some notations. We consider $0 \in \mathbb{N}$. We will use both $B_r(x)$ and $B(x,r)$ to denote the Euclidean ball in $\mathbb{R}^N$ centered in x and of radius r; with $B_r^{N-1}(x)$ we denote the Euclidean ball in $\mathbb{R}^{N-1}$ centered in x and of radius r; with S^{N-1} we denote the sphere $\partial B_1(0) \subseteq \mathbb{R}^N$; with Sym_N the symmetric real matrices of size $N \times N$. In the following, we will always speak about measurable sets and refer to a set as the union of all the points of density 1 of that set i.e. $E = E^{(1)}$. If not otherwise stated, we implicitly assume that the function spaces considered are defined on $\mathbb{R}^N$, e.g $L^\infty = L^\infty(\mathbb{R}^N)$; the space C^0 denotes the space of continuous functions. Moreover, we often drop the measure with respect to which we are integrating, if clear from the context.

Definition 2.1. We define *anisotropy* (sometimes defined as an *elliptic integrand*) a function ψ with the following properties: $\psi(x,p) : \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty)$ is a continuous function, which is convex and positively 1-homogeneous in the second variable, such that

$$\frac{1}{c_\psi}|p| \leq \psi(x,p) \leq c_\psi |p|$$

for any point $x \in \mathbb{R}^N$ and vector $p \in \mathbb{R}^N$.

We remark that, as standard, we define a real function f positively 1-homogeneous if for any $\lambda \geq 0$, it holds $f(\lambda x) = \lambda f(x)$. In particular, the anisotropies that we will consider are not symmetric. In the following, we will always denote the gradient of an anisotropy with respect to the first (respectively second) variable as $\nabla_x \psi$ (respectively $\nabla_p \psi$). We then recall the definition of some well-known quantities (see [8]). Define the polar function of an anisotropy ψ, denoted with ψ°, as

$$\psi^\circ(\cdot,\xi) := \sup_{p \in \mathbb{R}^N} \{\xi \cdot p \ : \ \psi(\cdot,p) \leq 1\}. \tag{9}$$

Using the definition it is easy to see that for all $p, \xi \in \mathbb{R}^N$ it holds

$$\psi(\cdot,p)\psi^\circ(\cdot,\xi) \geq p \cdot \xi, \quad -\psi(\cdot,-p)\psi^\circ(\cdot,\xi) \leq p \cdot \xi.$$

Furthermore, one can prove that (see [8]) for $p \neq 0$

$$\psi^\circ(\nabla_p \psi) = 1, \ \psi(\nabla_p \psi^\circ) = 1, \ (\psi^\circ)^\circ = \psi.$$

We define for any $x, y \in \mathbb{R}^N$ the geodesic distance induced by ψ, or $\psi-$distance in short, as

$$\mathrm{dist}^\psi(x,y) := \inf\left\{ \int_0^1 \psi^\circ(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \ : \ \gamma \in W^{1,1}([0,1];\mathbb{R}^N), \gamma(0)=x, \gamma(1)=y \right\}.$$

We remark that this function is not symmetric in general. We define the *signed distance function* from a closed set $E \subseteq \mathbb{R}^N$ as

$$\mathrm{sd}_E^\psi(x) := \inf_{y \in E} \mathrm{dist}^\psi(y,x) - \inf_{y \notin E} \mathrm{dist}^\psi(x,y), \tag{10}$$

so that $\mathrm{sd}_E^\psi \geq 0$ on E^c and $\mathrm{sd}_E^\psi \leq 0$ in E. We remark that the bounds stated in Definition 2.1 imply

$$\frac{1}{c_\psi}\mathrm{dist} \leq \mathrm{dist}^\psi \leq c_\psi \mathrm{dist}, \tag{11}$$

where here and in the following we will denote with $\mathrm{dist}, \mathrm{sd}$ the Euclidean distance and signed distance function respectively. We define the $\psi-$balls as the balls associated to the $\psi-$distance, that is

$$B^\psi_\rho(x) := \{y \in \mathbb{R}^N : \mathrm{dist}^\psi(y,x) < \rho\},$$

which in general are not convex nor symmetric.

Definition 2.2. We say that an anisotropy ϕ is a *regular elliptic integrand*, and write $\phi \in \mathscr{E}$, if there exists two constants $\lambda \geq 1$, $l \geq 0$ such that if $\phi(x,\cdot)\big|_{S^{N-1}} \in C^{2,1}(S^{N-1})$ and for every $x, y, e \in \mathbb{R}^N$, $\nu, \nu' \in S^{N-1}$ one has:

$$\frac{1}{\lambda} \leq \phi(x,\nu) \leq \lambda,$$
$$|\phi(x,\nu) - \phi(y,\nu)| + |\nabla_p \phi(x,\nu) - \nabla_p \phi(y,\nu)| \leq l|x-y|$$
$$|\nabla_p \phi(x,\nu)| + \|\nabla^2_p \phi(x,\nu)\| + \frac{\|\nabla^2_p \phi(x,\nu) - \nabla^2_p \phi(x,\nu')\|}{|\nu - \nu'|} \leq \lambda$$
$$e \cdot \nabla^2_p \phi(x,\nu)[e] \geq \frac{|e - (e\cdot\nu)\nu|^2}{\lambda}.$$

Given any set of finite perimeter E, one can define the $\phi-$perimeter P_ϕ as follows

$$P_\phi(E) := \int_{\partial^* E} \phi(x, \nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x),$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the measure-theoretic outer normal, see [39] for further references on sets of finite perimeter. The $\phi-$perimeter of a set of finite perimeter E in an open set A is defined as

$$P_\phi(E; A) := \int_{\partial^* E \cap A} \phi(x, \nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

We remark that, by definition of regular elliptic integrand, for any set E of finite perimeter it holds

$$\frac{1}{\lambda} P(E) \leq P_\phi(E) \leq \lambda P(E).$$

Some additional remarks on this definition can be found in [18]. We just recall the submodularity property of the $\phi-$perimeter, which can be proved for instance by using the formulae for the reduced boundary and measure-theoretic normal of union and intersection of sets of finite perimeter (see [39]).

Proposition 2.3 (Submodularity property)**.** *For any two sets $E, F \subseteq \mathbb{R}^N$ of finite perimeter, one has*

$$P_\phi(E \cup F) + P_\phi(E \cap F) \leq P_\phi(E) + P_\phi(F). \tag{12}$$

Moreover, by homogeneity, (9) and recalling that for any set E of finite perimeter it holds $D\chi_E = -\nu_E\,\mathrm{d}\mathcal{H}^{N-1}\big|_{\partial^* E}$ we have the following equivalent definitions

$$\begin{aligned} P_\phi(E) &= \sup\left\{ \int_{\mathbb{R}^N} -D\chi_E \cdot \xi \ : \ \xi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N), \phi^\circ(\cdot,\xi) \leq 1 \right\} \\ &= \sup\left\{ \int_E \mathrm{div}\,\xi\,\mathrm{d}\mathcal{H}^{N-1} \ : \ \xi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N), \phi^\circ(\cdot,\xi) \leq 1 \right\}. \end{aligned} \tag{13}$$

Concerning the regularity property of the $\phi-$perimeter minimizers, we refer to [42]. We just recall the following results. Given two anisotropies $\phi, \psi \in \mathscr{E}$, we define the "distance" between them as

$$\begin{aligned} &\mathrm{dist}_{\mathscr{E}}(\phi,\psi) := \sup\{|\phi(x,p) - \psi(x,p)| \\ &+ |\nabla_p \phi(x,p) - \psi(x,p)| + |\nabla^2_p \phi(x,p) - \nabla^2_p \psi(x,p)| : x \in \mathbb{R}^N, p \in S^{N-1}\}, \end{aligned}$$

where $|\cdot|$ denotes the Euclidian norm. Given $\phi \in \mathscr{E}$, we recall that E is a $0-$minimizer for the $\phi-$perimeter if for any $x \in \mathbb{R}^N, r > 0$

$$P_\phi(E; B_r(x)) \leq P_\phi(F; B_r(x))$$

for every $F \subset \mathbb{R}^N$ such that $F \triangle E \subset\subset B_r$. Then, some regularity properties of minimizers of $\phi-$perimeter can be found in the theorems of part $II.7$ and $II.8$ in [42], which are recalled below.

Theorem 2.4. *Assume $\phi \in \mathscr{E}$. Then, for any 0-minimizer E of the $\phi-$perimeter, the reduced boundary $\partial^* E$ of the set E is of class $C^{1,1/2}$ and the singular set $\Sigma := \partial E \setminus \partial^* E$ satisfies*

$$\mathcal{H}^{N-3}(\Sigma) = 0.$$

Theorem 2.5. *Let $m > 0, \alpha \in (0,1)$. Then, there exists $\varepsilon = \varepsilon(m, \alpha) > 0$ with the following property: let $\phi = \phi(p) \in \mathscr{E}$, $\phi \in C^{3,\alpha}(\mathbb{R}^N \setminus \{0\})$ with*

$$\|\phi|_{S^{N-1}}\|_{C^{3,\alpha}} \leq m \text{ and } \mathrm{dist}_{\mathscr{E}}(\phi, |\cdot|) \leq \varepsilon.$$

Then, for any 0-minimizer E of the $\phi-$perimeter, the reduced boundary $\partial^* E$ of the set E is of class $C^{1,1/2}$ and the singular set $\Sigma := \partial E \setminus \partial^* E$ satisfies

$$\mathcal{H}^{N-7}(\Sigma) = 0.$$

We sum up these hypotheses that yield the complete regularity of minimizers of parametric elliptic integrands:

$$\begin{aligned} &\text{either } \phi \in \mathscr{E} \text{ and } N \leq 3, \\ &\text{or } N \leq 7 \text{ and the hypotheses of Theorem 2.5 are satisfied.} \end{aligned} \tag{14}$$

2.1. **The first variation of the $\phi-$perimeter.** In this section we compute the first variation of the ϕ-perimeter and define some additional operators associated to it.

Assume E is of class C^2. Let X be a smooth and compactly supported vector field and assume $\Psi(x,t) =: \Psi_t(x)$ is the associated flow. To simplify the notation, we write

$$\nu(x,t) = \nabla_x \mathrm{sd}_{\Psi(E,t)}(x).$$

By classical formulae (see e.g. [9]) we can compute the following. For the sake of brevity, we avoid writing the evaluation $\phi = \phi(x, \nu_E(x))$, if not otherwise specified, and assume that all the integrals are made with respect to the Hausdorff $(N-1)$-dimensional measure $\mathcal{H}^{N-1}$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_\phi(E_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{\partial E} \phi(\Psi_t(x), \nu(\Psi_t(x), t)) J\Psi_t \\
&= \int_{\partial E} \nabla_x \phi \cdot X + \nabla_p \phi \cdot (-\nabla_\tau (X \cdot \nu) + D\nu[X]) + \phi \,\mathrm{div}_\tau X \\
&= \int_{\partial E} \nabla_x \phi \cdot X + \nabla_p \phi \cdot (-\nabla_\tau (X \cdot \nu) + D\nu[X]) + \mathrm{div}_\tau(\phi X) - \nabla \phi \cdot X + (\nabla \phi \cdot \nu)(X \cdot \nu) \\
&= \int_{\partial E} \nabla_x \phi \cdot X + \nabla_p \phi \cdot (-\nabla_\tau (X \cdot \nu) + D\nu[X]) - \nabla_x \phi \cdot X - D\nu[\nabla_p \phi] \cdot X \\
&\quad + \mathrm{div}_\tau(\phi X) + (\nabla \phi \cdot \nu)(X \cdot \nu) \\
&= \int_{\partial E} -\nabla_p \phi \cdot \nabla_\tau (X \cdot \nu) + (\nabla_x \phi \cdot \nu)(X \cdot \nu) + (D\nu[\nabla_p \phi] \cdot \nu)\,(X \cdot \nu) + \mathrm{div}_\tau(\phi X) \\
&= \int_{\partial E} \mathrm{div}_\tau \left(\nabla_p \phi (X \cdot \nu)\right) - \nabla_p \phi \cdot \nabla_\tau (X \cdot \nu) + (X \cdot \nu)(\nabla_x \phi \cdot \nu) \\
&= \int_{\partial E} (\mathrm{div}_\tau \nabla_p \phi)(X \cdot \nu) + \nabla_p \phi \cdot \nabla_\tau (X \cdot \nu) - \nabla_p \phi \cdot \nabla_\tau (X \cdot \nu) + (\nabla_x \phi \cdot \nu)(X \cdot \nu) \\
&= \int_{\partial E} (X \cdot \nu)\,(\mathrm{div}_\tau \nabla_p \phi + \nabla_x \phi \cdot \nu) = \int_{\partial E} (X \cdot \nu)\,\mathrm{div} \nabla_p \phi
\end{aligned} \tag{15}$$

where the last equality follows from the definition of div_τ and the fact that ϕ is $1-$homogeneous with respect to the p variable, since

$$\begin{aligned}\mathrm{div}\nabla_p\phi &= \mathrm{div}_\tau\nabla_p\phi + \sum_i \nu_i\left(\partial_{x_i}\nabla_p\phi\right)[\nu]\\ &= \mathrm{div}_\tau\nabla_p\phi + \sum_i \nu_i\nabla_p(\partial_{x_i}\phi)\cdot\nu + \nu\cdot\left(\nabla_p^2\phi D\nu\right)[\nu]\\ &= \mathrm{div}_\tau\nabla_p\phi + \nabla_x\phi\cdot\nu.\end{aligned}$$

Therefore, we define the first variation of a C^2-regular set E, induced by the vector field X, as

$$\delta P_\phi(E)[X\cdot\nu] := \int_{\partial E}(X(x)\cdot\nu(x))\,\mathrm{div}\nabla_p\phi(x,\nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) \tag{16}$$

and the $\phi-$curvature of the set E as

$$H_E^\phi(x) := \mathrm{div}\nabla_p\phi(x,\nu(x)). \tag{17}$$

If we now consider equation (15), we develop the tangential gradient to find

$$\nabla_p\phi\cdot(-\nabla_\tau(X\cdot\nu)+D\nu[X]) = \nabla_p\phi\cdot(-\nabla_\tau X[\nu]-D\nu[X]+D\nu[X]) = 0.$$

This shows that for any set E of class C^2 it holds

$$\delta P_\phi(E)[X\cdot\nu] := \int_{\partial E}\left(\nabla_x\phi\cdot X + \phi\,\mathrm{div}_\tau X\right)\,\mathrm{d}\mathcal{H}^{N-1},$$

where we dropped the evaluation of ϕ at $(x,\nu_E(x))$. We remark that the expression on the right hand side makes sense even if the set E is just of finite perimeter. Defining the $\phi-$divergence operator div_ϕ as

$$\mathrm{div}_\phi X := \nabla_x\phi\cdot X + \phi\,\mathrm{div}_\tau X, \tag{18}$$

we are led to define the distributional $\phi-$curvature of a set E of finite perimeter as an operator $H_E^\phi\in L^1(\partial E)$ (if it exists) such that the following representation formula holds

$$\int_{\partial E}\mathrm{div}_\phi X\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E}H_E^\phi\,\nu_E\cdot X\,\mathrm{d}\mathcal{H}^{N-1}, \qquad \forall X\in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N). \tag{19}$$

The previous computations allow to say that the distributional $\phi-$curvature can be expressed as (17) if the set is of class C^2. Finally, since ϕ is a regular elliptic integrand, one can prove the following monotonicity result.

Lemma 2.6. *Let E,F be two C^2 sets of finite $\phi-$perimeter with $E\subseteq F$, and assume that $x\in\partial F\cap\partial E$: then $H_F^\phi(x)\le H_E^\phi(x)$.*

Proof. Since the anisotropy is smooth, we can expand the curvature formula (17) as

$$H^\phi = \mathrm{tr}\left(\nabla_x\nabla_p\phi(x,\nu)+\nabla_p^2\phi(x,\nu)D\nu\right) \tag{20}$$

and compare H_E^ϕ with H_F^ϕ. We consider separately the two terms appearing in (20). The first one depends on ν just by the value it has at the point x. Therefore, since $\nu_E(x)=\nu_F(x)$ we have the equality. The second one falls in the classical framework of smooth anisotropies that do not depend on the space variable. Since $D\nu_F\le D\nu_E$ (as matrices) one concludes the proof. □

3. The minimizing movements approach

In this section we follow the work of [41] (see also [3, 38]) to prove the existence for the mean curvature flow via the *minimizing movements* approach. We recall that in the whole paper we will assume the hypothesis (H0).

3.1. **The discrete scheme.** In this subsection we will define the discrete scheme approximating the weak solution of the mean curvature flow, and we shall study some of its properties.

We define the following iterative scheme. Given $h > 0, f \in L^\infty(\mathbb{R}^N \times [0,\infty))$ and $t \geq h$, and given a bounded set of finite perimeter F, we minimize the energy functional

$$\mathscr{F}^F_{h,t}(E) = P_\phi(E) + \frac{1}{h}\int_E \mathrm{sd}^\psi_F(x)\,\mathrm{d}x - \int_E F_h(x,t)\,\mathrm{d}x \tag{21}$$

in the class of all measurable sets $E \subseteq \mathbb{R}^N$, and where we have set

$$F_h(x,t) := \fint_t^{t+h} f(x,s)\,\mathrm{d}s.$$

Equivalently, we could define the energy functional as

$$\mathscr{F}^F_{h,t}(E) = P_\phi(E) + \frac{1}{h}\int_{E\triangle F} |\mathrm{sd}^\psi_F| - \int_E F_h(x,t)\,\mathrm{d}x,$$

which agrees with (21) up to a constant. Then, we denote

$$T_{h,t}F = E \in \operatorname{argmin} \mathscr{F}^F_{h,t}.$$

We will refer to this minimizing procedure as the *incremental problem*. It is well-known (compare (16) and [39, Proposition 17.8]) that a minimimum of (21) of class C^2 satisfies the Euler-Lagrange equation

$$\int_{\partial E} H^\phi_E X\cdot\nu_E\,\mathrm{d}\mathcal{H}^{N-1} = -\int_{\partial E}\left(\frac{1}{h}\mathrm{sd}^\psi_F(x) - F_h(x,t)\right)X(x)\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \tag{22}$$

for all $X \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$. We can then define the *discrete flow*, which can be seen as a discrete-in-time approximation of the mean curvature flow starting from the initial set E_0. We define iteratively the *discrete flow* by setting $E^{(h)}_t = E_0$ for $t \in [0,h)$ and

$$E^{(h)}_t = T_{h,t-h}E^{(h)}_{t-h} = T_{h,([\frac{t}{h}]-1)h}E^{(h)}_{t-h}, \qquad t \in [h,+\infty), \tag{23}$$

where $[\cdot]$ denotes the integer part of a real number. This section is devoted to recall and prove some estimates on the discrete flow. The first one is a well-known existence result.

Lemma. *For any measurable function $g : \mathbb{R}^N \to \mathbb{R}$ such that $\min\{g,0\} \in L^1_{loc}$, the problem*

$$\min\left\{P(E) + \int_E g \;:\; E \text{ is of finite perimeter}\right\}$$

admits a solution.

Consider now F as a bounded set of finite perimeter. Then, the function $g = \mathrm{sd}^\psi_F/h - F_h$ is coercive, thus $\min\{g,0\} \in L^1$. Therefore, by the previous result and by classical arguments see [14, Proposition 6.1] for a proof, one can prove the following result.

Lemma 3.1. *For any given set F of finite perimeter, the problem* (21) *admits a solution E, which satisfies the discrete dissipation inequality*

$$P_\phi(E) + \frac{1}{h}\int_{E\triangle F}|\mathrm{sd}^\psi_F| \leq P_\phi(F) + \int_{E\setminus F} F_h(x,t)\,\mathrm{d}x - \int_{F\setminus E} F_h(x,t)\,\mathrm{d}x.$$

Moreover, the problem (21) *admits a minimal and a maximal solution.*

We define $T^+_{h,t}F$ (respectively $T^-_{h,t}F$) as the maximal (respectively minimal) solution to (21) having as initial datum F. In the following, whenever no confusion is possible, we shall write $T_{h,t}$ instead of $T^-_{h,t}$.

A comparison result holds. We will consider just bounded sets as datum for the problem (21), but the same result holds in general for unbounded sets (see also Section 4.1 for the case of unbounded sets with bounded boundary). The proof of this result is classical (see e.g. [14]) and it is based on the submodularity of the perimeter (12). We will omit it.

Lemma 3.2 (Weak comparison principle)**.** *Assume that F_1, F_2 are bounded sets with $F_1 \subset\subset F_2$ and consider $g_1, g_2 \in L^\infty$ with $g_1 \geq g_2$. Then, for any two solutions E_i, $i = 1, 2$ of the problems*

$$\min\left\{P_\phi(E) + \int_E \frac{\mathrm{sd}^\psi_{F_i}}{h} + g_i : E \text{ is of finite perimeter}\right\},$$

we have $E_1 \subseteq E_2$. If, instead, $F_1 \subseteq F_2$, then we have that the minimal (respectively maximal) solution to (21) *for $i = 1$ is contained in the minimal (respectively maximal) solution to* (21) *for $i = 2$.*

We now prove the volume-density estimates for minimizers of problem (21). This result is based on the minimality properties of almost-minimizers for perimeters induced by regular elliptic integrands (see [18, Remark 1.9] for further results). These estimates have the disadvantage that the smallness condition on the radius depends on the parameter h. Subsequently, we will recall a finer result in the spirit of [38], where we can drop this dependence by making some restrictions on the balls considered.

Lemma 3.3. *Let $g \in L^\infty$ and assume E minimizes the functional*

$$\mathscr{F}(F) = P_\phi(F) + \int_F g$$

among all measurable subsets of $\mathbb{R}^N$. Then the density estimate

$$\begin{aligned} \sigma\rho^N \leq |B_\rho(x) \cap E| \leq (1-\sigma)\rho^N \\ \sigma\rho^{N-1} \leq P_\phi(E; B_\rho(x)) \leq (1-\sigma)\rho^{N-1} \end{aligned} \tag{24}$$

holds for all $x \in \partial^* E$, $0 < \rho < (2\lambda\|g\|_\infty)^{-1} := \rho_0$, for a suitable $\sigma = \sigma(N, c_\psi, \lambda)$.

Proof. By minimality,

$$P_\phi(E) \leq P_\phi(F) + \|g\|_\infty |E \triangle F| \qquad \forall F \subseteq \mathbb{R}^N,$$

thus [18, Lemma 2.8] implies the thesis. □

Remark 3.4. We remark that the previous result allows us to choose the minimal solution to (21) to be an open set, and the maximal one to be a closed set. This follows from the fact that the density estimates imply that the boundary of any minimizer has zero measure.

We now recall [12, Lemma 3.7], which is an anisotropic version of [38, Remark 1.4]. It provides volume-density estimates for minimizers of (21) starting from E, uniform in ψ and h, holding in the exterior of E. We remark that, even if in the reference the anisotropy ϕ considered did not depend on x, all the arguments hold with minor modifications also in our case. We recall the proof of this result, as similar techniques will be used later on.

Lemma 3.5. *Let E be a bounded, closed set, $h > 0$, and $g \in L^\infty(\mathbb{R}^N)$. Let E' be a minimizer of*

$$P_\phi(F) + \int_F \frac{\mathrm{sd}^\psi_E}{h} + g.$$

Then, there exists $\sigma > 0$, depending on λ, and $r_0 \in (0, 1)$, depending only on $N, \lambda, G := \|g\|_{L^\infty(F)}$, with the following property: if $\bar{x}$ is such that $|E' \cap B_s(\bar{x})| > 0$ for all $s > 0$ and $B_r(\bar{x}) \cap E = \emptyset$ with $r \leq r_0$, then

$$|E' \cap B_r(\bar{x})| \geq \sigma r^N. \tag{25}$$

Analogously, if $\bar{x}$ is such that $|B_s(\bar{x}) \setminus E'| > 0$ for all $s > 0$ and $B_r(\bar{x}) \subseteq E$ with $r \leq r_0$, then

$$|B_r(\bar{x}) \setminus E'| \geq \sigma r^N.$$

Proof. For all $s \in (0, r)$, set $E'(s) := E' \setminus B_s(\bar{x})$. Note that, for a.e. s we have

$$P_\phi(E'(s)) = P_\phi(E') - P_\phi(E' \cap B_s(\bar{x})) + \int_{E' \cap \partial B_s(\bar{x})} (\phi(x, \nu(x)) + \phi(x, -\nu(x)))\, \mathrm{d}\mathcal{H}^{N-1}(x),$$

where ν denotes the outer normal vector of the set $E' \cap \partial B_s(\bar{x})$. Since $E' \cap B_s(\bar{x}) \subset E^c$ and $\mathrm{sd}_E^\psi \geq 0$ in E^c, one has $\int_{E' \cap B_s(\bar{x})} \mathrm{sd}_E^\psi \geq 0$, and therefore the minimality of E' implies

$$P_\phi(E' \cap B_s(\bar{x})) + \int_{E' \cap B_s(\bar{x})} g \leq \int_{E' \cap \partial B_s(\bar{x})} (\phi(x, \nu(x)) + \phi(x, -\nu(x)))\, \mathrm{d}\mathcal{H}^{N-1}(x).$$

By the bound on the $\phi-$perimeter and using the classical isoperimetric inequality (whose constant is denoted C_N) we obtain

$$\begin{aligned} 2\lambda \mathcal{H}^{N-1}(E' \cap \partial B_s(\bar{x})) &\geq \frac{1}{\lambda} P(E' \cap B_s(\bar{x})) + \int_{E' \cap B_s(\bar{x})} g \\ &\geq \frac{1}{\lambda} C_N |E' \cap B_s(\bar{x})|^{\frac{N-1}{N}} - \|g\|_\infty |E' \cap B_s(\bar{x})| \geq \frac{C_N}{2\lambda} |E' \cap B_s(\bar{x})|^{\frac{N-1}{N}}, \end{aligned}$$

provided $|E' \cap B_s(\bar{x})|^{1/N} \leq C_N/(2\lambda\|g\|_\infty)$, which is true if r_0 is small enough. Since the *rhs* is positive for every s, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}s} |E' \cap B_s(\bar{x})|^{\frac{1}{N}} \geq \frac{C_N}{4\lambda^2 N} \quad \text{for a.e. } s \in (0, r). \tag{26}$$

The thesis follows by integrating the above differential inequality. The other case is analogous. □

Remark 3.6. Requiring that the anisotropy ψ is bounded uniformly from above and below ensures that the results of the previous Lemmas 3.3 and 3.5 can be read in terms of the $\psi-$balls. For example, for any $r \geq 0$ and $x \in \mathbb{R}^N$, equation (25) could be read as $|E' \cap B_r^\psi(\bar{x})| \geq \sigma c_\psi^{-N} r^N$, provided $\bar{x}$ is such that $|E' \cap B_s^\psi(\bar{x})| > 0$ for all $s > 0$ and $B_r^\psi(\bar{x}) \cap E = \emptyset$, and holds for all $r \leq r_0/c_\psi$. Here, σ is as in Lemma 3.5 and depends only on λ. Analogous statements holds for Lemma 3.9.

We now provide some estimates on the evolution of balls under the discrete flow. We start by a simple remark concerning the boundedness of the evolving sets.

Remark 3.7. A simple estimate on the energies implies that the minimizers of (21) are bounded whenever F is bounded. Indeed, assume $F \subseteq B_R$ and consider $B_\rho(x) \cap (E \setminus B_R) \neq \emptyset$: testing the minimality of E against F we easily deduce

$$\frac{R}{2h} |B_\rho(x) \cap E| \leq \int_{E \cap B_\rho(x)} \frac{\mathrm{sd}_F^\psi}{h} \leq P_\phi(F) + \|F_h(\cdot, t)\|_\infty |E \triangle F| \leq P_\phi(F) + \|f\|_\infty (|F| + |E|).$$

Employing the density estimates of Lemma 3.5 and sending $R \to \infty$, we get a contradiction, as the isoperimetric inequality implies that $|E|$ is bounded since $\mathscr{F}_{h,t}^F(F) < \infty$.

We now want to prove finer estimates on the speed of evolution of balls. These estimates are classically a crucial step in order to prove existence of the flow. In the case under study, the main difficulties come from the inhomogeneity of the functionals considered, as in the homogeneous case convexity arguments easily yield the boundedness result, for example. We will use a "variational" approach in the spirit of [14] (but see also [41, Lemma 3.8] for a different proof relying more on the smoothness of the evolving set).

Lemma 3.8. *For every $R_0 > 0$ there exist $h_0(R_0) > 0$ and $C(R_0, \phi, \psi, f) > 0$ with the following property: For all $R \geq R_0$, $h \in (0, h_0)$, $t > 0$ and $x \in \mathbb{R}^N$ one has*

$$T_{h,t}(B_R(x)) \supset B_{R-Ch}(x). \tag{27}$$

Proof. We divide the proof into three steps. In the following, the constants σ, r_0 are those of Lemma 3.5. We will assume $x = 0$ for simplicity. We fix $R \geq R_0$ and denote $E := T_{h,t}B_R$.
Step 1. We prove that, given $a \in (0, \sigma), \varepsilon \in (0, 1)$, we can ensure $|B_{R(1-\varepsilon)} \setminus E| < a\, R^N(1-\varepsilon)^N$ for h small enough. Indeed, assume by contradiction $|B_{R(1-\varepsilon)} \setminus E| \geq a\, R^N(1-\varepsilon)^N$. Testing the minimality of E against B_R, we obtain

$$\int_{(B_{R(1-\varepsilon)}\setminus E)\cup(E\setminus B_R)} \frac{|\mathrm{sd}^{\psi}_{B_R}|}{h} \leq \frac{1}{h}\int_{B_R\triangle E} |\mathrm{sd}^{\psi}_{B_R}| \leq P_\phi(B_R) - \int_{B_R\setminus E} F_h + \int_{E\setminus B_R} F_h,$$

and estimating $|\mathrm{sd}^{\psi}_{B_R}| \geq R\varepsilon/c_\psi$ on $B_{R(1-\varepsilon)} \setminus E$, we get

$$\frac{R\varepsilon}{hc_\psi}|B_{R(1-\varepsilon)} \setminus E| \leq P_\phi(B_R) + \|f\|_\infty \left(\omega_N R^N + |B_{R(1+\varepsilon)} \setminus B_R|\right) + \int_{E\setminus B_{R(1+\varepsilon)}} \left(F_h - \frac{|\mathrm{sd}^{\psi}_{B_R}|}{h}\right).$$

Taking $h \leq \varepsilon/(c_\psi\|f\|_\infty)$, the last term on the *rhs* is negative, thus

$$\frac{R\varepsilon}{hc_\psi}|B_{R(1-\varepsilon)} \setminus E| \leq P_\phi(B_R) + \|f\|_\infty R^N(\omega_N + 2^{N+1}\varepsilon).$$

We employ the hypothesis to obtain

$$\frac{a}{hc_\psi}\,\varepsilon(1-\varepsilon)^N R^{N+1} \leq c_\psi N\omega_N R^{N-1} + cR^N,$$

a contradiction for $h \leq ca\,\varepsilon\,(1-\varepsilon)^N \min\{1, R^2\}$, where c is a constant depending on $N, \phi, \psi, \|f\|_\infty$.
Step 2. Using Step 1, we prove that $B_{R/2} \subset E$ for h small. Assume that $R \leq r_0$: by following the second part of the proof of Lemma 3.5 we obtain equation (26), which reads

$$\frac{\mathrm{d}}{\mathrm{d}s}|B_s \setminus E|^{1/N} \geq \frac{C_N}{4\lambda^2 N} = \sigma^{1/N} \quad \text{for a.e } s \in (0, R).$$

Applying the previous step with $\varepsilon = 1/4, a = \sigma/3^N$, it holds $|B_{3R/4} \setminus E| \leq \sigma R^N/4^N$ for all $h \leq c(N, \phi, \psi, f)R$. Therefore, one deduces the existence of a positive extinction radius

$$R^* = \frac{3R}{4} - \frac{|B_{3R/4} \setminus E|^{1/N}}{\sigma^{1/N}} \geq \frac{R}{2} \tag{28}$$

such that $|B_{R^*} \setminus E| = 0$, which proves the claim. Clearly, taking $h \leq cR_0$ the smallness assumption on h is uniform for $R \geq R_0$.

If $R \geq r_0$ one simply uses a covering argument. For any $x \in B_{R-r_0}$, applying the previous result to the ball $B_{r_0}(x)$ and using the comparison principle of Lemma 3.2, we conclude that $\forall h \leq c\,r_0$ it holds

$$\bigcup_{x\in B_{R-r_0}} B_{r_0/2}(x) \subset\subset E.$$

Step 3. We conclude the proof. By the previous two steps and Remark 3.7, taking h small enough, we see that

$$\rho := \sup\{r > 0 \ : \ |B_r \setminus E| = 0\} \in (R/2, +\infty).$$

We can assume $\rho \leq R$, otherwise the result of the lemma is trivial. Consider the vector field $\nabla_p\phi\left(x, \frac{x}{|x|}\right) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. Then, recalling (13), we get $P_\phi(G) \geq -\int_{\mathbb{R}^N} D\chi_G \cdot \nabla_p\phi(x, x/|x|)$ for all G set of finite perimeter and

$$P_\phi((1+\varepsilon)B_\rho) = \int_{\mathbb{R}^N} D\chi_{(1+\varepsilon)B_\rho} \cdot \left(-\nabla_p\phi\left(x, \frac{x}{|x|}\right)\right).$$

Setting $W_\varepsilon = (1+\varepsilon)B_\rho \setminus E$, by submodularity on $(1+\varepsilon)B_\rho, E$ and exploiting the minimality of E, we obtain

$$\int_{\mathbb{R}^N} \nabla_p\phi\left(x, \frac{x}{|x|}\right) \cdot D\chi_{W_\varepsilon} = \int_{\mathbb{R}^N} \nabla_p\phi\left(x, \frac{x}{|x|}\right) \cdot \left(D\chi_{(1+\varepsilon)B_\rho} - D\chi_{(1+\varepsilon)B_\rho\cap E}\right)$$

$$
\begin{aligned}
&\leq P_\phi((1+\varepsilon)B_\rho\cap E) - P_\phi((1+\varepsilon)B_\rho)\\
&\leq P_\phi(E) - P_\phi((1+\varepsilon)B_\rho\cup E)\\
&\leq \frac{1}{h}\int_{W^\varepsilon} \mathrm{sd}^\psi_{B_R} - \int_{W_\varepsilon} F_h(x,t)\,\mathrm{d}x.
\end{aligned}
$$

We conclude, using the divergence theorem ,

$$
\int_{W^\varepsilon} -\mathrm{div}\nabla_p\phi\left(x,\frac{x}{|x|}\right) \leq \frac{1}{h}\int_{W^\varepsilon}\mathrm{sd}^\psi_{B_R} + \|f\|_\infty |W_\varepsilon|.
$$

Dividing by $|W^\varepsilon|$ and sending $\varepsilon\to 0$ we obtain

$$
⨍_{\partial B_\rho\cap E} -\mathrm{div}\nabla_p\phi\left(x,\frac{x}{|x|}\right)\,\mathrm{d}\mathcal{H}^{N-1} \leq \frac{1}{c_\psi}\frac{\rho-R}{h} + \|f\|_\infty.
$$

Exploiting the regularity assumptions on ϕ, we remark that

$$
|\mathrm{div}\nabla_p\phi| = \left|\mathrm{tr}\left(\nabla_x\nabla_p\phi + \nabla^2_p\phi\nabla(x/|x|)\right)\right| \leq C\left(1+\frac{1}{|x|}\right).
$$

Thus, we obtain

$$
-C\left(1+\frac{1}{\rho}\right)\leq \frac{\rho-R}{h},
$$

which implies that $\rho\in(0,\rho_1)\cup(\rho_2,R)$ for $\rho_{1,2} = \left(R-Ch\mp\sqrt{(R-Ch)^2-4Ch}\right)/2$, as long as $h\leq R_0^2/(4C)$. Since the choice $\rho\leq\rho_1<R/2$ is not admissible, we conclude the proof by estimating

$$
\rho_2 = R-Ch+\frac{R-Ch}{2}\left(\sqrt{1-\frac{4Ch}{(R-Ch)^2}}-1\right)\geq R-Ch-\frac{Ch}{R-Ch},
$$

from which the thesis follows. □

The proof of the previous result can be employed to prove an estimate from above of the evolution speed of the flow, as the following result shows. Since the proof follows the same lines and is easier in this case, we only sketch it.

Lemma 3.9. *Fix $T>0$ and $R_0>0$. Then, there exist positive constants $C=C(\phi,\psi,f,R_0)$ and $h_0=h_0(R_0)$ such that, for every $R\geq R_0$ and $h\leq h_0$, if $E_0\subseteq B_R$, then $E^{(h)}_t\subseteq B_{R+CT}$ for all $t\in(0,T)$.*

Proof. Choose h small as in the previous result and set

$$
\rho = \inf\{r>0\ :\ |E\setminus B_r| = 0\}\in(R/2,+\infty).
$$

We can assume $\rho\geq R$, otherwise the result is trivial. Defining $W_\varepsilon = E\setminus(1-\varepsilon)B_\rho$ and reasoning as before we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N}\nabla_p\phi\left(x,\frac{x}{|x|}\right)\cdot D\chi_{W^\varepsilon} &= \int_{\mathbb{R}^N}\nabla_p\phi\left(x,\frac{x}{|x|}\right)\cdot\left(D\chi_{(1-\varepsilon)B_\rho\cup E} - D\chi_{(1-\varepsilon)B_\rho}\right)\\
&\geq -P_\phi((1-\varepsilon)B_\rho\cup E) + P_\phi((1-\varepsilon)B_\rho)\\
&\geq -P_\phi(E) + P_\phi((1-\varepsilon)B_\rho\cap E)\\
&\geq \frac{1}{h}\int_{W^\varepsilon}\mathrm{sd}^\psi_{B_R} - \int_{W_\varepsilon}F_h(x,t)\,\mathrm{d}x.
\end{aligned}
$$

As in the previous proof, we arrive at

$$
\frac{\rho-R}{h}\leq C\left(1+\frac{1}{\rho}\right),
$$

which implies that $\rho \le \rho_2 = \left(R + Ch + \sqrt{(R+Ch)^2 + 4Ch}\right)/2 \le R + Ch$, up to changing C. $\square$

3.2. **Existence of flat flows.** In the following, we will prove that the discrete flow (defined in (23)) defines a discrete-in-time approximation of a weak solution to the mean curvature flow, which is usually known as a "flat" flow (because the approximating surfaces $\partial^* E_t^{(h)}$ converge in the "flat" distance of Whitney to the limit $\partial^* E_t$, see [3]).

We start by proving uniform bounds on the distance between two consecutive sets of the discrete flow and on the symmetric difference between them. We introduce the time-discrete normal velocity: for all $t \ge 0$ and $x \in \mathbb{R}^N$, we set

$$v_h(x,t) := \begin{cases} \frac{1}{h}\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}(x) & \text{for } t \in [h, +\infty) \\ 0 & \text{for } t \in [0,h). \end{cases}$$

The following result provides a bound on the $L^\infty-$norm of the discrete velocity. Since the proof is essentially the same of [38, Lemma 2.1], we will omit it. The only difference is that we use the upper and lower bounds of (11) to work with Euclidean balls.

Lemma 3.10. *There exists a positive constant c_∞ depending only on N, ψ with the following property. Let E_0 be a bounded set of finite perimeter and let $\{E_t^{(h)}\}_{t \in (0,T)}$ be a discrete flow starting from E_0. Then,*

$$\sup_{E_t^{(h)} \triangle E_{t-h}^{(h)}} |v_h(\cdot, t)| \le c_\infty h^{-1/2}$$

for all h sufficiently small.

The following result can be found in [41, Proposition 3.4] (see also [25, Lemma 2.2]): it provides an estimate on the volume of the symmetric difference of two consecutive sets of the discrete flow. The proof is analogous to the one in the reference.

Lemma 3.11. *There exists a constant C such that for every $t \ge h$ the discrete flow $E_t^{(h)}$ satisfies for all h sufficiently small*

$$|E_{t+h}^{(h)} \triangle E_t^{(h)}| \le C \left(l P_\phi(E_t^{(h)}) + \frac{1}{l} \int_{E_t^{(h)} \triangle E_{t+h}^{(h)}} |\mathrm{sd}^{\psi}_{E_t^{(h)}}| \right) \quad \forall l \le c\sqrt{h}, \tag{29}$$

where c is a positive constant depending on N, ψ.

We are now able to prove an uniform bound on the perimeter of the evolving sets. The proof follows [25, Proposition 2.3].

Lemma 3.12. *For any initial bounded set E_0 of finite $\phi-$perimeter and h small enough, the discrete flow $\{E_t^{(h)}\}$ satisfies*

$$P_\phi(E_t^{(h)}) \le C_T \quad \forall t \in (0,T),$$

for a suitable constant $C_T = C_T(T, E_0, f, \phi, \psi)$.

Proof. By testing the minimality of $E_t^{(h)}$ against $E_{t-h}^{(h)}$ we obtain $\forall t \in [h,T)$

$$P_\phi(E_t^{(h)}) + \frac{1}{h}\int_{E_t^{(h)} \triangle E_{t-h}^{(h)}} |\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}| \le P_\phi(E_{t-h}^{(h)}) + \|f\|_\infty |E_t^{(h)} \triangle E_{t-h}^{(h)}|. \tag{30}$$

Combining this estimate with (29) for $l = 2Ch\|f\|_\infty \ll \sqrt{h}$, where C is the constant appearing in equation (29), we obtain for h sufficiently small

$$P_\phi(E_t^{(h)}) + \frac{1}{2h}\int_{E_t^{(h)} \triangle E_{t-h}^{(h)}} |\mathrm{sd}^{\psi}_{E^{(h)}_{t-h}}| \le \left(1 + 2C^2 h \|f\|_\infty^2\right) P_\phi(E_{t-h}^{(h)}) \tag{31}$$

Iterating the previous estimate, we find

$$P_\phi(E_t^{(h)}) \le (1 + 2C^2\|f\|_\infty h)^{\left[\frac{t}{h}\right]-1} P_\phi(E_h^{(h)}).$$

In order to estimate $P_\phi(E_h^{(h)})$ we start by observing that Remark 3.7, for $h = h(E_0)$ small enough, implies $E_h^{(h)} \subseteq B_{2r}$, where $E_0 \subseteq B_r$. Therefore, by (30) for $t = h$ we obtain $P_\phi(E_h^{(h)}) \le P_\phi(E_0) + c$ and we conclude $P_\phi(E_t^{(h)}) \le C_T(P_\phi(E_0) + 1)$. □

We then present a sketch of the proof of the local Hölder continuity in time of the discrete flow, uniformly in h, which can be deduced as in [25, Proposition 2.3]. We highlight the main differences.

Proposition 3.13. *Let E_0 be an initial bounded set of finite $\phi-$perimeter and $T > 0$. Then, for h small enough, for a discrete flow $\{E_t^{(h)}\}$ starting from E_0 it holds*

$$|E_t^{(h)} \triangle E_s^{(h)}| \le C_T |t-s|^{1/2} \quad \forall h \le t \le s < T,$$

for a suitable constant $C_T = C_T(T, E_0, f, \phi, \psi)$.

Proof. Following the previous proof, employing again (31) we find

$$\begin{aligned} P_\phi(E_{2h}^{(h)}) &+ \frac{1}{2}\int_{E_{2h}^{(h)} \triangle E_h^{(h)}} |v_h(\cdot, 2h)| + \frac{1}{2}\int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h(\cdot, h)| \\ &\le (1+ch) P_\phi(E_h^{(h)}) + \frac{1}{2}\int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h(\cdot, h)| \\ &\le (1+ch)\left(P_\phi(E_h^{(h)}) + \int_{E_h^{(h)} \triangle E_0^{(h)}} |v_h|(\cdot, h) \right) \le (1+ch)^2 P_\phi(E_0). \end{aligned}$$

Iterating, we conclude as before

$$\sum_{k=1}^{[T/h]} \int_{E_{kh}^{(h)} \triangle E_{(k-1)h}^{(h)}} |v_h(\cdot, kh)| \le C_T(P_\phi(E_0) + 1). \tag{32}$$

Therefore, combining the previous results and applying (29) with $l = h \ll \sqrt{h}$, we obtain

$$\int_h^T |E_t^{(h)} \triangle E_{t-h}^{(h)}| \le c \sum_{k=1}^{[T/h]} \left(h P_\phi(E_{kh}^{(h)}) + \int_{E_{kh}^{(h)} \triangle E_{(k-1)h}^{(h)}} |v_h(\cdot, kh)| \right) \le C_T \left(P_\phi(E_0) + 1 \right). \tag{33}$$

The proof then follows the one of [25, Proposition 2.3], from equation (2.5) onward. □

We finally prove the main result of this section, the existence of flat flows.

Proof of Theorem 1.1. The proof is classical and we only sketch it. By the uniform equicontinuity of the approximating sequence of Proposition 3.13 and compactness of sets of finite perimeter (by Lemma 3.9 and 3.12) we can use the Ascoli-Arzelà theorem to prove that the sequence $(E_t^{(h_k)})_{k\in\mathbb{N}}$ converges in L^1 to sets E_t for all times $t \ge 0$ and that the family $\{E_t\}_{t\ge 0}$ satisfies the $1/2-$Hölder continuity property, locally uniformly in time. The other property is then easily deduced. □

3.3. **Existence of distributional solutions.** From Theorem 1.1 we deduce the existence of a subsequence $(h_k)_{k\ge 0}$ such that

$$D\chi_{E_t^{(h_k)}} \overset{*}{\rightharpoonup} D\chi_{E_t} \qquad \forall t \ge 0. \tag{34}$$

We will also assume (4), remarking that it implies

$$\lim_{k\to\infty} P_\phi(E_t^{(h_k)}) = P_\phi(E_t) \qquad \text{for a.e. } t \in [0, +\infty). \tag{35}$$

Our aim is to derive (5) and (6) from the Euler-Lagrange equation (22) and passing to the limit $h \to 0$. To achieve this, we will prove that the discrete velocity is a good approximation (up to multiplicative factors) of the discrete evolution speed of the sets. Notice that (5) is a weak formulation of (1), while (6) establishes the link between v and the velocity of the boundaries of E_t. Indeed, law (1) can be interpreted as looking for a family $\{E_t\}_{t\geq 0}$ of sets, whose normal vector ν_{E_t} and $\phi-$curvature $H^\phi_{E_t}$ are well-defined objects and a function $v\ :\ [0,\infty)\times\mathbb{R}^N \to \mathbb{R}$ such that for every $t \in [0,+\infty)$ and $x \in \partial E_t$

$$\begin{cases} v(x,t) & = -H^\phi_{E_t}(x) + f(x,t) \\ V(x,t) & = \psi(x,\nu_{E_t}(x))v(x,t), \end{cases} \tag{36}$$

where V represents the normal velocity of evolution, obtained as the limit as $h \to 0$ (in a suitable sense) of the ratio

$$\frac{\chi_{E_t} - \chi_{E_{t-h}}}{h}.$$

In this whole section we will assume that hypothesis (14) holds. In particular, the sets defining the discrete flow are smooth hypersurfaces in $\mathbb{R}^N$. Moreover, we require hypotheses (H1) to hold.

We start by estimating in time the L^2-norm of the discrete velocity. The proof is the same as the one presented in [41, Lemma 3.6], up to using the density estimates on the $\phi-$perimeter of Lemma 3.3 and considering the $\psi-$balls instead of the Euclidean one.

Proposition 3.14. *Let $\{E^{(h)}_t\}_{t\geq 0}$ be a discrete flow starting from an initial bounded set E_0 of finite $\phi-$perimeter. Then, for any $T>0$ and for h small enough, it holds*

$$\int_0^T \int_{\partial E^{(h)}_t} v_h^2 \,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \leq C_T,$$

for a suitable constant $C_T = C_T(T, E_0, \phi, \psi, f)$.

Recalling now the Euler-Lagrange equation (22) and Lemma 3.12 we conclude

$$\int_0^T \int_{\partial E^{(h)}_t} \left(H^\phi_{E^{(h)}_t}\right)^2 = \int_0^T \int_{\partial E^{(h)}_t} (v_h - F_h)^2 \leq C_T, \tag{37}$$

We now prove an estimate on the error between the discrete velocity $\psi(\cdot,\nu_{E_t})v_h(\cdot,t)$ and the discrete time derivative of χ_h. The proof of this result is based on a double blow-up argument, and the smoothness of sets (locally) minimizing the $\phi-$perimeter is essential. We will split the proof in various lemmas: the first one concerns the composition of blow-ups, and is a well-known result to the experts. We present a simple proof since we could not find a reference.

Lemma 3.15 (Composition of blow-ups)**.** *Consider $0<\beta<\beta'<1$. Assume that $(A_h)_{h\in[0,1]}$ is a family of measurable sets such that the following blow-ups converge as $h \to 0$*

$$\frac{A_h - x_h}{h^\beta} \to A_1 \quad \textit{in } L^1_{loc}$$

$$h^{-(\beta'-\beta)}A_1 \to A_2 \quad \textit{in } L^1_{loc},$$

where $x_h \in \partial A_h$ for all $h \in [0,1]$. Then, if the composition of the blow-ups $h^{-\beta'}(A_h - x_h)$ converges in L^1_{loc}, the limit coincides with A_2.

Proof. We can assume *wlog* $x_h = 0$. Denote with $A_3 = L^1_{loc} - \lim_{h\to 0} h^{-\beta'}A_h$. We fix a ball B_M and $\varepsilon>0$. There exists h^* such that $\forall h \leq h^*$ it holds

$$|((h^{-\beta'}A_h)\triangle A_3)\cap B_M| \leq \varepsilon, \quad |((h^{-\beta'+\beta}A_1)\triangle A_2)\cap B_M| \leq \varepsilon.$$

We fix h and *wlog* assume $Mh^{\beta'-\beta} \le 1$. Taking $\tilde{h} < h$ suitably small (depending on h, ε), we can ensure

$$|((\tilde{h}^{-\beta}A_h)\triangle A_1) \cap B_1| \le \varepsilon h^{N(\beta'-\beta)}.$$

Since $\tilde{h}^{-\beta}h^{-(\beta'-\beta)} > h^{-\beta'}$, there exists $\bar{h} < h$ such that $\bar{h}^{-\beta'} = \tilde{h}^{-\beta}h^{-(\beta'-\beta)}$. We can then estimate

$$\begin{aligned}
|(A_3\triangle A_2) \cap B_M| &\le |(A_3\triangle \bar{h}^{-\beta'}A_h) \cap B_M| + |((h^{-\beta'+\beta})A_1\triangle(\bar{h}^{-\beta'}A_h)) \cap B_M| \\
&\quad + |((h^{-\beta'+\beta}A_1)\triangle A_2) \cap B_M| \\
&\le 2\varepsilon + h^{-N(\beta'-\beta)}|(A_1\triangle(\tilde{h}^{-\beta}A_h)) \cap B_{Mh^{\beta'-\beta}}| \\
&\le 2\varepsilon + h^{-N(\beta'-\beta)}|(A_1\triangle(\tilde{h}^{-\beta}A_h)) \cap B_1| \le 3\varepsilon.
\end{aligned}$$

□

We now compute some estimates on the normal vector on the boundary of the evolving sets, following the proof of [41, Lemma 4.2] (see also [38, Proposition 2.2]). We fix c_∞ as the constant appearing in Lemma 3.10.

In the sequel, we will denote by $\omega(h)$ a modulus of continuity, that is a continuous increasing function $\omega : [0,1] \to \mathbb{R}$ with $\omega(0) = 0$, which can possibly change from statement to statement and line to line to absorb constants independent of h.

Lemma 3.16. *Assume* (H0) *and* (H1). *For given constants* $1/2 < \beta' < \alpha < 1$ *and* $T > 2$, *there exists a modulus of continuity* ω *with the following property. Consider* $t \in [2h, T]$ *and* $x_h \in \partial E_t^{(h)}$ *such that*

$$|v_h(t,y)| \le h^{\alpha-1} \quad \forall y \in B_{c_\infty\sqrt{h}}(x_h) \cap (E_t^{(h)}\triangle E_{t-h}^{(h)}). \tag{38}$$

Then, there exists $\nu \in S^{N-1}$ *such that*

$$\begin{aligned}
&|\nu_{E_t^{(h)}}(\cdot) - \nu| \le \omega(h) \quad \text{in } B_{h^{\beta'}}(x_h) \cap \partial E_t^{(h)} \\
&|\nu_{E_{t-h}^{(h)}}(\cdot) - \nu| \le \omega(h) \quad \text{in } B_{h^{\beta'}}(x_h) \cap \partial E_{t-h}^{(h)}.
\end{aligned} \tag{39}$$

Proof. We fix $\frac{1}{2} < \beta < \beta' < \alpha$ and $0 < R < h^{\frac{1}{2}-\beta}/c_\psi$. Testing the minimality of $E_s^{(h)}$, $s = t, t-h$, we find

$$P_\phi(E_s^{(h)}, B_{Rh^\beta}(x_h)) \le P_\phi(G, B_{Rh^\beta}(x_h)) + \frac{1}{h}\int_{G\triangle E_s^{(h)}} |\mathrm{sd}^\psi_{E_{s-h}^{(h)}}| + \int_{G\triangle E_s^{(h)}} |F_h|, \tag{40}$$

for any set G of finite perimeter such that $G\triangle E_s^{(h)} \subset\subset B_{Rh^\beta}(x_h)$. Using Lemma 3.10, the $1-$Lipschitz regularity of sd^ψ and (38), we deduce $|v_h(s,y)| \le c_\psi Rh^{\beta-1} + c_\infty h^{-1/2} \le (1+c_\infty)h^{-1/2}$ for any $y \in B_{Rh^\beta}(x_h) \cap (E_s^{(h)}\triangle F)$. Plugging this inequality in (40), we find

$$P_\phi(E_s^{(h)}, B_{Rh^\beta}(x_h)) \le P_\phi(G, B_{Rh^\beta}(x_h)) + \frac{1+c}{\sqrt{h}}|F\triangle E_s^{(h)}| + \|f\|_\infty|G\triangle E_s^{(h)}|. \tag{41}$$

We then introduce the blown-up sets for $s = t, t-h$, defined as

$$E_s^{(h),\beta} := h^{-\beta}\left(E_s^{(h)} - x_h\right).$$

Rescaling equation (41), we easily find that $E_s^{(h),\beta}$ is a $(\Lambda_h, r_h)-$minimizer of the $\phi(x_h+h^\beta\cdot,\cdot)-$perimeter, with $\Lambda_h = (1+c)h^{\beta-1/2}$, $r_h = h^{1/2-\beta}$. Moreover, scaling the density estimates (24) we have a uniform bound on the perimeters of the sets $E_s^{(h),\beta}$ in each ball B_R. By compactness, there exist two sets E_1^β, E_2^β such that

$$E_t^{(h),\beta} \to E_1^\beta,\ E_{t-h}^{(h),\beta} \to E_2^\beta \qquad \text{in } L^1_{loc}.$$

Then, by scaling and (38) we find

$$|\mathrm{sd}^{\psi}_{E^{(h),\beta}_{t-h}}(\cdot)| \le c_\infty h^{\alpha-\beta} \qquad \text{on } B_{h^{1/2-\beta}}(0)\cap (E^{(h),\beta}_t \triangle E^{(h),\beta}_{t-h}),$$

thus we easily conclude that $E^\beta := E^\beta_1 = E^\beta_2$. By Lemma 3.9 we can assume that $x_h \to x_0$ as $h \to 0$, up to subsequences. Moreover, by closeness of Λ_h-minimizers under $L^1_{loc}-$convergence (see e.g. [18, Theorem 2.9]), one can see that E^β is a 0-minimizer for the $\phi(x_0,\cdot)-$perimeter. Thus, by complete regularity, it is a smooth C^2 set. We can then employ the classic blow-up theorem to deduce that, for a fixed $\beta' \in (\beta,\alpha)$, the blow-up $h^{-(\beta'-\beta)}E^\beta$ converges to a half-space $\mathbb{H} = \{x\cdot\nu \le 0\}$ as $h \to 0$. Finally, the blow-ups

$$E^{(h),\beta'}_s := \frac{E^{(h)}_s - x_h}{h^{\beta'}}$$

admit a converging subsequence by compactness of sets of finite perimeter and by rescaling equation (41). Thus, the previous Lemma 3.15 implies

$$E^{(h),\beta'}_s \to \mathbb{H} \quad \text{in } L^1_{loc}$$

as $h \to 0$. To conclude, the $\varepsilon-$regularity Theorem for $\Lambda-$minimizers (see e.g. [18, Theorem 3.1]) ensures that $E^{(h),\beta'}_s$ are uniformly $C^{1,\frac12}$ sets in $B_1(0)$ for $s = t, t-h$ as $h \to 0$. ☐

We recall here an approximation result proved in [38] (see also [41] for a more detailed proof). We remark that the proof of this result is purely geometric and does not rely on the variational problem satisfied by the sets $E^{(h)}_t, E^{(h)}_{t-h}$.

Corollary (Corollary 4.3 in [41]). *Under the hypotheses of Lemma 3.16, fix $0 < \beta < \alpha$ and let $\boldsymbol{C}_{h^\beta}$ be the open cylinder defined as*

$$\boldsymbol{C}_{h^\beta}(x_h,\nu) := \left\{ x \in \mathbb{R}^N : |(x-x_h)\cdot\nu| < \frac{h^\beta}{2},\ \left|(x-x_h) - ((x-x_h)\cdot\nu)\,\nu\right| < \frac{h^\beta}{2} \right\}.$$

Then, it holds

$$\Big|\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} (\chi_{E^{(h)}_t} - \chi_{E^{(h)}_{t-h}})\,\mathrm{d}x - \int_{\partial E^{(h)}_t \cap \boldsymbol{C}_{h^\beta/2}(x_h,\nu)} \mathrm{sd}_{E^{(h)}_{t-h}}\,\mathrm{d}\mathcal{H}^{N-1}\Big| \le \omega(h)\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} |\chi_{E^{(h)}_t} - \chi_{E^{(h)}_{t-h}}|.$$

Carefully inspecting the proof, one indeed proves that there exists a geometric constant C such that for any $y \in B^{N-1}_{h^\beta/2}(x_h)$

$$|\mathrm{sd}_{E^{(h)}_{t-h}}(y, f^{(h)}_t(y))\sqrt{1+|\nabla f^{(h)}_t(y)|^2} - \Big(f^{(h)}_t(y) - f^{(h)}_{t-h}(y)\Big)| \le \omega(h)|f^{(h)}_t(y) - f^{(h)}_{t-h}(y)|, \tag{42}$$

where we set

$$\partial E^{(h)}_s \cap \mathbf{C} = \{(y, f^{(h)}_s(y)) \in \mathbb{R}^{N-1}\times\mathbb{R},\ |y| \le h^\beta/2\},$$

for $s = t, t-h$.

We briefly recall some classical results. Consider an anisotropy ψ, independent of the position. It is well-known that, for any closed set $G \subseteq \mathbb{R}^N$, setting sd^ψ_G as the distance from G induced by ψ°, then the gradient of sd^ψ_G exists almost everywhere and satisfies the eikonal equation (for a proof see for instance [10, Remark 2.2])

$$\psi(\nabla \mathrm{sd}^\psi_G) = 1 \tag{43}$$

almost everywhere. Moreover, in this particular case, in the definition of dist^ψ we can consider just straight lines as follows from a simple application of Jensen's inequality: for any curve γ as in the definition of dist^ψ, we have

$$\int_0^1 \psi^\circ(\dot\gamma(t))\,\mathrm{d}t \geq \psi^\circ\left(\int_0^1 \dot\gamma\right) = \psi^\circ(y-x).$$

Proposition 3.17 (Estimate on almost flat sets)**.** *Under the hypotheses of Lemma 3.16 and with the same notation, fix $\beta \in (0,\alpha)$ and let $\boldsymbol{C}_{h^\beta}$ be the open cylinder defined as*

$$\boldsymbol{C}_{h^\beta}(x_h,\nu) := \left\{x \in \mathbb{R}^N : |(x-x_h)\cdot\nu| < \frac{h^\beta}{2},\ \Big|(x-x_h) - ((x-x_h)\cdot\nu)\,\nu\Big| < \frac{h^\beta}{2}\right\}.$$

Then, it holds

$$\begin{aligned}\Big|\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} (\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}})\,\mathrm{d}x - \int_{\partial E_t^{(h)}\cap \boldsymbol{C}_{h^\beta/2}(x_h,\nu)} \psi(x,\nu_{E_t^{(h)}})\,\mathrm{sd}^\psi_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\Big| \\ \leq \omega(h)\int_{\boldsymbol{C}_{h^\beta/2}(x_h,\nu)} |\chi_{E_t^{(h)}} - \chi_{E_{t-h}^{(h)}}|.\end{aligned}$$

Proof. We recall that the modulus of continuity ω may change from line to line to absorb constants independent of h.

From the previous Lemma 3.16 we know that, for h suitably small, both $\partial E_t^{(h)}$ and $\partial E_{t-h}^{(h)}$ in $\mathbf{C}_{h^\beta/2}(x_h,\nu)$ can be written as graphs of functions of class $C^{1,\frac{1}{2}}$. Up to a change of coordinates, we can assume *wlog* that $x_h = 0, \nu = e_N$. For simplicity, we set $\mathbf{C} = \mathbf{C}_{h^\beta/2}(0,e_N)$. We thus find

$$\partial E_s^{(h)} \cap \mathbf{C} = \{(y, f_s^{(h)}(y)) \in \mathbb{R}^{N-1}\times\mathbb{R},\ |y| \leq h^\beta/2\}$$

for $s = t, t-h$, where $f_s^{(h)} : B^{N-1}_{h^\beta/2} \to \mathbb{R}$ are $C^{1,\frac{1}{2}}$ functions with

$$\|\nabla f_s^{(h)}\|_{L^\infty(B_{h^\beta/2})} \leq \omega(h).$$

We want to prove the following slightly stronger pointwise inequality: namely, that for any point $x = (y, f_t^{(h)}(y)) \in \partial E_t^{(h)} \cap \mathbf{C}$, it holds

$$\left|\mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x)\,\psi(x,\nu_{E_t^{(h)}}(x))\sqrt{1+|\nabla f_t^{(h)}(y)|} - \Big(f_t^{(h)}(y) - f_{t-h}^{(h)}(y)\Big)\right| \leq \omega(h)|f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|. \tag{44}$$

Integrating the previous inequality over $\mathbf{C}$ yields the thesis. Clearly, it is enough to prove (44) at each point x such that $|\mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x)| > 0$. We thus fix $x = (y, f_t^{(h)}(y)) \in \partial E_t^{(h)} \cap \mathbf{C}$ and denote by $x' := (y, f_{t-h}^{(h)}(y))$. We remark that these points depend on h, but we drop the subscript to ease notation. It can be assumed without loss of generality that $x \notin E_{t-h}^{(h)}$, as the other case is analogous.

Step 1 We now prove that, with the notation previously introduced, it holds

$$|\mathrm{sd}'_{E_{t-h}^{(h)}}(x) - \mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x)| \leq \omega(h)|f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|, \tag{45}$$

where sd' denotes the signed distance function induced by the anisotropy $\psi(x',\cdot)$. Let γ be a smooth curve, with $\gamma(0) = x, \gamma(1) \in \partial E_{t-h}^{(h)}$ to be used in the definition of the geodesic distance $\mathrm{sd}^\psi_{E_{t-h}^{(h)}}$. Firstly, we remark that one could assume

$$\gamma([0,1]) \subseteq B(x, 2c_\psi^2|f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|) \tag{46}$$

Indeed, if it were not the case, the lower bounds contained in (11) and (42) allow us to estimate

$$\int_0^1 \psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \geq \frac{1}{c_\psi}\int_0^1 |\dot\gamma|\,\mathrm{d}t \geq 2c_\psi|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)| \geq 2c_\psi\,\mathrm{sd}_{E_{t-h}^{(h)}}(x) \geq 2\,\mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x), \tag{47}$$

a contradiction for h small. We can reason analogously for $\mathrm{sd}'_{E_{t-h}^{(h)}}$. In particular, we can consider just curves having length $\int_0^1 |\dot\gamma| \leq c|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|$. Therefore, we obtain (by homogeneity)

$$\begin{aligned}\mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x) &\leq \int_0^1 \psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \leq \int_0^1 \psi^\circ(x',\dot\gamma)\,\mathrm{d}t + \sup_{\nu\in S^{N-1},\,t\in[0,1]}|\psi(\gamma(t),\nu)-\psi(x',\nu)|\int_0^1|\dot\gamma|\\ &\leq \int_0^1 \psi^\circ(x',\dot\gamma)\,\mathrm{d}t + c\,\omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(h)|,\end{aligned}$$

and, taking the $\inf_\gamma$, we obtain $\mathrm{sd}^\psi_{E_{t-h}^{(h)}}(x) \leq \mathrm{sd}'_{E_{t-h}^{(h)}}(x) + \omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|$. The converse inequality can be proved analogously, yielding (45).

Therefore, in what follows we will consider always the anisotropy frozen in x', and use sd' instead of sd^ψ. Finally, let $p \in \partial E_{t-h}^{(h)}$ a minimizer for the definition of $\mathrm{sd}'_{E_{t-h}^{(h)}}(x)$. In the following, with $\Pi^v_{\mathbb{H}}z$, $\Pi_{\mathbb{H}}z$ we denote respectively the projection on the hyperplane $\mathbb{H}$ of z along the direction v and the orthogonal projection of z on $\mathbb{H}$.

Step 2. In this step we assume that $E_{t-h}^{(h)} \cap \mathbf{C}$ coincides with the halfspace $\mathbb{H} = p + \{z\cdot\nu \leq 0\}$ intersected with the same cylinder and prove claim (44).
To this aim, we start noticing that by translation we may assume $p = 0$ and that $\mathrm{sd}'_{\mathbb{H}}(z+\xi) = \mathrm{sd}'_{\mathbb{H}}(z)$ for all $z \in \mathbb{R}^N$ and for all ξ orthogonal to ν. Hence, in fact,

$$\mathrm{sd}'_{\mathbb{H}}(z) = \mathrm{sd}'_{\mathbb{H}}((z\cdot\nu)\nu) = (z\cdot\nu)\mathrm{sd}'_{\mathbb{H}}(\nu)\,. \tag{48}$$

Therefore, $\mathrm{sd}'_{\mathbb{H}}$ is differentiable everywhere, with $\nabla\mathrm{sd}'_{\mathbb{H}} = \mathrm{sd}'_{\mathbb{H}}(\nu)\nu$. Recalling the eikonal equation (43), it must hold $\mathrm{sd}'_{\mathbb{H}}(\nu) = 1/\psi(x',\nu)$ and in turn, from (48), and choosing $z = x$, we have

$$\mathrm{sd}'_{\mathbb{H}}(x)\psi(x',\nu) = x\cdot\nu = \mathrm{sd}_{\mathbb{H}}(x). \tag{49}$$

We remark that $\mathrm{sd}'_{\mathbb{H}}(x) = \mathrm{sd}'_{E_{t-h}^{(h)}}(x)$ by (38), thus we conclude (44) by combining (49) with (42).

Step 3. We now conclude in the general case. With the notation introduced at the end of Step 1, set $\nu = \nu_{E_{t-h}^{(h)}}(p)$, and consider the half-space $\mathbb{H} = p + \{z\cdot\nu \leq 0\}$ and $w := x' - \Pi_{\mathbb{H}}(x')$ as depicted in Figure 1. We shall prove that

$$|w| \leq \omega(h)|f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|.$$

To see this, we start by remarking that (39) implies

$$|e_N - e_N(e_N\cdot\nu_{E_{t-h}^{(h)}})| \leq \omega(h) \qquad \text{in } \partial E_{t-h}^{(h)}\cap\mathbf{C},$$

implying $e_N\cdot\nu_{E_{t-h}^{(h)}} \geq 1-\omega(h)$, and thus, for any versor v tangent to $\partial E_{t-h}^{(h)}\cap\mathbf{C}$ one has $|v\cdot e_N| \leq \omega(h)$. Therefore, we have $(x'-p)\cdot e_N \leq \omega(h)|x'-p|$ and also

$$\begin{aligned}\frac{x'-p}{|x'-p|}\cdot\nu &= \frac{x'-p}{|x'-p|}\cdot(e_N(\nu\cdot e_N) + \nu - e_N(\nu\cdot e_N))\\ &\leq \omega(h) + |\nu - e_N(\nu\cdot e_N)| = \omega(h) + \big(1-|\nu\cdot e_N|^2\big)^{1/2}\\ &\leq 3\sqrt{\omega(h)},\end{aligned}$$

by choosing h small. Up to defining $\sqrt{\omega}$ as ω, using the previous estimate and the bounds (46) we see that

$$|w| = |x'-p|\left(\frac{x'-p}{|x'-p|}\cdot\nu\right) \leq \omega(h)|x'-p| \leq \omega(h)|f_t^{(h)}(y)-f_{t-h}^{(h)}(y)|. \tag{50}$$

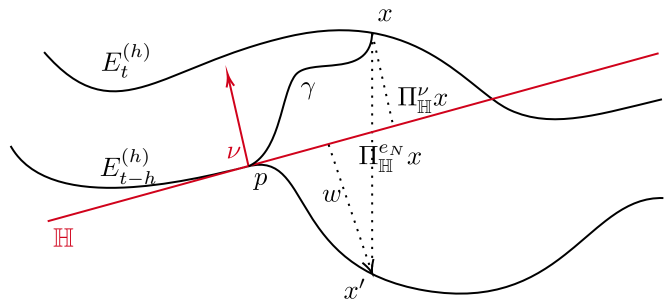

Figure 1. The situation in the proof of the lemma.

We now remark that $\mathrm{sd}'_{E_{t-h}^{(h)}}(x) = \mathrm{sd}'_{\mathbb{H}}(x)$ (by convexity of the anisotropy $\psi(x', \cdot)$) and so, applying the previous step to $\mathbb{H}$ and using also (45), we get

$$\Big|\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x)\, \psi(x, \nu_{E_t^{(h)}}(x)) \sqrt{1 + |\nabla f_t^{(h)}(y)|} - |x - \Pi_{\mathbb{H}}^{e_N} x|\Big| \leq \omega(h)|x - \Pi_{\mathbb{H}}^{e_N} x|.$$

We conclude (44) by estimating

$$\big||x - \Pi_{\mathbb{H}}^{e_N} x| - |x - x'|\big| \leq |x' - \Pi_{\mathbb{H}}^{e_N} x| = |w|/|\nu \cdot e_N| \leq \frac{\omega(h)}{1 - \omega(h)} |f_t^{(h)}(y) - f_{t-h}^{(h)}(y)|,$$

where we used (50). We conclude the proof by a simple change of coordinates and using (44) to find

$$\begin{aligned}
&\Big| \int_{\partial E_t^{(h)} \cap \mathbf{C}} \psi(x, \nu_{E_t^{(h)}}(x))\, \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(x)\, \mathrm{d}\mathcal{H}^{N-1} - \int_{B_{h^\beta/2}} f_t^{(h)}(y) - f_{t-h}^{(h)}(y)\, \mathrm{d}y \Big| \\
&= \Big| \int_{B_{h^\beta/2}} \psi((y, f_t^{(h)}(y)), \nu_{E_t^{(h)}}(y, f_t^{(h)}(y)))\, \mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}(y, f_t^{(h)}(y)) \sqrt{1 + |\nabla f_t^{(h)}(y)|^2} - (f_t^{(h)}(y) - f_{t-h}^{(h)}(y))\, \mathrm{d}y \Big| \\
&\leq \omega(h) \int_{B_{h^\beta/2}} |f_t^{(h)} - f_{t-h}^{(h)}|\, \mathrm{d}y.
\end{aligned}$$

□

Finally, we are able to prove that the error generated by approximating the discrete velocity with v_h goes to zero as $h \to 0$. We follow the lines of [38, Proposition 2.2].

Proposition 3.18 (Error estimate)**.** *Under the hypothesis of Lemma 3.16, the error in the discrete curvature equation vanishes in the limit $h \to 0$, namely*

$$\lim_{h \to 0} \left| \frac{1}{h} \int_0^T \left(\int_{E_t^{(h)}} \eta\, \mathrm{d}x - \int_{E_{t-h}^{(h)}} \eta\, \mathrm{d}x \right) \mathrm{d}t - \int_0^T \int_{\partial E_t^{(h)}} \psi(x, \nu_{E_t^{(h)}}) v_h \eta\, \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}t \right| = 0 \tag{51}$$

for all $\eta \in C_c^1(\mathbb{R}^N \times [0, T))$.

Proof. We fix $t \in [2h, \infty)$ and $\alpha \in (\frac{1}{2}, \frac{N+2}{2N+2})$. For any point $x_h \in \partial E_t^{(h)}$ we define the open set A_{x_h} defined as follows:

(i) if (38) holds, we set $A_{x_h} = \mathbf{C}_{h^\beta/2}(x_h, \nu)$, with the notations of Corollary 3.17;
(ii) otherwise we set $A_{x_h} = B(x_h, c_\infty \sqrt{h})$, where c_∞ is the constant of Lemma 3.10.

By Lemma 3.10, the family $\{A_{x_h} : x_h \in \partial E_t^{(h)}\}$ is a covering of $E_t^{(h)} \triangle E_{t-h}^{(h)}$. By a simple application of Besicovitch's theorem (see e.g. [39]), we find a finite collection of points $I \subseteq \partial E_t^{(h)}$ such that $\{A_{x_h}\}_{x_h \in I}$ is a covering of $E_t^{(h)} \triangle E_{t-h}^{(h)}$ with the finite intersection property. We proceed to estimate

(51) on each A_{x_h} belonging to this family.

Estimate in case (i) We use Proposition 3.17 to deduce

$$
\begin{aligned}
&\Big|\int_{A_{x_h}}(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})\eta\,\mathrm{d}x-\int_{\partial E_t^{(h)}\cap A_{x_h}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\eta\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&\le|\eta(x_h,t)|\Big|\int_{A_{x_h}}(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})-\int_{\partial E_t^{(h)}\cap A_{x_h}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&+\Big|\int_{A_{x_h}}(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})(\eta-\eta(x_h,t))-\int_{\partial E_t^{(h)}\cap A_{x_h}}(\eta-\eta(x_h,t))\,\psi(x,\nu_{E_t^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&\le C(\omega(h)\|\eta\|_\infty+h^\beta\|\nabla\eta\|_\infty)\int_{A_{x_h}}|\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1}+ch^\beta\|\nabla\eta\|_\infty P(E_t^{(h)},A_{x_h}).
\end{aligned}
\tag{52}
$$

Estimate in case (ii) By assumption $\exists y\in B_{c_\infty\sqrt{h}}(x_h)\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})$ such that $|v_h(t,y)|>h^{\alpha-1}$. We can assume *wlog* $y\in E_t^{(h)}$. We then have $B(y,h^\alpha/(2c_\psi))\subseteq\mathbb{R}^N\setminus E_{t-h}^{(h)}$ and $\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}>h^\alpha/(2c_\psi^2)$ on $B(y,h^\alpha/(2c_\psi))$. Since $h^\alpha<<h^{1/2}$, we can use the density estimates of Lemma 3.3 to deduce

$$
ch^{(N+1)\alpha-1}\le\int_{B(y,h^\alpha/(2c_\psi))\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})}|v_h|\,\mathrm{d}x.
$$

Analogously, recalling also Lemma 3.10, we deduce

$$
\int_{B(x_h,c_\infty\sqrt{h})\cap\partial E_t^{(h)}}|\psi(x,\nu_{E_{t-h}^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1}(x)\le ch^{\frac{N}{2}}.
$$

Combining the two previous equations and $B(y,h^\alpha/(2c_\psi))\subseteq B(y,c\sqrt{h})$, we infer

$$
\begin{aligned}
&\int_{A_{x_h}}|\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}}|+\int_{A_{x_h}\cap\partial E_t^{(h)}}|\psi(x,\nu_{E_{t-h}^{(h)}})\,\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}|\,\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad\le ch^{\frac{N}{2}-(N+1)\alpha+1}\int_{A_{x_h}\cap(E_t^{(h)}\triangle E_{t-h}^{(h)})}|\psi(x,\nu_{E_{t-h}^{(h)}})v_h|.
\end{aligned}
\tag{53}
$$

Summing over $x_h\in I$ both (52) and (53), and using the local finiteness of the covering, we get

$$
\begin{aligned}
&\Big|\int(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})\eta\,\mathrm{d}x-\int_{\partial E_t^{(h)}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\eta\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&\le\sum_{x_h\in I}\Big|\int_{A_{x_h}}(\chi_{E_t^{(h)}}-\chi_{E_{t-h}^{(h)}})\eta\,\mathrm{d}x-\int_{\partial E_t^{(h)}\cap A_{x_h}}\psi(x,\nu_{E_t^{(h)}})\mathrm{sd}^{\psi}_{E_{t-h}^{(h)}}\eta\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&\le c\Big(\omega(h)\|\eta\|_\infty+h^\beta\|\nabla\eta\|_\infty+h^{\frac{N}{2}-(n+1)\alpha+1}\|\eta\|_\infty\Big)\cdot\\
&\quad\cdot\Big(P(E_t^{(h)})+|E_t^{(h)}\triangle E_{t-h}^{(h)}|+\int_{E_t^{(h)}\triangle E_{t-h}^{(h)}}|v_h|\Big)
\end{aligned}
$$

where the last constant c depends on N,ψ. We then use Lemma 3.12, (32) and (33) to conclude

$$
\begin{aligned}
&\Big|\int_{2h}^T\frac{1}{h}\Big(\int_{E_t^{(h)}}\eta\,\mathrm{d}x-\int_{E_{t-h}^{(h)}}\eta\,\mathrm{d}x\Big)-\int_h^T\int_{\partial E_t^{(h)}}\psi(x,\nu_{E_t^{(h)}})v_h\eta\,\mathrm{d}\mathcal{H}^{N-1}\Big|\\
&\qquad\le c\Big(\omega(h)\|\eta\|_\infty+h^\beta\|\nabla\eta\|_\infty+h^{\frac{N}{2}-(n+1)\alpha+1}\|\eta\|_\infty\Big),
\end{aligned}
$$

where $c=c(E_0,f,T,\psi)$ and T is chosen such that $\operatorname{spt}\eta\subset\subset\mathbb{R}^N\times[0,T]$. The conclusion follows using the definition of α and taking the limit $h\to0$.

□

The proof of our main theorem of this section is now a consequence of the previous results. In particular, hypothesis (34) and (35) imply that the discrete flow converges to the flat flow in the sense of varifolds and this allows to prove (5), while (6) is a consequence of Proposition 3.18. In order to prove the convergence of the approximations in time of the forcing term, we need to require additionally that (H2) holds.

Proof of Theorem 1.2. Firstly, combining [32, Theorem 4.4.2] with the bounds contained in (37) and in Proposition 3.14, we conclude the existence of functions $v, H^\phi, \tilde{f} : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ satisfying

$$\int_0^T \int_{\partial E_t} |v|^2 + |H^\phi|^2 + |\tilde{f}|^2 \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \le C_T$$

and the following properties

$$\begin{aligned} \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} v_{h_k} \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta v \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \\ \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} F_{h_k}(x,t) \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta \tilde{f} \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t \\ \lim_k \int_0^T \int_{\partial E_t^{(h_k)}} H^\phi_{E_t^{(h_k)}} \eta \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t &= \int_0^T \int_{\partial E_t} \eta H^\phi \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}t, \end{aligned} \tag{54}$$

for any $\eta \in C^0_c(\mathbb{R}^N \times [0,T))$. We now employ an approximation procedure to prove that $H^\phi(\cdot,t)$ is the $\phi-$mean curvature of E_t for a.e. $t \in [0,\infty)$, following the lines of [38, 41]. Fixed $t \in [0,+\infty)$ and $\varepsilon > 0$, set ν_ε a continuous function such that $\int_{\partial E_t} (\nu_{E_t} - \nu_\varepsilon)^2 \, \mathrm{d}\mathcal{H}^{N-1} < \varepsilon$. Then, by (34) one could prove that $\lim_{k\to\infty} \int_{\partial E_t^{(h_k)}} (\nu_{E_t^{(h_k)}} - \nu_\varepsilon)^2 \, \mathrm{d}\mathcal{H}^{N-1} < \varepsilon$. Considering test functions in (54) of the form $\eta(x,t) = a(t)g(x)$, one has for a.e. $t \in [0,+\infty)$

$$\lim_k \int_{\partial E_t^{(h_k)}} H^\phi_{E_t^{(h_k)}} g \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E_t} H^\phi g \, \mathrm{d}\mathcal{H}^{N-1}.$$

Thus, for a.e. $t \in [0,+\infty)$ and for any $X \in C^0_c(\mathbb{R}^N;\mathbb{R}^N)$ it holds

$$\lim_k \int_{\partial E_t^{(h_k)}} H^\phi_{E_t^{(h_k)}} \nu_{E_t^{(h_k)}} \cdot X \, \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E_t} H^\phi \nu_{E_t} \cdot X \, \mathrm{d}\mathcal{H}^{N-1}$$

by approximating the normal vectors of $E_t^{(h_k)}$ with ν_ε. Furthermore, by the convergence (34) and the hypothesis (35) we can use the Reshetnyak's continuity theorem (see e.g. [4, Theorem 2.39]), ensuring

$$\int_{\partial E_t^{(h_k)}} L(x, \nu_{E_t^{(h_k)}}) \, \mathrm{d}\mathcal{H}^{N-1} \to \int_{E_t} L(x, \nu_{E_t}) \, \mathrm{d}\mathcal{H}^{N-1}$$

as $k \to \infty$, for any $L \in C^0_c(\mathbb{R}^N \times \mathbb{R}^N)$. We choose $L(x,\nu) = \mathrm{div}_\phi X$ for some $X \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$ to obtain

$$\begin{aligned} \int_{\partial E_t} \mathrm{div}_\phi X \, \mathrm{d}\mathcal{H}^{N-1} &= \lim_k \int_{\partial E_t^{(h_k)}} \mathrm{div}_\phi X \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \lim_k \int_{\partial E_t^{(h_k)}} X \cdot \nu_{E_t^{(h_k)}} H^\phi_{E_t^{(h_k)}} \, \mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{\partial E_t} X \cdot \nu_{E_t} H^\phi \, \mathrm{d}\mathcal{H}^{N-1}, \end{aligned}$$

which shows that $H^\phi(\cdot,t)$ is the $\phi-$mean curvature of the set E_t for a.e. $t\in[0,+\infty)$. Moreover, we remark that $F_{h_k}(x,t)\to f(x,t)$ for every (x,t), thus for any test function $\eta\in C^0_c(\mathbb{R}^N\times[0,+\infty))$ and $t\in[0,+\infty)$ we have

$$\begin{aligned}\left|\int_{\partial E_t^{(h)}} F_{h_k}(x,t)\eta(x,t)\,\mathrm{d}\mathcal{H}^{N-1}_x-\int_{\partial E} f\eta\,\mathrm{d}\mathcal{H}^{N-1}_x\right| &\le \left|\int_{\partial E_t^{(h)}} F_{h_k}\eta-\int_{\partial E_t}F_{h_k}\eta\right|+\int_{\partial E_t}|F_{h_k}-f|\eta\\ &\le \|f\|_\infty\|\eta\|_\infty\left(P(E_t^{(h)})-P(E_t)\right)+\int_{\partial E_t}|F_{h_k}-f|\eta\to 0\end{aligned}$$

applying the dominated convergence theorem and recalling Lemma 3.9. Thus, $\tilde f=f$. We then prove (5) by passing to the limit in the Euler-Lagrange equation (22).

To prove (6) we employ Proposition 3.18: for every $\eta\in C^0_c(\mathbb{R}^N\times[0,T))$, by a change of variables we have that

$$\int_h^T\left[\int_{E_t^{(h)}}\eta\,\mathrm{d}x-\int_{E_{t-h}^{(h)}}\eta\,\mathrm{d}x\right]\mathrm{d}t=\int_h^T\int_{E_t^{(h)}}\left(\eta(x,t)-\eta(x,t-h)\right)\mathrm{d}x\,\mathrm{d}t-h\int_{E_0}\eta\,\mathrm{d}x$$

where we have used that $E_t^{(h)}=E_0$ for $t\in[0,h)$. Therefore, a simple convergence argument yields

$$\lim_{h\to0}\frac1h\int_h^T\left[\int_{E_t^{(h)}}\eta\,\mathrm{d}x-\int_{E_{t-h}^{(h)}}\eta\,\mathrm{d}x\right]\mathrm{d}t=-\int_h^T\partial_t\eta(x,t)\,\mathrm{d}x\,\mathrm{d}t-\int_{E_0}\eta.$$

Combining the previous estimate with Proposition 3.18 and passing to the limit, we obtain (6). □

4. Viscosity solutions

In this section we will prove the existence of another weak notion of solution for the mean curvature flow starting from a compact set. We will follow the so-called level set approach based on the theory of viscosity solution. We recall that in the first part we work with the standing assumptions of the paper (H0). Additionally, we require (H1).

4.1. **The discrete scheme for unbounded sets.** In this short subsection we will define the discrete evolution scheme for unbounded sets having compact boundary. The idea would be to define this evolution simply as the complement of the evolution of the complementary set, but since the anisotropies we are considering are not symmetric, we need additional care.

We recall that, given an anisotropy ϕ, we define $\tilde\phi(x,\nu):=\phi(x,-\nu)$. This anisotropy has all the properties of the original one, concerning regularity and bounds. We start remarking the following simple fact. One can see that $\mathrm{dist}^\psi(x,y)=\mathrm{dist}^{\tilde\psi}(y,x)$, since for any curve $\gamma\in W^{1,1}([0,1];\mathbb{R}^N),\gamma(0)=x,\gamma(1)=y$, a simple change of variable yields

$$\int_0^1\psi^\circ(\gamma(t),\dot\gamma(t))\,\mathrm{d}t=\int_0^1\psi^\circ\left(\gamma(1-t),-\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(1-t)\right)\right)\mathrm{d}t=\int_0^1\widetilde{(\psi^\circ)}(\eta(t),\dot\eta(t))\,\mathrm{d}t,$$

for $\eta(t)=\gamma(1-t)$, once one sees that

$$\widetilde{(\psi^\circ)}(\cdot,\nu)=\sup_{\psi(\cdot,\xi)\le1}\xi\cdot(-\nu)=\sup_{\tilde\psi(\cdot,-\xi)\le1}(-\xi)\cdot\nu=(\tilde\psi)^\circ(\cdot,\nu).$$

Therefore, by definition of signed distance we have

$$\mathrm{sd}^\psi_E(x)=-\mathrm{sd}^{\tilde\psi}_{E^c}(x).\tag{55}$$

For every compact set F and $h>0,t\ge0$, we will denote by $\tilde T^\pm_{h,t}F$ the maximal and the minimal solution to problem (21), according to Lemma 3.1 with P_ϕ and sd^ψ, respectively, replaced by $P_{\tilde\phi}$

and $\mathrm{sd}^{\tilde\psi}$. Finally, for every set E with compact boundary we define

$$T^{\pm}_{h,t}E := \left(\tilde T^{\mp}_{h,t}E^c\right)^c. \tag{56}$$

As in the case for compact sets, we set $T_{h,t}E := T^-_{h,t}E$. Given an open, unbounded set E_0 having compact boundary, we can then define the discrete flow $\{E^{(h)}_t\}_{t\geq 0}$ as follows: $E^{(h)}_t := E_0$ for $t \in [0,h)$ and

$$E^{(h)}_t = T_{h,t}E^{(h)}_{t-h}, \quad \forall t \in [h,+\infty).$$

One easily checks that analogous results to Lemmas 3.2, 3.9 and 3.8 hold also for this problem. We state the corresponding results.

Lemma 4.1. *Let $F_1 \subseteq F_2$ be open, unbounded sets with compact boundary and fix $h > 0, t \geq 0$. Then, $T_{h,t}F_1 \subseteq T_{h,t}F_2$.*

Lemma 4.2. *For any $T > 0$ there exists a constant $C_T(\phi,\psi,f,T)$ such that for every $R > 0$ the following holds. If the initial open set $E \supset B^c_R$, then $E^{(h)}_t \supset B^c_{C_TR}$ for all $t \in [0,T]$.*

Lemma 4.3. *For every $R_0 > 0$ there exist $h_0(R_0) > 0$ and $C(R_0,\phi,\psi,f) > 0$ with the following property: For all $R \geq R_0$, $h \in (0,h_0)$, $t > 0$ and $x \in \mathbb{R}^N$ one has*

$$T_{h,t}((B_R(x))^c) \subseteq (B_{R-Ch}(x))^c.$$

We now state a comparison principle between bounded and unbounded sets, following the line of [14, Lemma 6.10].

Lemma 4.4. *Let E_1 be a compact set and let E_2 be an open, unbounded set, with compact boundary, and such that $E_1 \subseteq E_2$. Then, for every $h \in (0,1), t \geq 0$ it holds $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$.*

Proof. We fix $h \in (0,1), t \in [0,T]$ for $T > 0$. Set $R > 0$ such that $E_1, E^c_2 \subseteq B_R$ and note that by Lemmas 3.2 and 3.9 (applied to $P_{\tilde\phi}$ instead of P_ϕ) we get

$$\left(T^+_{h,t}E_2\right)^c \subseteq \tilde T^-_{h,t}E^c_2 \subseteq T^-_{h,t}B_R \subseteq B_{C_TR}, \tag{57}$$

for some $C_T(\phi,\psi,f,T)$. Since $\tilde T^-_{h,t}E^c_2$ is the minimal solution of

$$\min\left\{P_{\tilde\phi}(E) + \frac{1}{h}\int_E \mathrm{sd}^{\tilde\psi}_{E^c_2}(x)\,\mathrm{d}x - \int_E F_h(x,t)\,\mathrm{d}x\right\},$$

considering the change of variables $\tilde E = E^c$ and using (55), we easily conclude that $T^+_{h,t}E_2 = \left(\tilde T^-_{h,t}E^c_2\right)^c$ is the maximal solution of

$$\min\left\{P_\phi(\tilde E) + \frac{1}{h}\int_{B_{C_TR}} \mathrm{sd}^\psi_{E_2} - \frac{1}{h}\int_{\tilde E^c}\mathrm{sd}^\psi_{E_2} - \int_{\tilde E^c} F_h(x,t)\,\mathrm{d}x\right\} - \frac{1}{h}\int_{B_{C_TR}}\mathrm{sd}^\psi_{E_2}.$$

we then note that

$$\int_{B_{C_TR}}\mathrm{sd}^\psi_{E_2} = \int_{\tilde E}\mathrm{sd}^\psi_{E_2}\chi_{B_{C_TR}} + \int_{\tilde E^c}\mathrm{sd}^\psi_{E_2},$$

for every $\tilde E$ such that $\tilde E^c \subseteq B_{C_TR}$. By (57), we conclude that $T^+_{h,t}E_2$ is the maximal solution of

$$\min\left\{P_\phi(\tilde E) + \frac{1}{h}\int_{\tilde E}\mathrm{sd}^\psi_{E_2}\chi_{B_{C_TR}} - \int_{\tilde E^c}F_h(x,t)\,\mathrm{d}x \ : \ \tilde E^c \subseteq B_{C_TR}\right\}. \tag{58}$$

Analogously, one proves that $T^-_{h,t}E_2$ is the minimal solution of (58). Finally, we remark that $\mathrm{sd}^\psi_{E_s}\chi_{B_{C_TR}} \leq \mathrm{sd}^\psi_{E_1}$ and that $T^{\pm}_{h,t}E_1 \cup T^{\pm}_{h,t}E_2, T^{\pm}_{h,t}E_1 \cap T^{\pm}_{h,t}E_2$ are both admissible competitors for (58), one argues exactly as in the proof of Lemma 3.2 to conclude $T^{\pm}_{h,t}E_1 \subseteq T^{\pm}_{h,t}E_2$. $\square$

4.2. **The level set approach.** We recall that in this section we assume (H0), (H1). Consider a function $u : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ whose spatial superlevel sets $\{u(\cdot, t) \geq s\}$ evolve according to the mean curvature equation

$$V(x,t) = -\psi(x, \nu_{\{u(\cdot,t)\geq s\}}) \left(H^{\phi}_{\{u(\cdot,t)\geq s\}}(x) - f(x,t) \right) \quad \text{for } x \in \partial\{u(\cdot,t) \geq s\}.$$

The function u then satisfies (recalling that $-\nabla u/|\nabla u|$ is the *outer normal vector* to the superlevel set $\{u(\cdot, t) \geq u(x, t)\}$) the equation

$$\begin{aligned}
\partial_t u = |\nabla u| V(x) &= -\psi(x, -\nabla u) \left(H^{\phi}_{\{u(\cdot,t)\geq u(x,t)\}}(x) - f(x,t) \right) \\
&= -\psi(x, -\nabla u) \left(\mathrm{div} \nabla_p \phi(x, -\nabla u) - f(x,t) \right) \\
&= -\psi(x, -\nabla u) \left(\sum_i \partial_{x_i} \partial_p \phi(x, -\nabla u) - \nabla^2_p \phi(x, -\nabla u) \,:\, \nabla^2 u - f(x,t) \right) \\
:&= -\psi(x, -\nabla u) \left(H(x, \nabla u, \nabla^2 u) - f(x,t) \right),
\end{aligned}$$

where we defined the Hamiltonian $H \,:\, \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \times Sym_N \to \mathbb{R}$ as

$$H(x, p, X) := \sum_i \partial_{x_i} \partial_{p_i} \phi(x, -p) - \nabla^2_p \phi(x, -p) \,:\, X. \tag{59}$$

We therefore focus on solving the parabolic Cauchy problem

$$\begin{cases} \partial_t u + \psi(x, -\nabla u) \left(H(x, \nabla u, \nabla^2 u) - f(x,t) \right) = 0 \\ u(\cdot, t) = u_0. \end{cases} \tag{60}$$

The appropriate setting for this type of geometric evolution equations is the one of viscosity solutions, in the framework of [26, 35] (see also [14]). We will focus on the evolution of sets with compact boundary on compact time intervals of the form $[0, T]$. We now define the notion of admissible test function. In the following, with a small abuse of language, we will say that a function $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is constant outside a compact set if there exists a compact set $K \subset \mathbb{R}^N$ such that $u(\cdot, t)$ is constant in $\mathbb{R}^N \setminus K$ for every $t \in [0, T]$ (with the constant possibly depending on t).

Definition 4.5. Let $\hat{z} = (\hat{x}, \hat{t}) \in \mathbb{R}^N \times (0, T)$ and let $A \subseteq (0, T)$ be any open interval containing $\hat{t}$. We will say that $\eta \in C^0(\mathbb{R}^N \times \bar{A})$ is admissible at the point $\hat{z}$ if it is of class C^2 in a neighborhood of $\hat{z}$, if it is constant out of a compact set, and, in case $\nabla \eta(\hat{z}) = 0$, the following holds: for all $(x, t) \in \mathbb{R}^N \times A$, and there exist numbers $a, b > 0$ such that

$$|\eta(x,t) - \eta(\hat{z}) - \eta_t(\hat{z})(t - \hat{t})| \leq a|x - \hat{x}|^3 + b|t - \hat{t}|^2.$$

We then recall one of the equivalent definitions of viscosity solutions.

Definition 4.6. An upper semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ (in short, $u \in usc(\mathbb{R}^N \times [0, T]))$, constant outside a compact set, is a viscosity subsolution of the Cauchy problem (60) if $u(\cdot, 0) \leq u_0$ and for all $z := (x, t) \in \mathbb{R}^N \times (0, T)$ and all $C^\infty-$test functions η such that η is admissible at z and $u - \eta$ has a maximum at z (in the domain of definition of η) the following holds:

i) If $\nabla \eta(z) = 0$, then it holds

$$\eta_t(z) \leq 0 \tag{61}$$

ii) If $\nabla \eta(z) \neq 0$, then

$$\partial_t \eta(z) + \psi(z, -\nabla \eta(z)) \left(H(z, \nabla \eta(z), \nabla^2 \eta(z)) - f(z,t) \right) \leq 0. \tag{62}$$

A lower semicontinuous function $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ (in short, $u \in lsc(\mathbb{R}^N \times [0,T])$), constant outside a compact set, is a viscosity supersolution of the Cauchy problem (60) if $u(\cdot,0) \geq u_0$ and for all $z := (x,t) \in \mathbb{R}^N \times [0,T]$ and all $C^\infty-$test functions η such that η is admissible at z and $u - \eta$ has a minimum at z (in the domain of definition of η) the following holds:

i) If $\nabla\eta(z) = 0$, then $\eta_t(z) \geq 0$;
ii) If $\nabla\eta \neq 0$ then

$$\partial_t\eta(z) + \psi(z, -\nabla\eta(z))\left(H(z, \nabla\eta(z), \nabla^2\eta(z)) - f(z,t)\right) \leq 0.$$

Finally, a function u is a viscosity solution for the Cauchy problem (60) if it is both a subsolution and a supersolution of (60).

Remark. By classical arguments, one could assume that the maximum of $u - \eta$ is strict in the definition of subsolution above (an analogous remark holds for supersolutions).

Remark. We remark that, if $-u$ is a subsolution to (60) with initial datum $-u_0$, then u is a supersolution for (60) for the initial datum u_0 and where ϕ, ψ are replaced by $\tilde{\phi}, \tilde{\psi}$ respectively, as defined in Section 4.1.

We will first prove existence for viscosity solutions of (60) via an approximation-in-time technique, and then prove uniqueness of solutions to (60) to link the approximate solution to the mean curvature flow equation. We would like to proceed with the classical construction of e.g. [11, 14, 21], but in our case the lack of continuity of the evolving functions forces us to be particularly careful with the procedure.

We use the shorthand notation of *lsc* for lower semicontinuous and *usc* for upper semicontinuous. Given a bounded, *usc* function v which is constant outside a compact set, we define the transformation

$$T^+_{h,t}v(x) = \sup\left\{s \ : \ x \in T^+_{h,t}\{v \geq s\}\right\}. \tag{63}$$

Firstly, we see that $T^+_{h,t}v(x) \in \mathbb{R}$, as v is bounded. Moreover, it turns out that the function $T^+_{h,t}v$ is *usc*, bounded and constant outside a compact set. Indeed, definition (63) is equivalent to

$$T^+_{h,t}v(x) = \inf\left\{s \ : \ x \notin T^+_{h,t}\{v \geq s\}\right\} = \inf_{s\in\mathbb{R}}\left(s + \mathbb{1}_{\left(T^+_{h,t}\{v\geq s\}\right)^c}(x)\right),$$

where $\mathbb{1}_A(x)$ is the indicatrix function of a set A, being 0 on the set and $+\infty$ outside. By definition, $\mathbb{1}_A$ is an *usc* function for any open set A. Thus, recalling Remark 3.4, in the equation above we are taking the infimum of a family of *usc* functions, which is then a *usc* function. The other two properties follows from the previous study of the discrete evolution. Analogously, given a bounded *lsc* function g, we define

$$T^-_{h,t}g(x) = \sup\left\{s \ : \ x \in T^-_{h,t}\{g > s\}\right\} = \sup_{s\in\mathbb{R}}\left(s - \mathbb{1}_{T^-_{h,t}\{g>s\}}\right), \tag{64}$$

which is now a bounded *lsc* function (as sup of *lsc* functions), constant outside a compact set.

We are now ready to give the definition of the discrete-in-time approximations of sub - and super solution to (60). Given an initial compact set E_0, set u_0 as a (uniformly) continuous function, spatially constant outside a compact set, such that $\{u_0 \geq 0\} = E_0$. We remark that for every $s \in \mathbb{R}$, the superlevel set $\{u_0 \geq s\}$ is either compact or it is unbounded with compact boundary. Then, for $h > 0$ we introduce the following family of maps as $u^\pm_h(\cdot,t) = u_0$ for $t \in [0,h)$ and

$$u^\pm_h(\cdot,t) := T^\pm_{h,t-h}u^\pm_h(\cdot,t-h) \quad \text{for } t \geq h. \tag{65}$$

We easily see that the maps above are functions (as implied by the comparison principle contained in Lemmas 3.2, 4.1 and 4.4) piecewise constant in time (as $T^\pm_{h,t} = T^\pm_{h,[t/h]h}$). Moreover, by the previous remarks, we have that $u^+_h(\cdot,t)$ is an *usc* function, while $u^-_h(\cdot,t)$ is a *lsc* function, for every $t \in [0,+\infty)$. Some further properties of the approximating scheme are listed below.

Lemma 4.7. *For any $h > 0$, $t \geq 0$ we have the following. It holds*

$$u_h^-(\cdot, t) \leq u_h^+(\cdot, t). \tag{66}$$

Furthermore, given any $\lambda \in \mathbb{R}$ and $t \geq h$ it holds

$$\begin{aligned} &\{u_h^+(\cdot, t) > \lambda\} \subseteq T_{h,t-h}^+\{u_h^+(\cdot, t-h) \geq \lambda\} \subseteq \{u_h^+(\cdot, t) \geq \lambda\} \\ &\{u_h^-(\cdot, t) > \lambda\} \subseteq T_{h,t-h}^-\{u_h^-(\cdot, t-h) > \lambda\} \subseteq \{u_h^-(\cdot, t) \geq \lambda\}. \end{aligned} \tag{67}$$

Proof. Fix $x \in \mathbb{R}^N$, $t \in [0, h)$. For any given $\sigma < u_h^-(x, h)$ we have that there exists a sequence $(s_n) \nearrow \sigma$ so that $x \in T_{h,t-h}^-\{u_0 > s_n\} \subseteq T_{h,t-h}^+\{u_0 \geq s_n\}$. Thus, $u_h^+(x, t) \geq \sigma$. We then conclude by induction. Then, (67) follows easily by the definition (65). □

We then prove that the half-relaxed limits (in the spirit of [6], see also the references therein) of the families of functions $u_h^\pm$

$$\begin{aligned} u^+(x, t) &:= \sup_{(x_h, t_h) \to (x, t)} \limsup_{h \to 0} u_h^+(x_h, t_h) \\ u^-(x, t) &:= \inf_{(x_h, t_h) \to (x, t)} \liminf_{h \to 0} u_h^-(x_h, t_h), \end{aligned} \tag{68}$$

are (respectively) sub - and supersolutions in the viscosity sense of (60), see Theorem 1.3 (note that, by definition, u^+ is *usc*, while u^- is *lsc*). The proof of this result is the subject of the following section and we recall that the hypothesis required are (H0), (H1) and $f \in C^0(\mathbb{R}^N \times [0, \infty))$ only. Once the existence of sub - and super-solutions to the equation is settled, we need to properly define the notion of level-set solution to the mean curvature flow. To do so, we first prove uniqueness for (60) via a comparison principle and under additional hypothesis. Then, we show that the evolution of the zero superlevel set of the solution does not depend on the choice of the initial function u_0.

We start with a comparison result between u^+, u^- and u_0 at the initial time: it will ensure that the classical hypothesis for the comparison principle are satisfied. We first prove an estimate for the speed of decay of the level sets of the evolving functions. While it will only be needed in the following section, in the proof of the forthcoming Lemma 4.9 we will use similar techniques, so we preferred to state it here.

Lemma 4.8. *Let $u^+(x, t)$ be the function defined in* (68)*, let $\sigma \in \mathbb{R}$. Assume that, for a suitable x_0 and $R > 0$, it holds $B(x_0, R) \subseteq \{u^+(\cdot, t_0) \geq \sigma\}$. Then, there exists $C = C(R, \phi, \psi, f)$ such that $B(x_0, R - C(t - t_0)) \subseteq \{u^+(\cdot, t) \geq \sigma\}$ for every $t \leq t_0 + R/(2C)$. An analogous statement holds for u^- by considering its open sublevel sets.*

Proof. We focus on the case $\{u^+(\cdot, t_0) \geq \sigma\}$ bounded, the other case being analogous. By assumption, for any $R_0 < R$, if h is small enough, we have $B(x_0, R_0) \subseteq \{u_h^+(\cdot, t_0) \geq \sigma\}$. Set $C = C(R_0/2, \phi, \psi, f)$ as the constant of Lemma 3.8. Let R_n be defined recursively following law (27), that is $R_{n+1} = R_n - Ch$, as long as $R_n \geq R_0/2$. By simple iteration we find that $R_n = R_0 - nCh$, as long as $R_n \geq R_0/2$, which can be ensured enforcing $h\,n \leq R_0/(2C)$. Therefore, for any $t \geq t_0$ such that $t - t_0 \leq R_0/(2C)$, we set $n = [(t - t_0)/h]$ and send $h \to 0$ to deduce (recalling also Lemma 3.2)

$$\{u^+(\cdot, t) \geq \sigma\} \supset B(x_0, R_0 - C(t - t_0)).$$

Since the choice of R_0 is arbitrary, we conclude. □

We are now ready to prove a comparison result for the functions $u^\pm$ and a continuity estimate at the initial time $t = 0$.

Lemma 4.9. *For any $(x, t) \in \mathbb{R}^N \times [0, +\infty)$ it holds*

$$u^-(x, t) \leq u^+(x, t).$$

Moreover $u^-(\cdot,0) = u^+(\cdot,0) = u_0$, *so that there exists a modulus of continuity* ω *such that* $\forall x, y \in \mathbb{R}^N$

$$u^+(x,0) - u^-(y,0) \leq \omega(|x-y|).$$

Proof. The proof of the first inequality essentially follows from (66) and the definition of $u^\pm$. To prove the equality at the initial time $t=0$, we start by remarking that $u^+(\cdot,0) \geq u_0$ as can be seen taking sequences of the form $(x_h, 0)$ in (68). Then, consider ω as a continuous, strictly increasing modulus of continuity for u_0. We can also see that $\forall \varepsilon > 0$ $\{u_0 \leq u_0(x) + \varepsilon\} \supseteq B(x, \omega^{-1}(\varepsilon))$ by uniform continuity. Thus, reasoning iteratively as in Lemma 4.8 and using (67), we obtain that there exists $h_0(\varepsilon)$ such that $\forall h \leq h_0$ it holds

$$\{u_h^+(\cdot,t) \leq u_h^+(x,0)+\varepsilon\} \supseteq \left(T^+_{h,t-h}\{u_0 > u_0(x)+\varepsilon\}\right)^c = T^-_{h,t-h}\{u_0 \leq u_0(x)+\varepsilon\} \supseteq B(x, \omega^{-1}(\varepsilon/2)),$$

as long as $t \leq (\omega^{-1}(\varepsilon) - \omega^{-1}(\varepsilon/2))/(2C) =: t_\varepsilon$, and where we recalled that $u_h^\pm(\cdot,0) = u_0$. Now, fix $\sigma > 0, x \in \mathbb{R}^N$ such that $u(x,0) > \sigma$ and a sequence $(x_{h_k}, t_{h_k}) \to (x,0)$ such that $\lim_k u^+_{h_k}(x_{h_k}, t_{h_k}) > \sigma$. Then, for k large enough $(x_{h_k}, t_{h_k}) \in B(x, \omega^{-1}(\varepsilon/2)) \times [0, t_\varepsilon)$ and so we conclude

$$\sigma < \lim_k u_h^+(x_{h_k}, t_{h_k}) \leq u_0(x,0) + \varepsilon.$$

Letting $\varepsilon \to 0$ we conclude $u(\cdot,0)^+ \leq u_0$. The proof for u^- is essentially the same. The last claim follows from the previous one, recalling that ω is a modulus of uniform continuity for u_0. □

In order to prove a comparison principle for (60), we will need to assume (H3). Under these additional hypotheses, we are able to prove uniqueness for the parabolic Cauchy problem (60). The proof of this result follows from [26, Theorem 4.2]: we will just show in detail that the assumption of the aforementioned theorem hold in our case, following [8, Proposition 6.1] and [26, pag. 463].

Proof of Theorem 1.4. The proof of this result essentially follows from [26, Theorem 4.2], combined with the existence result of Theorem 1.3. Referring to the notation of [26], we firstly remark that in our case $\Omega = \mathbb{R}^N$, thus the parabolic boundary of $U = \Omega \times [0,T]$ is simply $\partial_p U = \mathbb{R}^N \times \{0\}$. Therefore, the initial conditions $(A1) - (A3)$ are all verified by Lemma 4.9. We then define the continuous Hamiltonian $F : [0,T] \times \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times M^{N\times N} \to \mathbb{R}$ as follows

$$F(t,x,p,X) := \psi(x,-p)\left(-\sum_i \partial_{x_i}\partial_{p_i}\phi(x,-p) + \nabla_p^2\phi(x,-p) : X + f(x,t)\right), \tag{69}$$

and focus on the conditions $(F1), (F3) - (F5), (F6'), (F7), (F9), (F10)$ that F must satisfy. The assumptions $(F1), (F3) - (F5), (F9)$ are easily checked. $(F6')$ follows from the Lipschitz regularity of ϕ and ψ, as $\forall t \in [0,T], x \in \mathbb{R}^N, |p| \geq \rho, |q| + |X| \leq R$ one has

$$\begin{aligned}
&|F(t,x,p,X) - F(t,x,q,X)| \leq c_\psi |p-q| \left| -\sum_i \partial_{x_i}\partial_{p_i}\phi(x,-p) + \nabla_p^2\phi(x,-p) : X \right| \\
&+ \psi(x,-q)\left| -\sum_i \left(\partial_{x_i}\partial_{p_i}\phi(x,-p) - \partial_{x_i}\partial_{p_i}\phi(x,-q)\right) + \left(\nabla_p^2\phi(x,-p) - \nabla_p^2\phi(x,-q)\right) : X \right| \\
&\leq c_R|p-q|\left(1 + \frac{1}{|p|}\right) + c_R|p-q| \leq c_{R,\rho}|p-q|.
\end{aligned}$$

For $(F7)$, we remark that the first term in the parenthesis in (69) is $0-$homogeneous in p, while the second one is $(-1)-$homogeneous in p but $1-$homogeneous in X. Lastly, we sketch how to prove $(F10)$. Since it concerns the X-terms, we focus simply on

$$\nabla_p^2\phi(x,-p) : X = \operatorname{tr}\left(\nabla_p^2\phi(x-,p)\, X^T\right).$$

Multiplying by $\phi(x,-p)$, we rewrite $\phi(x,-p)\mathrm{tr}\left(\nabla_p^2\phi(x-,p)\,X^T\right) = \mathrm{tr}(A(x,-p)X^T)$, where $A = B - (\nabla_p\phi\otimes\nabla_p\phi)$, with B being the uniformly elliptic operator $\frac{1}{2}\nabla_p^2\phi^2$. We can then factorize $B = \tilde{L}\tilde{L}^T$, with $\tilde{L}$ being a nondegenerate, lower triangular matrix. Then, following the proof of [8, Proposition 6.1] and [26, pg. 463], we obtain $(F10)$. □

Once uniqueness is settled, one can finally define the notion of *level set* solution to the mean curvature flow as follows.

Definition 4.10. Let E_0 be a compact initial set. Define a uniformly continuous, bounded function $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that $\{u_0 \geq 0\} = E_0$. Then, let $u : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ be the unique continuous viscosity solution to (60) given by Theorem 1.4. Then, the family $E_t := \{u^+(\cdot,t) \geq 0\}_{t\geq 0}$ will be called the level set solution to the mean curvature flow.

This definition is well posed since the Hamiltonian defined in (59) satisfies the so-called *geometricity condition*. Namely, one can easily check that for any $\lambda \neq 0, p \in \mathbb{R}^N \setminus 0, q \in \mathbb{R}^N$ and any symmetric $N\times N$ matrix X one has

$$H(x,\lambda p,\lambda X + p\otimes q + q\otimes p) = \frac{\lambda}{|\lambda|}H(x,p,X).$$

Thus, one can prove by classical arguments (see e.g. [14, Remark 3.9]) the following result.

Lemma 4.11. *Let $u_0, \tilde{u}_0$ two initial data for* (60) *such that $\{u_0 \geq 0\} = \{\tilde{u}_0 \geq 0\}$. Then, denoting by $u, \tilde{u}$ the corresponding solutions to* (60)*, one has*

$$\{u(\cdot,t)\geq 0\} = \{\tilde{u}(\cdot,t)\geq 0\} \quad \text{for all } t\in[0,T],$$

and the same identity holds for the open superlevel sets.

4.3. **Proof of Theorem 1.3.** In this section we will prove that the limiting functions $u^{\pm}$ are respectively a viscosity sub - and supersolutions to (60). We remark that we work assuming (H0), (H1) and that $f \in C^0(\mathbb{R}^N \times [0,+\infty))$. We will be following the structure of the proof of [14, Theorem 6.16], but taking into account the weaker definition of u^+ holding in our case. We will be using the O, o notations with respect to $h \to 0$ and focus on proving that u^+ is a subsolution. The proof for u^- is analogous.

Proof of Theorem 1.3. Consider u^+ as defined in (8): we need to prove that it is a subsolution. In the following, we will denote $u := u^+$ and $u_h := u_h^+$. Let $\eta(x,t)$ be an admissible test function in $\bar{z} := (\bar{x},\bar{t})\in \mathbb{R}^N \times (0,T)$ and assume that $(\bar{x},\bar{t})$ is a strict maximum point for $u-\eta$. Assume furthermore that $u-\eta = 0$ in such point. We need to show that either (61) or (62) holds at $\bar{z}$.
Case 1. Let us first assume that $\nabla\eta(\bar{z}) \neq 0$. By classical arguments, we can assume that $\bar{z}$ is a strict maximum point and that η is smooth. By the definition of u, there exists a sequence $\tilde{z}_k := (\tilde{x}_{h_k},\tilde{t}_{h_k}) \to \bar{z}$ such that $\lim_k u_{h_k}(\tilde{z}_k) = u(\bar{z})$. We remark that we can substitute the functions u_{h_k} for $t>0$ with their *usc* envelope in time, without changing the value of u. Indeed, the *usc* envelope of u_{h_k} is the function at all discrete times lh_k is given by

$$\max\{u_{h_k}(\cdot,(l-1)h_k), u_{h_k}(\cdot,lh_k)\}$$

and coincides with u_{h_k} elsewhere. Since now u_{h_k} is *usc* in time and space, by standard arguments (compare e.g. [5, Lemma 6.1]), there exists a radius $\rho > 0$ such that all functions $u_{h_k} - \eta$ achieve a local maximum in $B_\rho(\bar{z})$ at points $z_k = (x_k,t_k)$. Then, passing to a further subsequence we can ensure that $z_k \to w \in B_\rho(\bar{z})$, and we use the definition of u to obtain

$$(u-\eta)(w) \geq \limsup_k (u_{h_k}-\eta)(z_k) \geq \limsup_k (u_{h_k}-\eta)(\tilde{z}_k) = (u-\eta)(\bar{z}).$$

Therefore, $w=\bar{z}$ by maximality. Thus we can assume that each function $u_{h_k}-\eta$ achieves a local maximum in $B_\rho(\bar{z})$ at a point $z_{h_k} =: (x_k,t_k)$ and that $u_{h_k}(z_{h_k}) \to u(\bar{z})$ as $k\to\infty$. Finally, we can assume also that $\nabla\eta(x_k,t_k)\neq 0$ for k large enough.

Step 1. We start defining an appropriate set which is then used as a competitor for the minimality of the level sets of the functions u_h. From the previous computations, one has in particular that

$$u_h(x,t) \le \eta(x,t) + c_k \tag{70}$$

where $c_k := u_{h_k}(x_k,t_k) - \eta(x_k,t_k)$, with equality if $(x,t) = (x_k,t_k)$. Let $\sigma > 0$ and set

$$\eta^\sigma_{h_k}(x) := \eta(x,t_k) + c_k + \frac{\sigma}{2}|x - x_k|^2.$$

Then, for all $x \in \mathbb{R}^N$,

$$u_{h_k}(x,t_k) \le \eta^\sigma_{h_k}(x)$$

with equality if and only if $x = x_k$. We set $l_k = u_{h_k}(x_k,t_k) = \eta^\sigma_{h_k}(x_k)$. We fix $\varepsilon > 0$, to be chosen later, and write $E_{\varepsilon,k} := \{u_{h_k}(\cdot, t_k - h_k) \ge l_k - \varepsilon\}$. We define[1]

$$W_\varepsilon := \left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon\right\}. \tag{71}$$

We immediately see that $W_\varepsilon \to \{x_k\}$ in the Kuratowski sense as $\varepsilon \to 0$ since by (67)

$$\{u_{h_k}(\cdot,t_k) > l_k - \varepsilon\} \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon\right\} \subseteq W_\varepsilon \subseteq \{u_{h_k}(\cdot,t_k) \ge l_k - \varepsilon\} \setminus \left\{\eta^\sigma_{h_k}(\cdot) > l_k + \varepsilon\right\}, \tag{72}$$

see also (78) below. Then, we check that $|W_\varepsilon| > 0$ for all ε small enough. By the continuity of η^σ and $|\nabla \eta(\bar z)| \ne 0$, for any ε there exist a radius r_ε such that $W_\varepsilon \supseteq B(x_k, r_\varepsilon) \cap T^+_{h,t_k-h_k} E_{\varepsilon,k}$. Furthermore, for any $\varepsilon > 0$, using (67) again yields $x_k \in T^+_{h_k,t_k-h_k}\{u_{h_k}(\cdot, t_k - h_k) \ge l_k - \varepsilon\}$, and the latter set coincides with the closure of its points of density 1 by Lemma 3.3. Thus, x_k satisfies lower density estimates and so we conclude that $|W_\varepsilon| > 0$. Now, assume $E_{\varepsilon,k}$ is bounded. By minimality we have

$$\begin{aligned} &P_\phi(T^+_{h,t_k-h_k} E_{\varepsilon,k}) + \frac{1}{h_k} \int_{T^+_{h,t_k-h_k} E_{\varepsilon,k}} \mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x + \int_{W_\varepsilon} F_{h_k}(x, t_k - h_k)\,\mathrm{d}x \\ &\le P_\phi\left(\left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \cap \{\eta^\sigma_{h_k} > l_k + \varepsilon\}\right) + \frac{1}{h_k} \int_{\left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \cap \{\eta^\sigma_{h_k} > l_k\}} \mathrm{sd}^\psi_{E_{\varepsilon,k}}. \end{aligned} \tag{73}$$

Adding to both sides the term $P_\phi\left(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup T^+_{h,t_k-h_k} E_{\varepsilon,k}\right)$ and using the submodularity (12), we obtain

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\}) + \frac{1}{h_k} \int_{W_\varepsilon} \mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x \\ + \int_{W_\varepsilon} F_{h_k}(x, t_k - h_k)\,\mathrm{d}x \le 0. \end{aligned}$$

By (70), $\{u_{h_k}(\cdot, t_k - h_k) \ge l_k - \varepsilon\} \subseteq \{\eta(\cdot, t_k - h_k) \ge l_k - c_k - \varepsilon\}$, therefore it holds

$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\}) + \frac{1}{h_k} \int_{W_\varepsilon} \mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k) \ge l_k - c_k - \varepsilon\}}(x)\,\mathrm{d}x \\ + \int_{W_\varepsilon} F_{h_k}(x, t_k - h_k)\,\mathrm{d}x \le 0. \end{aligned} \tag{74}$$

If instead $E_{\varepsilon,k}$ is an unbounded set with compact boundary, we replace inequality (73) by

$$\begin{aligned} &P_\phi(T_{h,t_k-h_k} E_{\varepsilon,k}) + \frac{1}{h_k} \int_{\left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \cap B_R} \mathrm{sd}^\psi_{E_{\varepsilon,k}}(x)\,\mathrm{d}x + \int_{W_\varepsilon} F_{h_k}(x, t_k - h_k)\,\mathrm{d}x \\ &\le P_\phi(\left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \cap \{\eta^\sigma_{h_k} > l_k + \varepsilon\}) + \frac{1}{h_k} \int_{\left(T^+_{h,t_k-h_k} E_{\varepsilon,k}\right) \cap \{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cap B_R} \mathrm{sd}^\psi_{E_{\varepsilon,k}}, \end{aligned}$$

[1] We need to define the sets W_ε in this way (compare the different definition in [14]) since firstly, we can not rule out that the inclusions in (72) are strict, and secondly it is not clear if otherwise $|W_\varepsilon| > 0$.

for $R > 0$ sufficiently large, see (58). Then, one can argue as before to obtain (74).

Step 2. We estimate the first two terms in (74). The quantity $P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\})$ can be estimated as done in Lemma 3.8. Indeed, we consider the vector field $v = \nabla_p \phi(x, \nabla \eta^\sigma_{h_k})$ in (13) and we use the divergence theorem to get

(75)
$$\begin{aligned} P_\phi(\{\eta^\sigma_{h_k} > l_k + \varepsilon\} \cup W_\varepsilon) - P_\phi(\{\eta^\sigma_{h_k} \geq l_k + \varepsilon\}) &\geq \int_{\partial(\{\eta^\sigma_{h_k} > l_k+\varepsilon\} \cup W_\varepsilon)} v \cdot \nu - \int_{\partial\{\eta^\sigma_{h_k} > l_k+\varepsilon\}} v \cdot \nu \\ &= |W_\varepsilon| \fint_{W_\varepsilon} \operatorname{div} v, \end{aligned}$$

where ν denotes the unit outer vector to the set we are integrating on. We then remark that $\fint_{W_\varepsilon} \operatorname{div} v \to H^\phi_{\{\eta^\sigma_{h_k} > l_k\}}(x_k)$ and $\fint_{W_\varepsilon} F_{h_k}(x, t_k - h_k)\,\mathrm{d}x \to F_{h_k}(x_k, t_k - h_k)$ as $\varepsilon \to 0$ by continuity.

Step 3. We bound the distance term in (74) by showing that

$$\frac{1}{h_k}\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) \geq \frac{\partial_t \eta(z, t_k) - O(h_k)}{\psi(y, -\nabla\eta(y, t_k - h_k)) + O(h_k)}. \tag{76}$$

For any $z \in W_\varepsilon$, we have

$$\eta(z, t_k) + c_k + \frac{\sigma}{2}|z - x_k|^2 \leq l_k + \varepsilon. \tag{77}$$

Since, in turn, $\eta(z, t_k) + c_k > l_k - \varepsilon$ it follows that $\sigma|z - x_k|^2 < 4\varepsilon$ and thus, for ε small enough,

$$W_\varepsilon \subseteq B_{c\sqrt{\varepsilon}}(x_k). \tag{78}$$

By a Taylor expansion, for every $z \in W_\varepsilon$ we have

$$\eta(z, t_k - h_k) = \eta(z, t_k) - h_k \partial_t \eta(z, t_k) + h_k^2 \int_0^1 (1-s)\partial^2_{tt}\eta(z, t_k - sh_k)\,\mathrm{d}s. \tag{79}$$

Then, we consider $y, y_e \in \{\eta(\cdot, t_k - h_k)(y) = l_k - c_k - \varepsilon\}$ being respectively, a point of minimal $\psi-$distance and Euclidean distance from z.

Claim: We claim that it holds

$$|z - y| = O(h_k). \tag{80}$$

In order to prove this result, we start remarking that for $k \to \infty$ and choosing $\varepsilon \ll h_k$, one has $\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\geq l_k-c_k-\varepsilon\}}(z) \to 0$ (as $z \to x_k$ for $\varepsilon \to 0$ and $x_k \in \{\eta(\cdot, t_k) \geq l_k - c_k\}$). In particular, recalling the bounds (11) one has

$$|z - y_e| \leq c^2_\psi |z - y| \leq c^3_\psi |\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\geq l_k-c_k-\varepsilon\}}(z)| \to 0$$

as $k \to \infty$. By (77) we deduce in particular $\eta(z, t_k) + c_k < l_k + \varepsilon$, that is,

$$0 \leq \eta(z, t_k) - \eta(y, t_k - h_k) \leq 2\varepsilon, \tag{81}$$

and the same inequality substituting y_e to y. Thus, one has

$$\eta(z, t_k) - \eta(y_e, t_k - h_k) = \nabla\eta(y, t_k - h_k) \cdot (z - y_e) - h_k \partial_t \eta(y, t_k - h_k) + O(|z - y_e|^2 + h_k^2)$$

which we combine with $\nabla\eta(y, t_k - h_k) \cdot (z - y_e) = \pm|\nabla\eta(y, t_k - h_k)|\,|z - y_e|$ (see [14] for details) and (81) to get

$$|z - y_e|\,|\nabla\eta(y, t_k - h_k)| \leq 2\varepsilon + O(h_k) + O(|z - y_e|^2).$$

Recalling that $|\nabla\eta(y, t_k - h_k)| \geq c > 0$ for h_k small enough, we divide by $|\nabla\eta(y, t_k - h_k)|$ to conclude $|z - y_e| = O(h_k)$ as $\varepsilon \ll h_k$. Finally, employing again (11), we prove the claimed (80).

Then, we consider a geodesic curve for the definition of $\mathrm{sd}^\psi_{\{\eta(\cdot,t_k-h_k)\geq l_k-c_k-\varepsilon\}}(z)$: if this distance is positive, we choose $\gamma : [0, 1] \to \mathbb{R}^N$ with $\gamma(0) = z, \gamma(1) = y$, with y as before, otherwise we take

γ such that $\gamma(0) = y, \gamma(1) = z$. In the following, we will assume $\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)\geq l_k-c_k-\varepsilon\}}(z) > 0$, the other case being analogous. Recalling (9), we have

$$\begin{aligned}
\eta(z,t_k-h_k) &= \eta(y,t_k-h_k) + \int_0^1 \nabla\eta(\gamma,t_k-h_k)\cdot\dot\gamma\,\mathrm{d}t \\
&\geq \eta(y,t_k-h_k) - \int_0^1 \psi(\gamma,-\nabla\eta(\gamma,t_k-h_k))\psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \\
&\geq \eta(y,t_k-h_k) - \psi(y,-\nabla\eta(y,t_k-h_k))\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) \\
&\quad - \int_0^1 \left(\psi(\gamma,-\nabla\eta(\gamma,t_k-h_k)) - \psi(y,-\nabla\eta(y,t_k-h_k))\right)\psi^\circ(\gamma,\dot\gamma)\,\mathrm{d}t \\
&\geq \eta(y,t_k-h_k) - \big(\psi(y,-\nabla\eta(y,t_k-h_k)) + c|z-y|\big)\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z),
\end{aligned}$$

where in the last line we reasoned as in (47) to obtain the bound $\sup_t|\gamma(t)-y| \leq c|z-y|$. Recalling (80) one has

$$\eta(z,t_k-h_k) \geq \eta(y,t_k-h_k) - \psi(y,-\nabla\eta(y,t_k-h_k))\,\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) + o(h_k). \tag{82}$$

Combining (79) with (82) and using (81), we deduce

$$\begin{aligned}
&\mathrm{sd}^{\psi}{}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z)\,\psi(y,-\nabla\eta(y,t_k-h_k)) + o(h_k) \\
&\qquad\geq -2\varepsilon + h_k\partial_t\eta(z,t_k) - h_k^2\int_0^1(1-s)\partial^2_{tt}\eta(z,t_k-sh_k)\,\mathrm{d}s.
\end{aligned}$$

Note that, in view of (77) and (11), $|\eta(z,t_k)-\eta(y,t_k)| \leq c\varepsilon + ch_k = O(h_k)$, provided $\varepsilon \ll h_k$ and small enough. We then conlude (76) by combining the previous inequality with (78),(80) as

$$\begin{aligned}
\frac{1}{h_k}\mathrm{sd}^{\psi}_{\{\eta(\cdot,t_k-h_k)=l_k-c_k-\varepsilon\}}(z) &\geq \frac{\partial_t\eta(z,t_k) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(y,-\nabla\eta(y,t_k-h_k))} \\
&= \frac{\partial_t\eta(x_k,t_k) + O(\sqrt{\varepsilon}) - \frac{2\varepsilon}{h_k} - O(h_k) - O_{h_k}(1)}{\psi(x_k,-\nabla\eta(x_k,t_k-h_k)) + O(\sqrt{\varepsilon}) + O(h_k)}.
\end{aligned}$$

Step 4. We conclude the proof by employing (74), (75) and (76), dividing by $|W_\varepsilon|$ and sending $\varepsilon \to 0$ to obtain

$$\frac{\partial_t\eta(x_k,t_k) - O_{h_k}(1)}{\psi(x_k,-\nabla\eta(x_k,t_k)) + O(h_k)} + H^{\phi}_{\{\eta^\sigma_{h_k}\geq\eta^\sigma_{h_k}(x_k)\}}(x_k) - F_{h_k}(x_k,t_k-h_k) \leq 0.$$

Letting simultaneously $\sigma \to 0$ and $k \to \infty$, recalling the continuity properties of H^ϕ, we deduce (62). Indeed the sets $\{\eta^\sigma_{h_k} > \eta^\sigma_{h_k}(x_k)\}$ are converging in C^2 to the set $\{\eta > \eta(x)\}$, $x_k \to x$ and thus

$$H^{\phi}_{\{\eta^\sigma_{h_k}>\eta^\sigma_{h_k}(x_k)\}}(x_k) \to H^{\phi}_{\{\eta>\eta(x)\}}(x),$$

and we conclude the proof of this step.

Case 2. Now we consider the case $\nabla\eta(\bar x,\bar t) = 0$ and we show that $\partial_t\eta(\bar x,\bar t) \leq 0$. The proof follows the line of the one in [14], we just highlight the differences.

Since $\nabla\eta(\bar z) = 0$, there exist $a, b > 0$ such that

$$|\eta(x,t) - \eta(\bar z) - \partial_t\eta(\bar z)(t-\bar t)| \leq a|x-\bar x|^3 + b|t-\bar t|^2,$$

thus, we can define

$$\begin{aligned}
\tilde\eta(x,t) &= \partial_t\eta(\bar z)(t-\bar t) + 2a|x-\bar x|^3 + 2b|t-\bar t|^2 \\
\tilde\eta_k(x,t) &= \tilde\eta(x,t) + \frac{1}{k(\bar t - t)}.
\end{aligned}$$

We remark that $u-\tilde{\eta}$ achieves a strict maximum in $\bar{z}$ and the local maxima of $u-\tilde{\eta}_k$ in $\mathbb{R}^N\times[0,\bar{t}]$ are in points $(x_k,t_k)\to\bar{z}$ as $k\to\infty$, with $t_n\le\bar{t}$. From now on, the only difference from [14] is in the case $x_k=\bar{x}$ for an (unrelabeled) subsequence. We assume $x_k=\bar{x}$ $\forall k>0$ and define $b_k=\bar{t}-t_k>0$ and the radii

$$r_k:=2\sqrt{Cb_k},$$

where C is the constant of Lemma 4.8. Taking k large enough, by Lemma 4.8 the balls $B(\cdot,r_k)$ have an extinction time greater than $2(\bar{t}-t_k)$. We then have

$$\begin{aligned}B(\bar{x},r_k)&\subseteq\{\tilde{\eta}_k(\cdot,t_k)\le\tilde{\eta}_k(\bar{x},t_k)+2ar_k^3\}\\&\subseteq\{u(\cdot,t_k)\le u(\bar{x},t_k)+2ar_k^3\},\end{aligned}$$

by maximality of $u-\tilde{\eta}_k$ at z_k. Since the balls $B(\cdot,r_k)$ are not vanishing, we conclude

$$\bar{x}\in\{u(\cdot,\bar{t})\le u(\bar{x},t_k)+2ar_k^3\}.$$

Finally, we use again the maximality of $u-\eta$ at $\bar{z}$ and the choice of r_k to obtain

$$\frac{\eta(\bar{x},t_k)-\eta(\bar{z})}{t_k-\bar{t}}=\frac{\eta(\bar{x},t_k)-\eta(\bar{z})}{-b_k}\le\frac{u(\bar{x},t_k)-u(\bar{x},\bar{t})}{-b_k}\le\frac{-2ar_k^3}{-b_k}=c\sqrt{b_k}.$$

Passing to the limit $k\to\infty$, we conclude that $\partial_t\eta(\bar{z})\le 0$. □

We conclude with two remarks concerning some possible generalizations of the results presented.

Remark 4.12. The results presented in this work can be immediately extended to unbounded initial open sets E_0, whose boundary is compact. Indeed, defining the discrete flow as $E_t^{(h)}=E_0$ if $t\in[0,h)$, otherwise by induction $E_t^{(h)}=T^-_{h,t}E^{(h)}_{t-h}$, where the operator $T^-_{h,n}$ is the one defined in (56), this evolution is uniquely characterized by the one of the complement. Thus, all the results presented in this paper can be extended to this particular unbounded case.

Remark 4.13. Following the lines of [8] (in the spirit of [3]) one can see that the results of this paper may be extended to prove existence of flat flows and level set solutions to the mean curvature flow on $\mathbb{R}^N$ endowed with the geometric structure induced by a Finsler metric ϕ°. For example, the perimeter functional in this setting is defined as follows. Given a set E of finite perimeter, its (intrinsic) perimeter is

$$\mathcal{P}_{\phi^\circ}(E)=\int_{\partial^*E}\phi(x,\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}_{\phi^\circ}(x),$$

where the Hausdorff measure $\mathcal{H}^{N-1}_{\phi^\circ}$ is the one induced by the metric ϕ°. In particular, one can compute $\mathrm{d}\mathcal{H}^{N-1}_{\phi^\circ}(x)=\omega_N|B^{\phi^\circ}(x)|^{-1}\,\mathrm{d}\mathcal{H}^{N-1}(x)$ (see [8]), thus this approach is equivalent to consider in our framework a slightly different (but still regular) anisotropy, namely $\phi^*(x,\nu):=\omega_N|B^{\phi^\circ}(x)|^{-1}\phi(x,\nu)$. In particular, this approach leads to considering the evolution of hypersurfaces E_t moving according to the evolution law

$$V_{\phi^\circ}(x,t)=-\mathcal{H}_{E_t}(x)+f(x,t)\quad x\in\partial E_t,\ t\in(0,T)$$

where now V_{ϕ° represents the speed of evolution along the anisotropic normal outer vector $n_{\phi^\circ}(x)=\nabla_p\phi(x,\nu_E(x))$ and $\mathcal{H}$ is the "intrinsic" mean curvature, thus the first variation of the perimeter $\mathcal{P}_{\phi^\circ}$. Recalling that $n_{\phi^\circ}(x)\cdot\nu_E(x)=\phi(x,\nu_E(x))$, we see that the hypersurfaces are evolving with a normal (in the Euclidean sense) velocity given by the law

$$V(x,t)=\phi(x,\nu_{E_t}(x))\left(-H^{\phi^*}_{E_t}(x)+f(x,t)\right).$$

After this transformation, we can apply the results previously proved.

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